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Hamiltonian formulation for interfacial periodic waves propagating under an elastic sheet above stratified piecewise constant rotational flow

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A B S T R A C T

We present a Hamiltonian formulation of two-dimensional hydroelastic waves propagating at the free surface of a stratified rotational ideal fluid of finite depth, covered by a thin ice sheet. The flows considered exhibit a discontinuous stratification and piecewise constant vorticity, accommodating the presence of interfaces and of linearly sheared currents.

1. Introduction

The recent years have seen a tremendous development in the field of hydroelasticity, that is, the study of the deformations of elastic bodies responding to hydrodynamic excitations and simultaneously the modification of these excitations owing to the body deformation (see [1] for a review). Within this huge field, most mathematically analyzed seem to be the hydroelastic waves in the presence of an ice cover. Hydroelastic waves are of utmost importance in those cold regions where water is frozen in winter and where deep ice cover can be transformed into roads or aircraft runways, and where air-cushioned vehicles are used to break the ice, cf. [2].

In the polar regions the water under the floating ice plates is often stratified, due to difference in salinity and temperature [3,4]. Internal waves can be generated by tides in parts of the Arctic Ocean, and they are an important factor in the upper-layer mixing and transfer of nutrients in the surface layer [5]. During the summer months, when there is no ice cover, SAR observations have detected internal waves in various parts of the Arctic Ocean. A hotspot of internal waves is on the Kara Strait which connects Kara Sea and the Barents Sea in northern Russia, where the currents are also very strong [4]. The depth around the strait varies between 50 m and 200 m, the average current speed is between 6 cm/s and 26 cm/s and a back-flow of water near the bottom was also detected. The wavelenghts of the observed internal waves varies between 400 – 800 m for the short-internal waves to few kilometers [4]. Another place where a large number of internal solitary waves have been observed using SAR observations is the Laptev Sea [6], in the eastern part of the Arctic Ocean. The depth of Laptev Sea where these waves have been observed is between 50 m and 300 m

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and the typical wavelength of the solitary waves is between 1 km and 2 km, being observed to travel for hundreds of kilometers. In these regions the tidal current speed can be as high as 50cm/s and a vertical velocity shear was observed.

In winter the ice cover will suppress small-scale surface wave noise such as wave turbulence, and the signature of internal waves can be detected easier. There are many observation in the Arctic of internal waves interacting with the hydroelastic waves propagating along the floating ice plates [7,8]. Furthermore, recent experimental results [9] suggest that interactions between internal waves and sea ice may be an important mechanism for dissipation of internal wave energy in the Arctic Ocean.

The previously mentioned examples suggest the existence of a wide range of physically relevant regimes. For the rigorous and thorough treatment of these regimes from the perspective of analytical investigations and numerical simulations it is imperative to surpass the limitations inherent to a case-by-case analysis. The alternative that we offer here is the Hamiltonian formulation of the nonlinear governing equations (and of their boundary conditions). The Hamiltonian approach offers multiple benefits. Firstly, it elegantly reduces the number of variables, leading to significant simplifications. Secondly, identifying a system as Hamiltonian immediately grants access to conservation laws linked to the system's symmetry groups, such as changes to moving coordinate frames, spatial rotations, time translations, and scale changes, offering a deeper understanding of its dynamics. Lastly, the Hamiltonian viewpoint is particularly advantageous in perturbation theory, ensuring that, as long as the symmetries are retained during approximation, the conservation properties of the exact system are also preserved in the approximations. The latter aspect is illustrated by the recent work by Guyenne, Kairzhan & Sulem [10] where a Hamiltonian Dysthe equation is derived for nonlinear surface waves in the presence of constant vorticity (an aspect that is relevant to our scenario). In addition to preserving energy, the Hamiltonian model in [10] was found to be simpler and more accurate than the classical non-Hamiltonian version obtained by using the method of multiple scales.

From a historical perspective, the Hamiltonian formulation of the governing equations for two dimensional gravity water flows was pioneered in [11] for irrotational flows, and was extended to rotational flows with constant vorticity in [12–14]. The case with variable bottom was treated in [15–17]. The Hamiltonian formulation for two-dimensional irrotational two-layer gravity water flows with a free surface was developed in [18], and the rotational counterpart with constant vorticity and constant density in each layer was obtained in [19] for the case of periodic flows. Further studies on Hamiltonian methods in two- and multi-layer flows with piecewise constant vorticity were carried out in [20–22]. For flows that represent localized perturbations of an underlying pure current we refer the reader to [23] where also Coriolis effects in the equatorial f-plane approximation were included in the analysis. In line with the previous setting of geophysical effects, we would like to mention that, recently, a hamiltonian formalism for inertial waves in rotating fluid was derived in [24]. A multi-layer model based on the Green–Naghdi approximation has been proposed in [25].

While until recently most theoretical works in this subject used linear models, describing waves of small amplitude [26,27], recent nonlinear studies [28], although treating issues like stratification, did not allow for the presence of vorticity. Towards remedying this inconvenience we propose, as method of work a variational formulation of Hamiltonian type. More specifically, we derive the Hamiltonian formulation of the nonlinear governing equations modeling the propagation of hydroelastic waves admitting linearly sheared currents and stratification of discontinuous type. The derivation of approximate weakly-nonlinear models in a wide range of physical regimes is postponed to a future investigation since this endeavour requires extensive geophysical considerations. After presenting the governing equations in Section 2 we prove in Section 3 that they can be (equivalently) reformulated as an infinite-dimensional Hamiltonian system. The Hamiltonian formulation (in conjunction with the Dirichlet–Neumann operators) is then used to obtain the (linearized) dispersion relation which generalizes the dispersion relation for interfacial solitary waves propagating under an elastic sheet derived in the irrotational setting by Wang et al. [28]. We conclude in Section 4 by hinting to potential further perspectives on the subject.

2. The presentation of the problem

We examine here a two-dimensional periodic rotational stratified water flow, acted upon by gravity (in the bulk of the fluid) and by the flexural elasticity of a continuous (deformable) ice sheet that represents the free surface of the water flow whose vertical deformation is denoted by $y = h_1 + \eta_1(x,t)$. Here, $h_1 > 0$ is a constant, t stands for time and $x \to \eta_1(x,t)$ is a periodic function in the variable x, of principal period L, with mean zero, that is, $\int_0^L \eta_1(x,t)\,dx = 0$ for all $t \geq 0$. The free surface wave propagates in the positive x-direction, while the y axis points vertically upwards. Moreover, the bottom boundary of the fluid domain is written as $y = -h$, with $h > 0$ being a constant.

The stratification of the fluid is as follows: we assume that, neighboring the flat bed $y = -h$, the water domain consists of a layer

$$\Omega^* := \{(x,y,t) : x \in \mathbb{R}, t \geq 0, -h < y < \eta(x,t)\},$$

of constant density ρ, separated by the interface $y = \eta(x,t)$ from the free-surface adjacent layer

$$\Omega_1^* := \{(x,y,t) : x \in \mathbb{R}, t \geq 0, \eta(x,t) < y < h_1 + \eta_1(x,t)\},$$

of constant density $\rho_1 < \rho$. The interface $x \to \eta(x,t)$ is, at any fixed time t, an L periodic function with zero mean.

We assume the water flow to be incompressible and inviscid. The latter assumption is justified, since, as emphasized by daSilva and Peregrine [29], inviscid theory is suitable for the study of water waves that are not near breaking. Indeed, according to [29] the most appreciable effects of viscosity in the open sea are to produce wave-amplitude reduction, and diffusion of the deeper motions, over time scales and length scales that are far larger than those of the dynamical surface-processes. For results concerning wave-attenuation due to viscosity effects we refer the reader to [30–34].

Remark 2.1. A consequence of the presence of the interface $z = \eta(x,t)$ is that some physical variables may present discontinuities along it. To draw attention to this aspect, we use the subscript 1 for quantities pertaining to the upper layer. Whenever we refer to the overall physical variable without specification of the layer, we shall use the boldface writing style (as an example the density function $\boldsymbol{\rho}$ takes the value ρ in the lower layer and the value ρ_1 in the upper layer).

In line with the previous remark we denote with $(\boldsymbol{u}(x,y,t), \boldsymbol{v}(x,y,t))$ the velocity field and with $\boldsymbol{P} = \boldsymbol{P}(x,y,t)$ the pressure. Then, the equations governing the fluid motion, are Euler's equations

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u}\boldsymbol{u}_x + \boldsymbol{v}\boldsymbol{u}_y &= -\frac{1}{\boldsymbol{\rho}}\boldsymbol{P}_x, \\ \boldsymbol{v}_t + \boldsymbol{u}\boldsymbol{v}_x + \boldsymbol{v}\boldsymbol{v}_y &= -\frac{1}{\boldsymbol{\rho}}\boldsymbol{P}_y - g, \end{cases} \quad \text{in} \quad \Omega^* \cup \Omega_1^* \tag{2.1}$$

together with the equation of mass conservation

$$\boldsymbol{u}_x + \boldsymbol{v}_y = 0 \text{ in } \Omega^* \cup \Omega_1^*. \tag{2.2}$$

To select the water wave problem from a multitude of hydrodynamical ones we impose appropriate boundary and interface conditions. As such, at the free surface we have cf. [35,36] the elasticity condition

$$P(x, h_1 + \eta_1(x,t)) = \mathcal{D}\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right), \tag{2.3}$$

where $\mathcal{D} = \frac{Ed^3}{12(1-\nu^2)}$ is a constant representing the coefficient of the flexural rigidity of the ice sheet, d is the ice thickness, E denotes Young's modulus, ν is the Poisson's ratio for ice. κ is the curvature of the fluid-ice interface, and s is the arclength along this interface. More precisely, the right-hand side of Eq. (2.3) is the force arising under a deformation of a shell which has bending rigidity, see e.g. [37]. As in previous works we will neglect the stretching of the elastic shell and consider here only the effect of bending.

Remark. It is worth noting that if the effect of the tension force created by the stretching of the plate due to bending is not neglected, it will introduce a term of the form

$$-N\left\{\int_0^L \left(\sqrt{1+\eta_{1_x}^2} - 1\right)dx\right\}\kappa$$

on the right-hand side of Eq. (2.3), where $N = Ed/L$ (see the second term of equation (2.1) in [38] for a linearized form of this term, or the third term of equation (1.1g'') in [39] when choosing their parameter $\beta = 2$).

Another category of conditions ensure that a particle, once on one of the two boundaries, or on the interface, will remain there. They are called the kinematic boundary conditions and read as

$$v_1 = \eta_{1,t} + u_1\eta_{1,x} \text{ on } y = \eta_1(x,t) + h_1, \tag{2.4}$$

$$v_1 = \eta_t + u_1\eta_x \quad \text{and} \quad v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(x,t), \tag{2.5}$$

$$v = 0 \quad \text{on} \quad y = -h. \tag{2.6}$$

Moreover, the balance of forces at the interface $y = \eta(x,t)$ is expressed by the continuity of the pressure along this internal boundary, that is,

$$P = P_1 \quad \text{on} \quad y = \eta(x,t). \tag{2.7}$$

To capture the underlying currents and to keep track of the wave–current interactions the inclusion of the vorticity in the flow is needed. The vorticity is defined through

$$\boldsymbol{\gamma} := \boldsymbol{u}_y - \boldsymbol{v}_x, \tag{2.8}$$

and measures local rotation. In our setting the vorticity is constant throughout each layer, but discontinuous across the interface. Therefore, it takes the form

$$\boldsymbol{\gamma} = \begin{cases} \gamma & \text{in} \quad \Omega^*, \\ \gamma_1 & \text{in} \quad \Omega_1^*, \end{cases}$$

where $\gamma, \gamma_1 \in \mathbb{R}$ are constants with $\gamma \neq \gamma_1$.

2.1. The mathematical reformulation

The two-layer fluid occupies at a fixed time t the domain consisting of

$$\Omega = \Omega(t) := \{(x,y) : x \in (0,L), -h < y < \eta(x,t)\},$$

and

$$\Omega_1 = \Omega_1(t) := \{(x,y) : x \in (0,L), \eta(x,t) < y < h_1 + \eta_1(x,t)\},$$

respectively.

Due to (2.8) we are able to introduce (up to functions that depend only on time) in each layer a (generalized) velocity potential, denoted φ in Ω and φ_1 in Ω_1, respectively, through the formulas

$$\varphi(x,y,t) = \int_0^x (u(l,-h,t)+\gamma h)\,dl + \int_{-h}^y v(x,l,t)\,dl, \quad \text{for} \quad (x,y)\in\Omega, \tag{2.9}$$

and

$$\begin{aligned}\varphi_1(x,y,t) &= \int_0^x \left[u_1(l,\eta(l,t),t) - \gamma_1\eta(l,t) + v_1(l,\eta(l,t))\eta_x(l,t)\right] dl \\ &+ \int_{\eta(x,t)}^y v_1(x,l,t)\,dl, \quad \text{for} \quad (x,y)\in\Omega_1.\end{aligned} \tag{2.10}$$

The generalized velocity potentials satisfy

$$\begin{cases} u = \varphi_x + \gamma y \\ v = \varphi_y, \end{cases} \quad \text{in} \quad \Omega, \tag{2.11}$$

and

$$\begin{cases} u_1 = \varphi_{1,x} + \gamma_1 y \\ v_1 = \varphi_{1,y} \end{cases} \quad \text{in} \quad \Omega_1. \tag{2.12}$$

Moreover,

$$\varphi(x+L,y,t) - \varphi(x,y,t) = \kappa L, \tag{2.13}$$

$$\varphi_1(x+L,y,t) - \varphi_1(x,y,t) = \kappa_1 L, \tag{2.14}$$

where

$$\kappa := \frac{1}{L}\int_0^L [u(x,-h,t)+\gamma h]\,dx,$$

and

$$\kappa_1 := \frac{1}{L}\int_0^L [u_1(x,\eta(x,t),t) + v_1(x,\eta(x,t),t)\eta_x(x,t)]\,dx. \tag{2.15}$$

are averaged currents on the bed and on the interface $y=\eta(x,t)$, respectively, that are time-independent, cf. [19].

Another consequence of the previous formulas is that the functions

$$(x,y) \to \tilde{\varphi}(x,y) := \varphi(x,y) - \kappa x$$

and

$$(x,y) \to \tilde{\varphi}_1(x,y) := \varphi_1(x,y) - \kappa_1 x,$$

are periodic in the x-variable, of period L. The latter calls up for a splitting of the velocity into an underlying steady current component and a periodic harmonic wave velocity field as

$$\begin{cases} u = \tilde{\varphi}_x + \gamma y + \kappa \\ v = \tilde{\varphi}_y, \end{cases} \quad \text{in} \quad \Omega \tag{2.16}$$

and

$$\begin{cases} u_1 = \tilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1 \\ v_1 = \tilde{\varphi}_{1,y} \end{cases} \quad \text{in} \quad \Omega_1. \tag{2.17}$$

Also the kinematic boundary conditions (2.4) and (2.5) can be written in terms of the traces of the velocity potentials at the interface and the free surface, as

$$\eta_{1,t} = (\tilde{\varphi}_{1,y})_{s_1} - \eta_{1,x}[(\tilde{\varphi}_{1,x})_{s_1} + \gamma_1(h_1+\eta_1) + \kappa_1] \tag{2.18}$$

and

$$\eta_t = (\tilde{\varphi}_{1,y})_s - \eta_x[(\tilde{\varphi}_{1,x})_s + \gamma_1\eta + \kappa_1] = (\tilde{\varphi}_y)_s - \eta_x[(\tilde{\varphi}_x)_s + \gamma\eta + \kappa], \tag{2.19}$$

respectively; the subscripts s and s_1 stand for traces on the interface $y=\eta(x,t)$ and on the free surface $y=h_1+\eta_1(x,t)$, respectively.

The equation of mass conservation (2.2) entails the existence of two stream functions denoted ψ in Ω, and ψ_1 in Ω_1, satisfying

$$\begin{cases} u = \psi_y \\ v = -\psi_x \end{cases} \quad \text{in} \quad \Omega, \tag{2.20}$$

and

$$\begin{cases} u_1 = \psi_{1,y} \\ v_1 = -\psi_{1,x} \end{cases} \quad \text{in} \quad \Omega_1. \tag{2.21}$$

Arguing as in [19] is it possible to prove the existence of $\boldsymbol{\psi} \in C(\overline{\Omega \cup \Omega_1})$ with

$$\boldsymbol{\psi} = \begin{cases} \psi & \text{in} \quad \overline{\Omega}, \\ \psi_1 & \text{in} \quad \overline{\Omega}_1, \end{cases}$$

which implies the righteousness of the notation

$$\chi(x,t) := \psi(x,\eta(x,t),t) - \psi(0,\eta(0,t),t) = \psi_1(x,\eta(x,t),t) - \psi_1(0,\eta(0,t),t). \tag{2.22}$$

The kinematic boundary conditions (2.4) and (2.5) can be written also with the help of the stream functions as

$$\begin{aligned} \chi(x,t) &= -\textstyle\int_0^x \eta_t(x',t)dx', \\ \chi_1(x,t) &= -\textstyle\int_0^x \eta_{1,t}(x',t)dx', \end{aligned} \tag{2.23}$$

where

$$\chi_1(x,t) := \psi_1(x,h_1+\eta_1(x,t),t) - \psi_1(0,h_1+\eta_1(0,t),t). \tag{2.24}$$

The previous discussion about the (generalized) velocity potentials and the stream functions enable us to rewrite the Euler's equations as

$$\nabla\left[\tilde{\boldsymbol{\varphi}}_t + \frac{1}{2}|\nabla\boldsymbol{\psi}|^2 + \frac{\boldsymbol{P}}{\rho} - \boldsymbol{\gamma}\boldsymbol{\psi} + gy\right] = 0 \quad \text{in} \quad \Omega\cup\Omega_1, \tag{2.25}$$

and therefore

$$\tilde{\varphi}_t + \frac{1}{2}|\nabla\psi|^2 - \gamma\psi + \frac{P}{\rho} + gy = f(t) \quad \text{in} \quad \Omega, \tag{2.26}$$

and

$$\tilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - \gamma_1\psi_1 + \frac{P_1}{\rho_1} + gy = f_1(t) \quad \text{in} \quad \Omega_1, \tag{2.27}$$

for some arbitrary time-dependent functions f and f_1. Choosing

$$f_1(t) = -\gamma_1\psi_1(0,h_1+\eta_1(0,t),t), \tag{2.28}$$

and using the boundary condition (2.3) we obtain from (2.27) that the equation

$$\tilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - \gamma_1\psi_1 + \frac{\mathcal{D}}{\rho_1}\left(\Bbbk_{ss} + \frac{1}{2}\Bbbk^3\right) + g(h_1+\eta_1) = -\gamma_1\psi_1(0,h_1+\eta_1(0,t),t), \tag{2.29}$$

holds on $y = h_1 + \eta_1(x,t)$. The useful form of (2.29) that we shall need further is

$$\tilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - \gamma_1\chi_1 + \frac{\mathcal{D}}{\rho_1}\left(\Bbbk_{ss} + \frac{1}{2}\Bbbk^3\right) + g(h_1+\eta_1) = 0 \quad \text{on} \quad y = h_1 + \eta_1(x,t). \tag{2.30}$$

Moreover, the continuity of the pressure along the interface $y = \eta(x,t)$ together with (2.26)–(2.27) and the choice

$$f(t) = \frac{\rho_1}{\rho}(f_1(t) + \gamma_1\psi_1(0,\eta(0,t),t)) - \gamma\psi(0,\eta(0,t),t), \tag{2.31}$$

yield

$$\rho\left(\tilde{\varphi}_t + \frac{1}{2}|\nabla\psi|_s^2 - \gamma\chi + g\eta\right) = \rho_1\left(\tilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|_s^2 - \gamma_1\chi + g\eta\right). \tag{2.32}$$

3. The reformulation of the water wave problem

We provide here a choice of new dependent and independent variables that permits an equivalent reformulation of the water wave problem (2.1)–(2.7). In this new reformulation the equations assume a simpler form that mitigates the difficulties brought about by the rich structure of the nonlinear water wave problem. We start by indicating the choice for the Hamiltonian functional which consists of the kinetic and potential energy, respectively, and a suitable term that accounts for the hydrolelastic effects introduced by the condition (2.3). Another subtle aspect is the choice of suitable dynamical variables. We show here that the choice [40] (see (3.4) below) of dynamical variables for the Hamiltonian formulation of interfacial irrotational waves with a rigid lid is also relevant for our scenario which allows for piecewise constant vorticity, stratification, a free surface, and an interface.

3.1. The nearly Hamiltonian formulation

The candidate for the Hamiltonian functional is, see e.g. [11,36,41], the total energy of the flow

$$H = \int\int_{\Omega\cup\Omega_1} \rho\left\{\frac{\boldsymbol{u}^2 + \boldsymbol{v}^2}{2} + gy\right\} dydx + \frac{\mathcal{D}}{2}\int_0^L \frac{\eta_{1,xx}^2}{(1+\eta_{1,x}^2)^{\frac{5}{2}}}\,dx, \tag{3.1}$$

which, after taking into account the stratification of the fluid and the decomposition of the velocity field (2.16), (2.17), can be recast as

$$\begin{aligned} H =& \frac{\rho}{2}\int_0^L\int_{-h}^{\eta}|\nabla\tilde{\varphi}|^2\,dydx + \rho\gamma\int_0^L\int_{-h}^{\eta} y\tilde{\varphi}_x\,dydx + \rho\frac{\gamma^2}{6}\int_0^L(\eta^3+h^3)\,dx \\ &+ \rho\kappa\int_0^L\int_{-h}^{\eta}\tilde{\varphi}_x\,dydx + \frac{\rho\gamma\kappa}{2}\int_0^L(\eta^2-h^2)\,dx + \frac{\rho k^2}{2}\int_0^L(\eta+h)\,dx \\ &+ \frac{\rho_1}{2}\int_0^L\int_{\eta}^{h_1+\eta_1}|\nabla\tilde{\varphi}_1|^2\,dydx + \rho_1\gamma_1\int_0^L\int_{\eta}^{h_1+\eta_1} y\tilde{\varphi}_{1,x}\,dydx \\ &+ \rho_1\frac{\gamma_1^2}{2}\int_0^L\frac{(h_1+\eta_1)^3-\eta^3}{3}\,dx \\ &+ \rho_1\kappa_1\int_0^L\int_{\eta}^{h_1+\eta_1}\tilde{\varphi}_{1,x}\,dydx + \frac{\rho_1\gamma_1\kappa_1}{2}\int_0^L((h_1+\eta_1)^2-\eta^2)\,dx \\ &+ \frac{\rho_1 k_1^2}{2}\int_0^L(h_1+\eta_1-\eta)\,dx \\ &+ \frac{\rho g}{2}\int_0^L(\eta^2-h^2)\,dx + \frac{\rho_1 g}{2}\int_0^L\left((h_1+\eta_1)^2-\eta^2\right)\,dx \\ &+ \frac{\mathcal{D}}{2}\int_0^L\frac{\eta_{1,xx}^2}{(1+\eta_{1,x}^2)^{\frac{5}{2}}}\,dx. \end{aligned} \tag{3.2}$$

Remark. While, as mentioned before, we consider in this paper only the effect of bending and we neglect the stretching of the elastic plate, it is worth noting that the energy due to bending-induced stretching of the plate can be considered in future works by including a term of the form

$$+\frac{N}{2}\left(\int_0^L(\sqrt{1+\eta_{1_x}^2}-1)dx\right)^2,$$

in Eq. (3.1) as in Burton & Toland [39], equation (1.2c).

In the sequel we will compute the variations of H with respect η, η_1 and the new dynamical variables proposed by Benjamin & Bridges [40,42] (utilized also by Craig et al. [18])

$$\xi := \rho\Phi - \rho_1\Phi_1, \quad \xi_1 := \rho_1\Phi_2, \tag{3.3}$$

where

$$\begin{cases} \Phi(x,t) := \tilde{\varphi}(x,\eta(x,t),t), \\ \Phi_1(x,t) := \tilde{\varphi}_1(x,\eta(x,t),t), \\ \Phi_2(x,t) := \tilde{\varphi}_1(x,h_1+\eta_1(x,t),t). \end{cases} \tag{3.4}$$

The Hamiltonian formulation hinges upon the possibility to write the Hamiltonian functional (3.2) in the form

$$H = \int_0^L \mathcal{H}\,dx, \tag{3.5}$$

where $\mathcal{H}$ is a Hamiltonian density function which depends only on η, η_1, ξ, ξ_1 and their spatial derivatives. To establish the validity of formula (3.5) we will appeal to the Dirichlet–Neumann operators which we define now.

Definition 3.1. For smooth, L-periodic, real functions Φ, η with $\eta(x) > -h$ for all $x \in [0, L]$, we denote with $\tilde{\phi}$ the unique L-periodic solution of the boundary value problem

$$\begin{cases} \Delta\tilde{\varphi} = 0 \text{ in } \Omega^*(\eta), \\ \tilde{\varphi} = \Phi \quad \text{on } y = \eta(x), \\ \tilde{\varphi}_y = 0 \quad \text{on } y = -h. \end{cases} \tag{3.6}$$

Then, the Dirichlet–Neumann operator $G = G(\eta)$ associated to the lower layer $\Omega^*(\eta)$ of the fluid domain is defined by setting

$$G\Phi := \sqrt{1+\eta_x^2}\frac{\partial\tilde{\varphi}}{\partial\mathbf{n}}\Big|_{y=\eta(x)}, \tag{3.7}$$

where $\mathbf{n}$ denotes the outward pointing unit normal vector along $y = \eta(x)$, cf. Fig. 1. Similarly, for the upper layer, given smooth, L-periodic functions $\eta, \eta_1, \Phi, \Phi_1$, satisfying $\eta(x) < h_1 + \eta_1(x)$ for all $x \in [0, L]$, we denote with $\tilde{\varphi}_1$ the unique solution of the Dirichlet boundary value problem

$$\begin{cases} \Delta\tilde{\varphi}_1 = 0 \text{ in } \Omega_1^*(\eta,\eta_1), \\ \tilde{\varphi}_1 = \Phi_1 \quad \text{on } y = \eta(x), \\ \tilde{\varphi}_1 = \Phi_2 \quad \text{on } y = h_1+\eta_1(x). \end{cases} \tag{3.8}$$

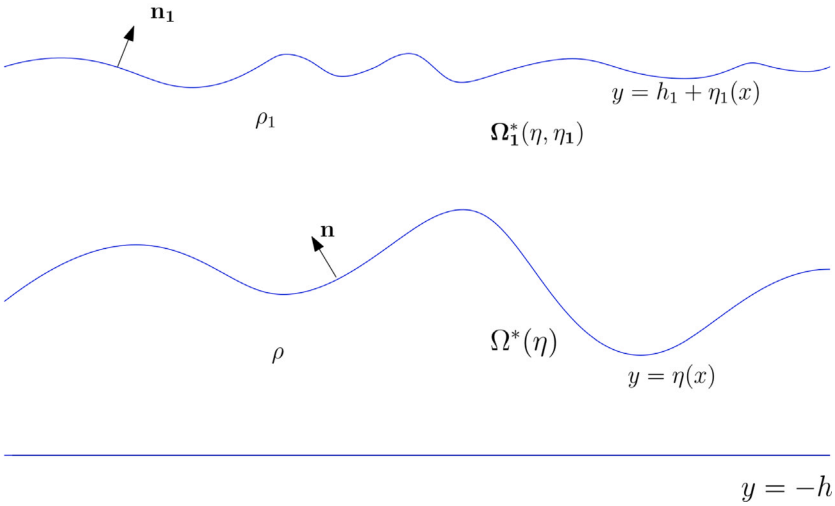

Fig. 1. Depiction of a periodicity cell connected to the definition of the Dirichlet–Neumann operators corresponding to the layer adjacent to the bottom, and to the layer adjacent to the surface, respectively.

The Dirichlet–Neumann operator $G_1 = G_1(\eta, \eta_1)$ associated to the upper layer $\Omega_1^*(\eta, \eta_1)$ of the fluid domain is defined to be the matrix operator

$$G_1(\Phi_1, \Phi_2) := \begin{pmatrix} -\sqrt{1+\eta_x^2}\,\frac{\partial\tilde{\varphi}_1}{\partial \mathbf{n}}\Big|_{y=\eta(x)} \\ \sqrt{1+\eta_{1,x}^2}\,\frac{\partial\tilde{\varphi}_1}{\partial \mathbf{n}_1}\Big|_{y=h_1+\eta_1(x)} \end{pmatrix}, \tag{3.9}$$

where $\mathbf{n}_1$ is the outward pointing unit normal vector along the upper boundary $y = h_1 + \eta_1(x)$; see Fig. 1.

Let us denote with $G_{ij}, i, j = 1, 2$ the entries of the matrix operator $G_1(\eta, \eta_1)$, that is

$$G_1 = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \tag{3.10}$$

More details on the operators G and G_1 emerging from Definition 3.1 are listed in the following Remark.

Remark 3.2. Detailing in the definition of G from (3.7), we find (using also (2.19)) that

$$G\Phi = \eta_t + (\gamma\eta + \kappa)\eta_x. \tag{3.11}$$

It also turns out from the definition of G_1 that

$$\begin{aligned} G_{11}\Phi_1 + G_{12}\Phi_2 &= -(\eta_t + \gamma_1\eta\eta_x) - \kappa_1\eta_x, \\ G_{21}\Phi_1 + G_{22}\Phi_2 &= \eta_{1,t} + [\gamma_1(h_1 + \eta_1) + \kappa_1]\eta_{1,x}. \end{aligned} \tag{3.12}$$

Toward proving the claim (3.5) we set for the kinetic part of the Hamiltonian expansion (3.2) the notation

$$\begin{aligned} K_1 &:= \frac{\rho}{2}\int_0^L \int_{-h}^{\eta} |\nabla\tilde{\varphi}|^2\, dy dx + \frac{\rho_1}{2}\int_0^L \int_{\eta}^{h_1+\eta_1} |\nabla\tilde{\varphi}_1|^2\, dy dx, \\ K_2 &:= \rho\gamma \int_0^L \int_{-h}^{\eta} y\tilde{\varphi}_x\, dy dx + \rho_1\gamma_1 \int_0^L \int_{\eta}^{h_1+\eta_1} y\tilde{\varphi}_{1,x}\, dy dx, \\ K_3 &:= \rho\kappa \int_0^L \int_{-h}^{\eta} \tilde{\varphi}_x\, dy dx + \rho_1\kappa_1 \int_0^L \int_{\eta}^{h_1+\eta_1} \tilde{\varphi}_{1,x}\, dy dx. \end{aligned} \tag{3.13}$$

Utilizing the definitions of the Dirichlet–Neumann operators (3.7),(3.10), Green's second identity and (3.11)–(3.12) we obtain

$$\begin{aligned} K_1+K_2+K_3 =&\frac{1}{2}\int_0^L \begin{pmatrix}\xi\\ \xi_1\end{pmatrix}^T \begin{pmatrix}-G_{11} & -G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}dx\\ &-\frac{1}{2}\int_0^L \left[(\rho\gamma\Phi-\rho_1\gamma_1\Phi_1)\eta+\rho\kappa\Phi-\rho_1\kappa_1\Phi_1\right]\eta_x\,dx\\ &-\int_0^L[\gamma_1(h_1+\eta_1)+\kappa_1]\xi_1\eta_{1,x}\,dx\\ &-\frac{1}{2}\int_0^L(\gamma_1\eta+\kappa_1)\xi\eta_x\,dx, \end{aligned} \tag{3.14}$$

which can be further transformed to

$$\begin{aligned} K_1+K_2+K_3 =&\frac{1}{2}\int_0^L \begin{pmatrix}\xi\\ \xi_1\end{pmatrix}^T \begin{pmatrix}G_{11}B^{-1}G & -GB^{-1}G_{12}\\ -G_{21}B^{-1}G & -\frac{\rho}{\rho_1}G_{21}B^{-1}G_{12}+\frac{1}{\rho_1}G_{22}\end{pmatrix}\begin{pmatrix}\xi\\ \xi_1\end{pmatrix}dx\\ &+\int_0^L \mu B^{-1}(\rho_1 G\xi+\rho G_{12}\xi_1)\,dx-\frac{\rho\rho_1}{2}\int_0^L\mu B^{-1}\mu\,dx\\ &-\int_0^L(\gamma\eta+\kappa)\xi\eta_x\,dx-\int_0^L[\gamma_1(h_1+\eta_1)+\kappa_1]\xi_1\eta_{1,x}\,dx, \end{aligned} \tag{3.15}$$

where

$$\mu=\mu(\eta):=[(\gamma-\gamma_1)\eta+\kappa-\kappa_1]\eta_x \quad \text{and} \quad B=B(\eta,\eta_1):=\rho_1 G+\rho G_{11}.$$

The previous considerations together with the assumption of zero mean for the free surface- and the interface defining functions, η_1 and η, respectively, prove the claim (3.5) about the functional dependence of the Hamiltonian H. More precisely, we have

$$\begin{aligned} \mathcal{H} =&\mathcal{H}_0+\mu B^{-1}(\rho_1 G\xi+\rho G_{12}\xi_1)-\frac{\rho\rho_1}{2}\mu B^{-1}\mu\\ &-(\gamma\eta+\kappa)\xi\eta_x-[\gamma_1(h_1+\eta_1)+\kappa_1]\xi_1\eta_{1,x}\\ &+\frac{\rho\gamma^2(h^3+\eta^3)+\rho_1\gamma_1^2\left((h_1+\eta_1)^3-\eta^3\right)}{6}\\ &+\frac{\rho\kappa\gamma(\eta^2-h^2)+\rho_1\kappa_1\gamma_1\left((h_1+\eta_1)^2-\eta^2\right)}{2}+\frac{\rho\kappa^2 h+\rho_1\kappa_1^2 h_1}{2}\\ &+\frac{\mathcal{D}\eta_{1,xx}^2}{2(1+\eta_{1,x}^2)^{\frac{5}{2}}}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \mathcal{H}_0 =&\frac{1}{2}\begin{pmatrix}\xi\\ \xi_1\end{pmatrix}^T \begin{pmatrix}G_{11}B^{-1}G & -GB^{-1}G_{12}\\ -G_{21}B^{-1}G & -\frac{\rho}{\rho_1}G_{21}B^{-1}G_{12}+\frac{1}{\rho_1}G_{22}\end{pmatrix}\begin{pmatrix}\xi\\ \xi_1\end{pmatrix}\\ &+\frac{\rho g(\eta^2-h^2)+\rho_1 g\left((h_1+\eta_1)^2-\eta^2\right)}{2} \end{aligned} \tag{3.17}$$

denotes the contribution from the irrotational component of the flow. With the previous preparations we can now state the following preliminary result.

Theorem 3.3. *The governing equations admit the following nearly-Hamiltonian formulation*

$$\begin{cases} \xi_t=-\frac{\delta H}{\delta\eta}+(\rho\gamma-\rho_1\gamma_1)\chi, & \eta_t=\frac{\delta H}{\delta\xi},\\ \xi_{1,t}=-\frac{\delta H}{\delta\eta_1}+\rho_1\gamma_1\chi_1, & \eta_{1,t}=\frac{\delta H}{\delta\xi_1}. \end{cases} \tag{3.18}$$

Proof. Collecting all the variations with respect to η as obtained in Appendix we obtain

$$\begin{aligned} \frac{\delta H}{\delta\eta} =&\rho(\tilde\varphi_y-\tilde\varphi_x\eta_x-\gamma\eta\eta_x)_s\cdot(-\tilde\varphi_y)_s+\rho\left(\frac{1}{2}|\nabla\tilde\varphi|_s^2+\gamma\eta(\tilde\varphi_x)_s+\frac{\gamma^2}{2}\eta^2+g\eta\right)\\ &+\rho_1(\eta_x\tilde\varphi_{1,x}-\tilde\varphi_{1,y}+\gamma_1\eta\eta_x)_s\cdot(-\tilde\varphi_{1,y})_s\\ &-\rho_1\left(\frac{1}{2}|\nabla\tilde\varphi_1|_s^2+\gamma_1\eta(\tilde\varphi_{1,x})_s+\frac{\gamma_1^2}{2}\eta^2+g\eta\right)\\ &+\rho\kappa\left[(\tilde\varphi_x)_s+\eta_x(\tilde\varphi_y)_s\right]-\rho_1\kappa_1\left[(\tilde\varphi_{1,x})_s+\eta_x(\tilde\varphi_{1,y})_s\right]\\ &+\eta(\rho\kappa\gamma-\rho_1\kappa_1\gamma_1)+\frac{\rho\kappa^2}{2}-\frac{\rho_1\kappa_1^2}{2}. \end{aligned} \tag{3.19}$$

In order to simplify the formula above we make use of formula (2.19) and of the identities

$$\frac{1}{2}|\nabla\tilde{\varphi}|_s^2 + \frac{\gamma^2\eta^2}{2} + \gamma\eta(\tilde{\varphi}_x)_s = \frac{1}{2}|\nabla\psi|_s^2 - \frac{\kappa^2}{2} - \kappa(\tilde{\varphi}_x)_s - \kappa\gamma\eta, \tag{3.20}$$

$$\frac{1}{2}|\nabla\tilde{\varphi}_1|_s^2 + \frac{\gamma_1^2\eta^2}{2} + \gamma_1\eta(\tilde{\varphi}_{1,x})_s = \frac{1}{2}|\nabla\psi_1|_s^2 - \frac{\kappa_1^2}{2} - \kappa_1(\tilde{\varphi}_{1,x})_s - \kappa_1\gamma_1\eta, \tag{3.21}$$

and obtain that

$$\begin{aligned}\frac{\delta H}{\delta\eta} =& -\rho(\eta_t+\kappa\eta_x)(\tilde{\varphi}_y)_s + \rho_1(\eta_t+\kappa_1\eta_x)(\tilde{\varphi}_{1,y})_s\\ &+\rho\left(\frac{1}{2}|\nabla\psi|_s^2 - \frac{\kappa^2}{2} - \kappa(\tilde{\varphi}_x)_s - \kappa\gamma\eta + g\eta\right)\\ &-\rho_1\left(\frac{1}{2}|\nabla\psi_1|_s^2 - \frac{\kappa_1^2}{2} - \kappa_1(\tilde{\varphi}_{1,x})_s - \kappa_1\gamma_1\eta + g\eta\right)\\ &+\rho\kappa\left[(\tilde{\varphi}_x)_s + \eta_x(\tilde{\varphi}_y)_s\right] - \rho_1\kappa_1\left[(\tilde{\varphi}_{1,x})_s + \eta_x(\tilde{\varphi}_{1,y})_s\right]\\ &+\eta(\rho\kappa\gamma - \rho_1\kappa_1\gamma_1) + \frac{\rho\kappa^2}{2} - \frac{\rho_1\kappa_1^2}{2}.\end{aligned} \tag{3.22}$$

Cancellations in the formula above and (2.32) lead further to

$$\begin{aligned}\frac{\delta H}{\delta\eta} =& -\rho\eta_t(\tilde{\varphi}_y)_s + \rho_1\eta_t(\tilde{\varphi}_{1,y})_s\\ &+\rho\left(\frac{|\nabla\psi|_s^2}{2} + g\eta\right) - \rho_1\left(\frac{|\nabla\psi_1|_s^2}{2} + g\eta\right)\\ =& -\rho\underbrace{\left[\eta_t(\tilde{\varphi}_y)_s + (\tilde{\varphi}_t)_s\right]}_{=\Phi_t \text{ by } (3.4)} + \rho_1\underbrace{\left[\eta_t(\tilde{\varphi}_{1,y})_s + (\tilde{\varphi}_{1,t}))_s\right]}_{=\Phi_{1,t} \text{ by } (3.4)} + \rho\gamma\chi - \rho_1\gamma_1\chi,\\ =& -\xi_t + (\rho\gamma - \rho_1\gamma_1)\chi\end{aligned} \tag{3.23}$$

We also have

$$\begin{aligned}\frac{\delta H}{\delta\eta_1} =& \rho_1(-\tilde{\varphi}_{1,y})_{s_1}\left[(\tilde{\varphi}_{1,y})_{s_1} - \eta_{1,x}(\tilde{\varphi}_{1,x})_{s_1} - \gamma_1(h_1+\eta_1)\eta_{1,x}\right]\\ &+\frac{\rho_1}{2}|\nabla\tilde{\varphi}_1|_{s_1}^2 + \rho_1\gamma_1(h_1+\eta_1)(\tilde{\varphi}_{1,x})_{s_1}\\ &+\frac{\rho_1\gamma_1^2}{2}(h_1+\eta_1)^2 + g\rho_1(h_1+\eta_1)\\ &+\rho_1\kappa_1\left[(\tilde{\varphi}_{1,x})_{s_1} + \eta_{1,x}(\tilde{\varphi}_{1,y})_{s_1}\right] + \rho_1\kappa_1\gamma_1(h_1+\eta_1) + \frac{\rho_1\kappa_1^2}{2}\\ &+\mathcal{D}\left(\mathcal{k}_{\mathfrak{s}\mathfrak{s}} + \frac{1}{2}\mathcal{k}^3\right)\end{aligned} \tag{3.24}$$

Using the kinematic boundary condition at the interface (2.18) and the formula

$$\begin{aligned}\frac{1}{2}|\nabla\tilde{\varphi}_1|_{s_1}^2 + \frac{\gamma_1^2(\eta_1+h_1)^2}{2} &+ \gamma_1(\eta_1+h_1)(\tilde{\varphi}_{1,x})_{s_1}\\ &= \frac{1}{2}|\nabla\psi_1|_{s_1}^2 - \frac{\kappa_1^2}{2} - \kappa_1(\tilde{\varphi}_{1,x})_{s_1} - \kappa_1\gamma_1(\eta_1+h_1),\end{aligned} \tag{3.25}$$

obtained from (2.17), we see that

$$\begin{aligned}\frac{\delta H}{\delta\eta_1} =& -\rho_1(\tilde{\varphi}_{1,y})_{s_1}(\eta_{1,t}+\kappa_1\eta_{1,x})\\ &+\rho_1\left(\frac{1}{2}|\nabla\psi_1|_{s_1}^2 - \frac{\kappa_1^2}{2} - \kappa_1(\tilde{\varphi}_{1,x})_{s_1} + (g-\kappa_1\gamma_1)(h_1+\eta_1)\right)\\ &+\rho_1\kappa_1\left[(\tilde{\varphi}_{1,x})_{s_1} + \eta_{1,x}(\tilde{\varphi}_{1,y})_{s_1}\right] + \rho_1\kappa_1\gamma_1(h_1+\eta_1)\\ &+\frac{\rho_1\kappa_1^2}{2} + \mathcal{D}\left(\mathcal{k}_{\mathfrak{s}\mathfrak{s}} + \frac{1}{2}\mathcal{k}^3\right)\\ =& -\rho_1(\tilde{\varphi}_{1,y})_{s_1}\eta_{1,t} + \underbrace{\frac{\rho_1}{2}|\nabla\psi_1|_{s_1}^2 + \mathcal{D}\left(\mathcal{k}_{\mathfrak{s}\mathfrak{s}} + \frac{1}{2}\mathcal{k}^3\right) + \rho_1 g(h_1+\eta_1)}_{=\rho_1\gamma_1\chi_1 - \rho_1(\tilde{\varphi}_{1,t})_{s_1} \text{ by formula } (2.30)}\\ =& -\rho_1\left[(\tilde{\varphi}_{1,y})_{s_1}\eta_{1,t} + (\tilde{\varphi}_{1,t})_{s_1}\right] + \rho_1\gamma_1\chi_1\\ =& -\xi_{1,t} + \rho_1\gamma_1\chi_1.\end{aligned} \tag{3.26}$$

Collecting now all the factors of $\delta\Phi$ and $\delta\Phi_1$ from (A.1)–(A.6) and using (2.19) and that $\xi = \rho\Phi - \rho_1\Phi_1$ we have

$$\frac{\delta H}{\delta \xi} = (\tilde{\varphi}_y)_s - \eta_x(\tilde{\varphi}_x)_s - \gamma\eta\eta_x - \kappa\eta_x = (\tilde{\varphi}_{1,y})_s - \eta_x(\tilde{\varphi}_{1,x})_s - \gamma_1\eta\eta_x - \kappa_1\eta_x = \eta_t. \tag{3.27}$$

Similarly, from (A.2), (A.4), (A.6) and using $\xi_1 = \rho_1\Phi_2$ we obtain that

$$\frac{\delta H}{\delta \xi_1} = (\tilde{\varphi}_{1,y})_{s_1} - \eta_{1,x}\left[(\tilde{\varphi}_{1,x})_{s_1} + \gamma_1(h_1+\eta_1) + \kappa_1\right] = \eta_{1,t}, \tag{3.28}$$

the last equality being in fact the formula (2.18).

The previous computations can be summarized as follows

$$\delta H = \int_0^L \left\{\eta_t\delta\xi + \eta_{1,t}\delta\xi_1 + [-\xi_t + (\rho\gamma - \rho_1\gamma_1)\chi]\delta\eta + [-\xi_{1,t} + \rho_1\gamma_1\chi_1]\delta\eta_1\right\}, \tag{3.29}$$

relation, which together with the definition of the variational derivative with respect to the inner product in the space $\mathcal{L}^2[0,L]$ of square integrable functions, enforces (3.18). □

After having proven the result in (3.18) a few remarks are in order.

(i) The two (constant) vorticities γ and γ_1 represent the extent by which the governing Eqs. (2.1)–(2.6) fail to be representable as a Hamiltonian system

$$\omega_t = J\frac{\delta H}{\delta \omega}.$$

Here $t \mapsto \omega(t)$ is a path in a Hilbert space $\mathfrak{H}$ equipped with an inner product, the associated Hamiltonian functional $H : \mathfrak{D} \subset \mathfrak{H} \to \mathbb{R}$ is defined on the dense subset $\mathfrak{D}$ of $\mathfrak{H}$, and J is a skew-adjoint (pseudo)-differential operator, cf. [43].

(ii) We would like to remark that the system (3.18) is automatically Hamiltonian in the absence of shear: that is, setting $\gamma = \gamma_1 = 0$ we have that

$$\begin{cases} \xi_t = -\frac{\delta H}{\delta \eta}, & \eta_t = \frac{\delta H}{\delta \xi}, \\ \xi_{1,t} = -\frac{\delta H}{\delta \eta_1}, & \eta_{1,t} = \frac{\delta H}{\delta \xi_1}, \end{cases} \tag{3.30}$$

recovering the result by Craig, Guyenne and Kalisch [18].

(iii) Setting now $\gamma = \gamma_1$ and $\rho = \rho_1$, we have $\Phi = \Phi_1$ and we are now in the situation of a single layer flow with a free surface and obtain the reduced system

$$\begin{cases} \eta_{1,t} = \frac{\delta H}{\delta \xi_1}, \\ \xi_{1,t} = -\frac{\delta H}{\delta \eta_1} + \rho_1\gamma_1\chi_1, \end{cases}$$

representing the (nearly)-Hamiltonian formulation of the governing equations in the single layer $-h < y < h_1 + \eta_1(x,t)$, obtained by Constantin, Ivanov & Prodanov, cf. [12]. Further, setting $\gamma_1 = 0$ above we recover the finite depth analogue of Zakharov's seminal Hamiltonian formulation for irrotational waves on infinitely deep fluids [11].

3.2. The Hamiltonian formulation

This section is devoted to showing that the (nearly)-Hamiltonian formulation (3.18) becomes Hamiltonian by a suitable change of variables.

Theorem 3.4. *The change of variables*

$$\begin{aligned} z &= \xi + \tfrac{\rho\gamma - \rho_1\gamma_1}{2}\int_0^x \eta(x',t)\,dx' \\ z_1 &= \xi_1 + \tfrac{\rho_1\gamma_1}{2}\int_0^x \eta_1(x',t)\,dx'. \end{aligned} \tag{3.31}$$

transforms the nearly-Hamiltonian system (3.18) in the Hamiltonian system

$$\begin{cases} z_t = -\frac{\delta H}{\delta \eta}, & \eta_t = \frac{\delta H}{\delta z}, \\ z_{1,t} = -\frac{\delta H}{\delta \eta_1}, & \eta_{1,t} = \frac{\delta H}{\delta z_1}, \end{cases} \tag{3.32}$$

which is a re-formulation of the governing equations.

Proof. Setting $\alpha := \rho\gamma - \rho_1\gamma_1$ and $\beta := \rho_1\gamma_1$ in (3.29), we obtain by means of (3.31)

$$\begin{aligned}\delta H =& \int_0^L \left[-z_t + \frac{\alpha}{2}\int_0^x \eta_t(x',t)\,dx' + \alpha\chi\right]\delta\eta\,dx\\ &+ \int_0^L \eta_t\left[\delta z - \frac{\alpha}{2}\int_0^x \delta\eta(x',t)dx'\right]dx\\ &+ \int_0^L \left[-z_{1,t} + \frac{\beta}{2}\int_0^x \eta_{1,t}(x',t)\,dx' + \beta\chi_1\right]\delta\eta_1\,dx\\ &+ \int_0^L \eta_{1,t}\left[\delta z_1 - \frac{\beta}{2}\int_0^x \delta\eta_1(x',t)dx'\right]dx.\end{aligned} \tag{3.33}$$

Note now that, due to (2.23) and since $\int_0^L \eta(x,t)dx = 0$, we have

$$\begin{aligned}\int_0^L \eta_t\left(\int_0^x \delta\eta(x',t)dx'\right)dx &= \int_0^L \frac{d}{dx}\left(\int_0^x \eta_t(x'',t)\,dx''\right)\left(\int_0^x \delta\eta(x',t)dx'\right)dx\\ &= -\int_0^L\left(\int_0^x \eta_t(x'',t)\,dx''\right)(\delta\eta)(x,t)\,dx\\ &= \int_0^L \chi\delta\eta\,dx,\end{aligned} \tag{3.34}$$

and, similarly

$$\begin{aligned}\int_0^L \eta_{1,t}\left(\int_0^x \delta\eta_1(x',t)dx'\right)dx &= \int_0^L \frac{d}{dx}\left(\int_0^x \eta_{1,t}(x'',t)\,dx''\right)\left(\int_0^x \delta\eta_1(x',t)dx'\right)dx\\ &= -\int_0^L\left(\int_0^x \eta_{1,t}(x'',t)\,dx''\right)(\delta\eta_1)(x,t)\,dx\\ &= \int_0^L \chi_1\delta\eta_1\,dx.\end{aligned} \tag{3.35}$$

With the help of the previous two relations we can rewrite (3.33) as

$$\delta H = \int_0^L (-z_t)\delta\eta\,dx + \int_0^L \eta_t\delta z\,dx + \int_0^L (-z_{1,t})\delta\eta_1\,dx + \int_0^L \eta_{1,t}\delta z_1\,dx, \tag{3.36}$$

from which our claim emerges. □

Remark 3.5. In terms of the variables η, η_1 and of the new variables z_1 we can write the Hamiltonian density as

$$\begin{aligned}\mathcal{H} =& \frac{1}{2}\begin{pmatrix} z \\ z_1\end{pmatrix}^T \mathcal{A}\begin{pmatrix} z \\ z_1\end{pmatrix} - \frac{1}{2}\begin{pmatrix} z \\ z_1\end{pmatrix}^T \mathcal{A}\begin{pmatrix} \Gamma\int_0^x \eta(l,t)\,dl \\ \Gamma_1\int_0^x \eta_1(l,t)\,dl\end{pmatrix} - \frac{1}{2}\begin{pmatrix} \Gamma\int_0^x \eta(l,t)\,dl \\ \Gamma_1\int_0^x \eta_1(l,t)\,dl\end{pmatrix}^T \mathcal{A}\begin{pmatrix} z \\ z_1\end{pmatrix}\\ &+ \frac{1}{2}\begin{pmatrix} \Gamma\int_0^x \eta(l,t)\,dl \\ \Gamma_1\int_0^x \eta_1(l,t)\,dl\end{pmatrix}^T \mathcal{A}\begin{pmatrix} \Gamma\int_0^x \eta(l,t)\,dl \\ \Gamma_1\int_0^x \eta_1(l,t)\,dl\end{pmatrix}\\ &+ \frac{\rho g(\eta^2 - h^2) + \rho_1 g\left((h_1+\eta_1)^2 - \eta^2\right)}{2} - \frac{\rho\rho_1}{2}\mu B^{-1}\mu\\ &+ \mu B^{-1}(\rho_1 G z + \rho G_{12} z_1) - \mu B^{-1}\left(\Gamma\rho_1 G\int_0^x \eta(l,t)\,dl + \Gamma_1\rho G_{12}\int_0^x \eta_1(l,t)\,dl\right)\\ &+ \left(\frac{\gamma\eta^2}{2} + \kappa\eta\right)(z_x - \Gamma\eta) + \left(\frac{\gamma_1\eta_1^2}{2} + (\gamma_1 h_1 + \kappa_1)\eta_1\right)(z_{1,x} - \Gamma_1\eta_1)\\ &+ \frac{\rho\gamma^2(h^3+\eta^3) + \rho_1\gamma_1^2\left((h_1+\eta_1)^3 - \eta^3\right)}{6}\\ &+ \frac{\rho\kappa\gamma(\eta^2 - h^2) + \rho_1\kappa_1\gamma_1\left((h_1+\eta_1)^2 - \eta^2\right)}{2} + \frac{\rho\kappa^2 h + \rho_1\kappa_1^2 h_1}{2}\\ &+ \frac{\mathcal{D}\eta_{1,xx}^2}{2(1+\eta_{1,x}^2)^{\frac{5}{2}}},\end{aligned} \tag{3.37}$$

where

$$\mathcal{A} = \begin{pmatrix} G_{11}B^{-1}G & -GB^{-1}G_{12} \\ -G_{21}B^{-1}G & -\frac{\rho}{\rho_1}G_{21}B^{-1}G_{12} + \frac{1}{\rho_1}G_{22}\end{pmatrix},$$

and $\Gamma = \rho\gamma - \rho_1\gamma_1$, $\Gamma_1 = \rho_1\gamma_1$.

3.3. The linear dispersion relation

We illustrate in this section the usefulness of Hamiltonian methods which in conjunction with the Dirichlet–Neumann operators allow the derivation of the dispersion relation. In connection with the latter aspect, we would like to remark that the understanding of many relevant aspects of the evolution of water waves begins with the investigation of the linearized equations about a trivial solution. In this case such a trivial solution is the state of rest, which entails that the free surface and the interface are given as $\eta = \eta_1 = 0$ and, moreover $\xi = \xi_1 = 0$. The derivation of the linearized equations of motion is done by truncating the Taylor expansion of the Hamiltonian density $\mathcal{H}$ at its quadratic term, which we denote with $\mathcal{H}^{(2)}$. To make the presentation more straightforward, we are omitting the currents κ and κ_1 from the context. To obtain $\mathcal{H}^{(2)}$ we note that the Dirichlet–Neumann operators $G(\eta)$ and $G_1(\eta, \eta_1)$ are analytic with respect to the dependence on η abd (η, η_1), respectively [18,44], and have convergent Taylor expansions

$$G(\eta) = \sum_{p=0}^{\infty} G^{(p)}(\eta), \tag{3.38}$$

$$G_1(\eta,\eta_1) = \sum_{p_0,p_1=0}^{\infty} \begin{pmatrix} G_{11}^{(p_0p_1)}(\eta,\eta_1) & G_{12}^{(p_0p_1)}(\eta,\eta_1) \\ G_{21}^{(p_0p_1)}(\eta,\eta_1) & G_{22}^{(p_0p_1)}(\eta,\eta_1) \end{pmatrix}, \tag{3.39}$$

where each linear operator $G^{(p)}(\eta)$ is homogeneous of degree p in η and each linear operator $G_{ij}^{(p_0p_1)}(\eta,\eta_1)$, $(i,j = 1,2)$, is homogeneous of degree p_0 in η and of degree p_1 in η_1, cf. [18,44]. Each of the operators $G^{(p)}(\eta)$ and $G_{ii}^{(p_0p_1)}(\eta,\eta_1)$, $i = 1,2$, is self-adjoint, while $\left(G_{12}^{(p_0p_1)}(\eta,\eta_1)\right)^* = G_{21}^{(p_0p_1)}(\eta,\eta_1)$.

Remark 3.6. In order to give more insights into the asymptotic structures (3.38) and (3.39) we need additional notation, definitions and the observation that the Dirichlet–Neumann can be understood as certain pseudodifferential operators. More precisely, let m be a complex-valued function of one real variable whose derivatives of any order have polynomial growth and setting $D := -i\partial_x$ we define

$$(m(D)f)(x) := \frac{1}{2\pi} \iint e^{ik(x-y)} m(k) f(y)\, dy\, dk. \tag{3.40}$$

The operator $m(D)$ is called a Fourier multiplier operator and maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. Furthermore,

1. $m(D)$ extends to a self-adjoint operator in $L^2(\mathbb{R})$ if and only if m is real valued, cf. [45].
2. $m(D)$ is bounded if and only if $m \in L^\infty(\mathbb{R})$.

Then, the leading order terms of the DN operators are given (cf. [18,44]) by means of Fourier multipliers as

$$G^{(0)}(\eta) = D\tanh(hD), \tag{3.41}$$

$$\begin{pmatrix} G_{11}^{(00)}(\eta,\eta_1) & G_{12}^{(00)}(\eta,\eta_1) \\ G_{21}^{(00)}(\eta,\eta_1) & G_{22}^{(00)}(\eta,\eta_1) \end{pmatrix} = \begin{pmatrix} D\coth(h_1D) & -D\mathrm{csch}(h_1D) \\ -D\mathrm{csch}(h_1D) & D\coth(h_1D) \end{pmatrix}. \tag{3.42}$$

Utilizing also that the homogeneous part of order 0 of B is $B^{(0)} := \rho_1 G^{(0)}(\eta) + \rho G_{11}^{(00)}(\eta,\eta_1)$ we obtain that the quadratic part of the Hamiltonian density (3.16) is

$$\begin{aligned} \mathcal{H}^{(2)} =& \frac{1}{2}\xi \frac{D\tanh(hD)\coth(h_1D)}{\rho\coth(h_1D) + \rho_1\tanh(hD)}\xi + \xi\frac{D\tanh(hD)\mathrm{csch}(h_1D)}{\rho\coth(h_1D)+\rho_1\tanh(hD)}\xi_1 \\ &+ \frac{1}{2}\xi_1 \frac{D\left(\tanh(hD)\coth(h_1D) + \frac{\rho}{\rho_1}\right)}{\rho\coth(h_1D)+\rho_1\tanh(hD)}\xi_1 \\ &+ \frac{1}{2}\left(g(\rho-\rho_1)\eta^2 + \rho_1(\gamma_1^2 h_1 + g)\eta_1^2\right) - \gamma_1 h_1 \xi_1 \eta_{1,x} + \frac{\mathcal{D}}{2}\eta_{1,xx}^2. \end{aligned} \tag{3.43}$$

Denoting with $H^{(2)}$ the quadratic part of the Hamiltonian H, the linearized equations of motion are

$$\begin{cases} \eta_t = \dfrac{\delta H^{(2)}}{\delta \xi}, \\ \eta_{1,t} = \dfrac{\delta H^{(2)}}{\delta \xi_1}, \\ \xi_t = -\dfrac{\delta H^{(2)}}{\delta \eta} + (\rho_1\gamma_1 - \rho\gamma)\displaystyle\int_0^x \eta_t(l,t)\, dl, \\ \xi_{1,t} = -\dfrac{\delta H^{(2)}}{\delta \eta_1} - \rho_1\gamma_1 \displaystyle\int_0^x \eta_{1,t}(l,t)\, dl, \end{cases} \tag{3.44}$$

which can be detailed as

$$\begin{cases} \eta_t = \dfrac{D\tanh(hD)\coth(h_1 D)}{\rho\coth(h_1 D)+\rho_1\tanh(hD)}\xi + \dfrac{D\tanh(hD)\mathrm{csch}(h_1 D)}{\rho\coth(h_1 D)+\rho_1\tanh(hD)}\xi_1, \\ \eta_{1,t} = \dfrac{D\tanh(hD)\mathrm{csch}(h_1 D)}{\rho\coth(h_1 D)+\rho_1\tanh(hD)}\xi + \dfrac{D\Big(\tanh(hD)\coth(h_1 D)+\frac{\rho}{\rho_1}\Big)}{\rho\coth(h_1 D)+\rho_1\tanh(hD)}\xi_1 - \gamma_1 h_1\eta_{1,x}, \\ \xi_t = -g(\rho-\rho_1)\eta + (\rho_1\gamma_1-\rho\gamma)\displaystyle\int_0^x \eta_t(l,t)\,dl, \\ \xi_{1,t} = -\gamma_1 h_1\xi_{1,x} - \rho_1(\gamma_1^2 h_1+g)\eta_1 - \rho_1\gamma_1\displaystyle\int_0^x \eta_{1,t}(l,t)\,dl - \mathcal{D}\eta_{1,xxxx}. \end{cases} \tag{3.45}$$

To derive the dispersion relation we will investigate monochromatic solutions of the leading order linear Eqs. (3.45), thus all components of the solutions (η,η_1,ξ,ξ_1) are taken to be proportional to $e^{i(kx-\omega(k)t)}$ where k denotes the wave number, and $\omega(k)=kc(k)$ represents the frequency, with $c(k)$ being the wave speed. Therefore, with the ansatz

$$\eta = \alpha(k)e^{i(kx-\omega(k)t)}, \qquad \eta_1(t) = \alpha_1(k)e^{i(kx-\omega(k)t)}, \tag{3.46}$$

(for some coefficients $\alpha(k),\alpha_1(k)$) we obtain from (3.45) the (linear) dispersion relation

$$\begin{aligned} &\frac{G^2(k)}{k^2}\left[\frac{\Gamma}{\rho}-\frac{\tilde g}{c}\right]\left[\frac{\Gamma_1}{\rho}+\frac{1}{c-\gamma_1 h_1}\left(\theta-\frac{\mathcal{D}k^4}{\rho}\right)\right] = \\ &\left[c+\frac{F(k)}{k}\left(\frac{\Gamma}{\rho}-\frac{\tilde g}{c}\right)\right]\left[c-\gamma_1 h_1+\frac{H(k)}{k}\left(\frac{\Gamma_1}{\rho}+\frac{1}{c-\gamma_1 h_1}\left(\theta-\frac{\mathcal{D}k^4}{\rho}\right)\right)\right], \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} F(k) &= \frac{\rho\tanh(kh)\coth(kh_1)}{\rho_1\tanh(kh)+\rho\coth(kh_1)}, \\ G(k) &= \frac{\rho\tanh(kh)\mathrm{csch}(kh_1)}{\rho_1\tanh(kh)+\rho\coth(kh_1)}, \\ H(k) &= \frac{\rho\left(\tanh(kh)\coth(kh_1)+\frac{\rho}{\rho_1}\right)}{\rho_1\tanh(kh)+\rho\coth(kh_1)}, \\ \theta &= \frac{\gamma_1 h_1\Gamma_1-\rho_1(\gamma_1^2 h_1+g)}{\rho}, \\ \tilde g &= \frac{g(\rho-\rho_1)}{\rho}. \end{aligned} \tag{3.48}$$

We conclude this section with two particular cases of the dispersion relation (3.47).

Remark 3.7. Setting $\gamma=\gamma_1=0$ in (3.47) we obtain that the wave speed c satisfies the equation

$$c^4 - \left[\left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{H(k)}{k}+\tilde g\frac{F(k)}{k}\right]c^2 + \tilde g\left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{F(k)H(k)-G^2(k)}{k^2} = 0, \tag{3.49}$$

which, seen as a second degree equation in c^2, has the discriminant equal to

$$\Delta = \left[\left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{H(k)}{k}-\tilde g\frac{F(k)}{k}\right]^2 + 4\tilde g\left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{G^2(k)}{k^2} > 0. \tag{3.50}$$

Since $F(k)H(k)-G^2(k) = \frac{\rho^2\tanh(kh)}{\rho_1\left(\rho_1\tanh(kh)+\rho\coth(kh_1)\right)} > 0$ we see that (3.49) has two positive solutions, representing c^2, that is

$$c_\pm^2 = \frac{B(k)\pm\sqrt{B^2(k)-4\tilde g\left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{F(k)H(k)-G^2(k)}{k^2}}}{2}, \tag{3.51}$$

with

$$\begin{aligned} B(k) &= \left(\frac{g\rho_1+\mathcal{D}k^4}{\rho}\right)\frac{H(k)}{k}+\tilde g\frac{F(k)}{k}, \\ &= \frac{g\rho\big(\tanh(kh)+\tanh(kh_1)\big)+\mathcal{D}k^4\left(\frac{\rho}{\rho_1}\tanh(kh_1)+\tanh(kh)\right)}{k(\rho_1\tanh(kh)\tanh(kh_1)+\rho)}. \end{aligned} \tag{3.52}$$

Since

$$F(k)H(k)-G^2(k) = \frac{\rho\tanh(kh)\tanh(kh_1)}{\rho_1(1+R\tanh(kh)\tanh(kh_1))},$$

with $R=\frac{\rho_1}{\rho}$, we see that formula (3.51) coincides with the dispersion relation found by Wang et al. [28], cf. formulas (2.9)-(2.12) therein, in the setting of interfacial solitary waves propagating under an elastic sheet in irrotational flows.

Remark 3.8. Setting now $\gamma = \gamma_1 = 0$ and $\mathcal{D} = 0$ in (3.47) we obtain that the frequency $\omega(k) = kc(k)$ satisfies the equation

$$\omega^4 - g\rho k \frac{1 + \tanh(kh)\coth(kh_1)}{\rho_1 \tanh(kh) + \rho \coth(kh_1)}\omega^2 + g^2(\rho - \rho_1)k^2 \frac{\tanh kh}{\rho_1 \tanh(kh) + \rho \coth(kh_1)} = 0, \tag{3.53}$$

which coincides with the dispersion relation found by Craig et al. [18], in the setting of linear free surfaces and interfaces in irrotational flows.

4. Conclusions and further perspectives

We have derived here a Hamiltonian formulation for the nonlinear equations governing the motion of two-dimensional hydroelastic waves propagating at the surface of a stratified rotational two-layer ideal fluid of finite depth, covered by a thin ice sheet. The setting we considered includes the combined effects of (discontinuous) stratification and of piecewise constant vorticity, permitting thus the presence of interfaces (playing the role of internal waves) and of linearly sheared currents. Appealing to the Dirichlet–Neumann operators we have derived linearized equations of motion for the free surface, the interface and for the traces of the (generalized) velocity potentials on the two surfaces. The analysis of the latter equations was concluded by the derivation of the (linear dispersion) relation.

Since the Hamiltonian variables $(\eta, \eta_1, \xi, \xi_1)$ give the boundary values of the (generalized) velocity potential, which extends analytically to the interior of the fluid domain, it follows that the Hamiltonian approach developed here has the potential to lead to future thorough investigations concerning the flow beneath pertaining to the velocity field, the pressure and the particle trajectories, cf. e.g. [46–48]. Moreover, a detailed analysis of the Dirichlet–Neumann operators makes it plausible that certain weakly-nonlinear models for various propagation regimes will be derived from the nonlinear governing equations. We would like to emphasize the importance of the Dirichlet–Neumann operators by noticing that the recent years have witnessed a significant increase in the study of both the full nonlinear water wave problem and of various approximate nonlinear water wave models. This advancement is propelled by a combination of methods centered on the Dirichlet–Neumann operators and Hamiltonian systems, cf. [23,49–55], as well as numerical computation techniques for quantities expressed by the Dirichlet–Neumann operators [56–58], building on the pioneering work [59].

Due to the fact that we start from the full nonlinear equations we expect a nonlinear Schrödinger equation (in some weakly-nonlinear regimes), which is different from the model based on a modification of the multi-layer Green–Naghdi equations [25]. The quantitative geophysical characteristics of the regions of the Arctic Ocean where internal waves have been observed (e.g. [4,6]) will help in deriving weakly-nonlinear equations from the Hamiltonian formulation in useful settings, which then can be used to predict and calculate properties of internal waves and compare them with field results. This is especially useful as the observations of internal waves cannot be performed during the winter months, when the sea is covered by ice.

Of further interest are investigations about the dispersion relation for waves at the elastic interface between fluids in multi-layered domains cf. [60,61]. Moreover, while inclusion of viscous effects would impede a Hamiltonian formulation by means of the dynamical variables defined in (3.3), of high relevance are future investigations concerning viscoelastic wave-ice interactions by means of computational methods, cf. [62–64].

CRediT authorship contribution statement

Călin-Iulian Martin: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Emilian I. Părău:** Writing – original draft, Supervision, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

We recall in this section a few formulas regarding the variations of the functional H with respect to η, η_1, ξ and ξ_1 as obtained in [19]. We start first with the kinetic terms from the representation of H. For the kinetic term corresponding to the lower layer we have

$$\begin{aligned}\delta\left(\int_0^L\int_{-h}^{\eta}|\nabla\tilde{\varphi}|^2\,dydx\right) =& 2\int_0^L[\tilde{\varphi}_y-\eta_x\tilde{\varphi}_x]_s[\delta\boldsymbol{\Phi}-(\tilde{\varphi}_y)_s\delta\eta]\,dx\\ &+\int_0^L|\nabla\tilde{\varphi}|_s^2\delta\eta\,dx.\end{aligned}\tag{A.1}$$

Analogously, for the kinetic term corresponding to the upper layer it holds that

$$\begin{aligned}\delta\left(\int_0^L\int_{\eta}^{h_1+\eta_1}|\nabla\tilde{\varphi}_1|^2\,dydx\right) =& 2\int_0^L[\eta_x\tilde{\varphi}_{1,x}-\tilde{\varphi}_{1,y}]_s[\delta\boldsymbol{\Phi}_1-(\tilde{\varphi}_{1,y})_s\delta\eta]\,dx\\ &+2\int_0^L[\tilde{\varphi}_{1,y}-\eta_{1,x}\tilde{\varphi}_{1,x}]_{s_1}\cdot[\delta\boldsymbol{\Phi}_2-(\tilde{\varphi}_{1,y})_{s_1}\delta\eta_1]\,dx\\ &+\int_0^L|\nabla\tilde{\varphi}_1|_{s_1}^2\delta\eta_1\,dx-\int_0^L|\nabla\tilde{\varphi}_1|_s^2\delta\eta\,dx.\end{aligned}\tag{A.2}$$

We also have that

$$\begin{aligned}&\delta\left(\int_0^L\int_{-h}^{\eta}y\tilde{\varphi}_x\,dydx\right)\\ &=-\int_0^L[\delta\boldsymbol{\Phi}-(\tilde{\varphi}_y)_s\delta\eta]\eta\eta_x+\int_0^L\eta(\tilde{\varphi}_x)_s\delta\eta\,dx,\end{aligned}\tag{A.3}$$

as well as

$$\begin{aligned}&\delta\left(\int_0^L\int_{\eta}^{h_1+\eta_1}y\tilde{\varphi}_{1,x}\,dydx\right)\\ &=-\int_0^L(h_1+\eta_1)[\delta\boldsymbol{\Phi}_2-(\tilde{\varphi}_{1,y})_{s_1}\delta\eta_1]\eta_{1,x}\,dx+\int_0^L\eta[\delta\boldsymbol{\Phi}_1-(\tilde{\varphi}_{1,y})_s\delta\eta]\eta_x\,dx\\ &\quad+\int_0^L(h_1+\eta_1)(\tilde{\varphi}_{1,x})_{s_1}\delta\eta_1\,dx-\int_0^L\eta(\tilde{\varphi}_{1,x})_s\delta\eta\,dx.\end{aligned}\tag{A.4}$$

The variation of the kinetic terms is concluded by noting that

$$\delta\left(\int_0^L\int_{-h}^{\eta}\tilde{\varphi}_x\,dydx\right)=\int_0^L(\tilde{\varphi}_x)_s\delta\eta\,dx-\int_0^L(\delta\boldsymbol{\Phi}-(\tilde{\varphi}_y)_s\delta\eta)\eta_x\,dx,\tag{A.5}$$

and

$$\begin{aligned}\delta\left(\int_0^L\int_{\eta}^{h_1+\eta_1}\tilde{\varphi}_{1,x}\,dydx\right)=&\int_0^L(\tilde{\varphi}_{1,x})_{s_1}\delta\eta_1\,dx-\int_0^L(\delta\boldsymbol{\Phi}_2-(\tilde{\varphi}_{1,y})_{s_1}\delta\eta_1)\eta_{1,x}\,dx\\ &+\int_0^L(\delta\boldsymbol{\Phi}_1-(\tilde{\varphi}_{1,y})_s\delta\eta)\eta_x\,dx-\int_0^L(\tilde{\varphi}_{1,x})_s\delta\eta\,dx.\end{aligned}\tag{A.6}$$

By the periodicity of η_1 and the chain rule for variational derivatives, we obtain

$$\frac{\delta}{\delta\eta_1}\left(\int_0^L\frac{\eta_{1,xx}^2}{2(1+\eta_x^2)^{\frac{5}{2}}}\,dx\right)=\mathcal{k}_{\mathfrak{s}\mathfrak{s}}+\frac{1}{2}\mathcal{k}^3.\tag{A.7}$$

Data availability

No data was used for the research described in the article.

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