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Cite this article: Keeler JS, Humphries BS, Alberello A, Părău E. 2025 Parameter-free higher-order Schrödinger systems with weak dissipation and forcing. *Proc. R. Soc. A* **481**: 20240967.
https://doi.org/10.1098/rspa.2024.0967

Received: 18 December 2024
Accepted: 3 June 2025

Subject Areas:
mathematical modelling, applied mathematics, differential equations

Keywords:
nonlinear waves, nonlinear Schrödinger equation, Dysthe equation, dispersive waves, envelope equations, dissipation

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Parameter-free higher-order Schrödinger systems with weak dissipation and forcing

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The higher-order nonlinear Schrödinger equation (NLS) (Dysthe's equation in the context of water waves) models the time evolution of the slowly modulated amplitude of a wave packet in physical systems described by dispersive partial differential equations (PDEs). These systems, of which water waves are a canonical example, require the presence of a small-valued ordering parameter so that a multi-scale expansion can be performed. However, often the resulting system itself contains this parameter. Thus, these models are difficult to interpret from a formal asymptotics perspective. This article describes a procedure to derive a parameter-free, higher-order evolution equation for a generic infinite-dimensional dispersive PDE with weak linear damping and/or forcing. This is achieved by placing the PDE in an infinite-dimensional Hilbert space and Taylor expanding with Fréchet derivatives. An attractive feature of this procedure is that it can be used in many different physical settings, including water waves, nonlinear optics and any dispersive system with weak dissipation or forcing and does not assume any additional structure to the governing PDE, for example its Hamiltonian nature. To complement this, two specific examples with accompanying symbolic algebra code are demonstrated that can be used as a template for other physical systems.

1. Introduction

The nonlinear Schrödinger equation, first derived in [1], and the higher-order version, or, in the context of water waves, Dysthe's equation, first derived in [2], describes slowly varying wave envelopes and is immensely

Electronic supplementary material is available online at https://doi.org/10.6084/m9.figshare.c.7879885.

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successful in modelling an array of physical phenomena, including water waves [3] and optics [4]. These evolution equations are cornerstone achievements of nonlinear science and are still an incredibly active research field. A key feature of the 'standard' nonlinear Schrödinger equation (hereafter denoted NLS) and the higher-order Schrödinger equation (hereafter denoted HNLS) is that they are Hamiltonian systems (in the context of water waves, for the HNLS, this has only been shown very recently [5,6]). In such systems, there is no dissipation mechanism. Therefore, dynamical attractors or repellers cannot exist and, crucially, energy cannot be gained or lost. In some physical situations, where energy loss/gain is an important part of the physics, this is undesirable [7]. For example, in the evolution of ocean swell [8,9] and wind forcing of ocean waves [10–12], the dissipation results in a constant decay rate of the wave attenuation. Extending this, in ice-infested ocean waves, experimental evidence suggests that the energy of a wave-packet is actually damped at a rate that depends on the wave frequency [13] and in the context of wave propagation in an ice pack, [14] investigated the effect of introducing viscosity to the NLS. The classical NLS and HNLS, which in these contexts are derived from the inviscid, irrotational and incompressible free-surface Euler system [3], cannot capture this energy attenuation. For the system to dissipate energy, an ad hoc modification to the form of the NLS/HNLS is necessary as the Euler system is derived from the underlying Navier–Stokes system under the assumption of inviscid flow [15,16]. A rational argument of how the Navier–Stokes equations can lead to the free-surface Euler system with dissipation was proposed in [17], and an HNLS with dissipation was derived in [18] but it is unclear how this analysis can be extended to more general dispersive systems with weak dissipation and/or forcing.

In early models of a modified-NLS [8,9,15] and modified-HNLS [18], an extra linear, dissipation term was added. More recently, in light of striking evidence that the damping is frequency dependent [13], a spatially dependent, linear dissipation term was added to the NLS and HNLS to model sea-ice [12,19,20]. In these models, the equation is often simply stated without derivation and the coefficients often vary from study to study, even for identical physical situations (see for example, in the context of water waves, [1,11,21,22]). However, attempts to derive the equations from the fully nonlinear Euler system can be unwieldy, laborious, and prone to inadvertent algebraic mistakes, making derivations difficult to follow. In this article, we derive a modified-HNLS system for a *general* dispersive evolution partial differential equation (PDE) with weak dissipation/forcing. A key feature is that, in contrast to the literature, the small-ordering parameter is absent from the final form of the resulting HNLS; which as we shall explain in this article, provides an asymptotically consistent form of the envelope equations.

The literature on the NLS and HNLS in terms of water waves is vast, and we now highlight two of the approaches to derive them. One approach is to perform a direct asymptotic expansion of the governing equations in terms of powers of a small parameter and solve a linear problem at each order of ε. Truncating the analysis at three orders of magnitude in ε results in the NLS and going to four orders of magnitude results in the HNLS, e.g. [3]. The other, more recently proposed method, in the specific case of water waves, uses the Hamiltonian nature of the system and performs an expansion on the Hamiltonian so that the resulting HNLS is itself Hamiltonian, see [5].

In both of these above approaches, the small parameter ε is often employed as a *control* parameter, particularly in the HNLS (see [5,11]) rather than an asymptotic *ordering* parameter. Even if ε can be scaled out of the PDE by suitable variable transforms, from a formal asymptotic perspective, this is undesirable as ε should only be used to order the expansion, not be defined as a specified finite value. Furthermore, while the literature on the HNLS is well developed for water waves and, we emphasize, has been incredibly successful at modelling water-wave modulation, we have been unable to identify an analysis that (i) can systematically be implemented in other physical systems; (ii) results in an HNLS that does not contain ε; and (iii) does not necessarily rely on the Hamiltonian structure of the system and thus can explicitly contain dissipative/forcing terms.

With these three observations in mind, a fundamental aim of this article is to develop and describe a friendly 'user-guide' and systematic method to derive higher-order evolution equations for a general dispersive system. With regard to the three points made at the end of the last paragraph, the three main features of this article are to describe a systematic procedure that (i) can be applied to a wide range of dispersive evolution PDEs; (ii) does not contain ε as a control parameter; and (iii) contains weak dissipation/forcing so that a Hamiltonian structure is not assumed and necessary. We aim for the description of our approach to be as constructive and transparent as possible. In direct asymptotic expansions, one often relies on the scaling of dependent/independent variables using the small parameter, ε. Sometimes in the literature, these scalings are made at the outset, and for the uninitiated reader, can appear mysterious and ad hoc. In this article, we make no *a priori* assumptions on these scalings; rather, they are transparently motivated as part of the analysis when the expansion 'goes wrong'. Therefore, we believe that this article will be a valuable tool for future derivations of modulation equations in other physical contexts and that an article of this type is long overdue in the nonlinear science community.

It is important to observe from the outset that in the literature, the nomenclature of higher-order NLS models and so-called 'modified' NLS models have become convoluted (for example the modified NLS in [16] refers to the damping, whereas in [2], it refers to the higher-order terms). To avoid any confusion, in this article, we shall use the term modified-HNLS model, where 'modified' refers to the dissipation/forcing term (similar to the NLS in [16]).

(a) Current approaches

Before proceeding, we now describe our broad strategy and the current approaches in the literature. Despite the success of the NLS (and associated systems) in describing the wave amplitude for water waves, from a formal asymptotic perspective, an undesirable aspect of the current models in the literature is the presence of a small, real parameter $\varepsilon \in \mathbb{R}$, where $|\varepsilon| \ll 1$. We highlight that ε is highly problem dependent and could also represent different physical quantities depending on the non-dimensionalization of the governing system. For example, in [23], ε represents the closeness of a parameter to a Hopf bifurcation. By contrast, in surface gravity waves, ε is usually defined as the wave steepness or wave amplitude [2,3]. However, there is no reason it cannot represent other small quantities, for example, in water waves ε could represent surface tension or dispersive effects; the analysis will yield different coefficients, yet the structure of the general envelope equations will remain the same.

To illustrate this approach, let $u(x,t)$ represent the solution of a nonlinear PDE in $O(1)$ time and space variables, x and t, written functionally as

$$\frac{\partial u}{\partial t} + \text{nonlinear operator }(u;\varepsilon) = 0, \tag{1.1}$$

where the nonlinear operator contains spatial derivatives and is dependent on ε. Usually, to derive the NLS and HNLS systems (and modified versions), a solution to equation (1.1) is proposed of the form

$$u = A(\xi,\tau) \times \text{periodic-function }(x,t) + \varepsilon \times \text{higher-order terms }(x,t,\xi,\tau), \tag{1.2}$$

where $A(\xi,\tau)$ is the complex-valued wave envelope depending on some slow space and time variables, ξ and τ (yet to be defined), and the periodic function is linear and periodic in both space and time, e.g. an exponential (figure 1). For the modified HNLS, the evolution of the wave-envelope, $A(\xi,\tau)$, is found by solving a nonlinear PDE of the form

$$\begin{aligned}\frac{\partial A}{\partial \tau} &+ \text{Schrödinger operator }(A) + \text{dissipative operator }(A)\\ &+ \varepsilon \times \left[\text{higher-order Schrödinger operator }(A) + \frac{\partial \text{ dissipative operator }(A)}{\partial x}\right] = 0,\end{aligned} \tag{1.3}$$

where the Schrödinger and higher-order Schrödinger operators contain linear spatial derivatives of A and nonlinear terms involving A and $\overline{A}$ (an over-bar indicates the complex conjugate) and the dissipation operator depends on the type of damping required [12,19,20]. In these traditional

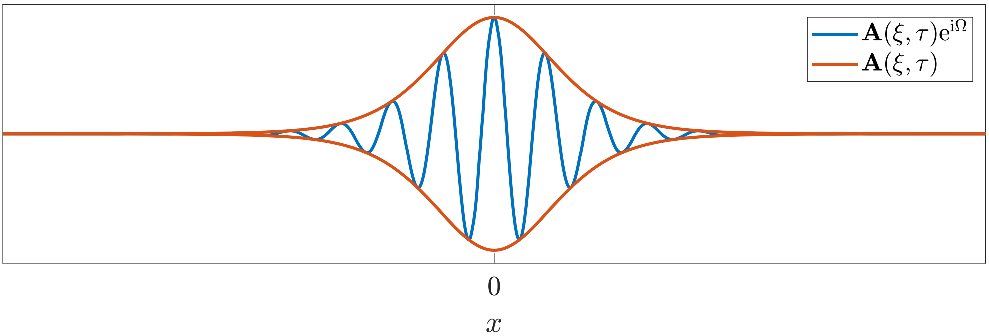

Figure 1. Sketch of a steadily translating wave packet, labelled $u = \mathbf{A}(\xi, \tau)e^{i\Omega} + \mathrm{c.c}$; $\Omega = kx - \omega t$, with oscillations on a fast time scale (blue) and an envelope, $A(\xi, \tau)$, varying on a slow space and time scale (red).

approaches, equation (1.3) reduces to the 'normal' nonlinear Schrödinger system, simply by setting $\varepsilon = 0$.

We repeat that, models of the form in equation (1.3) are extremely successful at modelling and predicting the evolution of the wave-packet [8,18] yet, from a purely formal asymptotics perspective, there is a troubling aspect of these equations in that the small-ordering parameter, ε, is present in the system as a coefficient of the higher-order Schrödinger operator. Furthermore, from a practical perspective, for numerical simulations, ε is a control parameter rather than an ordering parameter, which can result in unnecessary 'stiffness' as $\varepsilon \ll 1$ in the numerical implementation. Therefore, the aim of this article is to describe definite procedures for deriving *parameter-free*, higher-order evolution equations for a general dispersive system with weak dissipation.

(b) A parameter-free approach

We now sketch our general strategy to achieve a parameter-free modified HNLS. One way to achieve this is by introducing a succession of, potentially infinite, small time scales proportional to ε^n [24,25]. With the view of a numerical implementation, where the discretization of different time scales may be problematic, we expand the amplitude function instead, resulting in a coupled system of evolution equations. These are more amenable to numerical study as there is only one scale for the independent variable in the slow-time scale, τ. Our strategy involves expanding the amplitude function to higher orders so that instead of equation (1.2), we expand the solution using

$$u = [A(\xi,\tau) + \varepsilon B(\xi,\tau)\cdots] \times \text{periodic-function}\,(x,t) + \varepsilon \times \text{higher-order terms}\,(x,t,\xi,\tau), \tag{1.4}$$

where $B(\xi,\tau)$ is the slowly varying first-order envelope function. We show that an asymptotically consistent system for A and B, which is *parameter-free* is of the form

$$\frac{\partial A}{\partial \tau} + \text{linear derivative} + \text{nonlinear term} + \text{dissipation term} = 0 \tag{1.5}$$

and

$$\frac{\partial B}{\partial \tau} + \text{linear derivative} + \text{nonlinear terms} + \text{dissipation term} = 0. \tag{1.6}$$

This approach relies on the interpretation of an evolution PDE as an infinite-dimensional dynamical system where, instead of state variables belonging to $\mathbb{R}^d$, with d finite, the state variables, u, belong to an infinite-dimensional Hilbert space and is inspired by work where a similar formal asymptotic expansion was used in a problem involving the propagation of air-bubbles in a viscous fluid within a Hele–Shaw channel [23].

We proceed as follows. In §2, we devote the preliminary part of the article to describing the framework. Then, in §3, we proceed with a formal asymptotic analysis. First, we show how a naive asymptotic expansion fails and then provide an appropriate remedy. In our analysis, our main philosophy is to be constructive and transparent to the reader, so that, as well as deriving an asymptotically consistent form of the HNLS with dissipation/forcing, this article will also act as a user-friendly pedagogical guide. In §§4 and 5, we focus on the derivation from a toy system and the fully nonlinear water-wave problem, respectively. For the latter, the rationale for this is that although the evolution amplitude equations for water waves are well known in the case of simple geometries (infinite depth or flat finite depth) and for standard physics (i.e. gravity-capillary waves) [3], researchers may wish to determine the evolution equations for their particular problem, which may have different geometries (for example, a submerged body in the fluid or a variable bottom topography) and include different physics (for example, hydro-elastic waves and external pressure distributions). In both examples, we provide symbolic algebra MATLAB code that allows one to generate the coefficients efficiently and that can, in principle, be adapted to other toy PDEs and water-wave problems. Therefore, the secondary aim of this article is to provide a useful, user-friendly 'look-up' tool for researchers so they can make the necessary adjustments for their own physical models.

2. Preliminaries

We study the generic PDE for the evolution of u:

$$\frac{\partial u}{\partial t} + \mathcal{F}(u;\varepsilon) + \varepsilon^2 \mathcal{V}(u) = 0, \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}^+, \tag{2.1}$$

where $u(x,t) \in \mathcal{U}$, a Hilbert space, $\mathcal{F}:\mathcal{U} \mapsto \mathcal{U}$ is a nonlinear operator that depends on a parameter $\varepsilon \in \mathbb{R}$ and $\mathcal{V}:\mathcal{U} \mapsto \mathcal{U}$ is a linear dissipation operator. We note that $\mathcal{F}(u;\varepsilon)$ only contains spatial derivatives and in the rest of the article, for ease of exposition, we omit the explicit ε dependence from $\mathcal{F}$. We also assume that associated with equation (2.1) are the correct number of boundary conditions on u and spatial derivatives of u to make the system well-posed. We assume that $d=1$ (this will be relaxed in §5) and that when $\varepsilon = 0$, the linearized equation admits a real-valued dispersion relation. We start our analysis with several definitions involving Hilbert spaces and Fréchet derivatives and introduce a notation that simplifies the subsequent analysis. While at first glance, equation (2.1) may appear restrictive as only first-order time derivatives are present, it is possible to transform a PDE that is second order (for example, the D'Alembert or wave equation) in time to a system of PDEs that are first order in time and hence fall in the class of problems defined in equation (2.1). We will only seek bounded solutions to equation (2.1) that are not subject to secular growth.

To proceed, we first assume that $u = u_s(x)$ is a steady-state form of equation (2.1), independent of time, i.e.

$$\mathcal{F}(u_s) = 0. \tag{2.2}$$

The steady state, u_s, does not have to be identically zero; the analysis will hold for a generic steady base state u_s. Anticipating periodic solutions of equation (2.1) with temporal period $\tilde{T}$ and spatial period $\tilde{X}$, we define an inner product:

$$\langle u, v\rangle = \int_0^{\tilde{T}} \int_0^{\tilde{X}} u\bar{v}\,\mathrm{d}x\,\mathrm{d}t, \quad u, v \in \mathcal{U}. \tag{2.3}$$

(a) Taylor expansions in Hilbert spaces

An important feature of the subsequent analysis is the ability to Taylor expand the fully nonlinear operator, $\mathcal{F}$, within a Hilbert space (see, for example, §5.6 of [26]). We assume that $\mathcal{F}(u)$ is n-differentiable, in the Fréchet sense (cf. equation (2.5)), and admits a Taylor expansion about the steady base state, $u = u_s + v$, i.e.

$$\mathcal{F}(u) = \underbrace{\mathcal{F}(u_s)}_{=0} + D^1[\mathcal{F}(u_s)](v) + \frac{1}{2!}D^2[\mathcal{F}(u_s)](v,v) + \frac{1}{3!}D^3[\mathcal{F}(u_s)](v,v,v) + \cdots; \tag{2.4}$$

where the first-order Fréchet derivative is defined as

$$D^1[\mathcal{F}(u_s)](v_1) \equiv \lim_{h\to 0} \frac{\mathcal{F}(u_s + hv_1) - \mathcal{F}(u_s)}{h}. \tag{2.5}$$

Higher-order derivatives can be defined inductively;

$$D^n[\mathcal{F}(u_s)](v_1, \ldots, v_n) \equiv D[D^{n-1}(v_1, \ldots, v_{n-1})](v_n), \quad v_i \in \mathcal{U}. \tag{2.6}$$

The nth Fréchet derivative is an n-linear functional. For ease of exposition, we introduce specific symbols for the first four Fréchet derivatives about u_s as

$$\left.\begin{aligned} &\mathcal{J}(u_1) \equiv D^1[\mathcal{F}(u_s)](u_1), \quad \mathcal{H}(v_1,v_2) \equiv \frac{1}{2!}D^2[\mathcal{F}(u_s)](v_1,v_2), \\ &\mathcal{T}(v_1,v_2,v_3) \equiv \frac{1}{3!}D^3[\mathcal{F}(u_s)](v_1,v_2,v_3), \quad \mathcal{Q}(v_1,v_2,v_3,v_4) \equiv \frac{1}{4!}D^4[\mathcal{F}(u_s)](v_1,v_2,v_3,v_4). \end{aligned}\right\} \tag{2.7}$$

We emphasize that the Taylor expansion in equation (2.4) is for a general nonlinear operator, $\mathcal{F}(u)$.

(b) The linear operator

We now discuss the linear operator, $\mathcal{J}(u)$ in equation (2.7), that will perform an important role in the subsequent analysis. In what follows, we adopt the notation

$$\Omega_i = k_i x - \omega t, \quad \Omega_{i,j} = (k_i + k_j)x - \omega t \quad \text{and} \quad \Omega_{i,j,l} = (k_i + k_j + k_l)x - \omega t, \tag{2.8}$$

where k_i is a spatial wavenumber, ω is a temporal wavenumber or angular frequency and i is a labelling index. We can map our operator onto Fourier space as a pseudo-operator, i.e.

$$\mathcal{J}(Ae^{i\Omega_1}) \mapsto iAe^{i\Omega_1}J(k_1), \tag{2.9}$$

where $J : \mathbb{R} \mapsto \mathbb{R}$ is a function of a real variable.

(c) The bilinear and trilinear operators

In a similar way, we can map the bilinear and trilinear operators, $\mathcal{H}$, $\mathcal{T}$, in Fourier space so that

$$\mathcal{H}(A_1e^{i\Omega_1}, A_2e^{i\Omega_2}) \mapsto A_1A_2e^{i\Omega_{1,2}}H(k_1,k_2) \tag{2.10}$$

and

$$\mathcal{T}(A_1e^{i\Omega_1}, A_2e^{i\Omega_2}, A_3e^{i\Omega_3}) \mapsto A_1A_2A_3e^{i\Omega_{1,2,3}}T(k_1,k_2,k_3). \tag{2.11}$$

The bilinearity of $\mathcal{H}$ means the following identities hold:

$$\mathcal{H}(a+b, c+d) \equiv \mathcal{H}(a,c) + \mathcal{H}(a,d) + \mathcal{H}(b,c) + \mathcal{H}(b,d), \quad \mathcal{H}(\lambda a, \mu b) \equiv \lambda\mu\mathcal{H}(a,b), \tag{2.12}$$

$\forall\, a,b,c,d \in \mathcal{U}$ and $\forall\, \lambda, \mu \in \mathbb{C}$. Similar distributive properties hold for the trilinear operator, $\mathcal{T}$.

3. Formal asymptotic expansion

Now that the problem and associated operators are defined, we can attempt to construct an asymptotically valid expansion for the solution. First, we perform a naive asymptotic expansion to solve equation (2.1) and show how it fails, as is well known, to understand how we may find a remedy.

We now assume that $\varepsilon \ll 1$, and formally expand u in the form

$$u(x,t) = u_s(x) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \varepsilon^4 u_4(x,t) + \cdots, \quad |\varepsilon| \ll 1. \tag{3.1}$$

Substituting equation (3.1) into equation (2.1) and using equation (2.4) gives

$$\varepsilon \frac{\partial u_1}{\partial t} + \varepsilon^2 \frac{\partial u_2}{\partial t} + \varepsilon^3 \frac{\partial u_3}{\partial t} + \cdots \underbrace{[\varepsilon \mathcal{J}(u^*) + \varepsilon^2 \mathcal{H}(u^*,u^*) + \varepsilon^3 \mathcal{T}(u^*,u^*,u^*) + \cdots]}_{\text{Taylor expansion}} = 0, \tag{3.2}$$

where we explicitly highlight the higher-order terms resulting from the Taylor expansion, see equation (1.2), and $u^* = u_1 + \varepsilon u_2 + \varepsilon^2 u_3 + \cdots$.

(a) Leading order: the dispersion relation

In equation (3.2), at $O(\varepsilon)$, we have

$$\mathcal{L}(u_1) = 0, \quad \left[\mathcal{L}(\star) \equiv \frac{\partial(\star)}{\partial t} + \mathcal{J}(\star), \quad \star \in \mathcal{U}\right]. \tag{3.3}$$

Equation (3.3) has a solution:

$$u_1(x,t) = A\mathrm{e}^{\mathrm{i}\Omega} + \overline{A}\mathrm{e}^{-\mathrm{i}\Omega}, \quad \Omega = kx - \omega t, \tag{3.4}$$

for an, as yet, unknown constant $A \in \mathbb{C}$. Using our notation introduced in equation (2.9), k and ω are related through the linear dispersion relation

$$\omega - J(k) = 0. \tag{3.5}$$

(b) Disordered expansion

Continuing our naive expansion of equation (3.2), at $O(\varepsilon^2)$, we obtain

$$\mathcal{L}(u_2) = -\mathcal{H}(u_1,u_1). \tag{3.6}$$

The operator $\mathcal{H}$ is bilinear and examining the form of equation (3.4) means that $\mathcal{H}$ will contain two types of terms proportional to (i) $\mathrm{e}^{2\mathrm{i}\Omega}$ and (ii) $\mathrm{e}^{0\mathrm{i}\Omega}$. Therefore, the solution at this order will be of the form

$$u_2(x,t) = \varphi_0 A^2 \mathrm{e}^{2\mathrm{i}\Omega} + B\mathrm{e}^{\mathrm{i}\Omega} + \varphi_1 |A|^2 + \text{c.c}, \tag{3.7}$$

where $\varphi_{0,1}$ are known solutions to linear problems, $B \in \mathbb{C}$ is arbitrary and c.c. stands for complex conjugate. At this stage, u_1 and u_2 are bounded and periodic. At the next order $O(\varepsilon^3)$, we find

$$\mathcal{L}(u_3) = -\mathcal{H}(u_1,u_2) - \mathcal{H}(u_1,u_2) - \mathcal{T}(u_1,u_1,u_1). \tag{3.8}$$

Now, there are terms on the right-hand side that are proportional to $A|A|^2\mathrm{e}^{\mathrm{i}\Omega}$ and hence u_3 will experience secular growth unless we choose $A = 0$, which reduces the expansion to null and thus illustrates the failure of the naive expansion.

(c) Multiple-scales and two-timing

To avoid a trivial selection of our amplitude function, we introduce slow variables X,T that are related to the $O(1)$ quantities by

$$X = \varepsilon x \quad \text{and} \quad T = \varepsilon t. \tag{3.9}$$

With $u(x,t) \mapsto u(x,t,X,T)$, the differential operators get mapped to

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}. \tag{3.10}$$

It is important to be careful with how this ansatz modifies the multi-linear operators introduced in the previous section. Due to this ansatz, the wavenumber in Fourier space is mapped to

$$k \mapsto k + \varepsilon K, \tag{3.11}$$

where K is a slow wavenumber associated with the slow spatial scale, X. Using equation (3.11), expanding $\mathcal{J}$ as a Taylor series yields

$$\mathcal{J}(A(X,T)\mathrm{e}^{\mathrm{i}\Omega}) \mapsto \left[\mathrm{i}J(k) + \varepsilon\frac{\mathrm{d}J}{\mathrm{d}k}K - \frac{\mathrm{i}}{2}\varepsilon^2\frac{\mathrm{d}^2J}{\mathrm{d}k^2}K^2 + \cdots\right]A(X,T)\mathrm{e}^{\mathrm{i}\Omega}, \tag{3.12}$$

$$\mapsto \left[\mathrm{i}J(k) + \varepsilon\frac{\mathrm{d}J}{\mathrm{d}k}\frac{\partial A}{\partial X} - \frac{\mathrm{i}}{2}\varepsilon^2\frac{\mathrm{d}^2J}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial X^2} + \cdots\right]\mathrm{e}^{\mathrm{i}\Omega}. \tag{3.13}$$

For the bilinear operator, we have to be more careful. The wave numbers are mapped according to $k_1 \mapsto k_1 + \varepsilon K_1, k_2 \mapsto k_2 + \varepsilon K_2$ so that

$$\mathcal{H}(A_1\mathrm{e}^{\mathrm{i}\Omega_1}, A_2\mathrm{e}^{\mathrm{i}\Omega_2}) \mapsto \left[H(k_1,k_2) + \varepsilon \boldsymbol{K}t\nabla_k H + \frac{1}{2}\varepsilon^2\boldsymbol{K}^T\boldsymbol{H}\boldsymbol{K} + \cdots\right]A_1A_2\mathrm{e}^{\mathrm{i}\Omega_{1,2}}, \tag{3.14}$$

$$\mapsto \left[H(k_1,k_2) + \varepsilon\left(\frac{\partial H}{\partial k_1}\frac{\partial A_1}{\partial X}A_2 + \frac{\partial H}{\partial k_2}A_1\frac{\partial A_2}{\partial X}\right) + \frac{1}{2}\varepsilon^2\left(\frac{\partial^2 H}{\partial k_1^2}\frac{\partial^2 A_1}{\partial X^2}A_2 + 2\frac{\partial^2 H}{\partial k_1\partial k_2}\frac{\partial A_1}{\partial X}\frac{\partial A_2}{\partial X} + \frac{\partial^2 H}{\partial k_2^2}A_1\frac{\partial^2 A_2}{\partial X^2}\right) + \cdots\right]\mathrm{e}^{\mathrm{i}\Omega_{1,2}}, \tag{3.15}$$

where the slow wavenumber vector is defined as $\boldsymbol{K} = (K_1, K_2)^T$, wavenumber gradient $\nabla_k = (\partial/\partial k_1, \partial/\partial k_2)$ and $\boldsymbol{H}$ is the Hessian matrix of H with respect to k_1, k_2. When going from equation (3.14) to equation (3.15), we consider that in Fourier space $A_1 \mapsto \hat{A}_1(K_1, T)$ and $A_2 \mapsto \hat{A}_2(K_2, T)$.

Finally, we do the same to the trilinear operator, this time the slow wavenumber vector is $\boldsymbol{K} = (K_1, K_2, K_3)^T$ and gradient operator, $\nabla_k = (\partial/\partial k_1, \partial/\partial k_2, \partial/\partial k_3)^T$, where the ambiguity with the expressions above is noted, yet accepted as a reasonable abuse of notation. The trilinear operator is mapped to

$$\mathcal{T}(A_1\mathrm{e}^{\mathrm{i}\Omega_1}, A_2\mathrm{e}^{\mathrm{i}\Omega_2}, A_3\mathrm{e}^{\mathrm{i}\Omega_3}) \mapsto [T(k_1,k_2,k_3) + \varepsilon\boldsymbol{K}^T\nabla_k T + \cdots]A_1A_2A_3\mathrm{e}^{\mathrm{i}\Omega_{1,2,3}}. \tag{3.16}$$

It will be useful later to expand the gradient term above explicitly:

$$[\boldsymbol{K}^T\nabla_k T]A_1A_2A_3 = \frac{\partial T}{\partial k_1}\frac{\partial A_1}{\partial X}A_2A_3 + \frac{\partial T}{\partial k_1}A_1\frac{\partial A_2}{\partial X}A_3 + \frac{\partial T}{\partial k_1}A_1A_2\frac{\partial A_3}{\partial X}. \tag{3.17}$$

For ease of exposition, we make the following definitions:

$$\left.\begin{aligned} &\mathcal{S}^n_{\mathcal{J}}(u_1) \equiv (\mathrm{i})^{n-1}\frac{1}{n!}\frac{\mathrm{d}^nJ}{\mathrm{d}k_1^n}\frac{\partial A}{\partial X}\mathrm{e}^{\mathrm{i}\Omega}, \quad \mathcal{S}_{\mathcal{H}}(u_1,u_2) \equiv [\boldsymbol{K}^T\nabla_k H]A_1A_2\mathrm{e}^{\mathrm{i}\Omega_{1,2}} \\ \text{and} \quad &\mathcal{S}^2_{\mathcal{H}}(u_1,u_2) \equiv \frac{1}{2!}[\boldsymbol{K}^T\boldsymbol{H}\boldsymbol{K}]A_1A_2\mathrm{e}^{\mathrm{i}\Omega_{1,2}}, \mathcal{S}_{\mathcal{T}}(u_1,u_2,u_3) \equiv \frac{1}{3!}[\boldsymbol{K}^T\nabla_k T]A_1A_2A_3\mathrm{e}^{\mathrm{i}\Omega_{1,2,3}}. \end{aligned}\right\} \tag{3.18}$$

We have used the symbol $\mathcal{S}$ to denote an operator defined on the 'slow' scale. For a technical discussion of the multi-scale ansatz see [27].

(d) The two-timed system

Using the notation introduced in §3c, we can now write our system in a way that will let us pick out the terms at each order in a straightforward manner.

Summarizing the process so far, we have performed two sets of Taylor expansions. The first involved a Hilbert-space Taylor expansion of the fully nonlinear operator $\mathcal{F}$ about the base state, u_s. This procedure resulted in a succession of multi-linear operators, $\mathcal{J}, \mathcal{H}, \ldots$, at each order in ε. The second set of Taylor expansions involved expanding the individual multi-linear operators due to the multi-scale ansatz in equation (3.9).

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Bringing these expansions together, using the notation in equation (3.18), we can write equation (3.2) to an arbitrary order in ε. The new expansion is of the form

$$u = u_s + \varepsilon A(X,T)\mathrm{e}^{\mathrm{i}\Omega} + \varepsilon^2 u_2(X,T,x,t) + \varepsilon^3 u_3(X,T,x,t) + \varepsilon^4 u_4(X,T,x,t) + \cdots. \tag{3.19}$$

Substituting equation (3.19) into equation (3.2), and using the Taylor expansions in (3.18), yields

$$\begin{aligned}&\underbrace{\varepsilon\left(\frac{\partial u_1}{\partial t} + \varepsilon\frac{\partial u_1}{\partial T}\right) + \varepsilon^2\left(\frac{\partial u_2}{\partial t} + \varepsilon\frac{\partial u_2}{\partial T}\right) + \varepsilon^3\left(\frac{\partial u_3}{\partial t} + \varepsilon\frac{\partial u_3}{\partial T}\right)}_{\text{time derivatives}} \\ &\quad + \underbrace{\varepsilon\mathcal{J}(u^*) + \varepsilon^2\mathcal{S}_{\mathcal{J}}(u^*) + \varepsilon^3\mathcal{S}^2_{\mathcal{J}}(u^*) + \varepsilon^4\mathcal{S}^3_{\mathcal{J}}(u^*)}_{\text{linear dispersion, }\mathcal{J}} + \underbrace{\varepsilon^2\mathcal{H}(u^*,u^*) + \varepsilon^3\mathcal{S}_{\mathcal{H}}(u^*,u^*) + \varepsilon^4\mathcal{S}^2_{\mathcal{H}}(u^*,u^*)}_{\text{bilinear terms, }\mathcal{H}} \\ &\quad + \underbrace{\varepsilon^3\mathcal{T}(u^*,u^*,u^*) + \varepsilon^4\mathcal{S}_{\mathcal{T}}(u^*,u^*,u^*)}_{\text{trilinear terms, }\mathcal{T}} + \underbrace{\varepsilon^4\mathcal{Q}(u^*,u^*,u^*,u^*)}_{\text{quadrilinear term}} + \underbrace{\varepsilon^2\mathcal{V}(u^*)}_{\text{dissipation}} + O(\varepsilon^5) = 0.\end{aligned} \tag{3.20}$$

Each set of grouped terms corresponds to the Taylor expansion of each of the (multi-linear) operators in equation (3.2) due to the two-timing ansatz (equation (3.9)).

The system in equation (3.20) may seem unwieldy. Still, conceptually it is simple as it arises from Taylor expansions in Hilbert spaces, which have a strong correlation to standard Taylor expansions of functions of real variables, with which we assume the reader is familiar. In practice, all that is required are the (i) elementary linear dispersion relation and (ii) the forms of the bilinear and trilinear operators. For a given system, the linear dispersion relation is often a trivial exercise, but the form of the bilinear and trilinear operators often requires some work. However, once established, the subsequent analysis becomes straightforward as it simply involves repeated applications of these operators with different arguments.

(e) Two-time first order

We now proceed with the formal asymptotic expansion. As before, at $O(\varepsilon)$, we obtain

$$\mathcal{L}(u_1) = 0. \tag{3.21}$$

This the time solution for u_1 will contain an unknown slowly varying (in space and time) modulation function;

$$u_1(x,t,X,T) = \underbrace{A(X,T)\mathrm{e}^{\mathrm{i}\Omega} + \overline{A}(X,T)\mathrm{e}^{-\mathrm{i}\Omega}}_{\text{oscillating}}, \qquad \Omega = kx - \omega t \quad \text{(leading-order solution)}, \tag{3.22}$$

with the dispersion relation stated in equation (3.5).

(f) Two-time second order: the transport equation

We continue our expansion to $O(\varepsilon^2)$ where

$$\mathcal{L}(u_2) + \frac{\partial u_1}{\partial T} + \mathcal{S}_{\mathcal{J}}(u_1) + \mathcal{H}(u_1,u_1) = 0 \tag{3.23}$$

or

$$\mathcal{L}(u_2) = -\underbrace{\left[\frac{\partial A}{\partial T} + c_g\frac{\partial A}{\partial X}\right]}_{\text{resonant}}\mathrm{e}^{\mathrm{i}\Omega} - \underbrace{\mathcal{H}(u_1,u_1)}_{\text{non-resonant}} + \text{c.c}, \tag{3.24}$$

where $c_g = \mathrm{d}J/\mathrm{d}k$ is the group velocity of the travelling wave. To eliminate the resonant terms on the right-hand side and thus avoid secular growth of the harmonic modes for u_2, we invoke the Fredholm alternative [28]. This states that either i) the only solution of $\mathcal{L}(u) = 0$ is $u = 0$ and then $\mathcal{L}(u) = v$ has a unique solution *or* ii) there exists a non-zero solution of $\mathcal{L}(u) = 0$ in which case

$\mathcal{L}(u) = v$ has a solution only if the inner product of v with the solutions of $\mathcal{L}^\dagger(u^\dagger) = 0$ vanishes, where $\mathcal{L}^\dagger, u^\dagger$ are the adjoint problem and eigenmodes, respectively, of

$$\langle \mathcal{L}(u), v\rangle = \langle u, \mathcal{L}^\dagger(v)\rangle, \quad \forall\, u, v \in \mathcal{U}. \tag{3.25}$$

In what follows, we assume our problem is self-adjoint, but a non-self-adjoint problem can be handled similarly [23]. For self-adjoint operators, the adjoint eigenmodes are proportional to $\mathrm{e}^{\mathrm{i}\Omega}$ and hence in our problem, equation (3.24), this can be written as

$$\left\langle \left(\frac{\partial A}{\partial T} + c_g \frac{\partial A}{\partial X}\right)\mathrm{e}^{\mathrm{i}\Omega} + \mathcal{H}(u_1, u_1), u^\dagger \right\rangle = 0. \tag{3.26}$$

The non-resonant terms in $\mathcal{H}(u_1, u_1)$ will automatically be orthogonal to $u^\dagger$ due to periodicity and so the solvability condition in equation (3.26) becomes

$$\frac{\partial A}{\partial T} + c_g \frac{\partial A}{\partial X} = 0. \tag{3.27}$$

We can write an alternative form of equation (3.24) using the Fourier representation,

$$\mathcal{L}(u_2) = -\underbrace{A^2 H(k,k)\mathrm{e}^{2\mathrm{i}\Omega} - \overline{A}^2 H(-k,-k)\mathrm{e}^{-2\mathrm{i}\Omega}}_{\text{oscillating}} - \underbrace{|A|^2 H(k,-k) - |A|^2 H(-k,k)}_{\text{non-oscillating}} + \text{resonant terms.} \tag{3.28}$$

This form allows us to write down the solution, u_2, as

$$u_2 = \varphi_0 \underbrace{A^2\mathrm{e}^{2\mathrm{i}\Omega} + \overline{\varphi}_0\overline{A}^2\mathrm{e}^{-2\mathrm{i}\Omega} + B\mathrm{e}^{\mathrm{i}\Omega} + \overline{B}\mathrm{e}^{-\mathrm{i}\Omega}}_{\text{oscillating}} + \underbrace{\mathbb{G}(\varphi_1|A|^2)}_{\text{non-oscillating}} \quad \text{(first-order solution)}, \tag{3.29}$$

where $B(X, T)$ is an arbitrary modulation that can be considered as an order ε correction to A and $\varphi_0 = H(k, k)$ and $\varphi_1 = H(k, -k) + H(-k, k)$. Finally, the operator $\mathbb{G}$ is defined as

$$\mathbb{G}(u) = \mathbb{F}^{-1}\left(-\frac{\mathrm{i}}{\omega}\mathbb{F}(u)\right), \tag{3.30}$$

where $\mathbb{F}$ is the Fourier transform in the spatial derivatives x.

(g) Two-time third order: the modified-NLS system

Continuing our analysis, at $O(\varepsilon^3)$, we have

$$\begin{aligned}\mathcal{L}(u_3) = &-\left(\frac{\partial u_2}{\partial T} + c_g\frac{\partial u_2}{\partial X} + \frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 u_1}{\partial X^2}\right) - \mathcal{H}(u_1, u_2) - \mathcal{H}(u_2, u_1)\\ &- \mathcal{S}_{\mathcal{H}}(u_1, u_1) - \mathcal{T}(u_1, u_1, u_1) - \mathcal{V}(u_1).\end{aligned} \tag{3.31}$$

or in Fourier representation:

$$\begin{aligned}\mathcal{L}(u_3) = &-\underbrace{\left[\frac{\partial u_2}{\partial T} + c_g\frac{\partial u_2}{\partial X}\right]}_{\text{transport}} - \underbrace{\left[-\frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial X^2} + \lambda_1 A|A|^2 + V(k)A + \lambda_2\mathbb{G}(|A|^2)\right]\mathrm{e}^{\mathrm{i}\Omega}}_{\text{resonant}}\\ &- \underbrace{\left[\lambda_3 A^3\mathrm{e}^{3\mathrm{i}\Omega} + \left(\lambda_4 A\frac{\partial A}{\partial X} + \lambda_5 AB\right)\mathrm{e}^{2\mathrm{i}\Omega} + \lambda_6\frac{\partial |A|^2}{\partial X} + \lambda_7(\overline{A}B + A\overline{B})\right]}_{\text{non-resonant}} + \text{c.c},\end{aligned} \tag{3.32}$$

where λ_i are functions of k that, again, in principle, can be determined, and $V(k)$ is the Fourier symbol of the dissipation term. We note that the second term in the bracket containing the resonant terms arises due to the nonlinear interaction between u_1 and u_2 in those terms involving $\mathcal{H}$ and $\mathcal{T}$ in equation (3.31). A further important point is that a non-local term appears in the resonant part of the right-hand side of equation (3.32). We note that non-local terms have appeared before in NLS-type models, for example, water waves [22], a 'shallow-deep' approximation for stratified fluids [29] and in so-called generalized quasi-nonlinear models of shear flow [30].

The transport term will vanish if we move to the frame of reference moving with the group velocity of the wave, which the form of equation (3.27) suggests. Therefore, we introduce a travelling-wave coordinate, ξ, moving with speed c_g. In addition, the resonant term in equation (3.32) has zero time-derivative (in T), implying that A is constant in time. To ensure a slow modulation, in time, of the envelope, A, we introduce a further slow time scale τ, so that

$$\xi = X - c_g T, \quad \tau = \varepsilon^2 t, \qquad A(X,T) \mapsto A(\xi,\tau), \quad B(X,T) \mapsto B(\xi,\tau). \tag{3.33}$$

Therefore, equation (3.32) becomes

$$\mathcal{L}(u_3) = -\mathrm{e}^{\mathrm{i}\Omega}\left(\frac{\partial A}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial \xi^2} + \lambda_1 A|A|^2 + V(k)A + \lambda_2 A\mathbb{G}(|A|^2)\right) + \text{non-resonant terms}. \tag{3.34}$$

Now, to avoid secular growth, we repeat the imposition of the Fredholm alternative. Again, assuming $\mathcal{L}$ is self-adjoint, the solvability condition is the modified-NLS equation:

$$\boxed{\frac{\partial A}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial \xi^2} + \lambda_1 A|A|^2 + V(k)A + \lambda_2 A\mathbb{G}(|A|^2) = 0.} \tag{3.35}$$

We emphasize that this equation is valid for any system of the form equation (2.1) with small linear dissipation. We note that the non-local operator, $\mathbb{G}$, in the context of water waves is absent as we will show in §5.

As in the previous order, using Fourier transforms, the full particular integral for u_3 can be written as

$$u_3 = \underbrace{A^3\psi_0\mathrm{e}^{3\mathrm{i}\Omega} + \overline{A}^3\overline{\psi}_0\mathrm{e}^{-3\mathrm{i}\Omega} + \left(\psi_1 A\frac{\partial A}{\partial \xi} + \psi_2 AB\right)\mathrm{e}^{2\mathrm{i}\Omega} + \overline{\psi}_3\overline{A}\frac{\partial \overline{A}}{\partial \xi}\mathrm{e}^{-2\mathrm{i}\Omega} + C\mathrm{e}^{\mathrm{i}\Omega} + \overline{C}\mathrm{e}^{-\mathrm{i}\Omega}}_{\text{oscillating}}$$
$$\underbrace{+\,\psi_4\mathbb{G}\left(\frac{\partial|A|^2}{\partial \xi}\right) + \psi_5\mathbb{G}(\overline{A}B + A\overline{B})}_{\text{non-oscillating}}\ \text{(second-order solution)}, \tag{3.36}$$

where $C(\xi,\tau)$ is an arbitrary modulation function and ψ_i are functions of k.

(h) Two-time third order: the modified-HNLS system

The expansion at $O(\varepsilon^4)$ is

$$\begin{aligned}\mathcal{L}(u_4) &+ \frac{\partial u_2}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 u_2}{\partial \xi^2} - \frac{1}{6}\frac{\mathrm{d}^2 c_g}{\mathrm{d}k^2}\frac{\partial^2 u_1}{\partial \xi^3} + \mathcal{H}(u_1,u_3) + \mathcal{H}(u_3,u_1) + \mathcal{H}(u_2,u_2)\\ &+ [\mathcal{S}_{\mathcal{H},\xi}(u_1,u_2) + \mathcal{S}_{\mathcal{H},\xi}(u_2,u_1)] + \mathcal{S}^2_{\mathcal{H},\xi}(u_1,u_1) + \mathcal{T}(u_1,u_1,u_2) + \mathcal{T}(u_1,u_2,u_1)\\ &+ \mathcal{T}(u_2,u_1,u_1) + \mathcal{S}_{\mathcal{T},\xi}(u_1,u_1,u_1) + \mathcal{Q}(u_1,u_1,u_1,u_1) + \mathcal{V}'\left(\frac{\partial u_1}{\partial \xi}\right) = 0,\end{aligned} \tag{3.37}$$

where $\mathcal{V}'$ is the Fréchet derivative of the operator $\mathcal{V}$. In Fourier representation, this is

$$\mathcal{L}(u_4) = -\Bigg[\underbrace{\frac{\partial B}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 B}{\partial \xi^2}}_{\text{order } \varepsilon \text{ NLS}} \underbrace{+\mu_1 B\mathbb{G}(|A|^2) + \mu_2 A^2\overline{B} + \mu_3|A|^2 B}_{\text{coupling terms}}$$
$$\underbrace{-\frac{1}{6}\frac{\mathrm{d}^2 c_g}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial \xi^3} + \mu_4|A|^2\frac{\partial A}{\partial \xi} + \mu_5 A^2\frac{\partial \overline{A}}{\partial \xi} + \mu_6 A\,\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right) + \mathcal{V}'(k)\frac{\partial A}{\partial \xi}\Bigg]\mathrm{e}^{\mathrm{i}\Omega}}_{\text{resonant}}$$
$$\underbrace{+\mu_7 A^4\mathrm{e}^{4\mathrm{i}\Omega} + \mu_8 A^2\frac{\partial A}{\partial \xi}\mathrm{e}^{3\mathrm{i}\Omega}}_{\text{non-resonant}} + \underbrace{\left(\mu_9|A|^2A^2 + \mu_{10}A^2\mathbb{G}(|A|^2) + \mu_{11}A\frac{\partial^2 A}{\partial \xi^2} + \mu_{12}\left(\frac{\partial A}{\partial \xi}\right)^2\right)\mathrm{e}^{2\mathrm{i}\Omega}}_{\text{non-resonant}}$$
$$+\underbrace{\left(\mu_{13}|A|^4 + \mu_{14}|A|^2\mathbb{G}(|A|^2) + \mu_{15}\mathbb{G}(|A|^2)^2 + \mu_{16}\frac{\partial^2 |A|^2}{\partial \xi^2}\right)}_{\text{non-resonant}} + \text{c.c}, \tag{3.38}$$

where μ_i are functions of k and the transport terms vanish due to the travelling wave frame of reference (equation (3.33)). We note that the expressions on the first two lines in equation (3.38) are resonant.

Invoking the Fredholm alternative yields the solvability condition

$$\frac{\partial B}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 B}{\partial \xi^2} + \mu_1 B\mathbb{G}(|A|^2) + \mu_2 A^2\overline{B} + +\mu_3|A|^2 B$$
$$-\frac{1}{6}\frac{\mathrm{d}^2 c_g}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial \xi^3} + \mu_4|A|^2\frac{\partial A}{\partial \xi} + \mu_5 A^2\frac{\partial \overline{A}}{\partial \xi} + \mu_6 A\,\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right) + V'(k)\frac{\partial A}{\partial \xi} = 0. \tag{3.39}$$

Equation (3.39) is an evolution equation for $B(\xi,\tau)$ that is coupled to $A(\xi,\tau)$ through equation (3.35) in the modified-HNLS system:

$$\boxed{\begin{aligned}&\frac{\partial A}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial \xi^2} + \lambda_1 A|A|^2 + V(k)A + \lambda_5 A\mathbb{G}(|A|^2) = 0, \quad (\mathrm{A})\\ &\frac{\partial B}{\partial \tau} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 B}{\partial \xi^2} + \mu_1 B\mathbb{G}(|A|^2) + \mu_2 A^2\overline{B} + \mu_3|A|^2 B,\\ &-\frac{1}{6}\frac{\mathrm{d}^2 c_g}{\mathrm{d}k^2}\frac{\partial^2 A}{\partial \xi^3} + \mu_4|A|^2\frac{\partial A}{\partial \xi} + \mu_5 A^2\frac{\partial \overline{A}}{\partial \xi} + \mu_6 A\,\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right) + \mathcal{V}'(k)\frac{\partial A}{\partial \xi} = 0. \quad (\mathrm{B})\end{aligned}} \tag{3.40}$$

After solving equation (3.40) for A and B the asymptotically consistent solution to the original PDE, equation (3.2), is

$$u_\varepsilon = (A + \varepsilon B)\mathrm{e}^{\mathrm{i}\Omega} + \varepsilon u_2, \tag{3.41}$$

where u_2 is stated in equation (3.29). Although equation (3.40) is a system of equations, calculating (A) + $\varepsilon \times$ (B) in equation (3.40) results in a single PDE for $\mathcal{A} = A + \varepsilon B$ (with terms of $O(\varepsilon)$) that is similar to the standard single PDEs seen in the literature [2,11,31]. In equation (3.40), we emphasize that all quantities are $O(1)$ as we have truncated the expansion at $O(\varepsilon^4)$. Furthermore, equation (3.41) is the explicit second-order reconstruction of the solution u, once A and B have been found by solving equation (3.40).

(i) Discussion

We have derived a system of equations, equation (3.40), that determines an asymptotically consistent form of the wave-envelope. Crucially, the small parameter ε is absent from the final form in equation (3.40), thus fulfilling our main aim. An important feature is that this analysis has been on a general PDE, demonstrating the universality of the wave-envelope evolution equations in equation (3.40) for systems defined in equation (3.2).

Note that, equation (3.40) is *time-like* in that it is of the form:

$$\frac{\partial A}{\partial \tau} = \text{operator}\left(A, \frac{\partial A}{\partial x}, \ldots\right), \quad (\text{time-like}), \tag{3.42}$$

rather than *space-like*, where

$$\frac{\partial A}{\partial x} = \text{Operator}\left(A, \frac{\partial A}{\partial \tau}, \ldots\right), \quad (\text{space-like}). \tag{3.43}$$

In most physical situations, the governing PDEs involve a single time derivative and are time-like (for example, the Navier–Stokes equations and the free-surface Euler equations). However, when making a comparison with experiments, the space-like form of the HNLS is more useful and can be determined by making a Galilean transformation in t (in the context of water waves, [11,19,31]). In practice, this involves neglecting certain nonlinear terms that arise from transforming equation (3.40) to its space-like counterpart and great care has to be taken as it is not as straightforward as simply switching the spatial and temporal coordinates in equation (3.40).

A further point to mention is the numerical implementation of equation (3.40). In current derivations of the HNLS, where ε is present, the wavenumber k and ε have to be specified as control parameters and, therefore, the parameter space to explore numerically is two dimensional. In equation (3.40), all that needs to be specified is the wavenumber k so that for a given wavenumber, once equation (3.40) has been solved, the solution can be reconstructed immediately for all ε using equation (3.41), thus resulting in a significant reduction in computation time when exploring parameter space.

An important part of this procedure was Taylor expanding a nonlinear functional in terms of Fréchet derivatives. However, for specific systems, an expert knowledge of Fréchet derivatives is not needed as we shall now demonstrate for two specific systems: a toy fifth-order problem and the more complicated damped water-wave problem.

(j) Symbolic algebra code

The algebra involved can be challenging (especially in the example of water waves discussed later) but we attempt to omit as few details as possible to make the analysis easier to follow. The reader can generate all of the coefficients by running the `toy_example_modulation_equations.m` and `water_wave_modulation_equations.m` in the electronic supplementary material or running `driver.m` and generating the output as a PDF file. These symbolic MATLAB codes are designed to act as a template for other systems, in particular water-wave problems, where the operators present are intrinsically non-local and great care has to be taken.

4. Example 1: a toy fifth-order system

We now apply this analysis to the toy dispersive system (without any damping):

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^5} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial x^3} = 0, \tag{4.1}$$

which, with appropriately scaled coefficients, is related to the fifth-order Korteweg–de Vries equation [32]. We expand equation (4.1) using equation (3.19) with base state $u_s = 0$. It is instructive to omit the multi-scale expansion at first to make the form of the operators transparent. Therefore, our (single-scale) expansion is

$$u = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \varepsilon^4 u_4(x,t) + \cdots . \tag{4.2}$$

Substituting equation (4.2) into equation (4.1) gives

$$\begin{aligned}\varepsilon\mathcal{L}(u_1) + \varepsilon^2[\mathcal{L}(u_2) + \mathcal{H}(u_1,u_1)] + \varepsilon^3[\mathcal{L}(u_3) + \mathcal{H}(u_1,u_2) + \mathcal{H}(u_2,u_1)]\\ + \varepsilon^4[\mathcal{L}(u_4) + \mathcal{H}(u_2,u_2) + \mathcal{H}(u_1,u_3) + \mathcal{H}(u_3,u_1)] + \cdots = 0,\end{aligned} \tag{4.3}$$

where we have identified the operators:

$$\mathcal{L}(u) \equiv \frac{\partial u}{\partial t} + \mathcal{J}(u), \quad \mathcal{J}(u) \equiv \frac{\partial^2 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^5} \quad \text{and} \quad \mathcal{H}(u,v) \equiv u\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} + u\frac{\partial^3 v}{\partial x^3}. \tag{4.4}$$

The Fourier multipliers of the operators in equation (4.4) are

$$J(k_1) = \mathrm{i}k_1^3(k_1^2 - 1) \quad \text{and} \quad H(k_1,k_2) = \mathrm{i}k_2(1 - k_1k_2 - k_2^2). \tag{4.5}$$

We emphasize that we have *not* found the Fréchet derivatives explicitly; the definitions in equation (4.4) arise from simple algebra. We now expand the operators by simply substituting equation (4.5) into the multi-scale expansions in equations (3.13)–(3.16). Our problem then becomes

$$\begin{aligned}&\varepsilon\mathcal{L}(u_1) + \varepsilon^2\left[\mathcal{L}(u_2) + \mathcal{H}(u_1,u_1) + \mathcal{S}_{\mathcal{J}}(u_1) + \mathrm{i}\frac{\partial u_1}{\partial T}\right]\\ &\quad + \varepsilon^3\left[\mathcal{L}(u_3) + \mathcal{H}(u_1,u_2) + \mathcal{H}(u_2,u_1) + \mathcal{S}^2_{\mathcal{J}}(u_1) + \mathcal{S}_{\mathcal{H}}(u_1,u_1) + \mathcal{S}_{\mathcal{J}}(u_2) + \mathrm{i}\frac{\partial u_2}{\partial T}\right]\\ &\quad + \varepsilon^4[\mathcal{L}(u_4) + \mathcal{H}(u_2,u_2) + \mathcal{H}(u_1,u_3) + \mathcal{H}(u_3,u_1) + \mathcal{S}^3_{\mathcal{J}}(u_1) + \mathcal{S}^2_{\mathcal{H}}(u_1,u_1)\\ &\quad + \mathcal{S}^2_{\mathcal{J}}(u_2) + \mathcal{S}_{\mathcal{H}}(u_1,u_2) + \mathcal{S}_{\mathcal{H}}(u_2,u_1) + \mathrm{i}\frac{\partial u_3}{\partial T}\Bigg] + \cdots = 0.\end{aligned} \tag{4.6}$$

We are now in a position to simply 'pick' out the equations in each order.

(a) First order

The expansion at $O(\varepsilon)$ equation (4.6) is

$$\mathcal{L}(u_1) = 0, \tag{4.7}$$

which has the solution and dispersion relation

$$u_1 = A(X,T)\,\mathrm{e}^{\mathrm{i}\Omega} + \text{c.c}, \quad \omega = k^5 - k^3. \tag{4.8}$$

(b) Second order

Continuing, at $O(\varepsilon^2)$, equation (4.6) is

$$\mathcal{L}(u_2) = \mathrm{i}\left[\frac{\partial A}{\partial T} + c_g\frac{\partial A}{\partial X}\right]\mathrm{e}^{\mathrm{i}\Omega} + \mathrm{i}k(1-2k^2)A^2\mathrm{e}^{2\mathrm{i}\Omega}, \qquad c_g = 5k^4 - 3k^2. \tag{4.9}$$

Note, there are no non-oscillating terms in equation (4.9). We eliminate the resonant transport terms by moving to a frame of reference moving with the group velocity c_g by setting $\xi = X - c_gT$ and $\tau = \varepsilon^2 t$. By finding the particular integral of equation (4.9) and using the dispersion relation in equation (4.8), the solution at this order is therefore

$$u_2 = B(\xi,\tau)\,\mathrm{e}^{\mathrm{i}\Omega} + \varphi_0 A^2\mathrm{e}^{2\mathrm{i}\Omega} + \text{c.c.}, \quad \varphi_0 = \frac{2k^2-1}{6k^2(1-5k^2)}. \tag{4.10}$$

(c) Third order

The next order in equation (4.6) is $O(\varepsilon^3)$, where

$$\mathcal{L}(u_3) = -\mathrm{i}\left[\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{1-2k^2}{6k}A|A|^2 + \frac{\mathrm{d}c_g}{\mathrm{d}k}\frac{\partial^2 A}{\partial \xi^2}\right]\mathrm{e}^{\mathrm{i}\Omega}$$
$$+\left(\mathrm{i}(1-6k^2)\frac{\partial A}{\partial \xi}A + \mathrm{i}k(1-2k^2)AB\right)\mathrm{e}^{2\mathrm{i}\Omega} + 3\mathrm{i}k(1-5k^2)\varphi_0 A^3\mathrm{e}^{3\mathrm{i}\Omega} + \mathrm{i}(1-2k^2)\frac{\partial |A|^2}{\partial \xi}. \tag{4.11}$$

Using the Fredholm alternative, we get the modified-NLS equation for our toy system:

$$\boxed{\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{1-2k^2}{6k}A|A|^2 + 2k(10k^2-3)\frac{\partial^2 A}{\partial \xi^2} = 0.} \tag{4.12}$$

The solution at this order is

$$u_3 = C(\xi,\tau)\,\mathrm{e}^{\mathrm{i}\Omega} + \psi_1 A\frac{\partial A}{\partial \xi}\mathrm{e}^{2\mathrm{i}\Omega} + \psi_2 AB\mathrm{e}^{2\mathrm{i}\Omega} + \psi_3\mathrm{e}^{3\mathrm{i}\Omega} + \psi_4\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right) + \mathrm{c.c}, \tag{4.13}$$

where $C(\xi,\tau)$ is undetermined and by finding the particular integral of equation (4.11) and using the dispersion relation in equation (4.8):

$$\psi_1 = \frac{6k^2-1}{6k^3(1-5k^2)},\quad \psi_2 = \frac{2k^2-1}{6k^2(1-5k^2)},\quad \psi_3 = \frac{(2k^2-1)(3k^2-1)}{144k^4(1-10k^2)(1-5k^2)}\quad \text{and}\quad \psi_4 = 2\mathrm{i}(1-2k^2). \tag{4.14}$$

(d) Fourth order

Finally, at $O(\varepsilon^4)$, equation (4.6) is

$$\mathcal{L}(u_4) = \mathrm{i}\left[\mathrm{i}\frac{\partial B}{\partial \tau} + \frac{1-2k^2}{6k}B|B|^2 + 2k(10k^2-3)\frac{\partial^2 B}{\partial \xi^2} + \frac{\mathrm{i}(22k^4-11k^2+1)}{2k^2(5k^2-1)}|A|^2\frac{\partial A}{\partial \xi} + \right.$$
$$\frac{\mathrm{i}(2k^2-1)}{6k}|A|^2 B + \frac{\mathrm{i}(2k^2-1)}{6k}A^2\overline{B} + \frac{\mathrm{i}(2k^2-1)(3k^2-1)}{6k^2(5k^2-1)}A^2\frac{\partial \overline{A}}{\partial \xi} + 6k(10k-3)\frac{\partial^3 A}{\partial X^3}$$
$$\left. -2k^3(1-2k^2)A\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right)\right]\mathrm{e}^{\mathrm{i}\Omega} + \text{non-resonant terms.} \tag{4.15}$$

Using the Fredholm alternative, we get the evolution equation for B that can be coupled with equation (4.12) to form the complete HNLS for the toy problem in equation (4.1):

$$\boxed{\begin{aligned}
&\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{1-2k^2}{6k}A|A|^2 + 2k(10k^2-3)\frac{\partial^2 A}{\partial \xi^2} = 0\\
&\mathrm{i}\frac{\partial B}{\partial \tau} + \frac{1-2k^2}{6k}B|B|^2 + 2k(10k^2-3)\frac{\partial^2 B}{\partial \xi^2}\\
&+\frac{\mathrm{i}(22k^4-11k^2+1)}{2k^2(5k^2-1)}|A|^2\frac{\partial A}{\partial \xi} + \frac{\mathrm{i}(2k^2-1)}{6k}|A|^2 B + \frac{\mathrm{i}(2k^2-1)}{6k}A^2\overline{B}\\
&+\frac{\mathrm{i}(2k^2-1)(3k^2-1)}{6k^2(5k^2-1}A^2\frac{\partial \overline{A}}{\partial \xi} + 6k(10k-3)\frac{\partial^3 A}{\partial X^3} - 2k^3(1-2k^2)A\mathbb{G}\left(\frac{\partial |A|^2}{\partial \xi}\right) = 0.
\end{aligned}} \tag{4.16}$$

(e) A note on the boundary conditions

As a final note, equation (4.1) is fifth order in space and so five conditions on the x-derivatives of u are required to ensure the problem is well-posed. The system in equation (4.16) is third order in space and hence requires three boundary conditions and the other two are accounted for by the periodicity requirement. Therefore, going to higher orders for this toy system is ill-posed as more boundary conditions for the wave envelope are required than the original toy problem in equation (4.16).

5. Example 2: the damped water-wave problem

We now describe a more complicated example; determining the shape of the free surface, $y=\zeta$, bounding a body of irrotational, inviscid and incompressible fluid with damping. In the fluid, the unknown velocity potential, ϕ, must satisfy Laplace's equation, and in this instance, we neglect surface tension. As in [3,33], we concentrate on the three-dimensional problem with horizontal coordinates (x, y) and vertical coordinate z.

We non-dimensionalize the system in the same way as [3]. To replicate the general system in equation (3.2), the governing equations can be written as a dynamical system of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\mathcal{F}}(\boldsymbol{u}) + \varepsilon^2 \boldsymbol{\mathcal{V}}(\boldsymbol{u}) = 0, \tag{5.1}$$

where $\boldsymbol{u} = [\zeta, \psi]^T$, $\psi = \phi_{y=\zeta}$ and $\boldsymbol{\mathcal{V}}$ is the, now vectorial, linear damping/forcing term, which is consistent with the approach of [17]. The nonlinear functional can be written explicitly as

$$\boldsymbol{\mathcal{F}}(\boldsymbol{u}) = \left[-\mathcal{G}[\varepsilon\zeta]\psi, \zeta + \frac{1}{2}\varepsilon|\nabla\psi|^2 - \frac{\varepsilon(\mathcal{G}[\varepsilon\zeta](\psi) + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\varepsilon^2|\nabla\zeta|^2)} \right]^T, \tag{5.2}$$

where ε is the nonlinearity parameter, proportional to the ratio of a typical wave-height to wavelength [3]. The spatial boundary conditions are that φ and ζ and the first spatial derivative of ζ vanish as $x \to \pm\infty$. In addition, $\nabla = [\partial/\partial x, \partial/\partial y]^T$, $G[\zeta]$ is the Dirichlet-to-Neumann map (DtN) on deep water defined as

$$\mathcal{G}[\zeta](\psi) = (1 + |\nabla\zeta|^2)^{1/2}(\nabla\phi \cdot \boldsymbol{n}), \tag{5.3}$$

where $\boldsymbol{n}$ is the outwards-pointing normal of the free surface. The operator is a linear with respect to ψ (and hence the Fréchet derivatives with respect to ψ are trivial) but highly nonlinear with respect to ζ and more work is required to determine the Fréchet derivatives with respect to ζ.

As the dynamical system is two dimensional in the sense that there are two independent phase variables in $\boldsymbol{u}$, the coefficients of the modified-HNLS equation for this system are vectors in $\mathbb{R}^2$ and the envelope is also a vector, $\boldsymbol{A} = [A_\zeta, A_\psi]^T$. Furthermore, the domain is spatially three dimensional and ζ describes a co-dimension two surface. To reflect equation (3.19), we therefore look for a solution of the form

$$\boldsymbol{u} = \boldsymbol{A}\,\mathrm{e}^{\mathrm{i}\Omega} + \varepsilon\boldsymbol{u}_2 + \varepsilon^2\boldsymbol{u}_3 + \cdots + \text{c.c}, \tag{5.4}$$

where $\boldsymbol{u}_n = [\zeta_n, \psi_n]^T$, $\Omega = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ and $\boldsymbol{k} = [k_x, k_y]^T$.

A key part of the analysis is the expansion of the DtN in equation (5.3). The DtN is analytic in ζ (e.g. [34]) and hence we expand it as a Taylor series about the zero base state:

$$\mathcal{G}[\varepsilon\zeta](\psi) = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{G}_n[\zeta](\psi), \tag{5.5}$$

where $\mathcal{G}_n$ is an n-linear operator. The form of $\mathcal{G}_n$ is highly dependent on the *geometry* of the problem. For a finite-depth domain with a flat bottom and for an infinite-depth domain, the explicit forms of $\mathcal{G}_n$ are known, e.g. [33,35], respectively. While the DtN is well defined for more complex geometries, including varying bottom topography, the explicit calculation is more difficult [36,37]. As the main focus of this article is to show that the evolution equations emerge from general dispersive systems with weak dissipation, not just for water-wave problems, we do not explore these difficulties, which are well documented in the literature. With that in mind, we apply our methodology to waves over infinite depth, in which case the expansion in equation (5.5) is relatively straightforward. For infinite depth, the first few terms of the DtN are

$$\mathcal{G}_0(\psi) = |\mathcal{D}|(\psi), \qquad \mathcal{G}_1[\zeta](\psi) = -|\mathcal{D}|(\zeta(|\mathcal{D}|(\psi)) - \nabla\cdot(\zeta\nabla\psi) \tag{5.6}$$

and

$$\mathcal{G}_2[\zeta](\psi) = |\mathcal{D}|(\zeta|\mathcal{D}|(\zeta|\mathcal{D}|(\psi))) + \frac{1}{2}\nabla^2(\zeta^2|\mathcal{D}|(\psi)) + \frac{1}{2}|\mathcal{D}|(\zeta^2\nabla^2\psi), \tag{5.7}$$

where $\mathcal{D} = -\mathrm{i}\,\nabla$ with a Fourier multiplier of $\boldsymbol{k}$, and $|\mathcal{D}|$ has a Fourier multiplier of $|\boldsymbol{k}|$, see p. 88 of [3].

(a) Leading order

At $O(1)$, we obtain

$$\boldsymbol{\mathcal{L}}(\boldsymbol{u}_1) \equiv \frac{\partial \boldsymbol{u}_1}{\partial t} + \boldsymbol{\mathcal{J}}(\boldsymbol{u}_1) = 0, \quad \boldsymbol{\mathcal{J}} = \begin{pmatrix} 0 & -\mathcal{G}_0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{J}(\boldsymbol{k}_1) = \begin{pmatrix} 0 & -|\boldsymbol{k}| \\ 1 & 0 \end{pmatrix}, \tag{5.8}$$

where the bold symbols are matrix versions of $\mathcal{J}(u_1)$ and $J(k)$ introduced earlier. Writing $\boldsymbol{u}_1$ in component form,

$$\boldsymbol{u}_1 = [\zeta_1, \psi_1]^T = \boldsymbol{A}\mathrm{e}^{\mathrm{i}\Omega}, \tag{5.9}$$

results in an eigenvalue problem. To ensure that a non-zero solution exists the full dispersion relation for deep-water waves emerges:

$$\omega = J(\boldsymbol{k}) \equiv |\boldsymbol{k}|^{1/2}, \tag{5.10}$$

along with the eigenmodes and adjoint eigenmodes:

$$\boldsymbol{g} = \boldsymbol{v}\mathrm{e}^{\mathrm{i}\Omega}, \quad \boldsymbol{g}^\dagger = \boldsymbol{v}^\dagger\mathrm{e}^{\mathrm{i}\Omega}, \quad \text{where} \qquad \boldsymbol{v} = [\mathrm{i}\omega, 1]^T, \; \boldsymbol{v}^\dagger = [1, -\mathrm{i}\omega]^T. \tag{5.11}$$

The slowly varying modulation amplitude is now $A(\boldsymbol{X}, T)\boldsymbol{v}$, where A is a scalar function of $\boldsymbol{X} = [X_1, X_2]^T = \varepsilon\boldsymbol{x}$ and $T = \varepsilon t$. Therefore, we can write the first-order solution as

$$\boldsymbol{u}_1 = A\boldsymbol{v}\mathrm{e}^{\mathrm{i}\Omega} + \text{c.c} \quad \text{(first-order solution)}. \tag{5.12}$$

At this stage, it is also useful to Taylor expand the operator $\mathcal{G}_0$ in the slow variable. We find that

$$\mathcal{G}_0(A(\boldsymbol{X})\mathrm{e}^{\mathrm{i}\Omega}) \mapsto A(\boldsymbol{X})|\boldsymbol{k}|\mathrm{e}^{\mathrm{i}\Omega} - \varepsilon(2\mathrm{i}\omega\nabla_{\boldsymbol{k}}\omega(\boldsymbol{k})\cdot\nabla_{\boldsymbol{X}}A)\mathrm{e}^{\mathrm{i}\Omega} + \cdots, \tag{5.13}$$

where $\nabla_{\boldsymbol{k}} = [\partial/\partial k_1, \partial/\partial k_2]^T$ and $\nabla_{\boldsymbol{X}} = [\partial/\partial X_1, \partial/\partial X_2]^T$.

(b) First order

The expansion at $O(\varepsilon)$ is

$$\boldsymbol{\mathcal{L}}(\boldsymbol{u}_2) = -\underbrace{\left[\begin{pmatrix} \mathrm{i}\omega \\ 1 \end{pmatrix}\frac{\partial A}{\partial T} + \begin{pmatrix} 2\mathrm{i}\omega \\ 0 \end{pmatrix}\nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{X}}A\right]}_{\text{resonant terms}}\mathrm{e}^{\mathrm{i}\Omega} + \text{c.c} + \boldsymbol{\mathcal{H}}(\boldsymbol{u}_1, \boldsymbol{u}_1), \tag{5.14}$$

where

$$\boldsymbol{\mathcal{H}}(\boldsymbol{u}_i, \boldsymbol{u}_j) \equiv \begin{pmatrix} \mathcal{G}_1[\zeta_i](\psi_j) \\ -\frac{1}{2}\nabla\psi_i\cdot\nabla\psi_j + \frac{1}{2}\mathcal{G}_0(\psi_i)\mathcal{G}_0(\psi_j) \end{pmatrix}. \tag{5.15}$$

Now, to impose the Fredholm alternative, we multiply equation (5.14) by the conjugate transpose adjoint eigenmode in equation (5.11) and integrate over a period in t. To eliminate the secular terms that still arise, we set

$$\boldsymbol{\xi} = \boldsymbol{X} - \boldsymbol{c}_g T \qquad \boldsymbol{c}_g = \nabla_{\boldsymbol{k}}\omega. \tag{5.16}$$

The solution to $\boldsymbol{u}_2$ is

$$\boldsymbol{u}_2 = \boldsymbol{\varphi}_0\, A^2 \mathrm{e}^{2\mathrm{i}\Omega} + B\,\boldsymbol{v}\mathrm{e}^{\mathrm{i}\Omega} + \mathrm{c.c}, \tag{5.17}$$

where $B(\boldsymbol{\xi}, T)$ and $\bar{\varphi}_2(\boldsymbol{\xi})$ are both arbitrary at this order and $\boldsymbol{\varphi}_0$ solves the linear problem

$$\boldsymbol{L}_2\boldsymbol{\varphi}_0 = [0,\ |\boldsymbol{k}|^2]^T, \qquad \boldsymbol{L}_n \equiv \begin{pmatrix} -n\mathrm{i}\omega & -|n||\boldsymbol{k}| \\ 1 & -n\mathrm{i}\omega \end{pmatrix}, \tag{5.18}$$

yielding

$$\boldsymbol{\varphi}_0 = \frac{\mathrm{i}|\boldsymbol{k}|^2}{\omega}\,\boldsymbol{v} = \mathrm{i}\omega^3\boldsymbol{v}. \tag{5.19}$$

(c) Second order

Continuing the expansion, at $O(\varepsilon^2)$ (by moving to a frame of reference using equation (5.16) and using the small time scale, $\tau = \varepsilon^2 t$), we obtain

$$\begin{aligned}\boldsymbol{\mathcal{L}}(\boldsymbol{u}_3) = -&\left[\underbrace{\begin{pmatrix}\mathrm{i}\omega\\1\end{pmatrix}\frac{\partial A}{\partial\tau} - \begin{pmatrix}1\\0\end{pmatrix}\frac{1}{2\omega^2}\left(\nabla_{\xi}^2 A - \frac{1}{\omega^4}(\boldsymbol{k}\cdot\nabla_{\xi})^2 A\right) + \begin{pmatrix}1\\1\end{pmatrix}V(\boldsymbol{k})A}_{\equiv\mathcal{N}(A)}\right]\mathrm{e}^{\mathrm{i}\Omega} + \mathrm{c.c}\\ &+ \underbrace{\boldsymbol{\mathcal{S}}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{u}_1,\boldsymbol{u}_1) + \boldsymbol{\mathcal{H}}(\boldsymbol{u}_1,\boldsymbol{u}_2) + \boldsymbol{\mathcal{H}}(\boldsymbol{u}_2,\boldsymbol{u}_1) + \boldsymbol{\mathcal{T}}(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1)}_{\text{nonlinear interactions}},\end{aligned} \tag{5.20}$$

where we have defined the linear operator $\mathcal{N}(\star)$ as the operator in the square brackets and $V(\boldsymbol{k})$ is the pseudo-Fourier operator of $\boldsymbol{\mathcal{V}}$. The operator $\mathcal{T}$ is defined as

$$\boldsymbol{\mathcal{T}}(\boldsymbol{u}_i,\boldsymbol{u}_j,\boldsymbol{u}_k) \equiv \begin{pmatrix} \mathcal{G}_2[\zeta_i,\zeta_j](\psi_k) \\ \mathcal{G}_0(\psi_i)\mathcal{G}_1[\zeta_j](\psi_k) + \mathcal{G}_0(\psi_i)\nabla\zeta_j\cdot\nabla\psi_k \end{pmatrix}. \tag{5.21}$$

The nonlinear resonant term, that is proportional to $A|A|^2$, arises from interactions between $\bar{\boldsymbol{u}}_1$ and $\boldsymbol{u}_2$ in the $\boldsymbol{\mathcal{H}}$ terms and $\boldsymbol{u}_1, \boldsymbol{u}_1, \bar{\boldsymbol{u}}_1$ in the $\boldsymbol{\mathcal{T}}$ term. As at the previous order, the Fredholm alternative is imposed by multiplying the right-hand side of equation (5.20) by the conjugate transpose adjoint eigenmode (see equation (5.11)) that results in the modified NLS:

$$\boxed{\frac{\partial A}{\partial\tau} + \frac{1}{4\omega^3}\mathrm{i}\left(\nabla_{\xi}^2 A - \frac{1}{\omega^4}(\boldsymbol{k}\cdot\nabla_{\xi})^2 A\right) + \frac{5\mathrm{i}}{2}\frac{|\boldsymbol{k}|^4}{\omega}A|A|^2 - \frac{\mathrm{i}}{2\omega}V(\boldsymbol{k})A = 0.} \tag{5.22}$$

Note that, for this particular example, no non-local terms enter the modulation equation, unlike the general modified-NLS in equation (3.35). We also note that, the form of equation (5.22) is nearly identical to that of non-dissipative NLS with $\boldsymbol{V}(\boldsymbol{k}) = 0$, as derived in [3] (§8.3, eqns (8.50) and (8.51) in [3]). The important difference is that we obtain a *single* evolution equation (5.22) instead of separate evolution equations for the leading-order term in the modulation amplitude for ζ and ψ. Indeed, this highlights another advantage of our Fréchet approach.

By examining the non-resonant terms of equation (5.20) that arise from nonlinear interactions, the form of $\boldsymbol{u}_2$ can be determined. By solving a series of linear problems (see `water_wave_modulation_equations.m` in the electronic supplementary material), one can deduce that

$$u_3 = \underbrace{\varphi_2 A^3 e^{3i\Omega} + (\varphi_3 AB + (\varphi_4 \cdot A\nabla_\xi A))e^{2i\Omega} + Cve^{i\Omega} + \text{c.c}}_{\text{oscillating terms}} + \underbrace{\begin{pmatrix} -2i(A\boldsymbol{k}\cdot\nabla_\xi \overline{A} - \overline{A}\boldsymbol{k}\cdot\nabla_\xi A) \\ \frac{2}{|\mathcal{D}|}(\omega\boldsymbol{k}\cdot\nabla_\xi |A|^2) \end{pmatrix}}_{\text{mean-flow terms}}, \tag{5.23}$$

where

$$\varphi_2 = -\frac{1}{2}\omega^6[9i\omega, 5]^T, \quad \varphi_3 = 2i\omega^3[i\omega, 1]^T \quad \text{and} \quad \varphi_4 = \frac{4}{\omega}[2i\omega, 1]^T, \tag{5.24}$$

and $1/|\mathcal{D}|$ is the inverse leading-order DtN operator. The labelling in the subscripts above has been chosen to be consistent with the script `water_wave_modulation_equations.m` in the electronic supplementary material.

(d) Third order

At $O(\varepsilon^3)$, and recalling the definition of $\mathcal{N}(\star)$ from equation (5.20), the linear system to be solved is

$$\begin{aligned}\mathcal{L}(u_4) = &-\left[\mathcal{N}(B) - \begin{pmatrix}1\\0\end{pmatrix}\frac{1}{2\omega^{10}}i(\boldsymbol{k}\cdot\nabla_\xi)^3 A + \begin{pmatrix}1\\0\end{pmatrix}\frac{1}{2\omega^6}i(\boldsymbol{k}\cdot\nabla_\xi)\nabla_\xi^2 A + \nabla_\xi A\cdot\nabla_\kappa V\right]e^{i\Omega} \\ &+ \mathcal{S}_{\mathcal{H}}(u_1,u_2) + \mathcal{S}_{\mathcal{H}}(u_2,u_1) + \mathcal{S}_{\mathcal{T}}(u_1,u_1,u_1) + \mathcal{H}(u_1,u_3) + \mathcal{H}(u_3,u_1) + \mathcal{H}(u_2,u_2) \\ &+ \mathcal{T}(u_1,u_1,u_2) + \mathcal{T}(u_1,u_2,u_1) + \mathcal{T}(u_2,u_1,u_1) + \text{non-resonant terms.}\end{aligned} \tag{5.25}$$

As in the previous order, we first identify resonant terms on the right-hand side of equation (5.25) and then multiply by the conjugate transpose adjoint eigenmodes to form a solvability condition. At this stage of the analysis, the algebra involved is formidable; we omit the details that can be found in the symbolic algebra script `water_wave_modulation_equations.m` in the electronic supplementary material and simply state the final result:

$$\begin{aligned}&\underbrace{\mathcal{N}(B) + \frac{1}{4\omega^{11}}((\boldsymbol{k}\cdot\nabla_\xi)^3 A - |\boldsymbol{k}|^2(\boldsymbol{k}\cdot\nabla_\xi)\nabla_\xi^2 A)}_{\text{linear derivatives}} + \underbrace{\frac{5}{2}i\omega^7 A^2\overline{B} + 5i\omega^7|A|^2\overline{B}}_{\text{coupling terms}} \\ &+ \underbrace{\frac{1}{4}\omega^3 A^2\boldsymbol{k}\cdot\nabla_\xi\overline{A} + 2\omega^3|A|^2\boldsymbol{k}\cdot\nabla_\xi A}_{\text{nonlinear terms}} + \underbrace{2i\omega^3 A\frac{1}{|\mathcal{D}|}(\nabla_\xi|A|^2)}_{\text{non-local terms}} \\ &- \frac{i}{2\omega}\underbrace{\nabla_\xi A\cdot\nabla_k V}_{\text{dissipation terms}} = 0.\end{aligned} \tag{5.26}$$

Therefore, the coupled form for the slow modulation amplitudes, A and B is

$$\boxed{\begin{aligned}&\frac{\partial A}{\partial\tau} + \frac{1}{4\omega^3}i\left(\nabla_\xi^2 A - \frac{1}{\omega^4}(\boldsymbol{k}\cdot\nabla_\xi)^2 A\right) + \frac{5i}{2}\frac{|\boldsymbol{k}|^4}{\omega}A|A|^2 - \frac{i}{2\omega}V(\boldsymbol{k})A = 0. \\ &\frac{\partial B}{\partial\tau} + \frac{1}{4\omega^3}i\left(\nabla_\xi^2 B - \frac{1}{\omega^4}(\boldsymbol{k}\cdot\nabla_\xi)^2 B\right) - \frac{i}{2\omega}V(\boldsymbol{k})B + \frac{1}{4\omega^{11}}((\boldsymbol{k}\cdot\nabla_\xi)^3 A - |\boldsymbol{k}|^2(\boldsymbol{k}\cdot\nabla_\xi)\nabla_\xi^2 A) \\ &+ \frac{5}{2}i\omega^7 A^2\overline{B} + 5i\omega^7|A|^2\overline{B} + \frac{1}{4}\omega^3 A^2\boldsymbol{k}\cdot\nabla_\xi\overline{A} + 2\omega^3|A|^2\boldsymbol{k}\cdot\nabla_\xi A \\ &+ 2i\omega^3 A\frac{1}{|\mathcal{D}|}(\nabla_\xi|A|^2) - \frac{i}{2\omega}\nabla_\xi A\cdot\nabla_k V = 0.\end{aligned}} \tag{5.27}$$

(e) Comparison to existing literature

To make a comparison to existing models of the HNLS, we can combine the two equations in equation (5.27) by writing $\mathcal{A} = A + \varepsilon B$ and then truncating to $O(\varepsilon^2)$ to a single PDE (as discussed in equation (3.41)). By noting that in deep-water $\omega^2 = |\boldsymbol{k}|$, and setting the dissipative term to zero, we obtain (with a slight re-ordering of terms):

$$\begin{aligned} &\mathrm{i}\frac{\partial \mathcal{A}}{\partial \tau} - \frac{1}{4\omega^3}\left(\nabla_{\xi}^2\mathcal{A} - \frac{1}{\omega^4}(\boldsymbol{k}\cdot\nabla_{\xi})^2\mathcal{A}\right)\\ &-\frac{5}{2}\omega|\boldsymbol{k}|^3\mathcal{A}|\mathcal{A}|^2 + \frac{\mathrm{i}\varepsilon}{4\omega^{11}}((\boldsymbol{k}\cdot\nabla_{\xi})^3\mathcal{A} - |\boldsymbol{k}|^2(\boldsymbol{k}\cdot\nabla_{\xi})\nabla_{\xi}^2\mathcal{A})\\ &\frac{1}{4}\varepsilon\mathrm{i}\omega^3\mathcal{A}\boldsymbol{k}\cdot\nabla_{\xi}\overline{\mathcal{A}} + 2\varepsilon\omega^3\mathrm{i}|\mathcal{A}|^2\nabla_{\xi}\mathcal{A} - 2\varepsilon\omega^3\mathcal{A}\frac{1}{|\mathcal{D}|}(\nabla_{\xi}|\mathcal{A}|^2) + O(\varepsilon^2) = 0, \quad \text{(Keeler } et\ al.\text{)}. \end{aligned} \tag{5.28}$$

We can compare this directly with the recent formulation based on the Hamiltonian form of the governing equations, stated at the end of §5 in [33] as

$$\begin{aligned} &\mathrm{i}\frac{\partial A}{\partial \tau} - \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial x^2} + \frac{\omega}{4k^2}\frac{\partial^2 A}{\partial y^2} - k^3 A|A|^2 - \mathrm{i}\varepsilon\frac{\omega}{16k^3}\frac{\partial^3 A}{\partial x^3} + \mathrm{i}\varepsilon\frac{3\omega}{8k^3}\frac{\partial}{\partial x}\frac{\partial^2 A}{\partial y^2} + 3\mathrm{i}\varepsilon k^2|A|^2\frac{\partial A}{\partial x}\\ &\quad -\varepsilon k^2 A\frac{1}{|\mathcal{D}|}\frac{\partial |A|^2}{\partial x} = 0, \quad \text{(Guyenne } et\ al.\text{ three-dimensional deep-water gravity)}. \end{aligned} \tag{5.29}$$

Upon inspection of equations (5.28) and (5.29), one can see similar terms occurring in both equations. There are a number of important differences: (i) nonlinear terms in equation (5.28) have an additional ω factor in the coefficient (which agrees with the non-Hamiltonian form of the HNLS as stated in eqn. (7.3) of [33]); (ii) in equation (5.28), the preceding analysis has not assumed any preferred direction for the wavenumber $\boldsymbol{k}$, hence the linear spatial derivatives terms in equation (5.28) can be written more compactly; and (iii) although the numerical coefficients differ in equations (5.28) and (5.29), the sign and i prefactors of all the equivalent terms are identical.

Finally, due to the similarity of our modified-HNLS to the literature, we would expect equation (5.27) to have the same mathematical properties as existing models in the literature. A systematic study of the existence and uniqueness of solutions of equation (5.27), as well as how they behave, is out of the scope of this present study and a comprehensive numerical study is an interesting future research avenue.

(f) Choice of dissipation term

In the context of the water-wave problem, we have purposefully left the form of the dissipation operator, $\boldsymbol{\mathcal{V}}$, general. For flow over ice-infested waters (e.g. [19,20,31]), assuming the flow is two dimensional so that there is only one spatially horizontal coordinate, x, the dissipation term in Fourier space is of the form

$$V(k) = (-\mathrm{i})^n \alpha_n k^n, \qquad n \in \mathbb{R}. \tag{5.30}$$

Very recently, Humphries *et al.* [31] simulated a space-like HNLS with $n = 2$ in equation (5.30). It would be of significant interest to simulate the parameter-free equation (5.27) with a non-integer n representing non-local dissipation.

From a more theoretical perspective, a weakly damped water-wave system was derived from first principles from the Navier–Stokes equations in [17]. The form of this dissipation (again in two dimensions) is

$$\boldsymbol{V}(\boldsymbol{u}) = \left[2\nu\frac{\partial^2 \varphi}{\partial z^2}, -\frac{\partial a}{\partial x}\right]^T, \quad \text{where} \quad \frac{\partial^2 a}{\partial x \partial t} = 2\nu\frac{\partial^3 a}{\partial x^2 \partial y} \tag{5.31}$$

and ν is the kinematic viscosity. Again, as a suggestion for future study, it would be of great interest to include this form of dissipation in equation (5.27) for the two dimensional and fully three-dimensional systems and study the behaviour of the solutions.

6. Conclusion and perspective

In this article, we have formally derived the modified-NLS and -HNLS systems for a general set of dispersive PDEs with damping. We have achieved this by posing the system as an infinite-dimensional dynamical system and performing a series of Taylor expansions on the nonlinear functionals, allowing us to derive general evolution equations using the method of multiple scales. In particular, we have placed the recent work of [19,20] on a firmer theoretical footing where a modified-NLS equation damping was used to describe ocean waves in the marginal ice-zone of the southern ocean.

It is important to state that the derivation of the evolution equations for this problem has been achieved before by many different authors (see §1). Yet, we believe this general analytic framework provides a fresh perspective in that; (i) it makes no assumption on what the small parameter, ε, is; (ii) there is no assumption that the underlying base state is the zero state; (iii) there is no requirement that the particular form of solutions should be known *a priori*, as is often the case in existing derivations; and (iv) this framework can easily be applied to similar problems that do not have a Hamiltonian structure.

Although the evolution equations have been traditionally derived in the context of measuring the wave envelope of water waves and optics, there is an intriguing possibility that the framework we have derived here can be used to approximate general time-dependent periodic invariant solutions of infinite-dimensional PDEs in numerical formulations. An attractive feature of this formulation is that in any numerical computation of these systems, we can approximate the multi-linear operators using the numerically constructed Jacobian and Hessian operators. This substantially increases the potential scope of our approach to a wider class of problems. For example, a recently discovered periodic invariant solution of the water-wave problem for flow over a localized topographic forcing has been discovered numerically [38] and it is an intriguing possibility that the envelope equations developed here can be used to approximate such objects.

Data accessibility. The data are provided in the electronic supplementary material [39].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. J.S.K.: conceptualization, formal analysis, methodology, writing—original draft, writing—review and editing; A.A.: conceptualization, writing—review and editing; B.S.H.: conceptualization; E.P.: conceptualization, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. J.S.K. acknowledges support from the Leverhulme trust grant no. (ECF-2021-017) and A.A and E.P acknowledge funding from EPSRC grant no. (EP/Y02012X/1).

Acknowledgements. J.S.K. sincerely thanks Mark Cooker for discussions and detailed comments on an early draft of this work. The Isaac Newton Institute for Mathematical Sciences supported the Maths of Sea Ice network, which hosted fruitful discussions that contributed to this article and the authors also acknowledge the helpful comments and suggestions of Anthony Roberts and the reviewers.

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