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Comparing external and internal instruments for vector autoregressions ☆

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ABSTRACT

In conventional proxy VAR analysis, the shocks of interest are identified by external instruments. This is typically accomplished by considering the covariance of the instruments and the reduced-form residuals. Alternatively, the instruments may be internalized by augmenting the VAR process by the instruments or proxies. These alternative identification methods are compared and it is shown that the resulting shocks obtained with the alternative approaches differ in general. Conditions are provided under which their impulse responses are nevertheless identical. If the conditions are satisfied, identification of the shocks is ensured. An empirical example illustrates the theoretical results.

1. Introduction

Using external instruments or proxies to identify structural shocks has become an important tool in structural vector autoregressive (VAR) analysis. Nowadays some authors use several proxies to identify a set of shocks. In that case, the proxies generally identify only linear combinations of the shocks and typically additional assumptions are needed to identify the shocks individually (see, e.g., Mertens and Ravn, 2013, Piffer and Podstawski, 2017 or Jarociński and Karadi, 2020).

The dominant approaches for estimating the structural parameters and, hence, the shocks in proxy VAR analysis are based on the covariance of the instruments and the reduced-form residuals (see, e.g., Mertens and Ravn, 2013) or on augmenting the VAR model by the proxies and, hence, internalizing them (e.g., Kilian and Lütkepohl, 2017, Section 15.2, Jarociński and Karadi, 2020, Plagborg-Møller and Wolf, 2021, 2022). The distinction between external and internal instruments is also discussed by Stock and Watson (2018).

This study compares the two alternative approaches for using multiple proxies for identifying a set of shocks in structural VAR analysis and makes several contributions to the proxy VAR literature. (1) Conditions are derived under which the impulse response functions are identical in population for the external instruments and the augmented VAR approaches. (2) It is established that the shocks obtained with both approaches are different in population even if the conditions for identical impulse responses are satisfied. (3) It is shown that, if the conditions for identical impulse responses from both approaches are satisfied, the structural shocks are

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fully identified by the proxies such that additional restrictions for disentangling the shocks in the external proxy VAR approach are unnecessary.

Specifically, we show that, if the proxies are mutually uncorrelated, each proxy is correlated with exactly one shock only and the proxies are not Granger-causal for the variables of interest, then the shocks can be scaled such that the structural impulse responses obtained from the external instruments and the augmented VAR approaches are identical in population. However, the shocks obtained from the two approaches will still be different. If the external instruments approach is used, the shocks will be linear transformations of the reduced-form residuals. In this setup, the proxies used for identification need not be direct measurements of the shocks of interest, but can contain some measurement error. If instead the augmented VAR approach is used, the resulting shocks will in many situations be linear transformations of the proxies, i.e., there is no built-in correction for measurement errors in the proxies. We use an empirical example considering a model of the crude oil market based on a study by Känzig (2021) to illustrate our theoretical results.

The remainder of the paper is organized as follows. In the next section we present the model setup, compare the alternative identification and estimation methods formally, and present conditions for individually identified shocks when multiple proxies are used. In Section 3 we study the empirical example and Section 4 concludes. Some proofs are provided in the Appendix.

2. Model setup and identification

2.1. The model

Our point of departure is a K-dimensional reduced-form VAR process,

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t. \tag{1}$$

The error process, u_t, is zero-mean white noise with nonsingular covariance matrix Σ_u. In short, the u_t are serially uncorrelated and $u_t \sim (0, \Sigma_u)$. The structural shocks, denoted as $\mathbf{w}_t = (w_{1t}, \ldots, w_{Kt})'$, are assumed to be linear combinations of the u_t, $\mathbf{w}_t = B^{-1}u_t$, such that $u_t = B\mathbf{w}_t$. In the following we will refer to shocks that are linear combinations of the reduced-form residuals u_t as fundamental.[1] The $(K \times K)$ transformation matrix B contains the impact effects of the structural shocks. They are assumed to have a diagonal covariance matrix $\Sigma_\mathbf{w}$ such that $B\Sigma_\mathbf{w} B' = \Sigma_u$. If the shocks are normalized to have unit variances and, hence, $\Sigma_\mathbf{w} = I_K$, the transformation matrix B has to be such that $BB' = \Sigma_u$.

We assume further that the first K_1 shocks, $\mathbf{w}_{1t} = (w_{1t}, \ldots, w_{K_1 t})'$, are of primary interest and have to be properly identified as economic shocks, while the last $K - K_1$ shocks, $\mathbf{w}_{2t} = (w_{K_1+1,t}, \ldots, w_{Kt})'$, are not of interest. Accordingly, we partition the vector of shocks as $\mathbf{w}_t' = (\mathbf{w}_{1t}', \mathbf{w}_{2t}')$. The matrix of impact effects, B, is partitioned correspondingly as $B = [B_1 : B_2]$, B_1 being a $(K \times K_1)$ matrix and B_2 being of dimensions $(K \times (K - K_1))$.

The matrix B contains the structural parameters of the model. The k-th column of B, say b_k, respresents the impact effects of the k-th shock on all the K variables. Thus, the columns of B_1 contain the impact effects of the shocks of interest, $\mathbf{w}_{1t}$. Having B_1, the latter shocks can be obtained from the reduced-form residuals as[2]

$$\mathbf{w}_{1t} = (B_1'\Sigma_u^{-1}B_1)^{-1}B_1'\Sigma_u^{-1}u_t. \tag{2}$$

The structural impulse responses of the shocks of interest for propagation horizon h are known to be $\Theta_{1,h} = \Phi_h B_1$, where the Φ_h are reduced-form quantities obtained recursively from the $A_1, \ldots, A_p$ VAR slope coefficients as $\Phi_h = \sum_{j=1}^{h} \Phi_{h-j}A_j$, with $\Phi_0 = I_K$, for $h = 1, \ldots,$ and $A_j = 0$ for $j > p$ (e.g., Lütkepohl, 2005, Section 2.1.2).

2.2. Identification via proxy variables

Identification of the structural parameters and, hence, the structural shocks is assumed to be based on a set of N instrumental variables (proxies) $z_t = (z_{1t}, \ldots, z_{Nt})'$ satisfying

$$\mathbb{E}(\mathbf{w}_{1t}z_t') = \Sigma_{\mathbf{w}_1 z} \neq 0, \quad \Sigma_{\mathbf{w}_1 z}\ (K_1 \times N), \quad \mathrm{rk}(\Sigma_{\mathbf{w}_1 z}) = K_1 \quad \text{(relevance)}, \tag{3}$$

$$\mathbb{E}(\mathbf{w}_{2t}z_t') = 0 \quad \text{(exogeneity)}. \tag{4}$$

These conditions imply that

$$\mathbb{E}(u_t z_t') = B\mathbb{E}(\mathbf{w}_t z_t') = B_1\Sigma_{\mathbf{w}_1 z}. \tag{5}$$

In the following, we will refer to the approach based on these conditions and, hence, on the relation (5) as the external proxy VAR approach to distinguish it from the approach that internalizes the proxies by augmenting the VAR model to be discussed in Subsections 2.3 and 2.4. The result in (5) implies that B_1 has to satisfy

[1] In some of the recent literature, this property is referred to as invertibility (see, e.g., Plagborg-Møller and Wolf, 2021).

[2] The relation follows from the fact that $w_{kt} = b_k'\Sigma_u^{-1}u_t / b_k'\Sigma_u^{-1}b_k$ (see, e.g., Stock and Watson, 2018, Bruns and Lütkepohl, 2022, Appendix A.1) and $(B_1'\Sigma_u^{-1}B_1)^{-1} = \Sigma_{\mathbf{w}_1}$.

$$B_1^{ext} = \mathbb{E}(u_t z_t')\Sigma'_{\mathbf{w}_1 z}(\Sigma_{\mathbf{w}_1 z}\Sigma'_{\mathbf{w}_1 z})^{-1}, \tag{6}$$

where we reserve the notation B_1^{ext} for impact matrices satisfying (6). Unfortunately, this relation does not suffice to fully identify B_1 if $K_1 > 1$ because

$$B_1\mathbb{E}(\mathbf{w}_{1t} z_t') = B_1 Q\mathbb{E}(Q'\mathbf{w}_{1t} z_t')$$

for any $Q \in \mathcal{O}_{K_1}$, where $\mathcal{O}_{K_1}$ is the set of orthogonal $(K_1 \times K_1)$ matrices. Thus, for a given vector of proxies z_t, there are admissible shocks corresponding to $B_1 Q$ such that (6) holds. We denote the set of admissible impact effects matrices for a set of proxies z_t by $\mathcal{B}_z^{ext}$, i.e.,

$$\mathcal{B}_z^{ext} = \{ B_1 | \mathbb{E}(u_t z_t') = B_1 \Sigma_{\mathbf{w}_1 z}, \text{ with } \mathbf{w}_{1t} \text{ satisfying (2) and } \mathbb{E}(\mathbf{w}_{1t}\mathbf{w}_{1t}') \text{ diagonal} \}. \tag{7}$$

Thus, for a given matrix B_1 satisfying (5), $\mathcal{B}_z^{ext}$ contains all matrices $B_1 Q$, $Q \in \mathcal{O}_{K_1}$. An element of the set of admissible impact effects matrices is denoted by B_1^{ext}. The shocks obtained from B_1^{ext} via equation (2) are signified as $\mathbf{w}_{1t}^{ext}$. A related discussion on the identified set in proxy VAR analysis can be found in Giacomini et al. (2022).

Obviously, there must be at least as many proxies as there are shocks to be identified such that $N \geq K_1$, to satisfy the rank condition for $\Sigma_{\mathbf{w}_1 z}$ which ensures that $\Sigma_{\mathbf{w}_1 z}\Sigma'_{\mathbf{w}_1 z}$ in equation (6) is invertible and the N proxies contain identifying information for all shocks in $\mathbf{w}_{1t}$. As we can estimate $B_1\Sigma_{\mathbf{w}_1 z}$ by the usual covariance matrix estimator

$$\overline{\hat{u}z} = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t z_t', \tag{8}$$

where the $\hat{u}_t$ are reduced-form least squares (LS) residuals, the proxies contain identifying information for the first K_1 structural shocks collectively but the shocks are not necessarily individually identified. Further identifying information may be imposed on B_1 directly or on $\Sigma_{\mathbf{w}_1 z}$ or on both matrices. For example, Mertens and Ravn (2013) and Känzig (2021) impose exclusion restrictions on B_1, while Piffer and Podstawski (2017) use sign restrictions for B_1 for identification. Identifying restrictions on $\Sigma_{\mathbf{w}_1 z}$ are used, e.g., by Altavilla et al. (2019), Jarociński and Karadi (2020), Lakdawala (2019), and Bruns et al. (2025).

2.3. Population results for VAR models augmented by the proxies

In proxy VAR analysis, some authors augment the VAR model by the proxy variables (see, e.g., Caldara and Herbst, 2019, Arias et al., 2021, Angelini and Fanelli, 2019, Jarociński and Karadi, 2020, Plagborg-Møller and Wolf, 2021). In this case, there are often as many proxies as there are shocks of interest. Therefore, from now on, we assume that $N = K_1$. In practice, this assumption is not always satisfied (see, e.g., Hou, 2024, Arias et al., 2021) but it holds in many empirical studies using the approach that augments the VAR model by the proxies.

We consider the augmented reduced-form VAR(p) model

$$\begin{pmatrix} z_t \\ y_t \end{pmatrix} = \begin{pmatrix} \nu^z \\ \nu^y \end{pmatrix} + \begin{bmatrix} A_1^{zz} & A_1^{zy} \\ A_1^{yz} & A_1^{yy} \end{bmatrix}\begin{pmatrix} z_{t-1} \\ y_{t-1} \end{pmatrix} + \cdots + \begin{bmatrix} A_p^{zz} & A_p^{zy} \\ A_p^{yz} & A_p^{yy} \end{bmatrix}\begin{pmatrix} z_{t-p} \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} u_t^z \\ u_t^y \end{pmatrix}. \tag{9}$$

In the following, we assume that the $(K_1 + K)$-dimensional error process $u_t^{aug} = (u_t^{z\prime}, u_t^{y\prime})'$ is a zero-mean white noise process and denote its covariance matrix by Σ_u^{aug}. Furthermore, we assume that $\mathbb{E}(u_t z_t') = \mathbb{E}(u_t u_t^{z\prime})$. In other words, the covariance of the reduced-form residuals u_t from model (1) and the proxies is the same as the covariance between u_t and the residuals u_t^z from model (9). Clearly, this is not restrictive given that u_t^z is not serially correlated and uncorrelated with lagged z_t and y_t.

An augmented model such as (9) is often used for Bayesian proxy VAR analysis because it allows to use standard Bayesian VAR methods that place priors on the reduced-form parameters. For example, one could use a Gaussian-inverse-Wishart prior that results in a convenient Gaussian-inverse-Wishart posterior for the reduced-form parameters of the augmented model. For more sophisticated proposals see also Caldara and Herbst (2019) and Arias et al. (2021).

In large VARs, identification of the shocks is often based on a recursiveness assumption and we use this assumption for the augmented model (9). Hence, the shocks are identified based on a Cholesky decomposition of Σ_u^{aug} (e.g., Bańbura et al., 2010) such that

$$\mathbf{w}_t^{aug} = (B^{aug})^{-1} u_t^{aug}. \tag{10}$$

In that case, the first K_1 shocks are interpreted as the shocks identified by the K_1 proxies. The lower left-hand $(K \times K_1)$ submatrix of B^{aug} contains the impact effects of the K_1 shocks of interest on the variables y_t and will be denoted by B_1^{int} in the following. The corresponding shocks, e.g., the first K_1 elements of $\mathbf{w}_t^{aug}$ will be signified as $\mathbf{w}_{1t}^{int}$. In contrast to B_1^{ext} in Section 2.2, the impact effects obtained from the internal approach, B_1^{int}, are unique for a given set of proxies z_t and so are the shocks $\mathbf{w}_{1t}^{int}$ because the Cholesky decomposition is unique.[3]

[3] We always use the unique lower-triangular matrix with positive diagonal elements and not potentially different Cholesky decompositions as mentioned, e.g., by Lütkepohl (1996).

To understand the relationship of the shocks $\mathbf{w}_{1t}^{int}$ from the augmented model to the $\mathbf{w}_{1t}^{ext}$ shocks from the external proxy VAR approach, it may be worth mentioning that the inverse of the lower-triangular B^{aug} is also a lower-triangular matrix and, hence, the first component of $\mathbf{w}_t^{aug}$ is just a multiple of u_{1t}^z, the second component is a linear combination of u_{1t}^z and u_{2t}^z, and, more generally, the k-th component of $\mathbf{w}_t^{aug}$ is a linear combination of $u_{1t}^z, \dots, u_{kt}^z$ for $k = 1, \dots, K_1$. There are no linear combinations of the u_t^y involved in determining the first K_1 shocks, $\mathbf{w}_{1t}^{int}$, of the augmented model. This contrasts to the $\mathbf{w}_{1t}^{ext}$ which are linear transformations of the u_t from model (1). We state this result formally for future reference.

Result 1. The shocks $\mathbf{w}_{1t}^{int}$ are linear combinations of the u_t^z, while the $\mathbf{w}_{1t}^{ext}$ shocks are linear combinations of the u_t from model (1). $\square$

In other words, the shocks obtained from the external and internal approaches generally differ. In practice, proxies are often uncorrelated white noise by construction. For example, proxies for monetary policy shocks are sometimes based on changes in forward rates or futures contracts at the time of policy announcements of the central bank (e.g., Gürkaynak et al., 2007, Gertler and Karadi, 2015, Barakchian and Crowen, 2013, Miranda-Agrippino and Ricco, 2021, Cesa-Bianchi et al., 2020). Moreover, some authors explicitly remove autocorrelation in the proxies by using residual series from a dynamic model as proxies. For example, Lunsford (2015) constructs proxies for consumption TFP and investment TFP shocks as the residuals from a VAR model fitted to two relevant TFP series. In that case, there may be no lags of z_t and y_t on the right-hand side of the z_t equations in model (9). Hence, the $u_t^z = z_t - \mathbb{E}(z_t)$ are just the mean-adjusted proxies. According to Result 1, the shocks $\mathbf{w}_{1t}^{int}$ are then linear combinations of the mean-adjusted z_t.

In contrast, the $\mathbf{w}_{1t}^{ext}$ shocks, obtained from the external proxy VAR approach, which are linear combinations of the reduced-form residuals of the VAR model (1), are just correlated with the proxies such that the proxies are better thought of as shocks measured with error. In fact, the two sets of shocks, $\mathbf{w}_{1t}^{ext}$ and $\mathbf{w}_{1t}^{int}$, can be quite different although they may result in identical impulse response functions, as we will see below. An empirical example is provided in Section 3.

Given the way many proxies are constructed in practice, it may not be very appealing to view them directly as shocks. As an extreme case, consider for example the sign proxies proposed by Boer and Lütkepohl (2021) which are discrete variables with values $-1, 0$, and 1 only. A number of proxies used in the proxy VAR literature are explicitly constructed to be nonzero only for selected periods where shocks have occurred and set to zero for many other periods where no shock measurements are available (e.g., Gertler and Karadi, 2015, Piffer and Podstawski, 2017, Boer and Lütkepohl, 2021, Känzig, 2021). This can be problematic when the shocks are used directly for economic analysis as, for instance, in historical decompositions, as we will see in the example in Section 3. It is easy to picture the proxies as correlated with shocks of interest but perhaps less plausible to view them as the shocks of interest themselves. Thus, interpreting the proxies as shocks may not be natural and appealing in many situations.

It turns out, however, that, under suitable conditions, the impact effects of the shocks of interest of the external and internal proxy VAR approaches may be identical upon suitable rescaling of the shocks. In the following proposition, proven in Appendix A, we state conditions under which that equivalence of the impact effects of the shocks holds. The proposition presents a necessary and sufficient condition for equality of the impact effects obtained from the external and internal proxy VAR approaches.

Proposition 1. *Suppose the proxies, z_t, satisfy the relevance and exogeneity conditions (3) and (4) and, for the augmented VAR model (9), $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$ holds. Then there exists a $B_1^{ext} \in \mathcal{B}_z^{ext}$ such that*

$$B_1^{ext} = B_1^{int}$$

if and only if the proxies are such that

$$\Sigma_{\mathbf{w}_1 z} = chol(\Sigma_z)'. \quad \square \tag{11}$$

As B_1^{ext} is in general not unique for a given set of proxies z_t if $K_1 > 1$, the proposition makes a statement about one specific element of the set $\mathcal{B}_z^{ext}$ of admissible matrices. In contrast, B_1^{int} is a unique matrix. Hence, identifying B_1 recursively as in the internal proxy VAR approach, is more restrictive than under-identifying B_1 only by the relevance and exogeneity conditions. Clearly, condition (11) uniquely fixes also B_1^{ext} and shows that B_1^{ext} can be identified by restrictions on $\Sigma_{\mathbf{w}_1 z}$. Proposition 1 actually implies that $\Sigma_{\mathbf{w}_1 z}$ has to be an upper triangular matrix for the two approaches to result in equivalent impact effects of the structural shocks of interest. Restricting $\Sigma_{\mathbf{w}_1 z}$ to be upper-triangular just-identifies the B_1 matrix and, hence, the shocks of interest. Interestingly, Lakdawala (2019) uses triangularity of $\Sigma_{\mathbf{w}_1 z}$ to identify his shocks of interest. Also Altavilla et al. (2019) and Bruns et al. (2025) consider a diagonal (hence, a special triangular) $\Sigma_{\mathbf{w}_1 z}$ which is compatible with condition (11) but over-identifies B_1 due to the additional zero restrictions imposed on $\Sigma_{\mathbf{w}_1 z}$.

In Proposition 1, the condition $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$ can be replaced by the more restrictive sufficient condition $A_1^{yz} = \cdots = A_p^{yz} = 0$, i.e., no lags of the proxies appear in the y_t equations of model (9). The latter condition implies $u_t^y = u_t$ and may be more intuitive than the condition $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$.

While Proposition 1 is useful to understand the implications of restrictions placed on $\Sigma_{\mathbf{w}_1 z}$, it is less helpful if identifying restrictions are imposed directly on B_1 as, e.g., in Mertens and Ravn (2013) and Känzig (2021). However, it implies the following sufficient conditions for equivalence of the external and internal proxy VAR approaches which can be useful in applied work.

Corollary 1. *If the proxies, z_t, satisfy the relevance and exogeneity conditions (3) and (4), then the external and internal proxy VAR approaches result in equivalent impact effects matrices B_1^{ext} and B_1^{int} which differ only by a scalar multiple of each column, if the following three conditions hold:*

(a) *In the augmented VAR model (9), $A_1^{yz} = \cdots = A_p^{yz} = 0$, i.e., no lags of the proxies appear in the y_t equations.*
(b) *The covariance matrix of z_t, Σ_z, is a diagonal matrix, i.e., the proxies are instantaneously uncorrelated.*
(c) *$\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix, i.e., each shock in $\mathbf{w}_{1t}$ is correlated with one proxy only.* □

Proof of Corollary 1. As mentioned earlier, condition (a) implies the condition $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$ of Proposition 1. Thus, we can use Proposition 1 to prove the corollary. If Σ_z is a diagonal matrix, then $\mathrm{chol}(\Sigma_z)$ is also diagonal and equal to its transpose. Moreover, if $\mathrm{chol}(\Sigma_z)$ and $\Sigma_{\mathbf{w}_1 z}$ are diagonal matrices, we can choose the shocks $\mathbf{w}_{1t}$ such that the two matrices are identical because we can choose the size of the shocks arbitrarily. Thus, the conditions in Corollary 1 are sufficient for (11) to hold up to scale of the shocks. □

The corollary states that, if the proxies are contemporaneously uncorrelated and each of the proxies is correlated with one shock of interest only, then we can get impact effects of the first K_1 structural shocks, $\mathbf{w}_{1t}^{int}$, by considering $\mathrm{chol}(\Sigma_u^{aug})$. These impact effects will be scalar multiples of the impact effects of the $\mathbf{w}_{1t}^{ext}$ shocks on the y_t variables.

Corollary 1 is useful in practice because all three conditions can be assessed with statistical tools. Condition (a) just means that the proxies are not Granger-causal for the y_t variables which can be checked by a standard Granger-causality test. Condition (b) can be tested by checking the correlations between the proxies. Finally, if $K_1 > 1$, condition (c) can be assessed by an over-identification J-test as proposed by Bruns et al. (2025). As mentioned before, diagonality of $\Sigma_{\mathbf{w}_1 z}$ over-identifies the shocks because condition (11) shows that for just-identification it is sufficient for $\Sigma_{\mathbf{w}_1 z}$ to be triangular. Imposing diagonality hence means imposing additional over-identifying restrictions. Bruns et al. (2025) provide a set of moment conditions corresponding to these restrictions and show formally that they are over-identifying. They set up a GMM estimation approach and the corresponding J-test for over-identifying moment conditions.

If there is just one shock of interest ($K_1 = 1$), conditions (b) and (c) of Corollary 1 are automatically satisfied and, hence, provided $A_1^{yz} = \cdots = A_p^{yz} = 0$, the impact effects of the shock can be obtained directly from the relation $\mathbb{E}(u_t z_{1t}) = \sigma_1 b_1$, where σ_1 is a scalar, or, equivalently, by using the covariance matrix of the VAR residuals augmented by a single proxy. The latter fact follows from Corollary A.1 in Appendix A which implies that, if there is just one shock identified by a single proxy ($N = K_1 = 1$), the last K elements of the first column of the Cholesky decomposition of the covariance matrix Σ_u^{aug} of the augmented VAR residual vector are a multiple of $\mathbb{E}(u_t^y u_t^z) = \mathbb{E}(u_t z_t)$ and thus, upon standardization, are precisely the desired impact effects of the external proxy VAR approach, b_1^{ext}, where the latter quantity denotes the first column of B_1^{ext}. We can even get the slightly more general result for the case of $K_1 \geq 1$.

Corollary 2. *If in the augmented VAR model (9), $A_1^{yz} = \cdots = A_p^{yz} = 0$ and the proxies, z_t, satisfy the relevance and exogeneity conditions (3) and (4), then b_1^{int}, the first column of B_1^{int}, is a multiple of b_1^{ext}.* □

Proof of Corollary 2. As $A_1^{yz} = \cdots = A_p^{yz} = 0$ implies $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$, the corollary follows from Proposition 1. Note first that the last K elements in the first column of Σ_u^{aug} are equal to $\mathbb{E}(u_t u_{1t}^z) = \mathbb{E}(u_t z_{1t}) = b_1^{ext}$ under the condition $A_1^{yz} = \cdots = A_p^{yz} = 0$ of Corollary 2. Moreover, Corollary A.1 in Appendix A shows that the first column of $\mathrm{chol}(\Sigma_u^{aug})$ is a scalar multiple of the first column of Σ_u^{aug}. Thus, b_1^{int} is a scalar multiple of $b_1^{ext} = \mathbb{E}(u_t z_{1t})$ as claimed in Corollary 2. □

Note that this result always holds for the first shock, provided $\mathbb{E}(u_t^y u_t^{z\prime}) = \mathbb{E}(u_t u_t^{z\prime})$, e.g., if there are no lagged proxies in the y_t equations in model (9), even if there are several proxies and shocks of interest. However, it only holds for the first shock, not for the other shocks in $\mathbf{w}_{1t}$ in general. Only if there are several proxies which are instantaneously uncorrelated, i.e., Σ_z is a diagonal matrix as in Corollary 1, then, by Corollary A.2 given in Appendix A, we can get also the impact effects of all shocks from the external proxy VAR approach from the Cholesky decomposition of Σ_u^{aug}, provided $\Sigma_{\mathbf{w}_1 z}$ in (5) is a diagonal matrix, that is, if the i-th proxy is only correlated with the i-th shock and uncorrelated with all other shocks.

We note that condition (a) of Corollary 1 implies that the proxies have to be Granger-noncausal for the y_t to ensure equivalence of the impact effects of the external and internal approaches. There are, in fact, good reasons for including lags of z_t in the y_t equations. If the proxies contain information on some of the structural shocks $\mathbf{w}_{1t}$, which after all is why we use them as proxies, they may well be Granger-causal for y_t. In that case, at least some of the A_i^{yz}, $i = 1, \ldots, p$, are nonzero. Including them leads to different impulse responses. Note that, in general, the $((K_1 + K) \times (K_1 + K))$ dimensional reduced-form impulse response matrices, say Φ_h^{aug}, of the augmented VAR model (9) are quite different from the $(K \times K)$ Φ_h matrices from model (1). In contrast, if $A_1^{yz} = \cdots = A_p^{yz} = 0$, the reduced-form impulse response matrices of the augmented model will have the form

$$\Phi_h^{aug} = \begin{bmatrix} * & * \\ 0 & \Phi_h \end{bmatrix}, \quad h = 1, 2, \ldots, \tag{12}$$

where * stands for potentially nonzero elements (see, e.g., Lütkepohl, 2005, Section 2.3.1). Computing, as usual, the structural impulse responses from the augmented model as

$$\Theta_h^{aug} = \Phi_h^{aug} \mathrm{chol}(\Sigma_u^{aug}),$$

we get exactly the same structural impulse responses from the augmented model as from the external proxy VAR approach whenever $B_1^{ext} = B_1^{int}$. We state that insight as Result 2 for easy reference.

Result 2. If $B_1^{ext} = B_1^{int}$ and $A_1^{yz} = \cdots = A_p^{yz} = 0$ in the augmented VAR model (9), then the structural impulse responses of the y_t variables from the external proxy VAR and the augmented VAR approaches are identical. □

The structural impulse responses are not only of interest by themselves but they are also the basis for forecast error variance decompositions (FEVDs) and historical decompositions. As the FEVDs can be computed as nonlinear functions of the structural impulse responses (see, e.g., Lütkepohl, 2005, Section 2.3.3), FEVDs of the external and internal approaches will be identical when the conditions for identical structural impulse responses are fulfilled. The situation is a bit different for historical decompositions which are also often of interest in structural VAR analysis (see, e.g., Antolín-Díaz and Rubio-Ramírez, 2018). They are determined from the structural impulse responses and the structural shocks. Thus, even if the impulse responses from the two approaches are identical, the historical decompositions may differ because the shocks $\mathbf{w}_{1t}^{ext}$ and $\mathbf{w}_{1t}^{int}$ generally differ (see Result 1). Which of the historical decompositions are preferred will depend on which shocks are more plausible. In any case, our results do not directly suggest how narrative restrictions as considered, for example, by Antolín-Díaz and Rubio-Ramírez (2018) for historical decompositions, can be useful for identification in our setup.

If the lags of the proxies enter the y_t equation in (9), one may even wonder whether the shocks of interest are fundamental and can be obtained as linear transformations of the reduced-form residuals u_t, as assumed in our model setup in Section 2.1, and in the external proxy VAR approach. Plagborg-Møller and Wolf (2021) consider the situation where a single shock is to be identified by one proxy. They consider a more general setup as ours by allowing the DGP of the y_t and the proxy to have infinite order VAR representations and the shocks to be potentially nonfundamental. In that framework they show that local projection methods, where the unlagged z_t and lags of z_t appear on the right-hand side of the y_t equations, can be used to estimate impulse responses even of nonfundamental shocks properly in an augmented VAR model. Clearly, being able to determine the impulse responses of nonfundamental shocks in an augmented VAR model setup is an advantage of the latter model.

In summary, if $A_1^{yz} = \cdots = A_p^{yz} = 0$ and there is only one shock that is identified by a single proxy, then we can use a Cholesky decomposition of the residual covariance matrix of the augmented VAR model to compute the impact effects of the impulse responses. If there are several shocks identified by a set of proxies, then the impact effects of the first shock can be obtained from the first column of the Cholesky decomposition of the residual covariance matrix of the augmented VAR. The impact effects of the other shocks can be obtained from the Cholesky decomposition of the residual covariance matrix of the augmented VAR if the proxies are uncorrelated (the matrix Σ_z is a diagonal matrix) and the i-th proxy is correlated with the i-th shock and not with any of the other shocks of interest (the matrix $\Sigma_{\mathbf{w}_1 z}$ is a diagonal matrix).

2.4. Estimation of augmented VAR models

The previous discussion refers to population quantities. It is important to emphasize that some of the results will carry over to estimated quantities in small samples as long as standard estimation methods are used. For example, if the VAR model (1) is estimated by LS and the LS residuals together with the proxies can be used for estimating B_1^{ext}. Of course, that may require further identifying restrictions to be imposed.

If the conditions of Proposition 1 can be assumed to hold, we can alternatively estimate the augmented model (9) by LS and estimate the covariance matrix Σ_u^{aug} as

$$\widehat{\Sigma}_u^{aug} = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} \hat{u}_t^z \\ \hat{u}_t^y \end{pmatrix} (\hat{u}_t^{z\prime}, \hat{u}_t^{y\prime}), \tag{13}$$

where the $\hat{u}_t^y$ and $\hat{u}_t^z$ are LS residuals. Then the lower left-hand $(K \times K_1)$ block of $\mathrm{chol}(\widehat{\Sigma}_u^{aug})$ can be used to estimate the impact effects B_1^{int} of the structural shocks. This estimator will generally differ from the one from the external approach because it is based on a model with many more parameters and, hence, is potentially less precise.

However, if we consider an augmented model (9) without lags in the z_t equations and without lags of the z_t in the y_t equations and we restrict the parameter estimates such that $A_1^{zz} = \cdots = A_p^{zz} = 0$, $A_1^{zy} = \cdots = A_p^{zy} = 0$ and $A_1^{yz} = \cdots = A_p^{yz} = 0$, then the estimator of Σ_u^{aug} becomes

$$\widehat{\Sigma}_u^{aug} = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} z_t - \bar{z} \\ \hat{u}_t \end{pmatrix} \left((z_t - \bar{z})', \hat{u}_t'\right), \tag{14}$$

where $\bar{z} = T^{-1}\sum_{t=1}^{T} z_t$ and the $\hat{u}_t$ are LS residuals. In that case, the corresponding estimator $\hat{B}_1^{int}$ based on $\mathrm{chol}(\widehat{\Sigma}_u^{aug})$ is equivalent to $\hat{B}_1^{ext} = T^{-1}\sum_{t=1}^{T} \hat{u}_t z_t'$ which may be used if $\Sigma_{\mathbf{w}_1 z}$ is assumed to be diagonal. Thus, in this case, the internal and external estimator would be identical even in small samples if the columns are scaled appropriately. In fact, even the estimated structural impulse responses would be identical in small samples in this case.

Alternatively, one may include lags of the proxies in the y_t equations of the augmented model (9) and consider the shocks and impulse responses obtained from the augmented model, still maintaining no lags in the z_t equations. In that case, the estimated shocks, their interpretation and their impulse responses are different in small samples from those of the external proxy VAR approach. Of course, the additional parameters to be estimated in the augmented VAR may reduce the precision of the estimated impulse responses if that approach is used.

One issue that is often a concern in the estimation of the impact effects is that some proxies are only weakly related to the shocks of interest. In other words, the proxies may be weak instruments for which special estimation procedures are recommended (see Montiel Olea et al., 2021). Clearly, as the external proxy and augmented VAR estimates are identical under the conditions of Proposition 1, both approaches are equally affected by the weakness of an instrument.

Of course, there are a number of other estimation methods for proxy VAR models. The methods based on (13) or (14) do not account for the over-identifying restrictions from the diagonality of $\Sigma_{\mathbf{w}_1 z}$ and the uncorrelatedness of the structural shocks and are, hence, not efficient. To improve efficiency, one could use the GMM approach of Bruns et al. (2025), as mentioned earlier. There are also local projection (LP) and Bayesian estimation methods that could be applied. Moreover, if there is just one proxy that identifies a single shock ($K_1 = N = 1$), then an equivalent estimator of B_1^{ext} is obtained by including z_t as an additional regressor in the VAR model, i.e., by estimating the model

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_1 z_t + u_t^*$$

by LS. The LS estimator $\hat{B}_1$ is a multiple of $\hat{B}_1^{ext} = T^{-1} \sum_{t=1}^{T} \hat{u}_t z_t$ in this case (see Paul, 2020, Online Appendix A.2, A.3).

Generally, it may be useful to keep in mind the theoretical results of Proposition 1 and Corollaries 1 and 2 in the identification and estimation of proxy VAR models. In the present study we are interested in presenting and illustrating the theoretical results of this section and therefore do not consider alternative estimation methods.

3. Oil market shocks

Känzig (2021) considers a six-dimensional benchmark model to study the impact of oil market shocks on some key economic variables. In his benchmark analysis, he uses a single proxy to identify an oil supply news shock. To safeguard against distortions in his analysis, he also uses a setup with two proxies to identify two shocks, an oil supply news shock and an oil production shortfall shock. Given that our main contribution is to generalize results known for a single proxy to the case of multiple proxies, we consider the two proxy case to illustrate our results and we will denote the corresponding proxies by z_t(news) and z_t(ops), respectively.

Känzig uses a VAR(12) model with a constant term for the real price of oil (rp_t), world oil production ($prod_t$), world oil inventories (inv_t), world industrial production (ip_t^{World}), U.S. industrial production (ip_t^{US}), and the U.S. consumer price index (cpi_t^{US}) such that $y_t = (rp_t, prod_t, inv_t, ip_t^{World}, ip_t^{US}, cpi_t^{US})'$. All variables are in logs. Känzig uses monthly data from January 1974 to December 2017. Hence, his gross sample size is 528. Accounting for the presample values required for LS estimation of the VAR(12) model, we have a net sample size of $T = 516$. We use his sample period and data set to facilitate a comparison with his results although there is some evidence that the structural impulse responses may not be time-invariant across the full sample period (see Bruns and Lütkepohl, 2023).

Känzig (2021) constructs one proxy, z_t(news), based on OPEC announcements about their production plans. It is used to identify an 'oil supply news shock', w_t(news), while the other proxy, z_t(ops), is based on work by Kilian (2008) and Bastianin and Manera (2018) and captures the shortfall of OPEC oil production caused by exogenous political events such as wars or civil disturbances and, hence, may be related to the first proxy. Känzig considers the two proxies to exclude possible distortions due to omitting effects related to his oil supply news shock.

We have applied the external and internal proxy VAR approaches to Känzig's reduced-form VAR model using his two proxies. As discussed in Section 2.2, the external approach requires further restrictions to identify the shocks individually. Känzig (2021) uses the external approach and identifies the shocks with the additional restriction that oil production does not respond instantaneously to an oil supply news shock. In other words, he applies an external proxy VAR approach where he imposes a zero restriction on B_1 to ensure individually identified and uncorrelated shocks. We consider this identification condition as one alternative but we also identify the shocks by using the proxies one-by-one to identify the shocks individually which is more in line with the results presented in Section 2. In other words, we are identifying the impact effects by zero restrictions on $\Sigma_{\mathbf{w}_1 z}$. More precisely, we assume that this matrix is diagonal. Thus, Känzig's identification approach for the case of two proxies is different from ours. To distinguish the resulting shocks, we denote the shocks obtained with Känzig's identifying restriction by $\hat{w}_t^{ext,K}$(ops) and $\hat{w}_t^{ext,K}$(news) while we signify the shocks obtained by using the proxies one-by-one by $\hat{w}_t^{ext,1by1}$(ops) and $\hat{w}_t^{ext,1by1}$(news). Note, however, that for the news shock, our approach corresponds exactly to Känzig's baseline model where he considers a single proxy for the news shock. In other words, $\hat{w}_t^{ext,1by1}$(news) is Känzig's news shock in his one-proxy baseline case.

For computing the shocks with the internal approach, we first do not include lags of z_t and y_t in the z_t equations of the augmented VAR model to limit the number of estimated parameters. This seems plausible, given the way the proxies are constructed.[4] In Fig. 1 we show scatter plots of the shocks obtained from the external and internal approaches, where the shocks from the augmented VAR approach are signified as $\hat{w}_t^{int}$(ops) and $\hat{w}_t^{int}$(news). Note that we have computed the $\hat{\mathbf{w}}_{1t}^{int}$ using the relation (10) and that z_t(news)

[4] A Wald test of the null hypothesis $A_i^{zz} = 0$, $A_i^{zy} = 0$ for $i = 1, \ldots, p$, results in a p-value of 0.361. For detailed formulas of a related test see Footnote 5.

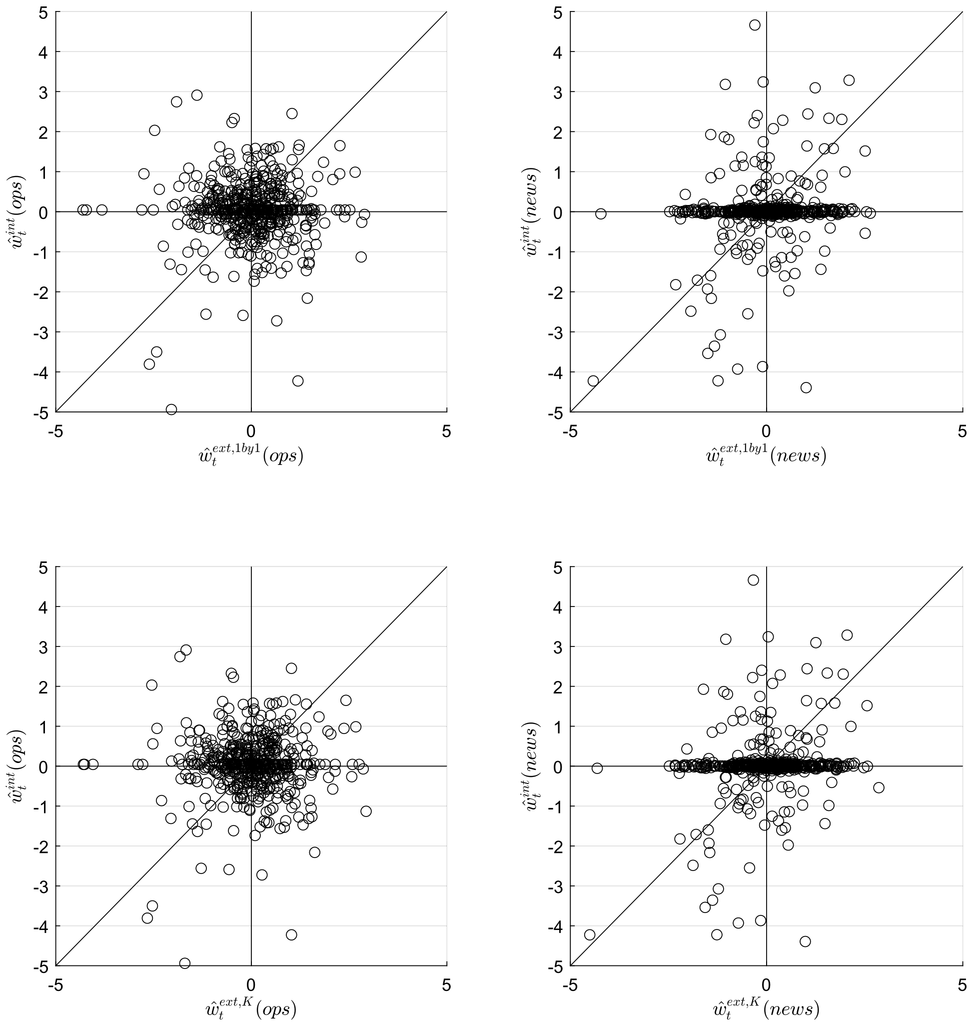

Fig. 1. Scatter plots of shocks for oil market example. $\mathbf{w}_{1t}^{int}$ obtained from model (9) restricting $A_i^{zz} = 0$, $A_i^{zy} = 0$, $i = 1, \ldots, p$.

contains many zero elements, implying $\hat{w}_t^{int}$(news) to take on constant values close to zero for much of the sample. In contrast, $\hat{w}_t^{ext,1by1}$(news) and $\hat{w}_t^{ext,K}$(news) vary across the full sample period. This outcome illustrates the external proxy VAR's ability to account for measurement errors in the proxies rather than intending to measure the shock directly. The two types of external shocks are in fact quite similar. Their correlation is close to one, as can be seen in the lower right-hand panel of Table 1.

Before we compare impulse responses based on the different approaches, we investigate the validity of the conditions (a), (b) and (c) of Corollary 1 to see whether similar empirical impulse responses from the one-by-one approach and the internal approach can be expected. We begin with a standard Wald test of $\mathbb{H}_0 : A_1^{yz} = \cdots = A_p^{yz} = 0$. It returns a value of 116.9 which corresponds to a p-value of 0.95 of the associated χ^2 limiting distribution with 144 degrees of freedom. Thus, the null hypothesis is clearly not rejected.[5] Although this suggests that the shocks may indeed be fundamental, it is clear that nonrejection of a null hypothesis can also mean that the test may not have enough power to reject. The null hypothesis may still be false. Actually, Plagborg-Møller and Wolf (2022) find that the Känzig shocks may not be fundamental. For illustrative purposes we nevertheless treat the shocks as fundamental in the following.

Condition (b) of Corollary 1 requires that the proxies are instantaneously uncorrelated. We show the correlation between the proxies in Table 1. Although the empirical correlation between the proxies is as small as -0.065, there is evidence that the proxies are correlated as zero is not in the 95% bootstrap confidence interval. Thus, condition (b) is approximately but possibly not strictly satisfied.

[5] Based on a model $y_t = [\nu, A_1^{yz}, \ldots, A_p^{yz}, A_1, \ldots, A_p]X_t + u_t = DX_t + u_t$ or, for $t = 1, \ldots, T$, $Y = DX + U$, we use the Wald statistic $W = \mathrm{vec}(\hat{D})' R' [R((XX')^{-1} \otimes \hat{\Sigma}_u)R']^{-1} R\mathrm{vec}(\hat{D})$ for testing $\mathbb{H}_0 : R\mathrm{vec}(D) = 0$. Here $\hat{D} = YX'(XX')^{-1}$ and $R = [0_{pKK_1 \times K}, I_{pKK_1}, 0_{pKK_1 \times pK^2}]$.

Table 1
Empirical Correlations of Proxies and Shocks for Oil Market Example with 95% Bootstrap Confidence Intervals.

	z_t(ops)	z_t(news)	$\hat{w}_t^{ext,1by1}$(ops)	$\hat{w}_t^{ext,1by1}$(news)
z_t(ops)	1			
z_t(news)	−0.065	1		
	(−0.131, −0.008)			
$\hat{w}_t^{ext,1by1}$(ops)	0.173	−0.020	1	
	(0.006, 0.319)	(−0.091, 0.050)		
$\hat{w}_t^{ext,1by1}$(news)	−0.015	0.226	−0.088	1
	(−0.130, 0.092)	(0.098, 0.346)	(−0.182, 0.005)	
$\hat{w}_t^{ext,K}$(ops)	0.170	0.019	0.985	0.084
	(0.007, 0.312)	(−0.049, 0.082)	(0.981, 0.988)	(−0.008, 0.174)
$\hat{w}_t^{ext,K}$(news)	−0.030	0.225	−0.172	0.997
	(−0.149, 0.082)	(0.096, 0.346)	(−0.262, −0.080)	(0.996, 0.997)

Note: The confidence intervals are obtained with a bootstrap suggested by Lunsford (2015) and presented in detail in the Appendix of Bruns et al. (2025).

Finally, condition (c) of Corollary 1 requires that $\Sigma_{\mathbf{w}_1 z}$ is diagonal. Using the estimated external shocks, obtained with the Känzig identifiying restriction and with the one-by-one approach, its off-diagonal elements are indeed very small and not significantly different from zero (see Table 1). The two covariance matrices $\Sigma_{\mathbf{w}_1 z}$ in Table 1 obtained with the one-by-one and Känzig identifying restrictions are actually very similar. That result is not surprising given the high correlation between the related shocks given in Table 1.

We also tested for diagonality of $\Sigma_{\mathbf{w}_1 z}$ by imposing the additional, over-identifying, restriction of orthogonality for the one-by-one shocks in a J-test, as proposed by Bruns et al. (2025). We find that the corresponding test statistic takes on a value of 0.0834 implying a p-value of 0.7727 of the corresponding asymptotic $\chi^2(1)$ distribution. Hence, we cannot reject that the moment conditions associated with a diagonal $\Sigma_{\mathbf{w}_1 z}$ jointly hold, supporting condition (c) of Corollary 1.

These results are at least indicative for the conditions of Corollary 1 to be approximately satisfied. Thus, the impulse responses obtained from the internal and external approaches are expected to be similar, despite the rather different shocks in Fig. 1. Actually, the impulse responses to the first shock, the ops shock, obtained by the one-by-one external proxy VAR approach and the augmented VAR approach restricting $A_i^{zz} = 0$, $A_i^{zy} = 0$, $A_i^{yz} = 0$, $i = 1, \dots, p$, are identical, as discussed in Section 2.3 (see Corollary 2). Therefore we focus on the impulse responses of the news shock and compare them in Fig. 2. We follow Känzig (2021) and consider shocks that increase the real oil price by 10% on impact.

In Fig. 2 we show the impulse responses corresponding to four different shocks: $\hat{w}_t^{ext,K}$(news), $\hat{w}_t^{ext,1by1}$(news) and $\hat{w}_t^{int}$(news) with and without lags of the proxies and endogenous variables in the proxy equations. The impulse responses to the two types of external shocks in the first two columns of Fig. 2 are very similar. The main difference is that the response of oil production on impact is exactly zero when a $\hat{w}_t^{ext,K}$(news) hits while it is only close to zero for a $\hat{w}_t^{ext,1by1}$(news) shock.

If A_i^{zz}, A_i^{zy} and A_i^{yz} are restricted to zero in the augmented model, the responses to a $\hat{w}_t^{int}$(news) are also very similar to the responses in the first two columns of Fig. 2, as expected, given that the conditions of Corollary 1 roughly hold. Although setting the A_i^{zz}, A_i^{zy} and A_i^{yz} to zero is not rejected by our tests, leaving them unrestricted has a substantial impact on the estimated impulse responses. First of all, the confidence intervals (in cyan color in Fig. 2, column 4) are much wider than the confidence intervals of the corresponding impulse responses estimated with the other three approaches (in blue, pink and red in Fig. 2) which may be due to the larger number of parameters in the model. The impulse responses also have partly very different shapes and, hence, are likely to lead to different interpretations of the dynamics of the system. For example, the response of the cpi^{US} index is much less persistent than the response estimated with the other three methods. Thus, augmenting the model with many additional insignificant parameters, may not be a good idea if they are actually not needed. On the other hand, the differences in the impulse responses may be indicative for the lags to be important in the equations in which case the impulse responses based on the external VAR approaches and from the augmented VAR with $A_i^{zz} = 0$, $A_i^{zy} = 0$, $A_i^{yz} = 0$, $i = 1, \dots, p$, would be distorted.

Clearly, the internal approach as discussed in the foregoing does not result in impulse responses satisfying Känzig's identifying restriction exactly. To impose such a restriction, one could also consider internalizing the proxies and use a different identification scheme by considering a B^{aug} matrix that satisfies $B^{aug} B^{aug\prime} = \Sigma_u^{aug}$ but is different from $\mathrm{chol}(\Sigma_u^{aug})$. As such alternative approaches are not common in the literature we do not elaborate on this possibility here but just mention that by a suitable choice of B^{aug} one could obtain the restricted impact effect of production considered by Känzig.

We have also reversed the order of the proxies and hence the shocks such that the news shock is first and the oil production shortfall shock is second. In Fig. B.1 in Appendix B we show the corresponding impulse responses of the oil production shortfall shock. The results are qualitatively similar to those in Fig. 2 in that the responses to $\hat{w}_t^{ext,1by1}$(ops), $\hat{w}_t^{ext,K}$(ops) and $\hat{w}_t^{int}$(ops) shocks restricting $A_i^{zz} = 0$, $A_i^{zy} = 0$, $A_i^{yz} = 0$, $i = 1, \dots, p$, are almost identical, while the responses to $\hat{w}_t^{int}$(ops) with A_i^{zz}, A_i^{zy}, and A_i^{yz} not restricted to zero have wider confidence intervals and are partly quite different from the corresponding other two impulse responses.

Given that historical decompositions involve both structural impulse responses and the shocks, they can be expected to be quite different for the external and internal proxy VAR approaches because the shocks are very different (see Fig. 1). To illustrate this point, we show the historical contributions of the news shocks to the real price of oil in Fig. 3. Obviously, the historical contributions

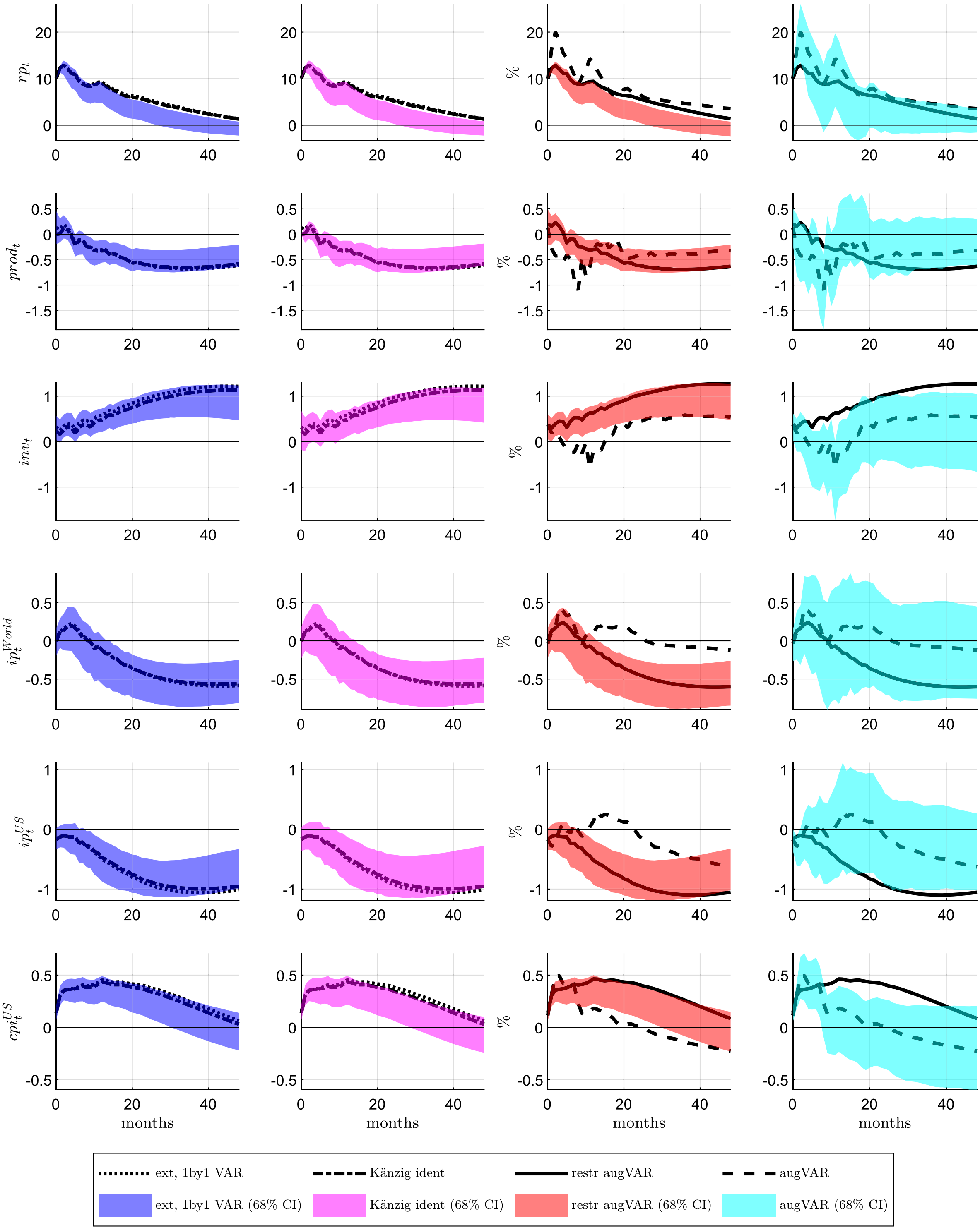

Fig. 2. Comparison of impulse responses of $\hat{w}_t^{ext,1by1}$ (news) shock (column 1), $\hat{w}_t^{ext,K}$ (news) shock (column 2), $\hat{w}_t^{int}$ (news) shock with restrictions $A_i^{zz}=0$, $A_i^{zy}=0$, $A_i^{yz}=0$, $i=1,\dots,p$, (column 3) and without restrictions (column 4). The shocks are normalized to increase oil prices by 10 percent on impact. The confidence intervals around the impulse responses are based on 5000 bootstrap samples. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

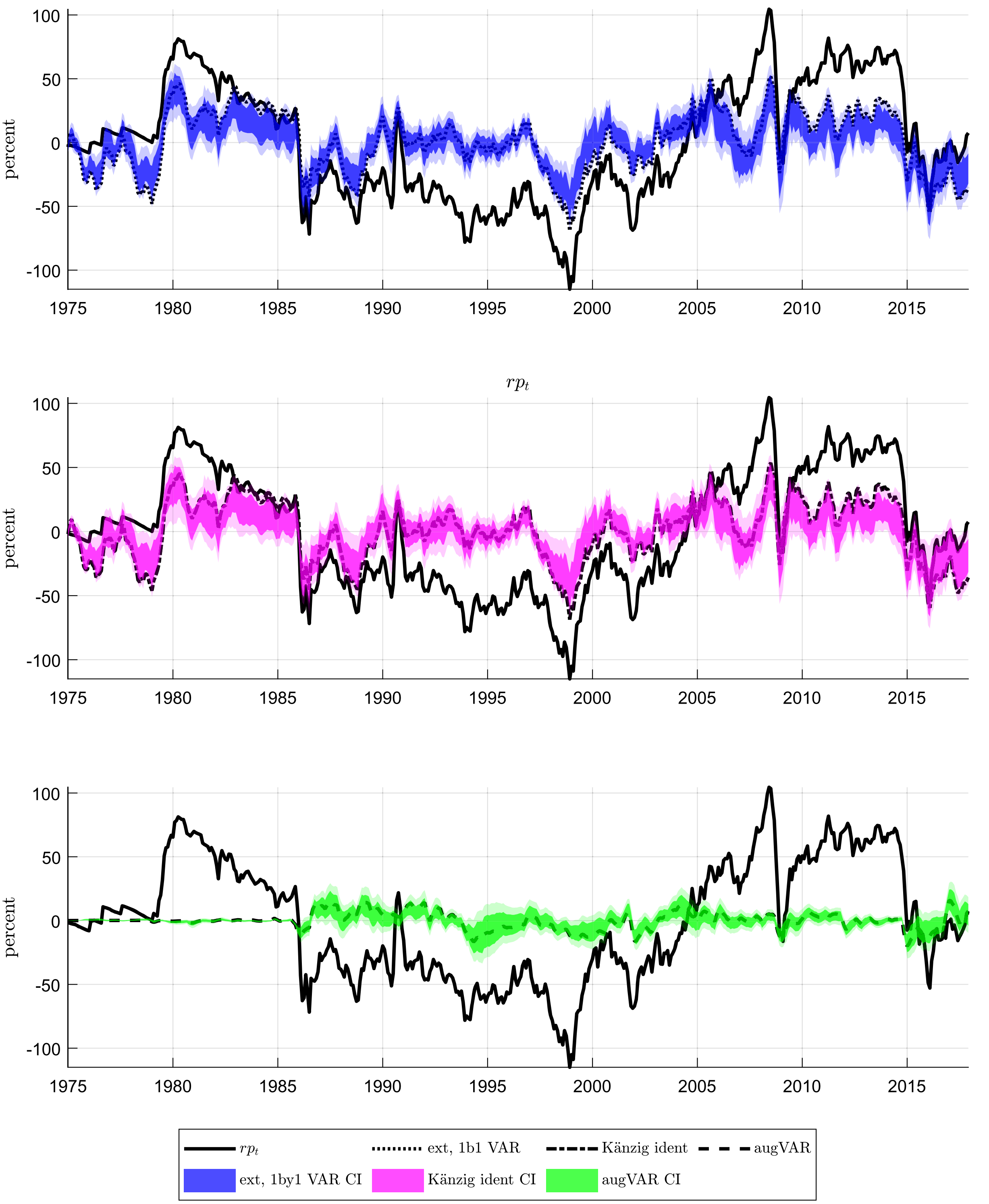

Fig. 3. Cumulative historical contributions of news shocks to the real price of oil. The top panel shows the historical decomposition based on $\hat{w}_t^{ext,1b1}$(news). The middle panel shows the contribution of $\hat{w}_t^{ext,K}$(news). The bottom panel shows the historical decomposition based on $\hat{w}_t^{int}$(news) obtained from equation (9) restricting $A_i^{zz} = 0,\ A_i^{zy} = 0,\ i = 1, \dots, p$. The 68% and 90% confidence intervals around the historical decomposition are based on 5000 bootstrap samples.

from the external and internal approaches differ substantially while the two external approaches deliver very similar results. If we use $\hat{w}_t^{int}$(news) and recover the structural shocks as linear combinations of the proxies, then the historical decompositions look very peculiar. The reason is that $\hat{w}_t^{int}$(news) is constant for much of the sample and can therefore not contribute to the historical evolution of the real price of oil. In the present example one might argue that the proxies are better thought of as noisy rather than direct measurements of the shocks themselves and that the historical decomposition is more readily interpretable if it is constructed using $\hat{w}_t^{ext}$(news) rather than $\hat{w}_t^{int}$(news).

4. Conclusions

When instruments are used in a proxy VAR study to identify a single shock or a set of shocks, a couple of alternative approaches for estimating the structural parameters are in common use. The first one is based on the covariance of the proxies and the reduced-form residuals and another one augments the VAR by the proxies. We have pointed out important differences and similarities between these approaches for the case of multiple proxies. Thereby we not only provide some new insights for researchers using multiple proxies to identify a set of shocks but also generalize results that were previously known for the case of identifying a single shock only.

In general, both approaches for identifying multiple shocks by a set of proxies have to be complemented with additional identifying assumptions in order to properly identify the shocks of interest individually. However, if there are exactly as many proxies as there are shocks to be identified and if the proxies are mutually uncorrelated, and each of them is correlated with a single shock only, then no additional information is needed to fully identify the shocks of interest individually. If the proxies are not Granger-causal for the variables of the VAR model, the impulse responses obtained with the two alternative estimation approaches are identical in population and may be very similar if in the augmented VAR no lags of the proxies are included in the VAR equations for the variables. If the lagged proxies are nevertheless included in the model, the increased estimation uncertainty due to the additional parameters in the model may distort the impulse responses. Even if the conditions for identical population impulse responses are satisfied, the shocks will generally not be identical. In the external proxy VAR approach, the shocks are linear combinations of the reduced-form residuals while the shocks are linear combinations of the mean-adjusted proxies or the residuals of the proxy equations in the augmented VAR approach.

We consider an empirical example model for the crude oil market to illustrate these theoretical results. In the example, the conditions for identical population impulse responses are nearly satisfied and, as expected, the estimated impulse responses turn out to be also very similar if lagged proxies are not included in the augmented model. Dropping the lagged proxies in this example is supported by the nonsignificant outcome of a Granger-causality test. If nevertheless lags of the proxies and the variables are included on the right-hand side of the augmented VAR model, the additional estimation uncertainty due to model augmentation is reflected in much wider confidence intervals around the impulse responses and quite different impulse response estimates than in the external proxy VAR approach that may lead the researcher to draw different conclusions regarding the dynamics of the model. Thus, the example illustrates that it may not be a good idea to include unnecessarily many parameters in the VAR model.

Our results imply the following strategies for applied work. If the shocks can be thought of as linear combinations of the proxies, then the augmented VAR approach may be useful. If, however, the proxies are better thought of as shocks measured with error, then the external proxy VAR approach is perhaps more suitable. If the conditions for identical population impulse responses from the two approaches are satisfied and the researcher is interested only in the impulse responses and FEVDs, then s/he is free to choose between the external proxy VAR and the augmented VAR approaches. As the shocks based on the different approaches still differ, it depends on the construction of the proxies whether using the internal approach can be recommended for a historical decomposition. Finally, if the proxies are Granger-causal for the variables of the model, then using the augmented VAR approach with lags of the proxies in the model is called for and the two approaches will provide different shocks and impulse responses.

In practice, one may also want to consider alternative estimation methods which may be more efficient or account for weak instruments. Also Bayesian methods can be considered instead of the frequentist methods mentioned in this study. We have not discussed those methods here because the objective of this study is to raise awareness for the theoretical relations between external proxies and internalizing them. Considering the implications for the various possible estimation methods for proxy VAR analysis may be an interesting topic for future research.

Appendix A. Proof of Proposition 1

Proposition 1 follows from the following matrix result.

Lemma 1. *Let Σ_{11} be a symmetric positive definite $(N \times N)$ matrix, Σ_{22} be a symmetric positive definite $(K \times K)$ matrix, and Σ_{21} a $(K \times N)$ matrix such that the $((N+K)\times(N+K))$ matrix*

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{21}' \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

is positive definite. Then the lower-triangular Cholesky decomposition of Σ is

$$chol(\Sigma) = \begin{bmatrix} chol(\Sigma_{11}) & 0 \\ \Sigma_{21} chol(\Sigma_{11})^{-1\prime} & G \end{bmatrix}, \tag{A.1}$$

where $G = chol(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}')$, *i.e.,* $GG' = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}'$. □

Proof of Lemma 1. The lemma follows by multiplying the right-hand side of equation (A.1) by its transpose and noting that $\text{chol}(\Sigma_{11})^{-1\prime}\text{chol}(\Sigma_{11})^{-1} = \Sigma_{11}^{-1}$. □

Lemma 1 is a useful tool for proving Proposition 1.

Proof of Proposition 1. From relation (5) we have $\mathbb{E}(u_t z_t') = B_1\Sigma_{\mathbf{w}_1 z}$. Hence, the external proxy VAR approach implies $B_1^{ext} = \mathbb{E}(u_t z_t')\Sigma_{\mathbf{w}_1 z}^{-1}$ because the proxies satisfy the relevance and exogeneity conditions and the rank condition implies invertibility of $\Sigma_{\mathbf{w}_1 z}$ given that $N = K_1$.

Setting $\Sigma_{11} = \Sigma_z$ and

$$\Sigma_{21} = \mathbb{E}(u_t z_t') = \mathbb{E}\left(u_t(z_t - \mathbb{E}(z_t))'\right) = \mathbb{E}(u_t u_t^{z\prime}) = \mathbb{E}(u_t^y u_t^{z\prime}),$$

Lemma 1 implies that the lower-left hand $(K \times K_1)$ block of $\text{chol}(\Sigma_u^{aug})$ is $B_1^{int} = \mathbb{E}(u_t z_t')\text{chol}(\Sigma_z)^{-1\prime}$. Thus, if B_1^{ext} is equal to the latter matrix, we have

$$\mathbb{E}(u_t z_t')\text{chol}(\Sigma_z)^{-1\prime} = \mathbb{E}(u_t z_t')\Sigma_{\mathbf{w}_1 z}^{-1}. \tag{A.2}$$

Multiplying from the left by $(\mathbb{E}(u_t z_t')'\mathbb{E}(u_t z_t'))^{-1}\mathbb{E}(u_t z_t')'$ gives $\Sigma_{\mathbf{w}_1 z} = \text{chol}(\Sigma_z)^{-1\prime}$ showing that $B_1^{ext} = B_1^{int}$ implies the result in (11).

Conversely, (11) implies (A.2) and, hence, equality of the impact effects from both approaches, as claimed in Proposition 1. □

We also state the following straightforward implications of Lemma 1 for future reference.

Corollary A.1. *The first column of* $chol(\Sigma)$ *is a multiple of the first column of* Σ. *More precisely, denoting the upper left-hand element of* Σ *by* σ_{11}, *the first column of* $chol(\Sigma)$ *is* $1/\sqrt{\sigma_{11}}$ *times the first column of* Σ. □

Proof of Corollary A.1. Obvious. □

Corollary A.2. *If* Σ_{11} *in Proposition 1 is a diagonal matrix, then the first* N *columns of* $chol(\Sigma)$ *are multiples of the corresponding columns of* Σ. *More precisely, denoting the* i*-th diagonal element of* Σ_{11} *by* σ_{ii}, *the* i*-th column of* $chol(\Sigma)$ *is* $1/\sqrt{\sigma_{ii}}$ *times the* i*-th column of* Σ *for* $i = 1, \dots, N$. □

Proof of Corollary A.2. The corollary follows by noting that, for a diagonal matrix $\Sigma_{11} = \text{diag}(\sigma_{11}, \dots, \sigma_{NN})$, $\sigma_{ii} > 0$, the Cholesky decomposition is $\text{chol}(\Sigma_{11}) = \text{diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{NN}})$. □

Appendix B. Additional results

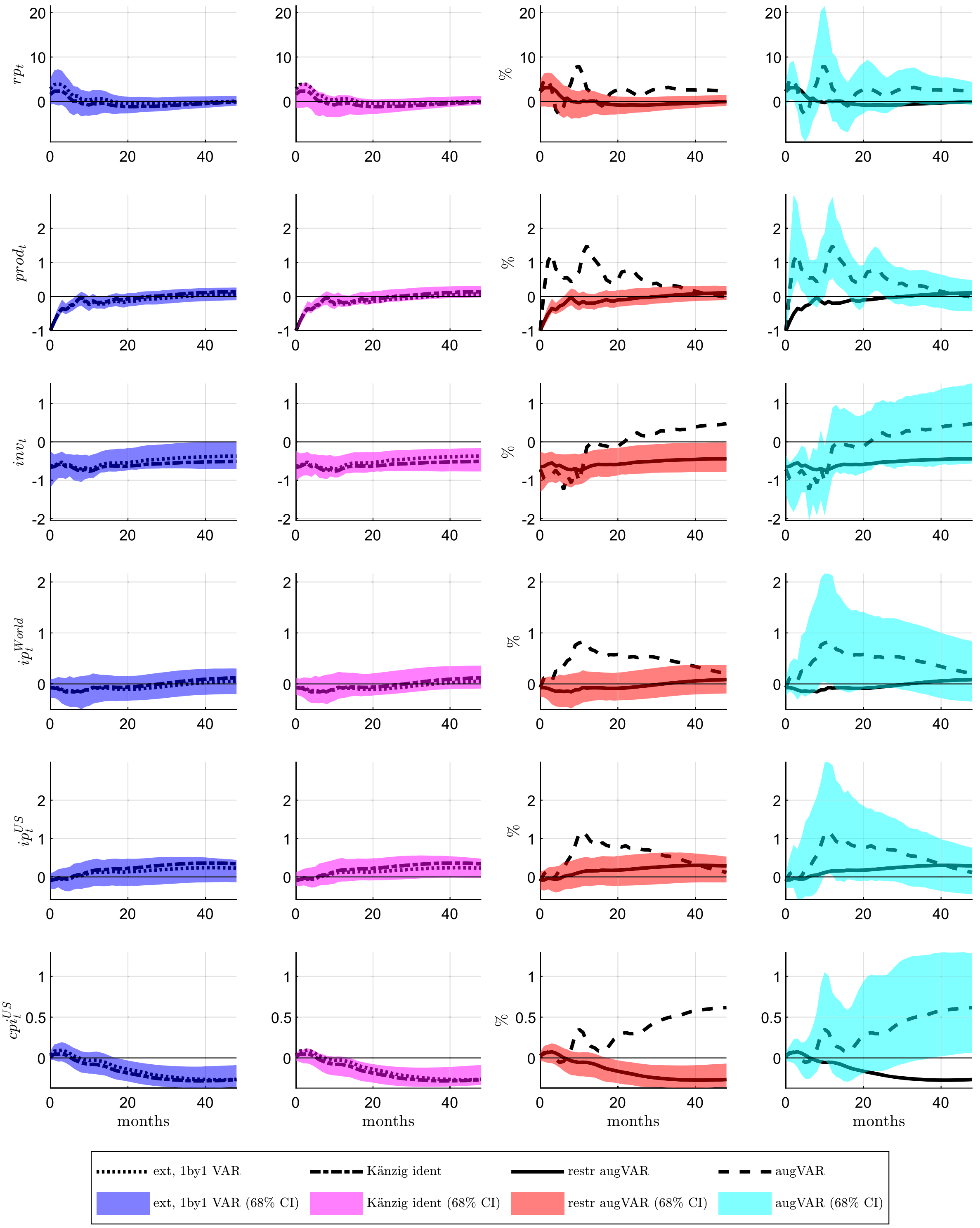

Fig. B.1. Comparison of impulse responses of $\hat{w}_t^{ext,1by1}$ (ops) shock (column 1), $\hat{w}_t^{ext,K}$ (ops) shock (column 2), $\hat{w}_t^{int}$ (ops) shock with restrictions $A_i^{zz}=0$, $A_i^{zy}=0$, $A_i^{yz}=0$, $i=1,\dots,p$, (column 3) and without restrictions (column 4). The shocks are normalized to increase oil prices by 10 percent on impact. The confidence intervals around the impulse responses are based on 5000 bootstrap samples.

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