ON THE NUMBER OF SUBDIRECT PRODUCTS INVOLVING SEMIGROUPS OF INTEGERS AND NATURAL NUMBERS

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Abstract. We extend a recent result that for the (additive) semigroup of positive integers $\mathbb{N}$, there are continuum many subdirect products of $\mathbb{N} \times \mathbb{N}$ up to isomorphism. We prove that for U, V each one of $\mathbb{Z}$ (the group of integers), $\mathbb{N}_0$ (the monoid of non-negative integers), or $\mathbb{N}$, the direct product $U \times V$ contains continuum many (semigroup) subdirect products up to isomorphism.

1. Introduction

In [5] it is proved that the direct product $\mathbb{N} \times \mathbb{N}$ of two copies of the free monogenic semigroup $\mathbb{N}$ contains uncountably many pairwise non-isomorphic subdirect products. This is perhaps somewhat surprising, given that the direct product $\mathbb{Z} \times \mathbb{Z}$ of two copies of the free cyclic group contains only two subdirect products up to isomorphism, namely $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ itself, and that the subsemigroup structure of $\mathbb{N}$ is not fundamentally different from the subgroup structure of $\mathbb{Z}$, in that both essentially depend on arithmetic progressions; see [9] for an explicit description.

The purpose of this paper is to extend the scope of the above-mentioned result from [5] and prove the following:

Main Theorem. *Let each of U and V be any of the following three additive semigroups: $\mathbb{Z}$, the group of integers; $\mathbb{N}_0$, the monoid of non-negative integers; $\mathbb{N}$, the semigroup of natural numbers. Then $U \times V$ contains continuum many non-isomorphic semigroup subdirect products of U and V.*

By a *subdirect product* of two semigroups U and V we mean any subsemigroup P of $U \times V$ which projects *onto* each of U and V, i.e. $\{u : (u, v) \in P \text{ for some } v\} = U$ and $\{v : (u, v) \in P \text{ for some } u\} = V$. Subdirect products are an important decomposition tool in algebra in general, due to Birkhoff's decomposition theorem [7, Theorem 4.44]. They also have many intriguing combinatorial properties. For some examples from group theory see [2, 3, 4], and for a discussion from the viewpoint of general algebra see [6].

The rest of the paper constitutes the proof of the Main Theorem, using the following outline. For reasons of symmetry, and keeping in mind that the case where $U = V = \mathbb{N}$ has been dealt with in [5], it is sufficient to prove the theorem for (U, V) in $\mathcal{P} = \{(\mathbb{N}_0, \mathbb{N}), (\mathbb{N}_0, \mathbb{N}_0), (\mathbb{Z}, \mathbb{N}), (\mathbb{Z}, \mathbb{N}_0), (\mathbb{Z}, \mathbb{Z})\}$. In Section 2 we construct a family of

2020 *Mathematics Subject Classification.* 20M13, 20M14.

Key words and phrases. Semigroup, natural number, integer, subdirect product, indecomposable element.

The third author acknowledges support from EPSRC EP/V003224/1.

subsemigroups S_σ of $\mathbb{Z} \times \mathbb{Z}$, where σ is a sequence of natural numbers with certain additional requirements. These requirements are sufficiently mild that the number of sequences satisfying them is uncountable. We begin Section 3 by proving that the intersection $S_\sigma \cap (U \times V)$ is a subdirect product in $U \times V$ for each $(U, V) \in \mathcal{P}$ (Lemma 3.1). In the remainder of Section 3 we consider each possibility for (U, V) in turn, starting with $(U, V) = (\mathbb{N}_0, \mathbb{N})$, and show that for $\sigma \neq \tau$ we have $S_\sigma \cap (U \times V) \not\cong S_\tau \cap (U \times V)$. Thus the subsemigroups $S_\sigma \cap (U \times V)$ constitute uncountably many pairwise non-isomorphic subdirect products in $U \times V$, and the Main Theorem is proved.

Of the several assertions encompassed by the Main Theorem, perhaps the one concerning $\mathbb{Z} \times \mathbb{Z}$ is worth highlighting as somewhat surprising. As mentioned earlier, $\mathbb{Z} \times \mathbb{Z}$ contains *countably many group* subdirect products. However, our result shows that it contains *uncountably many semigroup* subdirect products.

2. The semigroups S_σ

We begin our work towards proving the Main Theorem by exhibiting a family S_σ of subdirect products of $\mathbb{Z} \times \mathbb{Z}$ indexed by certain infinite sequences of natural numbers. We first define the sets $S_\sigma \subseteq \mathbb{Z} \times \mathbb{Z}$, then prove they are subsemigroups, and finally that they are subdirect products.

Construction 2.1. Given a sequence, $\sigma = (c_i)_{i \geq 2}$ of natural numbers satisfying

$$c_2 = 1 \quad \text{and} \quad c_{i+1} \geq 2c_i \text{ for all } i \geq 2, \tag{1}$$

define

$$S_\sigma := \{(x, y) : x \leq 0, y \geq x\} \cup \bigcup_{k=2}^{\infty} \{(x, x + k) : x = 1, \ldots, c_k\}.$$

The following comments and Figure 1 may be of help in understanding S_σ and how it will be treated subsequently.

- It is useful to consider S_σ as a union of 'vertical lines'. Specifically, $S_\sigma = \bigcup_{i \in \mathbb{Z}} L_i$, where $L_i := S_\sigma \cap (\{i\} \times \mathbb{Z})$.
- The lines L_i with $i \leq 0$ are the same for all S_σ, namely $L_i = \{(i, x) : x \geq i\}$.
- The remaining lines L_i, $i > 0$, depend on σ. Each such line L_i has a unique 'lowest point', denoted (i, l_i). The construction assures that $l_i > i$. The line contains all points above this lowest point, meaning that $(i, x) \in L_i$ for all integers $x \geq l_i$.
- The number $c_k = i$ indicates the rightmost line L_i for which the lowest point is $(i, i + k)$.
- In other words, for any $i > 0$ and $k \geq 2$, we have $L_i = \{(i, x) : x \geq i + k\}$ if and only if $c_{k-1} < i \leq c_k$ for all $i, k > 0$.
- The conditions $c_2 = 1$ and $c_{i+1} \geq 2c_i$ are technical, and are needed to facilitate the proofs of closure below and non-isomorphism later on.
- Due to the fixed requirement $c_2 = 1$, we have that $L_1 = \{(1, x) : x \geq 3\}$ is still the same for all S_σ.

The above terminology and notation will be used throughout the paper. In Figure 1 we visualise a typical example of S_σ.

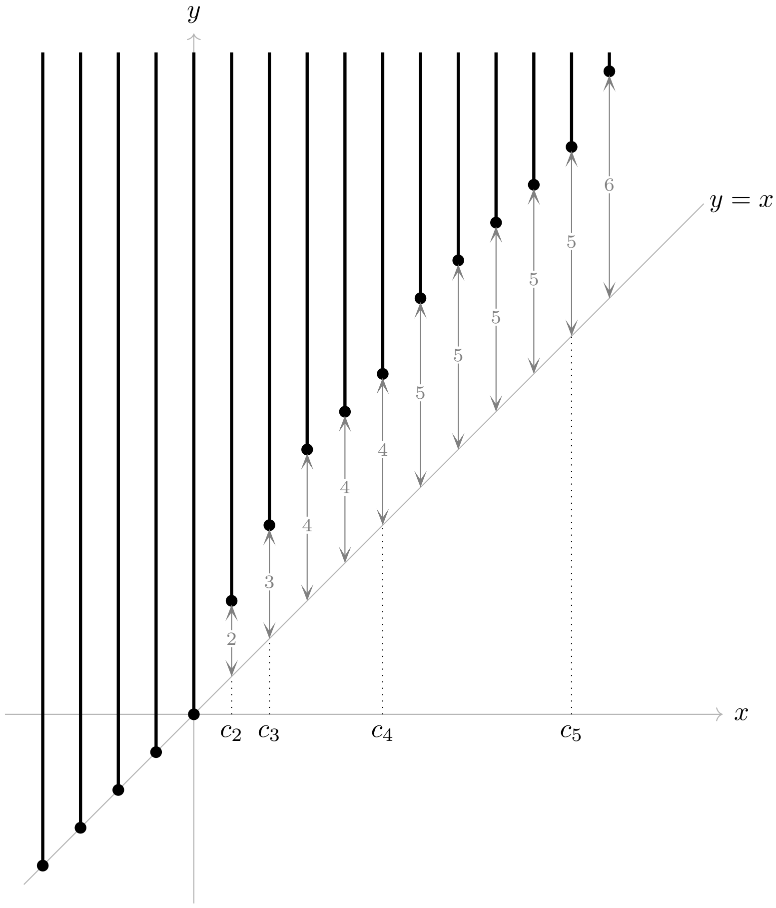

FIGURE 1. The semigroup S_σ, with $\sigma = (1, 2, 5, 10, \dots)$.

Lemma 2.2. *Each S_σ is a subsemigroup of $\mathbb{Z} \times \mathbb{Z}$.*

Proof. We show that S_σ is closed under pairwise addition. To this end, let $\mu, \nu \in S_\sigma$, with $\mu = (p, q)$, $\nu = (r, s)$. Without loss of generality we may suppose that $p \leq r$. We split the proof that $\mu + \nu = (p+r, q+s) \in S_\sigma$ into cases, depending on the sign of $p+r$.

Case 1: $p + r \leq 0$. In this instance, to show $(p + r, q + s) \in S_\sigma$, it suffices to show that $q+s \geq p+r$. This follows, as $(p, q), (r, s) \in S_\sigma$ implies $q \geq p$ and $s \geq r$ by construction, whence $q + s \geq p + r$.

Case 2: $p + r > 0$. From $p \leq r$ we have $r > 0$. Let $k, l \geq 2$ be the unique numbers such that

$$c_{k-1} < r \leq c_k, \tag{2}$$

$$c_{l-1} < p + r \leq c_l. \tag{3}$$

To show that $(p + r, q + s) \in S_\sigma$ it suffices to show that $q + s \geq p + r + l$.

If $p \leq 0$ then $p + r \leq r$, $c_l \leq c_k$ and $l \leq k$ follow in order, and then

$$q + s \geq p + r + k \geq p + r + l.$$

Suppose now that $p > 0$. Let $j \geq 2$ be the unique number such that

$$c_{j-1} < p \leq c_j,$$

whereby $q \geq p + j$. We have that

$$q + s \geq p + r + j + k,$$

from which it follows that

$$c_{l-1} \leq p + r \leq 2r \leq 2c_k \leq c_{k+1} \leq c_{j+k-1},$$

using (1)-(3) and $j - 1 \geq 1$. This implies $l \leq j + k$, and so

$$q + s \geq p + r + l$$

as required, completing the proof that $S_\sigma \leq \mathbb{Z} \times \mathbb{Z}$. □

Lemma 2.3. *Each S_σ is a subdirect product of $\mathbb{Z} \times \mathbb{Z}$.*

Proof. Any integer can be obtained as the first coordinate of a pair using the elements $(1, 3)$ and $(-1, -1)$, which are in S_σ for every σ. The same can be done in the second coordinate using $(0, 1), (-1, -1) \in S_\sigma$. □

3. Intersection of S_σ with some subsemigroups of $\mathbb{Z} \times \mathbb{Z}$

In this section, let $(U, V) \in \{(\mathbb{Z}, \mathbb{Z}), (\mathbb{Z}, \mathbb{N}_0), (\mathbb{Z}, \mathbb{N}), (\mathbb{N}_0, \mathbb{N}_0), (\mathbb{N}_0, \mathbb{N})\}$. Recall from the introduction that we need only consider such (U, V) to prove our Main Theorem.

Having constructed the semigroups S_σ in the preceding section as subsemigroups of $\mathbb{Z} \times \mathbb{Z}$, this gives us the following way of obtaining subdirect products of $U \times V$ from them.

Lemma 3.1. *The intersection $S_\sigma \cap (U \times V)$ is a subdirect product of $U \times V$.*

Proof. First, the intersection is a subsemigroup of $U \times V$, as $U \times V$ and S_σ are both subsemigroups of $\mathbb{Z} \times \mathbb{Z}$ (the latter by Lemma 2.2).

It then just remains to show that the projection maps onto U and V are surjective. For any $i \in U$ the line L_i has non-empty intersection with $S_\sigma \cap (U \times V)$, and any element of this line has first coordinate i. This gives surjectivity of the first projection map.

For the second projection map, if $j \in V$ is such that $j < 0$, it must be that $U = V = \mathbb{Z}$, in which case $S_\sigma \cap (U \times V) = S_\sigma$, which is a subdirect product by Lemma 2.3.

If $j = 0$, then V is one of $\mathbb{Z}$ or $\mathbb{N}_0$, and we have $(0, 0) \in S_\sigma \cap (U \times V)$.

Finally, if $j > 0$, then as $L_0 \setminus \{(0, 0)\} \subseteq S_\sigma \cap (U \times V)$, it follows that $(0, j) \in S_\sigma \cap (U \times V)$.

This completes the proof of surjectivity of the second projection map, and thus of the lemma. □

If we can show that different sequences σ and τ give non-isomorphic subdirect products $S_\sigma \cap (U \times V)$ and $S_\tau \cap (U \times V)$, this will be sufficient to prove our Main Theorem.

In the following subsections, we will use the notion of *indecomposability.* In fact, we will use this term in two different senses. Suppose W is a subsemigroup of $\mathbb{Z} \times \mathbb{Z}$. An element $(a, b) \in W$ is *semigroup indecomposable* if it cannot be written as the sum of any two elements from W. In case where W is a monoid, i.e. where W contains the element $(0, 0)$, we say that $(a, b) \in W$ is *monoid indecomposable* if it cannot be written as the sum of any two elements of $W \setminus \{(0, 0)\}$. Typically, we will omit the adjective 'semigroup' or 'monoid' when talking about indecomposability, but it will always be clear from context which one is meant.

3.1. **Intersection with $\mathbb{N}_0 \times \mathbb{N}$.** We will start with the case where $U \times V = \mathbb{N}_0 \times \mathbb{N}$. The semigroup $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ is just the union of the lines $\{L_i : i \geq 0\}$ from S_σ, but without the identity $(0, 0)$. Recall that the lowest point of a line L_i is denoted (i, l_i).

We describe the indecomposables of $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ in the following lemma, which will be useful in ruling out possible isomorphisms between these semigroups.

Lemma 3.2. *The set of indecomposable elements of $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ is exactly the set*

$$\{(0, 1)\} \cup \{(i, l_i) : i \geq 1\}.$$

Proof. As

$$(i, j) = (i, l_i) + (j - l_i)(0, 1)$$

for all $(i, j) \in S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$, then any element which is not the lowest point of its line is decomposable. Hence it remains to show that $(0, 1)$ and the lowest points of each line are indecomposable in $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$.

Firstly, $(0, 1)$ is indecomposable as 1 is indecomposable in $\mathbb{N}$.

Now suppose that some element (i, l_i) for $i \in \mathbb{N}$ is decomposable, say

$$(i, l_i) = (j, q) + (k, r) \tag{4}$$

for some $(j, q), (k, r) \in S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$.

Note that we cannot have $j = 0$ or $k = 0$ as that would contradict (i, l_i) being the lowest point of the line L_i; thus $j, k \geq 1$. Now let $x, y, z \geq 2$ be the smallest possible satisfying

(i) $i \leq c_x$, so that $l_i = i + x$;
(ii) $j \leq c_y$, so that $l_j = j + y \leq q$;
(iii) $k \leq c_z$, so that $l_k = k + z \leq r$.

From, (4), (i), (ii) and (iii), we have:

$$j + k + x = i + x = l_i = q + r \geq l_j + l_k = j + k + y + z,$$

and hence $x \geq y + z$. Recalling that $c_{n+1} \geq 2c_n$ for all $n \geq 2$, we have that $c_{y+z-1} \geq 2^{z-1}c_y$ and $c_{y+z-1} \geq 2^{y-1}c_z$. Using this, together with $y, z \geq 2$ and (ii) and (iii), we have:

$$i = j + k \leq c_y + c_z \leq (\frac{1}{2^{z-1}} + \frac{1}{2^{y-1}})c_{y+z-1} \leq c_{y+z-1} \leq c_{x-1}$$

a contradiction with minimality of x with respect to (i). Hence, the elements of the form (i, l_i) are all indecomposable. □

We can now prove the main result of this section – that there are continuum many subdirect products of $\mathbb{N}_0 \times \mathbb{N}$ up to isomorphism.

Proposition 3.3. *For any two sequences σ and τ satisfying the conditions of Construction 2.1, we have that*

$$\sigma \neq \tau \;\Rightarrow\; S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N}) \ncong S_\tau \cap (\mathbb{N}_0 \times \mathbb{N}).$$

Consequently, there are continuum many subdirect products of $\mathbb{N}_0 \times \mathbb{N}$ *up to isomorphism.*

Proof. We will prove the contrapositive. So suppose two subdirect products $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ and $S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$ are isomorphic, and let φ be an isomorphism between them.

This isomorphism must map the indecomposable elements of $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ bijectively onto indecomposable elements of $S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$.

Any indecomposable (i, l_i) in either semigroup has the property that $(i, l_i) + (i, l_i)$ has more than one decomposition into a sum of indecomposable elements, as

$$(i, l_i) + (i, l_i) = (2i, l_{2i}) + (2l_i - l_{2i})(0, 1).$$

By way of contrast, $(0, 1) + (0, 1)$ has only that one decomposition into a sum of indecomposables. Hence it must be that $\varphi(0, 1) = (0, 1)$.

Now consider the image of the indecomposable $(1, 3)$, say $\varphi(1, 3) = (j, l_j)$ for some $j \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, it must be that

$$(nj, nl_j) = \varphi(n, 3n) = \varphi\big((n, l_n) + (3n - l_n)(0, 1)\big) = \varphi(n, l_n) + (0, 3n - l_n).$$

It follows that $\varphi(n, l_n)$ belongs to the line L_{nj}. Furthermore, since it must be indecomposable, we have

$$\varphi(n, l_n) = (nj, l_{nj}). \tag{5}$$

For φ to be surjective on the set of indecomposables, it must be that $j = 1$. It follows that φ is the identity mapping, since $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ is generated by its indecomposable elements. Therefore $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N}) = S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$, and hence $\sigma = \tau$, proving the result. □

3.2. **Intersection with $\mathbb{N}_0 \times \mathbb{N}_0$.** We now consider the case where $U \times V = \mathbb{N}_0 \times \mathbb{N}_0$. The semigroup $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N}_0)$ is the union of lines $\{L_i : i \geq 0\}$ from S_σ. In fact, these semigroups are simply the semigroups $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$ with the identity element $(0, 0)$ adjoined. Therefore, as an immediate consequence of Proposition 3.3 we have

Proposition 3.4. $\mathbb{N}_0 \times \mathbb{N}_0$ *has continuum many subdirect products up to isomorphism.*

3.3. **Intersection with $\mathbb{Z} \times \mathbb{N}$.** Considering the case where $U \times V = \mathbb{Z} \times \mathbb{N}$, we have

$$S_\sigma \cap (\mathbb{Z} \times \mathbb{N}) = \{(i, j) \,:\, i \leq 0,\ j \geq 1\} \cup \bigcup_{i \geq 1} L_i.$$

We describe the indecomposable elements of $S_\sigma \cap (\mathbb{Z} \times \mathbb{N})$ in the following lemma, which is again used to rule out non-identity isomorphisms between these semigroups.

Lemma 3.5. *The set of indecomposable elements of* $S_\sigma \cap (\mathbb{Z} \times \mathbb{N})$ *is exactly the set*

$$\{(i, l_i) : i \geq 1\} \cup \{(i, 1) : i \leq 0\}.$$

Proof. Notice that

$$(i, j) = \begin{cases} (i, l_i) + (j - l_i)(0, 1) & \text{when } i \geq 1,\ j > l_i \\ (i, 1) + (j - 1)(0, 1) & \text{when } i \leq 0,\ j > 1. \end{cases}$$

Thus all of these elements are decomposable.

The elements $(i, 1)$ for $i \leq 0$ are indecomposable in $\mathbb{Z} \times \mathbb{N}$, as 1 is indecomposable in $\mathbb{N}$. It remains to consider (i, l_i) where $i \geq 1$. Suppose that (i, l_i) is decomposable, say $(i, l_i) = (a, x) + (b, y)$. We cannot have $a, b \geq 0$ by Lemma 3.2. Without loss of generality, suppose $a < 0$. Then $b = i - a > i$, and hence

$$y + x > y \geq l_b = b + c_b > i + c_i = l_i,$$

a contradiction. □

We can now move on to proving that there are continuum many subdirect products of $\mathbb{Z} \times \mathbb{N}$ up to isomorphism.

Proposition 3.6. *For any two sequences* σ *and* τ *satisfying the conditions of Construction 2.1, we have that*

$$\sigma \neq \tau \ \Rightarrow\ S_\sigma \cap (\mathbb{Z} \times \mathbb{N}) \not\cong S_\tau \cap (\mathbb{Z} \times \mathbb{N}).$$

Consequently, there are continuum many subdirect products of $\mathbb{Z} \times \mathbb{N}$ *up to isomorphism.*

Proof. Suppose that $\varphi : S_\sigma \cap (\mathbb{Z} \times \mathbb{N}) \to S_\tau \cap (\mathbb{Z} \times \mathbb{N})$ is an isomorphism. We proceed via a sequence of claims, aiming to show that $\varphi(S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})) = S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$ and then use Proposition 3.3 to obtain $\sigma = \tau$.

Claim 1. $\varphi(0, 1) = (0, 1)$.

Proof. We claim that $(0, 1)$ is the only indecomposable element (x, y) such that $(x, y) + (x, y)$ cannot be expressed as a sum of indecomposables in any other way, and the assertion then follows from this. That $(0, 1)$ has this property follows from Lemma 3.5. For any other indecomposable we have

$$(i, l_i) + (i, l_i) = (2i, l_{2i}) + (2l_i - l_{2i})(0, 1),$$

an alternative decomposition as a sum of indecomposables. □

Claim 2. $\varphi(i, l_i) \in L_p$ *for each* $i \geq 1$, *where* p *is* i *times the first coordinate of* $\varphi(1, 3)$.

Proof. As $\varphi(i, 3i) = \varphi((i, l_i) + (3i - l_i)(0, 1))$, then

$$i\varphi(1, 3) = \varphi(i, l_i) + (3i - l_i)(0, 1)$$

by Claim 1, and hence $\varphi(i, l_i)$ and $i\varphi(1, 3)$ must have the same first coordinate. □

Claim 3. $\varphi(-i, 1) \in L_q$ *for each* $i \geq 0$, *where* q *is* i *times the first coordinate of* $\varphi(-1, 1)$.

Proof. As $\varphi(-i,i) = \varphi((-i,1)+(i-1)(0,1))$, then

$$i\varphi(-1,1) = \varphi(-i,1)+(i-1)(0,1)$$

by Claim 1, and hence $\varphi(-i,1)$ and $i\varphi(-1,1)$ must have the same first coordinate. □

Claim 4. *$\varphi(1,3), \varphi(-1,1) \in L_{-1} \cup L_1$, and hence either*

$$\begin{aligned} \varphi(1,3) &= (1,3) && \text{and} && \varphi(-1,1) = (-1,1); \text{ or} \\ \varphi(1,3) &= (-1,1) && \text{and} && \varphi(-1,1) = (1,3). \end{aligned}$$

Proof. Suppose $\varphi(1,3) \in L_m$ and $\varphi(-1,1) \in L_n$ for some $m,n \in \mathbb{Z}$. Then by Claim 2 and Claim 3, it would follow that $\varphi(i,l_i) \in L_{mi}$ and $\varphi(-i,1) \in L_{in}$ for all $i \geq 0$.

As the set of indecomposables of $S_\sigma \cap (\mathbb{Z} \times \mathbb{N})$ must map bijectively to the set of indecomposables of $S_\tau \cap (\mathbb{Z} \times \mathbb{N})$, then by Lemma 3.5, it must be that $\{m,n\} = \{-1,1\}$ for φ to be surjective.

The last part of the claim follows as either $\varphi(1,3) \in L_1$, $\varphi(-1,1) \in L_{-1}$ or $\varphi(1,3) \in L_{-1}$, $\varphi(-1,1) \in L_1$, and noting that each of $(1,3)$, $(-1,1)$ must map to the unique indecomposable of the given line. □

Claim 5. *$\varphi(1,3) = (1,3)$, $\varphi(-1,1) = (-1,1)$.*

Proof. Suppose otherwise, which by Claim 4 would force $\varphi(1,3) = (-1,1)$, $\varphi(-1,1) = (1,3)$. On one hand, for $n \in \mathbb{N}$, we have

$$\varphi(-n,n) = \varphi(n(-1,1)) = (n,3n) = (n,l_n) + j(0,1) \tag{6}$$

for $j = 3n - l_n \in \mathbb{N}$. On the other hand,

$$\varphi(-n,n) = \varphi((-n,1)+(n-1)(0,1)) = \varphi(-n,1)+(n-1)(0,1).$$

Hence

$$(n,l_n) + j(0,1) = \varphi(-n,1) + (n-1)(0,1).$$

It must therefore be that $\varphi(-n,1)$ and (n,l_n) share the same first coordinate, and hence that $\varphi(-n,1) = (n,l_n)$ and $j = n-1$. Now as

$$3n = l_n + j$$

from considering second coordinates in (6), it follows that $l_n = 2n+1$ for any $n \in \mathbb{N}$. As $2n+1 = n+(n+1)$, it must follow that $k = n+1$ is the unique index such that $c_{k-1} < n \leq c_k$. In particular, $c_n < n$ for all $n \in \mathbb{N}$, which is a contradiction, as the sequence (c_k) grows at least exponentially by assumption (1) from Construction 2.1. □

Claim 6. *$\varphi((-\mathbb{N}) \times \mathbb{N}) = (-\mathbb{N}) \times \mathbb{N}$.*

Proof. Notice that every $(a,b) \in (-\mathbb{N}) \times \mathbb{N}$ can be decomposed as $(a,1)+(b-1)(0,1)$. Hence

$$\varphi(a,b) = \varphi(a,1) + (b-1)\varphi(0,1) = \varphi(a,1) + (b-1)(0,1) \tag{7}$$

by Claim 1. By Claims 3 and 5, it follows that $\varphi(a,1) = (a,1)$, and hence (7) is equal to (a,b). □

Returning to the main proof of the theorem, it must be that $\varphi(S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})) = S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$ by the last claim above. We now apply Proposition 3.3 and the result follows. $\square$

3.4. **Intersection with $\mathbb{Z} \times \mathbb{N}_0$.** We now consider the case $U \times V = \mathbb{Z} \times \mathbb{N}_0$. Each semigroup $S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0)$ is the union of lines $\{L_i : i \geq 1\} \cup \{(x, y) : x \leq 0, y \geq 0\}$.

In what follows we will repeatedly use the following observation, which follows immediately from Construction 2.1:

Lemma 3.7. *If $(a, x) \in S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0)$ and $x \in \{0, 1, 2\}$ then $a \leq 0$.* $\square$

We proceed to describe the indecomposable elements of $S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0)$ in order to prove that there are uncountably many subdirect products of $\mathbb{Z} \times \mathbb{N}_0$. Since $S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0)$ is a monoid, indecomposability will be understood to be in the monoid sense.

Lemma 3.8. *The set of indecomposable elements of $S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0)$ is exactly the set*

$$\{(i, l_i) : i \geq 1\} \cup \{(0, 0), (0, 1), (-1, 0)\}.$$

Proof. As $(i, j) = (i, l_i) + (j - l_i)(0, 1)$ when $i \geq 1$ and $(i, j) = (-i)(-1, 0) + j(0, 1)$ for $i \leq 0$, we only need to prove that the elements of the set $\{(i, l_i) : i \geq 1\} \cup \{(0, 0), (0, 1), (-1, 0)\}$ are all indecomposable. For $(0, 0)$, $(0, 1)$, $(-1, 0)$ this follows from Lemma 3.7 and indecomposability of 0 and 1 in $\mathbb{N}_0$.

Now, consider an element (i, l_i) for some $i > 0$ and suppose it has decomposition

$$(i, l_i) = (a, x) + (b, y).$$

By Lemma 3.2, we cannot have $a, b \geq 0$. Suppose without loss of generality that $a < 0$, thus $b > i$. Hence,

$$l_i = x + y \geq y \geq l_b > l_i,$$

a contradiction. Therefore the elements $\{(i, l_i) : i \geq 1\}$ are indecomposable. $\square$

Now we can prove that $\mathbb{Z} \times \mathbb{N}_0$ has continuum many subdirect products up to isomorphism.

Proposition 3.9. *For any two sequences σ and τ satisfying the conditions of Construction 2.1, we have that*

$$\sigma \neq \tau \Rightarrow S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0) \ncong S_\tau \cap (\mathbb{Z} \times \mathbb{N}_0).$$

Consequently, there are continuum many subdirect products of $\mathbb{Z} \times \mathbb{N}_0$ up to isomorphism.

Proof. Suppose that $\varphi : S_\sigma \cap (\mathbb{Z} \times \mathbb{N}_0) \to S_\tau \cap (\mathbb{Z} \times \mathbb{N}_0)$ is an isomorphism. We will proceed via a series of claims, aiming to show that $\varphi(S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})) = S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$, and then use Proposition 3.3.

Claim 1. $\varphi(0, 0) = (0, 0)$.

Proof. This follows from the fact that $(0, 0)$ is the unique identity element. $\square$

Claim 2. *$(0, 1) + (0, 1)$ is the unique decomposition of $(0, 2)$.*

Proof. If $(0,2) = (a,x) + (b,y)$ with $(a,x) + (b,y) \in S_\sigma \cap (\mathbb{Z} \cap \mathbb{N}_0)$ then $x, y \in \{0,1,2\}$, and hence $a = b = 0$ by Lemma 3.7, from which the claim follows readily. □

Claim 3. $(-1,0) + (-1,0)$ *is the unique decomposition of* $(-2,0)$.

Proof. Suppose that

$$(-2,0) = (a,x) + (b,y).$$

Since $x, y \in \mathbb{N}_0$, we must have $x = y = 0$.

By Lemma 3.7 we have $a, b \leq 0$, and the claim follows. □

Claim 4. $\varphi(0,1) = (0,1)$ *and* $\varphi(-1,0) = (-1,0)$.

Proof. If we consider (i, l_i) for some $i > 0$, then

$$(i, l_i) + (i, l_i) = (2i, 2l_i) = (2i, l_{2i}) + (2l_i - l_{2i})(0,1).$$

This tells us that $(0,1)$ and $(-1,0)$ are the unique indecomposables x with the property that $2x = x + x$ is the only decomposition into a sum of indecomposables, and hence

$$\varphi(\{(0,1), (-1,0)\}) = \{(0,1), (-1,0)\}.$$

In particular, we can also deduce that $\varphi(\{(0,3), (-3,0)\}) = \{(0,3), (-3,0)\}$. The element $(-3,0)$ has a unique decomposition into a sum of indecomposables, namely $(3,0) = 3(-1,0)$. By way of contrast, the element $(0,3)$ has more than one such decomposition, namely

$$(0,3) = 3(0,1) = (1,3) + (-1,0).$$

This is now sufficient to prove the claim, since it must therefore be that $\varphi(0,3) = (0,3)$ and $\varphi(-3,0) = (-3,0)$. □

Claim 5. $\varphi((-\mathbb{N}) \times \mathbb{N}) = (-\mathbb{N}) \times \mathbb{N}$.

Proof. Using Claim 4, for any $(i,j) \in (-\mathbb{N}) \times \mathbb{N}$ we have

$$\begin{aligned}\varphi(i,j) &= \varphi((-i)(-1,0) + j(0,1)) = (-i)\varphi(-1,0) + j\varphi(0,1) \\ &= (-i)(-1,0) + j(0,1) = (i,j),\end{aligned}$$

and the claim follows. □

Returning to the proof of the proposition, from Claim 5 it follows that $\varphi(S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})) = S_\tau \cap (\mathbb{N}_0 \times \mathbb{N})$, and the result follows by Proposition 3.3. □

3.5. **Intersection with $\mathbb{Z} \times \mathbb{Z}$.** We will now consider the final case where $U \times V = \mathbb{Z} \times \mathbb{Z}$. Notice that $S_\sigma \cap (\mathbb{Z} \times \mathbb{Z})$ is simply just S_σ from Construction 2.1.

We determine the indecomposable elements in S_σ and use this to describe the isomorphisms between these semigroups.

Lemma 3.10. *The set of indecomposable elements of* S_σ *is exactly the set*

$$\{(c_k, c_k + k) : k \geq 2\} \cup \{(0,0), (0,1), (-1,-1)\}.$$

Proof. Denote the above set by I. First we consider $(i, j) \in S_\sigma \setminus I$ and show that it is decomposable. Notice that

$$(i, j) = (-i)(-1, -1) + (j - i)(0, 1),$$

and this is a non-trivial decomposition of (i, j) in the following cases:

- $i < -1$, because $j \geq i$, so that $-i \geq 2$;
- $i = -1$, because $j \geq 0$, so that $-i = 1$ and $j - i > 0$;
- $i = 0$, because $-i = 0$ and $j \geq 2$.

Now suppose $i > 0$. Let $k \geq 2$ be the smallest index such that $i \leq c_k$. Define $a := c_k - i \geq 0$. Since $(i, j) \in S_\sigma$ we have $j \geq i + k$, and we let $b := j - (i + k) = j - (c_k - a + k) \geq 0$. Moreover, as $(i, j) \notin I$ by assumption, we cannot have both $a = 0$ and $b = 0$. Hence

$$(i, j) = (c_k - a, c_k - a + k + b) = (c_k, c_k + k) + a(-1, -1) + b(0, 1)$$

is a non-trivial decomposition of (i, j).

Now we show that each $(i, j) \in I$ is indecomposable. We consider separately the cases where $(i, j) \in \{(0, 0), (0, 1)\}$, $(i, j) = (-1, -1)$, and $(i, j) = (c_k, c_k + k)$. In each of these cases we assume that

$$(i, j) = (a, x) + (b, y) \quad \text{for some } (a, x), (b, y) \in S_\sigma \setminus \{(0, 0)\}$$

and proceed to derive a contradiction.

Case 1: $(i, j) \in \{(0, 0), (0, 1)\}$. We cannot have $a = b = 0$ because $0, 1$ are both indecomposable in $\mathbb{N}_0$, and this would imply that one of (a, x), (b, y) equals $(0, 0)$. Now, without loss of generality suppose that $a < 0$, so that $x \geq a$. Then $b > 0$, so that $y \geq b + 2$. Hence $1 \geq j = x + y \geq a + b + 2 = 2$, a contradiction.

Case 2: $(i, j) = (-1, -1)$. Suppose $a = -1$, $b = 0$, with $x \geq -1$, $y \geq 1$. Then $-1 = j = x + y \geq -1 + 1 > -1$. Next, without loss of generality suppose that $a < -1$, $b > 0$. Reasoning as in the previous case, $-1 = j = x + y \geq a + b + 2 = 1$, a contradiction.

Case 3: $(i, j) = (c_k, c_k + k)$. By Lemma 3.2, precisely one of a, b must be negative, as $(c_k, c_k + k)$ is indecomposable in $S_\sigma \cap (\mathbb{N}_0 \times \mathbb{N})$. Assume that $a < 0$ without loss. We have $b = c_k - a$ and $y = c_k + k - x$. As $x \geq a$, then $y \leq c_k + k - a$, i.e $y \leq b + k$. But as $b = c_k - a > c_k$, we know that $l_b \geq b + k + 1$. This gives a contradiction, as if $(b, y) \in S_\sigma$, then

$$b + k \geq y > l_b \geq b + k + 1. \qquad \square$$

Now we can prove that $\mathbb{Z} \times \mathbb{Z}$ has continuum many subdirect products up to isomorphism.

Proposition 3.11. *For any two sequences σ and τ satisfying the conditions of Construction 2.1, we have that*

$$\sigma \neq \tau \Rightarrow S_\sigma \ncong S_\tau.$$

Consequently, there are continuum many (semigroup) subdirect products of $\mathbb{Z} \times \mathbb{Z}$ up to isomorphism.

Proof. Suppose that $\varphi : S_\sigma \to S_\tau$ is an isomorphism. We will proceed via a series of claims, aiming to show that $\sigma = \tau$.

Claim 1. $(-2,-2)$ *has precisely one decomposition into a sum of non-zero indecomposables, namely* $(-2,-2) = (-1,-1) + (-1,-1)$.

Proof. Suppose that

$$(-2,-2) = \sum_{i=1}^{n} (a_i, x_i)$$

is a sum of $n \geq 2$ non-zero indecomposables. Consider the differences $d_i := x_i - a_i$. Noting that

$$0 = -2 - (-2) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} d_i,$$

and that

$$d_i = \begin{cases} 0 & \text{if } (a_i, x_i) = (-1,-1), \\ 1 & \text{if } (a_i, x_i) = (0,1), \\ k & \text{if } (a_i, x_i) = (c_k, c_k + k), \end{cases}$$

it follows that $(a_i, x_i) = (-1,-1)$ for all i, and the claim follows. □

Claim 2. $(0,2)$ *has precisely two decompositions into a sum of non-zero indecomposables, namely* $(0,1) + (0,1)$ *and* $(-1,-1) + (1,3)$.

Proof. Suppose that

$$(0,2) = \sum_{i=1}^{n} (a_i, x_i), \tag{8}$$

a sum of $n \geq 2$ non-zero indecomposables. Let $d_i := x_i - a_i$, as in the previous claim. This time, $\sum_{i=1}^{n} d_i = 2$. Hence the only choices for (a_i, x_i) appearing in (8) are $(-1,-1), (0,1)$ and $(c_2, c_2 + 2) = (1,3)$. Furthermore, there are either precisely two occurrences of $(0,1)$, or precisely one occurrence of $(1,3)$. In the former case we obtain the decomposition $(0,2) = (0,1) + (0,1)$, and in the latter $(0,2) = (1,2) + (-1,-1)$ as the only options. □

Claim 3. *For* $k \geq 2$*, the element* $(2c_k, 2(c_k + k))$ *has a decomposition into a sum of three or more non-zero indecomposables.*

Proof. Let $d := c_{k+2} - 2c_k$. Recalling (1) from Construction 2.1 we have

$$d \geq 2c_{k+1} - 2c_k \geq 4c_k - 2c_k = 2c_k \geq 2.$$

Also let $n := k - 2 \geq 0$. Then

$$\begin{aligned} &(c_{k+2}, c_{k+2} + k + 2) + d(-1,-1) + n(0,1) \\ =& (c_{k+2} - d, c_{k+2} + k + 2 - d + n) \\ =& \big(c_{k+2} - (c_{k+2} - 2c_k), c_{k+2} + k + 2 - (c_{k+2} - 2c_k) + k - 2\big) \\ =& (2c_k, 2c_k + 2k), \end{aligned}$$

a decomposition of $(2c_k, 2c_k + 2k)$ into $1 + d + n \geq 3$ non-zero indecomposables, as required. □

Having finished the series of claims, we can now proceed to directly prove that $\sigma = \tau$. As every element of S_σ is a sum of the indecomposables described in Lemma 3.10, we consider the images of these, each of which will be indecomposable in S_τ. To distinguish between the sequences σ and τ, we will let $\sigma = (c_k)_{k\geq 2}$, and $\tau = (C_k)_{k\geq 2}$.

Clearly $\varphi(0,0) = (0,0)$, being the identity of both monoids. By considering possible decompositions of $\varphi(2(-1,-1))$, $\varphi(2(0,1))$ and $\varphi(2(c_k, c_k+k))$, Claims 1 to 3 assert that we must have $\varphi(-1,-1) = (-1,-1)$, $\varphi(0,1) = (0,1)$ and for any $k \geq 2$, $\varphi(c_k, c_k + k) = (C_j, C_j + j)$ for some $j \geq 2$.

Noting that

$$(c_k, c_k + k) + c_k(-1,-1) = k(0,1),$$

then applying φ to the above shows that

$$(C_j, C_j + j) + c_k(-1,-1) = k(0,1),$$

and hence that $c_k = C_j$ and $k = j$. Therefore $c_k = C_k$ for all $k \geq 2$, and we conclude $\sigma = \tau$ as required. □

4. Concluding remarks

As stated, our Main Theorem subsumes Theorem A and Theorem C for $k = 2$ from [5] dealing with $\mathbb{N} \times \mathbb{N}$. However, within our proof we appeal to these results, rather than reprove them. In fact it is unclear whether our present methods could be modified to cover the $\mathbb{N} \times \mathbb{N}$ case. For starters, the intersection $S_\sigma \cap (\mathbb{N} \times \mathbb{N})$ is not subdirect: the elements 1 and 2 are missing from the first projection. And secondly, our way of proving non-isomorphisms for different σ via analysis of indecomposable elements would not work, as $S_\sigma \cap (\mathbb{N} \times \mathbb{N})$ has a lot of indecomposables, which moreover depend on σ.

It is known that in groups, and more generally congruence permutable varieties, subdirect products of two factors coincide with the so called fiber products; this is known as Goursat's Lemma for groups (see [1, Theorem 4]) and Fleischer's Lemma in general (see [7, Theorem 4.74]). For two algebraic structures A_1, A_2 of the same type, a fiber product of A_1 and A_2 is a substructure of their direct product $A_1 \times A_2$ of the form $\{(a_1, a_2) \in A_1 \times A_2 \,:\, \varphi(a_1) = \varphi(a_2)\}$, where $\varphi_i : A_i \to Q$ $(i = 1, 2)$ are *onto* homomorphisms to a common quotient Q. It is well-known that Goursat's/Fleischer's Lemma does not extend to semigroups. The Main Theorem offers a glimpse of just how badly it fails. It is easy to see that each of $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ has only countably many quotients (which are, respectively, all monogenic semigroups, all monogenic monoids, and all cyclic groups). It therefore follows that for any $U, V \in \{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}\}$ there are only countably many fiber products of U and V. Combining with the Main Theorem we conclude that uncountably many subdirect products of U and V are not fiber products.

The Main Theorem seems to suggest that it is rather hard for the direct product $U \times V$ of two infinite semigroups to contain only countably many subdirect products up to isomorphism. But it is not impossible. One trivial example can be obtained by taking U and V to be two copies of an infinite zero semigroup Z ($zu = 0$ for all $z, u \in Z$). Then $Z \times Z$ is again a countable zero semigroup, i.e. $Z \times Z \cong Z$, as is every infinite subsemigroup of $Z \times Z$. Thus Z is the only subdirect product of $Z \times Z$ up to isomorphism. Another, less trivial, example is obtained by taking U and V to be two copies of a

Tarski Monster M – an infinite simple group in which every proper subgroup has order p, where p is a fixed prime [8]. Since $M \times M$ is periodic, its subsemigroups are in fact subgroups. Therefore semigroup subdirect products in $M \times M$ coincide with group subdirect products. And then it follows from simplicity and Goursat's Lemma that there are only two such subdirect products up to isomorphisms, namely M and $M \times M$.

Motivated by the above discussion, we ask:

Question 4.1. Do there exist infinite non-periodic semigroups U and V such that $U \times V$ contains only countably many pairwise non-isomorphic subdirect products? Do there exist such U and V which are commutative?

We conjecture that the answer to the first question is affirmative and negative for the second.

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