Side Conditions of Models of Two Types and High Forcing Axioms

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Abstract

The present dissertation is a contribution to the areas of combinatorial set theory and high forcing axioms through the technique of forcing with side conditions. We introduce new forcing notions consisting of symmetric systems of models of two types, which can be seen as generalizations of both Neeman's chains of elementary submodels of two types and Asperó and Mota's symmetric systems of countable elementary submodels. After a preliminary chapter in which we establish the notation and cover the background material required for what follows, we develop the theory of the pure side condition forcings and prove their main properties. The first application of this technique is in the area of combinatorial set theory. We partially answer a question of Hajnal and Szentmiklóssy from the 1990s, by forcing a strong chain of subsets of ω_1 of length ω_3, improving earlier results of Koszmider and Veličković-Venturi. In the final chapter we introduce finite support forcing iterations with symmetric systems of models of two types as side conditions in the sense of Asperó and Mota. We isolate a class of forcing notions naturally associated with these iterations and prove the consistency of its forcing axiom, which is compatible with $2^{\aleph_0} > \aleph_2$. This class of posets, which is a subclass of Neeman's high analog of the class of proper forcings, can be seen as a generalization of Asperó and Mota's classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-chain condition.

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Introduction

The present thesis is a contribution to set theory, the area of mathematics devoted to the study of infinity and the foundations of mathematics. More generally, set theory can be described as the mathematical theory of certain well-defined collections of objects called *sets* with respect to the relation of *membership*.

Set theory starts with the work of Georg Cantor in the 1870s (possibly influenced by the work of Dedekind). While searching for a classification of infinite sets according to their cardinality, he made the amazing discovery that the set of real numbers $\mathbb{R}$ has a strictly greater cardinality than the set of natural numbers $\mathbb{N}$. In other words, the elements of $\mathbb{R}$ cannot be put in a one-to-one correspondence with the elements of $\mathbb{N}$. A consequence of this result, which Cantor showed later, is that for every infinite set, there is another one of larger cardinality. This motivated him to introduce a system of numbers to measure the cardinality of infinite sets, the *infinite cardinals*. While it is relatively easy to show that the set of natural numbers has the least possible infinite cardinality ($\aleph_0$), finding the cardinality of the set of real numbers ($2^{\aleph_0}$) turned out to be a much harder problem. This led Cantor, in 1878, to formulate his famous *Continuum Hypothesis* (CH), that asserts that there are no sets of real numbers of cardinality lying strictly between the size of $\mathbb{N}$ and the size of $\mathbb{R}$[1]. In other words, CH asserts that the set of real numbers has the second least possible infinite cardinality, that is, it asserts that $2^{\aleph_0} = \aleph_1$. This problem, which troubled Cantor for the rest of his life, appeared

[1]It is common practice in set theory to refer, indistinctly, to the set of real numbers (sometimes regarded as a topological space) and to its cardinality, as the *continuum*.

as the first problem in David Hilbert's famous list of 23 problems from 1900, and the search for its solution has been (and still is) one of the main driving forces of development of set theory.

Early on, some inconsistencies arised from the naive use of the notion of set proposed by Cantor as a "collection of objects that share some property". The most famous one being *Russell's Paradox*, discovered by the mathematician and philosopher Bertrand Russell. This lead to a foundational crisis in set theory and in mathematics, and the way to overcome this obstacle was to adopt an axiomatic system on which set theory could be built upon. The first axiomatization of set theory is due to Ernst Zermelo, in 1908, and the final version, improved by Thoralf Skolem, Abraham Fraenkel, and Zermelo himself, appeared in the 1920s. The resulting theory, known as ZFC (Zermelo-Fraenkel with the Axiom of Choice), is still today the most commonly used axiomatization of set theory.

Set theory has the very particular status of being a two-faced mathematical theory. On one hand, as we have explained, it is the mathematical theory of infinity and infinite sets. But on the other hand, set theory, and ZFC in particular, serve as a foundation for all of mathematics. The remarkable fact that ZFC is a formal system in which virtually all of mathematics can be interpreted, makes possible a mathematical study of mathematics itself. However, by Kurt Gödel's Incompleteness Theorems ([33]), any consistent theory which interprets Peano Arithmetic and whose axioms are presentable as a recursively enumerable set of sentences, such as ZFC, will be incomplete. That is, there will be statements, expressible in the language of the system, whose truth or falsity cannot be proven within the system. Such unprovable statements are called *independent* (of ZFC), and far from being mere artificial self-referential logical paradoxes, hundreds, if not thousands, of concrete example of independent statements have been discovered over the years.

The first and most famous example was precisely the Continuum Hypothesis. In 1938, Gödel [34] introduced his constructible universe L, and showed that

ZFC plus CH held in his model. Thus, proving that assuming the consistency of ZFC, there is no counterexample to the CH in ZFC. Twenty-five years later, in 1963, Paul Cohen [24] introduced the method of forcing and, assuming the consistency of ZFC, obtained a model of ZFC plus the negation of CH. Therefore, by combining Gödel and Cantor's results, we can deduce that the truth or falsity of the CH cannot be decided from the ZFC axioms.

The idea behind the technique of forcing is to start with a model of ZFC (the *ground model*) and extend it to a new model of ZFC (the *generic extension*) by adding, in a very controlled way, new *generic* objects. This new model of ZFC has the same ordinals as the ground model, (most often) has the same cardinals, and satisfies some desired formula in which we are interested.

After some refinement of this tremendously powerful method, a vast amount of consistency results in all areas of mathematics have been obtained. For instance, *Suslin's Hypothesis* [80], *Whitehead's problem* [72], the *Borel Conjecture* [46], or more recently, the *Brown-Douglas-Fillmore problem* [27]. The search for independent statements has become one of the central topics in set theory, and exploring the different models of ZFC (and fragments and extensions thereof) obtainable by forcing has become one of the main sources of development of the area.

Even if Gödel's Theorems tell us that the independence phenomenon is unavoidable, the search for the "right" truth value of statements such as the ones from the last paragraph hasn't stopped. In fact, this has motivated set theorists to search for "natural" extensions of ZFC that settle these questions. There are multiple candidates that have been studied over the years, but the three main lineages of axioms that have been seriously considered are the *Determinacy Axioms*, the *Large Cardinal Axioms*, and the *Forcing Axioms*. In this thesis we will focus on the third group.

In Cohen's proof of the consistency of ZFC+¬CH, forcing was used to extend a model of ZFC by adding $\aleph_2$-many new reals, thus violating the CH in the new model. However, it is known that models of ZFC+CH are also obtainable by means of the technique of forcing. Therefore, models satisfying incompatible statements can be obtained by forcing. In fact, there is a multitude of other examples in the literature of incompatible statements whose consistency can be proved by this technique. Hence, since no other effective method for extending models of ZFC is known, one of the main directions pursued by set theorists has been the search for new axioms that "get rid" of the relativity of truth with respect to the different models of ZFC that are obtainable by forcing. In other words, we want to eliminate, as much as possible, the independence derived from the technique of forcing.

Forcing axioms are principles occurring naturally in set theory that fit into this category, although they can be stated in purely combinatorial terms:

Definition 0.0.1. If $\mathcal{K}$ is a class of forcing notions[2] and κ is an infinite cardinal, $FA_\kappa(\mathcal{K})$ states that for every $\mathbb{P} \in \mathcal{K}$ and every collection $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ such that $G \cap D \neq \varnothing$ for every $D \in \mathcal{D}$.

Forcing axioms can be seen as generalizations of the Baire Category Theorem, but their true power comes from the fact that they also assert that the universe of set theory is saturated with respect to forcing extensions, in the sense mentioned above. By this we mean that forcing axioms are equivalent to certain statements of the form "if we can force, with a forcing in $\mathcal{K}$, a formula of a certain complexity with parameters in $H(\kappa^+)$[3], then this formula is in fact true" [16]. Therefore, forcing axioms for κ-many dense sets decide, to a large extent, the theory of $H(\kappa^+)$.

In 1965 (although the result was published in 1971), Solovay and Tennenbaum [80] developed the theory of forcing iterations of posets with the *countable chain condition* (c.c.c.) to prove the consistency of Suslin's Hypothesis. However, it was

[2] A forcing notion is simply a partially ordered set with a top element.

[3] If θ is an infinite cardinal, $H(\theta)$ denotes the set of all sets having transitive closure of cardinality $< \theta$. If θ is an uncountable regular cardinal, then $H(\theta)$ is a model of ZFC minus the Power Set axiom.

realised by Martin and Solovay [48] that they could use this technique to build a model of ZFC with arbitrarily large values of the continuum and satisfying a much stronger condition. Namely, what came to be known as *Martin's Axiom*[4] (MA), the first forcing axiom to be isolated.

Iterated forcing is quite a complicated technique, which requires a deep understanding of forcing theory. The point of forcing axioms is that they capture part of the combinatorial content of forcing iterations and detach it from the logical one. For instance, Martin's Axiom captures the combinatorial content of finite support iterations of c.c.c. forcings. This is exactly where the great success of forcing axioms stems from. Since no knowledge of logic or set theory is required to state them, mathematicians interested in the independence phenomenon have used forcing axioms to prove consistency results in other areas of mathematics. There are abundant concrete examples in areas such as measure theory, topology, group theory, Boolean algebras, Banach space theory and C*-algebras (see [29] for many examples). Most notable is Shelah's solution to the Whitehead problem in group theory, already mentioned above:

Theorem 0.0.2 (Shelah,[72])**.** *In a model of ZFC where MA and the negation of CH hold there is a non-free Whitehead group.*

Nevertheless, there are many natural statements in mathematics known to be independent of ZFC, whose independence is known not to be provable by MA. Indeed, the class of c.c.c. forcings is quite small, and many nice forcing notions lie outside of it. This motivated set theorists to search for larger iterable classes of posets with consistent forcing axioms.

Inspired by Laver's proof of the consistency of the Borel Conjecture [46], which used a countable support iteration of Laver forcing, Shelah isolated the class of *proper* forcings [73] (see also [74] and [76]). The class of proper forcings is much larger than the class of c.c.c. forcings, but the posets in this class are still quite

[4]Martin's Axiom is the forcing axiom for the class of posets with the c.c.c. and families of less than continuum-many dense sets.

well-behaved. Indeed, proper forcings preserve ω_1 and countable support iterations of proper forcings are themselves proper (it was also shown by Shelah that this fact is not true if you replace countable supports by finite ones). Baumgartner showed the consistency of the forcing axiom for the class of proper forcings and for collections of $\aleph_1$-many dense sets, also known as the *Proper Forcing Axiom* (PFA), by building a countable support iteration of length a supercompact cardinal [18]. Proper forcing and the PFA have been extremely successful in uncovering the combinatorial structure of ω_1 and the possible combinatorics of sets of real numbers, but they have also found many applications in topology, algebra and analysis (see [19]). The most remarkable consequences in set theory being that PFA solves the continuum problem by deciding the value of $|\mathbb{R}|$ to be $\aleph_2$ (see [20] and [87]), and it implies the *Singular Cardinals Hypothesis*[5] (see [89]) and the failure of $\square_\lambda$ for all uncountable cardinals λ (see [83]).

In 1988, Foreman, Magidor and Shelah [28] proved the consistency of a maximal forcing axiom for $\aleph_1$-many dense sets, known as *Martin's Maximum* (MM). The proof uses revised countable support iterations and it is much more involved than the proof of the PFA. This forcing axiom is maximal in the sense that any strictly stronger forcing axiom for $\aleph_1$-many dense sets is inconsistent. Hence, we cannot hope to find stronger consistent forcing axioms by considering bigger classes of forcing notions, or at least not in the obvious way. Having exhausted this direction, for many years set theorists focused on studying weakenings of these forcing axioms, especially weakenings of PFA. One major direction has been on measuring the strength of some consequences of PFA by studying their relative consistency with different values of the continuum. There are essentially three groups: those consequences of PFA that are consistent with CH, those that decide the value of the continuum to be $\aleph_2$, and those that are consistent with large values of the continuum (i.e., larger than $\aleph_2$). Of special interest are the following:

[5] The Singular Cardinals Hypothesis (SCH) is the assertion that if κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$.

- Abraham-Rubin-Shelah's and Todorčević's *Open Coloring Axioms*, OCA_{ARS} and OCA_T, respectively. Introduced independently in [4] and [84]. Moore [60] showed that the conjunction of both coloring axioms imply that the continuum is $\aleph_2$. Gilton and Neeman [31] showed that OCA_{ARS} is consistent with the continuum being $\aleph_3$, and Neeman in an unpublished work has shown that OCA_T restricted to colorings of sets of reals of size less than the continuum is compatible with large values of $2^{\aleph_0}$. Showing that OCA_T is consistent together with the continuum large is still an open question.

- Todorčević's *P-Ideal Dichotomy* (PID) [85]. Abraham and Todorčević [6] showed that the PID is consistent with the CH. It is a very important open problem whether PID is consistent with large continuum.

- Moore's *Mapping Reflection Principle* (MRP) [61]. In the same paper it was shown that MRP implies that the continuum is $\aleph_2$.

- Goldstern and Shelah's *Bounded Proper Forcing Axiom* (BPFA) [36]. Moore in [61] introduced another principle, called υ_{AC}, which implies that the continuum is $\aleph_2$ and follows from both BPFA and MRP.

There is still of course another possible direction, which hasn't been fully explored due to the lack of an appropriate iteration theory. That is, the search for consistent forcing axioms for more than $\aleph_1$-many dense sets, known as *high forcing axioms*. We have already mentioned that models of these principles are naturally produced by suitable iterated forcing extensions. However, due to technical limitations, the known techniques of iterated forcing are not suitable for proving the consistency of high forcing axioms. The fact that $2^{\aleph_0} = \aleph_2$ is a consequence of PFA implies that the forcing axiom for the class of proper forcings for more than $\aleph_1$-many dense sets would be inconsistent. Therefore, we cannot hope for a straightforward high analog of properness. It is worth mentioning Rosłanowski and Shelah's series of papers (see for example [66], [67], [68], [69], [70]) on high versions of properness. Although, it is fair to say the right generalization of properness hasn't been found yet. The program aiming to find consistent high forcing axiom can be summarised in the following very general question, which has been one of the main driving forces in the development of modern set theory and, especially, (iterated) forcing theory:

Question. *Can we find reasonable classes of forcing notions $\mathcal{K}$ for which $FA_\kappa(\mathcal{K})$ is consistent for $\kappa > \aleph_1$?*

By "reasonable" we mean, first of all, iterable classes of forcing notions that preserve cardinals. Additionally, we would like these classes to include high versions of the posets that belong to the classes of forcing notions $\mathcal{K}$ for which $FA_{\aleph_1}(\mathcal{K})$ is known to be consistent. Finally, we would like the corresponding forcing axiom to have strong set-theoretic consequences, such as deciding the value of the continuum. In particular, a high analog of the PFA should decide the value of the continuum to be $\aleph_3$. More generally, we want forcing axioms that decide the theory of $H(\omega_3)$, in the same way that strong forcing axioms for $\aleph_1$-many dense sets, such as PFA and MM, largely decide the theory of $H(\omega_2)$.

Recent developments in the technique of forcing with side conditions have opened the door to new approaches to overcome the technical limitations of classical iteration theory, and it has become the most promising candidate in the search for consistent high forcing axioms. About ten years ago, Neeman [62] found a new and revolutionary way to iterate proper forcings with finite support (although, not in the classical way), which he used to obtain an alternative proof of the consistency of PFA. The breakthrough in Neeman's work is the use of chains of elementary submodels of two types to ensure the preservation of two cardinals. The idea of adding elementary submodels in the conditions of your forcing notions to preserve cardinals, known as *forcing with side conditions*, goes back to Todorčević (see [83] and [81]). This technique has been exploited by set theorists to build nice notions of forcing (see [82] for many applications), most notably Mitchell [50] and Friedman [30] showed independently that you can add clubs on ω_2 with finite conditions. In all these applications countable elementary submodels are used to ensure that the forcing preserves one cardinal (typically ω_1). Neeman, building on Mitchell and Friedman's work, introduced a general framework for building forcing notions with side conditions of models of two types, countable and transitive, thus ensuring the preservation of two cardinals (typically ω_1 and ω_2). Applications of this method include very elegant proofs of new and old consistency results (see [63], [88], and [59]).

Some of the technical obstacles that prevented set theorists from tackling the problem of finding consistent high forcing axioms disappear when countable support iterations are replaced by finite support ones. This prompted set theorists to generalise Neeman's new iteration theory for proper forcings to other classes of posets. Moti Gitik and Menachem Magidor [32], and Boban Veličković [86], independently found alternative finite support proofs of the consistency of the *semiproper forcing axiom* (SPFA)[6]. However, although Neeman announced (see [64] and [65]) that a generalization of his method could be used to obtain a high analog of PFA, these results have not yet been published, and in fact, there hasn't been much work on high forcing axioms. Rahman Mohammadpour's PhD thesis [58] is one of the very few works in this direction.

Independently of Neeman, but around the same time, David Asperó and Miguel Ángel Mota ([11], [12]) developed a new method for building finite support forcing iterations with symmetric systems of countable structures as side conditions. These iterations were used to prove the consistency of certain fragments of PFA, namely the restriction of PFA to the classes of *finitely proper forcings* and forcings with the $\aleph_{1.5}$-*chain condition*, together with arbitrarily large values of the continuum. This was later applied to show that many known consequences of PFA are consistent with large continuum. Especially, very strong failures of Shelah's *club guessing* (CG) principle.

In this thesis we generalise Neeman's chains of elementary submodels of two types

[6]The class of semiproper forcings is an extension of the class of proper forcings, and SPFA refers to the forcing axiom $FA_{\aleph_1}$(Semiproper).

by naturally combining them with Asperó and Mota's symmetric systems. We will introduce the forcing consisting of symmetric systems of models of two types, we will prove their main properties, and we will use them in two very different applications.

The thesis is organised as follows:

In chapter 1 we will establish some of the notation that will be used throughout the thesis and cover the necessary background material about forcing, forcing axioms, large cardinals and elementary submodels.

In chapter 2 we will introduce symmetric systems of elementary submodels of two types. We will start by reviewing Neeman's chains of elementary submodels of two types and Asperó and Mota's symmetric systems. Then we will define symmetric systems of models of two types and show their main properties. In the last section we will introduce a variant of the two-type symmetric systems from the previous section, which will be used in chapter 4. The following theorem summarises the main properties of the pure side condition forcings from these last two sections, namely the forcing notions consisting of two-type symmetric systems, ordered by reverse inclusion:

Theorem 0.0.3. *Let $\mathbb{M}$ denote any of the two forcing notions from sections 2.3 and 2.4. Then, $\mathbb{M}$ has the following properties:*

(1) $\mathbb{M}$ is strongly proper with respect to countable elementary submodels.

(2) $\mathbb{M}$ is strongly proper with respect to an appropriate class of $\aleph_1$-sized elementary submodels.

(3) If $2^{\aleph_1} = \aleph_2$ holds, then $\mathbb{M}$ has the $\aleph_3$-Knaster condition.

(4) It follows from the last three items that if $2^{\aleph_1} = \aleph_2$ holds, then $\mathbb{M}$ preserves all cardinals.

(5) $\mathbb{M}$ preserves $2^{\aleph_1} = \aleph_2$.

In chapter 3 we will force the existence of a strong chain of subsets of ω_1 of length ω_3. This partially answers a question of Hajnal and Szentmiklóssy from the 1990s in combinatorial set theory. The construction involves symmetric systems of models of two types as side conditions to ensure the preservation of cardinals. This is proof that two-type symmetric systems can be very useful in forcing objects of size $\aleph_3$ with finite approximations, which are usually out of reach of Neeman's two-type side conditions. The following theorem is the main result of chapter 3:

Theorem 0.0.4. *Assuming the* Generalized Continuum Hypothesis[7] *(*GCH*) there is a forcing notion $\mathbb{P}$ with the following properties:*

(1) $\mathbb{P}$ is proper with respect to countable elementary submodels.

(2) $\mathbb{P}$ is proper with respect to an appropriate class of $\aleph_1$-sized elementary submodels.

(3) $\mathbb{P}$ has the $\aleph_3$-chain condition.

(4) $\mathbb{P}$ forces the existence of a strong chain of subsets of ω_1 of length ω_3.

In chapter 4 we will define the class of $(\mathcal{S}, \mathcal{L})$-*finitely proper forcings*, which is a subclass of Neeman's high analog of the class of proper forcings ([64], [65]), and can be seen as a natural generalization of Asperó and Mota's classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-chain condition. We will develop a high version of Asperó and Mota's finite support iterations ([11], [12]) in which we will incorporate symmetric systems of models of two types as side conditions. This construction, which consists of a sequence of forcing notions $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$, has the following properties for every $\alpha < \beta \leq \kappa$:

- $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$.
- $\mathbb{P}_\beta$ is proper for countable elementary submodels.

[7] The GCH asserts that $2^\kappa = \kappa^+$ holds for every infinite cardinal κ.

- $\mathbb{P}_\beta$ is proper for an appropriate class of $\aleph_1$-sized elementary submodels.
- $\mathbb{P}_\beta$ has the $\aleph_3$-chain condition.

As an application, these iterations will be used to build a generic extension where the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings holds. More precisely, the forcing notion $\mathbb{P}_\kappa$ will be a witness of the following theorem:

Theorem 0.0.5. *If κ is a supercompact cardinal and $2^{\aleph_1} = \aleph_2$ holds, then $\mathbb{P}_\kappa$ is a cardinal-preserving forcing notion, which forces $2^{\aleph_0} = \kappa > \aleph_2$ and the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings and $< \kappa$-many dense sets.*

In the last section we will speculate about some possible extensions of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings and their potential applications.

In appendix A we will list some open problems, and possible future lines of research and applications of the techniques developed in this thesis.

1

Preliminaries

The standard references for set-theoretic notions are [44] and [38]. We refer the reader to these two sources for any undefined notions. Most of the results in this chapter are well-known and can be found in either of the two books. We will include specific references in each section when required.

Our notation will be standard and will also follow [44] and [38]. Unless otherwise specified, lower case Greek letters $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \eta$ will be used to denote ordinals, while $\kappa, \lambda, \mu, \nu, \theta$ will be used to denote infinite cardinals. We will denote by OR the class of all ordinals. Let X be any set. We will denote by $\mathcal{P}(X)$ the power set of X. If μ is a cardinal, we will denote by $[X]^\mu$ the set of all subsets of X of size μ. The sets $[X]^{<\mu}$ and $[X]^{\leq\mu}$ are defined in the obvious way. If f is a function and $X \subseteq \mathrm{dom}(f)$, then $f"(X)$ denotes the set $\{f(x) : x \in X\}$. If λ is an infinite regular cardinal and $\mu < \mathrm{cf}(\lambda)$, we denote the set $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \mu\}$ by S^λ_μ.

1.1 Forcing

In this section we recall some basic facts about the technique of forcing and fix some of the notation that will be used throughout the thesis. Our standard reference is [44], but we will also follow [1] and [35] when dealing with proper forcing and related concepts.

1.1.1 Basic facts and notation

Forcing notions or *forcing posets* (or simply *forcings*) are triples $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, where $\leq_{\mathbb{P}}$ is a preorder on $\mathbb{P}$ (transitive and reflexive binary relation) and $\mathbb{1}_{\mathbb{P}} \in \mathbb{P}$ is the largest element with respect to $\leq_{\mathbb{P}}$. We will abuse notation by identifying a forcing notion with its universe $\mathbb{P}$, and we will usually omit the subscripts from $\leq_{\mathbb{P}}$ and $\mathbb{1}_{\mathbb{P}}$ if $\mathbb{P}$ is clear from the context. The elements of $\mathbb{P}$ are called *conditions*, and if $p, q \in \mathbb{P}$, we will read $p \leq_{\mathbb{P}} q$ as "p extends q" or "p is stronger than q". If $p, q \in \mathbb{P}$, we say that *p and q are compatible* if there is another condition r stronger than p and q. If two conditions are not compatible, we say that they are *incompatbile*.

An *antichain* is a subset $A \subseteq \mathbb{P}$ consisting of pairwise incompatible conditions. A subset $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $d \in D$ such that $d \leq p$, and it is *predense* if for every $p \in \mathbb{P}$ there is $d \in D$ compatible with p. If $q \in \mathbb{P}$, we say that $D \subseteq \mathbb{P}$ is *dense below* q if for every $p \leq q$ there is $d \in D$ such that $d \leq p$, and it is *predense below* q if for every $p \leq q$ there is $d \in D$ compatible with p. A subset $O \subseteq \mathbb{P}$ is *open* if it is downwards closed.

Let M be a countable transitive model of a big enough fragment of ZFC[1]. We also require that $\mathbb{P} \in M$. This model, namely the model over which we do forcing, is called a *ground model*. A non-empty subset $G \subseteq \mathbb{P}$ is called a *filter on* $\mathbb{P}$ if it is upwards closed and any two elements of G are compatible in G. A filter G on $\mathbb{P}$ is called $\mathbb{P}$-*generic over* M (or simply, *generic over* M) if $G \cap D \neq \varnothing$, for every dense subset $D \subseteq \mathbb{P}$ such that $D \in M$. The Rasiowa–Sikorski lemma ensures that generic filters over countable transitive models exist. In fact, for every condition $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G over M such that $p \in G$.

A forcing notion $\mathbb{P}$ is called *separative* if whenever $p \not\leq q$, then there is an extension r of p incompatible with q. There are many reasons to consider separative posets in the context of forcing. First of all, if $\mathbb{P}$ is separative and $\mathbb{P} \in M$, then no

[1]This theory should at least include Kripke-Platek set theory.

generic filter $G \subseteq \mathbb{P}$ over M is a member of M. This is crucial, since we want to extend M to a new model of set theory by means of this generic G, so it needs to live outside of M. Moreover, any poset can be embedded (there is a map that preserves order and incompatibility) onto a separative partial order.

Definition 1.1.1. The class $M^{\mathbb{P}}$ of $\mathbb{P}$-*names* is defined in M by transfinite recursion on the ordinals as follows:

(1) τ is a $\mathbb{P}$-name of rank 0 if $\tau = \varnothing$.

(2) τ is $\mathbb{P}$-name of rank $\leq \alpha$ if its elements are of the form $\langle \sigma, p \rangle$, where σ is a $\mathbb{P}$-name of rank $< \alpha$ and $p \in \mathbb{P}$.

(3) τ is a $\mathbb{P}$-name if it is a $\mathbb{P}$-name of rank $\leq \alpha$ for some $\alpha \in OR \cap M$.

The class $M^{\mathbb{P}}$ is Σ_1-definable in M with $\mathbb{P}$ as a parameter. In fact, it is Δ_1-definable, since the complement of $M^{\mathbb{P}}$ is also Σ_1-definable in M. We will use the Greek letters τ, σ and π to denote arbitrary $\mathbb{P}$-names and dotted letters $(\dot{x}, \dot{y}, \dot{z}, \dot{f}, \dot{g}, \dots)$ to denote $\mathbb{P}$-names that name specific objects. The *forcing language* is the language of set theory with names added as constants.

Definition 1.1.2. For every $x \in M$, we define the *standard name of* x, denoted $\check{x}$, as the $\mathbb{P}$-name

$$\check{x} = \{\langle \check{y}, \mathbb{1} \rangle : y \in x\}$$

Similarly, we define the *standard name for the generic filter* G, denoted $\dot{G}$, as the $\mathbb{P}$-name

$$\dot{G} = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}.$$

Definition 1.1.3. If $G \subseteq \mathbb{P}$ is a generic filter over M and τ is a $\mathbb{P}$-name, we define the *interpretation of* τ *by* G, denoted τ_G, by transfinite recursion as follows:

(1) $\tau_G = \varnothing$ if τ is a $\mathbb{P}$-name of rank 0.

(2) $\tau_G = \{\sigma_G : \langle \sigma, p \rangle \in \tau, p \in G\}$

For each $x \in M$, $\check{x}_G = x$, and similarly, $(\dot{G})_G = G$. We will usually omit inverted circumflexes and write x instead of $\check{x}$, when it is clear from the context that we are dealing with standard names.

Definition 1.1.4. If $G \subseteq \mathbb{P}$ is a generic filter over M, we define the *generic extension of* M *by* G, denoted $M[G]$, as the set

$$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}.$$

If $\mathbb{P}$ is a forcing notion, the *forcing relation for* $\mathbb{P}$, denoted by $\Vdash_{\mathbb{P}}$ (or simply $\Vdash$, if $\mathbb{P}$ is clear from the context), is a binary relation between conditions of $\mathbb{P}$ and formulas taking $\mathbb{P}$-names as parameters. The purpose of the forcing relation is to encode the truth predicate of the generic extension $M[G]$ within M. The definition of $\Vdash_{\mathbb{P}}$ (which we won't include here, but can be found in [44]) is by recursion on the complexity of the formulas in the forcing language, using $\mathbb{P}$ as a parameter. The idea is to define the forcing relation so that if $\tau_0, \ldots, \tau_n$ are $\mathbb{P}$-names and $\varphi(x_0, \ldots, x_n)$ is a formula in the language of set theory, then $p \Vdash_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_n)$ (read "p forces $\varphi(\tau_0, \ldots, \tau_n)$") if and only if for every generic filter $G \subseteq \mathbb{P}$ such that $p \in G$, $M[G] \models \varphi((\tau_0)_G, \ldots, (\tau_n)_G)$.

Remark 1.1.5. Another important feature of separative posets, related to the forcing relation, is that for any two $p, q \in \mathbb{P}$, $p \leq q$ iff $p \Vdash_{\mathbb{P}} q \in \dot{G}$.

The following are some of the basic (but necessary) properties of the forcing relation:

(1) If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.

(2) No condition p forces both φ and $\neg\varphi$.

(3) For every formula φ, the set $D(\varphi) = \{p \in \mathbb{P} : p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ is a dense open subset of $\mathbb{P}$. If M satisfies ZF minus the Power Set axiom, then $D(\varphi)$ is definable in M with $\mathbb{P}$ as a parameter. Hence, if G is $\mathbb{P}$-generic over M, then φ is decided (i.e., forced to be true or false) by some condition in G.

If φ is a formula in the language of forcing, "$\mathbb{P}$ forces φ" stands for $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi$. Often, $\Vdash_{\mathbb{P}} \varphi$ or $\mathbb{P} \Vdash_{\mathbb{P}} \varphi$ are written, instead of $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi$. Note that by item (1) above, if $\mathbb{1}_{\mathbb{P}}$ forces φ, then every condition in $\mathbb{P}$ forces φ. Therefore, by definition of the forcing relation, any formula forced by $\mathbb{P}$ will hold in any generic extension of M. Since for any condition $p \in \mathbb{P}$ we can find a $\mathbb{P}$-generic filter G over M such that $p \in G$, then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\tau)$ if and only if $V[G] \models \varphi(\tau_G)$, for every $\mathbb{P}$-generic filter G over V, where τ is a tuple of $\mathbb{P}$-names of the same arity as the number of free variables of the formula φ.

In some situations we will use expressions of the form "τ is the $\mathbb{P}$-name of an ordinal" or "τ is the $\mathbb{P}$-name of a subset of ω_1". In general, if φ is a formula in the language of set theory, the sentence "τ is a $\mathbb{P}$-name such that $\varphi(\tau)$" means that $\Vdash_{\mathbb{P}} \varphi(\tau)$.

The whole machinery of forcing can be summarised in the following three theorems, which are known as the *forcing theorems.*

Theorem 1.1.6. *For every $n \geq 1$, the forcing relation $\Vdash_{\mathbb{P}}$ restricted to Σ_n-formulas in the language of set theory is Σ_n-definable in M with $\mathbb{P}$ as a parameter.*

Theorem 1.1.7. *If G is a $\mathbb{P}$-generic filter over M, then for every formula $\varphi(x_0, \ldots, x_n)$ in the language of set theory and $\tau_0, \ldots, \tau_n \in M^{\mathbb{P}}$,*

$$M[G] \models \varphi((\tau_0)_G, \ldots, (\tau_n)_G) \textit{ iff } \exists p \in G\big(M \models p \Vdash_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_n)\big).$$

Theorem 1.1.8. *If M is a countable transitive model of ZFC, $\mathbb{P} \in M$ is a forcing notion and G a $\mathbb{P}$-generic filter over M, then $M[G]$ is the least countable transitive model of ZFC such that $M \cup \{G\} \subseteq M[G]$ and $M \cap OR = M[G] \cap OR$.*

Even though the above exposition of forcing has been made over countable transitive models of big enough fragments of ZFC, it is common practice to use V, the universe of set theory, as the ground model for our forcing constructions. More about the technicalities of such considerations can be found in [44].

1.1.2 Equivalence of generic extensions

Let $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ be forcing posets. Let M be a countable transitive model of a big enough fragment of ZFC.

Definition 1.1.9. A map $e : \mathbb{P} \to \mathbb{Q}$ is said to be an *embedding* if the following hold:

(1) $e(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$.

(2) For every $p_0, p_1 \in \mathbb{P}$, if $p_0 \leq_{\mathbb{P}} p_1$, then $e(p_0) \leq_{\mathbb{Q}} e(p_1)$.

(3) For every $p_0, p_1 \in \mathbb{P}$, if p_0 and p_1 are incompatible in $\mathbb{P}$, then $e(p_0)$ and $e(p_1)$ are incompatible in $\mathbb{Q}$.

The map e is said to be a *complete embedding* if along with (1)-(3) the following holds:

(4) If $A \subseteq \mathbb{P}$ is a maximal antichain, then $e(A)$ is a maximal antichain in $\mathbb{Q}$.

The map e is said to be a *dense embedding* if along with (1)-(3) the following holds:

(5) $e(\mathbb{P})$ is a dense subset of $\mathbb{Q}$.

Definition 1.1.10. We say that $\mathbb{P}$ is a *complete suborder* of $\mathbb{Q}$, and denote it by $\mathbb{P} \lessdot \mathbb{Q}$, if the inclusion map $\mathbb{P} \subseteq \mathbb{Q}$ is a complete embedding.

Lemma 1.1.11. *If $e : \mathbb{P} \to \mathbb{Q}$ is an isomorphism (between the two structures $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$), then e is a dense embedding.*

Lemma 1.1.12. *If $e : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then e is a complete embedding.*

Lemma 1.1.13. *Suppose that $\mathbb{P}, \mathbb{Q} \in M$ and that $e : \mathbb{P} \to \mathbb{Q}$ is a complete embedding such that $e \in M$. If H is $\mathbb{Q}$-generic over M, then $G = i^{-1}(H)$ is $\mathbb{P}$-generic over M, and $M[G] \subseteq M[H]$.*

Lemma 1.1.14. *Suppose that* $\mathbb{P}, \mathbb{Q} \in M$ *and that* $e : \mathbb{P} \to \mathbb{Q}$ *is a dense embedding such that* $e \in M$.

(1) If G *is* $\mathbb{P}$*-generic over* M*, then* $H = \{q \in \mathbb{Q} : \exists p \in G(e(p) \leq_{\mathbb{Q}} q)\}$ *is* $\mathbb{Q}$*-generic over* M*, and* $M[G] = M[H]$.

(2) If H *is* $\mathbb{Q}$*-generic over* M*, then* $G = \{p \in \mathbb{P} : \exists q \in H(e^{-1}(q) = p)\}$ *is* $\mathbb{P}$*-generic over* M*, and* $M[H] = M[G]$.

1.1.3 Some classes of forcing notions

When going from a ground model V to a generic extension $V[G]$ via the forcing $\mathbb{P}$, many new objects may appear in $V[G]$, which weren't in V. However, if we are not careful enough, some of these objects could be troublesome, depending on the model that we were initially aiming for. For instance, it could happen that κ is a cardinal in V, but $\mathbb{P}$ has added a surjection from α to κ, where α is an ordinal smaller than κ, and hence, we have collapsed κ in $V[G]$. Assuming certain additional properties on the forcing poset $\mathbb{P}$, we can prevent these unwanted situations to happen. Let us recall some of the most important classes of forcing notions which will be relevant to us, and mention their main properties. We will mostly follow [44] and [76], although [1] and [35] are two great sources for proper forcing.

From now on (countable) elementary submodels, not necessarily ground models, will play a central role in the thesis. Hence, in order to avoid confusion, from now on we will assume that V is our ground model.

Definition 1.1.15. A forcing notion $\mathbb{P}$ *preserves a cardinal* κ if for every generic filter $G \subseteq \mathbb{P}$ over V, κ is a cardinal in V if and only if κ is a cardinal in $V[G]$.

Definition 1.1.16. Let κ be an uncountable cardinal and let $\mathbb{P}$ be a forcing notion.

(1) $\mathbb{P}$ is *κ-closed* if every descending sequence

$$p_0 \geq p_1 \geq \cdots \geq p_\alpha \geq \ldots$$

of length $\beta < \kappa$ has a lower bound $q \in \mathbb{P}$, i.e., $q \leq p_\alpha$ for all $\alpha < \beta$.

(2) $\aleph_1$-closed forcings are usually called *σ-closed.*

(3) $\mathbb{P}$ has the *κ-chain condition* (or *κ-c.c.* for short) if every antichain of $\mathbb{P}$ has cardinality $< \kappa$.

(4) The $\aleph_1$-chain condition is commonly referred to as the *countable chain condition* (or *c.c.c.* for short).

(5) $\mathbb{P}$ has the *κ-Knaster condition* if for every $A \subseteq \mathbb{P}$ of size κ, there is $B \subseteq A$ of the same size consisting of pairwise compatible conditions.

It's almost straightforward to check that a forcing notion with the κ-Knaster condition has the κ-chain condition.

Lemma 1.1.17. *If $\mathbb{P}$ is a κ-closed forcing notion, then every cardinal $\lambda \leq \kappa$ is preserved after forcing with $\mathbb{P}$.*

Lemma 1.1.18. *If $\mathbb{P}$ is a forcing notion with the κ-c.c., then every cardinal $\lambda \geq \kappa$ is preserved after forcing with $\mathbb{P}$.*

For lack of a better place, we will include here the statement of the Δ-system lemma, which is an extremely important result in combinatorial set theory, with very important consequences in forcing theory. One of its main applications is in showing that some forcing notions have certain forms of chain conditions.

Lemma 1.1.19 (Δ-system lemma)**.** *Suppose that μ is an infinite cardinal and λ is a regular cardinal greater than μ such that $\forall \alpha < \lambda$, $|\alpha^{<\mu}| < \lambda$. If A is a collection of sets, each of cardinality less than μ, and $|A| = \lambda$, then there is a subcollection $B \subseteq A$ of cardinality λ that forms a Δ-system, i.e., there is r such that for all $a, b \in B$, $a \cap b = r$.*

Recall that if θ is an infinite cardinal, $H(\theta)$ is the set of all sets having transitive closure of cardinality $< \theta$. Moreover, if θ is an uncountable regular cardinal, then $(H(\theta); \in)$ is a model of ZFC minus the Power Set axiom. Therefore, as a structure, $H(\theta)$ is a model of a big enough fragment of ZFC to develop the general theory of forcing. The expressions "let θ be a big enough cardinal" or "let θ be sufficiently large" or "let $H(\theta)$ be big enough" will be constantly used throughout the thesis. These expressions depend on the context, but they mean that θ is a big enough regular cardinal so that $H(\theta)$ contains all the objects that are relevant to us in any given situation. For instance, when dealing with a forcing notion $\mathbb{P}$ of size κ, then $\theta > 2^\kappa$ will usually be big enough for our purposes. Moreover, in many cases, additional predicates $T \subseteq H(\theta)$ are added to the structure $H(\theta)$. For instance, it is common practice to add a well-ordering of $H(\theta)$ as a predicate, the forcing notion $\mathbb{P}$, the forcing relation for $\mathbb{P}$, and other relevant objects. We will normally identify the structure $(H(\theta); \in, T)$ with its universe $H(\theta)$. Elementary submodels of $H(\theta)$ play a very important role in forcing theory, and specially, in the theory of proper forcing. It is understood that if we let M be an elementary submodel of $H(\theta)$ (written "$M \preceq H(\theta)$"), we actually mean that M is an elementary submodel of the structure $H(\theta)$ including all the extra parameters that are relevant.

Definition 1.1.20. Let M be an elementary submodel of a big enough $H(\theta)$ and let $\mathbb{P} \in M$ be a forcing notion. A condition $p \in \mathbb{P}$ is said to be $(M, \mathbb{P})$-*generic* if for every dense subset $D \subseteq \mathbb{P}$ such that $D \in M$, the set $D \cap M$ is predense below p.

It is not too hard to see that if we replace "dense subset" by "maximal antichain", "dense open subset" or "predense subset", we get an equivalent definition.

Remark 1.1.21. If M is an elementary submodel of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$, and p is an $(M, \mathbb{P})$-generic condition, then any condition $q \in \mathbb{P}$ extending p is $(M, \mathbb{P})$-generic.

Lemma 1.1.22. *Let M be an elementary submodel of a big enough $H(\theta)$, let*

$\mathbb{P} \in M$ be a forcing notion and let $p \in \mathbb{P}$. The following are equivalent:

(1) p is $(M, \mathbb{P})$-generic.

(2) For every dense subset $D \subseteq \mathbb{P}$ such that $D \in M$ there is $\dot{q} \in V^{\mathbb{P}}$ such that

$$p \Vdash_{\mathbb{P}} \dot{q} \in M \cap D \cap \dot{G}.$$

(3) For every $\mathbb{P}$-name $\tau \in M$ such that $\Vdash_{\mathbb{P}} \tau \in V$, $p \Vdash_{\mathbb{P}} \tau \in M$.

(4) $p \Vdash_{\mathbb{P}} M[\dot{G}] \cap V = M \cap V$.

(5) For every $\mathbb{P}$-name $\tau \in M$ for an ordinal, $p \Vdash_{\mathbb{P}} \exists \alpha \in M \cap OR(\tau = \alpha)$.

(6) $p \Vdash_{\mathbb{P}} M[\dot{G}] \cap OR = M \cap OR$.

Definition 1.1.23. Let $\mathcal{B}$ be a set of elementary submodels of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$. We say that $\mathbb{P}$ is *$\mathcal{B}$-proper* if for every $M \in \mathcal{B}$ and every $p \in \mathbb{P} \cap M$ there is an $(M, \mathbb{P})$-generic condition $q \leq p$.

If we let $\mathcal{B}$ be the set of all countable elementary submodels of a big enough $H(\theta)$, then $\mathcal{B}$-proper forcings are simply called *proper*.

Example 1.1.24. For every infinite cardinal κ, if $\mathbb{P}$ has the κ^+-c.c., then for every big enough θ and every $M \preceq H(\theta)$ such that $|M| \geq \kappa$, every $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic. Therefore, any c.c.c. forcing is proper.

Recall that for every set X, a subset $C \subseteq \mathcal{P}(X)$ is *club in* $\mathcal{P}(X)$ (or *club in* X) if it is closed and unbounded with respect to inclusion. A subset $S \subseteq \mathcal{P}(X)$ is *stationary in* $\mathcal{P}(X)$ (or *stationary in* X) if it has non-empty intersection with all clubs in $\mathcal{P}(X)$. More generally, $C \subseteq \mathcal{P}(X)$ is *club in* $\mathcal{P}(X)$ if there exists a function $f : [X]^{<\omega} \to X$ such that for every $x \in \mathcal{P}(X)$, $x \in C$ if and only if $f"([x]^{<\omega}) \subseteq x$. Therefore, a subset $S \subseteq \mathcal{P}(X)$ is *stationary in* $\mathcal{P}(X)$ if for every function $f : [X]^{<\omega} \to X$, there is $x \in S$ which is closed under f, i.e., $f"([x]^{<\omega}) \subseteq x$. The two notions of club coincide when replacing $\mathcal{P}(X)$ with $[X]^{\aleph_0}$.

Lemma 1.1.25. *Let $\mathcal{B}$ be a set of elementary submodels of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$. Suppose that $\mathbb{P}$ is $\mathcal{B}$-proper. Let λ be a cardinal and suppose that for each $\alpha < \lambda$, the set $\{M \in \mathcal{B} : \alpha \subseteq M, |M| < \lambda\}$ is stationary in $H(\lambda)$. Then, the forcing $\mathbb{P}$ preserves λ.*

In this thesis we will be mostly interested in the classes $\mathcal{S}$ of countable elementary submodels, which is always stationary, and classes $\mathcal{L}$ of $\aleph_1$-sized elementary submodels that, under the right assumptions, are stationary. Most of the forcing notions that we will define throughout the thesis will be $\mathcal{S}$-proper and $\mathcal{L}$-proper. Thereofore, in light of the lemma above, they will preserve ω_1 and ω_2.

We will also be interested in a certain strengthening of the notion of generic condition, and hence, of proper forcing, which is due to Mitchell ([50]). We include here the definitions and some of their basic properties without proofs, which can be found in Mitchell's paper and in [62].

Definition 1.1.26. Let M be an elementary submodel of a big enough $H(\theta)$ and let $\mathbb{P} \in M$ be a forcing notion. A condition $p \in \mathbb{P}$ is said to be *strongly $(M, \mathbb{P})$-generic* if for every dense subset $D \subseteq \mathbb{P} \cap M$, the set D is predense below p.

Remark 1.1.27. Any strongly $(M, \mathbb{P})$-generic condition is $(M, \mathbb{P})$-generic. To see this, it's enough to note that if $D \in M$ is a dense subset of $\mathbb{P}$, then by elementarity $D \cap M$ is a dense subset of $\mathbb{P} \cap M$.

Similarly to lemma 1.1.22, we have the following characterisation of strongly generic conditions.

Lemma 1.1.28. *Let M be an elementary submodel of a big enough $H(\theta)$, let $\mathbb{P} \in M$ be a forcing notion and let $p \in \mathbb{P}$. The following are equivalent:*

(1) p is strongly $(M, \mathbb{P})$-generic.

(2) For every dense subset $D \subseteq \mathbb{P} \cap M$ there is $\dot{q} \in V^{\mathbb{P}}$ such that

$$p \Vdash_{\mathbb{P}} \dot{q} \in M \cap D \cap \dot{G} = D \cap \dot{G}.$$

Definition 1.1.29. Let $\mathcal{B}$ be a set of elementary submodels of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$. We say that $\mathbb{P}$ is *strongly $\mathcal{B}$-proper* if for every $M \in \mathcal{B}$ and every $p \in \mathbb{P} \cap M$ there is a strongly $(M, \mathbb{P})$-generic condition $q \leq p$.

If we let $\mathcal{B}$ be the set of all countable elementary submodels of a big enough $H(\theta)$, strongly $\mathcal{B}$-proper forcings are simply called *strongly proper*.

It follows from the last remark that any strongly $\mathcal{B}$-proper forcing is $\mathcal{B}$-proper. Therefore, strongly $\mathcal{B}$-proper forcings also ensure the preservation of cardinals (lemma 1.1.25), but they do in fact have stronger properties than proper forcings.

Remark 1.1.30. The statements "p is a strongly $(M, \mathbb{P})$-generic condition" and "$\mathbb{P}$ is a strongly $\{M\}$-proper forcing" are Σ_0-formulas with parameters $p, \mathbb{P}$, and $\mathbb{P} \cap M$. Therefore, these two statements are absolute between transitive models of set theory containing the parameters.

Remark 1.1.31. Let M be an elementary submodel of a big enough $H(\theta)$ containing the forcing notion $\mathbb{P}$. Suppose that $\mathbb{P}$ is strongly $\{M\}$-proper. If M^* is an elementary submodel of $H(\theta^*)$, for $\theta^* > \theta$, such that $M^* \cap H(\theta) = M$, then $\mathbb{P}$ is also strongly $\{M^*\}$-proper.

Lemma 1.1.32 ([62])**.** *Let $\mathcal{B}$ be a set of elementary submodels of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$. Suppose that $\mathbb{P}$ is strongly $\mathcal{B}$-proper. Let λ be a regular cardinal and suppose that the set $\{M \in \mathcal{B} : |M| < \lambda\}$ is stationary in $H(\lambda)$. Then, forcing with $\mathbb{P}$ does not add branches of length λ to trees in V.*

One might wonder why we consider elementary submodels of $H(\theta)$ instead of say V_α, for some ordinal α. First of all, as we have already mentioned, if θ is an uncountable regular cardinal, then $H(\theta)$ is a model of ZFC minus the Power

Set axiom, and a model satisfying Replacement is a better model in the sense of forcing. For instance, if $\mathbb{P} \in H(\theta)$ is a forcing notion, then most of the essential properties of $\mathbb{P}$ are absolute between V and $H(\theta)$:

- $\mathbb{P}$ is a forcing notion in V if and only if $\mathbb{P}$ is a forcing notion in $H(\theta)$.
- For every $G \subseteq \mathbb{P}$, G is a $\mathbb{P}$-generic filter over V if and only if G is a $\mathbb{P}$-generic filter over $H(\theta)$.
- For every $A \subseteq \mathbb{P}$, A is a maximal antichain in V if and only if A is a maximal antichain in $H(\theta)$.

Proposition 1.1.33. *If $2^\theta = \theta^+$, then there is a bijection from $H(\theta^+)$ to θ^+, which is definable in $(H(\theta^+); \in)$.*

By equipping $H(\theta)$ with a well-order $\lhd$, we also have definable Skolem hulls for subsets of $H(\theta)$. The point is that, for every existential formula we can find the $\lhd$-least witness for this formula, and this makes Skolem functions definable. Moreover, well-orders of $H(\theta)$ induce well-orders of their generic extensions.

Proposition 1.1.34. *Let $\mathbb{P} \in H(\theta)$ be a forcing notion and let $\lhd$ be a well-order of $H(\theta)$. If G is a $\mathbb{P}$-generic filter over V, then $\lhd$ induces a well-order on $H(\theta)[G]$.*

Proof. If $x, y \in H(\theta)[G]$, we define the binary relation $\lhd_G$ on $H(\theta)[G]$ by $x \lhd_G y$ if and only if $\tau \lhd \sigma$, where τ and σ are the $\lhd$-least $\mathbb{P}$-names in $H(\theta)$ such that $\tau_G = x$ and $\sigma_G = y$. It's not too hard to see that $\lhd_G$ is in fact a well-order of $H(\theta)[G]$. □

Lastly, we also have the following two well-known facts. For a proof see, for example, [35].

Lemma 1.1.35. *If $\mathbb{P} \in H(\theta)$ is a forcing notion, then $\mathbb{P}$ forces that $H(\theta)^V[\dot{G}] = H(\theta)^{V[\dot{G}]}$. In other words, if τ is a $\mathbb{P}$-name, then $\tau \in H(\theta)$ if and only if $\mathbb{P}$ forces that $\tau \in H(\theta)^{V[\dot{G}]}$.*

Lemma 1.1.36. *Let M be an elementary submodel of a big enough $H(\theta)$ containing a forcing notion $\mathbb{P}$. Then, $\mathbb{P}$ forces that $M[\dot{G}] \preceq H(\theta)[\dot{G}]$.*

We will finish this section by defining the classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-chain condition. These two classes were introduced by Asperó and Mota ([11], [12]), and as it has already been mentioned in the introduction, our class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings can be seen as a high analog of these two classes.

Definition 1.1.37. Given a forcing notion $\mathbb{P}$, we will say that $\mathbb{P}$ is *finitely proper* if and only if for every big enough regular cardinal θ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ of elementary submodels of $H(\theta)$ such that for every finite set $\mathcal{N} \subseteq D$ and every $p \in \mathbb{P} \cap \bigcap \mathcal{N}$ there is $q \in \mathbb{P}$ extending p, which is $(M, \mathbb{P})$-generic for every $M \in \mathcal{N}$.

Definition 1.1.38. Given a forcing notion $\mathbb{P}$, we will say that $\mathbb{P}$ has the $\aleph_{1.5}$-*chain condition* ($\aleph_{1.5}$-*c.c.* for short) if and only if for every big enough regular cardinal θ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ of elementary submodels of $H(\theta)$ such that for every finite set $\mathcal{N} \subseteq D$ and every $p \in \mathbb{P} \cap N$, for some $N \in \mathcal{N}$ such that $N \cap \omega_1 = \min\{M \cap \omega_1 : M \in \mathcal{N}\}$, there is $q \in \mathbb{P}$ extending p, which is $(M, \mathbb{P})$-generic for every $M \in \mathcal{N}$.

It's obvious that finitely proper forcings have the $\aleph_{1.5}$-chain condition, and that c.c.c. forcings are included in the class of finitely proper forcings, in light of example 1.1.24. Moreover, both of Asperó and Mota classes of forcings are subclasses of the class of proper forcings with the $\aleph_2$-chain condition. They are clearly included in the class of proper forcings. Let us see that $\aleph_{1.5}$-c.c. forcings have the $\aleph_2$-c.c.

Lemma 1.1.39 (Asperó-Mota, [12])**.** *If $\mathbb{P}$ has the $\aleph_{1.5}$-c.c., then $\mathbb{P}$ has the $\aleph_2$-c.c.*

Proof. Suppose that A is a maximal antichain of $\mathbb{P}$ of size $\geq \aleph_2$. Let θ and $\mathcal{B}$ be as in definition 1.1.38 and let M_p, for every $p \in A$, a countable elementary submodel of $H(\theta)$ such that $M_p \in \mathcal{B}$ and $p, A \in M_p$. Since $M_p \cap \omega_1 < \omega_1$ for each $p \in A$, there are $A' \subseteq A$ and $\delta < \omega_1$ such that $|A'| = |A|$ and for every $p \in A'$, $M_p \cap \omega_1 = \delta$. Since $|A'| > \aleph_1$ and the models in $\mathcal{B}$ are countable, we can find two different conditions $p, p' \in A'$ such that $p' \notin M_p$. Note that as $p' \in M_{p'}$ and $M_p \cap \omega_1 = M_{p'} \cap \omega_1 = \delta$, we can find an extension q of p' that is $(M_p, \mathbb{P})$-generic, by the $\aleph_{1.5}$-c.c. of $\mathbb{P}$. Therefore, there must be a condition $q^* \in M_p \cap A$ compatible with q, and thus, also compatible with p'. But this is impossible because $p' \neq q^*$, as $p' \notin M_p$ and $q^* \in M_p$, and p' and q^* are both members of the maximal antichain A. □

An interesting feature of these two classes is that, unlike the class of proper forcings, they can be iterated in arbitrarily long length. Therefore, any statement forceable with a forcing from one of these two classes is compatible with arbitrarily large values of the continuum.

1.1.4 Iterated forcing

Let $\mathbb{P}$ be a forcing poset in V and suppose that G is a $\mathbb{P}$-generic filter over V. Let $\mathbb{Q}$ be another forcing notion in the generic extension $V[G]$. Suppose that H is a $\mathbb{Q}$-generic filter over $V[G]$, and force with $\mathbb{Q}$ over $V[G]$ to obtain a further generic extension $V[G][H]$. The idea behind iterated forcing is that we can define a single forcing notion in V such that if we force with it, we can go from the model V to the generic extension $V[G][H]$ in just one step.

Let $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ be a triple of $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \text{“}(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}}) \text{ is a forcing poset”}.$$

We denote $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ by $\dot{\mathbb{Q}}$.

Definition 1.1.40. We define the *two-step iteration of* $\mathbb{P}$ *and* $\dot{\mathbb{Q}}$, denoted $\mathbb{P} * \dot{\mathbb{Q}}$, as the forcing notion $(\mathbb{P} * \dot{\mathbb{Q}}, \leq_{\mathbb{P}*\dot{\mathbb{Q}}}, \mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}})$ with the following properties:

(1) Conditions of $\mathbb{P} * \dot{\mathbb{Q}}$ are pairs $(p, \dot{q})$ such that $p \in \mathbb{P}$, $\dot{q} \in \text{dom}(\dot{\mathbb{Q}})$, and

$p \Vdash \dot{q} \in \dot{\mathbb{Q}}$.

(2) $(p_1, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{Q}}} (p_0, \dot{q}_0)$ if and only if $p_1 \leq_{\mathbb{P}} p_0$ and $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\dot{\mathbb{Q}}} \dot{q}_0$.

(3) $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} = (\mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\mathbb{Q}})$.

Let G be a $\mathbb{P}$-generic filter over V. Let H be a $\dot{\mathbb{Q}}_G$-generic filter over $V[G]$. Then,

$$G * \dot{H} = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G, \dot{q}_G \in H\}$$

is a $\mathbb{P} * \dot{Q}$-generic filter over V. Moreover, $V[G * \dot{H}] = V[G][H]$.

Conversely, let K is a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter over V. Let G be the projection of K on the first coordinate and let $H = \{\dot{q}_G : \exists p(p, \dot{q}) \in K\}$. Then, G is a $\mathbb{P}$-generic filter over V and H is a $\dot{\mathbb{Q}}_G$-generic filter over $V[G]$. Moreover, $V[K] = V[G][H]$.

Lemma 1.1.41. *Let κ be an uncountable regular cardinal. Let $\mathbb{P}$ be a forcing notion and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing notion. If $\mathbb{P}$ is κ-c.c. and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-c.c.", then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c.*

We can consider forcing iterations of infinite length. We won't include the details here, but let us briefly mention that a *λ-stage forcing iteration* is a pair of sequences $\langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle$ and $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$, where $\mathbb{P}_{\alpha+1}$ is essentially the two step iteration $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Finite support iterations, countable support iterations, and other forms of iterations differ on their definition at limit stages. We call $\mathbb{P}_\alpha$ the *α-th stage of the iteration* and $\dot{\mathbb{Q}}_\alpha$ the *α-th iterand*. When we say that a certain class of forcing notions is iterable and the like, what we actually mean is that the iterands belong to that class of forcings and that every stage of the iteration preserves cardinals. Normally, the preservation of cardinals is achieved thanks to the specific properties of the class of forcing notions that we are iterating.

Example 1.1.42. (1) Finite support iterations of c.c.c. forcings are themselves c.c.c., and hence, every stage of the iteration preserves all cardinals.

(2) Countable support iterations of proper forcings are themselves proper, and hence, they preserve ω_1 at every stage of the iteration. Moreover, starting from a model of CH, countable support iterations of length $\leq \aleph_2$ of proper forcings of size $\aleph_1$ have the $\aleph_2$-c.c. Therefore, they preserve all cardinals.

One key property of forcing iterations, which we will crucially use in chapter 4, is that for every $\alpha < \beta \leq \lambda$, typically $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$. Therefore, if G_β is a $\mathbb{P}_\beta$-generic filter over V, then $\mathbb{P}_\alpha \cap G_\beta$ is a $\mathbb{P}_\alpha$-generic filter over V.

1.2 Forcing axioms

Let $\mathcal{K}$ be a class of forcing notions and let κ be an uncountable cardinal. Recall from the introduction that the *forcing axiom for the class* $\mathcal{K}$ *and* κ*-many dense sets* is the following statement:

$FA_\kappa(\mathcal{K})$: For every $\mathbb{P} \in \mathcal{K}$ and every collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| = \kappa$, there is a filter $G \subseteq \mathbb{P}$ such that for every $D \in \mathcal{D}$, $G \cap D \neq \varnothing$.

In order to obtain a model of a forcing axiom, the strategy is to force generic filters for all the posets and all the collections of dense subsets of these posets, via a forcing iteration. Start with a model V. Force with a poset from the class $\mathcal{K}$ in V to produce a generic filter for a collection of dense subsets that are in V. We want to add a generic filter for every partial order $\mathbb{P} \in \mathcal{K}$ and every collection of prescribed size κ of dense subsets. Therefore, we need to force with all the forcing posets from the class $\mathcal{K}$. Namely, we need to define a forcing iteration in V that goes over all the posets from the class $\mathcal{K}$. But there are two issues that need to be addressed. First of all, when forcing with a poset from $\mathcal{K}$ we may create new forcing notions. Hence, we not only have to take care of the posets from $\mathcal{K}$ in V, but also those posets that may appear along the iteration. The so-called *bookkeeping function* is a device that keeps track of all these new posets. Secondly, the class $\mathcal{K}$ of forcing notions might be a proper class, so we need to come up with a way to capture all of these posets in an iteration of length an infinite cardinal λ. There are different ways to achieve this. In some cases a Löwenheim-Skolem argument allows us to restrict the proper class $\mathcal{K}$ to a subclass, which is in fact a set. In other cases stronger assumptions are needed, even beyond the ZFC axioms, such as some form of $\diamondsuit$ or large cardinals. For instance, in order to force PFA, or the forcing axiom for our class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings, we need to assume a supercompact cardinal. We will give more details about this in the next section.

For instance, using Asperó and Mota's finite support iterations with symmetric systems as side conditions we can show, from very mild cardinal arithmetic assumptions, the consistency of the forcing axioms for the classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-c.c. together with arbitrarily large values of the continuum. Let us denote by $\text{PFA}^{fin}(\aleph_1)$ the forcing axiom for the class of finitely proper forcings of size $\aleph_1$ and $\aleph_1$-many dense sets.

Theorem 1.2.1 (Asperó-Mota, [11]). *(CH) If κ is a cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$, then there is a proper forcing notion $\mathbb{P}$ with the $\aleph_2$-chain condition such that the following statements hold in the generic extension by $\mathbb{P}$:*

(1) $2^{\aleph_0} = \kappa$.

(2) $PFA^{fin}(\aleph_1)$.

Theorem 1.2.2 (Asperó-Mota, [12]). *(CH) If $\kappa \geq \omega_2$ is a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa : \alpha \geq \omega_1\})$[2] holds, then there is a proper forcing notion $\mathbb{P}$ of size κ with the $\aleph_2$-chain condition such that the following statements hold in the generic extension by $\mathbb{P}$:*

(1) $2^{\aleph_0} = \kappa$.

(2) $FA_{<2^{\aleph_0}}(\aleph_{1.5}\text{-}c.c.)$.

[2]If $S \subseteq \kappa$ is stationary, $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that for every $A \subseteq \kappa$, the set $\{\alpha \in S : A_\alpha = A \cap \alpha\}$ is stationary.

1.3 Large cardinals

Let us introduce the notion of supercompact cardinal, which will be relevant in the proof of the consistency of the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings in chapter 4. The standard reference for large cardinal notions is [40].

Definition 1.3.1. M is an *inner model* if and only if $M \subseteq V$ is a transitive $\in$-model of ZF with $OR \subseteq M$.

Definition 1.3.2. Let $\kappa \leq \lambda$ be two cardinals. The cardinal κ is called λ-*supercompact* if and only if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$[3], $\lambda < j(\kappa)$ and ${}^{\lambda}M \subseteq M$. We call such an embedding a (κ, λ)-*supercompact embedding.*

A cardinal κ is *supercompact* if it is λ-supercompact for every $\lambda \geq \kappa$.

Remark 1.3.3. Let $j : V \to M$ be a (κ, λ)-supercompact embedding. Then, the following hold:

- $j \restriction V_\kappa = id \restriction V_\kappa$.
- $H(\lambda^+) \subseteq M$.

Theorem 1.3.4 (Laver, [47])**.** *Let κ be a supercompact cardinal. There exists a function $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \geq \kappa$ such that $\lambda \geq |trcl(x)|$ there exists a (κ, λ)-supercompact embedding $j : V \to M$ such that $j(f)(\kappa) = x$.*

The function f from the statement of the last theorem is usually called a *Laver function*, and it is crucially used, for instance, in the proof of the consistency of PFA and other strong forcing axioms.

[3] κ being the *critical point* of the elementary embedding j, written $\text{crit}(j) = \kappa$, means that κ is the least ordinal such that $j(\kappa) > \kappa$.

1.4 Elementary submodels

It has already been stressed in the previous sections how important elementary submodels are in forcing theory, and specially in forcing with side conditions. In this section we will recall some basic facts about elementary submodels and fix the notation that will be used in subsequent chapters of the thesis. As it has already been stated, the standard references for set-theoretic notions are [44] and [38], and most of the results of this section can be found there. Additional information about elementary submodels and their applications in set theory can be found in [26] and [39].

Given a model Q, we will denote $Q \cap \omega_1$ by δ_Q and $\sup(Q \cap \omega_2)$ by ε_Q, and we will call δ_Q the *ω_1-height of Q* and ε_Q the *ω_2-height of Q*.

Given two $\in$-isomorphic models of the Axiom of Extensionality Q_0 and Q_1, we write Ψ_{Q_0,Q_1} to denote the unique isomorphism $\Psi : (Q_0; \in) \to (Q_1; \in)$.

Let $\kappa > \omega_2$ be a cardinal and let $T \subseteq H(\kappa)$. Recall that we will usually refer to the structure $(H(\kappa); \in, T)$ simply by $H(\kappa)$. Let $\mathcal{S}$ be the collection of countable $M \preceq (H(\kappa); \in, T)$. We will tend to use the capital letter M to refer to models in $\mathcal{S}$, which we will call *countable elementary* or *small models.*

Definition 1.4.1. We will call a collection $\mathcal{L}$ of $\aleph_1$-sized elementary submodels $N \preceq (H(\kappa); \in, T)$ *appropriate for $\mathcal{S}$*[4] if for every $N \in \mathcal{L}$ and every $M \in \mathcal{S}$ such that $N \in M$, then $N \cap M \in N \cap \mathcal{S}$.

We will tend to use $\mathcal{L}$ to denote arbitrary collections of $\aleph_1$-sized appropriate models for $\mathcal{S}$, and the capital letter N to refer to models in $\mathcal{L}$, which we will call *uncountable* or *large models.* We will use $\mathcal{S}$ and $\mathcal{L}$ throughout the thesis, independently of the parameters κ and T. At the beginning of each subsequent chapter, κ and T will be properly specified, so that there is no possible ambiguity. To refer to models of arbitrary size we will use Q, as well as other capital letters

[4]This naming comes from [62], although it has a sligthly different meaning here.

further down in the alphabet. If Q is an elementary submodel of $H(\kappa)$, we will usually refer to the structure $(Q; \in, T \cap Q)$ by $(Q; \in, T)$. Moreover, we might indistinctly use Q to refer to the structure $(Q; \in, T)$ or its universe. It will be clear from the context which one we are referring to.

The following are some basic facts about elementary submodels which we will constantly use throughout the thesis, sometimes without mention.

Theorem 1.4.2 (Tarski-Vaught's Test)**.** *Let M be a structure and let $A \subseteq M$. Then, A is the domain of a structure $N \preceq M$ iff for every formula $\varphi(y, \bar{x})$ and every tuple $\bar{a}$ of A such that $M \models \exists y \varphi(y, \bar{a})$, there exists an element b in A such that $M \models \varphi(b, \bar{a})$.*

Proposition 1.4.3. *If $Q_0, Q_1 \preceq H(\kappa)$ are such that $Q_0 \subseteq Q_1$, then $Q_0 \preceq Q_1$.*

Proposition 1.4.4. *Let $Q \preceq H(\kappa)$, $\mu < \kappa$, and $\mu \subseteq Q$. Then for every $A \in Q$, if $H(\kappa) \models |A| = \mu$, then $A \subseteq Q$.*

Proposition 1.4.5. *Let $Q \preceq H(\kappa)$. If A is definable over $H(\kappa)$ with parameters in Q, then $A \in Q$.*

Proposition 1.4.6. *Let $Q \preceq H(\kappa)$ such that $|Q| = \mu < \mu^+ < \kappa$. Then, $Q \cap \mu^+$ is a limit ordinal.*

Proposition 1.4.7. *Let $Q_0, Q_1 \preceq H(\kappa)$ such that $|Q_0| = |Q_1| = \mu < \mu^+ < \kappa$ and let Ψ be an isomorphism between $(Q_0; \in, T)$ and $(Q_1; \in, T)$. Then, Ψ is the identity on $Q_0 \cap \mu^+$. In particular, $Q_0 \cap \mu^+ = Q_1 \cap \mu^+$.*

Proposition 1.4.8. *Let $\kappa < \theta$, $Q \preceq H(\theta)$ and $\kappa \in Q$. Then $Q \cap H(\kappa) \preceq H(\kappa)$.*

Proposition 1.4.9. *Let Q, Q' and P be elementary substructures of $H(\kappa)$. Suppose that $P \in Q$, $P \subseteq Q$, and that $\Psi : (Q; \in, T) \to (Q'; \in, T)$ is an isomorphism. Then $\Psi(P)$ is an elementary substructure of $(H(\kappa); \in, T)$.*

Proof. It's easy to see that $\Psi \restriction P$ is an isomorphism between $(P; \in, T)$ and $(\Psi(P); \in, T)$. Assume now that $\varphi(y, \bar{x})$ is a first-order formula in the language of set theory and let $\Psi(\bar{a})$ be a tuple of elements of $\Psi(P)$ such that $H(\kappa)$ satisfies the formula $\exists y\varphi\big(y, \Psi(\bar{a})\big)$. Since $Q' \preceq H(\kappa)$ and $\Psi(P) \subseteq Q'$, Q' also satisfies the formula $\exists y\varphi\big(y, \Psi(\bar{a})\big)$, and since Ψ is an isomorphism, Q satisfies the formula $\exists y\varphi(y, \bar{a})$. Hence, again by elementarity, $H(\kappa) \models \exists y\varphi(y, \bar{a})$, and since $\bar{a}$ is a tuple of elements in P, by the Tarski-Vaught test there is $b \in P$ such that $H(\kappa) \models \varphi(b, \bar{a})$. Now it's easy to see with a similar argument, using elementarity of the models Q and Q' and the isomorphism Ψ, that $H(\kappa) \models \varphi\big(\Psi(b), \Psi(\bar{a})\big)$. Therefore, by the Tarski-Vaught test we can conclude that $\Psi(P)$ is an elementary substructure of $(H(\kappa); \in, T)$. □

Lemma 1.4.10. *$\mathcal{S}$ contains a club in $[H(\kappa)]^\omega$.*

Lemma 1.4.11. *Let Q be an elementary submodel of $H(\kappa)$. If α is any limit ordinal such that $\alpha \in Q \cap \kappa$, then $\sup(Q \cap \alpha) = \alpha$ if and only if $\mathrm{cf}(\alpha) \leq |Q|$.*

Proof. The left-to-right implication is clear. Assume that $\mathrm{cf}(\alpha) \leq |Q|$. Then, since $\alpha \in Q$, there must be a function $f \in Q$ on $\mathrm{cf}(\alpha)$ whose range is unbounded in α. But since $|f| = |\mathrm{cf}(\alpha)| \leq |Q|$, by proposition 1.4.4 we have that $f \subseteq Q$. □

Proposition 1.4.12. *Let $Q_0, Q_1 \preceq H(\kappa)$ such that $Q_0 \in Q_1$. If $\alpha \notin Q_1$, then $\sup(Q_0 \cap \alpha) < \sup(Q_1 \cap \alpha)$.*

Proposition 1.4.13. *Let $M \in \mathcal{S}$ and $N \in \mathcal{L} \cap M$. Then, for every $Q \in \mathcal{S} \cup \mathcal{L}$ such that $\varepsilon_{N\cap M} \leq \varepsilon_Q < \varepsilon_N$, $Q \notin M$.*

Proof. Suppose, towards a contradiction, that there is some $Q \in \mathcal{S} \cup \mathcal{L}$ such that $\varepsilon_{N\cap M} \leq \varepsilon_Q < \varepsilon_N$ and $Q \in M$. First, note that since $N \in \mathcal{L}$, then $\varepsilon_N = N \cap \omega_2$ is an ordinal in ω_2. Therefore, $\varepsilon_Q = \sup(Q \cap \omega_2) \in N \cap \omega_2$, and hence, $\varepsilon_Q \in M \cap (N \cap \omega_2)$. So we can conclude that $\varepsilon_Q < \varepsilon_{N\cap M}$, which contradicts our assumption. □

Definition 1.4.14. Let $\mu < \kappa$ be an infinite cardinal. An elementary submodel $Q \preceq H(\kappa)$ is said to be *μ-closed* if ${}^{\mu}Q \subseteq Q$[5].

[5]$\aleph_0$-closed models are also called *countably-closed*.

We will denote by $\mathcal{L}^{\omega\text{-c}}$ the collection of all $\aleph_1$-sized countably-closed elementary submodels of $(H(\kappa); \in, T)$.

Lemma 1.4.15. *Let ν be a regular cardinal and let μ be another cardinal such that $\mu^{<\nu} = \mu < \kappa$. Then, for every $A \in [H(\kappa)]^{\leq\mu}$ there is some $Q \preceq H(\kappa)$ such that $A \subseteq Q$, $|Q| = \mu$ and ${}^{<\nu}Q \subseteq Q$.*

Corollary 1.4.16. *If* CH *holds, then $\mathcal{L}^{\omega\text{-c}}$ is stationary in $[H(\kappa)]^{\omega_1}$.*

Proposition 1.4.17. *The class $\mathcal{L}^{\omega\text{-c}}$ is appropriate for $\mathcal{S}$.*

Proof. Let $N \in \mathcal{L}^{\omega\text{-c}}$ and $M \in \mathcal{S}$ such that $N \in M$. It's clear that $N \cap M$ is a member of N, because N is countably-closed. Now we need to check that $N \cap M \in \mathcal{S}$. We use the Tarski-Vaught test. Let $\varphi(y, \bar{x})$ be a first-order formula in the language of set theory. Let $\bar{a}$ be a tuple of elements of $N \cap M$ such that $H(\kappa) \models \exists y \varphi(y, \bar{a})$. Since $N \preceq H(\kappa)$, there is $b \in N$ such that $H(\kappa) \models \varphi(b, \bar{a})$, and as N is an element of $H(\kappa)$, we can rewrite this as $H(\kappa) \models \exists y(y \in N \wedge \varphi(y, \bar{a}))$. Now, as $N \in M$, by elementarity of M there must be some $d \in M$ such that $H(\kappa) \models d \in N \wedge \varphi(d, \bar{a})$. Therefore, we have found $d \in N \cap M$ such that $H(\kappa) \models \varphi(d, \bar{a})$, and thus, we can conclude that $N \cap M \preceq H(\kappa)$. □

Definition 1.4.18. An $\aleph_1$-sized elementary submodel $N \preceq H(\kappa)$ is said to be *internally club* if N is the union of a continuous $\in$-increasing sequence of small models $\langle M_\xi : \xi < \omega_1 \rangle$. In this case we call $\langle M_\xi : \xi < \omega_1 \rangle$ an *IC-sequence* for N.

We will denote by $\mathcal{L}^{IC}$ the collection of all internally club models.

Lemma 1.4.19. *The class $\mathcal{L}^{IC}$ is stationary in $[H(\kappa)]^{\omega_1}$.*

Proof. Fix a function $f : [H(\kappa)]^{<\omega} \to H(\kappa)$. Using the clubness of $\mathcal{S}$ build a continuous $\in$-increasing sequence of small models $\langle M_\xi : \xi < \omega_1 \rangle$ such that $f \in M_0$. Then, $N := \bigcup_{\xi<\omega_1} M_\xi$ is an internally club model closed under f. □

Proposition 1.4.20. *The class $\mathcal{L}^{IC}$ is appropriate for $\mathcal{S}$.*

Proof. Let $N \in \mathcal{L}^{IC}$ and let $M \in \mathcal{S}$ such that $N \in M$. By elementarity there exists an IC-sequence $\langle M_\xi : \xi < \omega_1 \rangle$ for N in M. It's not too hard to see that $N \cap M = M_{\delta_M}$. □

2

Symmetric systems of elementary submodels

Suppose that we want to force an uncountable set with some specific properties. The natural thing would be to force with the poset of all finite approximations, but if we are not careful enough, this poset might collapse cardinals. The method of forcing with side conditions, invented by Todorčević ([83]), consists on adding finite systems of models to the conditions of a forcing notion to ensure the preservation of cardinals.

Typically, a condition of a forcing with side conditions $\mathbb{P}$ is a pair (x, Δ) where:

- x, the *working part*, is the finite approximation of the object that we want to add generically.
- Δ, the *side condition*, is a finite system of elementary submodels.
- Normally, x and Δ are related in such a way that we can prove that x is $(M, \mathbb{P})$-generic for every $M \in \Delta$.

We can also consider the forcing with side conditions whose working part is empty. Namely, the poset which consists only of the side condition. This poset is interesting in its own right, and we refer to it as the *pure side condition forcing*.

Suppose that $\mathcal{S}_0$ is a set of countable elementary submodels of a big enough $H(\theta)$. Todorčević's original pure side condition forcing, sometimes called *Todorčević's* *collapse*, is defined as follows:

Definition 2.0.1. $\mathbb{C}(\mathcal{S}_0)$ is the poset of finite $\in$-chains of models from $\mathcal{S}_0$, ordered by reverse inclusion.

We will usually refer to the conditions of $\mathbb{C}(\mathcal{S}_0)$ as $\mathcal{S}_0$-*chains*, and the reason to add them as side conditions is to ensure $\mathcal{S}_0$-properness.

There are different lineages of side conditions, which vary on the structure of the system of models. In this chapter we will focus on *symmetric systems* (also called *matrices of elementary submodels*). These are finite sets of models, not necessarily linearly ordered by $\in$, but exhibiting some form of symmetry. Typically, models of the same rank are isomorphic and the system is closed under these isomorphisms. Symmetric systems were also invented by Todorčević ([81]), and later rediscovered by Asperó and Mota ([11], [12]). Finite symmetric systems of countable elementary submodels, on top of ensuring properness, also grant the $\aleph_2$-chain condition, and the pure side condition forcing preserves CH. These side conditions have been used in many different contexts. Especially, in situations in which we need to preserve all cardinals. Let us list some of the most important objects known to be forceable using symmetric systems of models of one type:

- Clubs of ω_1 ([11], [12], [8]).
- Kurepa trees and almost Souslin Kurepa trees ([45]).
- The $\diamondsuit$ principle ([45]).
- Certain colourings of $[\omega_2]^2$, which give negative polychromatic partition relations ([2], [3]).
- The principles $\square_{\omega_1}$ and $\square_{\omega_2}$ ([51], [55]).
- Kurepa trees of height ω_1 ([51]).

- $(\omega_1, 1)$-morasses, simplified $(\omega_1, 1)$-morasses, (ω_1, λ)-semimorasses for $\lambda \geq \omega_2$, and simplified $(\omega_2, 1)$-morasses with linear limits ([51], [53], [56],[54], [55]).
- ω_2-Souslin trees ([51], [54]).
- Countable fast functions ([52]).
- Strong almost disjoint families of functions from ω_1 to ω of arbitrary size ([91]).

The chapter is organised as follows. We will start by introducing Neeman's two-type version of Todorčević's collapse and we will state its main properties. As we will see, the conditions in Neeman's pure side condition forcing are models of two different size, which ensure the preservation of two cardinals. Then, we will define a variant of this forcing, which includes a third type of "models". The last three sections will focus on symmetric systems of elementary submodels. First, we will give an overview of symmetric systems of countable elementary submodels. Then, building on the ideas of the previous sections, we will introduce the pure side condition forcing with symmetric systems of models of two-types and we will prove their main properties. We will finish this chapter by also considering a variant of the two-type symmetric systems, which include "models" of a third non-elementary type.

Let $\mathbb{M}$ denote either of the two forcing notions $\mathbb{M}(\mathcal{S}, \mathcal{L})$ or $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$, which will be properly defined in sections 2.3 and 2.4, respectively. The main properties of these forcing notions, which constitute the main results of this chapter, can be summarised as follows:

(1) $\mathbb{M}$ is strongly $\mathcal{S}$-proper.

(2) $\mathbb{M}$ is strongly $\mathcal{L}$-proper.

(3) If $2^{\aleph_1} = \aleph_2$ holds, then $\mathbb{M}$ has the $\aleph_3$-Knaster condition.

(4) $\mathbb{M}$ preserves $2^{\aleph_1} = \aleph_2$.

We let $\kappa > \omega_2$ be a cardinal and we fix a predicate $T \subseteq H(\kappa)$. Throughout this chapter, $\mathcal{S}$ will be the set of countable $M \preceq (H(\kappa); \in, T)$ and $\mathcal{L}$ will be a collection of $\aleph_1$-sized elementary submodels $N \preceq (H(\kappa); \in, T)$ appropriate for $\mathcal{S}$. Furthermore, we will assume that $\mathcal{L}$ is stationary in $[H(\kappa)]^{\aleph_1}$.

2.1 Chains of elementary submodels of two types

This section is devoted to introduce Neeman's side conditions with models of two types and review their main properties. These were introduced in [62][1] (see also [63]), and all the results from this section can be found in said paper. We decided to include them here because they inspire many of the ideas from sections 2.3 and 2.4. The variant of Neeman's two-type side conditions which include models of non-elementary type were presented in [64] and [65]. Since this work hasn't been yet published, we have decided to include the proof of some of the results.

2.1.1 Neeman's two-type side conditions

Definition 2.1.1. Let $\mathcal{C}$ be a finite set $\{Q_i : i \leq n\}$ of members of $H(\kappa)$. We say that $\mathcal{C}$ is an $(\mathcal{S}, \mathcal{L})$-*chain* if and only if the following holds:

(1) Every Q_i is a member of $\mathcal{S} \cup \mathcal{L}$.

(2) $\mathcal{C}$ is $\in$-increasing. That is, $Q_i \in Q_{i+1}$ for each $i < n$.

(3) $\mathcal{C}$ is closed under intersections in the following sense. If $N \in \mathcal{C} \cap \mathcal{L}$ and $M \in \mathcal{C} \cap \mathcal{S}$ are such that $N \in M$, then $N \cap M \in \mathcal{C}$.

We will naturally identify a given nonempty $(\mathcal{S}, \mathcal{L})$-chain $\mathcal{C}$ with the unique enumeration $\langle Q_i : i \leq n \rangle$, given by the strictly increasing sequence $\langle \varepsilon_{Q_i} : i \leq n \rangle$ of ω_2-heights of the models in $\mathcal{C}$. If $i < j \leq n$, we say that Q_i *occurs before* Q_j in $\mathcal{C}$ and we will denote it by $Q_i < Q_j$. Moreover, (Q_i, Q_j) is the interval of

[1]With a slightly different notation.

models lying strictly between Q_i and Q_j in $\mathcal{C}$. We define half-open and closed intervals in the natural way. We let $\mathbb{C}(\mathcal{S},\mathcal{L})$ be the forcing notion whose conditions are $(\mathcal{S},\mathcal{L})$-chains and the order is reverse inclusion.

Lemma 2.1.2. *Let $Q \in \mathcal{S} \cup \mathcal{L}$ and let $\mathcal{C}$ be an $(\mathcal{S},\mathcal{L})$-chain such that $\mathcal{C} \in Q$. Then, there is an $(\mathcal{S},\mathcal{L})$-chain $\mathcal{C}^*$ such that $\mathcal{C} \cup \{Q\} \subseteq \mathcal{C}^*$.*

Proposition 2.1.3. *Let $\mathcal{C}$ be an $(\mathcal{S},\mathcal{L})$-chain. Let $M \in \mathcal{C}$ be a small model, and let $N \in \mathcal{C}$ be a large model such that $N \in M$. If $Q \in M \cap \mathcal{C}$, then $Q \notin [N \cap M, N)$.*

If M and N are as in the statement of the last proposition, we call the interval $[N \cap M, N)$ a *residue gap*, or simply a *gap*, of $\mathcal{C}$ in M.

Proposition 2.1.4. *Let $\mathcal{C}$ be an $(\mathcal{S},\mathcal{L})$-chain and let $Q \in \mathcal{C}$. Then, the following hold:*

(1) If $Q \in \mathcal{L}$, then $\mathcal{C} \cap Q$ consists of all models of $\mathcal{C}$ that occur before Q.

(2) If $Q \in \mathcal{S}$, then $\mathcal{C} \cap Q$ consists of all models of $\mathcal{C}$ that occur before Q and do not belong to residue gaps of $\mathcal{C}$ in Q.

Lemma 2.1.5. *Let $\mathcal{C}$ be an $(\mathcal{S},\mathcal{L})$-chain and let $Q \in \mathcal{C}$. Then, $\mathcal{C} \cap Q$ is an $(\mathcal{S},\mathcal{L})$-chain such that $\mathcal{C} \cap Q \in Q$.*

Lemma 2.1.6. *Let $\mathcal{C}$ be an $(\mathcal{S},\mathcal{L})$-chain and let $Q \in \mathcal{C}$. Let $\mathcal{D}$ be another $(\mathcal{S},\mathcal{L})$-chain such that $\mathcal{C} \cap Q \subseteq \mathcal{D} \subseteq Q$. Then, there is an $(\mathcal{S},\mathcal{L})$-chain $\mathcal{R}$ such that $\mathcal{C} \cup \mathcal{D} \subseteq \mathcal{R}$.*

Corollary 2.1.7. *$\mathbb{C}(\mathcal{S},\mathcal{L})$ is strongly $\mathcal{S}$-proper and strongly $\mathcal{L}$-proper.*

2.1.2 Neeman's two-type side conditions with non-elementary submodels

If $\overline{M}$ is a countable set of elements of $\mathcal{L}$ (possibly finite), we will denote by $\varepsilon_{\overline{M}}$ the ordinal $\bigcup\{\varepsilon_N + 1 : N \in \overline{M}\}$, and we will call it the *ω_2-height of $\overline{M}$*. Note that if $\overline{M}$ is infinite, then $\varepsilon_{\overline{M}} = \sup_{N \in \overline{M}} \varepsilon_N$, and hence, $\mathrm{cf}(\varepsilon_{\mathcal{N}}) = \omega$. The reason to define $\varepsilon_{\overline{M}}$ this way is to prevent $\varepsilon_{\overline{M}}$ from being equal to the ω_2-height of one of its models of maximal ω_2-height in case $\overline{M}$ is finite. This will be relevant when dealing with $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains and $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems in section 2.4.

Definition 2.1.8. A countable collection $\overline{M}$ of elementary submodels of $H(\kappa)$ is called an *$\mathcal{L}$-tower* if it is linearly ordered by $\in$ and every $N \in \overline{M}$ is a member of $\mathcal{L}$.

We will also refer to $\mathcal{L}$-towers as *tower-type models* or *models of non-elementary type*, even though they are not really models. We let $\mathcal{T}$ denote the collection of all $\mathcal{L}$-towers[2].

Definition 2.1.9. Let $\mathcal{C}$ be a finite set $\{Q_i : i \leq n\}$ of members of $H(\kappa)$. We say that $\mathcal{C}$ is an *$(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain* if and only if the following holds:

(A) Every Q_i is a member of $\mathcal{S} \cup \mathcal{L} \cup \mathcal{T}$.

(B) $\mathcal{C}$ is $\in$-increasing. That is, $Q_i \in Q_{i+1}$ for each $i < n$.

(C) $\mathcal{C}$ is closed under intersections in the following sense. If $N \in \mathcal{C} \cap \mathcal{L}$ and $M \in \mathcal{C} \cap (\mathcal{S} \cup \mathcal{T})$ such that $N \in M$, then the following holds:

  (C.a) If $M \in \mathcal{S}$, then $N \cap M \in \mathcal{C}$. Hence, $N \cap M$ equals some Q_i appearing before N and M in $\mathcal{C}$.

  (C.b) If $M \in \mathcal{T}$ and $N \cap M \neq \varnothing$, then there is another $\mathcal{L}$-tower $\overline{M} \in \mathcal{C}$ such that $N \cap M \subseteq \overline{M} \in N$.

As with $(\mathcal{S}, \mathcal{L})$-chains, we will identify a nonempty $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain $\mathcal{C}$ with the unique enumeration given by the strictly increasing sequence of ω_2-heights of its models. A model $Q_i \in \mathcal{C}$ is said to occur before $Q_j \in \mathcal{C}$ if $i < j \leq n$, and intervals of models are defined in the same way as with $(\mathcal{S}, \mathcal{L})$-chains. We let $\mathbb{C}(\mathcal{S}, \mathcal{L}, \mathcal{T})$ be

[2] Not to be confused with the $\mathcal{T}$ used in Neeman's papers [62] and [63], which denotes a collection of transitive elementary submodels of some $H(\theta)$.

the forcing notion whose conditions are $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains and the order is reverse inclusion.

Note that it follows from the definition of $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains that non-elementary models are always preceded by a large model.

All of the results from last section translate word by word to the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains, and their proofs are almost exactly the same as the ones given in [62][3]. We have decided to only include the analog of lemma 2.1.6 (see lemma 2.1.11) to familiarise ourselves with arguments involving tower-type nodes.

Proposition 2.1.10. *Let $\mathcal{C}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain and let $M \in \mathcal{C}$ be a small model. If $\overline{M}$ is a tower-type model in $\mathcal{C} \cap M$, then its immediate predecessor is a large model N that belongs to $\mathcal{C} \cap M$.*

Lemma 2.1.11. *Let $\mathcal{C}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain and let $M \in \mathcal{C} \cap \mathcal{S}$. Let $\mathcal{D}$ be another $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain such that $\mathcal{C} \cap M \subseteq \mathcal{D} \subseteq M$. Then, there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain $\mathcal{R}$ such that $\mathcal{C} \cup \mathcal{D} \subseteq \mathcal{R}$.*

Proof. For each uncountable model W in $\mathcal{D} \setminus \mathcal{C}$ we define E_W and F_W exactly as in the proof of lemma 2.21 of [62]. Let N be the least uncountable model in $\mathcal{C} \cap M$ above W, if it exists, and let N^* be the least uncountable model above $N \cap M$. Then, we let E_W be the set of all models in the interval $[N \cap M, N^*)$. If there is no uncountable model in $\mathcal{C} \cap M$ above W, we let N^* be the least uncountable model above M. Then, E_W is defined as the set of all models in the interval $[M, N^*)$. Let F_W be the set of all models of the form $W \cap Q$, where $Q \in E_W$. Note that it makes sense to define E_W and F_W exactly as in [62] because all the models in intervals $[N \cap M, N^*)$ and $[M, N^*)$ as above must be elements of $\mathcal{C}$ and of countable elementary type. Indeed, this follows from the fact that, in $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains, tower-type nodes need to be preceded by uncountable-type nodes. Let $\mathcal{R}$ be the result of adding all the models in F_W to $\mathcal{C} \cup \mathcal{D}$, placing them in order, right before W, for each uncountable model $W \in \mathcal{D} \setminus \mathcal{C}$. We claim that $\mathcal{R}$ is a condition.

[3] Modulo some easy arguments about non-elementary models

To see that $\mathcal{R}$ is $\in$-increasing, we can argue as in claims 2.22 and 2.25 from [62]. First we need to see that the intervals F_W are increasing. Then, we show that $\mathcal{D} \cup \mathcal{C}$ is $\in$-increasing. Lastly, we add the intervals F_W and check that the models at the borders of F_W are $\in$-increasing.

Clause (C.a) from definition 2.1.9 for $\mathcal{R}$ is proven exactly as in [62] (claim 2.26).

Clause (C.b) is the only one that needs an argument. Let $N \in \mathcal{R}$ be an uncountable model. Note that $N \in \mathcal{C} \cup \mathcal{D}$ because all the models added to $\mathcal{C} \cup \mathcal{D}$ to form $\mathcal{R}$ are countable and elementary. Let $\overline{M} \in \mathcal{R}$ be a model of tower type such that $N \in \overline{M}$ and $N \cap \overline{M} \neq \varnothing$, and note that $\overline{M} \in \mathcal{C} \cup \mathcal{D}$ because of the same reason as above. We need to find a tower model $\overline{M}^* \in \mathcal{R}$ such that $N \cap \overline{M} \subseteq \overline{M}^* \in N$. Assume first that $N \in \mathcal{C}$. If N occurs above M, then both N and $\overline{M}$ must be members of $\mathcal{C}$ and the result is clear, because $\mathcal{C}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain. Hence, assume that N occurs below M. We may assume that $\overline{M} \in \mathcal{D} \setminus \mathcal{C}$. Note that in this case $\overline{M} \in M$, so $N \in \overline{M} \subseteq M$. Therefore, both N and $\overline{M}$ belong to $\mathcal{D}$, and we get the result immediately from the fact that $\mathcal{D}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain. Assume now that $N \in \mathcal{D} \setminus \mathcal{C}$. If $\overline{M} \in \mathcal{D}$ there is nothing to check. Hence, we may assume that $\overline{M} \in \mathcal{C} \setminus M$. Note that $\overline{M}$ must be in a gap $[N' \cap M, N')$, where $N' \in \mathcal{C} \cap M$ is an uncountable model occurring above N. Hence, since we don't add new models in this gap of $\mathcal{C}$ when forming $\mathcal{R}$, $\overline{M}$ needs to be immediately preceded by an uncountable model of $\mathcal{C}$, which has to belong to the same gap. Let $N^+ \in \mathcal{C} \cap \overline{M}$ be such an uncountable model, which also occurs above N. Then, as $N \in N^+ \cap \overline{M}$ and $N^+, \overline{M} \in \mathcal{C}$, there must be some tower-type model $\overline{M}^+ \in \mathcal{C}$ such that $N \in N^+ \cap \overline{M} \subseteq \overline{M}^+ \in N^+$. Note that $N \cap \overline{M} \subseteq N \cap \overline{M}^+$. If $\overline{M}^+ \in \mathcal{C} \setminus M$, we can replace $\overline{M}$ by $\overline{M}^+$ and repeat the argument. Otherwise, $\overline{M}^+ \in \mathcal{C} \cap M \subseteq \mathcal{D}$. Therefore, as $\varnothing \neq N \cap \overline{M} \subseteq N \cap \overline{M}^+$, $N \in \overline{M}^+$ and $N, \overline{M}^+ \in \mathcal{D}$, there is another tower-type model $\overline{M}^* \in \mathcal{D}$ such that $N \cap \overline{M}^+ \subseteq \overline{M}^* \in N$. But then we are done, because $N \cap \overline{M} \subseteq N \cap \overline{M}^+ \subseteq \overline{M}^*$. □

Lemma 2.1.12. *Let $\mathcal{C}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain and let $\overline{M} \in \mathcal{C} \cap \mathcal{T}$. Then, for every*

$N \in \overline{M}$ there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain $\mathcal{C}_N \supseteq \mathcal{C}$ such that $N \in \mathcal{C}_N$.

Proof. Start by assuming that $N \notin \mathcal{C}$, otherwise there is nothing to check. Since tower-type models need to be preceded by large models in $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains, there has to be some $N' \in \overline{M}$ in $\mathcal{C}$. Note that if N' lies above N, by clause (C.b) of definition 2.1.9 there must be another tower-type model $\overline{M}' \in \mathcal{C}$ such that $N \in N' \cap \overline{M} \subseteq \overline{M}' \in N'$. Therefore, we may start by assuming that $\overline{M}$ is a minimal tower-type model containing N. That is, we assume that all $N' \in \overline{M} \cap \mathcal{C}$ lie below N. Let N^* be the predecessor of $\overline{M}$ in $\mathcal{C}$, which is an element of N because of the minimality of $\overline{M}$. We claim that

$$\mathcal{C}_N = \mathcal{C} \cup \{N\} \cup \{N \cap \overline{M}\} \cup \{N \cap M : M \in \mathcal{C} \cap \mathcal{S}, N \in M\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain. We will denote by S_N the set of all countable elementary $M \in \mathcal{C}$ such that $N \in M$.

Let us check first that $\mathcal{C}_N$ is $\in$-increasing. We will show that the new models added to $\mathcal{C}$ are ordered in the following way in $\mathcal{C}_N$:

(1) $N \cap \overline{M}$ is the immediate successor of N^* in $\mathcal{C}_N$.

(2) $N \cap \overline{M}$ is followed by all the models of the form $N \cap M$, for each $M \in S_N$.

(3) N lies on top of the models $N \cap \overline{M}$ and $N \cap M$, for all $M \in S_N$, and it is the immediate predecessor of $\overline{M}$ in $\mathcal{C}_N$.

Note that $N^* \in N \cap \overline{M} \in N \in \overline{M}$ and that $N \cap M \in N$, for all $M \in S_N$. So, it's enough to check two things. First, that $N \cap \overline{M} \in N \cap M$ for all $M \in S_N$, and then, that for any two different $M_0, M_1 \in S_N$, either $N \cap M_0 \in N \cap M_1$, $N \cap M_1 \in N \cap M_0$, or $N \cap M_0 = N \cap M_1$.

Note first that all the models in S_N lie above $\overline{M}$, by the minimality of $\overline{M}$. Suppose that $M \in \mathcal{C}$ is a model of countable elementary type lying above $\overline{M}$ and such that $\overline{M} \notin M$. Then, there must be some uncountable elementary node $N' \in \mathcal{C} \cap M$

such that $\overline{M}$ lies in the gap $[N' \cap M, N')$. Since N^* is the immediate predecessor of $\overline{M}$ in $\mathcal{C}$, $N' \cap M$ needs to occur below N^*, and this implies that N also belongs to the gap $[N' \cap M, N')$. Therefore, $N \notin M$. Hence, by contraposition, if $M \in S_N$, then $\overline{M} \in M$. In fact, this implies that for every countable elementary node $M \in \mathcal{C}$, $N \in M$ if and only if $\overline{M} \in M$. So we can conclude that $N \cap \overline{M} \in N \cap M$, for every $M \in S_N$.

Now, let $M_0, M_1 \in S_N$. We will show, by induction, that either $N \cap M_0 \in N \cap M_1$, $N \cap M_1 \in N \cap M_0$, or $N \cap M_0 = N \cap M_1$. Suppose that M_0 occurs below M_1 in $\mathcal{C}$. If $M_0 \in M_1$, then $N \cap M_0 \in M_1$, and we are done because it implies that $N \cap M_0 \in N \cap M_1$. Hence, we may assume that $M_0 \notin M_1$. In this case, there has to be some uncountable elementary $N^+ \in \mathcal{C}$ such that M_0 lies in the gap $[N^+ \cap M_1, N^+)$. If $M_0 = N^+ \cap M_1$, then $N \cap M_0 = N \cap (N^+ \cap M_1) = N \cap M_1$, because $N \in N^+$, and we are done. Hence, suppose that $N^+ \cap M_1$ lies strictly below M_0. There are two possibilities now. The first one is that $N^+ \cap M_1 \in M_0$. In this case, again because $N \in N^+$, we have that $N \cap (N^+ \cap M_1) = N \cap M_1 \in M_0$. But then we are done, because as $M_0 \in S_N$ implies that $N \cap M_1 \in N \cap M_0$. The second possibility is that $N \cap M_1 \notin M_0$. In this case we can repeat the same argument as above, after the assumption that $M_0 \notin M_1$, but arguing with respect to $N \cap M_1$ and M_0, instead of M_0 and M_1, respectively. Since $\mathcal{C}$ is finite, in finitely many iterations of the argument we will get the conclusion that we were looking for. This finishes the proof that $\mathcal{C}_N$ is $\in$-increasing.

Let us show now that $\mathcal{C}_N$ is closed under intersections. Assume first that N' and M' are two models in $\mathcal{C}_N$ of uncountable and countable elementary type, respectively, such that $N' \in M'$. We only need to check two cases. If $N' \in \mathcal{C}$ and $M' = N \cap M$, for some $M \in S_N$, then N' must lie below N, and hence, $N' \cap M' = N' \cap (N \cap M) = N' \cap M$, which belongs to $\mathcal{C}$ because $N', M \in \mathcal{C}$. If $N' = N$ and $M' \in \mathcal{C}$, then $M' \in S_N$, and $N' \cap M' \in \mathcal{C}_N$ by construction.

Assume now that $N', \overline{M}' \in \mathcal{C}_N$ are uncountable elementary and tower-type models, respectively, such that $N' \in \overline{M}'$ and $N' \cap \overline{M}' \neq \varnothing$. We need to find a tower-type node $\overline{M}^* \in \mathcal{C}_N$ such that $N' \cap \overline{M}' \subseteq \overline{M}^* \in N'$. Again, we only need to check two cases. First, suppose that $N' \in \mathcal{C}$ and $\overline{M}' = N \cap \overline{M}$. Since $N' \in \overline{M}$ and $N', \overline{M} \in \mathcal{C}$, there must be another tower-type model $\overline{M}^* \in \mathcal{C}$ such that $N' \cap \overline{M} \subseteq \overline{M}^* \in N'$. But note that as $N' \in N$, we have that $N' \cap \overline{M}' = N' \cap (N \cap \overline{M}) = N' \cap \overline{M}$. Therefore, $\overline{M}^*$ is such that $N' \cap M' \subseteq \overline{M}^* \in N'$, as we wanted. As for the second case, suppose that $N' = N$ and that $\overline{M}' \in \mathcal{C}$. If $\overline{M}' = \overline{M}$, there is nothing to check because $N \cap \overline{M} \in \mathcal{C}_N$ by construction. Hence, we may assume that $\overline{M}' \neq \overline{M}$. We will show by induction that there is a model of non-elementary type $\overline{M}^* \in \mathcal{C}_N$ such that $N \cap \overline{M}' \subseteq \overline{M}^* \in N$. By the minimality of $\overline{M}$ and since $\overline{M}'$ must occur above $\overline{M}$, there must be a model of uncountable elementary type $N_0 \in \mathcal{C}$ such that $\overline{M} \in N_0 \in \overline{M}'$. Since $N \in N_0 \cap \overline{M}'$ and $N_0, \overline{M}' \in \mathcal{C}$, there must be a model of non-elementary type $\overline{M}_0 \in \mathcal{C}$ such that $N_0 \cap \overline{M}' \subseteq \overline{M}_0 \in N_0$. By the minimality of $\overline{M}$, either $\overline{M} = \overline{M}_0$ or $\overline{M}_0$ lies above $\overline{M}$. If $\overline{M} = \overline{M}_0$, then we have $N \cap \overline{M}' \subseteq N_0 \cap \overline{M}' \subseteq \overline{M}$, and thus, $N \cap \overline{M}' \subseteq N \cap \overline{M} \in N$. Therefore, in this case letting $\overline{M}^* = N \cap \overline{M}$, which belongs to $\mathcal{C}_N$ by construction, gives us the result. If $\overline{M}_0$ occurs after $\overline{M}$, we can argue as above, with respect to $\overline{M}_0$ instead of $\overline{M}'$. It should be clear that working by induction, we will get the result after finitely many repetitions of the same argument. $\square$

The forcing $\mathbb{C}(\mathcal{S}, \mathcal{L}, \mathcal{T})$ is strongly $\mathcal{S}$-proper and strongly $\mathcal{L}$-proper, but moreover, it has the following property, which follows directly from the last lemma.

Corollary 2.1.13. *Let $\mathcal{C}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chain and let $\overline{M} \in \mathcal{C} \cap \mathcal{T}$. Then, $\mathcal{C}$ is strongly $(N, \mathbb{C}(\mathcal{S}, \mathcal{L}, \mathcal{T}))$-generic for every $N \in \overline{M}$.*

2.2 Symmetric systems of models of one type

We have divided this section in two parts. First, we will review some basic results about elementary submodels and the properties of their isomorphisms, some of them without proofs. All these facts are well-known and can be found in any of the sources cited in section 1.4, but let us add [23], which might be more fitting for our current purposes. In the second part we will introduce symmetric systems of models of one type and review their main properties. All the results from the second part can be found in [11], [12], or [45].

2.2.1 Elementary submodels and isomorphisms

Proposition 2.2.1. *Let Q_0, Q_1 be two elementary submodels of $H(\kappa)$. If Ψ and Φ are two isomorphisms between $(Q_0; \in, T)$ and $(Q_1; \in, T)$, then $\Psi = \Phi$.*

Proposition 2.2.2. *Let $Q_0, Q_1 \preceq H(\kappa)$. If $Q_0 \in Q_1$, then $(Q_0; \in)$ and $(Q_1; \in)$ are not isomorphic.*

Proposition 2.2.3. *Let Q_0, Q_1 be two elementary submodels of $H(\kappa)$, let Ψ be an isomorphism between the structures $(Q_0; \in, T)$ and $(Q_1; \in, T)$, and let $\mathrm{fp}(\Psi) = \{x \in Q_0 : \Psi(x) = x\}$ be the set of fixed points of Ψ. Then, the following hold:*

(1) $\mathrm{fp}(\Psi) \subseteq Q_0 \cap Q_1$.

(2) $[\mathrm{fp}(\Psi)]^\omega \cap Q_0 \subseteq \mathrm{fp}(\Psi)$.

(3) $\Psi \restriction (Q_0 \cap H(\omega_1))$ *is the identity.*

(4) $Q_0 \cap H(\omega_1) = Q_1 \cap H(\omega_1)$.

Proposition 2.2.4. *Let $Q_0, Q_1 \preceq H(\kappa)$ such that $|Q_0| = |Q_1| = \mu < \mu^{++} < \kappa$, and suppose that Ψ is an isomorphism between $(Q_0; \in, T)$ and $(Q_1; \in, T)$. Then, $Q_0 \cap Q_1 \cap \mu^{++}$ is an initial segment of both $Q_0 \cap \mu^{++}$ and $Q_1 \cap \mu^{++}$.*

Proof. Without loss of generality, we may assume that we have added a sequence $\vec{e} = (e_\alpha : \alpha < \mu^{++})$, where each e_α is a bijection between α and $|\alpha|$, as a predicate to the structure $H(\kappa)$. We will show that for every $\beta \in Q_0 \cap Q_1 \cap \mu^{++}$, if $\alpha \in Q_0 \cap \beta$, then $\alpha \in Q_1 \cap \beta$. Note that there is some $\xi \in Q_0 \cap \mu^+$ such that $e_\beta(\xi) = \alpha$ in Q_0. But since $Q_0 \cap \mu^+ = Q_1 \cap \mu^+$ by proposition 1.4.7, we have that $\xi, e_\beta \in Q_1$. Therefore, $e_\beta(\xi) = \alpha \in Q_1$. $\square$

2.2.2 The pure side condition forcing

Let μ be a cardinal such that $\mu^+ < \kappa$ and let $\mathcal{B}$ be the collection of elementary submodels $Q \preceq (H(\kappa); \in, P)$ such that $|Q| = \mu$ and ${}^{<\mu}Q \subseteq Q$.

Definition 2.2.5 ([11],[12])**.** Let $\mathcal{N}$ be a set of size $< \mu$ of subsets of $H(\kappa)$. We say that $\mathcal{N}$ is a $\mathcal{B}$-*symmetric system* if the following holds:

(a) Every $Q \in \mathcal{N}$ is an element of $\mathcal{B}$.

(b) Given $Q_0, Q_1 \in \mathcal{N}$, if $Q_0 \cap \mu^+ = Q_1 \cap \mu^+$, then there is a (unique) isomorphism $\Psi_{Q_0,Q_1} : (Q_0; \in, T) \to (Q_1; \in, T)$, which is the identity on $Q_0 \cap Q_1$.

(c) Given $Q_0, Q_1 \in \mathcal{N}$, if $Q_0 \cap \mu^+ < Q_1 \cap \mu^+$, then there is $Q_2 \in \mathcal{N}$ such that $Q_2 \cap \mu^+ = Q_1 \cap \mu^+$ and $Q_0 \in Q_2$.

(d) Given $Q_0, Q_1, Q_1' \in \mathcal{N}$, if $Q_0 \in Q_1$ and $Q_1 \cap \mu^+ = Q_1' \cap \mu^+$, then $\Psi_{Q_1,Q_1'}(Q_0) \in \mathcal{N}$.

We will usually refer to condition (c) as the *shoulder axiom for* $\mathcal{N}$. We let $\mathbb{M}(\mathcal{B})$ be the forcing notion whose conditions are $\mathcal{B}$-symmetric systems and the order is reverse inclusion.

Proposition 2.2.6. *Let $\mathcal{N}$ be a $\mathcal{B}$-symmetric system. Let $M_0, M_1 \in \mathcal{B}$ such that Ψ_{M_0,M_1} is the unique isomorphism between $(M_0; \in)$ and $(M_1; \in)$. If $\mathcal{N} \in M_0$, then $\Psi_{M_0,M_1}(\mathcal{N})$ is a $\mathcal{B}$-symmetric system.*

Lemma 2.2.7. *Let $Q \in \mathcal{B}$ and let $\mathcal{N}$ be a $\mathcal{B}$-symmetric system such that $\mathcal{N} \in Q$. Then, there is a $\mathcal{B}$-symmetric system $\mathcal{N}^*$ such that $\mathcal{N} \cup \{Q\} \subseteq \mathcal{N}^*$.*

Lemma 2.2.8. *Let $\mathcal{N}$ be a $\mathcal{B}$-symmetric system and let $Q \in \mathcal{N}$. Then, $\mathcal{N} \cap Q$ is a $\mathcal{B}$-symmetric system such that $\mathcal{N} \cap Q \in Q$.*

Lemma 2.2.9. *Let $\mathcal{N}$ be a $\mathcal{B}$-symmetric system and let $Q \in \mathcal{N}$. Let $\mathcal{W}$ be another $\mathcal{B}$-symmetric system such that $\mathcal{N} \cap Q \subseteq \mathcal{W} \subseteq Q$. Then, there is a $\mathcal{B}$-symmetric system $\mathcal{U}$ such that $\mathcal{N} \cup \mathcal{W} \subseteq \mathcal{U}$.*

Corollary 2.2.10. *$\mathbb{M}(\mathcal{B})$ is strongly $\mathcal{B}$-proper.*

Lemma 2.2.11. *Let $\mathcal{N}_0$ and $\mathcal{N}_1$ be two $\mathcal{B}$-symmetric systems. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = X$ and that there is an isomorphism Ψ between the structures $(\bigcup \mathcal{N}_0; \in, T, X, M_0)_{M_0 \in \mathcal{N}_0}$ and $(\bigcup \mathcal{N}_1; \in, T, X, M_1)_{M_1 \in \mathcal{N}_1}$ fixing X. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $\mathcal{B}$-symmetric system.*

Corollary 2.2.12. *If $2^\mu = \mu^+$ holds, then $\mathbb{M}(\mathcal{B})$ has the μ^{++}-chain condition.*

Theorem 2.2.13. *$\mathbb{M}(\mathcal{B})$ is μ-closed.*

Theorem 2.2.14. *$\mathbb{M}(\mathcal{B})$ preserves $2^\mu = \mu^+$.*

Countable symmetric systems of models in $\mathcal{L}$, namely $\mathcal{L}$-symmetric systems, are directly involved in our variant of the two-type symmetric systems that include models of non-elementary type. As we will see in section 2.4, $\mathcal{L}$-symmetric systems will play the same role that tower type models played in the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains.

We let $\mathcal{T}^+$ denote the collection of all $\mathcal{L}$-symmetric systems. In the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems, we will also refer to the elements of $\mathcal{T}^+$ as *non-elementary models* or *models of non-elementary type*, although they are not really models. We will tend to use $\overline{M}$ to denote the elements of $\mathcal{T}^+$.

2.3 $(\mathcal{S}, \mathcal{L})$-symmetric systems

In this section we will introduce the forcing with symmetric systems of models of two types and prove their main properties. These side conditions can be seen as a natural combination of $(\mathcal{S}, \mathcal{L})$-chains from section 2.1 and symmetric systems from section 2.2. We will start by defining the notion of an ω_1-hull of an elementary submodel, and prove some basic results about the interaction between the operations of taking intersections and taking isomorphic copies of elementary submodels. We will see that, in our context, these operations commute. Then we will define $(\mathcal{S}, \mathcal{L})$-symmetric systems and show their main properties. Not surprisingly, we will see that the pure side condition forcing is strongly $\mathcal{S}$-proper, strongly $\mathcal{L}$-proper, that it has the $\aleph_3$-Knaster condition (assuming $2^{\aleph_1} = \aleph_2$), and that it preserves $2^{\aleph_1} = \aleph_2$.

2.3.1 The ω_1-hull and isomorphisms

Definition 2.3.1. Given a model Q, we let

$$Q[\omega_1] := \{f(\alpha) : f \in Q, f \text{ a function with } \operatorname{dom}(f) = \omega_1, \alpha \in \omega_1\},$$

and we call it the ω_1-*hull of* Q, or simply the *hull of* Q.

It is worth noting that if $Q \in \mathcal{S} \cup \mathcal{L}$ and $M \in \mathcal{S} \cap Q$, then $M[\omega_1]$ is definable in Q, and thus, $M[\omega_1] \in Q$. Consequently, if $M \in \mathcal{S}$, then $M[\omega_1]$ cannot be countably-closed, otherwise M would be an element of $M[\omega_1]$, and thus, we would be able to define $M[\omega_1]$ in $M[\omega_1]$ itself. Therefore, if $M \in \mathcal{S}$, then $M[\omega_1] \notin \mathcal{L}^{\omega\text{-c}}$.

It's also easy to see that if $N \in \mathcal{L}$, then $N[\omega_1] = N$.

Proposition 2.3.2. *Let $M \in \mathcal{S}$. Then, $M[\omega_1]$ is the smallest elementary submodel of $H(\kappa)$ that contains $M \cup \omega_1$ as a subset.*

Proof. First note that since $\omega_1 \in M$, the identity function $id : \omega_1 \to \omega_1$ is definable in M, and thus, $\omega_1 \subseteq M[\omega_1]$. Moreover, for each $a \in M$, the constant function sending all $\alpha \in \omega_1$ to a is definable in M, so $M \subseteq M[\omega_1]$.

We use the Tarski-Vaught test to show that $M[\omega_1]$ is an elementary submodel of $H(\kappa)$. Let $\varphi(y, x_0, \ldots, x_n)$ be a first-order formula in the language of set theory. Let $a_0, \ldots, a_n \in M[\omega_1]$ such that $H(\kappa) \models \exists y \varphi(y, a_0, \ldots, a_n)$. We have to find $b \in M[\omega_1]$ such that $H(\kappa) \models \varphi(b, a_0, \ldots, a_n)$. Let $f_i \in M$ and $\alpha_i \in \omega_1$ be such that $a_i = f_i(\alpha_i)$, for all $i \leq n$. Fix a bijection $F : \omega_1^{<\omega} \to \omega_1$. We can define a function g in $H(\kappa)$ by

$$g\big(F(\beta_0, \ldots, \beta_n)\big) = d \iff H(\kappa) \models \varphi\big(d, f_0(\beta_0), \ldots, f_n(\beta_n)\big),$$

for any $\beta_0, \ldots, \beta_n \in \omega_1$. Note that $g \circ F$ is defined with $f_0, \ldots, f_n$ and ω_1 as parameters. Hence, we can assume that $g \circ F \in M$. Now, let $b = g\big(F(\alpha_0, \ldots, \alpha_n)\big)$. Then, $b \in M$ and $H(\kappa) \models \varphi(b, a_0, \ldots, a_n)$. Therefore, $M[\omega_1] \preceq H(\kappa)$.

To prove the minimality of $M[\omega_1]$, let N be an elementary submodel of $H(\kappa)$ such that $N \subseteq M[\omega_1]$ and $M \cup \omega_1 \subseteq N$. On one hand, since N has size $\aleph_1$, $N[\omega_1] = N$ and hence $N[\omega_1] \subseteq M[\omega_1]$. On the other hand, since $M \subseteq N$, every function that belongs to M also belongs to N, so $M[\omega_1] \subseteq N[\omega_1] = N$. □

Proposition 2.3.3. *Let $M_0, M_1 \in \mathcal{S}$ be such that Ψ is the unique isomorphism between the structures $(M_0[\omega_1]; \in, M_0, T)$ and $(M_1[\omega_1]; \in, M_1, T)$. Then, $\Psi \restriction M_0$ is the unique isomorphism between the structures $(M_0; \in, T)$ and $(M_1; \in, T)$.*

Proof. It's clear that $\Psi \restriction M_0$ is a bijection between M_0 and M_1. Let φ be a first-order formula in the language of set theory and let $\bar{a}$ be a tuple of elements of M_0. Then, since $M_0 \preceq M_0[\omega_1]$ and $M_1 \preceq M_1[\omega_1]$ by proposition 1.4.3,

$$\begin{aligned} M_0 \models \varphi(\bar{a}) &\iff M_0[\omega_1] \models \varphi(\bar{a}) \\ &\iff M_1[\omega_1] \models \varphi\big(\Psi(\bar{a})\big) \\ &\iff M_1 \models \varphi\big(\Psi(\bar{a})\big) \end{aligned}$$

Hence, $\Psi \restriction M_0$ is an isomorphism between $(M_0; \in, T)$ and $(M_1; \in, T)$. □

Proposition 2.3.4. *Let $M \in \mathcal{S}$ and let $\alpha \in M$ be an ordinal with $cf(\alpha) > \omega_1$. Then,* $\sup(M[\omega_1] \cap \alpha) = \sup(M \cap \alpha)$.

Proof. The inequality $\geq$ is clear. For the direction $\leq$ let $\xi \in M[\omega_1] \cap \alpha$. Then, there are a function $f \in M$ and an ordinal $\beta \in \omega_1$ such that $f(\beta) = \xi$. Define a function g on ω_1 by $g(\gamma) = f(\gamma)$ if $f(\gamma) \in \alpha$, and $g(\gamma) = 0$ otherwise, for each $\gamma \in \omega_1$. Note that since $f, \omega_1, \alpha \in M$, the function g is definable in M. Therefore, as $\mathrm{dom}(g) = \omega_1$ and $cf(\alpha) > \omega_1$, we have that $\sup(\mathrm{ran}(g)) \in M \cap \alpha$. Hence,

$$\xi = f(\beta) = g(\beta) < \sup(M \cap \alpha).$$

$\square$

Corollary 2.3.5. *If $M \in \mathcal{S}$, then $\varepsilon_M = \varepsilon_{M[\omega_1]}$, and hence, $cf(\varepsilon_{M[\omega_1]}) = \omega$.*

Corollary 2.3.6. *Let $M_0, M_1 \in \mathcal{S}$ be such that Ψ is the unique isomorphism between $(M_0[\omega_1]; \in, M_0, T)$ and $(M_1[\omega_1]; \in, M_1, T)$. Then, $\varepsilon_{M_0} = \varepsilon_{M_1}$.*

The following result is an immediate consequence of lemma 1.4.11.

Proposition 2.3.7. *Let $M \in \mathcal{S}$ and let $\alpha \in M$ be an ordinal with $cf(\alpha) = \omega_1$. Then, $\sup(M \cap \alpha) < \sup(M[\omega_1] \cap \alpha) = \alpha$.*

The following results tell us that, in some way, in the context of $(\mathcal{S}, \mathcal{L})$-symmetric systems the operations of taking intersections and taking isomorphic copies of models in $\mathcal{S} \cup \mathcal{L}$ commute.

Proposition 2.3.8. *Let $Q_0, Q_1, Q_1' \in \mathcal{S} \cup \mathcal{L}$ such that $Q_0 \in Q_1[\omega_1]$, and let Ψ be an isomorphism between $(Q_1[\omega_1]; \in, Q_1, T)$ and $(Q_1'[\omega_1]; \in, Q_1', T)$, which is the identity on $Q_1[\omega_1] \cap Q_1'[\omega_1]$. Then, $\Psi(Q_0[\omega_1]) = \Psi(Q_0)[\omega_1]$, and $\Psi \restriction Q_0[\omega_1]$ is the unique isomorphism between $(Q_0[\omega_1]; \in, Q_0, T)$ and $(\Psi(Q_0)[\omega_1]; \in, \Psi(Q_0), T)$, and it is the identity on $Q_0[\omega_1] \cap \Psi(Q_0)[\omega_1]$.*

Proof. First we show that $\Psi(Q_0[\omega_1]) = \Psi(Q_0)[\omega_1]$. By proposition 2.3.2 we only need to show that $\Psi(Q_0[\omega_1])$ is the minimal elementary submodel of $H(\kappa)$ that contains $\Psi(Q_0) \cup \omega_1$ as a subset. It's not hard to see that $\Psi(Q_0) \cup \omega_1 \subseteq \Psi(Q_0[\omega_1])$, and it follows from proposition 1.4.9 that $\Psi(Q_0[\omega_1]) \preceq H(\kappa)$. Now, let R be a subset of $\Psi(Q_0[\omega_1])$ such that $\Psi(Q_0) \cup \omega_1 \subseteq R \preceq H(\kappa)$. Then, there is a subset $P \subseteq Q_0[\omega_1]$ such that $R = \Psi(P)$. Note that P is an elementary submodel of $H(\kappa)$ by proposition 1.4.9, and as $\Psi(Q_0) \cup \omega_1 \subseteq R$, we have that $Q_0 \cup \omega_1 \subseteq P$. Therefore, by proposition 2.3.2, $P = Q_0[\omega_1]$, and thus, $R = \Psi(P) = \Psi(Q_0[\omega_1])$.

This shows the minimality of $\Psi(Q_0[\omega_1])$, and thus, that $\Psi(Q_0[\omega_1]) = \Psi(Q_0)[\omega_1]$, as we wanted.

It follows easily that $\Psi \restriction Q_0[\omega_1]$ is an isomorphism between $(Q_0[\omega_1]; \in, Q_0, T)$ and $(\Psi(Q_0)[\omega_1]; \in, \Psi(Q_0), T)$. To see that it is the identity on $Q_0[\omega_1] \cap \Psi(Q_0)[\omega_1]$ we only need to note that $Q_0[\omega_1] \cap \Psi(Q_0)[\omega_1]$ is contained in $Q_1[\omega_1] \cap Q_1'[\omega_1]$. □

Corollary 2.3.9. *Let $M_0, M_1 \in \mathcal{S}$, and let Ψ be an isomorphism between the structures $(M_0[\omega_1]; \in, M_0, T)$ and $(M_1[\omega_1]; \in, M_1, T)$, which is the identity on $M_0[\omega_1] \cap M_1[\omega_1]$. Let $N_0 \in \mathcal{L}$ be such that $N_0 \in M_0$ and denote $\Psi(N_0)$ by N_1. Then, $\Psi \restriction \big((N_0 \cap M_0)[\omega_1]\big)$ is the unique isomorphism between the structures $\big((N_0 \cap M_0)[\omega_1]; \in, N_0 \cap M_0, T\big)$ and $\big((N_1 \cap M_1)[\omega_1]; \in, N_1 \cap M_1, T\big)$, and it is the identity on $(N_0 \cap M_0)[\omega_1] \cap (N_1 \cap M_1)[\omega_1]$.*

Proposition 2.3.10. *Let $M \in \mathcal{S}$ and let $N_0, N_1 \in \mathcal{L}$ such that $N_0, N_1 \in M$. Suppose that Ψ_{N_0,N_1} is an isomorphism between the structures $(N_0; \in, T)$ and $(N_1; \in, T)$, which is the identity on $N_0 \cap N_1$. Then, $\Psi_{N_0,N_1} \restriction \big((N_0 \cap M)[\omega_1]\big)$ is the unique isomorphism between the structures $\big((N_0 \cap M)[\omega_1]; \in, N_0 \cap M, T\big)$ and $\big((N_1 \cap M)[\omega_1]; \in, N_1 \cap M, T\big)$, and it is the identity on $(N_0 \cap M)[\omega_1] \cap (N_1 \cap M)[\omega_1]$.*

Proof. Note that, by proposition 2.3.8 it's enough to check that $\Psi_{N_0,N_1}(N_0 \cap M)$ equals $N_1 \cap M$. First note that Ψ_{N_0,N_1} is definable in M with parameters N_0 and N_1, hence $\Psi_{N_0,N_1} \in M$. On one hand, it follows that for all $x \in N_0 \cap M$, $\Psi_{N_0,N_1}(x) \in N_1 \cap M$. So, $\Psi_{N_0,N_1}(N_0 \cap M) \subseteq N_1 \cap M$. On the other hand, for all $y \in N_1 \cap M$ there is a unique $x \in N_0$ such that $\Psi_{N_0,N_1}(x) = y$. But note that x is definable in M from y and Ψ_{N_0,N_1}, so $x \in M$. Therefore, $x \in N_0 \cap M$, and thus, $N_1 \cap M \subseteq \Psi_{N_0,N_1}(N_0 \cap M)$. This shows that $\Psi_{N_0,N_1}(N_0 \cap M) = N_1 \cap M$. □

Proposition 2.3.11. *Let $N_0, N_1 \in \mathcal{L}$ and $M_0, M_1 \in \mathcal{S}$ such that $N_0 \in M_0$ and $N_1 \in M_1$. Suppose that Ψ_{N_0,N_1} is an isomorphism between $(N_0; \in, T)$ and $(N_1; \in, T)$, which is the identity on $N_0 \cap N_1$. Suppose that $\Psi_{M_0[\omega_1],M_1[\omega_1]}$ is an isomorphism between the structures $(M_0[\omega_1]; \in, M_0, T)$ and $(M_1[\omega_1]; \in, M_1, T)$, which is the identity on $M_0[\omega_1] \cap M_1[\omega_1]$. Then, $\Psi_{N_0,N_1} \restriction \big((N_0 \cap M_0)[\omega_1]\big)$ is the* *unique isomorphism between the structures* $\big((N_0 \cap M_0)[\omega_1]; \in, N_0 \cap M_0, T\big)$ *and* $\big((N_1 \cap M_1)[\omega_1]; \in, N_1 \cap M_1, T\big)$*, and* $\Psi_{N_0,N_1} \restriction \big((N_0 \cap M_0)[\omega_1]\big)$ *is the identity on* $(N_0 \cap M_0)[\omega_1] \cap (N_1 \cap M_1)[\omega_1]$.

Proof. Let N_0' denote $\Psi_{M_0[\omega_1],M_1[\omega_1]}(N_0)$. Note that $N_0' \in M_1$ and that $\Psi_{N_0,N_0'} := \Psi_{M_0[\omega_1],M_1[\omega_1]} \restriction N_0$ is the unique isomorphism between $(N_0; \in, T)$ and $(N_0'; \in, T)$. Hence, $\Psi_{N_0,N_1} = \Psi_{N_0',N_1} \circ \Psi_{N_0,N_0'}$. By corollary 2.3.9, $\Psi_{N_0,N_0'} \restriction \big((N_0 \cap M_0)[\omega_1]\big) = \Psi_{M_0[\omega_1],M_1[\omega_1]} \restriction \big((N_0 \cap M_0)[\omega_1]\big)$ is the unique isomorphism between the structures $\big((N_0 \cap M_0)[\omega_1]; \in, N_0 \cap M_0\big)$ and $\big((N_0' \cap M_1)[\omega_1]; \in, N_0' \cap M_1\big)$. Hence, we only need to show that $\Psi_{N_0',N_1} \restriction \big((N_0' \cap M_1)[\omega_1]\big)$ is the unique isomorphism between $(N_0' \cap M_1)[\omega_1]$ and $(N_1 \cap M_1)[\omega_1]$, but this follows from the last proposition. □

Proposition 2.3.12. *Let* $N \in \mathcal{L}$ *and* $M_0, M_1 \in \mathcal{S}$ *such that* $N \in M_0[\omega_1] \cap M_1[\omega_1]$. *If* $(M_0[\omega_1]; \in, M_0, T)$ *and* $(M_1[\omega_1]; \in, M_1, T)$ *are isomorphic through the unique isomorphism* Ψ *that fixes* $M_0[\omega_1] \cap M_1[\omega_1]$*, then* $N \cap M_0 = N \cap M_1$.

Proof. First note that since $N \in M_0[\omega_1] \cap M_1[\omega_1]$, then $\Psi(N) = N$, which in turn implies that $\Psi(N \cap M_0) = N \cap M_1$. Therefore, as $N \cap M_0 \in N$, and thus, $N \cap M_0 \in M_0[\omega_1] \cap M_1[\omega_1]$, we have that $N \cap M_0 = \Psi(N \cap M_0) = N \cap M_1$. □

2.3.2 The pure side condition forcing

The following is the natural version of symmetric system of models of two types, which combines the notion of $(\mathcal{S}, \mathcal{L})$-chains with that of symmetric systems. In fact, every $(\mathcal{S}, \mathcal{L})$-chain is, in particular, an $(\mathcal{S}, \mathcal{L})$-symmetric system.

Definition 2.3.13. Let $\mathcal{N}$ be a finite set of members of $H(\kappa)$. We say that $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-*symmetric system* if and only if the following holds:

(A) Every $Q \in \mathcal{N}$ is an element of $\mathcal{S} \cup \mathcal{L}$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, then $Q_0[\omega_1] \cong Q_1[\omega_1]$. Furthermore, $\Psi_{Q_0[\omega_1],Q_1[\omega_1]}$ is an isomorphism between the structures

$(Q_0[\omega_1]; \in, Q_0, T)$ and $(Q_1[\omega_1]; \in, Q_1, T)$, and $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}$ is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is $Q_1' \in \mathcal{N}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in Q_1'[\omega_1]$.

(D) For every $Q \in \mathcal{N}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $Q \in M[\omega_1]$ and there is no $Q' \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_{Q'} < \varepsilon_M$, then in fact $Q \in M$.

(E) For all $Q_0, Q_1, Q_1' \in \mathcal{N}$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, $\Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0) \in \mathcal{N}$.

(F) For every $N \in \mathcal{N} \cap \mathcal{L}$ and every $M \in \mathcal{N} \cap \mathcal{S}$, if $N \in M$, then $N \cap M \in \mathcal{N}$.

A finite set $\mathcal{N}$ of members of $H(\kappa)$ is a *pre-$(\mathcal{S}, \mathcal{L})$-symmetric system* if it satisfies clauses (A)-(E). As in the case of symmetric systems of models of one type, we will refer to clause (C) as the *shoulder axiom for $\mathcal{N}$*. Let $\mathbb{M}(\mathcal{S}, \mathcal{L})$ be the forcing notion whose conditions are $(\mathcal{S}, \mathcal{L})$-symmetric systems and the order is reverse inclusion.

As we will see later, in $(\mathcal{S}, \mathcal{L})$-symmetric systems there are also residue gaps, similar to the ones in $(\mathcal{S}, \mathcal{L})$-chains. The reason to ask for isomorphisms between the ω_1-hulls of the models in an $(\mathcal{S}, \mathcal{L})$-symmetric system is to have full symmetry, even for the models inside the residue gaps. This is also the reason why, in general, $\mathcal{S}$-symmetric systems are not $(\mathcal{S}, \mathcal{L})$-symmetric systems.

It's not too hard to see that if $\mathcal{N}$ is a finite set of members of $H(\kappa)$ satisfying clauses (A) and (B) of definition 2.3.13, then $\mathcal{N}$ satisfies clauses (C) and (D) if and only if it satisfies the following clause:

(C+D) For any two distinct $Q_0, Q_1 \in \mathcal{N}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$ and there is no $P \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$, then there is $Q_2 \in \mathcal{N}$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $Q_0 \in Q_2$.

In most cases, when showing that a finite set of members of $H(\kappa)$ is an $(\mathcal{S}, \mathcal{L})$-

symmetric system, showing that it satisfies clause (C+D) will be easier than showing that it satisfies clauses (C) and (D) separately.

2.3.3 Basic properties

The following proposition is an immediate consequence of the basic facts about elementary submodels presented at the beginning of section 1.4 in the preliminaries.

Proposition 2.3.14. *Let $\mathcal{N}$ be a pre-$(\mathcal{S},\mathcal{L})$-symmetric system. If $Q_0, Q_1 \in \mathcal{N}$, then the following holds:*

(1) If $Q_0 \in Q_1$ and $|Q_0| < |Q_1|$, then $Q_0 \subseteq Q_1$.

(2) If Q_1 is a small model, $Q_0 \in Q_1[\omega_1]$, and there is no large model $N \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_N < \varepsilon_{Q_1}$, then $Q_0 \in Q_1$.

Proposition 2.3.15. *Let $\mathcal{N}$ be a pre-$(\mathcal{S},\mathcal{L})$-symmetric system and let $Q_0, Q_1, Q_1' \in \mathcal{N}$ such that $Q_0 \in Q_1[\omega_1]$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$. Then, $\Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(Q_0)$ belongs to $\mathcal{N}$.*

Proof. If $Q_1, Q_1' \in \mathcal{L}$, then the result follows directly from clause (E). Hence, we may assume that Q_1 and Q_1' are countable. If there are no uncountable models $N \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_N < \varepsilon_{Q_1}$, then $Q_0 \in Q_1$ by the last proposition, and the result follows from clause (E). Suppose the contrary and let ε be the greatest ω_2-height of any uncountable model in $\mathcal{N}$ lying strictly between ε_{Q_0} and ε_{Q_1}. By two applications of the shoulder axiom we can find an uncountable $N \in \mathcal{N}$ and a model $R \in \mathcal{N}$ such that $\varepsilon_N = \varepsilon$, $\varepsilon_R = \varepsilon_{Q_1}$, $Q_0 \in N$ and $N \in R[\omega_1]$. Hence, again by the last proposition, $N \in R$. Therefore, we can apply clause (E) to get $N_1 := \Psi_{R[\omega_1],Q_1[\omega_1]}(N) \in \mathcal{N} \cap Q_1$ and $N_1' := \Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(N_1) \in \mathcal{N} \cap Q_1'$. Note that $Q_0 \in R[\omega_1] \cap Q_1[\omega_1]$. Hence, $Q_0 = \Psi_{R[\omega_1],Q_1[\omega_1]}(Q_0) = \Psi_{N[\omega_1],N_1[\omega_1]}(Q_0)$. Thus, as $Q_0 \in N_1$, we have that $\Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(Q_0) = \Psi_{N_1[\omega_1],N_1'\omega_1}(Q_0) \in \mathcal{N}$, again by clause (E). □

Proposition 2.3.16. *Let $\mathcal{N}$ be a pre-$(\mathcal{S},\mathcal{L})$-symmetric system. Let $Q_0, Q_1 \in \mathcal{N}$ such that $Q_0 \in Q_1[\omega_1]$. If there is $P \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$, then there is some $R \in \mathcal{N}$ such that $\varepsilon_R = \varepsilon_P$, $Q_0 \in R[\omega_1]$ and $R \in Q_1[\omega_1]$.*

Proof. By two applications of the shoulder axiom we can find $P', Q'_1 \in \mathcal{N}$ such that $\varepsilon_{P'} = \varepsilon_P$, $\varepsilon_{Q'_1} = \varepsilon_{Q_1}$, $Q_0 \in P'[\omega_1]$ and $P' \in Q'_1[\omega_1]$. Let R be the image of P' under the isomorphism $\Psi_{Q'_1[\omega_1],Q_1[\omega_1]}$, which belongs to $\mathcal{N}$ by proposition 2.3.15. On one hand, note that $R \in Q_1[\omega_1]$. And on the other hand, note that as $Q_0 \in Q'_1[\omega_1] \cap Q_1[\omega_1]$, by clause (B), $Q_0 = \Psi_{Q'_1[\omega_1],Q_1[\omega_1]}(Q_0)$. Therefore, as $\Psi_{P'[\omega_1],R[\omega_1]} = \Psi_{Q'_1[\omega_1],Q_1[\omega_1]} \restriction P'[\omega_1]$, we have that $Q_0 \in R[\omega_1]$. □

The following proposition, which tells us that an isomorphic copy of an $(\mathcal{S},\mathcal{L})$-symmetric system is again an $(\mathcal{S},\mathcal{L})$-symmetric system, is an easy exercise.

Proposition 2.3.17. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system. Let N_0, N_1 be two elementary submodels of $H(\kappa)$ of size $\aleph_1$ such that Ψ_{N_0,N_1} is the unique isomorphism between $(N_0;\in)$ and $(N_1;\in)$. If $\mathcal{N} \subseteq N_0$, then $\Psi_{N_0,N_1}(\mathcal{N})$ is an $(\mathcal{S},\mathcal{L})$-symmetric system.*

Lemma 2.3.18. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system and let $N \in \mathcal{L}$ such that $\mathcal{N} \subseteq N$. Then $\mathcal{N} \cup \{N\}$ is an $(\mathcal{S},\mathcal{L})$-symmetric system.*

Proof. Every $Q \in \mathcal{N}$ is an element of N, and thus $\varepsilon_Q < \varepsilon_N$. This shows that clause (B) from definition 2.3.13 is satisfied. The other clauses are obvious. □

Lemma 2.3.19. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system and let $M \in \mathcal{S}$ such that $\mathcal{N} \subseteq M$. Then, there is an $(\mathcal{S},\mathcal{L})$-symmetric system $\mathcal{N}^*$ such that $\mathcal{N} \cup \{M\} \subseteq \mathcal{N}^*$.*

Proof. We claim that the set

$$\mathcal{N}^* = \mathcal{N} \cup \{M\} \cup \{N \cap M : N \in \mathcal{N} \cap \mathcal{L}\}$$

is an $(\mathcal{S},\mathcal{L})$-symmetric system. It is worth noting first that if $N \in \mathcal{N} \cap \mathcal{L}$, then there is no $Q \in \mathcal{N}^*$ such that $\varepsilon_{N \cap M} < \varepsilon_Q < \varepsilon_N$. Suppose first that $Q \in \mathcal{N}$

is such that $\varepsilon_Q < \varepsilon_N$. Then there is some $N' \in \mathcal{N}$ such that $\varepsilon_{N'} = \varepsilon_N$ and $Q \in N'$. Hence, as $\mathcal{N} \subseteq M$, the model Q is an element of $N' \cap M$, and as $\varepsilon_{N' \cap M} = \varepsilon_{N \cap M}$ by proposition 2.3.10, $\varepsilon_Q < \varepsilon_{N \cap M}$. Suppose now that Q is of the form $N' \cap M$ for some $N' \in \mathcal{N} \cap \mathcal{L}$, and $\varepsilon_Q < \varepsilon_N$. If $\varepsilon_{N'} > \varepsilon_N$, then there is some $N'' \in \mathcal{N}$ such that $\varepsilon_{N''} = \varepsilon_{N'}$ and $N \in N''$. Therefore, $N \in N'' \cap M$, and as $\varepsilon_{N'' \cap M} = \varepsilon_{N' \cap M} = \varepsilon_Q$ by proposition 2.3.10, we get a contradiction. If $\varepsilon_{N'} = \varepsilon_N$, then $\varepsilon_Q = \varepsilon_{N \cap M}$, again by proposition 2.3.10. Finally, if $\varepsilon_{N'} < \varepsilon_N$, there must be some $N^* \in \mathcal{N}$ such that $\varepsilon_{N^*} = \varepsilon_N$ and $N' \in N^*$. Hence, $N' \in N^* \cap M$, and we can conclude that $\varepsilon_Q < \varepsilon_{N'} < \varepsilon_{N^* \cap M} = \varepsilon_{N \cap M}$. Now we are ready to show that $\mathcal{N}^*$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

Clause (A) is clear.

To show clause (B) we only need to check the case $M_0, M_1 \in \mathcal{N}^* \cap \mathcal{S}$, where $\varepsilon_{M_0} = \varepsilon_{M_1}$ and M_0 is of the form $N_0 \cap M$ for some $N_0 \in \mathcal{N} \cap \mathcal{L}$. It follows from the argument above that M_1 must be of the form $N_1 \cap M$, for some $N_1 \in \mathcal{N} \cap \mathcal{L}$ with $\varepsilon_{N_1} = \varepsilon_{N_0}$. Hence, the result follows from proposition 2.3.10.

Let us show now that $\mathcal{N}^*$ satisfies clause (C+D). Let $Q_0, Q_1 \in \mathcal{N}^*$ such that $\varepsilon_{Q_0} < \varepsilon_{Q_1}$ and so that there is no $P \in \mathcal{N}^*$ with $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$. We need to find $Q_2 \in \mathcal{N}^*$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $Q_0 \in Q_2$. If both Q_0 and Q_1 belong to $\mathcal{N}$ there is nothing to check. Suppose first that $Q_1 = N_1 \cap M$ for some $N_1 \in \mathcal{N} \cap \mathcal{L}$. By the observation at the beginning of the proof, Q_0 cannot be of the form $N_0 \cap M$ for any $N_0 \in \mathcal{N} \cap \mathcal{L}$. Hence, $Q_0 \in \mathcal{N}$, and thus, there must be some $N_2 \in \mathcal{N}$ such that $\varepsilon_{N_2} = \varepsilon_{N_1}$ and $Q_0 \in N_2$. Note that as $\mathcal{N} \subseteq M$, we have that $Q_0 \in N_2 \cap M$. Hence, if we let $Q_2 = N_2 \cap M$, since $\varepsilon_{N_2 \cap M} = \varepsilon_{N_1 \cap M}$, we are done. Suppose now that Q_0 is of the form $N_0 \cap M$ for some $N_0 \in \mathcal{N} \cap \mathcal{L}$. Then, by the observation above, Q_1 must have ω_2-height ε_{N_0}. But then we are done, as $Q_0 \in N_0$.

Let $Q_1, Q_1' \in \mathcal{N}^*$ be such that $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, let $Q_0 \in \mathcal{N}^* \cap Q_1$, and define $Q_0' := \Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0)$. To show clause (E) we have to check that $Q_0' \in \mathcal{N}^*$. First of all, note that $Q_1 \in \mathcal{N}^* \setminus \mathcal{N}$ if and only if $Q_1' \in \mathcal{N}^* \setminus \mathcal{N}$. If $Q_0, Q_1, Q_1' \in \mathcal{N}$ there is nothing to check. If $Q_0 = N_0 \cap M$ and $Q_1 = N_1 \cap M$, for $N_0, N_1 \in \mathcal{N} \cap \mathcal{L}$, then $N_0 \cap M \in N_1 \cap M$, which is impossible by proposition 1.4.13. If $Q_0 \in \mathcal{N}$, $Q_1 = N_1 \cap M$ and $Q_1' = N_1' \cap M$, for some $N_1, N_1' \in \mathcal{N} \cap \mathcal{L}$, then $Q_0' = \Psi_{N_1,N_1'}(Q_0)$ by proposition 2.3.10, which is clearly an element of $\mathcal{N}$. Lastly, suppose that $Q_0 = N_0 \cap M$, for some $N_0 \in \mathcal{N} \cap \mathcal{L}$, and $Q_1, Q_1' \in \mathcal{N}$. Note that by the argument at the beginning of the proof, $\varepsilon_{N_0} \leq \varepsilon_{Q_1}$. If $\varepsilon_{N_0} = \varepsilon_{Q_1}$, as $Q_0 \in N_0 \cap Q_1$, we have that $Q_0' = \Psi_{Q_1,Q_1'}(Q_0) = \Psi_{Q_1,Q_1'}(\Psi_{N_0,Q_1}(Q_0)) = \Psi_{N_0,Q_1'}(Q_0) = Q_1' \cap M$, which is an element of $\mathcal{N}^*$. If $\varepsilon_{N_0} < \varepsilon_{Q_1}$, there is some $Q_1'' \in \mathcal{N}$ such that $\varepsilon_{Q_1''} = \varepsilon_{Q_1}$ and $N_0 \in Q_1''[\omega_1]$. Let $N_0' = \Psi_{Q_1''[\omega_1],Q_1'[\omega_1]}(N_0)$, which is an element of $\mathcal{N}$, and note that since $Q_0 \in Q_1''[\omega_1] \cap Q_1[\omega_1]$, we have that $Q_0' = \Psi_{Q_1''[\omega_1],Q_1'[\omega_1]}(Q_0) = \Psi_{N_0,N_0'}(Q_0) = N_0' \cap M$. Hence, $Q_0' \in \mathcal{N}^*$.

Lastly, we show that $\mathcal{N}^*$ satisfies clause (F). Note that all the uncountable models in $\mathcal{N}^*$ belong to $\mathcal{N}$. Hence, the only non-trivial case that needs to be checked is the following one. Let $N_0, N_1 \in \mathcal{N}$ be large models such that $N_0 \in N_1 \cap M$. We need to show that $N_0 \cap (N_1 \cap M) \in \mathcal{N}^*$. But note that since $N_0 \in N_1$ and both models are uncountable, $N_0 \subseteq N_1$. Therefore, $N_0 \cap (N_1 \cap M) = N_0 \cap M$, which clearly belongs to $\mathcal{N}^*$. □

Lemma 2.3.20. *Let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{L})$-symmetric system and let $N \in \mathcal{N} \cap \mathcal{L}$. Then, $\mathcal{N} \cap N$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.*

Proof. Clauses (A) and (B) are straightforward.

Let us check that $\mathcal{N} \cap N$ satisfies clause (C+D). Let $Q_0, Q_1 \in \mathcal{N} \cap N$ such that $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, and suppose that there is no $P \in \mathcal{N} \cap N$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$. We need to find $Q_2 \in \mathcal{N} \cap N$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $Q_0 \in Q_2$. Note that, in fact, there is no $R \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_R < \varepsilon_{Q_1}$, otherwise by proposition 2.3.16 there would be some $R' \in \mathcal{N}$ such that $\varepsilon_{R'} = \varepsilon_R$, $Q_0 \in R'[\omega_1]$ and $R' \in N$. Hence, by an application of clause (C+D) followed by an application of clause (C) of $\mathcal{N}$, there must be $Q_2', N' \in \mathcal{N}$ such that $\varepsilon_{Q_2'} = \varepsilon_{Q_1}$, $\varepsilon_{N'} = \varepsilon_N$ and $Q_0 \in Q_2' \in N'$. But then we are done. If we let $Q_2 = \Psi_{N',N}(Q_2')$, since $Q_0 \in N' \cap N$, we have that $Q_0 = \Psi_{N',N}(Q_0) \in Q_2$.

Let $Q_0, Q_1, Q_1' \in \mathcal{N} \cap N$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$. Then, $\Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0) \in \mathcal{N} \cap Q_1'[\omega_1] \subseteq \mathcal{N} \cap N$. Hence, clause (E) is also satisfied.

Lastly, let $N' \in \mathcal{N} \cap N \cap \mathcal{L}$ and let $M \in \mathcal{N} \cap N \cap \mathcal{S}$ such that $N' \in M$. Then, $N' \cap M \in N' \cap \mathcal{N}$, and since $N' \subseteq N$, $N' \cap M \in N$. Therefore, $\mathcal{N} \cap N$ satisfies clause (F) as well. □

Let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{L})$-symmetric system and let $M \in \mathcal{N} \cap \mathcal{S}$. If $\mathcal{N} \cap \mathcal{L} \cap M$ is nonempty, we can fix a maximal increasing $\in$-chain $\langle N_i : i \leq n \rangle$ of elements of $\mathcal{N} \cap \mathcal{L} \cap M$. We clearly have $N_i \in N_{i+1} \cap M \in N_{i+1}$, for each $i < n$. Denote ε_{N_i} by $\varepsilon_i^{\mathcal{N} \cap \mathcal{L} \cap M}$ and $\varepsilon_{N_i \cap M}$ by $\varepsilon_i^{\mathcal{N} \cap \mathcal{S} \cap M}$, or simply $\varepsilon_i^{\mathcal{L}}$ and $\varepsilon_i^{\mathcal{S}}$, respectively, if $\mathcal{N}$ and M are clear from the context, for every $i \leq n$. We call $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ *the residue sequence of* $\mathcal{N} \cap M$. Note that this sequence doesn't depend on the choice of the sequence of models $\langle N_i : i \leq n \rangle$ because of proposition 2.3.10.

Remark 2.3.21. Note that every model $Q \in \mathcal{N}$ such that either $\varepsilon_Q < \varepsilon_0^{\mathcal{S}}$, or $\varepsilon_Q \in \bigcup_{i<n} (\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$, or $\varepsilon_Q \in (\varepsilon_n^{\mathcal{L}}, \varepsilon_M)$, is a countable model. Therefore, by proposition 2.3.14, if $P \in \mathcal{N}$ is such that $\varepsilon_P \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ and $P \in (N \cap M)[\omega_1]$, for some $N \in \mathcal{N} \cap M$ such that $\varepsilon_N = \varepsilon_{i+1}^{\mathcal{L}}$, then $P \in N \cap M$. Similarly, if $\varepsilon_P < \varepsilon_0^{\mathcal{S}}$ and $P \in (N \cap M)[\omega_1]$ for some $N \in \mathcal{N} \cap M$ such that $\varepsilon_N = \varepsilon_0^{\mathcal{L}}$, then $P \in N \cap M$.

The following proposition is analogous to proposition 2.1.4 for $(\mathcal{S}, \mathcal{L})$-chains. It tells us exactly which models belong to $\mathcal{N} \cap M$, when M is a small model that belongs to an $(\mathcal{S}, \mathcal{L})$-symmetric system $\mathcal{N}$.

Proposition 2.3.22. *Let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{L})$-symmetric system, let $M \in \mathcal{N} \cap \mathcal{S}$, and let $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ be the residue sequence of $\mathcal{N} \cap M$. Then, $Q \in \mathcal{N} \cap M$ if and only if $Q \in \mathcal{N} \cap M[\omega_1]$ and either,*

(1) $\varepsilon_Q \in [\varepsilon_n^{\mathcal{L}}, \varepsilon_M)$, or

(2) $\varepsilon_Q \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ and $Q \in (Z_{i+1} \cap M)[\omega_1]$, for some $i < n$ and some large model $Z_{i+1} \in \mathcal{N} \cap M$ such that $\varepsilon_{Z_{i+1}} = \varepsilon_{i+1}^{\mathcal{L}}$, or

(3) $\varepsilon_Q < \varepsilon_0^{\mathcal{S}}$ and $Q \in (Z_0 \cap M)[\omega_1]$, for some large model $Z_0 \in \mathcal{N} \cap M$ such that $\varepsilon_{Z_0} = \varepsilon_0^{\mathcal{L}}$.

Proof. Let us check first the right-to-left implication. Let $Q \in \mathcal{N} \cap M[\omega_1]$. Note that all the models $R \in \mathcal{N}$ such that $\varepsilon_R \in (\varepsilon_n^{\mathcal{L}}, \varepsilon_M)$ are countable. Therefore, if Q is as in (1), then $Q \in M$ by proposition 2.3.14. If Q is as in (2) or (3), the conclusion follows from the last remark.

Let us show the other direction by contraposition. Let $Q \in \mathcal{N} \cap M[\omega_1]$. If $\varepsilon_Q \in \bigcup_{i \leq n} [\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}})$, then $Q \notin M$ by proposition 1.4.13. Hence, assume that Q is such that $\varepsilon_Q \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$, for some $i < n$. Since we are arguing by contraposition, $Q \notin (N \cap M)[\omega_1]$, for all $N \in \mathcal{N} \cap \mathcal{L} \cap M$ such that $\varepsilon_N = \varepsilon_j^{\mathcal{L}}$, for some $j \leq n$. Note that by the last remark, since $\varepsilon_Q \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$, we have that $Q \notin N \cap M$, for all $N \in \mathcal{N} \cap \mathcal{L} \cap M$ such that $\varepsilon_N = \varepsilon_j^{\mathcal{L}}$, for some $j \leq n$. Now, by proposition 2.3.16, since $\varepsilon_Q < \varepsilon_n^{\mathcal{S}} < \varepsilon_n^{\mathcal{L}}$ and $Q \in M[\omega_1]$, there must be some $N' \in \mathcal{N} \cap \mathcal{L} \cap M[\omega_1]$ such that $Q \in N'$ and $\varepsilon_{N'} = \varepsilon_n^{\mathcal{L}}$. By item (1), $N' \in M$, and therefore, $N' \cap M \in \mathcal{N}$. Hence, $Q \notin M$, otherwise Q would be an element of $N' \cap M$, which is impossible by assumption. If Q is as in (3), we can argue in the exact same way. □

Under the assumptions of the last proposition, $\mathcal{N} \cap M$ consists exactly of those models $Q \in \mathcal{N} \cap M[\omega_1]$ such that $Q \in N \cap M$ for some large model $N \in \mathcal{N} \cap M$, and those $Q \in \mathcal{N} \cap M[\omega_1]$ for which there is no $N \in \mathcal{N} \cap \mathcal{L} \cap M$ such that $\varepsilon_Q < \varepsilon_N$. This resembles the situation of Neeman's $(\mathcal{S}, \mathcal{L})$-chains from section 2.1. Suppose that $\mathcal{C}$ is an $(\mathcal{S}, \mathcal{L})$-chain, that $M \in \mathcal{C} \cap \mathcal{S}$, and that $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ is the residue sequence of $\mathcal{C} \cap M$. Then, we can reformulate item (2) from proposition 2.1.4 by saying that $\mathcal{C} \cap M$ consists exactly of those models $Q \in \mathcal{C}$ such that either

- $\varepsilon_Q \in [\varepsilon_n^{\mathcal{L}}, \varepsilon_M)$, or
- $\varepsilon_Q \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ for some $i < n$, or
- $\varepsilon_Q < \varepsilon_0^{\mathcal{S}}$.

There is a major difference between $(\mathcal{S},\mathcal{L})$-symmetric systems and $(\mathcal{S},\mathcal{L})$-chains, though. Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system, let $M \in \mathcal{N}$ be a small model, and let $\langle N_i : i \leq m\rangle$ be a maximal increasing $\in$-chain of large models in $\mathcal{N} \cap M$. Then, for any $i \leq m$, there can be small models $P \in \mathcal{N} \cap N_i$ such that $P \in N_i$ and $\varepsilon_P = \varepsilon_{N_i \cap M}$, and P not being of the form $N_i' \cap M$, for some $N_i' \in \mathcal{N} \cap \mathcal{L}$ such that $\varepsilon_{N_i'} = \varepsilon_{N_i}$. In particular this implies that there can be models $Q \in \mathcal{N}$ such that $\varepsilon_Q \in [\varepsilon_{N_i}, \varepsilon_{N_{i+1}\cap M})$ and $Q \in P$, but $Q \notin M$.

Lemma 2.3.23. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system and let $M \in \mathcal{N} \cap \mathcal{S}$. Then, $\mathcal{N} \cap M$ is an $(\mathcal{S},\mathcal{L})$-symmetric system.*

Proof. Let $\langle(\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n\rangle$ be the residue sequence of $\mathcal{N} \cap M$. Note that clauses (A) and (B) are straightforward.

Let us show that $\mathcal{N} \cap M$ satisfies clause (C+D). Let $Q_0, Q_1 \in \mathcal{N} \cap M$ such that $\varepsilon_{Q_0} < \varepsilon_{Q_1}$ and suppose that there is no $P \in \mathcal{N} \cap M$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$. We need to find $Q_2 \in \mathcal{N} \cap M$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $Q_0 \in Q_2$. If $\varepsilon_n^{\mathcal{L}} < \varepsilon_{Q_0}$, then this is clear by proposition 2.3.16 and the analysis of $\mathcal{N} \cap M$ from proposition 2.3.22. Suppose that $\varepsilon_{Q_0} \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ and $Q_0 \in N_{i+1} \cap M$, for some $i < n$ and some large model $N_{i+1} \in \mathcal{N} \cap M$ such that $\varepsilon_{N_{i+1}} = \varepsilon_{i+1}^{\mathcal{L}}$. If $\varepsilon_{Q_1} < \varepsilon_{i+1}^{\mathcal{S}}$, then there can't be any model $P' \in \mathcal{N}$ such that $\varepsilon_{Q_0} < \varepsilon_{P'} < \varepsilon_{Q_1}$. Indeed, if there was such a model, by proposition 2.3.16 there would be some $R \in \mathcal{N}$ such that $\varepsilon_R = \varepsilon_{P'}$, $Q_0 \in R[\omega_1]$ and $R \in (N_{i+1} \cap M)[\omega_1]$. But then, since all the models in $\mathcal{N}$ of ω_2-height in the interval $(\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ are countable, R would be a member of M, contradicting our assumption. Hence, by an application of clause (C+D) followed by an application of clause (C) of $\mathcal{N}$, there must be some $Q_2', P_{i+1} \in \mathcal{N}$ such that $\varepsilon_{Q_2'} = \varepsilon_{Q_1}$, $\varepsilon_{P_{i+1}} = \varepsilon_{i+1}^{\mathcal{S}}$, and $Q_0 \in Q_2' \in P_{i+1}[\omega_1]$. Therefore, since Q_0 is a member of $(N_{i+1} \cap M) \cap P_{i+1}[\omega_1]$, we are done by letting $Q_2 = \Psi_{P_{i+1}[\omega_1],(N_{i+1}\cap M)[\omega_1]}(Q_2')$. Suppose now that $\varepsilon_{Q_1} \geq \varepsilon_{i+1}^{\mathcal{S}}$. Then, by proposition 2.3.22, Q_1 must have ω_2-height $\varepsilon_{i+1}^{\mathcal{L}}$, and if we let Q_2 be N_{i+1}, we are done. The case $\varepsilon_{Q_0} < \varepsilon_0^{\mathcal{S}}$ is proven exactly as the case $\varepsilon_{Q_0} \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ by arguing with $\varepsilon_0^{\mathcal{S}}$ instead of $\varepsilon_{i+1}^{\mathcal{S}}$ and $\varepsilon_0^{\mathcal{L}}$ instead of $\varepsilon_{i+1}^{\mathcal{L}}$.

Now we check that $\mathcal{N} \cap M$ satisfies clause (E). Let $Q_0, Q_1, Q_1' \in \mathcal{N} \cap M$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, and let $Q_0' := \Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0)$. It's clear that $Q_0' \in \mathcal{N}$, hence we only need to show that $Q_0' \in M$. If Q_1' is countable, then $Q_1' \subseteq M$, and as $Q_0' \in Q_1'$, we are done. If Q_1 and Q_1' are uncountable, then both $Q_1 \cap M$ and $Q_1' \cap M$ belong to $\mathcal{N}$. By proposition 2.3.10, $Q_1' \cap M = \Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_1 \cap M)$, and as $Q_0 \in Q_1 \cap M$, we conclude that $Q_0' \in Q_1' \cap M$.

Finally, let $M' \in \mathcal{N} \cap M$ be a small model and let $N \in \mathcal{N} \cap M$ be a large model such that $N \in M'$. Then, $N \cap M' \in \mathcal{N} \cap M$ because $M \preceq H(\kappa)$. Hence, $\mathcal{N} \cap M$ also satisfies clause (F). □

2.3.4 Amalgamation lemmas

If $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-symetric system, we let $\mathcal{N}[\omega_1]$ denote the set $\{Q[\omega_1] : Q \in \mathcal{N}\}$.

Lemma 2.3.24. *Let $n < \omega$, let $\mathcal{N}_0, \ldots, \mathcal{N}_n$ be $(\mathcal{S}, \mathcal{L})$-symmetric systems, and let $X_{i,j}$ denote $\bigcup \mathcal{N}_i[\omega_1] \cap \bigcup \mathcal{N}_j[\omega_1]$, for all $i, j \leq n$. Suppose that there are isomorphisms $\Psi_{i,j}$ between the structures $(\bigcup \mathcal{N}_i[\omega_1]; \in, X_{i,j}, Q^i)_{Q^i \in \mathcal{N}_i}$ and $(\bigcup \mathcal{N}_j[\omega_1]; \in, X_{i,j}, Q^j)_{Q^j \in \mathcal{N}_j}$ fixing $X_{i,j}$, for all $i, j \leq n$. Then, $\bigcup_{i \leq n} \mathcal{N}_i$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.*

Proof. Clause (A) is obviously satisfied by $\bigcup_{i \leq n} \mathcal{N}_i$.

Let us show clause (B). Let $Q_0 \in \mathcal{N}_i$ and $Q_1 \in \mathcal{N}_j$ such that $\varepsilon_{Q_0} = \varepsilon_{Q_1}$, for some $i, j \leq n$. Let $Q_2 = \Psi_{i,j}(Q_0)$, which is an element of $\mathcal{N}_j$. Then, the structures $(Q_0[\omega_1]; \in, Q_0)$ and $(Q_1[\omega_1]; \in, Q_1)$ are isomorphic through the isomorphism $\Psi_{Q_2[\omega_1], Q_1[\omega_1]} \circ \Psi_{i,j} \restriction Q_0[\omega_1]$. Now, if $x \in Q_0[\omega_1] \cap Q_1[\omega_1]$, then x must be an element of $X_{i,j}$, which implies that $\Psi_{i,j}(x) = x$, and hence, $x \in Q_2[\omega_2]$. Therefore, $x \in Q_2[\omega_1] \cap Q_1[\omega_1]$, and as $\mathcal{N}_j$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system, $\Psi_{Q_2[\omega_1], Q_1[\omega_1]}(x) = x$. Thus, $\Psi_{Q_0[\omega_1], Q_1[\omega_1]}(x) = \Psi_{Q_2[\omega_1], Q_1[\omega_1]}(\Psi_{i,j}(x)) = x$.

To show clause (C+D) it suffices to note that the existence of $\Psi_{i,j}$ implies that $\{\varepsilon_{Q_0} : Q_0 \in \mathcal{N}_i\} = \{\varepsilon_{Q_1} : Q_1 \in \mathcal{N}_j\}$, for any two $i, j \leq n$.

Let $i_0, i_1, i_2 \leq n$ and let $Q_0 \in \mathcal{N}_{i_0}$, $Q_1 \in \mathcal{N}_{i_1}$ and $Q_2 \in \mathcal{N}_{i_2}$, such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_2}$. In order to show clause (E) we must verify that $\Psi_{Q_1[\omega_1],Q_2[\omega_1]}(Q_0)$ belongs to $\bigcup_{i\leq n} \mathcal{N}_i$. Since $Q_0 \in Q_1$, it follows that $Q_0 \in X_{i_0,i_1}$, and hence $Q_0 = \Psi_{i_0,i_1}(Q_0) \in \mathcal{N}_{i_1}$. Let $Q_0' = \Psi_{i_1,i_2}(Q_0)$ and $Q_1' = \Psi_{i_1,i_2}(Q_1)$, which are both elements of $\mathcal{N}_{i_2}$. Then, $\Psi_{Q_1[\omega_1],Q_2[\omega_1]}(Q_0) = \Psi_{Q_1'[\omega_1],Q_2[\omega_1]}(Q_0')$, which is an element of $\mathcal{N}_{i_2}$ because $Q_0', Q_1', Q_2 \in \mathcal{N}_{i_2}$ and $\mathcal{N}_{i_2}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

Let $N \in \mathcal{N}_i \cap \mathcal{L}$ and $M \in \mathcal{N}_j \cap \mathcal{S}$ such that $N \in M$, for some $i, j \leq n$. We need to check that $N \cap M \in \bigcup_{k\leq n} \mathcal{N}_k$ to show clause (F). Note that as $N \in \mathcal{N}_i \cap M$, in particular, $N \in X_{i,j}$. Hence, $\Psi_{i,j}(N) = N$, and if we let $M_0 \in \mathcal{N}_i \cap \mathcal{S}$ be such that $\Psi_{i,j}(M_0) = M$, then $N \in M_0$. As $\mathcal{N}_i$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system, $N \cap M_0 \in \mathcal{N}_i$, and thus, $\Psi_{i,j}(N \cap M_0) = N \cap M \in \mathcal{N}_j$. □

Definition 2.3.25. Let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{L})$-symmetric system and let $R \in \mathcal{N}$. Then, an $(\mathcal{S}, \mathcal{L})$-symmetric system $\mathcal{V}$ is called a *virtual* $(\mathcal{N}, R)$*-reflection* if it satisfies the following properties:

(VR.1) $\mathcal{N} \cap R[\omega_1] \subseteq \mathcal{V} \subseteq R[\omega_1]$.

(VR.2) If $R \in \mathcal{S}$ and $V \in \mathcal{V}$ is such that $\varepsilon_V = \max\{\varepsilon_{V'} : V' \in \mathcal{V}\}$, then $V \in R$.

(VR.3) If $R \in \mathcal{S}$, let $\varepsilon^+ = \max\{\varepsilon_N : N \in \mathcal{V} \cap \mathcal{L}\}$, and define ε^- as the ordinal $\min\{\varepsilon_N : N \in \mathcal{N} \cap \mathcal{L}, \varepsilon_N > \varepsilon_R\}$, in case it exists, otherwise let $\varepsilon^- = \max\{\varepsilon_Q : Q \in \mathcal{N}\} + 1$. Let $N \in \mathcal{V} \cap \mathcal{L}$ such that $\varepsilon_N = \varepsilon^+$. Then, the following hold:

- $N \cap R \in \mathcal{V}$.
- For every $\varepsilon \in \{\varepsilon_{M'} : M' \in \mathcal{N}, \varepsilon_{M'} \in (\varepsilon_R, \varepsilon^-)\}$, there is some small model $M' \in \mathcal{N}$ such that $R \in M'$, $\varepsilon_{M'} = \varepsilon$, and $N \cap M' \in \mathcal{V}$.

Proposition 2.3.26. *Let $\mathcal{N}$ be an $(\mathcal{S}, \mathcal{L})$-symmetric system and let $R \in \mathcal{N}$. Let $\mathcal{V}$ be a virtual $(\mathcal{N}, R)$-reflection. Then,*

$$\mathcal{U} = \{Q \in \mathcal{N} : \varepsilon_Q \geq \varepsilon_R\} \cup \bigcup\{\Psi_{R[\omega_1],R'[\omega_1]}``(\mathcal{V}) : R' \in \mathcal{N}, \varepsilon_{R'} = \varepsilon_R\}$$

is an $(\mathcal{S},\mathcal{L})$-symmetric system extending $\mathcal{N}$ and $\mathcal{V}$.

Proof. First of all note that by proposition 2.3.17, for each $R' \in \mathcal{N}$ such that $\varepsilon_{R'} = \varepsilon_R$, it holds that $\Psi_{R[\omega_1],R'[\omega_1]}``(\mathcal{V})$ is an $(\mathcal{S},\mathcal{L})$-symmetric system which extends $\mathcal{N} \cap R'[\omega_1]$ and is contained in $R'[\omega_1]$. Hence,

$$\mathcal{V}^* := \bigcup\{\Psi_{R[\omega_1],R'[\omega_1]}``(\mathcal{V}) : R' \in \mathcal{N}, \varepsilon_{R'} = \varepsilon_R\},$$

is an $(\mathcal{S},\mathcal{L})$-symmetric system by lemma 2.3.24. Note that all models $Q \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_R$ belong to $\mathcal{V}^*$. Indeed, if $Q \in \mathcal{N}$ is such that $\varepsilon_Q < \varepsilon_R$, then by the shoulder axiom for $\mathcal{N}$ there must be some $R' \in \mathcal{N}$ such that $\varepsilon_{R'} = \varepsilon_R$ and $Q \in R'[\omega_1]$. Therefore, $\Psi_{R'[\omega_1],R[\omega_1]}(Q)$ is a member of $\mathcal{N} \cap R[\omega_1]$, which is a subset of $\mathcal{V}$ by (VR.1) of definition 2.3.25. Hence, any model in $\mathcal{U}$ of ω_2-height $< \varepsilon_R$ is an element of $\mathcal{V}^*$. Let us denote the set $\{Q \in \mathcal{N} : \varepsilon_Q \geq \varepsilon_R\}$ by $\mathcal{N}^*$. It's clear that $\mathcal{U}$ satisfies clauses (A) and (B) from definition 2.3.13, and clause (C+D) follows from (VR.2) of definition 2.3.25. Let us show that it satisfies clause (E) now. Note that since $\mathcal{V}^*$ and $\mathcal{N}$ are both $(\mathcal{S},\mathcal{L})$-symmetric systems, it's enough to show that if $Q_1, Q_1' \in \mathcal{N}^*$ are such that $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, and $Q_0 \in \mathcal{V}^* \cap Q_1$, then $\Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(Q_0) \in \mathcal{V}$. By definition of $\mathcal{U}$ there must be some $R_0 \in \mathcal{N}$ and some $Q_0^* \in \mathcal{V}$ such that $\varepsilon_{R_0} = \varepsilon_R$ and $Q_0 = \Psi_{R[\omega_1],R_0[\omega_1]}(Q_0^*)$. Suppose first that $\varepsilon_{Q_1} = \varepsilon_{Q_1'} = \varepsilon_R$. Since $Q_0 \in R_0[\omega_1] \cap Q_1$ and $\mathcal{N}$ is an $(\mathcal{S},\mathcal{L})$-symmetric system, $Q_0 = \Psi_{R_0[\omega_1],Q_1[\omega_1]}(Q_0)$. Therefore,

$$\begin{aligned}
\Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(Q_0) &= \Psi_{R_0[\omega_1],Q_1'[\omega_1]}(Q_0) \\
&= \Psi_{R_0[\omega_1],Q_1'[\omega_1]}(\Psi_{R[\omega_1],R_0[\omega_1]}(Q_0^*)) \\
&= \Psi_{R[\omega_1],Q_1'[\omega_1]}(Q_0^*),
\end{aligned}$$

which is clearly an element of $\mathcal{U}$. Suppose now that $\varepsilon_{Q_1} = \varepsilon_{Q_1'} > \varepsilon_R$. We can find $Q_2 \in \mathcal{N}$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $R_0 \in Q_2[\omega_1]$, by the shoulder axiom for $\mathcal{N}$. Let $R_1 = \Psi_{Q_2[\omega_1],Q_1[\omega_1]}(R_0)$ and $R_1' = \Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(R_1)$, which are both elements of $\mathcal{N}$ by proposition 2.3.15. Note that $Q_0 \in Q_2[\omega_1] \cap Q_1$. Hence, as $\mathcal{N}$ is an

$(\mathcal{S},\mathcal{L})$-symmetric system, $Q_0 = \Psi_{Q_2[\omega_1],Q_1[\omega_1]}(Q_0) = \Psi_{R_0[\omega_1],R_1[\omega_1]}(Q_0) \in R_1[\omega_1]$. Therefore, $\Psi_{Q_1[\omega_1],Q_1'[\omega_1]}(Q_0) = \Psi_{R_1[\omega_1],R_1'[\omega_1]}(Q_0)$, which is an element of $\mathcal{U}$ by the last case. Thus, $\mathcal{U}$ is a pre-$(\mathcal{S},\mathcal{L})$-symmetric system. Lastly, we check that $\mathcal{U}$ satisfies clause (F). Let $N^* \in \mathcal{U} \cap \mathcal{L}$ and $M^* \in \mathcal{U} \cap \mathcal{S}$ such that $N^* \in M^*$. Note that if $N^*, M^* \in \mathcal{V}^*$, then $N^* \cap M^* \in \mathcal{V}^*$ because $\mathcal{V}^*$ is an $(\mathcal{S},\mathcal{L})$-symmetric system. If $N^*, M^* \in \mathcal{N}^*$, then $N^* \cap M^* \in \mathcal{N}$, and it follows from the observations at the beginning of the proof that if $\varepsilon_{N^* \cap M^*} \geq \varepsilon_R$, then $N^* \cap M^* \in \mathcal{N}^*$, and if $\varepsilon_{N^* \cap M^*} < \varepsilon_R$, then $N^* \cap M^* \in \mathcal{V}^*$. Let us check now the last possible case, $N^* \in \mathcal{V}^*$ and $M^* \in \mathcal{N}^*$. Assume first that $R \in \mathcal{L}$. Let $N_0 \in \mathcal{N}^* \cap \mathcal{L}$ be a large model of $\mathcal{N}^*$ of minimal ω_2-height such that $N^* \in N_0 \in M^*$, by appealing to proposition 2.3.16. Then, $N_0 \cap M^* \in \mathcal{N}$, and by the minimality of the ω_2-height of N_0 and since R is a large model by assumption, $\varepsilon_{N_0 \cap M^*} < \varepsilon_R$. Therefore, $N_0 \cap M^* \in \mathcal{V}^*$ and $N^* \in N_0 \cap M^*$. Hence, as $\mathcal{V}^*$ is an $(\mathcal{S},\mathcal{L})$-symmetric system, $N^* \cap (N_0 \cap M^*) = N^* \cap M^* \in \mathcal{V}^*$. Assume now that $R \in \mathcal{S}$. Recall from definition 2.3.25, that in this case we define ε^+ as $\max\{\varepsilon_N : N \in \mathcal{V} \cap \mathcal{L}\}$ and ε^- as $\min\{\varepsilon_N : N \in \mathcal{N} \cap \mathcal{L}, \varepsilon_N > \varepsilon_R\}$, in case it exists, otherwise we let ε^- be $\max\{\varepsilon_Q : Q \in \mathcal{N}\} + 1$. We divide the proof in two cases:

Case 1. Suppose that $\varepsilon_{N^*} = \varepsilon^+$. If there is some $N \in \mathcal{N}^* \cap \mathcal{L}$ such that $\varepsilon_R < \varepsilon_N < \varepsilon_{M^*}$, let $N_0 \in \mathcal{N}^*$ be a large model of minimal ω_2-height such that $N^* \in N_0 \in M^*$, again by proposition 2.3.16. Note that $N^* \in N_0 \cap M^* \in \mathcal{N}$ and that $N^* \cap (N_0 \cap M^*) = N^* \cap M^*$. Hence, in this case it's enough to show that $N^* \cap (N_0 \cap M^*)$ belongs to $\mathcal{U}$. Therefore, we may assume that there is no large model $N \in \mathcal{N}^*$ such that $\varepsilon_R < \varepsilon_N < \varepsilon_{M^*}$. Suppose that $N^* \in R[\omega_1]$ and $R \in M^*[\omega_1]$. Then $N^* \in \mathcal{V}$, and by proposition 2.3.14, $N^* \in R \in M^*$. Hence, by (VR.3) of definition 2.3.25 and proposition 2.3.12, $N^* \cap M^* \in \mathcal{V}$. Suppose now that $N^* \notin R[\omega_1]$. Then, there must be some $N' \in \mathcal{V}$, and hence $N' \in R[\omega_1]$, such that $N^* = \Psi_{R[\omega_1],R'[\omega_1]}(N')$, for some $R' \in \mathcal{N}$ such that $\varepsilon_{R'} = \varepsilon_R$. Let $M' \in \mathcal{N}^*$ such that $\varepsilon_{M'} = \varepsilon_{M^*}$ and $R \in M'[\omega_1]$, given by the shoulder axiom for $\mathcal{N}$. Note that $\varepsilon_{N'} = \varepsilon^+$ and $\varepsilon_{M'} < \varepsilon^-$, and hence, there are no large models of ω_2-height in the interval $(\varepsilon_{N'}, \varepsilon_{M'})$. Therefore, by (VR.2) of definition 2.3.25 we have that

$N' \in M'[\omega_1]$, and hence, by proposition 2.3.14, $N' \in M'$. Therefore, $N' \cap M' \in \mathcal{V}$ by the last case, and hence, we can conclude that $\Psi_{N'[\omega_1],N^*[\omega_1]}(N' \cap M') = N^* \cap M^* \in \mathcal{V}^*$, by proposition 2.3.11 and the fact that $\mathcal{V}^*$ is an $(\mathcal{S},\mathcal{L})$-symmetric system.

Case 2. Suppose that $\varepsilon_{N^*} < \varepsilon^+$. As in the last case, we may assume that there are no large models $N \in \mathcal{N}^*$ such that $\varepsilon_R < \varepsilon_N < \varepsilon_{M^*}$. Since $\mathcal{U}$ is a pre-$(\mathcal{S},\mathcal{L})$-symmetric system, we can find a model $N^+ \in \mathcal{U}$ such that $\varepsilon_{N^+} = \varepsilon^+$ (so $N^+ \in \mathcal{V}^*$) and $N^* \in N^+ \in M^*[\omega_1]$, by proposition 2.3.16. Since there are no uncountable models of ω_2-height in the interval $(\varepsilon^+, \varepsilon_{M^*})$, in fact, $N^+ \in M^*$ by proposition 2.3.14. Therefore, by case 1, $N^+ \cap M^* \in \mathcal{V}^*$. Hence, as $N^* \in N^+ \cap M^*$ and $\mathcal{V}^*$ is an $(\mathcal{S},\mathcal{L})$-symmetric system, $N^* \cap (N^+ \cap M^*) = N^* \cap M^* \in \mathcal{V}^*$. □

Lemma 2.3.27. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system and let $M \in \mathcal{N} \cap \mathcal{S}$. Let $\mathcal{W}$ be another $(\mathcal{S},\mathcal{L})$-symmetric system such that $\mathcal{N} \cap M \subseteq \mathcal{W} \subseteq M$. Then, there is an $(\mathcal{S},\mathcal{L})$-symmetric system $\mathcal{U}$ such that $\mathcal{N} \cup \mathcal{W} \subseteq \mathcal{U}$.*

Proof. The construction of $\mathcal{U}$ can be reduced to the closure of $\mathcal{N} \cup \mathcal{W}$ under intersections and isomorphisms. We will first add models to $(\mathcal{N} \cap M[\omega_1]) \cup \mathcal{W}$ that ensure that a certain fragment of the symmetric system is closed under intersections. Then we will close under the relevant isomorphisms, and the isomorphic copies of the models that we have added first will guarantee that the resulting system is closed under intersections, and thus, forms an $(\mathcal{S},\mathcal{L})$-symmetric system. We can do this thanks to the propositions from section 2.3.1 that ensure that the operations of taking intersections and taking isomorphic copies commute in our context.

Let $\langle(\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n\rangle$ be the residue sequence of $\mathcal{N} \cap M$ and fix an increasing $\in$-chain of large models $\langle N_i : i \leq n\rangle$ such that $N_i \in \mathcal{N} \cap M$ and $\varepsilon_{N_i} = \varepsilon_i^{\mathcal{L}}$ for every $i \leq n$. First of all, note that since $\mathcal{W} \subseteq M$, it follows from proposition 1.4.13 that for every $W \in \mathcal{W}$, either $\varepsilon_W < \varepsilon_0^{\mathcal{S}}$, or $\varepsilon_i^{\mathcal{L}} \leq \varepsilon_W < \varepsilon_{i+1}^{\mathcal{S}}$, or $\varepsilon_n^{\mathcal{L}} \leq \varepsilon_W < \varepsilon_M$, for all $i < n$. Let $W \in \mathcal{W} \cap \mathcal{L}$ such that $\varepsilon_W \neq \varepsilon_i^{\mathcal{L}}$ for all $i \leq n$. The closure under intersections is based on the proof of lemma 2.21 of [62] and uses a similar notation.

If $\varepsilon_i^{\mathcal{L}} < \varepsilon_W < \varepsilon_{i+1}^{\mathcal{S}}$ for some $i < n$, since $N_i, N_{i+1} \in \mathcal{W}$, we can find some $X \in \mathcal{W}$ such that $\varepsilon_X = \varepsilon_W$ and $N_i \in X \in N_{i+1}$, by proposition 2.3.16. Note that $X \in N_{i+1} \cap M$, as $\mathcal{W} \subseteq M$. Let $\varepsilon^* < \omega_2$ be the least ω_2-height of a model from $\mathcal{N} \cap \mathcal{L}$ such that $\varepsilon_{i+1}^{\mathcal{S}} < \varepsilon^*$ (possibly $\varepsilon^* = \varepsilon_{i+1}^{\mathcal{L}}$). Let E_X be any set of models of $\mathcal{N}$ of ω_2-height in the interval $[\varepsilon_{i+1}^{\mathcal{S}}, \varepsilon^*)$ forming a maximal $\in$-chain with minimal element $N_{i+1} \cap M$. Note that since $X \in N_{i+1} \cap M$ and all models in E_X are countable, for every $Q \in E_X$, we have that $X \in Q$. Let F_X be the set of all models of the form $X \cap Q$, for $Q \in E_X$. Note that if $Q, Q' \in E_X$ are such that $Q \in Q'$, then $X \cap Q \in X \cap Q'$. Moreover, note that $X \cap (N_{i+1} \cap M) = X \cap M$ is the least element in F_X, and by proposition 1.4.17, $X \cap Q \in X$ for all $Q \in E_X$. Note that F_X doesn't depend on the choice of the elements of E_X by proposition 2.3.12.

If $\varepsilon_W < \varepsilon_0^{\mathcal{S}}$, we let $X \in \mathcal{W}$ be such that $\varepsilon_X = \varepsilon_W$ and $X \in N_0$, and we define E_X and F_X exactly as in the last paragraph.

If $\varepsilon_n^{\mathcal{L}} < \varepsilon_W < \varepsilon_M$, we let $X \in \mathcal{W}$ such that $\varepsilon_X = \varepsilon_W$ and $N_n \in X$. Let $\varepsilon^* < \omega_2$ be the least ω_2-height of a model from $\mathcal{N} \cap \mathcal{L}$ such that $\varepsilon_M < \varepsilon^*$, if it exists. Otherwise, let ε^* be any ordinal of ω_2 greater than all the ω_2-heights of models in $\mathcal{N}$. Let E_X be any set of models of $\mathcal{N}$ of ω_2-height in the interval $[\varepsilon_M, \varepsilon^*)$ forming a maximal $\in$-chain with minimal element M. By the same reason as above, all models $Q \in E_X$ are countable and $X \in Q$. Let F_X be the set of all models of the form $X \cap Q$, for $Q \in E_X$, which again forms an $\in$-chain, since for any two $Q, Q' \in E_X$, if $Q \in Q'$, then $X \cap Q \in X \cap Q'$. The least element in F_X is $X \cap M$ and, for the same reason as before, all the elements in F_X belong to X. Again, F_X doesn't depend on the choice of the elements of E_X.

For every $W \in \mathcal{W} \cap \mathcal{L}$ such that $\varepsilon_W \neq \varepsilon_i^{\mathcal{L}}$ for all $i \leq n$, fix a single model $X \in \mathcal{W}$ as in the preceding paragraphs so that they form an $\in$-chain. Let $\mathcal{W}^*$ be the result of adding all the models from each F_X to $(\mathcal{N} \cap M[\omega_1]) \cup \mathcal{W}$. Denote by

$\langle Z_i : i \leq m\rangle$, $m \geq n$, the $\in$-sequence of models that results from adding all the models X as above to the sequence $\langle N_i : i \leq n\rangle$. Denote by $\langle(\bar{\varepsilon}_i^{\mathcal{S}}, \bar{\varepsilon}_i^{\mathcal{L}}) : i \leq m\rangle$ the sequence of pairs $(\varepsilon_{Z_i \cap M}, \varepsilon_{Z_i})$, for $i \leq m$, ordered in the obvious way. The following claim tells us that, for each $Z_i \in \mathcal{W} \setminus \mathcal{N}$, $i \leq m$, the chain of models that conforms F_{Z_i} is placed right before Z_i.

Claim 2.3.28. *Let $i \leq m$ and $Z_i \in \mathcal{W} \setminus \mathcal{N}$. If $Q \in \mathcal{W}^*$ belongs to Z_i, then either $Q \in F_{Z_i}$, or $\varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{S}}$. Moreover, if $\varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{S}}$, then $Q \in (Z_i \cap M)[\omega_1]$.*

Proof. Suppose that $Q \notin F_{Z_i}$. Then, $Q \in (\mathcal{N} \cap M[\omega_1]) \cup \mathcal{W}$. We will show that $\varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{S}}$. Note that it's enough to show the result for models Q for which there is no $R \in (\mathcal{N} \cap M[\omega_1]) \cup \mathcal{W}$ such that $\varepsilon_Q < \varepsilon_R < \bar{\varepsilon}_i^{\mathcal{L}}$. In this case, we will show that $Q \in Z_i \cap M$, which is a stronger result. Aiming for a contradiction, suppose that $Q \notin M$. First of all, note that since $\mathcal{W} \subseteq M$, Q must be an element of $\mathcal{N}$. If every Z_j was a member of $\mathcal{W} \setminus \mathcal{N}$, for all $j > i$, then all the models in $\mathcal{N} \cap M[\omega_1]$ of ω_2-height greater than ε_Q would have been of countable type, and hence, by proposition 2.3.14, $Q \in M$. Therefore, there must be a minimal $j > i$ for which $Z_j \in \mathcal{N}$. By the shoulder axiom for $\mathcal{N}$, there must be a model $P_j \in \mathcal{N}$ such that $\varepsilon_{P_j} = \bar{\varepsilon}_j^{\mathcal{S}}$ and $Q \in P_j[\omega_1]$. But note that all the models in $\mathcal{N}$ of ω_2-height in the interval $(\varepsilon_Q, \bar{\varepsilon}_j^{\mathcal{S}})$ must be of countable type because of the minimality of j. Hence, $Q \in P_j$, again by proposition 2.3.14. Now, since $Q \in Z_i \in Z_j \cap M$, the model Q must be an element of $(Z_j \cap M)[\omega_1]$. Therefore, as $Q \in P_j \cap (Z_j \cap M)[\omega_1]$ and $Q, P_j, Z_j \cap M \in \mathcal{N}$, by clause (B) for $\mathcal{N}$, it must be the case that Q is fixed by the isomorphism $\Psi_{P_j[\omega_1],(Z_j \cap M)[\omega_1]}$. But then we are done, because $Q \in Z_j \cap M$, which contradicts our initial assumption, as we wanted.

Let us show the second part of the statement now. Hence, assume that $\varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{S}}$. Note that if $Q \in F_{Z_l}$, for some $l < i$, the conclusion is clear because $Z_l \in Z_i \cap M$, and hence $Q \in Z_l \subseteq (Z_i \cap M)[\omega_1]$. Moreover, if $Q \in \mathcal{W}$, since $\mathcal{W} \subseteq M$, the result follows trivially. Hence, we may assume that $Q \in \mathcal{N} \setminus M$. We divide the proof in two cases.

Case 1. Suppose that there is no $j > i$ for which $Z_j \in \mathcal{N}$. First of all note that if for all $k < i$, every Z_k such that $\varepsilon_{Z_k} \in (\varepsilon_Q, \overline{\varepsilon}_i^{\mathcal{S}})$ was a member of $\mathcal{W} \setminus \mathcal{N}$, then Q would be an element of M. Indeed, since all the models of $\mathcal{N}$ of ω_2-height in the interval $(\varepsilon_Q, \varepsilon_M)$ would be countable, it would follow from proposition 2.3.14 that $Q \in M$, contradicting our assumption $Q \in \mathcal{N} \setminus M$. Fix the maximal $k < i$ such that $Z_k \in \mathcal{N}$ and $\varepsilon_Q < \overline{\varepsilon}_k^{\mathcal{L}}$. Since $Q, N_k, M \in \mathcal{N}$, by proposition 2.3.16 we can find a model $N_k \in \mathcal{N}$ such that $\varepsilon_{N_k} = \overline{\varepsilon}_k^{\mathcal{L}}$ and $Q \in N_k \in M[\omega_1]$. By the maximality of k and since $Z_j \in \mathcal{W} \setminus \mathcal{N}$ for all $j > i$, it follows from proposition 2.3.14 that all models of ω_2-height $\overline{\varepsilon}_k^{\mathcal{L}}$ in $M[\omega_1]$, and in particular N_k, are elements of M, and consequently, elements of $\mathcal{W}$. Now, since $Z_i \in \mathcal{W}$, by the shoulder axiom for $\mathcal{W}$ there is a model $N_i \in \mathcal{W}$ such that $\varepsilon_{N_i} = \overline{\varepsilon}_i^{\mathcal{L}}$ and $N_k \in N_i$. Note that Q is an element of $N_i \cap Z_i$, hence if we let $N_k' = \Psi_{N_i,Z_i}(N_k)$, which is an element of $\mathcal{W}$ because of clause (E) for $\mathcal{W}$, the model Q belongs to N_k' because it is fixed by the isomorphism Ψ_{N_i,Z_i}. But then we are done. Note that since $N_k' \in Z_i$ and $N_k' \in \mathcal{W}$, the model N_k' must be a member of $Z_i \cap M$, and therefore, $Q \in N_k' \subseteq (Z_i \cap M)[\omega_1]$.

Case 2. Suppose that there is some $j > i$ for which $Z_j \in \mathcal{N}$. Assume that j is minimal with this property. Again, we claim that if for all $k < i$, every Z_k such that $\varepsilon_{Z_k} \in (\varepsilon_Q, \overline{\varepsilon}_i^{\mathcal{L}})$ was a member of $\mathcal{W} \setminus \mathcal{N}$, then Q would be an element of M. Note that all the models in $\mathcal{N}$ of ω_2-height in the interval $(\varepsilon_Q, \overline{\varepsilon}_j^{\mathcal{S}})$ would be countable. Therefore, since $Q \in Z_i \in Z_j \cap M$, the model Q would be an element of $(Z_i \cap M)[\omega_1]$, but then, by proposition 2.3.14, Q would be a member of $Z_i \cap M$, which would contradict our assumption $Q \in \mathcal{N} \setminus M$. Fix the maximal $k < i$ such that $Z_k \in \mathcal{N}$ and $\varepsilon_Q < \overline{\varepsilon}_k^{\mathcal{L}}$. Since $Q \in Z_i \subseteq Z_j$ and $Q, Z_k, Z_j \cap M \in \mathcal{N}$, we can find $N_k \in \mathcal{N}$ such that $Q \in N_k \in (Z_j \cap M)[\omega_1]$ and $\varepsilon_{N_k} = \overline{\varepsilon}_k^{\mathcal{L}}$, by proposition 2.3.16. By the minimality of j and the maximality of k, it follows that all the models in $\mathcal{N}$ of ω_2-height in the interval $(\overline{\varepsilon}_k^{\mathcal{L}}, \overline{\varepsilon}_j^{\mathcal{S}})$ are countable. Therefore, by proposition 2.3.14, N_k is in fact an element of $Z_j \cap M$, and thus, $N_k \in \mathcal{N} \cap M \subseteq \mathcal{W}$. Now, since both N_k and Z_i are elements of $\mathcal{W}$, we can find $N_i \in \mathcal{W}$ such that $\varepsilon_{N_i} = \overline{\varepsilon}_i^{\mathcal{L}}$ and $N_k \in N_i$, by the shoulder axiom for $\mathcal{W}$. Let $N_k' = \Psi_{N_i,Z_i}(N_k)$, which is an element of $\mathcal{W}$ by symmetry, and hence, $N_k' \in Z_i \cap M$. But now, since $Q \in N_k \subseteq N_i$, the model Q must be an element of $N_i \cap Z_i$, and hence, it must be fixed by the isomorphism Ψ_{N_i,Z_i}, by clause (B) for $\mathcal{W}$. Therefore, $Q \in N'_k \subseteq (Z_i \cap M)[\omega_1]$, as we wanted. □

Claim 2.3.29. *There is a virtual $(\mathcal{N}, M)$-reflection $\mathcal{V}$ extending $\mathcal{W}^*$.*

Proof. We will build $\mathcal{V}$ by induction on $i \leq m$. First, note that $(Z_0 \cap M)[\omega_1] \cap \mathcal{W}^*$, which equals $(Z_0 \cap M) \cap \mathcal{W}$, is an $(\mathcal{S}, \mathcal{L})$-symmetric system. It's straightforward to check that it is in fact a virtual $(Z_0 \cap \mathcal{N}, Z_0 \cap M)$-reflection. Indeed, (VR.2) holds because $\mathcal{W} \subseteq M$, and (VR.3) holds vacuously because $(Z_0 \cap M) \cap \mathcal{W}$ only contains small models. Let $\mathcal{V}_0^{\mathcal{S}}$ be $(Z_0 \cap M) \cap \mathcal{W}$. This is the base case of our induction. We will build now two sequences of $(\mathcal{S}, \mathcal{L})$-symmetric systems $(\mathcal{V}_i^{\mathcal{S}})_{i \leq m+1}$ and $(\mathcal{V}_i^{\mathcal{L}})_{i \leq m}$ with the following properties:

- $\mathcal{V}_i^{\mathcal{S}} \subseteq \mathcal{V}_i^{\mathcal{L}} \subseteq \mathcal{V}_{i+1}^{\mathcal{S}}$ for all $i \leq m$.
- $(Z_i \cap M)[\omega_1] \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{S}}$ for all $i \leq m$.
- $Z_i \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{L}}$ for all $i \leq m$.
- $M[\omega_1] \cap \mathcal{W}^* = \mathcal{W}^* \subseteq \mathcal{V}_{m+1}^{\mathcal{S}}$.

Let us explain first the idea behind the proof. Since we want to build a virtual $(\mathcal{N}, M)$-reflection $\mathcal{V}$ extending $\mathcal{W}^*$, in particular $\mathcal{V}$ needs to be an $(\mathcal{S}, \mathcal{L})$-symmetric system, and hence, it needs to be closed under isomorphisms. Let us assume that $Z_0 \in \mathcal{N}$. Note that there might be models Q in $(Z_0 \cap M) \cap \mathcal{W}$ which are not elements of $\mathcal{N}$. Let now $P \in \mathcal{N} \cap Z_0$ be such that $\varepsilon_P = \varepsilon_{Z_0 \cap M} = \overline{\varepsilon}_0^{\mathcal{S}}$, and recall that P doesn't need to be of the form $N_0 \cap M$, for some $N_0 \in \mathcal{N} \cap \mathcal{L}$ such that $\varepsilon_{N_0} = \overline{\varepsilon}_0^{\mathcal{L}}$. Therefore, there might not be an isomorphic copy of Q in P. Hence, we need to add all models of this form to $\mathcal{V}$. This will be the first step of the induction, and $\mathcal{V}_0^{\mathcal{L}}$ will be the result of adding all the models of this form. Now, suppose that $Z_0 \in \mathcal{W} \setminus \mathcal{N}$. In the construction of $\mathcal{W}^*$ we have added all the models from F_{Z_0} to $(\mathcal{N} \cap M[\omega_1]) \cap \mathcal{W}$. Suppose that $W_0 \in \mathcal{W} \cap (Z_1 \cap M)$ is such that $\varepsilon_{W_0} = \varepsilon_{Z_0} = \overline{\varepsilon}_0^{\mathcal{L}}$ and $W_0 \neq Z_0$. Note that there might not be isomorphic copies in W_0 of the models from F_{Z_0}. Hence, we need to add them all. This is the second step of the induction, and $\mathcal{V}_1^{\mathcal{S}}$ will be the result of adding all these models. The idea is to build $\mathcal{V}$ inductively by adding all these missing models. We will start by copying $\mathcal{V}_0^{\mathcal{S}}$ through the relevant isomorphism to obtain $\mathcal{V}_0^{\mathcal{L}}$. Then, we will copy $\mathcal{V}_0^{\mathcal{L}}$ through the relevant isomorphisms to obtain $\mathcal{V}_1^{\mathcal{S}}$. And we will keep repeating this process until we obtain $\mathcal{V}$. The notion of virtual reflection was isolated precisely so that we can apply proposition 2.3.26 at each step of the induction to obtain the $(\mathcal{S}, \mathcal{L})$-symmetric system that will be copied in the next step.

Throughout the induction below we will keep referring to the models $Z_{i+1} \cap M$ and their ω_2-heights $\overline{\varepsilon}_{i+1}^{\mathcal{S}}$, for $i \leq m$. Since the model $Z_{m+1} \cap M$ and its ω_2-height $\overline{\varepsilon}_{m+1}^{\mathcal{S}}$ are undefined, we will make the convention that when $i = m$ and we refer to $Z_{m+1} \cap M$ and $\overline{\varepsilon}_{m+1}^{\mathcal{S}}$, we actually mean M and ε_M, respectively.

Inductive step 1. Let $i \leq m$. Suppose that we have obtained $\mathcal{V}_i^{\mathcal{S}}$, which has the following properties:

(IH.1) $(Z_i \cap M)[\omega_1] \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{S}} \subseteq (Z_i \cap M)[\omega_1]$.

(IH.2) If $Z_i \in \mathcal{N}$, then $\mathcal{V}_i^{\mathcal{S}}$ is a virtual $(Z_i \cap \mathcal{N}, Z_i \cap M)$-reflection.

(IH.3) If $Z_i \in \mathcal{W} \setminus \mathcal{N}$, then $\mathcal{V}_i^{\mathcal{S}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system. If moreover $i > 0$, then $N' \cap M' \in \mathcal{V}_i^{\mathcal{S}}$, for all $M' \in E_{Z_i}$ and all $N' \in \mathcal{V}_i^{\mathcal{S}} \cap \mathcal{L}$ such that $\varepsilon_{N'} = \overline{\varepsilon}_{i-1}^{\mathcal{L}}$.

(IH.4) If $i = 0$, then

$$\{Q \in \mathcal{V}_i^{\mathcal{S}} : \varepsilon_Q < \overline{\varepsilon}_0^{\mathcal{S}}\} = \{Q \in (Z_0 \cap M)[\omega_1] \cap \mathcal{W}^* : \varepsilon_Q < \overline{\varepsilon}_0^{\mathcal{S}}\}.$$

(IH.5) If $i > 0$, then

$$\{Q \in \mathcal{V}_i^{\mathcal{S}} : \varepsilon_Q \in [\overline{\varepsilon}_{i-1}^{\mathcal{L}}, \overline{\varepsilon}_i^{\mathcal{S}})\} = \{Q \in (Z_i \cap M)[\omega_1] \cap \mathcal{W}^* : \varepsilon_Q \in [\overline{\varepsilon}_{i-1}^{\mathcal{L}}, \overline{\varepsilon}_i^{\mathcal{S}})\}.$$

(IH.6) $\{\varepsilon_Q : Q \in \mathcal{V}_i^{\mathcal{S}}\} = \{\varepsilon_Q : Q \in (Z_i \cap M)[\omega_1] \cap \mathcal{W}^*\}$.

From this we will define $\mathcal{V}_i^{\mathcal{L}}$ and we will show that it has the following properties, which will allow us to continue the induction:

(C.1) $Z_i \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{L}} \subseteq Z_i$.

(C.2) $\mathcal{V}_i^{\mathcal{L}}$ is a virtual $((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, Z_i)$-reflection.

(C.3) $\{Q \in \mathcal{V}_i^{\mathcal{L}} : \bar{\varepsilon}_i^{\mathcal{S}} \leq \varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{L}}\} = \{Q \in Z_i \cap \mathcal{W}^* : \bar{\varepsilon}_i^{\mathcal{S}} \leq \varepsilon_Q < \bar{\varepsilon}_i^{\mathcal{L}}\}$.

(C.4) $\{\varepsilon_Q : Q \in \mathcal{V}_i^{\mathcal{L}}\} = \{\varepsilon_Q : Q \in Z_i \cap \mathcal{W}^*\}$.

We will assume that $i > 0$. The case $i = 0$ is proved in the exact same way, modulo some little notational changes.

We need to distinguish the two cases $Z_i \in \mathcal{W} \setminus \mathcal{N}$ and $Z_i \in \mathcal{N}$. If $Z_i \in \mathcal{W} \setminus \mathcal{N}$, it follows from claim 2.3.28 that all the models in $\mathcal{W}^*$ of ω_2-height in the interval $[\bar{\varepsilon}_i^{\mathcal{S}}, \bar{\varepsilon}_i^{\mathcal{L}})$ are exactly the models in F_{Z_i}. Recall that these models are all countable and form an $\in$-chain below Z_i with minimal element $Z_i \cap M$. In this case we simply define $\mathcal{V}_i^{\mathcal{L}}$ as $F_{Z_i \cap M} \cup \mathcal{V}_i^{\mathcal{S}}$. If $Z_i \in \mathcal{N}$, all the models in $\mathcal{W}^*$ of ω_2-height in the interval $[\bar{\varepsilon}_i^{\mathcal{S}}, \bar{\varepsilon}_i^{\mathcal{L}})$ belong to $\mathcal{N}$. In this case we let $\mathcal{V}_i^{\mathcal{L}}$ be the amalgamation of $Z_i \cap \mathcal{N}$ and $\mathcal{V}_i^{\mathcal{S}}$ given by proposition 2.3.26. That is,

$$\begin{aligned}\mathcal{V}_i^{\mathcal{L}} = &\{Q \in Z_i \cap \mathcal{N} : \varepsilon_Q \geq \bar{\varepsilon}_i^{\mathcal{S}}\} \\ &\cup \bigcup \{\Psi_{(Z_i \cap M)[\omega_1], P_i[\omega_1]}\text{"}(\mathcal{V}_i^{\mathcal{S}}) : P_i \in Z_i \cap \mathcal{N}, \varepsilon_{P_i} = \bar{\varepsilon}_i^{\mathcal{S}}\}.\end{aligned}$$

Let us show now that, in both cases, $\mathcal{V}_i^{\mathcal{L}}$ satisfies clauses (C.1)-(C.4).

Assume first that $Z_i \in \mathcal{W} \setminus \mathcal{N}$. Clauses (C.3) and (C.4) follow easily from the induction hypothesis and the definition of $\mathcal{V}_i^{\mathcal{L}}$, and clause (C.1) follows from claim 2.3.28. Hence, it's enough to show that $\mathcal{V}_i^{\mathcal{L}}$ is a virtual $((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, Z_i)$-reflection. It's not too hard to see that $\mathcal{V}_i^{\mathcal{L}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system. Clauses (A)-(E) follow easily from the way we have defined $\mathcal{V}_i^{\mathcal{L}}$, from claim 2.3.28, and from the induction hypothesis. Clause (F) follows from (IH.3) by the same argument for case 2 of the proof of proposition 2.3.26. Hence, it only remains to show that $\mathcal{V}_i^{\mathcal{L}}$ satisfies (VR.1)-(VR.3) from definition 2.3.25. Note that

$$\big((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}\big) \cap Z_i = Z_i \cap \mathcal{W} = (Z_i \cap M)[\omega_1] \cap \mathcal{W},$$

where the second equality follows from claim 2.3.28 and the fact that all the models in $\mathcal{W}^*$ of ω_2-height in the interval $[\bar{\varepsilon}_i^{\mathcal{S}}, \bar{\varepsilon}_i^{\mathcal{L}})$ are exactly the models in F_{Z_i}. Hence, as $(Z_i \cap M)[\omega_1] \cap \mathcal{W} \subseteq \mathcal{V}_i^{\mathcal{S}} \subseteq \mathcal{V}_i^{\mathcal{L}} \subseteq Z_i$ by (IH.1), $\mathcal{V}_i^{\mathcal{L}}$ satisfies (VR.1). Clauses (VR.2) and (VR.3) hold vacuously because Z_i is a large model.

Assume now that $Z_i \in \mathcal{N}$. Clauses (C.3) and (C.4) follow easily from induction hypothesis and the way we have defined $\mathcal{V}_i^{\mathcal{L}}$. In order to see that $\mathcal{V}_i^{\mathcal{L}}$ satisfies clause (C.1) it's enough to show that for every $P_i \in \mathcal{N} \cap Z_i$ such that $\varepsilon_{P_i} = \bar{\varepsilon}_i^{\mathcal{S}}$, $P_i[\omega_1] \cap \mathcal{W}^*$ is a subset of $\Psi_{(Z_i \cap M)[\omega_1], P_i[\omega_1]}``(\mathcal{V}_i^{\mathcal{S}})$. Let $Q \in P_i[\omega_1] \cap \mathcal{W}^*$. By (IH.1), we only need to check that $\Psi_{P_i[\omega_1],(Z_i \cap M)[\omega_1]}(Q) \in \mathcal{W}^*$. If $Q \in \mathcal{N}$, then $\Psi_{P_i[\omega_1],(Z_i \cap M)[\omega_1]}(Q) \in \mathcal{N}$ because both P_i and $Z_i \cap M$ are elements of $\mathcal{N}$, which is an $(\mathcal{S}, \mathcal{L})$-symmetric system. If $Q \in F_{Z_j}$ for some $j < i$, then Q belongs to Z_j, which is a subset of $(Z_i \cap M)[\omega_1] \cap \mathcal{W}^*$. Hence, $\Psi_{P_i[\omega_1],(Z_i \cap M)[\omega_1]}(Q) = Q$ is a member of $(Z_i \cap M)[\omega_1] \cap \mathcal{W}^*$, again using the fact that both P_i and $Z_i \cap M$ are elements of $\mathcal{N}$, which is an $(\mathcal{S}, \mathcal{L})$-symmetric system, and hence, the intersection $P_i[\omega_1] \cap (Z_i \cap M)[\omega_1]$ is fixed by the isomorphism $\Psi_{P_i[\omega_1],(Z_i \cap M)[\omega_1]}$. Lastly, let us assume that $Q \in \mathcal{W} \setminus \mathcal{N}$. Since $\mathcal{W} \subseteq M$, Q must be an element of M as well. Therefore, as $Q \in P_i[\omega_1] \subseteq Z_i$, we have that $Q \in Z_i \cap M$. Hence, as $Q \in P_i[\omega_1] \cap (Z_i \cap M)$, then $\Psi_{P_i[\omega_1],(Z_i \cap M)[\omega_1]}(Q) = Q$, which is an element of $\mathcal{W}^*$ by assumption. Now, it only remains to check clause (C.2). That is, we need to show that $\mathcal{V}_i^{\mathcal{L}}$ is a virtual $((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, Z_i)$-reflection. We know that $\mathcal{V}_i^{\mathcal{L}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system by proposition 2.3.26, and that it satisfies (VR.2) and (VR.3) from definition 2.3.25 vacuously because Z_i is a large model. Hence, we only need to check that it satisfies (VR.1). But note that

$$\big((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}\big) \cap Z_i = Z_i \cap \mathcal{W} \subseteq \mathcal{V}_i^{\mathcal{L}} \subseteq Z_i,$$

where the first inclusion follows from (C.1). This finishes the first inductive step.

Inductive step 2. Let $i \leq m$. Suppose that we have obtained $\mathcal{V}_i^{\mathcal{L}}$, which has the following properties:

(IH.1) $Z_i \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{L}} \subseteq Z_i$.

(IH.2) $\mathcal{V}_i^{\mathcal{L}}$ is a virtual $((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, Z_i)$-reflection.

(IH.3) $\{Q \in \mathcal{V}_i^{\mathcal{L}} : \overline{\varepsilon}_i^{\mathcal{S}} \leq \varepsilon_Q < \overline{\varepsilon}_i^{\mathcal{L}}\} = \{Q \in Z_i \cap \mathcal{W}^* : \overline{\varepsilon}_i^{\mathcal{S}} \leq \varepsilon_Q < \overline{\varepsilon}_i^{\mathcal{L}}\}$.

(IH.4) $\{\varepsilon_Q : Q \in \mathcal{V}_i^{\mathcal{L}}\} = \{\varepsilon_Q : Q \in Z_i \cap \mathcal{W}^*\}$.

From this we will define $\mathcal{V}_{i+1}^{\mathcal{S}}$, and we will show that it has the following properties:

(C.1) $(Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^* \subseteq \mathcal{V}_{i+1}^{\mathcal{S}} \subseteq (Z_{i+1} \cap M)[\omega_1]$.

(C.2) If $i < m$ and $Z_{i+1} \in \mathcal{N}$, then $\mathcal{V}_{i+1}^{\mathcal{S}}$ is a virtual $(Z_{i+1} \cap \mathcal{N}, Z_{i+1} \cap M)$-reflection.

(C.3) If $i < m$ and $Z_{i+1} \in \mathcal{W} \setminus \mathcal{N}$, then $\mathcal{V}_{i+1}^{\mathcal{S}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system and $N' \cap M' \in \mathcal{V}_{i+1}^{\mathcal{S}}$, for all $M' \in E_{Z_{i+1}}$ and all $N' \in \mathcal{V}_{i+1}^{\mathcal{S}} \cap \mathcal{L}$ such that $\varepsilon_{N'} = \overline{\varepsilon}_i^{\mathcal{L}}$.

(C.4) If $i = m$, then $\mathcal{V}_{i+1}^{\mathcal{S}}$ is a virtual $(\mathcal{N}, M)$-reflection.

(C.5) $\{Q \in \mathcal{V}_{i+1}^{\mathcal{S}} : \varepsilon_Q \in [\overline{\varepsilon}_i^{\mathcal{L}}, \overline{\varepsilon}_{i+1}^{\mathcal{S}})\} = \{Q \in (Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^* : \varepsilon_Q \in [\overline{\varepsilon}_i^{\mathcal{L}}, \overline{\varepsilon}_{i+1}^{\mathcal{S}})\}$.

(C.6) $\{\varepsilon_Q : Q \in \mathcal{V}_{i+1}^{\mathcal{S}}\} = \{\varepsilon_Q : Q \in (Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^*\}$.

Recall that in case $i = m$, when referring to $Z_{m+1} \cap M$ and $\overline{\varepsilon}_{m+1}$, we actually mean M and ε_M, respectively.

Note that all the models in $(Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^*$ of ω_2-height in the interval $[\overline{\varepsilon}_i^{\mathcal{L}}, \overline{\varepsilon}_{i+1}^{\mathcal{S}})$ are members of $\mathcal{W}$ by construction of $\mathcal{W}^*$. Define $\mathcal{V}_{i+1}^{\mathcal{S}}$ as the amalgamation of $(Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}$ and $\mathcal{V}_i^{\mathcal{L}}$ given by proposition 2.3.26. That is,

$$\begin{aligned}\mathcal{V}^{\mathcal{S}}_{i+1} = &\{Q \in (Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W} : \varepsilon_Q \geq \bar{\varepsilon}^{\mathcal{L}}_i\} \\ &\cup \bigcup \{\Psi_{Z_i,W_i}"(\mathcal{V}^{\mathcal{L}}_i) : W_i \in (Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, \varepsilon_{W_i} = \bar{\varepsilon}^{\mathcal{L}}_i\}.\end{aligned}$$

Let us show that $\mathcal{V}^{\mathcal{S}}_{i+1}$ satisfies clauses (C.1)-(C.6).

Clauses (C.5) and (C.6) are clear by induction hypothesis and the way we have defined $\mathcal{V}^{\mathcal{S}}_{i+1}$. In order to see that $\mathcal{V}^{\mathcal{S}}_{i+1}$ satisfies clause (C.1) it's enough to show that for every $W_i \in \mathcal{W} \cap (Z_{i+1} \cap M)[\omega_1]$ such that $\varepsilon_{W_i} = \bar{\varepsilon}^{\mathcal{L}}_i$, $W_i \cap \mathcal{W}^*$ is a subset of $\Psi_{Z_i,W_i}"(\mathcal{V}^{\mathcal{L}}_i)$. Let $Q \in W_i \cap \mathcal{W}^*$. We will show that $\Psi_{W_i,Z_i}(Q) \in \mathcal{V}^{\mathcal{L}}_i$. First of all, note that both W_i and Z_i are elements of $Z_{i+1} \cap M$, and thus, $W_i, Z_i \in \mathcal{W}$. Hence, if $Q \in \mathcal{W} \setminus \mathcal{N}$, then $\Psi_{W_i,Z_i}(Q) \in \mathcal{W} \subseteq \mathcal{W}^*$, because $\mathcal{W}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system. Hence, $\Psi_{W_i,Z_i}(Q)$ belongs to $\mathcal{V}^{\mathcal{L}}_i$ by (IH.1). If $Q \in F_{Z_k}$ for some $k \leq i$, then $Q \in Z_k \subseteq Z_i \cap \mathcal{W}^*$. Hence, $\Psi_{W_i,Z_i}(Q) = Q \in Z_i \cap \mathcal{W}^*$, again using the fact that both W_i and Z_i are elements of $\mathcal{W}$, which is an $(\mathcal{S}, \mathcal{L})$-symmetric system, and hence, the intersection $W_i \cap Z_i$ is fixed by the isomorphism Ψ_{W_i,Z_i}. Lastly, let us assume that $Q \in \mathcal{N} \setminus M$. Note that if there was no $k < i$ for which $Z_k \in \mathcal{N}$ and $\varepsilon_Q < \bar{\varepsilon}^{\mathcal{L}}_k$, since $Q \in W_i \subseteq (Z_{i+1} \cap M)[\omega_1]$, the model Q would be a member of M. Indeed, either Q would be an element of the least model of the form $Z_j \cap M$, where $Z_j \in \mathcal{N}$ and $j > i$, or there would be no large model in $\mathcal{N}$ of ω_2-height in the interval $(\varepsilon_Q, \varepsilon_M)$, and hence, Q would be an element of M by proposition 2.3.14. In both cases we get a contradiction with our assumption $Q \in \mathcal{N} \setminus M$. Hence, we can fix the maximal $k \leq i$ for which $Z_k \in \mathcal{N}$ and $\varepsilon_Q < \bar{\varepsilon}^{\mathcal{L}}_k$. We intend to find a model $N_k \in \mathcal{N} \cap M \subseteq \mathcal{W}$ such that $\varepsilon_{N_k} = \bar{\varepsilon}^{\mathcal{L}}_k$ and $Q \in N_k$ now. If there is no $j > i$ such that $Z_j \in \mathcal{N}$, since the models Q, Z_k, and M are members of $\mathcal{N}$, we can appeal to proposition 2.3.16 to find $N_k \in \mathcal{N}$ such that $\varepsilon_{N_k} = \bar{\varepsilon}^{\mathcal{L}}_k$ and $Q \in N_k \in M[\omega_1]$. But note that by the maximality of k and proposition 2.3.14, we have that $N_k \in \mathcal{N} \cap M \subseteq \mathcal{W}$. If there is some $j > i$ such that $Z_j \in \mathcal{N}$, and we let this j be minimal, since the models Q, Z_k, and $Z_j \cap M$ are members of $\mathcal{N}$ and $Q \in (Z_{i+1} \cap M)[\omega_1] \subseteq (Z_j \cap M)[\omega_1]$, we can find $N_k \in \mathcal{N}$ such that $\varepsilon_{N_k} = \bar{\varepsilon}^{\mathcal{L}}_k$

and $Q \in N_k \in (Z_j \cap M)[\omega_1]$, by appealing to proposition 2.3.16 once more. But then, by the minimality of j and the maximality of k, $N_k \in Z_j \cap M$, again by proposition 2.3.14, and hence, $N_k \in \mathcal{N} \cap M \subseteq \mathcal{W}$. In any case, there is a model $N_k \in \mathcal{N} \cap M \subseteq \mathcal{W}$ such that $\varepsilon_{N_k} = \overline{\varepsilon}_k^{\mathcal{L}}$ and $Q \in N_k$, as we wanted. Now, since both N_k and W_i are elements of $\mathcal{W}$, there has to be some $N_i \in \mathcal{W}$ such that $\varepsilon_{N_i} = \overline{\varepsilon}_i^{\mathcal{L}}$ and $N_k \in N_i$, by the shoulder axiom for $\mathcal{W}$. Since $Q \in N_k \subsetneq N_i$, and hence $Q \in W_i \cap N_i$, the model Q must be fixed by the isomorphism Ψ_{W_i,N_i}, and thus,

$$\Psi_{W_i,Z_i}(Q) = \Psi_{N_i,Z_i}(\Psi_{W_i,N_i}(Q)) = \Psi_{N_i,Z_i}(Q).$$

On one hand, note that if we let $N_k' = \Psi_{N_i,Z_i}(N_k)$, which is an element of $\mathcal{W}$ because $N_k, N_i, Z_i \in \mathcal{W}$, then $\Psi_{N_i,Z_i}(Q) = \Psi_{N_k,N_k'}(Q)$. On the other hand, note that since $Q, N_k, Z_k \in \mathcal{N}$, the model $Q^* = \Psi_{N_k,Z_k}(Q)$ must be an element of $\mathcal{N}$, by the symmetry of $\mathcal{N}$. Therefore, by the transitivity of the composition of isomorphisms, $\Psi_{N_k,N_k'}(Q) = \Psi_{Z_k,N_k'}(Q^*)$. But note that, by (IH.1), on one hand both models Q^* and Z_k are members of $Z_i \cap \mathcal{N} \subseteq Z_i \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{L}}$, and on the other hand, $N_k' \in Z_i \cap \mathcal{W} \subseteq Z_i \cap \mathcal{W}^* \subseteq \mathcal{V}_i^{\mathcal{L}}$. Hence, as $\mathcal{V}_i^{\mathcal{L}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system by (IH.2),

$$\Psi_{W_i,Z_i}(Q) = \Psi_{Z_k,N_k'}(Q^*) \in \mathcal{V}_i^{\mathcal{S}},$$

as we wanted. This finishes the proof of clause (C.1). We end the argument by showing that $\mathcal{V}_{i+1}^{\mathcal{S}}$ satisfies clauses (C.2)-(C.4). First of all, note that since $\mathcal{V}_i^{\mathcal{L}}$ is a virtual $((Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}, Z_i)$-reflection, $\mathcal{V}_{i+1}^{\mathcal{S}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system by proposition 2.3.26. We divide the rest of the proof in three cases:

Case 1. Suppose that $i < m$ and that $Z_{i+1} \in \mathcal{N}$. Let us note that in this case all models in $Z_{i+1} \cap \mathcal{W}^*$ of ω_2-height in the interval $[\overline{\varepsilon}_{i+1}^{\mathcal{S}}, \overline{\varepsilon}_{i+1}^{\mathcal{L}})$ belong to $\mathcal{N}$. We need to check the conclusion of (C.2). Namely, that $\mathcal{V}_{i+1}^{\mathcal{S}}$ is a virtual $(Z_{i+1} \cap \mathcal{N}, Z_{i+1} \cap M)$-reflection. Clause (VR.1) from definition 2.3.25 follows from (C.1). Clause (VR.2) follows from (C.5). Let us show clause (VR.3) now. Let ε^- be as in the statement of (VR.3) from definition 2.3.25, and note that ε^+ coincides with $\overline{\varepsilon}_i^{\mathcal{L}}$. Let $M' \in Z_{i+1} \cap \mathcal{N} \cap \mathcal{S}$ such that $Z_{i+1} \cap M \in M'$ and $\overline{\varepsilon}_{i+1}^{\mathcal{S}} \leq \varepsilon_{M'} < \varepsilon^+$.

We claim that $Z_i \cap M' \in \mathcal{V}_i^{\mathcal{L}}$. First of all note that as $Z_i \in Z_{i+1} \cap M \in M'$ and since there are no large models of ω_2-height in the interval $(\bar{\varepsilon}_i^{\mathcal{L}}, \varepsilon^-)$, we have that $Z_i \in M'$. Hence, if $Z_i \in \mathcal{N}$, then $Z_i \cap M' \in \mathcal{N}$, because $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system, and thus, $Z_i \cap M' \in (Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^* \subseteq \mathcal{V}_{i+1}^{\mathcal{S}}$ by (C.1). Otherwise, $Z_i \in \mathcal{W} \backslash \mathcal{N}$, and hence, $Z_i \cap M' \in F_{Z_i}$ by definition of $\mathcal{W}^*$ and proposition 2.3.12. Therefore, the conclusion of (VR.3) follows from the fact that $Z_i \cap M' \in \mathcal{V}_{i+1}^{\mathcal{S}}$, that $\mathcal{V}_{i+1}^{\mathcal{S}}$ is an $(\mathcal{L}, \mathcal{L})$-symmetric system, and proposition 2.3.10.

Case 2. Suppose that $i < m$ and that $Z_{i+1} \in \mathcal{W} \setminus \mathcal{N}$. In this case all models in $Z_{i+1} \cap \mathcal{W}^*$ of ω_2-height in the interval $[\bar{\varepsilon}_{i+1}^{\mathcal{S}}, \bar{\varepsilon}_{i+1}^{\mathcal{L}})$ belong to $F_{Z_{i+1}}$. We need to check the conclusion of (C.3).

Let $Z_{i+1} \cap M' \in F_{Z_{i+1}}$. Note that $Z_i \in Z_{i+1} \cap M'$ and that $Z_i \cap (Z_{i+1} \cap M') = Z_i \cap M'$. If $Z_i \in \mathcal{N}$, as $\mathcal{N}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system, then $Z_i \cap M' \in \mathcal{N}$, because $Z_i \in M'$ and $M' \in E_{Z_{i+1}} \subseteq \mathcal{N}$. Otherwise, $Z_i \in \mathcal{W} \setminus \mathcal{N}$, and hence $E_{Z_i} = E_{Z_{i+1}}$ by definition of $\mathcal{W}^*$. So $Z_i \cap M' \in F_{Z_i}$, and thus, $Z_i \cap M'$ is a member of $(Z_{i+1} \cap M)[\omega_1] \cap \mathcal{W}^* \subseteq \mathcal{V}_{i+1}^{\mathcal{S}}$ by (C.1). Therefore, the conclusion of (C.3) follows from the fact that $Z_i \cap M' \in \mathcal{V}_{i+1}^{\mathcal{S}}$, that $\mathcal{V}_{i+1}^{\mathcal{S}}$ is an $(\mathcal{L}, \mathcal{L})$-symmetric system, and proposition 2.3.10.

Case 3. Suppose that $i = m$. This case is a straightforward translation word by word of the proof of case 1.

This finishes the induction, and by simply letting $\mathcal{V}$ be $\mathcal{V}_{m+1}^{\mathcal{S}}$, we finish the proof of claim 2.3.29. $\square$

Now, since $\mathcal{V}$ is a virtual $(\mathcal{N}, M)$-reflection, we can define the amalgamation $\mathcal{U}$ of $\mathcal{V}$ and $\mathcal{N}$ given by proposition 2.3.26,

$$\mathcal{U} = \{Q \in \mathcal{N} : \varepsilon_Q \geq \varepsilon_M\} \cup \bigcup \{\Psi_{M[\omega_1], M'[\omega_1]}"(\mathcal{V}) : M' \in \mathcal{N}, \varepsilon_{M'} = \varepsilon_M\},$$

which is an $(\mathcal{S}, \mathcal{L})$-symmetric system extending $\mathcal{N}$, and since $\mathcal{W}^* \subseteq \mathcal{V}$, it also extends $\mathcal{W}$. $\square$

The proof of the following lemma is a simpler version of the proof of lemma 2.3.27.

Lemma 2.3.30. *Let $\mathcal{N}$ be an $(\mathcal{S},\mathcal{L})$-symmetric system and let $N \in \mathcal{N} \cap \mathcal{L}$. Let $\mathcal{W}$ be another $(\mathcal{S},\mathcal{L})$-symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{W} \subseteq N$. Then, there is an $(\mathcal{S},\mathcal{L})$-symmetric system such that $\mathcal{N} \cup \mathcal{W} \subseteq \mathcal{U}$.*

Proof. It's obvious that $\mathcal{W}$ is a virtual $(\mathcal{N}, N)$-reflection. Therefore, the amalgamation $\mathcal{U}$ of $\mathcal{N}$ and $\mathcal{W}$ given by proposition 2.3.26 is an $(\mathcal{S},\mathcal{L})$-symmetric system extending $\mathcal{N}$ and $\mathcal{W}$. □

2.3.5 Preservation lemmas

Theorem 2.3.31. *The forcing $\mathbb{M}(\mathcal{S},\mathcal{L})$ is strongly $\mathcal{S}$-proper.*

Proof. Let $M \in \mathcal{S}$ such that $\mathbb{M}(\mathcal{S},\mathcal{L}) \in M$, and let $\mathcal{M} \in \mathbb{M}(\mathcal{S},\mathcal{L}) \cap M$. Then, by lemma 2.3.19, there is $\mathcal{M}^* \in \mathbb{M}(\mathcal{S},\mathcal{L})$ such that $M \in \mathcal{M}^*$ and $\mathcal{M}^* \supseteq \mathcal{M}$. We claim that $\mathcal{M}^*$ is strongly $(M, \mathbb{M}(\mathcal{S},\mathcal{L}))$-generic. Let $D \subseteq \mathbb{M}(\mathcal{S},\mathcal{L}) \cap M$ be a dense subset, and let $\mathcal{N} \in \mathbb{M}(\mathcal{S},\mathcal{L})$ such that $\mathcal{N} \supseteq \mathcal{M}^*$. It follows from lemma 2.3.23 that $\mathcal{N} \cap M$ is a condition in $\mathbb{M}(\mathcal{S},\mathcal{L})$. Since D si dense, there is some $\mathcal{W} \in \mathbb{M}(\mathcal{S},\mathcal{L}) \cap D$ such that $\mathcal{W} \supseteq \mathcal{N} \cap M$, and as $D \subseteq M$, we have that $\mathcal{W} \in M$ (and $\mathcal{W} \subseteq M$ because $\mathcal{W}$ is finite). Therefore, $\mathcal{N}$ and $\mathcal{W}$ are compatible by lemma 2.3.27, and hence, we can conclude that $\mathcal{M}^*$ is strongly $(M, \mathbb{M}(\mathcal{S},\mathcal{L}))$-generic as we wanted. □

The proof of the following lemma is exactly the same as the proof of lemma 2.3.31, but uses lemma 2.3.18 instead of lemma 2.3.19, lemma 2.3.20 instead of lemma 2.3.23, and lemma 2.3.30 instead of lemma 2.3.27.

Theorem 2.3.32. *The forcing $\mathbb{M}(\mathcal{S},\mathcal{L})$ is strongly $\mathcal{L}$-proper.*

Theorem 2.3.33. *If $2^{\aleph_1} = \aleph_2$ holds, then the forcing $\mathbb{M}(\mathcal{S},\mathcal{L})$ has the $\aleph_3$-Knaster condition.*

Proof. Let $\mathcal{N}_\alpha$ be an $(\mathcal{S},\mathcal{L})$-symmetric system for every $\alpha<\omega_3$. Since $2^{\aleph_1}=\aleph_2$ and $\bigcup\mathcal{N}_\alpha[\omega_1]$ has size less than or equal $\aleph_1$ for each $\alpha<\omega_3$, we may assume that the set $\{\bigcup\mathcal{N}_\alpha[\omega_1]:\alpha<\omega_3\}$ forms a Δ-system with root X, by lemma 1.1.19. Moreover, also by $2^{\aleph_1}=\aleph_2$, there are only $\aleph_2$-many isomorphism types for structures of the form $(\bigcup\mathcal{N}_\alpha[\omega_1];\in,X,Q^\alpha)_{Q^\alpha\in\mathcal{N}_\alpha}$. Hence, there is a set $I\in[\omega_3]^{\omega_3}$ such that for any two different $\alpha,\beta\in I$, the structures $(\bigcup\mathcal{N}_\alpha[\omega_1];\in,X,Q^\alpha)_{Q^\alpha\in\mathcal{N}_\alpha}$ and $(\bigcup\mathcal{N}_\beta[\omega_1];\in,X,Q^\beta)_{Q^\beta\in\mathcal{N}_\beta}$ are isomorphic. Moreover, note the unique isomorphism Ψ between $(\bigcup\mathcal{N}_\alpha[\omega_1];\in,X,Q^\alpha)_{Q^\alpha\in\mathcal{N}_\alpha}$ and $(\bigcup\mathcal{N}_\beta[\omega_1];\in,X,Q^\beta)_{Q^\beta\in\mathcal{N}_\beta}$ is the identity on X. The reason is that since there is a definable bijection between $H(\omega_2)$ and ω_2, by lemma 1.1.33, then Ψ fixes X if and only if it fixes $X\cap\omega_2$. Therefore, $\mathcal{N}_\alpha\cup\mathcal{N}_\beta$ is an $(\mathcal{S},\mathcal{L})$-symmetric system by lemma 2.3.24, or in other words, $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are compatible in $\mathbb{M}(\mathcal{S},\mathcal{L})$. □

Theorem 2.3.34. *If $2^{\aleph_1}=\aleph_2$ holds, then $\mathbb{M}(\mathcal{S},\mathcal{L})$ preserves all cardinals.*

Proof. Recall that we have assumed that $\mathcal{L}$ is stationary in $[H(\kappa)]^{\aleph_1}$. Therefore, the conclusion follows from the previous results and lemma 1.1.25. □

If $\mathcal{L}$ was $\mathcal{L}^{\omega\text{-c}}$, we would have to assume CH to ensure the preservation of $\aleph_2$.

Theorem 2.3.35. *If $2^{\aleph_1}=\aleph_2$ holds, then $\mathbb{M}(\mathcal{S},\mathcal{L})$ preserves $2^{\aleph_1}=\aleph_2$.*

Proof. Let $\langle\tau_\alpha:\alpha<\omega_3\rangle$ be a sequence of $\mathbb{M}(\mathcal{S},\mathcal{L})$-names for subsets of ω_1, and suppose that $\mathcal{N}\in\mathbb{M}(\mathcal{S},\mathcal{L})$ is a condition forcing that $\langle\tau_\alpha:\alpha<\omega_3\rangle$ is a sequence of pairwise different subsets of ω_1. For every $\alpha<\omega_3$, let N_α be a large model such that $\mathbb{M}(\mathcal{S},\mathcal{L}),\tau_\alpha,\mathcal{N}\in N_\alpha$. We may assume that there are two different $\alpha,\beta<\omega_3$ for which the structures $(N_\alpha;\in,\tau_\alpha)$ and $(N_\beta;\in,\tau_\beta)$ are isomorphic, and the corresponding isomorphism fixes $N_\alpha\cap N_\beta$ and sends τ_α to τ_β. This follows from the fact that, as $2^{\aleph_1}=\aleph_2$, there are only $\aleph_2$-many isomorphism types for such structures. Let $\mathcal{N}_{\alpha,\beta}:=\mathcal{N}\cup\{N_\alpha,N_\beta\}$, which is easily seen to be an $(\mathcal{S},\mathcal{L})$-symmetric system. We only need to notice that since $\mathcal{N}\in N_\alpha\cap N_\beta$, then $\Psi_{N_\alpha,N_\beta}(\mathcal{N}) = \mathcal{N}$. Moreover, the same argument from the proof of lemma 2.3.32 shows that $\mathcal{N}_{\alpha,\beta}$ is strongly $(N_\alpha, \mathbb{P})$-generic and strongly $(N_\beta, \mathbb{P})$-generic. Recall that $\mathcal{N}$ forces that τ_α and τ_β are different. Hence, there must be an $(\mathcal{S}, \mathcal{L})$-symmetric system $\mathcal{M} \supseteq \mathcal{N}_{\alpha,\beta}$ and an ordinal $\gamma < \omega_1$, such that $\mathcal{M} \Vdash \check{\gamma} \in \tau_\alpha \setminus \tau_\beta$. Let $D \subseteq \mathbb{M}(\mathcal{S}, \mathcal{L}) \cap N_\alpha$ be the dense set of conditions deciding whether $\check{\gamma}$ is an element of τ_α or not. Since $\mathcal{N}_{\alpha,\beta}$ is strongly $(N_\alpha, \mathbb{M}(\mathcal{S}, \mathcal{L}))$-generic, then $\mathcal{M}$ is also strongly $(N_\alpha, \mathbb{M}(\mathcal{S}, \mathcal{L}))$-generic, and hence, D must be predense below $\mathcal{M}$. Therefore, there are conditions $\mathcal{W} \in D$ and $\mathcal{U} \in \mathbb{M}(\mathcal{S}, \mathcal{L})$ such that $\mathcal{U}$ extends $\mathcal{W} \cup \mathcal{M}$. On one hand, note that $\mathcal{W}$ forces that $\check{\gamma} \in \tau_\alpha$. Hence, since Ψ_{N_α,N_β} is an isomorphism and $\mathcal{W} \in N_\alpha$, we have that $\Psi_{N_\alpha,N_\beta}(\mathcal{W}) \Vdash \check{\gamma} \in \Psi_{N_\alpha,N_\beta}(\tau_\alpha) = \tau_\beta$. On the other hand, note that since $\mathcal{U}$ extends $\mathcal{M}$, the models N_α and N_β must be members of $\mathcal{U}$. Therefore, since $\mathcal{W} \subseteq \mathcal{U}$ and $\mathcal{W} \in N_\alpha$, then $\Psi_{N_\alpha,N_\beta}(\mathcal{W}) \subseteq \mathcal{U}$ by the symmetry of $\mathcal{U}$. Therefore, $\mathcal{U} \Vdash \check{\gamma} \in \tau_\beta$. But this is impossible because $\mathcal{U}$ is an extension of $\mathcal{M}$, and $\mathcal{M}$ forces that $\check{\gamma} \in \tau_\alpha \setminus \tau_\beta$. Therefore, we can conclude that there is no condition $\mathcal{N}$ forcing that $\langle \tau_\alpha : \alpha < \omega_3 \rangle$ is a sequence of pairwise different subsets of ω_1. $\square$

2.4 $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems

This section is devoted to the variant of the $(\mathcal{S}, \mathcal{L})$-symmetric systems that includes models of non-elementary type, the $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems. These non-elementary countable models are exactly $\mathcal{L}$-symmetric systems, and they play a similar role to that of tower-type models in $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains. In fact, the notion of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system is inspired, in a strong sense, by that of $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains, in the same way that $(\mathcal{S}, \mathcal{L})$-chains inspired $(\mathcal{S}, \mathcal{L})$-symmetric systems. The organization of this section is very similar to that of section 2.3. We will start by proving some general results about isomorphisms between non-elementary models, which are analogous to the results from section 2.3.1, and then we will move to the definition of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems. The rest of the section is devoted to prove their main properties. Most results in this section are proven in the exact same way as their counterparts from section 2.3. Hence, in most cases we will simply refer to those results and add only the necessary details.

The reason to add non-elementary models to our two-type symmetric systems will be made explicit in chapter 4. We can advance that $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems are the right notion of two-type symmetric systems if we want to use them as side conditions in high versions of Asperó and Mota's finite support iterations from [11] and [12].

2.4.1 $\mathcal{L}$-symmetric systems and isomorphisms

Recall from section 2.2 that $\mathcal{T}^+$ denotes the collection of all $\mathcal{L}$-symmetric systems, and that if $\overline{M} \in \mathcal{T}^+$, then we say that $\overline{M}$ is a *non-elementary model* or a *model of non-elementary type*.

Recall that if $\overline{M}$ is a countable set of elements of $\mathcal{L}$ (possibly finite), we denote by $\varepsilon_{\overline{M}}$ the ordinal $\bigcup\{\varepsilon_N + 1 : N \in \overline{M}\}$, which we call the *$\omega_2$-height of $\overline{M}$*. We gave in section 2.1 the reasons to define $\varepsilon_{\overline{M}}$ this way.

Let $\overline{M}_0$ and $\overline{M}_1$ be two $\mathcal{L}$-symmetric systems. We will denote the intersection $\bigcup \overline{M}_0 \cap \bigcup \overline{M}_1$ by $X_{\overline{M}_0,\overline{M}_1}$. We will use the notation $\Psi_{\overline{M}_0,\overline{M}_1}$ to denote the unique isomorphism Ψ between the structures $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, \bigcup \overline{M}_0 \cap T)_{N_0 \in \overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, \bigcup \overline{M}_1 \cap T)_{N_1 \in \overline{M}_1}$, instead of the more cumbersome $\Psi_{\bigcup \overline{M}_0, \bigcup \overline{M}_1}$. Moreover, for every $\overline{M} \in \mathcal{T}^+$, we will simply write T instead of $\bigcup \overline{M} \cap T$.

Proposition 2.4.1. *Let $N \in \mathcal{L}$, and let $\overline{M} \in \mathcal{T}^+$ such that $N \in \overline{M}$ and $N \cap \overline{M} \neq \varnothing$. Then, $N \cap \overline{M}$ is a member of $N \cap \mathcal{T}^+$.*

Proof. This is exactly lemma 2.2.8. □

Proposition 2.4.2. *Let $M \in \mathcal{S}$, and let $N \in \mathcal{L}$ such that $N \in M$. Then, for every $\overline{M} \in \mathcal{T}^+$ such that $\varepsilon_{N \cap M} < \varepsilon_{\overline{M}} < \varepsilon_N$, $\overline{M} \notin M$.*

Proof. If for some $N' \in \overline{M}$ we have that $\varepsilon_{N\cap M} < \varepsilon_{N'} < \varepsilon_N$, then by proposition 1.4.13, the model N' can't be an element of M. Hence, $\overline{M} \notin M$, otherwise, $\overline{M}$ would be a subset of M and, in particular, N' would be an element of M. $\square$

The proofs of the following two propositions are easy exercises.

Proposition 2.4.3. *Let $\overline{M}_0, \overline{M}_1 \in \mathcal{T}^+$. Let $\Psi_{\overline{M}_0,\overline{M}_1}$ be an isomorphism between $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, T)_{N_0\in\overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, T)_{N_1\in\overline{M}_1}$. Then, $\Psi_{\overline{M}_0,\overline{M}_1} \restriction N_0$ is the unique isomorphism between the models $(N_0; \in, T)$ and $(\Psi_{\overline{M}_0,\overline{M}_1}(N_0); \in, T)$, for every $N_0 \in \overline{M}_0$.*

Proposition 2.4.4. *Let N_0 and N_1 be elementary submodels of $H(\kappa)$ of size $\aleph_1$, and let Ψ_{N_0,N_1} be the unique isomorphism between $(N_0; \in, T)$ and $(N_1; \in, T)$ fixing $N_0 \cap N_1$. Let $\overline{M} \in \mathcal{T}^+$ such that $\overline{M} \in N_0$. Let $\overline{M}'$ be the image of $\overline{M}$ under Ψ_{N_0,N_1}. Then, $\overline{M}' \in \mathcal{T}^+$, and $\Psi_{N_0,N_1} \restriction \bigcup \overline{M}$ is the unique isomorphism between $(\bigcup \overline{M}; \in, N, X_{\overline{M},\overline{M}'}, T)_{N\in\overline{M}}$ and $(\bigcup \overline{M}'; \in, N', X_{\overline{M},\overline{M}'}, T)_{N'\in\overline{M}'}$, which is the identity on $X_{\overline{M},\overline{M}'}$.*

Proposition 2.4.5. *Let $\overline{M}_0$ and $\overline{M}_1$ be two $\mathcal{L}$-symmetric systems and let $\Psi_{\overline{M}_0,\overline{M}_1}$ be the unique isomorphism between $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, T)_{N_0\in\overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, T)_{N_1\in\overline{M}_1}$. For every $Q \in \bigcup \overline{M}_0$ the following holds:*

- *If $Q \in \mathcal{S}$, then $\Psi_{\overline{M}_0,\overline{M}_1}(Q) \in \mathcal{S}$.*
- *If $Q \in \mathcal{L}$, then $\Psi_{\overline{M}_0,\overline{M}_1}(Q) \in \mathcal{L}$.*
- *If $Q \in \mathcal{T}^+$, then $\Psi_{\overline{M}_0,\overline{M}_1}(Q) \in \mathcal{T}^+$.*

Proof. Note that there is some $N_0 \in \overline{M}_0$ such that $Q \in N_0$. Therefore, $\Psi_{\overline{M}_0,\overline{M}_1}(Q) = \Psi_{N_0,N_1}(Q)$, where $N_1 = \Psi_{\overline{M}_0,\overline{M}_1}(N_0)$, by proposition 2.4.3. Hence, the result follows from proposition 1.4.9, if $Q \in \mathcal{S} \cup \mathcal{L}$, and from proposition 2.4.4, if $Q \in \mathcal{T}^+$. $\square$

The following three results are analogous to propositions 2.3.10, 2.3.11 and 2.3.12, respectively. Therefore, they describe the interactions between the elements of

$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems with respect to the operations of taking isomorphic copies and taking intersections.

Proposition 2.4.6. *Let $\overline{M} \in \mathcal{T}^+$ and $N_0, N_1 \in \mathcal{L} \cap \overline{M}$. Suppose that $N_0 \cap \overline{M}$ and $N_1 \cap \overline{M}$ are nonempty and that Ψ_{N_0,N_1} is the unique isomorphism between $(N_0; \in, T)$ and $(N_1; \in, T)$, which is the identity on $N_0 \cap N_1$. Let $\overline{M}_0 = N_0 \cap \overline{M}$ and $\overline{M}_1 = N_1 \cap \overline{M}$. Then, $\Psi_{N_0,N_1} \restriction \bigcup \overline{M}_0$ is the unique isomorphism between the structures $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, T)_{N_0 \in \overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, T)_{N_1 \in \overline{M}_1}$, and it is the identity on $X_{\overline{M}_0,\overline{M}_1}$.*

Proof. Let $Q_0 \in N_0 \cap \overline{M}$. Then, $\Psi_{N_0,N_1}(Q_0)$ is clearly a member of N_1, and it is a member of $\overline{M}$ because $\overline{M}$ is an $\mathcal{L}$-symmetric system. Hence, $\Psi_{N_0,N_1}(N_0 \cap \overline{M})$ is included in $N_1 \cap \overline{M}$. A similar argument shows the other inclusion. The rest of the proof is an easy exercise. □

Proposition 2.4.7. *Let $\overline{M}_0, \overline{M}_1 \in \mathcal{T}^+$ and $N_0, N_1 \in \mathcal{L}$ such that $N_0 \in \overline{M}_0$ and $N_1 \in \overline{M}_1$. Suppose that $N_0 \cap \overline{M}_0$ and $N_1 \cap \overline{M}_1$ are nonempty. Let Ψ_{N_0,N_1} be an isomorphism between $(N_0; \in, T)$ and $(N_1; \in, T)$, which is the identity on $N_0 \cap N_1$. Let $\Psi_{\overline{M}_0,\overline{M}_1}$ be an isomorphism between $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, T)_{N_0 \in \overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, T)_{N_1 \in \overline{M}_1}$, which is the identity on $X_{\overline{M}_0,\overline{M}_1}$. Let $\overline{M}_0^* = N_0 \cap \overline{M}_0$ and $\overline{M}_1^* = N_1 \cap \overline{M}_1$. Then, $\Psi_{N_0,N_1} \restriction \bigcup \overline{M}_0^*$ is the unique isomorphism between $(\bigcup \overline{M}_0^*; \in, N_0^*, X_{\overline{M}_0^*,\overline{M}_1^*}, T)_{N_0^* \in \overline{M}_0^*}$ and $(\bigcup \overline{M}_1^*; \in, N_1^*, X_{\overline{M}_0^*,\overline{M}_1^*}, T)_{N_1^* \in \overline{M}_1^*}$, and it is the identity on $X_{\overline{M}_0^*,\overline{M}_1^*}$.*

Proof. Let N_0' and $\overline{M}_0'$ be the images of N_0 and $\overline{M}_0^*$ under $\Psi_{\overline{M}_0,\overline{M}_1}$, respectively. Note that $N_0' \cap \overline{M}_1 = \Psi_{\overline{M}_0,\overline{M}_1}(N_0 \cap \overline{M}_0) = \overline{M}_0'$. Hence, $\Psi_{N_0',N_1}(N_0' \cap \overline{M}_1) = N_1 \cap \overline{M}_1$ by the last proposition. Therefore, since $\Psi_{N_0,N_0'} = \Psi_{\overline{M}_0,\overline{M}_1} \restriction N_0$, we have

$$\begin{aligned} N_1 \cap \overline{M}_1 &= \Psi_{N_0',N_1}(N_0' \cap \overline{M}_1) = \Psi_{N_0',N_1}\big(\Psi_{\overline{M}_0,\overline{M}_1}(N_0 \cap \overline{M}_0)\big) \\ &= \Psi_{N_0',N_1}\big(\Psi_{N_0,N_0'}(N_0 \cap \overline{M}_0)\big) = \Psi_{N_0,N_1}(N_0 \cap \overline{M}_0). \end{aligned}$$

Thus, $\Psi_{N_0,N_1}(\overline{M}_0^*) = \overline{M}_1^*$. The rest of the proof is an easy exercise. □

Proposition 2.4.8. *Let $N \in \mathcal{L}$ and $\overline{M}_0, \overline{M}_1 \in \mathcal{T}^+$ such that $N \in \overline{M}_0 \cap \overline{M}_1$, and so that $N \cap \overline{M}_0$ and $N \cap \overline{M}_1$ are nonempty. Suppose that $\Psi_{\overline{M}_0,\overline{M}_1}$ is the unique isomorphism between the structures $(\bigcup \overline{M}_0; \in, N_0, X_{\overline{M}_0,\overline{M}_1}, T)_{N_0 \in \overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1, X_{\overline{M}_0,\overline{M}_1}, T)_{N_1 \in \overline{M}_1}$ fixing $X_{\overline{M}_0,\overline{M}_1}$. Then, $N \cap \overline{M}_0 = N \cap \overline{M}_1$.*

Proof. Note that since $\Psi_{\overline{M}_0,\overline{M}_1}(N) = N$, then $\Psi_{\overline{M}_0,\overline{M}_1}(N \cap \overline{M}_0) = N \cap \overline{M}_1$. Therefore, as $N \cap \overline{M}_0 \in N$, and thus, $N \cap \overline{M}_0 \in \bigcup \overline{M}_0 \cap \bigcup \overline{M}_1$, we have that $N \cap \overline{M}_0 = \Psi_{\overline{M}_0,\overline{M}_1}(N \cap \overline{M}_0) = N \cap \overline{M}_1$. □

2.4.2 The pure side condition forcing

Notation 2.4.9. Let $Q_0, Q_1 \in \mathcal{S} \cup \mathcal{L} \cup \mathcal{T}^+$. Then, we denote $Q_0 \in^* Q_1$ if and only if either

- $Q_0 \in Q_1[\omega_1]$, whenever $Q_1 \in \mathcal{S}$,
- $Q_0 \in Q_1$, whenever $Q_1 \in \mathcal{L}$, or
- $Q_0 \in \bigcup Q_1$, whenever $Q_1 \in \mathcal{T}^+$.

Definition 2.4.10. Let $\mathcal{M}$ be a finite set of subsets of $H(\kappa)$. We say that $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-*symmetric system* if and only if the following holds:

(A) Every $Q \in \mathcal{M}$ is an an element of $\mathcal{S} \cup \mathcal{L} \cup \mathcal{T}^+$. Moreover, if $Q \in \mathcal{M}$ is such that $\varepsilon_Q = \min\{\varepsilon_R : R \in \mathcal{M}\}$, then $Q \notin \mathcal{T}^+$.

(B) For any two distinct $Q_0, Q_1 \in \mathcal{M}$ such that $\varepsilon_{Q_0} = \varepsilon_{Q_1}$:

(B.1) If $Q_0, Q_1 \in \mathcal{S} \cup \mathcal{L}$, then there is a (unique) isomorphism $\Psi_{Q_0[\omega_1],Q_1[\omega_1]}$ between $(Q_0[\omega_1]; \in, Q_0, T)$ and $(Q_1[\omega_1]; \in, Q_1, T)$, which is the identity on $Q_0[\omega_1] \cap Q_1[\omega_1]$.

(B.2) If $Q_0, Q_1 \in \mathcal{T}^+$, then there is a (unique) isomorphism Ψ_{Q_0,Q_1} between the two structures $(\bigcup Q_0; \in, N_0, X_{Q_0,Q_1}, T)_{N_0 \in Q_0}$ and $(\bigcup Q_1; \in, N_1, X_{Q_0,Q_1}, T)_{N_1 \in Q_1}$, which is the identity on X_{Q_0,Q_1}.

(C) For any two distinct $Q_0, Q_1 \in \mathcal{M}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then there is some $Q_1' \in \mathcal{M}$ such that $\varepsilon_{Q_1'} = \varepsilon_{Q_1}$ and $Q_0 \in^* Q_1'$.

(D) For every $Q \in \mathcal{M}$ and every $M \in \mathcal{M} \cap (\mathcal{S} \cup \mathcal{T}^+)$, if $Q \in^* M$ and there is no $P \in \mathcal{M}$ such that $\varepsilon_Q < \varepsilon_P < \varepsilon_M$, then in fact $Q \in M$.

(E) For all $Q_0, Q_1, Q_1' \in \mathcal{M}$ such that $Q_0 \in Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q_1'}$, the following holds:

(E.1) If $Q_1, Q_1' \in \mathcal{S} \cup \mathcal{L}$, then $\Psi_{Q_1[\omega_1], Q_1'[\omega_1]}(Q_0) \in \mathcal{M}$.

(E.2) If $Q_1, Q_1' \in \mathcal{T}^+$, then $\Psi_{Q_1, Q_1'}(Q_0) \in \mathcal{M}$.

(F) For every $N \in \mathcal{M} \cap \mathcal{L}$ and every $Q \in \mathcal{M} \cap (\mathcal{S} \cup \mathcal{T}^+)$ such that $N \in Q$,

(F.1) if $Q \in \mathcal{S}$, then $N \cap Q \in \mathcal{M}$, and

(F.2) if $Q \in \mathcal{T}^+$ and $N \cap Q \neq \varnothing$, then $N \cap Q \in \mathcal{M}$.

A finite set $\mathcal{M}$ of subsets of $H(\kappa)$ is a *pre-*$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$*-symmetric system* if it satisfies clauses (A)-(E). As in the previous sections, we will refer to clause (C) as the *shoulder axiom for* $\mathcal{M}$. Let $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ be the forcing notion whose conditions are $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems and the order is reverse inclusion.

It should be obvious from the definition that any $(\mathcal{S}, \mathcal{L})$-symmetric system is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.

It is worth pointing out that the closure under intersections of non-elementary models (clause (F.2) from the definition) is somewhat weaker than the closure under intersections of tower-type nodes in $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains (clause (C.b) from definition 2.1.9). We have adopted this weaker closure to ensure that if $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, $M \in \mathcal{M}$ is a small model, and $N, \overline{M} \in \mathcal{M} \cap M$ are a large model and a non-elementary model, respectively, such that $N \in \overline{M}$, then $N \cap \overline{M} \in M$.

It's not hard to see that if $\mathcal{M}$ is a finite set of members of $H(\kappa)$ satisfying clauses (A) and (B) of definition 2.4.10, then $\mathcal{M}$ satisfies clauses (C) and (D) if and only if it satisfies the following clause:

(C+D) For any two distinct $Q_0, Q_1 \in \mathcal{M}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$ and there is no $P \in \mathcal{M}$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$, then there is $Q_2 \in \mathcal{M}$ such that $\varepsilon_{Q_2} = \varepsilon_{Q_1}$ and $Q_0 \in Q_2$.

In most cases, when showing that a finite set of members of $H(\kappa)$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, showing that it satisfies clause (C+D) will be easier than showing that it satisfies clauses (C) and (D) separately.

Proposition 2.4.11. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and $Q_0, Q_1 \in \mathcal{M}$. Then, $Q_0 \in^* Q_1$ if and only if there is an $\in$-chain of models of $\mathcal{M}$ from Q_0 to Q_1. That is, there are $P_0, \ldots, P_n \in \mathcal{M}$ such that $Q_0 \in P_0 \in \cdots \in P_n \in Q_1$.*

Remark 2.4.12. Note that if $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and $\overline{M} \in \mathcal{M} \cap \mathcal{T}^+$, then $\overline{M}$ needs to be preceded by large models. More precisely, if $Q \in \mathcal{M}$ is such that $\varepsilon_Q < \varepsilon_{\overline{M}}$ and there is no $P \in \mathcal{M}$ such that $\varepsilon_Q < \varepsilon_P < \varepsilon_{\overline{M}}$, then $Q \in \mathcal{L}$. Moreover, by the symmetry of $\mathcal{M}$ there must be some $Q' \in \mathcal{M} \cap \mathcal{L}$ such that $\varepsilon_{Q'} = \varepsilon_Q$ and $Q' \in \overline{M}$.

2.4.3 Basic properties

As we have already mentioned at the beginning of the section, except for a few exceptions, most of the results from section 2.3 can be translated pretty straightforwardly to the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems. The arguments are almost exactly the same, so in many cases we will simply omit them and just add a comment pointing out where they differ. Showing that clause (F.2) holds will be, in most cases, the only place that requires some extra clarification.

The following four propositions are analogous to propositions 2.3.14, 2.3.15, 2.3.16, and 2.4.16, from last section.

Proposition 2.4.13. *Let $\mathcal{M}$ be a pre-$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $Q_0, Q_1 \in \mathcal{M}$. Then, the following holds:*

(1) If $Q_1 \in \mathcal{S} \cup \mathcal{L}$, $Q_0 \in Q_1$, and $|Q_0| < |Q_1|$, then $Q_0 \subseteq Q_1$.

(2) If Q_1 is a small model, $Q_0 \in^* Q_1$, and there is no $N \in \mathcal{M} \cap \mathcal{L}$ such that $\varepsilon_{Q_0} < \varepsilon_N < \varepsilon_{Q_1}$, then $Q_0 \in Q_1$.

Proof. Clause (1) follows from the basic facts about elementary submodels in the preliminaries. Let us show clause (2). If there is no model $\overline{M} \in \mathcal{M} \cap \mathcal{T}^+$ such that $\varepsilon_{Q_0} < \varepsilon_{\overline{M}} < \varepsilon_{Q_1}$, the conclusion follows for the same reasons as clause (2) of proposition 2.3.14. Hence, suppose that there is some $\overline{M} \in \mathcal{M}$ such that $\varepsilon_{Q_0} < \varepsilon_{\overline{M}} < \varepsilon_{Q_1}$. By two applications of the shoulder axiom for $\mathcal{M}$ it can be further assumed that $Q_0 \in^* \overline{M}$ and that there is some $Q'_1 \in \mathcal{M}$ such that $\varepsilon_{Q'_1} = \varepsilon_{Q_1}$ and $\overline{M} \in^* Q'_1$. Since $\overline{M}$ needs to be preceded by large models and there are no large models in $\mathcal{M}$ of ω_2-height lying strictly between ε_{Q_0} and ε_{Q_1} by assumption, Q_0 has to be one of these predecessors. Hence, every $P \in \mathcal{M}$ such that $\varepsilon_{\overline{M}} < \varepsilon_P < \varepsilon_{Q_1}$ has to be a small model. Therefore, $Q_0 \in \overline{M} \in Q'_1$, and thus, as $\overline{M}$ is countable, by (1) we have that $\overline{M} \subseteq Q'_1$, which in turn implies that $Q_0 \in Q'_1$. Hence, $Q_0 \in Q'_1 \cap Q_1[\omega_1]$, and by clause (B.1), we can conclude that $Q_0 = \Psi_{Q'_1[\omega_1], Q_1[\omega_1]}(Q_0) \in Q_1$. □

Proposition 2.4.14. *Let $\mathcal{M}$ be a pre-$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, and let $Q_0, Q_1, Q'_1 \in \mathcal{M}$ such that $Q_0 \in^* Q_1$ and $\varepsilon_{Q_1} = \varepsilon_{Q'_1}$. Then the following holds:*

- *If $Q_1, Q'_1 \in \mathcal{S} \cup \mathcal{L}$, then $\Psi_{Q_1[\omega_1], Q'_1[\omega_1]}(Q_0) \in \mathcal{M}$.*
- *If $Q_1, Q'_1 \in \mathcal{T}^+$, then $\Psi_{Q_1, Q'_1}(Q_0) \in \mathcal{M}$.*

Proof. The first item is proven exactly as proposition 2.3.15, hence we may assume that $Q_1, Q'_1 \in \mathcal{T}^+$. We may further assume that $Q_0 \notin Q_1$, otherwise the conclusion follows directly from clause (E.2). Since non-elementary models in $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems need to be preceded by large models, there must be some large model N such that $N \in Q_1$ and $\varepsilon_{Q_0} < \varepsilon_N < \varepsilon_{Q_1}$. By two applications of the shoulder axiom, we can find $Q_2 \in \mathcal{M} \cap \mathcal{T}^+$ and $N_2 \in \mathcal{M} \cap Q_2$ such that $Q_0 \in N_2 \in Q_2$ and $\varepsilon_{Q_2} = \varepsilon_{Q_1}$. Since $Q_0 \in X_{Q_2, Q_1}$, the model Q_0 must be fixed by the isomorphism Ψ_{Q_2,Q_1} by clause (B.2), and hence, if we let $N = \Psi_{Q_2,Q_1}(N_2)$, we have $Q_0 \in N \in Q_1$. Let $N' = \Psi_{Q_1,Q_1'}(N)$, which is an element of $\mathcal{M}$ by clause (E.2). Then, by proposition 2.4.3 and clause (E.1), $\Psi_{Q_1,Q_1'}(Q_0) = \Psi_{N,N'}(Q_0) \in \mathcal{M}$. □

The following proposition is proven exactly as proposition 2.3.15, but using the last proposition instead of proposition 2.3.16.

Proposition 2.4.15. *Let $\mathcal{M}$ be a pre-$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, and let $Q_0, Q_1 \in \mathcal{M}$ such that $Q_0 \in^* Q_1$. If there is a model $P \in \mathcal{M}$ such that $\varepsilon_{Q_0} < \varepsilon_P < \varepsilon_{Q_1}$, then there is $R \in \mathcal{M}$ such that $\varepsilon_R = \varepsilon_P$ and $Q_0 \in^* R \in^* Q_1$.*

Proposition 2.4.16. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.*

(1) Let N_0, N_1 be two elementary submodels of $H(\kappa)$ of size $\aleph_1$ such that Ψ_{N_0,N_1} is the unique isomorphism between $(N_0; \in)$ and $(N_1; \in)$. If $\mathcal{M} \in N_0$, then $\Psi_{N_0,N_1}(\mathcal{M})$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.

(2) Let $\overline{M}_0, \overline{M}_1$ be two $\mathcal{L}$-symmetric systems such that $\Psi_{\overline{M}_0,\overline{M}_1}$ is the unique isomorphism between $(\bigcup \overline{M}_0; \in, N_0)_{N_0 \in \overline{M}_0}$ and $(\bigcup \overline{M}_1; \in, N_1)_{N_1 \in \overline{M}_1}$. If $\mathcal{M} \in \bigcup \overline{M}_0$, then $\Psi_{\overline{M}_0,\overline{M}_1}(\mathcal{M})$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.

The following result is a local form of the shoulder axiom, which holds for large models that are elements of non-elementary models, even if they are not members of the $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Although we don't need it here, the following form of symmetry will be crucially used in chapter 4, when dealing with finite support iterations with symmetric systems as side conditions.

Proposition 2.4.17. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Let $Q \in \mathcal{M}$, $\overline{M} \in \mathcal{M} \cap \mathcal{T}^+$, and $N \in \overline{M}$ (possibly such that $N \notin \mathcal{M}$) such that $\varepsilon_Q < \varepsilon_N$. Then, there is some $\overline{M}^* \in \mathcal{M} \cap \mathcal{T}^+$ and some $N^* \in \overline{M}^*$ such that $Q \in N^*$ and there is an isomorphism $\Psi_{N^*,N}$ between N^* and N that fixes $N^* \cap N$.*

Proof. We will show by induction that there is some $\overline{M}^+ \in \mathcal{M} \cap \mathcal{T}^+$ with the following properties:

- $\varepsilon_{\overline{M}^+} \leq \varepsilon_{\overline{M}}$.
- There is some $N' \in \overline{M}^+$ and an isomorphism $\Psi_{N',N}$ between N' and N that fixes $N' \cap N$.
- Either
  (a) $Q \in \overline{M}^+$, or
  (b) there is some $N^+ \in \mathcal{M} \cap \overline{M}^+$ such that $Q \in N^+$ and $\varepsilon_{N^+} \leq \varepsilon_N$.

Let us show first that this is enough to get the conclusion of the statement. If (a) holds, and hence, $Q \in \overline{M}^+$, since $\varepsilon_Q < \varepsilon_{N'} = \varepsilon_N$, by the shoulder axiom for $\overline{M}^+$ there must be some $N^* \in \overline{M}^+$ such that $Q \in N^*$ and $\varepsilon_{N^*} = \varepsilon_{N'}$. Otherwise, (b) must hold, and thus, there has to be some $N^+ \in \mathcal{M} \cap \overline{M}^+$ such that $Q \in N^+$ and $\varepsilon_{N^+} \leq \varepsilon_N$. If $\varepsilon_{N^+} = \varepsilon_N$, we let N^* be N^+ and we are done. If $\varepsilon_{N^+} < \varepsilon_N$, by the shoulder axiom for $\overline{M}^+$ there is some $N^* \in \overline{M}^+$ such that $N^+ \in N^*$ and $\varepsilon_{N^*} = \varepsilon_{N'} = \varepsilon_N$, and hence, as $Q \in N^+ \subseteq N^*$, we are done. The isomorphism between N^* and N is clear.

Let us start the induction. By the shoulder axiom for $\mathcal{M}$ there must be some $\overline{M}_0 \in \mathcal{M} \cap \mathcal{T}^+$ such that $\varepsilon_{\overline{M}_0} = \varepsilon_{\overline{M}}$ and $Q \in^* \overline{M}_0$. If $Q \in \overline{M}_0$, we let $\overline{M}^+$ be $\overline{M}_0$, and as (a) holds, the induction ends. Suppose that $Q \notin \overline{M}_0$. Then, since non-elementary models in $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems must be preceded by large models, there must be some $N_0 \in \mathcal{M} \cap \overline{M}_0$, which by proposition 2.4.15, we may assume that $Q \in N_0$. By the symmetry of $\mathcal{M}$ there is some $N_0' \in \overline{M}_0$ such that $\varepsilon_{N_0'} = \varepsilon_N$. If $\varepsilon_{N_0} \leq \varepsilon_N$, we let $\overline{M}^+ = \overline{M}_0$, $N^+ = N_0$, and $N' = N_0'$, and as (b) holds, the induction ends. Otherwise, $\varepsilon_{N_0} > \varepsilon_N$. In this case, by the symmetry of $\overline{M}_0$, there has to be some $N_0'' \in \overline{M}_0$ such that $N_0'' \in N_0$ and $\varepsilon_{N_0''} = \varepsilon_{N_0'} = \varepsilon_N$. Hence, $N_0 \cap \overline{M}_0 \in \mathcal{M}$ by clause (F.2). Note that $\varepsilon_Q < \varepsilon_{N_0''} = \varepsilon_N < \varepsilon_{N_0 \cap \overline{M}_0}$, as $N_0'' \in N_0 \cap \overline{M}_0$. Therefore, by the shoulder axiom for $\mathcal{M}$ there must be some $\overline{M}_1 \in \mathcal{M} \cap \mathcal{T}^+$ such that $Q \in^* \overline{M}_1$ and $\varepsilon_{\overline{M}_1} = \varepsilon_{N_0 \cap \overline{M}_0}$.

It should be clear that we can repeat the same argument from the last paragraph with respect to $\overline{M}_1$ instead of $\overline{M}_0$ and continue the induction. Since $\mathcal{M}$ is finite, this induction has to end in finitely many steps, and in the end there must be some $\overline{M}^+ \in \mathcal{M} \cap \mathcal{T}^+$ with the properties mentioned at the beginning of the proof. □

The proof of the next lemma is straightforward.

Lemma 2.4.18. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $N \in \mathcal{L}$ such that $\mathcal{M} \in N$. Then $\mathcal{M} \cup \{N\}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.*

Lemma 2.4.19. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $M \in \mathcal{S}$ such that $\mathcal{M} \in M$. Then, there exists an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{M}^*$ such that $\mathcal{M} \cup \{M\} \subseteq \mathcal{M}^*$.*

Proof. The argument from the proof of lemma 2.3.19 mostly shows that the set

$$\mathcal{M}^* = \mathcal{M} \cup \{M\} \cup \{N \cap M : N \in \mathcal{M} \cap \mathcal{L}\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Clauses (E) and (F) are easily seen to follow from the fact that $\mathcal{M}^*$ only adds small models to $\mathcal{M}$, using similar ideas to the ones used in the proof of lemma 2.3.19. □

Lemma 2.4.20. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $N \in \mathcal{M} \cap \mathcal{L}$. Then, $\mathcal{M} \cap N$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.*

Proof. It's a routine matter to check that $\mathcal{M} \cap N$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system using the same ideas from the proof of lemma 2.3.20, which translate easily to this situation. In particular, if $N' \in \mathcal{L}$ and $\overline{M} \in \mathcal{T}^+$ are models in $\mathcal{M} \cap N$ such that $N' \in \overline{M}$ and $N' \cap \overline{M} \neq \varnothing$, then it's clear that $N' \cap \overline{M} \in \mathcal{M} \cap N$. □

Towards the proof of the fact that the restriction of an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{M}$ to a small model M is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, we need to analyse the structure of $\mathcal{M} \cap M$ in the same way that we did for $(\mathcal{S}, \mathcal{L})$-symmetric systems, captured mostly in proposition 2.3.22.

If $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and $M \in \mathcal{M}$ is a small model, we will define the residue sequence of $\mathcal{M} \cap M$ exactly as we did for $(\mathcal{S}, \mathcal{L})$-symmetric systems. Let us recall that if $\mathcal{M} \cap \mathcal{L} \cap M$ is nonempty, we fix a maximal increasing $\in$-chain $\langle N_i : i \leq n \rangle$ of elements of $\mathcal{M} \cap \mathcal{L} \cap M$, and we denote ε_{N_i} by $\varepsilon_i^{\mathcal{M} \cap \mathcal{L} \cap M}$ and $\varepsilon_{N_i \cap M}$ by $\varepsilon_i^{\mathcal{M} \cap \mathcal{S} \cap M}$, or simply $\varepsilon_i^{\mathcal{L}}$ and $\varepsilon_i^{\mathcal{S}}$, respectively, if $\mathcal{M}$ and M are clear from the context, for every $i \leq n$. Then, $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ is called *the residue sequence of* $\mathcal{M} \cap M$.

It is worth noting that if $\overline{M} \in \mathcal{M} \cap M$ is a non-elementary model, it follows from the fact that non-elementary models in $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems need to be preceded by large models, that $\overline{M}$ must lie right above a large model whose ω_2-height equals $\varepsilon_i^{\mathcal{L}}$, for some $i \leq n$. To be more precise, $\varepsilon_{\overline{M}}$ must be greater than $\varepsilon_i^{\mathcal{L}}$, and there can't be any $P \in \mathcal{M}$ such that $\varepsilon_i^{\mathcal{L}} < \varepsilon_P < \varepsilon_{\overline{M}}$. Moreover, note that since $\overline{M} \in M$, all the models in $\overline{M}$ need to be elements of M by proposition 1.4.4.

The following proposition, which describes $\mathcal{M} \cap M$ in terms of the ω_2-heights of the models in $\mathcal{M}$, is proven exactly as proposition 2.3.22 with a few adjustments. We use propositions 2.4.13 and 2.4.15 instead of propositions 2.3.14 and 2.3.16, respectively, and we need to use proposition 2.4.2 in combination with proposition 1.4.13.

Proposition 2.4.21. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, let $M \in \mathcal{M} \cap \mathcal{S}$, and let $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ be the residue sequence of $\mathcal{M} \cap M$. Then, $Q \in \mathcal{M} \cap M$ if and only if $Q \in \mathcal{M} \cap M[\omega_1]$ and either,*

(1) $\varepsilon_Q \in [\varepsilon_n^{\mathcal{L}}, \varepsilon_M)$, *or*

(2) $\varepsilon_Q \in [\varepsilon_i^{\mathcal{L}}, \varepsilon_{i+1}^{\mathcal{S}})$ *and* $Q \in (Z_{i+1} \cap M)[\omega_1]$, *for some* $i < n$ *and some large model* $Z_{i+1} \in \mathcal{M} \cap M$ *such that* $\varepsilon_{Z_{i+1}} = \varepsilon_{i+1}^{\mathcal{L}}$, *or*

(3) $\varepsilon_Q < \varepsilon_0^{\mathcal{S}}$ *and* $Q \in (Z_0 \cap M)[\omega_1]$, *for some large model* $Z_0 \in \mathcal{M} \cap M$ *such that* $\varepsilon_{Z_0} = \varepsilon_0^{\mathcal{L}}$.

Lemma 2.4.22. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $M \in \mathcal{M} \cap \mathcal{S}$. Then, $\mathcal{M} \cap M$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.*

Proof. Similarly to the proof of lemma 2.4.20, checking that $\mathcal{M} \cap M$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system is a routine matter if we translate the ideas from the proof of lemma 2.3.23 to the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems. Again, if $N \in \mathcal{L}$ and $\overline{M} \in \mathcal{T}^+$ are models in $\mathcal{M} \cap M$ such that $N \in \overline{M}$ and $N \cap \overline{M} \neq \varnothing$, then clearly $N \cap \overline{M} \in \mathcal{M} \cap M$. □

2.4.4 Amalgamation lemmas

If $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, we let $\mathcal{M}^*$ be the set

$$\mathcal{M}^* = \{Q[\omega_1] : Q \in (\mathcal{S} \cup \mathcal{L}) \cap \mathcal{N}\} \cup \{\bigcup \overline{M} : \overline{M} \in \mathcal{T}^+ \cap \mathcal{M}\}.$$

Lemma 2.4.23. *Let $n < \omega$, let $\mathcal{M}_0, \ldots, \mathcal{M}_n$ be $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems and let $X_{i,j} = \bigcup \mathcal{M}_i^* \cap \bigcup \mathcal{M}_j^*$ for all $i, j \leq n$. Suppose that there are isomorphisms $\Psi_{i,j}$ between the pairs of structures $(\bigcup \mathcal{M}_i^*; \in, X_{i,j}, Q^i)_{Q^i \in \mathcal{M}_i}$ and $(\bigcup \mathcal{M}_j^*; \in, X_{i,j}, Q^j)_{Q^j \in \mathcal{M}_j}$ fixing $X_{i,j}$, for all $i, j \leq n$. Then, $\bigcup_{i \leq n} \mathcal{M}_i$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system.*

Proof. It's not too hard to check that $\bigcup_{i \leq n} \mathcal{M}_i$ satisfies all the clauses from definition 2.4.10 by translating the ideas from the proof of lemma 2.3.24 to the context of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems, except possibly for the following case of clause (F.2). Let N and $\overline{M}$ be a large and a non-elementary model, respectively, both in $\bigcup_{i \leq n} \mathcal{M}_i$, such that $N \in \overline{M}$ and $N \cap \overline{M} \neq \varnothing$. Let $i, j \leq n$ such that $N \in \mathcal{M}_i$ and $\overline{M} \in \mathcal{M}_j$. We need to show that $N \cap \overline{M} \in \bigcup_{i \leq n} \mathcal{M}_i$. Note that since $N \in \overline{M} \in \mathcal{M}_j$, we have that $N \in \bigcup \mathcal{M}_j^*$. Hence, N is fixed by the isomorphism $\Psi_{j,i}$. Let $\overline{M}' = \Psi_{j,i}(\overline{M})$, which is clearly an element of $\mathcal{M}_i$ such that $N \in \overline{M}'$. By proposition 2.4.8, we have $N \cap \overline{M}' = N \cap \overline{M}$. Hence, we can conclude that $N \cap \overline{M} \in \mathcal{M}_i$. □

Definition 2.4.24. Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $R \in \mathcal{M}$. Then, an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{V}$ is called a *two-size virtual* $(\mathcal{M}, R)$-*reflection* if it satisfies the following properties:

(VR.1) If $R \in \mathcal{S} \cup \mathcal{L}$, then $\mathcal{M} \cap R[\omega_1] \subseteq \mathcal{V} \subseteq R[\omega_1]$.

(VR.2) If $R \in \mathcal{T}^+$, then $\mathcal{M} \cap \bigcup R \subseteq \mathcal{V} \subseteq \bigcup R$.

(VR.3) If $R \in \mathcal{S} \cup \mathcal{T}^+$ and $V \in \mathcal{V}$ is such that $\varepsilon_V = \max\{\varepsilon_{V'} : V' \in \mathcal{V}\}$, then $V \in R$.

(VR.4) If $R \in \mathcal{S} \cup \mathcal{T}^+$, let $\varepsilon^+ = \max\{\varepsilon_N : N \in \mathcal{V} \cap \mathcal{L}\}$, and let ε^- be the ordinal $\min\{\varepsilon_N : N \in \mathcal{M} \cap \mathcal{L}, \varepsilon_N > \varepsilon_R\}$, in case it exists, otherwise let $\varepsilon^- = \max\{\varepsilon_Q : Q \in \mathcal{M}\} + 1$. Let $N \in \mathcal{V} \cap \mathcal{L}$ such that $\varepsilon_N = \varepsilon^+$. Then, the following hold:

- $N \cap R \in \mathcal{V}$.
- For every $\varepsilon \in \{\varepsilon_{M'} : M' \in \mathcal{M}, \varepsilon_{M'} \in (\varepsilon_R, \varepsilon^-)\}$, there is some small model $M' \in \mathcal{M}$ such that $R \in M'$, $\varepsilon_{M'} = \varepsilon$, and $N \cap M' \in \mathcal{V}$.

Proposition 2.4.25. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let R be a model in $\mathcal{M} \cap (\mathcal{S} \cup \mathcal{L})$. Let $\mathcal{V}$ be a two-size virtual $(\mathcal{M}, R)$-reflection. Then,*

$$\mathcal{U} = \{Q \in \mathcal{M} : \varepsilon_Q \geq \varepsilon_R\} \cup \bigcup \{\Psi_{R[\omega_1], R'[\omega_1]}``(\mathcal{V}) : R' \in \mathcal{M}, \varepsilon_{R'} = \varepsilon_R\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending $\mathcal{M}$ and $\mathcal{V}$.

Proof. As in the proof of proposition 2.3.26, the analogous result for $(\mathcal{S}, \mathcal{L})$-symmetric systems, it follows from proposition 2.4.16 and lemma 2.4.23, that

$$\bigcup \{\Psi_{R[\omega_1], R'[\omega_1]}``(\mathcal{V}) : R' \in \mathcal{M}, \varepsilon_{R'} = \varepsilon_R\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. We will denote this $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system by $\mathcal{V}^*$, and the set $\{Q \in \mathcal{N} : \varepsilon_Q \geq \varepsilon_R\}$ by $\mathcal{N}^*$. Note that all the models $Q \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_R$ belong to $\mathcal{V}^*$ by the same reasons as in the proof of proposition 2.3.26. Moreover, arguing exactly as in the proof of that proposition, it's straightforward to verify that $\mathcal{U}$ satisfies all the clauses from the definition of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, except for the following case of clause (F.2). Suppose that there are $N \in \mathcal{U} \cap \mathcal{L}$ and $\overline{M} \in \mathcal{U} \cap \mathcal{T}^+$ such that $N \in \overline{M}$ and $N \cap \overline{M} \neq \varnothing$. We want to show that $N \cap \overline{M} \in \mathcal{U}$. By similar considerations as in the proof of proposition 2.3.26, we may assume that $N \in \mathcal{V}^*$ and $\overline{M} \in \mathcal{N}^*$. If $R \in \mathcal{L}$, we let N_0 be a large model of $\mathcal{N}^*$ of minimal ω_2-height such that $N \in N_0 \in \overline{M}$, by appealing to proposition 2.4.15. Then, $N_0 \cap \overline{M} \in \mathcal{N}$, and by the minimality of N_0 and since R is a large model, $\varepsilon_{N_0 \cap \overline{M}} < \varepsilon_R$. Therefore, $N_0 \cap \overline{M} \in \mathcal{V}^*$ and $N \in N_0 \cap \overline{M}$. Hence, as $\mathcal{V}^*$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, $N \cap (N_0 \cap \overline{M}) = N \cap \overline{M} \in \mathcal{V}^*$. Assume now that $R \in \mathcal{S}$. Since non-elementary models need to be preceded by large models in $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems, there have to be large models $N_0 \in \mathcal{N}^*$ such that $N \in N_0 \in \overline{M}$, again by proposition 2.4.15. Note that ε_{N_0} must be strictly greater than ε_R. Hence, if we pick N_0 of minimal ω_2-height, then $N_0 \cap \overline{M} \in \mathcal{N}$ and $\varepsilon_{N_0 \cap \overline{M}} < \varepsilon_R$. Therefore, by the same reasons as above, $N \cap (N_0 \cap \overline{M}) = N \cap \overline{M} \in \mathcal{V}^*$. □

Proposition 2.4.26. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $\overline{M} \in \mathcal{M}$ be a non-elementary model. Let $\mathcal{V}$ be a two-size virtual $(\mathcal{M}, \overline{M})$-reflection. Then,*

$$\mathcal{U} = \{Q \in \mathcal{M} : \varepsilon_Q \geq \varepsilon_{\overline{M}}\} \cup \bigcup \{\Psi_{\overline{M}, \overline{M}'}``(\mathcal{V}) : \overline{M}' \in \mathcal{M}, \varepsilon_{\overline{M}'} = \varepsilon_{\overline{M}}\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending $\mathcal{M}$ and $\mathcal{V}$.

Proof. As in the proof of the last proposition, it follows from proposition 2.4.16 and lemma 2.4.23, that

$$\bigcup \{\Psi_{\overline{M}, \overline{M}'}``(\mathcal{V}) : \overline{M}' \in \mathcal{M}, \varepsilon_{\overline{M}'} = \varepsilon_{\overline{M}}\}$$

is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, which we denote by $\mathcal{V}^*$. Again, we can argue that all the models $Q \in \mathcal{N}$ such that $\varepsilon_Q < \varepsilon_{\overline{M}}$ belong to $\mathcal{V}^*$, and we will denote the set $\{Q \in \mathcal{N} : \varepsilon_Q \geq \varepsilon_{\overline{M}}\}$ by $\mathcal{N}^*$. The same argument from the proof of proposition

2.3.26 shows that $\mathcal{U}$ is a pre-$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Let us show that it is also closed under intersections. Let $N^* \in \mathcal{U} \cap \mathcal{L}$ and $M^* \in \mathcal{U} \cap (\mathcal{S} \cup \mathcal{T}^+)$ such that $N^* \in M^*$ and $N^* \cap M^* \neq \varnothing$. As usual, since $\mathcal{V}^*$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and $\mathcal{N}^* \subseteq \mathcal{N}$, we may assume that $N^* \in \mathcal{V}^*$ and $M^* \in \mathcal{N}^*$. Define ε^+ and ε^- as in definition 2.4.24. Note that by (VR.3), the models $V \in \mathcal{V}^*$ of maximal ω_2-height must be large models and elements of some $\overline{M}' \in \mathcal{N}^*$ such that $\varepsilon_{\overline{M}'} = \varepsilon_{\overline{M}}$. Hence, $\varepsilon^+ = \max\{\varepsilon_V : V \in \mathcal{V}^*\}$. Moreover, note that the models in $\mathcal{N}^*$ of ω_2-height in the interval $[\varepsilon_{\overline{M}}, \varepsilon^-)$ are either non-elementary, and have ω_2-height $\varepsilon_{\overline{M}}$, or countable elementary, and have ω_2-height in the interval $(\varepsilon_{\overline{M}}, \varepsilon^-)$. If there is some $N \in \mathcal{N}^* \cap \mathcal{L}$ such that $\varepsilon_N < \varepsilon_{M^*}$, let $N_0 \in \mathcal{N}^*$ be a model of minimal ω_2-height such that $N_0 \in \mathcal{L}$ and $N^* \in N_0 \in M^*$, using proposition 2.4.15. Note that $N^* \in N_0 \cap M^* \in \mathcal{N}$ and that $N^* \cap (N_0 \cap M^*) = N^* \cap M^*$. Hence, in this case it's enough to show that $N^* \cap (N_0 \cap M^*) \in \mathcal{U}$. Therefore, we may assume that there is no large model $N \in N^*$ such that $\varepsilon_N < \varepsilon_{M^*}$, which translates to $\varepsilon_{\overline{M}} \leq \varepsilon_{M^*} < \varepsilon^-$. Let $N^+ \in \mathcal{V}^*$ such that $N^* \in N^+ \in^* M^*$ and $\varepsilon_{N^+} = \varepsilon^+$, using proposition 2.4.15. Then, $N^+ \in M^*$ by the observations above. Therefore, if we show that $N^+ \cap M^* \in \mathcal{U}$, then in fact $N^+ \cap M^* \in \mathcal{V}^*$, and since $N^* \in N^+ \cap M^*$, we will have that $N^* \cap (N^+ \cap M^*) = N^* \cap M^* \in \mathcal{V}^*$. Therefore, we may assume that N^* is such that $\varepsilon_{N^*} = \varepsilon^+$. But note that, by (VR.4), for every $N' \in \mathcal{V}$ such that $\varepsilon_{N'} = \varepsilon^+$, $N' \cap M' \in \mathcal{V}$ for some $M' \in \mathcal{N}^*$ such that $\varepsilon_{M'} = \varepsilon_{M^*}$, and this is enough to get the conclusion $N^* \cap M^* \in \mathcal{U}$. Indeed, on one hand, $\Psi_{N', N^*}(N' \cap M') \in \mathcal{V}^*$ because $\mathcal{V}^*$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, and on the other hand, $\Psi_{N', N^*}(N' \cap M') = N^* \cap M^*$ by proposition 2.3.11, if $M^* \in \mathcal{S}$, and proposition 2.4.7, if $M^* \in \mathcal{T}^+$. □

Lemma 2.4.27. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $M \in \mathcal{M} \cap \mathcal{S}$. Let $\mathcal{W}$ be another $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system such that $\mathcal{M} \cap M \subseteq \mathcal{W} \subseteq M$. Then, there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{U}$ such that $\mathcal{M} \cup \mathcal{W} \subseteq \mathcal{U}$.*

Proof. The proof of this lemma is a translation, almost word by word, of the proof of lemma 2.3.27. We will just add a few words to clarify that the same construction works here.

The construction of $\mathcal{W}^*$, which can be summarised as the closure of a certain fragment of $(\mathcal{M} \cap M[\omega_1]) \cup \mathcal{W}$ under intersections, is exactly the same as the one for $(\mathcal{S}, \mathcal{L})$-symmetric systems. Let us recall some of the notation that we used. Let $\langle (\varepsilon_i^{\mathcal{S}}, \varepsilon_i^{\mathcal{L}}) : i \leq n \rangle$ be the residue sequence of $\mathcal{M} \cap M$ and fix an $\in$-increasing sequence of large models $\langle N_i : i \leq n \rangle$ such that $N_i \in \mathcal{M} \cap M$ and $\varepsilon_{N_i} = \varepsilon_i^{\mathcal{L}}$ for every $i \leq n$. Note that since $\mathcal{W} \subseteq M$, it follows from propositions 1.4.13 and 2.4.2 that for every $W \in \mathcal{W}$ either $\varepsilon_W < \varepsilon_0^{\mathcal{S}}$, or $\varepsilon_i^{\mathcal{L}} \leq \varepsilon_W < \varepsilon_{i+1}^{\mathcal{S}}$, or $\varepsilon_n^{\mathcal{L}} \leq \varepsilon_W < \varepsilon_M$, for all $i < n$. Moreover, note that if $\overline{M} \in \mathcal{W} \cap \mathcal{T}^+$, since $\overline{M} \in M$, every model $N \in \overline{M}$ must be a member of M. Therefore, by the same reason as above, for all $N \in \overline{M}$, either $\varepsilon_N < \varepsilon_0^{\mathcal{S}}$, or $\varepsilon_i^{\mathcal{L}} \leq \varepsilon_N < \varepsilon_{i+1}^{\mathcal{S}}$, or $\varepsilon_n^{\mathcal{L}} \leq \varepsilon_N < \varepsilon_M$, for all $i < n$. Let $W \in \mathcal{W} \cap \mathcal{L}$ such that $\varepsilon_W \neq \varepsilon_i^{\mathcal{L}}$ for all $i \leq n$. For every $X \in \mathcal{W}$ such that $\varepsilon_X = \varepsilon_W$, we define E_X and F_X exactly as we did in the proof of lemma 2.3.27. Recall that if $\varepsilon^* < \omega_2$ was the least ω_2-height of a large model in $\mathcal{M}$ such that $\varepsilon_i^{\mathcal{S}} < \varepsilon^*$, for some $i \leq n$, then E_X consisted of an $\in$-chain of models of ω_2-height in the interval $[\varepsilon_i^{\mathcal{S}}, \varepsilon^*)$, and the main point was that all those models were members of $\mathcal{M} \cap \mathcal{S}$. This is still the case here, since models from $\mathcal{T}^+$ need to be preceded by large models in a $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Hence, the definition of E_X and F_X as in the proof of lemma 2.3.27 makes sense here. Now, recall that we obtain the $\in$-increasing sequence of large models $\langle Z_j : j \leq m \rangle$ by adding an $\in$-increasing maximal sequence of models X as above to the sequence $\langle N_i : i \leq n \rangle$. We denote by $\langle (\bar{\varepsilon}_j^{\mathcal{S}}, \bar{\varepsilon}_j^{\mathcal{L}}) : j \leq m \rangle$ the sequence of pairs $(\varepsilon_{Z_j \cap M}, \varepsilon_{N_j})_{j \leq m}$, in componentwise increasing order. Lastly, we define $\mathcal{W}^*$ as the result of adding to $(\mathcal{M} \cap M[\omega_1]) \cup \mathcal{W}$ all the models in F_{Z_j}, for each $Z_j \in (\mathcal{W} \setminus \mathcal{M}) \cap \mathcal{L}$.

Recall that the idea of the proof of lemma 2.3.27 was to close a certain fragment of $(\mathcal{M} \cap M[\omega_1]) \cup \mathcal{W}$ under intersections by adding all the models from the sets of the form F_{Z_i} as above, and then close the resulting system $\mathcal{W}^*$ under isomorphisms. The exact same argument works here. Namely, we build by induction two sequences of two-size virtual reflections $(\mathcal{V}_i^{\mathcal{S}})_{i \leq m+1}$ and $(\mathcal{V}_i^{\mathcal{L}})_{i \leq m}$, which cover bigger and bigger initial segments of $\mathcal{W}^*$, and so that $\mathcal{V}_{m+1}^{\mathcal{S}}$ is a two-size virtual $(\mathcal{M}, M)$-reflection fully covering $\mathcal{W}^*$. Then, the amalgamation $\mathcal{U}$ of $\mathcal{V}^{\mathcal{S}}_{m+1}$ and $\mathcal{N}$ given by proposition 2.4.25 is the $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending $\mathcal{N}$ and $\mathcal{W}$ that we were looking for. □

Lemma 2.4.28. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $N \in \mathcal{M} \cap \mathcal{L}$. Let $\mathcal{W}$ be another $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system such that $\mathcal{M} \cap N \subseteq \mathcal{W} \subseteq N$. Then, there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{U}$ such that $\mathcal{M} \cup \mathcal{W} \subseteq \mathcal{U}$.*

Proof. It's obvious that $\mathcal{W}$ is a two-size virtual $(\mathcal{M}, N)$-reflection. Therefore, the amalgamation $\mathcal{U}$ of $\mathcal{M}$ and $\mathcal{W}$ given by proposition 2.4.25 is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending both $\mathcal{M}$ and $\mathcal{W}$. □

Lemma 2.4.29. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $\overline{M} \in \mathcal{M} \cap \mathcal{T}^+$. Then, for every $N \in \overline{M}$ there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{M}_N \supseteq \mathcal{M}$ such that $N \in \mathcal{M}_N$.*

Proof. Our argument is inspired by the proof of lemma 2.1.12, which is the analogous result for $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains. It's a good idea to keep it in mind throughout the proof, but specially in case 2. Let ε^* be the ω_2-height of the models $N^* \in \mathcal{M} \cap \overline{M}$ for which there is no $R \in \mathcal{M}$ such that $\varepsilon_{N^*} < \varepsilon_R < \varepsilon_{\overline{M}}$. We will divide the proof in three cases:

Case 1. Suppose that $\varepsilon_N = \varepsilon^*$. Let N^* be any model in $\mathcal{M} \cap \overline{M}$ such that $\varepsilon_{N^*} = \varepsilon^*$. Let $\mathcal{M}^\circ_N$ be the set

$$\bigcup \left\{ \{N'\} \cup (N' \cap \mathcal{M}) : N' \in \mathcal{M} \cap \overline{M}, \varepsilon_{N'} = \varepsilon^* \right\} \cup \{N\} \cup \Psi_{N^*,N}“(N^* \cap \mathcal{M}).$$

By propositions 2.4.20, 2.4.18, 2.4.16, and lemma 2.4.23, $\mathcal{M}^\circ_N$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. We claim that $\mathcal{M}^\circ_N$ is in fact a two-size virtual $(\mathcal{M}, \overline{M})$-reflection. Clauses (VR.2) and (VR.3) form definition 2.4.24 are clear. Let us check (VR.4). Let ε^+ and ε^- be defined as in the definition of two-size virtual reflection, and note that $\varepsilon^+ = \varepsilon^*$. Moreover, note that $N^* \cap \overline{M} \in N^* \cap \mathcal{M}$ and that for every $M \in \mathcal{M}$ such that $\varepsilon_{\overline{M}} < \varepsilon^-$ and $\overline{M} \in M$, then $N^* \cap M \in N^* \cap \mathcal{M}$.

Therefore, since $\mathcal{M}_N^\circ$ is an $(\mathcal{S},\mathcal{L},\mathcal{T}^+)$-symmetric system, $\Psi_{N^*,N}(N^*\cap\overline{M})\in\mathcal{M}_N^\circ$ and $\Psi_{N^*,N}(N^*\cap M)\in\mathcal{M}_N^\circ$, for every $M\in\mathcal{M}$ as above. Hence, in light of proposition 2.4.7, we can conclude that $N\cap\overline{M}\in\mathcal{M}_N^\circ$ and $N\cap M\in\mathcal{M}_N^\circ$, and thus, that $\mathcal{M}_N^\circ$ is a two-size virtual $(\mathcal{M},\overline{M})$-reflection, as we wanted. Now we simply let $\mathcal{M}_N$ be the amalgamation of $\mathcal{M}$ and $\mathcal{M}_N^\circ$ given by proposition 2.4.26.

Case 2. Suppose that $\varepsilon_N>\varepsilon^*$. Let $(N_j^*)_{j\leq m}$ be an enumeration of all the models $N^*\in\mathcal{M}\cap\overline{M}$ such that $\varepsilon_{N^*}=\varepsilon^*$. For all $j\leq m$, use the shoulder axiom for $\overline{M}$ to find $N_j\in\overline{M}$ such that $N_j^*\in N_j$ and $\varepsilon_{N_j}=\varepsilon_N$. If we show that there is an $(\mathcal{S},\mathcal{L},\mathcal{T}^+)$-symmetric system $\mathcal{M}^*$ extending $\mathcal{M}$ and containing N_j for all $j\leq m$, then the existence of $\mathcal{M}_N$ as in the statement of the lemma follows directly from case 1. Let ε^- be the least ω_2-height of any large model in $\mathcal{M}$ above $\overline{M}$, in case it exists. Otherwise, let $\varepsilon=\max\{\varepsilon_Q : Q\in\mathcal{M}\}+1$. Fix an $\in$-increasing sequence of countable elementary submodels $S_{\overline{M}}\subseteq\mathcal{M}$ such that

- $\{\varepsilon_M : M\in S_{\overline{M}}\}=\{\varepsilon_M : M\in\mathcal{M},\varepsilon_{\overline{M}}<\varepsilon_M<\varepsilon^-\}$, and
- $\overline{M}\in M$, for every $M\in S_{\overline{M}}$.

Our plan is to define a two-size virtual $(\mathcal{M},\overline{M})$-reflection $\mathcal{M}^\circ$ containing N_j, for every $j\leq m$, which then we will amalgamate with $\mathcal{M}$ to get $\mathcal{M}^*$. The idea behind the definition of $\mathcal{M}^\circ$ is to add all the models of the form N_j, $N_j\cap\overline{M}$ and $N_j\cap M$, for all $j\leq m$ and all $M\in S_{\overline{M}}$, mimicking the argument of the proof of lemma 2.1.12, and then closing the resulting system by the relevant isomorphisms. Let us define $\mathcal{M}^\circ$ step-by-step as follows:

(i) Let $\mathcal{M}_0^\circ$ be the result of adding N_j to $\bigcup\overline{M}\cap\mathcal{M}$, for all $j\leq m$.

(ii) Now, let $\mathcal{M}_1^\circ$ be the result of adding $N_j\cap\overline{M}$ and $N_j\cap M$, for all $j\leq m$ and all $M\in S_{\overline{M}}$, to $\mathcal{M}_0^\circ$.

(iii) Lastly, add all the models $\Psi_{N_{j_0},N_{j_1}}(Q)$, for every $j_0,j_1\leq m$ and every $Q\in N_{j_0}\cap\mathcal{M}_1^\circ$, to $\mathcal{M}_1^\circ$ and call the resulting set $\mathcal{M}^\circ$.

From the definition of $\mathcal{M}^\circ$ (specially (ii)), it's clear that in order to see that it is a two-size virtual $(\mathcal{M}, \overline{M})$-reflection, we only need to check that $\mathcal{M}^\circ$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Note that for any two $j_0, j_1 \leq m$, $N_{j_0} \cap \mathcal{M}^\circ$ and $N_{j_1} \cap \mathcal{M}^\circ$ are isomorphic. Therefore, in light of proposition 2.4.18 and lemma 2.4.23, in order to see that $\mathcal{M}^\circ$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system it's enough to check it for $N_j \cap \mathcal{M}^\circ$, for every $j \leq m$. It's almost straightforward to check that $N_j \cap \mathcal{M}^\circ$ equals

$$\bigcup_{i \leq m} \{\Psi_{N_i,N_j}(N_i^*)\} \cup \bigcup_{i \leq m} \Psi_{N_i,N_j}\text{"}(N_i^* \cap \mathcal{M}) \cup \{N_j \cap \overline{M}\} \cup \{N_j \cap M : M \in S_{\overline{M}}\}.$$

Let us make a couple observations about the structure of $N_j \cap \mathcal{M}^\circ$:

- Since the set $S_{\overline{M}}$ forms an $\in$-chain of countable elementary submodels, the set $\{N_j \cap \overline{M}\} \cup \{N_j \cap M : M \in S_{\overline{M}}\}$ also forms an $\in$-chain of models with minimal element $N_j \cap \overline{M}$. Let us call this set $\mathcal{M}_j^+$.
- For every $i \leq m$, $\Psi_{N_i,N_j}(N_i^*) \in N_j \cap \overline{M}$ by the symmetry of $\overline{M}$, which is an $\mathcal{L}$-symmetric system.
- By propositions 2.4.20 and 2.4.18, for any two $i_0, i_1 \leq m$, $\{N_{i_0}^*\} \cup \Psi_{N_{i_0},j}\text{"}(N_{i_0}^* \cap \mathcal{M})$ and $\{N_{i_1}^*\} \cup \Psi_{N_{i_1},j}\text{"}(N_{i_1}^* \cap \mathcal{M})$ are two isomorphic $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems. Therefore, by lemma 2.3.24,

  $$\bigcup_{i \leq m} \{\Psi_{N_i,N_j}(N_i^*)\} \cup \bigcup_{i \leq m} \Psi_{N_i,N_j}\text{"}(N_i^* \cap \mathcal{M})$$

  is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, which we will denote by $\mathcal{M}_j^-$. Also note that, by the last point, $\mathcal{M}_j^- \in \bigcup(N_j \cap \overline{M})$.

From these observations, it's obvious that $N_j \cap \mathcal{M}^\circ$ is a pre-$(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. In order to see that it is also closed under intersections let $N' \in \mathcal{M}_j^+ \cap \mathcal{L}$ and $N_j \cap M' \in \mathcal{M}_j^+ \cap (\mathcal{S} \cup \mathcal{T}^+)$ such that $N' \in N_j \cap M'$ and $N' \cap (N_j \cap M') \neq \varnothing$, where either $M' = \overline{M}$ or $M' \in S_{\overline{M}}$. But note that $N' \cap (N_j \cap M') = N' \cap M'$, and that $N' \cap M' \in \mathcal{M}$, because both N' and M' are elements of $\mathcal{M}$. Therefore,

$N_j \cap \mathcal{M}^\circ$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, as we wanted.

We have shown that $\mathcal{M}^\circ$ is a two-size virtual $(\mathcal{M}, \overline{M})$-reflection containing N_j, for all $j \leq m$. Hence, the amalgamation $\mathcal{M}^*$ of $\mathcal{M}$ and $\mathcal{M}^\circ$ given by proposition 2.4.26 is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending $\mathcal{M}$ and containing N_j, for every $j \leq m$. Therefore, as we have mentioned at the beginning of case 2, now the existence of $\mathcal{M}_N$ as in the statement of the lemma follows by applying case 1 to $\mathcal{M}^*$.

Case 3. Suppose that $\varepsilon_N < \varepsilon^*$. Let $\langle \varepsilon_i^* : i \leq n \rangle$ be the strictly decreasing enumeration of the set $\{\varepsilon_{N'} : N' \in \overline{M} \cap \mathcal{M}, \varepsilon_{N'} > \varepsilon_N\}$, where $\varepsilon_0^* = \varepsilon^*$. By the shoulder axiom for $\overline{M}$ there must be some $N_0 \in \overline{M}$ such that $N \in N_0$ and $\varepsilon_{N_0} = \varepsilon_0^*$. Apply case 1 to obtain an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system $\mathcal{M}_0$ extending $\mathcal{M}$ and such that $N_0 \in \mathcal{M}_0$ and $\{\varepsilon_Q : Q \in \mathcal{M}_0\} = \{\varepsilon_Q : Q \in \mathcal{M}\}$. Note that $N \in N_0 \cap \overline{M} \in \mathcal{M}_0$ and that there is some $N_1^* \in N_0 \cap \overline{M}$ such that $\varepsilon_{N_1^*} = \varepsilon_1^*$. It should be clear that we can repeat the argument with respect to $\mathcal{M}_0$, $N_0 \cap \overline{M}$, and ε_1^*, instead of $\mathcal{M}$, $\overline{M}$, and ε_0^*, respectively. Hence, we can build $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems $\mathcal{M}_i$ by induction on $i \leq n$ with the following properties:

(1) $\mathcal{M} \subseteq \mathcal{M}_0$ and $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$, for each $i < n$.

(2) $\{\varepsilon_Q : Q \in \mathcal{M}_i\} = \{\varepsilon_Q : Q \in \mathcal{M}\}$ for all $i \leq n$.

(3) There is some $N_n \in \overline{M} \cap \mathcal{M}_n$ such that $N \in N_n \cap \overline{M}$ and $\varepsilon_{N_n} = \varepsilon_n^*$.

(4) There is some $N_{n+1} \in (N_n \cap \overline{M}) \cap \mathcal{M}_n$ such that there is no $R \in \mathcal{M}_n$ such that $\varepsilon_{N_{n+1}} < \varepsilon_R < \varepsilon_{N_n \cap \overline{M}}$ and $\varepsilon_{N_{n+1}} \leq \varepsilon_N$.

Let us explain item (4). By the symmetry of $\overline{M}$, for every $N_n^* \in \overline{M} \cap \mathcal{M}$ such that $\varepsilon_{N_n^*} = \varepsilon_n^*$, there must be some $N' \in \overline{M}$ such that $N' \in N_n^*$ and $\varepsilon_{N'} = \varepsilon_N$. Therefore, $N_n^* \cap \overline{M} \neq \varnothing$, and by the closure of $\mathcal{M}$ under intersections, $N_n^* \cap \overline{M}$ must be an element of $\mathcal{M}$. Hence, as N' is a member of $N_n^* \cap \overline{M}$, and as non-elementary models must be preceded by large models in $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems, there has to be some large model $N_{n+1}^* \in (N_n^* \cap \overline{M}) \cap \mathcal{M}$ such that

$\varepsilon_{N^*_{n+1}} \leq \varepsilon_N$. We may even assume that N^*_{n+1} is an immediate predecessor of $N^*_n \cap \overline{M}$ in $\mathcal{M}$. The existence of N_{n+1} as in (4) follows from the existence of this model N^*_{n+1}.

But now we are done. If $\varepsilon_N = \varepsilon_{N_{n+1}}$, the conclusion follows from case 1, and if $\varepsilon_N > \varepsilon_{N_{n+1}}$, the conclusion follows from case 2, arguing with respect to $\mathcal{M}_n$, $N_n \cap \overline{M}$ and $\varepsilon_{N_{n+1}}$, instead of $\mathcal{M}$, $\overline{M}$ and ε^*, respectively. □

2.4.5 Preservation lemmas

Lemma 2.4.30. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Then, $\mathcal{M}$ is strongly $(Q, \mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+))$-generic for every $Q \in \mathcal{M} \cap (\mathcal{S} \cup \mathcal{L})$.*

Proof. Let $D \subseteq Q \cap \mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ be a dense subset. Let $\mathcal{M}^*$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system such that $\mathcal{M}^* \supseteq \mathcal{M}$. We need to find $\mathcal{W} \in D$ compatible with $\mathcal{M}^*$. It follows from lemmas 2.4.20 and 2.4.22 that $\mathcal{M}^* \cap Q$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system in Q. Since D is dense, there is some $\mathcal{W} \in D$ such that $\mathcal{W} \supseteq \mathcal{M}^* \cap Q$, and as $D \subseteq Q$, then $\mathcal{W} \in Q$. Hence, by lemmas 2.4.27 and 2.4.28, the $(\mathcal{S}, \mathcal{L})$-symmetric systems $\mathcal{M}^*$ and $\mathcal{W}$ are compatible. □

Lemma 2.4.31. *Let $\mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and let $\overline{M} \in \mathcal{M} \cap \mathcal{T}^+$. Then, $\mathcal{M}$ is strongly $(N, \mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+))$-generic for every $N \in \overline{M}$.*

Proof. Let $D \subseteq N \cap \mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ be a dense subset. Let $\mathcal{M}^* \supseteq \mathcal{M}$ be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. In light of lemma 2.4.29, we may assume that $N \in \mathcal{M}^*$. Hence, the result follows from exactly the same argument as in the proof of lemma 2.4.30. □

The following two theorems follow from lemmas 2.4.18, 2.4.19, and 2.4.30.

Theorem 2.4.32. *The forcing $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ is strongly $\mathcal{S}$-proper.*

Theorem 2.4.33. *The forcing $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ is strongly $\mathcal{L}$-proper.*

The following theorem follows from lemma 2.4.23, by the same argument as in the proof of lemma 2.3.33, the analogous result for $(\mathcal{S}, \mathcal{L})$-symmetric systems.

Theorem 2.4.34. *If* $2^{\aleph_1} = \aleph_2$ *holds, then* $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ *has the* $\aleph_3$*-Knaster condition.*

Theorem 2.4.35. *If* $2^{\aleph_1} = \aleph_2$ *holds, then* $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ *preserves all cardinals.*

The following theorem is proven exactly as theorem 2.3.35.

Theorem 2.4.36. *If* $2^{\aleph_1} = \aleph_2$ *holds, then* $\mathbb{M}(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$ *preserves* $2^{\aleph_1} = \aleph_2$.

3

Strong chains of subsets of ω_1

The entirety of this chapter is the result of a joint work between the author and his PhD advisor, David Asperó.

The spaces ${}^{\omega}2$, ${}^{\omega}\omega$ and $[\omega]^{\omega}$ have been a central topic of study in set theory since its beginnings. In fact, it is fair to say that a great majority of the developments in set theory were fuelled by the willingness of better understanding these spaces. Their importance comes from the fact that when equipped with the right topology, they are homeomorphic to $\mathbb{R}$ minus some countable subset. Therefore, they are in many regards equivalent to $\mathbb{R}$. It is even common practice in set theory to refer to the elements of these spaces as *reals*. Moreover, for many reasons which we won't include here, they are more suitable than $\mathbb{R}$ from the point of view of forcing theory. Of great importance is the area of *cardinal characteristics of the continuum*, which deals with the possible cardinalities of certain subsets of the the real line when the continuum is assumed to be larger than $\aleph_1$ (see [21] for a survey on the topic). In the past decades there has been an increasing interest for the spaces of *higher reals* ${}^{\lambda}2$, ${}^{\lambda}\lambda$ and $[\lambda]^{\lambda}$, where λ is an uncountable cardinal, and the *higher cardinal characteristics* associated to these spaces (see for example [41], [25], or [22]). What makes this topic so interesting is that, in many cases, it turns out that the classical theory doesn't generalise to the uncountable case as straightforwardly as one might expect. In fact, in some cases many results are only known to work at the level of some large cardinal. Hence, this area has motivated the development of many deep and powerful ideas in forcing theory, and moreover, it has given completely new characterisations of known large cardinal notions.

In the area of higher cardinal characteristics one usually considers spaces such as $[\lambda]^\lambda$ quotiented by the ideal of subsets of λ of size $< \lambda$. This fact is crucial when generalising the classical theory of cardinal characteristics to the uncountable. In fact, the space $[\lambda]^\lambda$ quotiented by the ideal of subsets of λ of size $< \mu$, for some cardinal $\mu < \lambda$, is much harder to deal with. Indeed, the known techniques coming from the area of cardinal characteristics don't seem to work in this context, so our understanding of this space is much more scarce. In this chapter we will focus on the space $[\omega_1]^{\omega_1}$ quotiented by the ideal of finite subsets of ω_1, but before we present our results, let us give a brief overview of the main results in this area.

Let λ be an infinite cardinal, let $\nu \leq \lambda$ be a cardinal, let $\mu \leq \lambda$ be a regular cardinal, and let δ be an infinite ordinal.

Definition 3.0.1. A *$(< \mu)$-strongly almost disjoint family of subsets of λ of length δ* is a sequence $\langle A_\alpha : \alpha < \delta \rangle$ of subsets of λ such that for all $\alpha < \beta < \delta$,

(1) $|A_\alpha| = \lambda$, and

(2) $|A_\alpha \cap A_\beta| < \mu$.

Baumgartner in 1978 showed that you can consistently have arbitrarily long strongly almost disjoint families of subsets of any infinite cardinal.

Theorem 3.0.2 ([17])**.** *If GCH holds, then there is a cardinal-preserving forcing notion $\mathbb{P}$ that forces the existence of a $(< \mu)$-strongly almost disjoint family of subsets of λ of length δ.*

The next big result in the area came with Zapletal's thesis, in which he showed that you can get a stronger result at the level of ω_1. Interestingly enough, the result was proven using a forcing with symmetric systems of models of one type as side conditions.

Definition 3.0.3. A $(< \mu)$-*strongly almost disjoint family of functions from* λ *to* ν *of length* δ is a sequence $\langle f_\alpha : \alpha < \delta \rangle$ of functions such that for all $\alpha < \beta < \delta$,

(1) $f_\alpha \in {}^\lambda\nu$, and

(2) $|\{\gamma \in \lambda : f_\alpha(\gamma) = f_\beta(\gamma)\}| < \mu$.

Theorem 3.0.4 ([91])**.** *If CH holds, then there is a cardinal-preserving forcing notion* $\mathbb{P}$ *that forces the existence of a* $(< \aleph_0)$*-strongly almost disjoint family of functions from* ω_1 *to* ω *of length* δ*.*

You can define an even stronger notion for families of functions, which will be the focus of this chapter.

Definition 3.0.5. A $(< \mu)$-*strong chain of functions from* λ *to* ν *of length* δ is a sequence $\langle f_\alpha : \alpha < \delta \rangle$ of functions such that for all $\alpha < \beta < \delta$,

(1) $f_\alpha \in {}^\lambda\nu$, and

(2) $|\{\gamma \in \lambda : f_\alpha(\gamma) \geq f_\beta(\gamma)\}| < \mu$.

Note that by letting $\nu = 2$ and identifying each subset of λ with its characteristic function, you can isolate the following strictly weaker notion.

Definition 3.0.6. A $(< \mu)$-*strong chain of subsets of* λ *of length* δ is a sequence $\langle X_\alpha : \alpha < \delta \rangle$ of subsets of λ such that for all $\alpha < \beta < \delta$,

(1) $|X_\beta \setminus X_\alpha| = \lambda$, and

(2) $|X_\alpha \setminus X_\beta| < \mu$.

From now on, when $\mu = \aleph_0$ we will omit the particle "$(< \aleph_0)$" from the names of the objects defined so far in this chapter.

It's clear that the existence of a strong chain of functions from λ to ν implies the existence of a strong chain of subsets of λ. Moreover, Baumgartner in [17] showed that the existence of a strong chain of subsets of λ implies the existence of a strongly almost disjoint family of subsets of λ. More precisely, if $\langle X_\alpha : \alpha < \delta \rangle$ is a $(< \mu)$-strong chain of subsets of λ and we define $A_\alpha := X_{\alpha+1} \setminus X_\alpha$, for every $\alpha < \delta$, then $\langle A_\alpha : \alpha < \delta \rangle$ is a $(< \mu)$-strongly almost disjoint family of subsets of λ of length δ.

It was known that strong chains on ω_1 of length ω_1 existed in ZFC. However, it was not known if longer strong chains could exist consistently. So Hajnal and Szentmiklóssy asked in [71] the natural questions about strong chains.

Question 3.0.7. *Is it consistent that there exists a strong chain of subsets of ω_1 of length ω_2?*

And the harder version of the question.

Question 3.0.8. *Is it consistent that there exists a strong chain of functions from ω_1 to ω_1 of length ω_2?*

Both questions were answered affirmatively by Koszmider in [42] and [43], respectively. In the later paper Koszmider introduced a cardinal-preserving forcing with side conditions organised along morasses. These side conditions capture the combinatorial content of the set $\{M \cap \omega_2 : M \preceq H(\omega_3), |M| = \aleph_0\}$ and, in some way, they seem to be closely related to symmetric systems. After the introduction of Neeman's side conditions of models of two types ([62], [63]), Veličković and Venturi [88] obtained a much simpler proof of Koszmider's result, by defining a forcing with $(\mathcal{S}, \mathcal{L})$-chains (see section 2.1).

On a completely different direction, Shelah and Inamdar proved some impossibility results about the existence of strong chains. First, Shelah [79] showed that, in particular, you cannot have strong chains of functions from ω_2 to ω_2 of length ω_3. Later, Inamdar [37] improved Shelah's result by showing that in fact you cannot have strong chains of subsets of ω_2 of length ω_3.

But this is of course not the end of the story, because we can still ask the following question.

Question 3.0.9. *How long can strong chains of functions from ω_1 to ω_1 be?*

The problem of finding strong chains of functions from ω_1 to ω_1 of length ω_3 is of course a very important question from the point of view of combinatorial set theory, but moreover, it has been a very important question in the area of forcing with side conditions. The reason is that it has been regarded as a test question for finding side conditions of models of three types. However, in this chapter we will show that side conditions of models of two types seem to be enough to answer this question. We will define a forcing with symmetric systems of models of two types as side condition to give a partial answer to the following easier question.

Question 3.0.10. *How long can strong chains of subsets of ω_1 be?*

The following theorem is the main result of this chapter.

Theorem 3.0.11. *If GCH holds, then there is a cardinal-preserving forcing notion $\mathbb{P}$ forcing the existence of a strong chain of subsets of ω_1 of length ω_3.*

We believe that a very mild modification of the forcing, using some ideas from [88], should allow us to get the same result for strong chains of functions from ω_1 to ω_1.

3.1 Definition of the forcing

Let us assume GCH holds throughout this chapter. We start out by fixing a sequence $\vec{e} = (e_\alpha : \alpha \in \omega_3)$ such that $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_3$. Throughout this chapter we will let $\kappa = \omega_3$ and $T = \vec{e}$. Hence, $\mathcal{S}$ is the collection of countable $M \preceq (H(\omega_3); \in, \vec{e})$. We will let $\mathcal{L}$ be the collection of all $N \preceq (H(\omega_3); \in, \vec{e})$ such that $|N| = \aleph_1$ and ${}^\omega N \subseteq N$.

The following standard fact, which is a particular case of proposition 2.2.4, will be crucially used in the proof of theorem 3.0.11.

Lemma 3.1.1. *For all countable $M_0, M_1 \preceq (H(\omega_3); \in, \vec{e})$, if $(M_0[\omega_1]; \in, M_0, \vec{e}) \cong (M_1[\omega_1]; \in, M_1, \vec{e})$, then $M_0 \cap M_1 \cap \omega_3$ is an initial segment of both $M_0 \cap \omega_3$ and $M_1 \cap \omega_3$.*

Proof. We will show that for every $\beta \in M_0 \cap M_1 \cap \omega_3$, if $\alpha \in M_0 \cap \beta$, then $\alpha \in M_1 \cap \beta$. Note that there is some $\xi \in M_0 \cap \omega_2$ such that $e_\beta(\xi) = \alpha$, and in fact, this is seen by $M_0[\omega_1]$. Therefore,

$$M_1[\omega_1] \models \Psi_{M_0[\omega_1], M_1[\omega_1]}(\xi) \in M_1 \cap \omega_2.$$

But note that since $\Psi_{M_0[\omega_1], M_1[\omega_1]}$ is the identity on $M_0[\omega_1] \cap \omega_2$, we have that $\Psi_{M_0[\omega_1], M_1[\omega_1]}(\xi) = \xi \in M_1 \cap \omega_2$. Hence, $\alpha = e_\beta(\xi) \in M_1$ as we wanted. □

Definition 3.1.2. Let $\mathcal{A}$ be a finite subset of $\mathcal{S}$, let $\nu \in \omega_1$, and let $\alpha, \beta \in \omega_3$. Then, $\alpha <_{\mathcal{A},\nu} \beta$ holds if and only if $\alpha < \beta$ and there are $M_0, \ldots, M_n \in \mathcal{A}$ and $\gamma_0 < \cdots < \gamma_{n-1} < \omega_3$ such that

(a) $\sup_{i \leq n} \delta_{M_i} \leq \nu$,

(b) $\alpha \in M_0$ and $\beta \in M_n$, and

(c) $\gamma_i \in M_i \cap M_{i+1} \cap (\alpha, \beta)$, for every $i < n$.

Note that the order $<_{\mathcal{A},\nu}$ is transitive. We will denote the set $\{M \in \mathcal{A} : \delta_M \leq \nu\}$ by $\mathcal{A}^\nu$, and if α and β are as in the definition above, we will say that they are *$\mathcal{A}^\nu$-connected through* $\{M_0, \ldots, M_n\}$.

The forcing $\mathbb{P}$ witnessing theorem 3.0.11 is defined as follows. Conditions in $\mathbb{P}$ are tuples $p = (\mathcal{N}_p, \mathcal{A}_p, a_p, d_p, u_p)$ such that:

(C1) $\mathcal{N}_p$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system.

(C2) $\mathcal{A}_p \subseteq \mathcal{N}_p \cap \mathcal{S}$ is such that $M \cap N \in \mathcal{A}_p$ for all $M \in \mathcal{A}_p$ and $N \in \mathcal{N}_p \cap \mathcal{L} \cap M$.

(C3) $a_p \in [\omega_3]^{<\omega}$.

(C4) $d_p \in [\omega_1]^{<\omega}$.

(C5) $u_p = (u_p^\alpha : \alpha \in a_p)$ and for each $\alpha \in a_p$, $u_p^\alpha : d_p \to 2$ is a function.

(C6) For every $\nu \in d_p$ and all $\alpha, \beta \in a_p$, if $\alpha <_{\mathcal{A}_p,\nu} \beta$, then $u_p^\alpha(\nu) \leq u_p^\beta(\nu)$.

Given $\mathbb{P}$-conditions p and q, q extends p (which we also denote by $q \leq p$) if and only if

(O1) $\mathcal{N}_q \supseteq \mathcal{N}_p$;

(O2) $\mathcal{A}_q \supseteq \mathcal{A}_p$;

(O3) $a_q \supseteq a_p$;

(O4) $d_q \supseteq d_p$;

(O5) for all $\alpha \in a_p$, $u_q^\alpha \supseteq u_p^\alpha$.

If p is a condition in $\mathbb{P}$, we will write $<_{p,\nu}$ instead of $<_{\mathcal{A}_p,\nu}$.

Let us give some intuition about the definition of the forcing $\mathbb{P}$. Recall that our goal is to force a strong chain $\langle X_\alpha : \alpha < \omega_3 \rangle$ of subsets of ω_1. Suppose that p is a condition in $\mathbb{P}$. The small models M in the distinguished set $\mathcal{A}_p$, which we will call the set of *active models* of p, are exactly the models for which we will show that p is $(M, \mathbb{P})$-generic. The fact that the set of active models is not closed under isomorphism, unlike the $(\mathcal{S}, \mathcal{L})$-symmetric system $\mathcal{N}_p$, is crucial in many proofs so that the arguments go through. For every $\alpha \in a_p$, the function $u_p^\alpha : d_p \to 2$ is a finite approximation of the characteristic function of the α-th set X_α of the strong chain that we want add generically. Therefore, d_p gives us an ordinal $\nu < \omega_1$, and u_p^α decides whether ν will be an element of X_α (i.e., $u_p^\alpha(\nu) = 1$), or it won't (i.e., $u_p^\alpha(\nu) = 0$).

Clause (C6) is the main ingredient of the definition. Let $\alpha, \beta \in a_p$ such that $\alpha < \beta$. Suppose that $\alpha, \beta \in M$, for some $M \in \mathcal{A}_p$. We want to promise that $u_p^\alpha(\nu) \leq u_p^\beta(\nu)$ for all $\nu \in d_p$ such that $\delta_M \leq \nu$. In other words, if u_p^α puts ν

in X_α, then u_p^β must put ν in X_β as well, for all $\nu < \omega_1$, save perhaps those ν that belong to M. However, the situation is a little bit more complicated. Let us momentarily denote by *naive-(C6)* this weak version of clause (C6). We need to anticipate some potential issues that may appear when amalgamating two different conditions $p, q \in \mathbb{P}$. Let $\nu \in d_p$ and suppose that $\alpha, \beta \in a_p$ are such that $\alpha < \beta$. Moreover, suppose that $\alpha \in M_0$ and $\beta \in M_1$ for some $M_0, M_1 \in \mathcal{A}_p^\nu$, but there is no $M \in \mathcal{A}_p^\nu$ such that $\alpha, \beta \in M$. In this case, clause naive-(C6) doesn't impose any requirement on the values of $u_p^\alpha(\nu)$ and $u_p^\beta(\nu)$. So it might very well happen that $u_p^\alpha(\nu) > u_p^\beta(\nu)$. Suppose now that $\gamma \in a_q$ such that $\alpha < \gamma < \beta$ and $\gamma \in M_0 \cap M_1$. Note that in this case it would be impossible to amalgamate p and q. Indeed, suppose that $r \in \mathbb{P}$ was a condition extending both p and q. Then, on one hand, $u_r^\alpha(\nu) = u_p^\alpha(\nu) > u_p^\beta(\nu) = u_r^\beta(\nu)$. But on the other hand, naive-(C6) would impose $u_r^\alpha(\nu) \leq u_r^\gamma(\nu) = u_q^\gamma(\nu) \leq u_r^\beta(\nu)$, which is impossible. The order $<_{\mathcal{A},\nu}$ is designed precisely to account for these potential issues that might appear in some of the proofs of the amalgamation lemmas, and clause (C6) is defined accordingly.

3.2 Density lemmas and basic properties

Lemma 3.2.1. *For every $p \in \mathbb{P}$ and every $\alpha \in \omega_3$ there is some $p^* \in \mathbb{P}$ extending p, and some $\alpha^* < \omega_3$, $\alpha^* > \alpha$, such that $\alpha^* \in a_{p^*}$.*

Proof. Let $\alpha^* < \omega_3$ such that $\alpha^* > \alpha$ and $\alpha^* > \sup(M \cap \omega_3)$ for each $M \in \mathcal{A}_p$. Hence, α^* is not $\mathcal{A}_p^\nu$-connected to any $\beta \in \mathcal{A}_p$, for any $\nu \in d_p$. For every $\nu \in d_p$, we let $u_{p^*}^{\alpha^*}(\nu) = 0$, and $u_{p^*}^\beta(\nu) = u_p^\beta(\nu)$ for each $\beta \in a_p$. Then,

$$p^* = (\mathcal{N}_p, \mathcal{A}_p, a_p \cup \{\alpha^*\}, d_p, (u_{p^*}^\beta : \beta \in a_p \cup \{\alpha^*\}))$$

is an extension of p in $\mathbb{P}$ as desired. □

Lemma 3.2.2. *For every $p \in \mathbb{P}$ and every $\nu \in \omega_1 \setminus d_p$, there exists some $p^* \in \mathbb{P}$*

such that $p^* \leq p$, $\nu \in d_{p^*}$, and for any $i \in \{0, 1\}$, $u^\alpha_{p^*}(\nu) = i$ for all $\alpha \in a_{p^*}$.

Lemma 3.2.3. *Let $p \in \mathbb{P}$, $\alpha \in a_p$, and $\nu < \omega_1$. For each $\beta \in a_p$ such that $\beta > \alpha$ there is some $p^* \in \mathbb{P}$, $p^* \leq p$, and some $\nu^* < \omega_1$ above ν such that $\nu^* \in d_{p^*}$, $u^\alpha_{p^*}(\nu^*) = 0$, and $u^\beta_{p^*}(\nu^*) = 1$.*

Proof. Let $\nu^* \in \omega_1 \setminus d_p$ be higher than both ν and δ_M for every $M \in \mathcal{A}_p$. Let us define $d_{p^*} = d_p \cup \{\nu^*\}$. For every $\gamma \in a_p$, we extend u^γ_p to a function $u^\gamma_{p^*}$ with domain d_{p^*}, such that $u^\gamma_{p^*}(\nu^*) = 0$ if $\gamma < \beta$, and $u^\gamma_{p^*}(\nu^*) = 1$ if $\gamma \geq \beta$. Then,

$$p^* = (\mathcal{N}_p, \mathcal{A}_p, a_p, d_{p^*}, (u^\gamma_{p^*} : \gamma \in a_p))$$

is an extension in $\mathbb{P}$ of p as desired. □

Lemma 3.2.4. *Let $Q \in \mathcal{S} \cup \mathcal{L}$ and let $p \in \mathbb{P} \cap Q$. Then there is a condition $q \in \mathbb{P}$ extending p and such that*

(1) $Q \in \mathcal{N}_q$ and

(2) $Q \in \mathcal{A}_q$ if Q is countable.

Proof. Suppose first that $Q \in \mathcal{S}$ and let $q = (\mathcal{N}_q, \mathcal{A}_q, a_p, d_p, u_p)$ be such that

$$\mathcal{N}_q = \mathcal{N}_p \cup \{Q\} \cup \{N \cap Q \,:\, N \in \mathcal{N}_p \cap \mathcal{L}\}$$

and

$$\mathcal{A}_q = \mathcal{A}_p \cup \{Q\} \cup \{N \cap Q \,:\, N \in \mathcal{N}_p \cap \mathcal{L}\}.$$

The only clause in the definition of $\mathbb{P}$ that we need to check is (C6), as clause (C1) follows from lemma 2.3.19, and the other ones are obvious. On one hand, note that since $p \in Q$, in particular $d_p \subseteq Q$, and hence, $\nu < \delta_Q$ for all $\nu \in d_p$. On the other hand, note that $\delta_Q = \delta_{N \cap Q}$ for all $N \in \mathcal{N}_p \cap \mathcal{L}$. Therefore, neither Q nor the models $N \cap Q$, where $N \in \mathcal{N}_p \cap \mathcal{L}$, add new $\mathcal{A}^\nu_q$-connections between the elements of a_p, for any $\nu \in d_p$. Hence, q satisfies clause (C6).

The proof for the uncountable case is very similar to the countable one, but uses lemma 2.3.18 instead of lemma 2.3.19, with the only change in the definition of q being that $\mathcal{N}_q$ is now $\mathcal{N}_p \cup \{Q\}$. □

Definition 3.2.5. Given a condition $p \in \mathbb{P}$ and a model $Q \in \mathcal{N}_p$, we define $p \restriction Q = (\mathcal{N}_{p \restriction Q}, \mathcal{A}_{p \restriction Q}, a_{p \restriction Q}, d_{p \restriction Q}, u_{p \restriction Q})$ by letting

(1) $\mathcal{N}_{p \restriction Q} = \mathcal{N}_p \cap Q$,

(2) $\mathcal{A}_{p \restriction Q} = \mathcal{A}_p \cap Q$,

(3) $a_{p \restriction Q} = a_p \cap Q$,

(4) $d_{p \restriction Q} = d_p \cap Q$, and

(5) $u^\alpha_{p \restriction Q}(\nu) = u^\alpha_p(\nu)$, for all $\alpha \in a_{p \restriction Q}$ and all $\nu \in d_{p \restriction Q}$,

and we call it *the restriction of p to Q* .

Lemma 3.2.6. *If $p \in \mathbb{P}$ and $Q \in \mathcal{N}_p$, then $p \restriction Q$ is a condition in $\mathbb{P} \cap Q$ such that $p \leq p \restriction Q$ if*

(1) $Q \in \mathcal{L}$ or

(2) $Q \in \mathcal{S}$ and $Q \in \mathcal{A}_p$.

Proof. We only prove the case when Q is countable and $Q \in \mathcal{A}_p$, as the uncountable case is straightforward. It is easily seen that $p \restriction Q$ is a condition in $\mathbb{P}$. Indeed, clause (C1) in the definition of $\mathbb{P}$ follows from lemma 2.3.23 and the other clauses are immediate (in the uncountable case we use lemma 2.3.20 instead of lemma 2.3.23). Since $p \restriction Q$ is finite, it clearly belongs to Q. Lastly, $p \leq p \restriction Q$ follows easily from the definition of $p \restriction Q$. □

3.3 $\mathcal{S}$-properness

Definition 3.3.1. Let $p \in \mathbb{P}$ and $M \in \mathcal{A}_p$. A condition $q \in \mathbb{P}$ is called an *$(\mathcal{S}, M)$-reflection of p* if it satisfies the following properties:

(1) $q \leq p \restriction M$.

(2) If $\mathcal{A}_p = \{M_0, \ldots, M_n\}$, then there are $M'_0, \ldots M'_n \in \mathcal{S}$ such that $\mathcal{A}_q = \{M'_0, \ldots, M'_n\}$ with the following properties:

(2.a) $M_i = M'_i$, for all $i \leq n$ such that $M_i \in M$.

(2.b) $M_i \cap M \cap \omega_3 = M'_i \cap M \cap \omega_3$, for all $i \leq n$ such that $\delta_{M_i} < \delta_M$.

(2.c) For each $\nu \in M \cap \omega_1$ and all $i \leq n$ such that $\delta_{M_i} \leq \nu$, $\delta_{M_i} = \delta_{M'_i}$.

(2.d) If $\alpha < \beta$ are in $M \cap \omega_3$, $\nu \in M \cap \omega_1$, and there is $A \subseteq \mathcal{A}_p^\nu$ such that α and β are $\mathcal{A}_p^\nu$-connected through A, then α and β are $\mathcal{A}_q^\nu$-connected through $A' = \{M'_i : M_i \in A\}$.

The following result is straightforward:

Lemma 3.3.2. *Let $p \in \mathbb{P}$ and $M \in \mathcal{A}_p$. Then, p is an $(\mathcal{S}, M)$-reflection of p.*

Lemma 3.3.3. *Let $p \in \mathbb{P}$ and let $M \in \mathcal{A}_p$. Let $q \in \mathbb{P} \cap M$ be an $(\mathcal{S}, M)$-reflection of p. Then there is a condition $r \in \mathbb{P}$ extending both p and q.*

Proof. Let $\mathcal{N}_r$ be the $(\mathcal{S}, \mathcal{L})$-symmetric system extending both $\mathcal{N}_p$ and $\mathcal{N}_q$ given by lemma 2.3.27. Let also

$$\mathcal{A}_r = \mathcal{A}_p \cup \mathcal{A}_q \cup \{M' \cap N \,:\, M' \in \mathcal{A}_p \cup \mathcal{A}_q,\ N \in \mathcal{N}_r \cap \mathcal{L} \cap M'\},$$

and let $a_r = a_p \cup a_q$ and $d_r = d_p \cup d_q$.

Fix some $\nu \in d_r$ and suppose that $a_r = \{\alpha_0, \ldots, \alpha_n\}$, where $\alpha_i < \alpha_{i+1}$ for each $i < n$. We will define $u_r^{\alpha_i}(\nu)$ by induction on $i \leq n$, while making sure that clause (C6) of the definition of $\mathbb{P}$ is satisfied.

Case 1. Suppose first that $\nu \in d_{p\restriction M}$. Then, for every $i \leq n$, we define $u_r^{\alpha_i}(\nu)$ as $u_q^{\alpha_i}(\nu)$ if $\alpha_i \in a_q$, and as $u_p^{\alpha_i}(\nu)$ if $\alpha_i \in a_p \setminus M$.

Case 2. Suppose now that $\nu \in d_q \setminus d_p$. Since r has to extend q, we will simply let $u_r^{\alpha_i}(\nu) = u_q^{\alpha_i}(\nu)$, for all $\alpha_i \in a_q$. Hence, we only need to define $u_r^{\alpha_i}(\nu)$ for $\alpha_i \in a_p \setminus M$. Assume first that $i = 0$. Then, we let $u_r^{\alpha_0} = u_q^{\alpha_0}(\nu)$ if $\alpha_0 \in a_q$, and $u_r^{\alpha_0} = 0$ if $\alpha_0 \in a_p \setminus M$. Now, let $k \leq n$ and assume that we have defined $u_r^{\alpha_i}(\nu)$ for every $i < k$, and that clause (C6) of the definition of $\mathbb{P}$ holds for every pair $\alpha_{i_0}, \alpha_{i_1} \in a_r$, where $i_0 < i_1 < k$. Namely, if $\alpha_{i_0} <_{r,\nu} \alpha_{i_1}$, then $u_r^{\alpha_{i_0}}(\nu) \leq u_r^{\alpha_{i_1}}(\nu)$. Moreover, if $\alpha_{i_1} \in a_p \setminus M$, we assume that $u_r^{\alpha_{i_1}}(\nu) = 0$ if α_{i_1} doesn't have any $<_{r,\nu}$-predecessor in a_r, and $u_r^{\alpha_{i_1}}(\nu) = u_r^{\alpha_{i_0}}(\nu)$ if α_{i_0} is the immediate $<_{r,\nu}$-predecessor of α_{i_1} in a_r. First, if α_k doesn't have any $<_{r,\nu}$-predecessor in a_r, then we define $u_r^{\alpha_k}(\nu)$ exactly as we did in the case $i = 0$. Otherwise, suppose that α_j is the greatest $<_{r,\nu}$-predecessor of α_k in a_r. If $\alpha_k \in a_p \setminus M$, we can simply let $u_r^{\alpha_k}(\nu) = u_r^{\alpha_j}(\nu)$. If $\alpha_k \in a_q$ we define $u_r^{\alpha_k}(\nu)$ as $u_q^{\alpha_k}(\nu)$, but we need to make sure that clause (C6) of the definition of $\mathbb{P}$ holds for $\{\alpha_0, \dots, \alpha_k\}$. Note that by the induction hypothesis and the transitivity of $<_{r,\nu}$, it's enough to check that $u_r^{\alpha_j}(\nu) \leq u_r^{\alpha_k}(\nu)$. If all the $<_{r,\nu}$-predecessors of α_k in a_r belong to $a_p \setminus M$, then $u_r^{\alpha_j}(\nu) = 0 \leq u_r^{\alpha_k}(\nu)$ and we are done. Otherwise, we let α_{j^*} be the greatest $<_{r,\nu}$-predecessor of α_j that belongs to a_q. Note that $\alpha_{j^*} <_{r,\nu} \alpha_k$ by the transitivity of $<_{r,\nu}$.

Claim 3.3.4. $\alpha_{j^*} <_{q,\nu} \alpha_k$.

Proof. Let $M_0, \dots, M_m \in \mathcal{A}_r^\nu$ and $\gamma_0 < \dots < \gamma_{m-1} < \omega_3$ such that $\alpha_{j^*} \in M_0$, $\alpha_k \in M_m$, and $\gamma_i \in M_i \cap M_{i+1} \cap (\alpha_{j^*}, \alpha_k)$ for each $i < m$. First of all note that if M_i is of the form $Q \cap N$, where $Q \in \mathcal{A}_p^\nu \cup \mathcal{A}_q^\nu$ and $N \in \mathcal{N}_r \cap \mathcal{L} \cap Q$, then $\delta_{M_i} = \delta_Q$ and $\gamma_i, \gamma_{i+1} \in M_i \subseteq Q$. Hence, α_{j^*} and α_k remain $\mathcal{A}_r^\nu$-connected if we substitute all the models M_i as above by Q. Thus, we may assume that $M_i \in \mathcal{A}_p^\nu \cup \mathcal{A}_q^\nu$ for all $i \leq m$. It is also worth noting that $\delta_{M_i} \leq \nu < \delta_M$ for every $i \leq m$, since $M_i \in \mathcal{A}_r^\nu$ and $\nu \in d_q$. We will show that there are $M_0', \dots, M_m' \in \mathcal{A}_q^\nu$ such that α_{j^*} and α_k are $\mathcal{A}_q^\nu$-connected through $\{M_0', \dots, M_m'\}$. If there is no $i \leq m$ such that

$M_i \in \mathcal{A}_q^\nu$, then $\alpha_{j^*} <_{\mathcal{A}_p,\nu} \alpha_k$, and by item (2) of definition 3.3.1, since $\alpha_{j^*}, \alpha_k \in a_q$ and $\nu \in d_q$, we have that $\alpha_{j^*} <_{q,\nu} \alpha_k$ and we are done. Otherwise, we let i_0 be the least $i \leq m$ for which $M_i \in \mathcal{A}_q^\nu$. If $i_0 = 0$, we let $M'_0 = M_0$. If $i_0 > 0$, we have that $\gamma_{i_0} \in M_{i_0} \subseteq M$, and thus, by item (2) of the definition of $(\mathcal{S}, M)$-reflection there are $M'_0, \ldots, M'_{i_0-1} \in \mathcal{A}_q^\nu$ such that α_{j^*} and γ_{i_0} are $\mathcal{A}_q^\nu$-connected through $\{M'_0, \ldots, M'_{i_0-1}\}$. Moreover, we let $M'_{i_0} = M_{i_0}$. Now, if there is no $i \leq m$ such that $i_0 < i$ and $M_i \in \mathcal{A}_q^\nu$, we can apply item (2) of definition 3.3.1 again to show that γ_{i_0+1} and α_k are $\mathcal{A}_q^\nu$-connected. Otherwise, we let i_1 be the least $i \leq m$ such that $i > i_0$ and $M_i \in \mathcal{A}_q^\nu$. By the same argument as above, using item (2) of the definition of $(\mathcal{S}, M)$-reflection, we can prove that γ_{i_0+1} and γ_{i_1} are $\mathcal{A}_q^\nu$-connected. At this point it should be clear that we can repeat this argument, which will finish eventually because $\alpha_k \in M \cap \omega_3$, and which will give us $M'_0, \ldots, M'_m \in \mathcal{A}_q^\nu$ such that α_{j^*} and α_k are $\mathcal{A}_q^\nu$-connected through $\{M'_0, \ldots, M'_m\}$, as we wanted. □

It follows that $u_r^{\alpha_{j^*}}(\nu) = u_q^{\alpha_{j^*}}(\nu) \leq u_q^{\alpha_k}(\nu) = u_r^{\alpha_k}(\nu)$ by clause (C6) applied to q. Hence, we will be done if $j^* = j$. If $j^* < j$, note that $\alpha_l \in a_p \setminus M$ for every $l < j$ such that $\alpha_{j^*} <_{r,\nu} \alpha_l <_{r,\nu} \alpha_j$, so $u_r^{j^*}(\nu) = u_r^{\alpha_l}(\nu) = u_r^{\alpha_j}(\nu)$, by induction hypothesis. Therefore, we can conclude that $u_r^{\alpha_j}(\nu) \leq u_r^{\alpha_k}(\nu)$.

Case 3. Finally, suppose that $\nu \in d_p \setminus d_q = d_p \setminus M$. Since r has to extend p, we will let $u_r^{\alpha_i}(\nu) = u_p^{\alpha_i}(\nu)$, for all $\alpha_i \in a_p$. Hence, we only need to define $u_r^{\alpha_i}(\nu)$ for $\alpha_i \in a_q \setminus a_p$. Assume first that $i = 0$. Then we let $u_r^{\alpha_0}(\nu) = u_p^{\alpha_0}(\nu)$ if $\alpha_0 \in a_p$, and $u_r^{\alpha_0}(\nu) = 0$ if $\alpha_0 \in a_q \setminus a_p$. As in the last case, let $k \leq n$ and assume that we have defined $u_r^{\alpha_i}(\nu)$ for every $i < k$, and that $u_r^{\alpha_{i_0}}(\nu) \leq u_r^{\alpha_{i_1}}(\nu)$ for all $i_0 < i_1 < k$ such that $\alpha_{i_0} <_{r,\nu} \alpha_{i_1}$. Moreover, if $\alpha_{i_1} \in a_q \setminus a_p$, assume that $u_r^{\alpha_{i_1}}(\nu) = 0$ if α_{i_1} doesn't have any $<_{r,\nu}$-predecessor in a_r, and $u_r^{\alpha_{i_1}}(\nu) = u_r^{\alpha_{i_0}}(\nu)$ if α_{i_0} is the immediate $<_{r,\nu}$-predecessor of α_{i_1} in a_r. First, if α_k doesn't have any $<_{r,\nu}$-predecessors in a_r, we define $u_r^{\alpha_k}$ in the same way as we did in the case $i = 0$. Otherwise, we let α_j be the greatest $<_{r,\nu}$-predecessor of α_k in a_r. If $\alpha_k \in a_q \setminus a_p$, we let $u_r^{\alpha_k}(\nu) = u_r^{\alpha_j}(\nu)$, and if $\alpha_k \in a_p$, we let $u_r^{\alpha_k}(\nu) = u_p^{\alpha_k}(\nu)$. In this second case we need to make sure that clause (C6) holds, and again, by induction hypothesis, this simply means that we need to check that $u_r^{\alpha_j}(\nu) \leq u_r^{\alpha_k}(\nu)$. If all the $<_{r,\nu}$-predecessors of α_k in a_r belong to $a_q \setminus a_p$, then $u_r^{\alpha_j}(\nu) = 0 \leq u_r^{\alpha_k}(\nu)$ and we are done. Otherwise, there is a greatest $<_{r,\nu}$-predecessor α_{j^*} of α_j in a_p. By the transitivity of $<_{r,\nu}$, we have that $\alpha_{j^*} <_{r,\nu} \alpha_k$. Suppose that α_{j^*} and α_k are $\mathcal{A}_r^\nu$-connected through $M_0, \ldots, M_m$, for some $m < \omega$. Note that for every $i \leq m$, if $M_i \in \mathcal{A}_q^\nu$, or if M_i is of the form $M_i^* \cap N$ for some $M_i^* \in \mathcal{A}_q^\nu$ and some $N \in \mathcal{N} \cap \mathcal{L} \cap M_i^*$, then $M_i \in M$, and thus $M_i \subseteq M$. Since $M \in \mathcal{A}_p^\nu$, if we substitute M_i by M, α_{j^*} and α_k remain $\mathcal{A}_r^\nu$-connected. Therefore, $\alpha_{j^*} <_{p,\nu} \alpha_k$, and by clause (C6) applied to p, we have that $u_r^{\alpha_{j^*}}(\nu) \leq u_r^{\alpha_k}(\nu)$. By the same reason as in case 2, $u_r^{\alpha_{j^*}}(\nu) = u_r^{\alpha_j}(\nu)$, and hence, we can conclude that $u_r^{\alpha_j}(\nu) \leq u_r^{\alpha_k}(\nu)$, as we wanted.

It should be clear from the way we have defined each $u_r^{\alpha_i}$ that $r = (\mathcal{N}_r, \mathcal{A}_r, a_r, d_r, (u_r^\alpha : \alpha \in a_r))$ is a condition in $\mathbb{P}$ extending both p and q. □

Lemma 3.3.5. *The forcing $\mathbb{P}$ is $\mathcal{S}$-proper.*

Proof. Let $p \in \mathbb{P}$ and let M^* be a countable elementary submodel of some large $H(\theta)$ containing p and $\mathbb{P}$. Let $M = M^* \cap H(\omega_3)$. By lemma 3.2.4, we may find a condition p' extending p and such that $M \in \mathcal{A}_{p'}$. We will show that p' is $(M^*, \mathbb{P})$-generic. Let $D \in M^*$ be a dense subset of $\mathbb{P}$, and let $p^* \in D$ be such that $p^* \leq p'$. Note that, in light of lemma 3.3.3, it's enough to find an $(\mathcal{S}, M)$-reflection q of p^* such that $q \in D \cap M^*$. Since p^* is an $(\mathcal{S}, M)$-reflection of itself by lemma 3.3.2, it's enough to check that all the parameters in the definition of $(\mathcal{S}, M)$-reflection are in M^* so that we can find a condition q as above by elementarity. Item (1) of definition 3.3.1 has parameters in M by lemma 3.2.6. Items (2.a), (2.c) and (2.d) are clear. The following claim shows item (2.b) and finishes the proof of the lemma.

Claim 3.3.6. *$M^* \cap M \cap \omega_3 \in M$ for every $M^* \in \mathcal{N}_{p^*} \cap \mathcal{S}$ such that $\delta_{M^*} < \delta_M$.*

Proof. We will use proposition 2.3.16 repeatedly throughout the proof of the claim without mention. Assume that $\varepsilon_{M^*} > \varepsilon_M$. The case $\varepsilon_{M^*} \leq \varepsilon_M$ is proven in the exact same way. We will find, by induction, two small models $\overline{M}$ and $\overline{M}^*$ in $\mathcal{N}_p$ such that $M \cap M^* \cap \omega_3 = \overline{M} \cap \overline{M}^* \cap \omega_3$ and that $\overline{M} \cap \overline{M}^* \cap \omega_3 \in M$. The model $\overline{M}^*$ will be of the form $N^* \cap M^*$ for some large model $N^* \in \mathcal{N}_p \cap M^*$, and the model $\overline{M}$ will be an isomorphic copy of $\overline{M}^*$ such that $\overline{M} \in M$. We will need the following result.

Subclaim 3.3.7. *Let $M_0, M_1 \in \mathcal{N}_{p^*} \cap \mathcal{S}$ such that $M_0 \in M_1[\omega_1] \setminus M_1$. Then, there is $N \in \mathcal{N}_{p^*} \cap \mathcal{L}$ such that the following hold:*

(1) $N \in M_1$ and $\varepsilon_{M_0} < \varepsilon_N$.

(2) $M_0 \cap M_1 \subseteq N$.

(3) One of the following two items holds:

- *$\varepsilon_{M_0} < \varepsilon_{N \cap M_1}$ and there is no large model $N' \in \mathcal{N}_{p^*}$ such that $\varepsilon_{M_0} < \varepsilon_{N'} < \varepsilon_{N \cap M_1}$, or*
- *$\varepsilon_{N \cap M_1} < \varepsilon_{M_0} < \varepsilon_N$.*

Proof. We will find models $N_0, \dots, N_n \in \mathcal{N}_{p^*} \cap \mathcal{L}$ by induction such that the following hold:

(i) $N_n \in N_{n-1} \cap M_1 \in \cdots \in N_1 \in N_0 \cap M_1 \in N_0 \in M_1$ and $\varepsilon_{M_0} < \varepsilon_{N_n}$.

(ii) $M_0 \cap M_1 \subseteq N_i$, for all $i \leq n$.

(iii) One of the two following items holds:

- $\varepsilon_{M_0} < \varepsilon_{N_n \cap M_1}$ and there is no large model $N \in \mathcal{N}_{p^*}$ such that $\varepsilon_{M_0} < \varepsilon_N < \varepsilon_{N_n \cap M_1}$, or
- $\varepsilon_{N_n \cap M_1} < \varepsilon_{M_0} < \varepsilon_{N_n}$.

We will be done by letting N be N_n. Since $M_0 \in M_1[\omega_1] \setminus M_1$, there has to be a large model $N_0 \in \mathcal{N}_{p^*}$ of minimal ω_2-height such that $M_0 \in N_0 \in M_1$. Therefore, $N_0 \cap M_1 \in \mathcal{N}_{p^*}$. Suppose that we have found $N_0, \dots, N_i \in \mathcal{N}_{p^*} \cap \mathcal{L}$ as above, for some $i < n$. Moreover, suppose that $\varepsilon_{M_0} < \varepsilon_{N_i \cap M_1}$ and that there are large models $N' \in \mathcal{N}_{p^*}$ such that $\varepsilon_{M_0} < \varepsilon_{N'} < \varepsilon_{N_i \cap M_1}$. Let $P_i \in \mathcal{N}_{p^*}$ such that $M_0 \in P_i[\omega_1]$, $P_i \in N_i$ and $\varepsilon_{P_i} = \varepsilon_{N_i \cap M_1}$, and let $N'_{i+1} \in \mathcal{N}_{p^*}$ be a large model of minimal ω_2-height such that $M_0 \in N'_{i+1} \in P_i$. Define N_{i+1} as the image of N'_{i+1} under the isomorphism $\Psi_{P_i[\omega_1],(N_i \cap M)[\omega_1]}$. Note that this induction must end in finitely many steps, and hence, there has to be a model N_n as above. Clause (i) clearly holds. We show clause (ii) by induction. Let $x \in M_0 \cap M_1$. Note that $x \in N_0$, because $M_0 \in N_0$, so $x \in N_0 \cap M_1$. Suppose that $x \in N_i \cap M_1$ for some $i < n$. Note that as $M_0 \in P_i[\omega_1]$, then $x \in P_i[\omega_1] \cap (N_i \cap M_1)$, and hence, $x = \Psi_{P_i[\omega_1],(N_i \cap M_1)[\omega_1]}(x)$. Therefore, as $x \in N'_{i+1}$, because $M_0 \in N'_{i+1}$, we have that $x \in N_{i+1}$. Finally, clause (iii) follows from the fact that the induction must end in finitely many steps. □

Note that if $\delta_{M_1} < \delta_{M_0}$, then the second item in (3) of the last subclaim must hold. Let $M^+ \in \mathcal{N}_{p^*}$ such that $M \in M^+[\omega_1]$ and $\varepsilon_{M^+} = \varepsilon_{M^*}$, by the shoulder axiom for $\mathcal{N}_p$. Note that $M \notin M^+$, otherwise we would contradict $\delta_{M^+} = \delta_{M^*} < \delta_M$. Hence, by subclaim 3.3.7 there is $N_0^+ \in \mathcal{N}_{p^*} \cap \mathcal{L}$ such that

- $N_0^+ \in M^+$,
- $M \cap M^+ \subseteq N_0^+$, and
- $\varepsilon_{N_0^+ \cap M^+} < \varepsilon_M < \varepsilon_{N_0^+}$.

We claim that $M \cap M^* \subseteq \Psi_{M^+[\omega_1],M^*[\omega_1]}(N_0^+)$. Let $x \in M \cap M^*$, and note that as $M \in M^+[\omega_1]$, the isomorphism $\Psi_{M^+[\omega_1],M^*[\omega_1]}$ fixes x. Therefore, $x \in M^+$, and since $M \cap M^+ \subseteq N_0^+$, then $x \in N_0^+$. But then $x \in \Psi_{M^+[\omega_1],M^*[\omega_1]}(N_0^+)$, again because x is fixed by the isomorphism $\Psi_{M^+[\omega_1],M^*[\omega_1]}$. Denote $\Psi_{M^+[\omega_1],M^*[\omega_1]}(N_0^+)$ by N_0^*, and note that $N_0^* \in M^*$, so $N_0^* \cap M^* \in \mathcal{N}_{p^*}$.

Now, let $M' \in \mathcal{N}_{p^*}$ such that $\varepsilon_{M'} = \varepsilon_M$ and $N_0^* \cap M^* \in M'[\omega_1]$, by the shoulder axiom for $\mathcal{N}_p$. If $N_0^* \cap M^* \in M'$, we let $\overline{M}^*$ and $\overline{M}$ be $N_0^* \cap M^*$ and $\Psi_{M'[\omega_1],M[\omega_1]}(\overline{M}^*)$, respectively. Otherwise, by subclaim 3.3.7 there must be some $N_0' \in \mathcal{N}_{p^*} \cap \mathcal{L}$ such that

- $N_0' \in M'$,
- $\varepsilon_{N_0^* \cap M^*} < \varepsilon_{N_0'}$,
- $(N_0^* \cap M^*) \cap M' \subseteq N_0'$, and
- N_0' must satisfy one of the two items of clause (3) from the statement of subclaim 3.3.7.

Let N_0 be the image of N_0' under the isomorphism $\Psi_{M'[\omega_1],M[\omega_1]}(N_0')$. Note that $M \cap M^* \subseteq N_0$. Indeed, if $x \in M \cap M^*$, we have seen that $x \in N_0^*$. Moreover, since x is fixed by the isomorphism $\Psi_{M[\omega_1],M'[\omega_1]}$, because $x \in N_0^* \cap M^* \subseteq M'[\omega_1]$ and $x \in M$, then x must be a member of M'. Consequently, x must be an element of $(N_0^* \cap M^*) \cap M' \subseteq N_0'$ as well. Now, if the first item of clause (3) of subclaim 3.3.7 for N_0' holds, we let $M'' \in \mathcal{N}_{p^*}$ such that $\varepsilon_{M''} = \varepsilon_{N_0 \cap M}$ and $N_0^* \cap M^* \in M''[\omega_1]$, by the shoulder axiom for $\mathcal{N}_p$. Note that since there are no large models of ω_2-height in the interval $(\varepsilon_{N_0^* \cap M^*}, \varepsilon_{M''})$, we have that $N_0^* \cap M^* \in M''$. In this case we let $\overline{M}^*$ and $\overline{M}$ be the models $N_0^* \cap M^*$ and $\Psi_{M''[\omega_1],(N_0 \cap M)[\omega_1]}(\overline{M}^*)$, respectively. If the second item of clause (3) of subclaim 3.3.7 holds, we can repeat the same argument as above with $N_0^* \cap M^*$ and $N_0 \cap M$ instead of M^* and M, respectively.

Note that if we keep repeating this argument, at some point the first item of clause (3) of subclaim 3.3.7 will have to hold. This means that there will be some $N^*, N \in \mathcal{N}_{p^*} \cap \mathcal{L}$ such that

- $N^* \in M^*$,
- $N \in M$,
- $M \cap M^* \subseteq N \cap N^*$,
- $\varepsilon_{N^* \cap M^*} < \varepsilon_{N \cap M}$, and
- there are no large models $N' \in \mathcal{N}_{p^*}$ such that $\varepsilon_{N^* \cap M^*} < \varepsilon_{N'} < \varepsilon_{N \cap M}$.

Let $M' \in \mathcal{N}_{p^*}$ such that $N^* \cap M^* \in M'[\omega_1]$ and $\varepsilon_{M'} = \varepsilon_{N \cap M}$, by the shoulder axiom for $\mathcal{N}_p$. Note that, by proposition 2.3.14, $N^* \cap M^* \in M'$. In this case we define $\overline{M}^*$ as $N^* \cap M^*$ and $\overline{M}$ as $\Psi_{M'[\omega_1],(N \cap M)[\omega_1]}(\overline{M}^*)$.

Independently from the number of iterations of the argument above, $\overline{M}^*$ and $\overline{M}$ have the following properties:

(a) $M \cap M^* \subseteq \overline{M}^*$,

(b) $\overline{M}^* \subseteq M^*$, and

(c) $\overline{M} \in M$.

It's not too hard to see from the way we have defined $\overline{M}$, that $M \cap M^* \subseteq \overline{M}$ too. If we combine this with (b) and (c) we have that $M \cap M^* = \overline{M} \cap \overline{M}^*$, and in particular $M \cap M^* \cap \omega_3 = \overline{M} \cap \overline{M}^* \cap \omega_3$. Note that $\overline{M} \cap \overline{M}^* \cap \omega_3$ is an initial segment of $\overline{M} \cap \omega_3$ by fact 3.1.1. Hence, as $\overline{M} \cap \omega_3 \in M$, we also have $\overline{M} \cap \overline{M}^* \cap \omega_3 \in M$. Therefore, $M \cap M^* \cap \omega_3 \in M$, as we wanted.

The case $\varepsilon_{M^*} \leq \varepsilon_M$ is proven in the exact same way. □

□

Remark 3.3.8. Lemma 3.3.5, which crucially depends on claim 3.3.6, which in turn depends on fact 3.1.1, is the only lemma in the proof of theorem 3.0.11 which does not go through if we try to force the same object consisting of partial functions indexed by ordinals not in ω_3 but on ω_4 (or anything higher, of course). The reason is that the version of claim 3.3.6 in this situation (i.e., replacing ω_3 with ω_4) does not hold.

3.4 $\mathcal{L}$-properness

Definition 3.4.1. Let $p \in \mathbb{P}$ and $N \in \mathcal{N}_p \cap \mathcal{L}$. A condition $q \in \mathbb{P}$ is called an *$(\mathcal{L}, N)$-reflection of p* if it satisfies the following properties:

(1) $q \leq p \upharpoonright N$.

(2) $d_p = d_q$.

(3) If $\mathcal{A}_p = \{M_0, \dots, M_n\}$, then there are $M'_0, \dots M'_n \in \mathcal{S}$ such that $\mathcal{A}_q = \{M'_0, \dots, M'_n\}$, and for every $i \leq n$,

(3.a) $M_i = M'_i$ if $M_i \in \mathcal{A}_p \cap N$,

(3.b) $\delta_{M'_i} = \delta_{M_i}$, and

(3.c) $N \cap M_i \cap \omega_3 = N \cap M'_i \cap \omega_3$.

(4) There is an order-preserving bijection $\pi : a_p \to a_q$ with the following properties:

(4.a) π is the identity on $a_p \cap N$.

(4.b) For every $M_i \in \mathcal{A}_p$ and every $\alpha \in a_p$, $\alpha \in M_i$ if and only if $\pi(\alpha) \in M'_i$.

(4.c) $u^\alpha_p(\nu) = u^{\pi(\alpha)}_q(\nu)$, for each $\alpha \in a_p$ and every $\nu \in d_p$.

Lemma 3.4.2. *Let $p \in \mathbb{P}$ and $N \in \mathcal{N}_p \cap \mathcal{L}$. Then, p is an $(\mathcal{L}, N)$-reflection of p.*

Lemma 3.4.3. *Let $p \in \mathbb{P}$ and let $N \in \mathcal{N}_p \cap \mathcal{L}$. Let $q \in \mathbb{P} \cap N$ be an $(\mathcal{L}, N)$-reflection of p. Then there is a condition $r \in \mathbb{P}$ extending both p and q.*

Proof. Let $\mathcal{N}_r$ be the $(\mathcal{S}, \mathcal{L})$-symmetric system extending both $\mathcal{N}_p$ and $\mathcal{N}_q$ given by lemma 2.3.30. Let also

$$\mathcal{A}_r = \mathcal{A}_p \cup \mathcal{A}_q \cup \{M \cap N' : M \in \mathcal{A}_p \cup \mathcal{A}_q,\ N' \in \mathcal{N}_r \cap \mathcal{L} \cap M\},$$

and let $a_r = a_p \cup a_q$ and $d_r = d_p = d_q$. For every $\nu \in d_r$ and every $\alpha \in a_r$, we define $u^\alpha_r(\nu) = u^\alpha_p(\nu)$ if $\alpha \in a_p$, and $u^\alpha_r(\nu) = u^\alpha_q(\nu)$ if $\alpha \in a_q$, and we let $u_r = (u^\alpha_r : \alpha \in a_r)$. It's clear that $r = (\mathcal{N}_r, \mathcal{A}_r, a_r, d_r, u_r)$ extends p and q and satisfies clauses (C1)-(C5) from the definition of $\mathbb{P}$, hence we only need to check that it also satisfies clause (C6).

Fix some $\nu \in d_r$ and let $\alpha < \beta$ in a_r such that $\alpha <_{r,\nu} \beta$. We will show that $u^\alpha_r(\nu) \leq u^\beta_r(\nu)$. Note that if $\alpha, \beta \in a_p$ or $\alpha, \beta \in a_q$, then we get the result directly from clause (C6) applied to p or q, respectively. Hence, we only need to check this for the case $\alpha \in a_q \setminus a_p$ and $\beta \in a_p \setminus N$, and for the case $\alpha \in a_p \setminus N$ and $\beta \in a_q \setminus a_p$. Let $\overline{M}_0, \dots, \overline{M}_m \in \mathcal{A}_r^\nu$ and $\gamma_0, \dots, \gamma_{m-1} < \omega_3$ such that $\alpha \in \overline{M}_0$, $\beta \in \overline{M}_m$, and $\gamma_i \in \overline{M}_i \cap \overline{M}_{i+1}$ for each $i < m$. Note that if $\overline{M}_i$ is of the form $M \cap N'$, where $M \in \mathcal{A}_p \cup \mathcal{A}_q$ and $N' \in \mathcal{N}_r \cap \mathcal{L} \cap M$, we can substitute $\overline{M}_i$ by M, and α and β remain $\mathcal{A}_r^\nu$-connected, witnessed by the same sequence $\gamma_0, \dots, \gamma_{m-1}$ of ordinals of ω_3. Hence, we may assume that $\overline{M}_0, \dots, \overline{M}_m \in \mathcal{A}_p^\nu \cup \mathcal{A}_q^\nu$.

Case 1. $\alpha \in a_q \setminus a_p$ and $\beta \in a_p \setminus N$. Let $\alpha^* \in a_p$ such that $\alpha = \pi(\alpha^*)$. First, note that if for some $i < m$, $\overline{M}_i \in \mathcal{A}_q^\nu$, then there is some $j \leq n$ for which $\overline{M}_i = M_j'$, and hence, by clause (3) of the definition of $(\mathcal{L}, N)$-reflection, $\gamma_i, \gamma_{i+1} \in N \cap M_j \cap \omega_3$. On one hand, if $\overline{M}_0 \in \mathcal{A}_q^\nu$, then $\overline{M}_0 = M_k'$ for some $k \leq n$, and by clause (4) of definition 3.4.1, $\alpha^* \in M_k$. On the other hand, if $\overline{M}_0 \in \mathcal{A}_p^\nu$, then $\overline{M}_0 = M_l$ for some $l \leq n$, and by clause (3) of the definition of $(\mathcal{L}, N)$-reflection, $\alpha \in M_l'$. Hence, by clause (4) of the same definition, $\alpha^* \in M_l$. Therefore, in both cases $\alpha^* <_{p,\nu} \beta$, and from clause (C6) applied to p and clause (4) of definition 3.4.1, we get $u_r^\alpha(\nu) = u_q^\alpha(\nu) = u_p^{\alpha^*}(\nu) \leq u_p^\beta(\nu) = u_r^\beta(\nu)$.

Case 2. $\alpha \in a_p \setminus N$ and $\beta \in a_q \setminus a_p$. Let $\beta^* \in a_p$ such that $\beta = \pi(\beta^*)$. We can argue as in case 1 with respect to β instead of α to show that $\alpha <_{r,\nu} \beta^*$, and conclude that $u_r^\alpha(\nu) = u_p^\alpha(\nu) \leq u_p^{\beta^*}(\nu) = u_q^\beta(\nu) = u_r^\beta(\nu)$. □

Lemma 3.4.4. *The forcing $\mathbb{P}$ is $\mathcal{L}$-proper.*

Proof. Let $p \in \mathbb{P}$ and let N^* be a countably closed $\aleph_1$-sized elementary submodel of some large $H(\theta)$ containing p and $\mathbb{P}$. Let $N = N^* \cap H(\omega_3)$. By lemma 3.2.4, we may find a condition p' extending p and such that $N \in \mathcal{N}_{p'}$. We will show that p' is $(N^*, \mathbb{P})$-generic. Let $D \in N^*$ be a dense subset of $\mathbb{P}$, and let $p^* \in D$ be such that $p^* \leq p'$. Similarly to the proof of lemma 3.3.5, since p^* is an $(\mathcal{L}, N)$-reflection of itself by lemma 3.4.2, if we show that all the parameters in the definition of $(\mathcal{L}, N)$-reflection are in N^*, and then argue by elementarity, we can find a condition $q \in D \cap N^*$ which is an $(\mathcal{L}, N)$-reflection of p^*. Indeed, note that the parameters in item (3.c) of the definition of $(\mathcal{L}, N)$-reflection are in N because it is a countably closed model, and the other items are obvious or follow from the fact that $\omega_1 \subseteq N$. But now we are done because p^* and q are compatible by lemma 3.4.3. □

3.5 The chain condition

In our final lemma we show that $\mathbb{P}$ has the $\aleph_3$-chain condition. We will in fact prove that $\mathbb{P}$ has the $\aleph_3$-Knaster condition; in other words, that for every sequence $(p_\xi : \xi < \omega_3)$ of conditions in $\mathbb{P}$ there is $X \subseteq \omega_3$ of size $\aleph_3$ such that p_{ξ_0} and p_{ξ_1} are compatible for al $\xi_0, \xi_1 < \omega_3$ in $\mathbb{P}$.

Lemma 3.5.1. *$\mathbb{P}$ has the $\aleph_3$-Knaster condition.*

Proof. Let $p_\xi \in \mathbb{P}$ for all $\xi < \omega_3$. For each ξ, let N_ξ be an elementary submodel of $(H(\omega_3); \in, \vec{e})$ of size $\aleph_1$ such that $p_\xi \in N_\xi$. By $2^{\aleph_1} = \aleph_2$ we may find $X \in [\omega_3]^{\aleph_3}$ such that for all $\xi_0 < \xi_1$ in X, the structures $(N_{\xi_0}; \in, p_{\xi_0})$ and $(N_{\xi_1}; \in, p_{\xi_1})$ are isomorphic, and the isomorphism $\Psi_{N_{\xi_0}, N_{\xi_1}}$ is the identity on $N_{\xi_0} \cap N_{\xi_1}$. We will now prove that for all $\xi_0 < \xi_1$ in X, the conditions p_{ξ_0} and p_{ξ_1} are compatible.

By lemma 2.3.24, we have that $\mathcal{N}_{p_{\xi_0}} \cup \mathcal{N}_{p_{\xi_1}}$ is an $(\mathcal{S}, \mathcal{L})$-symmetric system. Moreover, note that if $M \in \mathcal{A}_{p_{\xi_1}} \cap \mathcal{S}$ and $N \in \mathcal{N}_{p_{\xi_0}} \cap \mathcal{L} \cap M$, then $N \in N_{\xi_1}$, and hence, $\Psi_{N_{\xi_0}, N_{\xi_1}}(N) = N$. Therefore, $N \in \mathcal{N}_{p_{\xi_1}}$, and by clause (C2) applied to p_{ξ_1}, we have that $N \cap M \in \mathcal{A}_{p_{\xi_1}}$. Hence, we can conclude that $\mathcal{A}_{p_{\xi_0}} \cup \mathcal{A}_{p_{\xi_1}}$ satisfies clause (C2). Denote $\mathcal{A}_{p_{\xi_0}} \cup \mathcal{A}_{p_{\xi_1}}$ by $\mathcal{A}_q$ from now on. Since N_{ξ_1} is an uncountable model, $\omega_1 \subseteq N_{\xi_1}$. Hence, in particular, $d_{p_{\xi_0}} \subseteq N_{\xi_1}$, and thus, $\Psi_{N_{\xi_0}, N_{\xi_1}}{}''d_{p_{\xi_0}} = d_{p_{\xi_0}}$, which implies that $d_{p_{\xi_0}} = d_{p_{\xi_1}}$. Define $a_q = a_{p_{\xi_0}} \cup a_{p_{\xi_1}}$ and $d_q = d_{p_{\xi_0}} = d_{p_{\xi_1}}$. Note that for every $\nu \in d_q$, if $\alpha \in a_{p_{\xi_0}} \cap a_{p_{\xi_1}}$, then $u^\alpha_{p_{\xi_0}}(\nu) = u^\alpha_{p_{\xi_1}}(\nu)$. For every $\nu \in d_q$, each $i \in \{0, 1\}$, and every $\alpha \in a_{p_{\xi_i}}$, define $u^\alpha_q(\nu) = u^\alpha_{p_{\xi_i}}(\nu)$, and let u_q be the sequence $(u^\alpha_q : \alpha \in a_q)$. It's clear that u^α_q is a well-defined function extending $u^\alpha_{p_{\xi_i}}$, for all $i \in \{0, 1\}$ and all $\alpha \in a_{p_{\xi_i}}$.

Therefore, if we show that

$$q = (\mathcal{N}_{p_{\xi_0}} \cup \mathcal{N}_{p_{\xi_1}}, \mathcal{A}_q, a_q, d_q, u_q)$$

satisfies clause (C6), then q will be a common extension of p_{ξ_0} and p_{ξ_1} in $\mathbb{P}$, as we wanted.

Let $\nu \in d_q$, and $\alpha < \beta$ in a_q such that $u_q^\alpha(\nu) > u_q^\beta(\nu)$. We need to show that $\alpha \not<_{\mathcal{A}_q,\nu} \beta$. Suppose towards a contradiction that $\alpha <_{\mathcal{A}_q,\nu} \beta$. Let $M_0, \ldots, M_n$ be active models in $\mathcal{A}_q$ and let $\gamma_0 < \cdots < \gamma_{n-1} < \omega_3$ such that

- $\sup_{i \leq n} \delta_{M_i} \leq \nu$,
- $\alpha \in M_0$ and $\beta \in M_n$, and
- $\gamma_i \in (\alpha, \beta) \cap M_i \cap M_{i+1}$ for each $i < n$.

Now, for every $i \leq n$, if $M_i \in \mathcal{A}_{p_{\xi_0}}$, we let M_i' denote the model M_i, and if $M_i \in \mathcal{A}_{p_{\xi_1}}$, we let M_i' denote the model $\Psi_{N_{\xi_1}, N_{\xi_0}}(M_i)$. Moreover, for every $i < n$, if either M_i or M_{i+1} belong to $\mathcal{A}_{p_{\xi_1}}$, we let $\gamma_i' = \Psi_{N_{\xi_1}, N_{\xi_0}}(\gamma_i)$. Otherwise, we let $\gamma_i' = \gamma_i$. Finally, if $\alpha \in a_{p_{\xi_0}}$, let $\alpha' = \alpha$, and if $\alpha \in a_{p_{\xi_1}}$, let $\alpha' = \Psi_{N_{\xi_1}, N_{\xi_0}}(\alpha)$, and similarly for β. We claim that

- $\gamma_0' < \cdots < \gamma_{n-1}'$,
- $\sup_{i \leq n} \delta_{M_i'} \leq \nu$,
- $\alpha' \in M_0'$ and $\beta' \in M_n'$, and
- $\gamma_i' \in (\alpha', \beta') \cap M_i' \cap M_{i+1}'$ for each $i < n$,

or in other words, that $\alpha' <_{\mathcal{A}_{p_{\xi_0}},\nu} \beta'$. This will lead to a contradiction because $u_{p_{\xi_0}}^{\alpha'}(\nu) = u_q^\alpha(\nu) > u_q^\beta(\nu) = u_{p_{\xi_0}}^{\beta'}(\nu)$, by assumption and the fact that $\Psi_{N_{\xi_1}, N_{\xi_0}}$ is an isomorphism.

It's clear that $\sup_{i \leq n} \delta_{M_i'} \leq \nu$ and that $\gamma_i' \in M_i' \cap M_{i+1}'$, for each $i < n$. It obviously holds that $\alpha' \in M_0'$ in the cases $\alpha \in a_{p_{\xi_0}}$ and $M_0 \in \mathcal{A}_{p_{\xi_0}}$, and $\alpha \in a_{p_{\xi_1}}$

and $M_1 \in \mathcal{A}_{p_{\xi_1}}$. If $\alpha \in a_{p_{\xi_0}}$ and $M_0 \in \mathcal{A}_{p_{\xi_1}}$, then $\alpha \in N_{\xi_0} \cap N_{\xi_1}$, and hence, $\alpha' = \alpha = \Psi_{N_{\xi_1},N_{\xi_0}}(\alpha) \in \Psi_{N_{\xi_1},N_{\xi_0}}(M_0) = M_0'$. If $\alpha \in a_{p_{\xi_1}}$ and $M_0 \in \mathcal{A}_{p_{\xi_0}}$, then $\alpha \in N_{\xi_0} \cap N_{\xi_1}$, and thus, $\alpha' = \Psi_{N_{\xi_1},N_{\xi_0}}(\alpha) = \alpha \in M_0 = M_0'$. The same argument shows that $\beta' \in M_n'$. Hence, we only need to check that $\alpha' < \gamma_0' < \cdots < \gamma_{n-1}' < \beta'$. First we show that $\alpha' < \gamma_0'$. If $M_0 \in \mathcal{A}_{p_{\xi_0}}$, then $\alpha' = \alpha$. Moreover, if $M_1 \in \mathcal{A}_{p_{\xi_0}}$, then $\gamma_0' = \gamma_0$, and if $M_1 \in \mathcal{A}_{p_{\xi_1}}$, then $\gamma_0 \in N_{\xi_0} \cap N_{\xi_1}$, and thus, $\gamma_0' = \Psi_{N_{\xi_1},N_{\xi_0}}(\gamma_0) = \gamma_0$. In both cases $\alpha' = \alpha$ and $\gamma_0' = \gamma_0$, so we have that $\alpha' < \gamma_0'$. Now, if $M_0 \in \mathcal{A}_{p_{\xi_1}}$, then $\alpha' = \Psi_{N_{\xi_1},N_{\xi_0}}(\alpha)$ and $\gamma_0' = \Psi_{N_{\xi_1},N_{\xi_0}}(\gamma_0)$, so $\alpha' < \gamma_0'$. Next, we check that $\gamma_i' < \gamma_{i+1}'$ for every $i < n$. By similar reasons as above, it's not too hard to see that if $M_{i+1} \in \mathcal{A}_{p_{\xi_0}}$, then $\gamma_i' = \gamma_i$ and $\gamma_{i+1}' = \gamma_{i+1}$, and if $M_{i+1} \in \mathcal{A}_{p_{\xi_1}}$, then $\gamma_i' = \Psi_{N_{\xi_1},N_{\xi_0}}(\gamma_i)$ and $\gamma_{i+1}' = \Psi_{N_{\xi_1},N_{\xi_0}}(\gamma_{i+1})$. In both cases $\gamma_i' < \gamma_{i+1}'$. Lastly, we check that $\gamma_{n-1}' < \beta'$. But again, by the same reasons as before, if $M_n \in \mathcal{A}_{p_{\xi_0}}$, then $\gamma_{n-1}' = \gamma_{n-1}$ and $\beta' = \beta$, and if $M_n \in \mathcal{A}_{p_{\xi_1}}$, then $\gamma_{n-1}' = \Psi_{N_{\xi_1},N_{\xi_0}}(\gamma_{n-1})$ and $\beta' = \Psi_{N_{\xi_1},N_{\xi_0}}(\beta)$. So, $\gamma_{n-1}' < \beta'$. This finishes the proof of $\alpha' <_{\mathcal{A}_{p_{\xi_0}},\nu} \beta'$, which contradicts our assumption, as we mentioned above, and thus, shows that q satisfies clause (C6) as we wanted. □

3.6 Conclusions

Recall that we have assumed GCH throughout this chapter. Therefore, $\mathcal{L}$ is stationary in $[H(\omega_3)]^{\aleph_1}$, and hence, lemmas 1.1.25, 3.3.5, 3.4.4 and 3.5.1 ensure that the forcing $\mathbb{P}$ preserves all cardinals.

We finish this chapter by showing that $\mathbb{P}$ does in fact force a strong chain of subsets of ω_1 of length ω_3, and thus proving theorem 3.0.11.

Let G be a $\mathbb{P}$-generic filter over V and let us work in $V[G]$. For all $\alpha < \omega_3^V = \omega_3^{V[G]}$, if there is any $p \in G$ such that $\alpha \in a_p$, we let $C_\alpha := \{p \in \mathbb{P} : p \in G, \alpha \in a_p\}$, and

$$u_G^\alpha = \bigcup \{u_p^\alpha : p \in C_\alpha\}.$$

By lemma 3.2.1, the set of B of $\alpha < \omega_3$ for which u_G^α is defined has size $\aleph_3$.

Note that for all $\alpha < \beta$ in B, $\mathrm{dom}(u_G^\alpha) = \mathrm{dom}(u_G^\beta)$ (it follows from the fact that all $p \in C_\alpha$ are in G). Let now $A = \mathrm{dom}(u_G^\alpha)$ for any $\alpha \in B$, and let us note that $A \in [\omega_1]^{\omega_1}$ by lemma 3.2.2. Let $(\alpha_i : i < \omega_3)$ be the strictly increasing enumeration of B and $(\nu_\tau : \tau < \omega_1)$ the strictly increasing enumeration of A. For each $i < \omega_3$ let

$$X_i = \{\tau \in \omega_1 : u_G^{\alpha_i}(\nu_\tau) = 1\}$$

Using our density lemmas 3.2.2 and 3.2.3 it is now a routine matter to check that $(X_i : i < \omega_3)$ is a strong ω_3-chain of subsets of ω_1. This finishes the proof of theorem 3.0.11.

4

Finite support iterations with two-type symmetric systems

This chapter is devoted to introduce a high version of Asperó and Mota's finite support iterations with symmetric systems from [11] and [12]. The idea is to define a forcing iteration in which we incorporate symmetric systems to ensure the preservation of cardinals at every stage of the iteration. We can consider iterations of length an arbitrarily large uncountable cardinal κ. We will start by defining the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings, which is one of the natural classes of forcing notions which can be iterated in this sense. We will compare it with Neeman's high analog of properness and Asperó and Mota's classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-chain condition. We will then define the iterations and prove their main properties. These are sequences of cardinal-preserving forcing notions $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$, which can be seen as a forcing iteration in a broad sense. More precisely, for every $\alpha < \beta \leq \kappa$ the following hold:

(1) $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$.

(2) $\mathbb{P}_\beta$ is proper for countable elementary submodels.

(3) $\mathbb{P}_\beta$ is proper for an appropriate class of $\aleph_1$-sized elementary submodels.

(4) $\mathbb{P}_\beta$ has the $\aleph_3$-chain condition.

We will finish this chapter by building a model of the forcing axiom for the class of $(\mathcal{S},\mathcal{L})$-finitely proper forcings. The construction requires a supercompact cardinal and involves finite support iterations with two-type symmetric systems. In the final model the supercompact cardinal is collapsed to become $\aleph_3$, which coincides with the value of the continuum. We will also prove a restricted version of the forcing axiom which doesn't require any large cardinal assumptions. The last section is devoted to study different possible extensions of the class of $(\mathcal{S},\mathcal{L})$-finitely proper forcings and speculate about potential applications.

Let us fix a cardinal $\kappa > \omega_2$ and a predicate $T \subseteq H(\kappa)$. Throughout this chapter, $\mathcal{S}$ will be the set of countable $M \preceq (H(\kappa);\in,T)$ and $\mathcal{L}$ will be a collection of $\aleph_1$-sized elementary submodels $N \preceq (H(\kappa);\in,T)$ appropriate for $\mathcal{S}$. Furthermore, we will assume that $\mathcal{L}$ is stationary in $[H(\kappa)]^{\aleph_1}$.

4.1 The class of $(\mathcal{S},\mathcal{L})$-finitely proper forcings

We will devote this section to introducing the class of $(\mathcal{S},\mathcal{L})$-finitely proper forcings. First, we will define Neeman's high analog of properness, which stems from his work on forcing with two-type side conditions and finite support iterations of proper forcings ([62], [63]). Then, we will define our class of posets, which is partially inspired by Neeman's class, and can be seen as a high analog of Asperó and Mota's class of $\aleph_{1.5}$-c.c. forcings (see definition 1.1.38), and we will try to justify its definition.

Notation 4.1.1. Throughout the rest of the chapter, if $\mathbb{P} \subseteq H(\kappa)$ is a forcing notion, G is a $\mathbb{P}$-generic filter, and $Q \in \mathcal{S} \cup \mathcal{L}$, we will make the technical convention that $Q[G]$ denotes the set $\{\tau_G : \tau \in Q, \tau \text{ a } \mathbb{P}\text{-name}\}$. Note that $\mathbb{P}$ may fail to be a member of Q, and might not even be definable there. In general, if $\mathbb{P} \notin Q$, the set $Q[G]$ will fail to be the generic extension of Q, in the sense that it might not even be a a model of set theory (or a fragment thereof).

4.1.1 Neeman's $(\mathcal{S}, \mathcal{L})$-proper forcing

In Neeman's unpublished work presented in [64] and [65], we can see the seed of a new family of classes of forcing notions, which seem to be more amenable to high analogs of classical forcing axioms (specifically, of PFA). The idea comes from the following characterisation of properness in terms of Todorčević's collapse $\mathbb{C}(\mathcal{S})$. Recall that the conditions of $\mathbb{C}(\mathcal{S})$ are finite $\in$-chains of $\mathcal{S}$-models.

Proposition 4.1.2 ([64], [65])**.** *A forcing notion $\mathbb{P}$ is $\mathcal{S}$-proper if and only if for every $\mathcal{C} \in \mathbb{C}(\mathcal{S})$ and every $M \in \mathcal{C}$, if $p \in \mathbb{P} \cap M$ and p is $(M', \mathbb{P})$-generic for every $M' \in \mathcal{C} \cap M$, then there is an extension $q \leq p$ which is $(M', \mathbb{P})$-generic for every $M' \in \mathcal{C}$.*

You could try to generalize the notion of properness by replacing the finite chains of countable models $\mathcal{C}$ from last proposition by Neeman's $(\mathcal{S}, \mathcal{L})$-chains. However, as it was observed by Veličković, the straightforward generalization is not iterable.

Definition 4.1.3. We say that a forcing notion $\mathbb{P}$ is *naively* $(\mathcal{S}, \mathcal{L})$*-proper* if for every $\mathcal{C} \in \mathbb{C}(\mathcal{S}, \mathcal{L})$ and every $Q \in \mathcal{C}$, if $p \in \mathbb{P} \cap Q$ is $(Q', \mathbb{P})$-generic for every $Q' \in \mathcal{C} \cap Q$, then there is an extension $q \leq p$ which is $(Q', \mathbb{P})$-generic for every $Q' \in \mathcal{C}$.

Example 4.1.4. Neeman's decorated poset $\mathbb{C}(\mathcal{S}, \mathcal{L})^{dec}$ from [62] is naively $(\mathcal{S}, \mathcal{L})$-proper. Conditions are pairs $p = (\mathcal{C}_p, d_p)$, where $\mathcal{C}_p \in \mathbb{C}(\mathcal{S}, \mathcal{L})$ and d_p is a function $d_p : \mathcal{C}_p \to [H(\kappa)]^{<\omega}$ such that if $Q_0, Q_1 \in \mathcal{C}_p$ are so that $Q_0 \in Q_1$, then $d_p(Q_0)$ is an element of Q_1. The order is defined by $q \leq p$ if and only if $\mathcal{C}_q \supseteq \mathcal{C}_p$ and $d_q(Q) \supseteq d_p(Q)$ for all $Q \in \mathcal{C}_p$.

This poset adds a club on ω_2 which doesn't have infinite subsets from the ground model. Therefore, if we could iterate this forcing notion while preserving ω_1 and ω_2, we would contradict club guessing on ω_2. More precisely, we would force a club of ω_2 contradicting the following theorem of ZFC, due to Shelah:

Theorem 4.1.5 ([75])**.** *For every stationary $S \subseteq S^{\omega_2}_{\omega}$ there exists a sequence*

$\langle C_\delta : \delta \in S\rangle$ such that every C_δ is a club subset of δ of order-type ω, and for every club $D \subseteq \omega_2$ there exists some $\delta \in S$ such that $C_\delta \subseteq D$.

However, Neeman observed that he could avoid destroying club guessing by replacing $(\mathcal{S}, \mathcal{L})$-chains by $(\mathcal{S}, \mathcal{L}, \mathcal{T})$-chains in the definition of naive $(\mathcal{S}, \mathcal{L})$-properness. He defined the following class of forcing notions, which we have renamed as $(\mathcal{S}, \mathcal{L})$-*proper* for the sake of coherence with our own notation, but in Neeman's slides it appears as *two-size proper* or *baby-*$\{\omega, \omega_1\}$*-proper*. Let us make the convention that if p is a condition in a forcing notion $\mathbb{P}$ and $\overline{M} \in \mathcal{T}$, then we say that p is $(\overline{M}, \mathbb{P})$-generic if it is $(N, \mathbb{P})$-generic for every $N \in \overline{M}$.

Definition 4.1.6 ([64], [65])**.** We say that a forcing notion $\mathbb{P}$ is $(\mathcal{S}, \mathcal{L})$-*proper* if for every $\mathcal{C} \in \mathbb{C}(\mathcal{S}, \mathcal{L}, \mathcal{T})$ and every $Q \in \mathcal{C} \cap (\mathcal{S} \cup \mathcal{L})$, if $p \in \mathbb{P} \cap Q$ is $(Q', \mathbb{P})$-generic for every $Q' \in \mathcal{C} \cap Q$, then there is an extension $q \leq p$ which is $(Q', \mathbb{P})$-generic for every $Q' \in \mathcal{C}$.

This class excludes posets which kill club guessing, such as the one in example 4.1.4, and gives us the chance of defining a high forcing axiom for $\aleph_2$-many dense sets. Neeman claims that the class of $(\mathcal{S}, \mathcal{L})$-proper forcings can be iterated with a variant of his iterations from [62], and that he can obtain a model of the forcing axiom $FA_{\aleph_2}\big((\mathcal{S}, \mathcal{L})\text{-proper}\big)$ from a supercompact cardinal. In this model the continuum is $\aleph_3$, although it is not known whether the forcing axiom decides its value or not. This class of posets is closed under compositions, and it includes c.c.c. forcings and forcings for collapsing cardinals to ω_2 with finite conditions (the pure side condition forcing).

Let us finish this section by including a mildly relaxed version of the class of $(\mathcal{S}, \mathcal{L})$-proper forcings, which is closer to the class of forcing notions that we will define in the next section.

Definition 4.1.7. We say that a forcing notion $\mathbb{P}$ is *relaxed* $(\mathcal{S}, \mathcal{L})$-*proper* if for every $\mathcal{C} \in \mathbb{C}(\mathcal{S}, \mathcal{L}, \mathcal{T})$ and every $Q \in \mathcal{C} \cap (\mathcal{S} \cup \mathcal{L})$, if $p \in \mathbb{P} \cap Q$ is such that either

(1) $\mathcal{C} \cap Q = \varnothing$, or

(2) $Q \in \mathcal{S}$ and $\mathcal{C} \cap Q \subseteq \mathcal{L}$, and p is $(N, \mathbb{P})$-generic for every $N \in \mathcal{C} \cap Q$,

then there is an extension $q \leq p$ which is $(R, \mathbb{P})$-generic for every $R \in \mathcal{C}$.

Some of the consequences of the forcing axiom for the relaxed $(\mathcal{S}, \mathcal{L})$-proper forcings include the square principle $\square_{\omega_1, fin}$ and a certain high analogue of Moore's Mapping Reflection Principle (MRP), which is strong enough to imply the failure of $\square_\lambda$ for all $\lambda \geq \omega_2$, but not strong enough to decide the value of the continuum to be $\aleph_3$[1].

4.1.2 $(\mathcal{S}, \mathcal{L})$-finitely proper forcing

Our first approach to developing Asperó and Mota's iterations with symmetric systems of models of two types was to simply replace symmetric systems of models of one type in the definition of the iteration by two-types ones, and try to iterate the naive high analog of the class of $\aleph_{1.5}$-c.c. forcings. However, this first approach didn't work, since you run into problems when showing properness at stages of cofinality ω_1 of the iteration. We will give more details about the exact problems that you would run into in section 4.4. Neeman probably encountered similar problems, so it shouldn't be surprising that the class of forcing notions that we ended up defining could be seen, on one hand, as a high analog of the class of $\aleph_{1.5}$-c.c. forcings, and on the other hand, as a subclass of Neeman's class of $(\mathcal{S}, \mathcal{L})$-proper forcings.

Using Neeman's characterisation of properness you could try replacing the $\mathcal{S}$-chains of models by $\mathcal{S}$-symmetric systems to define the following class of posets.

Definition 4.1.8. We say that a forcing notion $\mathbb{P}$ is *naively $\mathcal{S}$-finitely proper* if for every $\mathcal{M} \in \mathbb{M}(\mathcal{S})$ and every $M \in \mathcal{M}$, if $p \in \mathbb{P} \cap M$ is $(M', \mathbb{P})$-generic for every $M' \in \mathcal{M} \cap M$, then there is an extension $q \leq p$ which is $(M', \mathbb{P})$-generic for every $M' \in \mathcal{M}$.

[1]We refer the reader to [38] for all the undefined notions.

However, you cannot iterate this class of forcings using Asperó and Mota's finite support iterations with $\mathcal{S}$-symmetric systems. Suppose that $\mathbb{P}_\alpha$ is the α-th stage of this iteration, where $\alpha \leq \kappa$. This iteration incorporates symmetric systems of $\mathcal{S}$-models that come from the ground model V to ensure the preservation of cardinals at every stage of the iteration. The reason why we can't iterate the above class of posets using these iterations is that, if G_α is a $\mathbb{P}_\alpha$-generic filter over V and $\mathcal{M}$ is an $\mathcal{S}$-symmetric system from the ground model V, then $\{M[G_\alpha] : M \in \mathcal{M}\}$ doesn't need to be an $\mathcal{S}$-symmetric system. If M_0, M_1 are two elementary submodels and they are isomorphic, then $M_0[G_\alpha]$ and $M_1[G_\alpha]$ need not be isomorphic. In fact, preserving isomorphisms is the only obstacle in preserving the symmetric system structure.

In the next section we will introduce the two-type version of Asperó and Mota's finite support iterations with symmetric systems as side condition. If you considered the same class of posets from above, but asking for $\mathcal{M}$ to be an $(\mathcal{S}, \mathcal{L})$-symmetric system, or even an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, rather than an $\mathcal{S}$-symmetric system, and you tried to iterate it using these iterations, you would of course run into the same problems described above. Therefore, the fact that the symmetric system structure is not preserved throughout the iterations puts a limitation on the class of forcings that you can define in the above way (which is iterable).

Keeping all these technical limitations in mind, we found that the following class of forcings is the most natural one that you can iterate using the two-type version of Asperó and Mota's iterations. It is worth pointing out the resemblance with Neeman's class of relaxed $(\mathcal{S}, \mathcal{L})$-proper forcings.

Definition 4.1.9. We say that a forcing notion $\mathbb{P} \in H(\kappa)$ is $(\mathcal{S}, \mathcal{L})$-*finitely proper* if and only if there is a club $D \subseteq [H(\kappa)]^{\leq \aleph_1}$ such that for every countable subset $\mathcal{M} \subseteq D$ such that $|\mathcal{M} \cap \mathcal{S}| < \aleph_0$ and $|\mathcal{M} \cap \mathcal{L}| \leq \aleph_0$, if $p \in \mathbb{P} \cap Q$ for some $Q \in \mathcal{M}$ such that either

(1) $\mathcal{M} \cap Q = \varnothing$, or

(2) $Q \in \mathcal{S}$ is such that $\varepsilon_Q = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$, and p is $(N, \mathbb{P})$-generic for every $N \in \mathcal{M} \cap Q$,

then there is an extension $q \leq p$ which is $(R, \mathbb{P})$-generic for every $R \in \mathcal{M}$.

4.1.3 Basic facts about $(\mathcal{S}, \mathcal{L})$-finitely proper forcings

The following result is a straightforward consequence from the fact that every condition of a c.c.c. forcing is generic with respect to every elementary submodel of every big enough $H(\theta)$.

Lemma 4.1.10. *Every c.c.c. forcing is $(\mathcal{S}, \mathcal{L})$-finitely proper.*

Moreover, the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings is a subclass of the class of $\mathcal{S}$-proper+ $\mathcal{L}$-proper+$\aleph_3$-c.c. forcings.

Lemma 4.1.11. *If $\mathbb{P}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper, then $\mathbb{P}$ is $\mathcal{S}$-proper.*

Lemma 4.1.12. *If $\mathbb{P}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper, then $\mathbb{P}$ is $\mathcal{L}$-proper.*

Lemma 4.1.13. *If $\mathbb{P}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper, then $\mathbb{P}$ has the $\aleph_3$-chain condition.*

Proof. Let A be a maximal antichain of $\mathbb{P}$ such that $|A| \geq \aleph_3$. Let D be as in definition 4.1.9 and let R_p, for every $p \in A$, be such that $R_p \in D \cap (\mathcal{S} \cup \mathcal{L})$ and $p, A \in R_p$. Since $\varepsilon_{R_p} < \omega_2$ for each $p \in A$, there are $A' \subseteq A$ and $\varepsilon < \omega_2$ such that $|A'| = |A|$ and for every $p \in A'$, $\varepsilon_{R_p} = \varepsilon$. Since the models R_p have size less than or equal $\aleph_1$, we can find two different conditions $p, p' \in A'$ such that $p' \notin R_p$. Note that as $p' \in R_{p'}$ and $\varepsilon_{R_p} = \varepsilon_{R_{p'}} = \varepsilon$, we can find an extension q of p' that is $(R_p, \mathbb{P})$-generic, by the $(\mathcal{S}, \mathcal{L})$-finite properness of $\mathbb{P}$. Therefore, there must be a condition $q^* \in R_p \cap A$ compatible with q, and thus, also compatible with p'. But this is impossible because $p' \neq q^*$, as $p' \notin R_p$ and $q^* \in R_p$, and p' and q^* are both members of the maximal antichain A. □

Lastly, our class of forcings is included in Neeman's relaxed version of $(\mathcal{S}, \mathcal{L})$-proper forcings.

Lemma 4.1.14. *Every $(\mathcal{S}, \mathcal{L})$-finitely proper forcing is relaxed $(\mathcal{S}, \mathcal{L})$-proper.*

Proof. Note that if $Q \in \mathcal{S}$ is such that $\varepsilon_Q = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$, in particular $\mathcal{M} \cap Q \subseteq \mathcal{L}$. □

Even though we have mentioned that the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings can be seen as a high analog of the class of forcings with the $\aleph_{1.5}$-c.c., these two classes don't seem to be comparable. It's clear that, in general, a forcing with the $\aleph_{1.5}$-c.c. is not $(\mathcal{S}, \mathcal{L})$-finitely proper. In order to show the other direction it would be enough to exhibit an example of an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing which doesn't have the $\aleph_{1.5}$-c.c. For instance, a poset forcing the failure of club guessing at $S^{\omega_2}_{\omega_1}$ would do the trick. Due to time constraints, we haven't been able to explore this direction, which we want to pursue in the future, so we don't have an explicit proof of the non-comparability of these two classes. Let us give, however, an argument which sustains this view.

Suppose that $\mathbb{P}$ is an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing and suppose that we want to show that it has the $\aleph_{1.5}$-c.c. Hence, fix a finite set of small models $\mathcal{M}$ and suppose that $p \in \mathbb{P}$ belongs to some $M \in \mathcal{M}$ such that $\delta_M = \min\{\delta_{M'} : M' \in \mathcal{M}\}$. Note, however, that we cannot ensure that $\varepsilon_M = \min\{\varepsilon_{M'} : M' \in \mathcal{M}\}$, and hence, in general we cannot use $(\mathcal{S}, \mathcal{L})$-finite properness to extend p to an $(M', \mathbb{P})$-generic condition for every $M' \in \mathcal{M}$.

4.2 The iteration

For the rest of the chapter we will assume that V is a ground model for the GCH, and that κ is such that $2^{<\kappa} = \kappa$.

We will describe what it means for a κ-sequence $\langle(\mathbb{P}_\alpha, \leq_\alpha) : \alpha \leq \kappa\rangle$ of posets to be *the finite support iteration with two-type symmetric systems as side conditions with respect to the sequence* $\langle\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle$, such that each $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset on κ. Our notation will agree with the notation of [11] and [12] to a large degree.

We will denote the forcing relation $\Vdash_{\mathbb{P}_\alpha}$ of $\mathbb{P}_\alpha$ by $\Vdash_\alpha$, for each $\alpha \leq \kappa$. We let $\Phi : \kappa \to H(\kappa)$ be a surjection such that for each $x \in H(\kappa)$, $\Phi^{-1}(\{x\})$ is unbounded in κ. The map Φ, which will act as the bookkeeping function of our iteration, exists by $2^{<\kappa} = \kappa$. Let also $\lhd$ be a well-order of $H(\kappa^+)$ in order type 2^κ. The well-order $\lhd$ exists by $|H(\kappa^+)| = 2^\kappa$.

Let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be a strictly increasing sequence of regular cardinals that grows fast enough to ensure that $H(\kappa) \in H(\theta_0)$ and $\langle H(\theta_\beta), \lhd_\beta \colon \beta < \alpha \rangle \in H(\theta_\alpha)$, for each $\alpha \leq \kappa$, where $\lhd_\beta$ is a well-ordering of $H(\theta_\beta)$. For example, we can define this sequence as $\theta_0 = |2^\kappa|^+$ and $\theta_\alpha = |2^{\sup\{\theta_\beta : \beta < \alpha\}}|^+$ for all $\alpha > 0$.

For every $\alpha \leq \kappa$, let $\mathcal{S}^*_\alpha$ be the collection of all countable elementary submodels of $H(\theta_\alpha)$ containing $\Phi, \lhd$ and $\langle \theta_\beta : \beta < \alpha \rangle$, and let $\mathcal{S}_\alpha = \{M^* \cap H(\kappa) : M^* \in \mathcal{S}^*_\alpha\}$, which is a subset of $\mathcal{S}$. Similarly, for every $\alpha \leq \kappa$, let $\mathcal{L}^*_\alpha$ be the collection of all $\aleph_1$-sized elementary submodels $N \preceq H(\theta_\alpha)$ containing $\Phi, \lhd$ and $\langle \theta_\beta : \beta < \alpha \rangle$, and such that $N^* \cap H(\kappa) \in \mathcal{L}$. Let $\mathcal{L}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{L}^*_\alpha\}$, which is a subset of $\mathcal{L}$.

The definition of the posets $\mathbb{P}_\alpha$ will be by induction on $\alpha \leq \kappa$. The bookkeeping function will give us $\mathbb{P}_\alpha$-names $\dot{\mathbb{Q}}_\alpha$ of $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notions, and for each $\alpha \leq \kappa$, the side condition will make sure that $\mathbb{P}_\alpha$ has the $\aleph_3$-chain condition and that it is $\mathcal{S}^*_\alpha$-proper and $\mathcal{L}^*_\alpha$-proper. Before going into the definition of the iteration, let us review some basic facts about the collections of models introduced in the last paragraph:

- For each $Q \in \mathcal{S}_\alpha \cup \mathcal{L}_\alpha$, $\alpha \in Q$.
- If $N \in \mathcal{L}_\alpha$ and $M \in \mathcal{S}_\alpha$ such that $N \in M$, it follows from proposition 1.4.17, that $N \cap M \in \mathcal{S}_\alpha$.
- The classes $\mathcal{S}_\alpha$ and $\mathcal{L}_\alpha$ are closed under taking isomorphic copies.

- If $\alpha < \beta$, then $\mathcal{S}^*_\alpha$ belongs to all members of $\mathcal{S}^*_\beta$ containing the ordinal α, and similarly for $\mathcal{L}^*_\alpha$ and $\mathcal{L}^*_\beta$.

- If $\alpha < \beta$ and $Q^* \in \mathcal{S}^*_\beta \cup \mathcal{L}^*_\beta$ such that $\alpha \in Q^*$, then $Q^* \cap H(\kappa) \in \mathcal{S}_\alpha \cup \mathcal{L}_\alpha$. Hence, in particular, if $Q \in \mathcal{S}_\beta \cup \mathcal{L}_\beta$ and $\alpha \in Q$, then $Q \in \mathcal{S}_\alpha \cup \mathcal{L}_\alpha$. To see this note that, by proposition 1.4.8, $Q^* \cap H(\theta_\alpha) \preceq H(\theta_\alpha)$, that is, $Q^* \cap H(\theta_\alpha) \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$, and hence, $(Q^* \cap H(\theta_\alpha)) \cap H(\kappa) = Q^* \cap H(\kappa) \in \mathcal{S}_\alpha \cup \mathcal{L}_\alpha$.

4.2.1 Definition of the iteration

Let us proceed to the definition of $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ now by induction on $\alpha \leq \kappa$.

Conditions in $\mathbb{P}_0$ are pairs of the form $(\varnothing, \Delta)$, where Δ is a finite set of ordered pairs of the form $(Q, 0)$, where $\mathrm{dom}(\Delta) = \{Q : (Q, 0) \in \Delta\}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. Given two conditions $p = (\varnothing, \Delta_p)$ and $q = (\varnothing, \Delta_q)$, we define the order on $\mathbb{P}_0$ by $p \leq_0 q$ if and only if $\mathrm{dom}(\Delta_p) \supseteq \mathrm{dom}(\Delta_q)$.

Notation 4.2.1. Let p be an ordered pair (F, Δ) such that F is a function and Δ is a binary relation.

(1) We denote F by F_p and Δ by Δ_p.

(2) If ξ is an ordinal, *the restriction of* p *to* ξ, denoted by $p|_\xi$, is defined as the pair

$$p|_\xi := \big(F_p \restriction \xi, \{(R, \min\{\beta, \xi\}) : (R, \beta) \in \Delta_p\}\big).$$

(3) If $\xi \in \mathrm{ran}(\Delta_p)$, we denote by $\Delta_p^{-1}(\xi)$ the set of all $R \in \mathrm{dom}(\Delta_p)$ such that $(R, \xi) \in \Delta_p$.

Let $\alpha \leq \kappa$ greater than 0, and suppose that we have defined $\mathbb{P}_\xi$ for all $\xi < \alpha$. Suppose also that if $\xi < \alpha$, then $\mathbb{P}_\xi \subseteq H(\kappa)$, and if $p \in \mathbb{P}_\xi$, then p is an ordered pair of the form (F_p, Δ_p), where

- F_p is a finite function with $\mathrm{dom}(F_p) \subseteq \xi$.

- Δ_p is a finite set of ordered pairs (Q, γ), where $\mathrm{dom}(\Delta_p) = \{Q : (Q, \gamma) \in \Delta_p\}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, and γ is an ordinal such that $\gamma \leq \xi$.

For every $\xi < \alpha$, we define $\mathbb{P}_\alpha|_\xi := \{p|_\xi : p \in \mathbb{P}_\alpha\}$.

Definition 4.2.2. For every $\xi < \alpha$, if we let G_ξ be a $\mathbb{P}_\xi$-generic filter over V, and we let P be a subset of $\mathbb{P}_\alpha$, then we define the *quotient of P by G_ξ* as the set $P/G_\xi := \{p \in P : p|_\xi \in G_\xi\}$.

Notation 4.2.3. If $\mathcal{M}$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system and $\mathcal{Z}$ is a subset of $\mathcal{M}$, we will denote by $\overline{\mathcal{Z}}$ the set

$$\overline{\mathcal{Z}} = (\mathcal{Z} \cap \mathcal{L}) \cup \bigcup(\mathcal{Z} \cap \mathcal{T}^+).$$

Definition 4.2.4. Let G_ξ be a $\mathbb{P}_\xi$-generic filter over V and let $M \in \mathcal{S}$. We will denote by $\mathcal{N}^M_{G_\xi}$ the set

$$\mathcal{N}^M_{G_\xi} = \{N \in M : N \in \overline{\mathrm{dom}(\Delta_u)}, u \in G_\xi\}.$$

Moreover, we will denote by $\mathcal{A}^M_{G_\xi}$ the subset of $\mathcal{N}^M_{G_\xi}$,

$$\mathcal{A}^M_{G_\xi} = \{N \in M \cap \mathcal{L}_{\xi+1} : N \in \overline{\Delta_u^{-1}(\xi)}, u \in G_\xi\}.$$

Let us unravel the definition of $\mathcal{A}^M_{G_\xi}$. A model $N \in M \cap \mathcal{L}_{\xi+1}$ belongs to $\mathcal{A}^M_{G_\xi}$ if there is a condition $u \in G_\xi$ such that either

- $N \in \mathrm{dom}(\Delta_u) \cap \mathcal{L}$ and $(N, \xi) \in \mathrm{dom}(\Delta_u)$, or
- $N \in \overline{M}$ for some $\overline{M} \in \mathrm{dom}(\Delta_u) \cap \mathcal{T}^+$ such that $(\overline{M}, \xi) \in \Delta_u$.

The proof of the following lemma is an easy exercise. It uses proposition 2.4.17.

Lemma 4.2.5. *Let G_ξ be a $\mathbb{P}_\xi$-generic filter over V and let $p \in G_\xi$. If M is a small model in* $\mathrm{dom}(\Delta_p)$*, then $\mathcal{N}^M_{G_\xi}$ is an $\mathcal{L}$-symmetric system.*

It will be convenient to consider the following technical variant of the class of $(\mathcal{S},\mathcal{L})$-finitely proper forcings in the context of our construction.

Definition 4.2.6. (In $V[G_\xi]$ for a $\mathbb{P}_\xi$-generic filter G_ξ over V, $\xi<\alpha$) A forcing notion $\mathbb{Q}\subseteq H(\kappa)^V$ is *$(\mathcal{S},\mathcal{L})$-finitely proper with respect to V and G_ξ* if and only if there is a club $D\subseteq [H(\kappa)^V]^{\leq\aleph_1}$ in V with the following property:

Suppose that $\mathcal{M}$ and ν are as follows.

(1) $\mathcal{M}$ is a countable subset of D such that $|\mathcal{M}\cap\mathcal{S}|<\aleph_0$ and $|\mathcal{M}\cap\mathcal{L}|\leq\aleph_0$,

(2) there is $u\in G_\xi$ such that for every $R\in\mathcal{M}$, either

(2.a) $(R,\xi)\in\Delta_u$,

(2.b) $R\in\overline{M}$, where $\overline{M}\in\mathcal{T}^+$ is such that $(\overline{M},\xi)\in\Delta_u$, or

(2.c) $R\in\mathcal{A}^M_{G_\xi}$, where $M\in\mathcal{S}_\delta$ is such that $\sup(M\cap\delta)\leq\xi<\delta$, for some limit ordinal $\delta\leq\kappa$ of cofinality ω_1 and $(M,\xi)\in\Delta_u$,

(3) $\nu\in\mathbb{Q}$, and $\mathbb{Q}\in R[G_\xi]$ for all $R\in\mathcal{M}$, and

(4) $\nu\in R^+[G_\xi]$ for some $R^+\in\mathcal{M}$ such that either,

(4.a) $\mathcal{M}\cap R^+=\varnothing$, or

(4.b) $R^+\in\mathcal{S}$ is such that $\varepsilon_{R^+}=\min\{\varepsilon_M : M\in\mathcal{M}\cap\mathcal{S}\}$, and ν is $(N[G_\xi],\mathbb{Q})$-generic for every N in $\mathcal{M}\cap R^+$.

Then there is an extension $\nu^*\in\mathbb{Q}$ of ν, which is $(R[G_\xi],\mathbb{Q})$-generic for all $R\in\mathcal{M}$.

If $\Phi(\alpha)=\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name in $H(\kappa)$ for a nontrivial $(\mathcal{S},\mathcal{L})$-finitely proper forcing relative to V and $\dot{G}_\alpha$ (the standard name for the $\mathbb{P}_\alpha$-generic filter), then we let $\dot{\mathbb{Q}}_\alpha=\dot{\mathbb{Q}}$. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial forcing on $\{0\}$.

Conditions in $\mathbb{P}_\alpha$ are pairs of the form $p=(F_p,\Delta_p)$ with the following properties.

(C0) F_p is a finite function such that $\mathrm{dom}(F_p)\subseteq\alpha$. We call F_p *the working part of* p and $\mathrm{dom}(F_p)$ *the support of* p.

(C1) Δ_p is a finite set of pairs (Q, γ) with

- $\gamma \leq \min\{\alpha, \sup(Q \cap \kappa)\}$, if $Q \in \mathcal{S} \cup \mathcal{L}$, and
- $\gamma \leq \min\{\alpha, \sup\{\sup(N \cap \kappa) : N \in Q\}\}$, if $Q \in \mathcal{T}^+$.

Moreover, if $(\overline{M}, \gamma) \in \Delta_p$ for some $\overline{M} \in \mathcal{T}^+$, then $(N, \gamma') \in \Delta_p$, for every $N \in \overline{M} \cap \mathrm{dom}(\Delta_p)$ and some $\gamma' \geq \gamma$. We call Δ_p *the side condition of* p.

(C2) For every $\xi < \alpha$, the restriction $p|_\xi$ of p to ξ is a condition in $\mathbb{P}_\xi$.

(C3) If $\xi \in \mathrm{dom}(F_p)$, then $F_p(\xi) \in H(\kappa)$ and $p|_\xi \Vdash_\xi F_p(\xi) \in \dot{\mathbb{Q}}_\xi$.[2]

(C4) If $\xi \in \mathrm{dom}(F_p)$, $(R, \beta) \in \Delta_p$ for some $\beta \geq \xi + 1$, and $R \in \mathcal{S}_{\xi+1} \cup \mathcal{L}_{\xi+1}$, then

$$p|_\xi \Vdash_\xi F_p(\xi) \text{ is } (R[\dot{G}_\xi], \dot{\mathbb{Q}}_\xi)\text{-generic.}$$

(C5) If $\xi \in \mathrm{dom}(F_p)$, $(\overline{M}, \beta) \in \Delta_p$ for some $\beta \geq \xi + 1$, and $\overline{M} \in \mathcal{T}^+$, then for every $N \in \overline{M} \cap \mathcal{L}_{\xi+1}$,

$$p|_\xi \Vdash_\xi F_p(\xi) \text{ is } (N[\dot{G}_\xi], \dot{\mathbb{Q}}_\xi)\text{-generic.}$$

(C6) Suppose that $\eta \in \mathrm{dom}(F_p)$. Suppose that $(M, \beta) \in \Delta_p$ for some $M \in \mathcal{S}_\delta$, where $\beta \geq \eta + 1$ and $\delta \leq \kappa$ is a limit ordinal such that $cf(\delta) = \omega_1$ and $\sup(M \cap \delta) \leq \eta < \delta$. Then, $p|_\eta$ forces that $F_p(\eta)$ is $(N[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic for every $N \in \dot{\mathcal{A}}^M_{\dot{G}_\eta}$.

Given conditions $p = (F_p, \Delta_p)$ and $q = (F_q, \Delta_q)$ in $\mathbb{P}_\alpha$, we define the order $\leq_\alpha$ on $\mathbb{P}_\alpha$ by $q \leq_\alpha p$ if and only if the following holds.

(D1) $q|_\xi \leq_\xi p|_\xi$ for all $\xi < \alpha$.

(D2) $\mathrm{dom}(F_q) \supseteq \mathrm{dom}(F_p)$ and, for all $\xi \in \mathrm{dom}(F_p)$,

$$q|_\xi \Vdash_\xi F_q(\xi) \leq_{\dot{\mathbb{Q}}_\xi} F_p(\xi).$$

[2] Recall from the preliminaries that we will omit inverted circumflexes when dealing with standard names.

(D3) For all $(Q, \beta) \in \Delta_p$ there is some $\beta^* \geq \beta$ such that $(Q, \beta^*) \in \Delta_q$.

The pairs (Q, γ) from the side conditions are called *models with markers*. The reason why we use models with markers is twofold. On one hand, markers grant complete embeddability between intermediate stages of the iteration (see corollary 4.3.4. On the other hand, the markers tell us at what stages of the iteration the models will be generic. More precisely, if $(Q, \gamma) \in \Delta_p$ and $Q \in \mathcal{S} \cup \mathcal{L}$, for some condition $p \in \mathbb{P}_\alpha$, we want to force the working part $F_p(\xi)$, for $\xi \in Q \cap \gamma$, to be generic for $Q[\dot{G}_\xi]$. Similarly, if $\overline{M} \in \mathcal{T}^+$ and $(\overline{M}, \gamma) \in \Delta_p$, we want to force the working part $F_p(\xi)$ to be generic for $N[\dot{G}_\xi]$, for all $N \in \overline{M}$ such that $\xi \in N$. These are clauses (C4) and (C5), respectively. Hence, the marker γ for the pair (Q, γ) is a device that tells us up to which stage is Q to be seen as "active" as a model in the side condition. This is why we will say that a model $Q \in \mathrm{dom}(\Delta_p) \cup \bigcup \mathrm{dom}(\Delta_p)$ is *active at* $\xi + 1$ if either

- $(Q, \xi + 1) \in \Delta_{p|_{\xi+1}}$ and $Q \in \mathcal{S}_{\xi+1} \cup \mathcal{L}_{\xi+1}$, or
- $Q \in \overline{M} \cap \mathcal{L}_{\xi+1}$ and $(\overline{M}, \xi + 1) \in \Delta_{p|_{\xi+1}}$, for some $\overline{M} \in \mathcal{T}^+$.

Remark 4.2.7. Let $p \in \mathbb{P}_\alpha$ for some $\alpha \leq \kappa$, and let $Q \in \mathcal{S}_\beta \cap \mathcal{L}_\beta$. If $(Q, \gamma) \in \Delta_p$ or $Q \in \overline{M}$ for some $\overline{M} \in \mathrm{dom}(\Delta_p) \cap \mathcal{T}^+$ such that $(\overline{M}, \gamma) \in \Delta_p$, then Q is active at all $\xi \in Q \cap \min\{\gamma, \beta\}$.

It is also worth noting that given a condition $p \in \mathbb{P}_\alpha$, a model $Q \in \mathrm{dom}(\Delta_p)$ might have multiple markers. However, in practice we only care about the largest one. In the next section we will see exactly what we mean by this.

There is one significant difference between Asperó and Mota's finite support iterations with one-type symmetric systems and our two-type version. This difference, which is reflected in clause (C6) above, is a consequence of the coexistence of models of two different types in our side conditions. Suppose that we have a condition p and a stage α of cofinality ω_1. Suppose that M and Q are, respectively, a small model and a model of arbitrary type in $\mathrm{dom}(\Delta_p)$, such that $\varepsilon_Q < \varepsilon_M$. Note that, in light of lemma 1.4.11, $M \cap \alpha$ is bounded below α, and thus, it cannot be active at stages β such that $\sup(M \cap \alpha) \leq \beta < \alpha$. If Q is a small model, we can argue by symmetry that $\sup(Q \cap M \cap \alpha) < \sup(M \cap \alpha)$, and hence, Q cannot be active at stages beyond $\sup(M \cap \alpha)$ either. However, if Q is a large model, then $Q \cap \alpha$ is unbounded in α, and thus, it may be active at stages $\beta \geq \sup(M \cap \alpha)$. This is in fact one of the reasons why we are forced to use $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems as side conditions in our iterations, instead of the more natural $(\mathcal{S}, \mathcal{L})$-symmetric systems. We will get back to this matter in the proof of $(\mathrm{P2})_\alpha$ in section 4.4, where we will explain the exact reasons why we need to add non-elementary models to our symmetric systems.

4.3 General facts and amalgamation lemmas

All the results appearing in this section are essentially a translation of lemmas 2.9-2.15 from [12]. It is worth noting that they are all general facts about the iteration, and that they are independent of the class of forcings that we are iterating. That is, we won't use anywhere the fact that the iterands are names for $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notions. Let us start with some basic observations about the iterations.

Note that for every $\alpha \leq \kappa$, the poset $\mathbb{P}_\alpha$ is a subset of $H(\kappa)$. Hence, as $|H(\kappa)| = 2^{<\kappa} = \kappa$ by assumption, $\mathbb{P}_\alpha \in H(\kappa^+)$.

For all $\alpha < \beta \leq \kappa$ and every $R^* \in \mathcal{S}^*_\beta \cup \mathcal{L}^*_\beta$, if $\alpha \in R^*$, then $\mathcal{S}^*_\alpha, \mathcal{L}^*_\alpha, \mathbb{P}_\alpha, \Vdash_\alpha \in R^*$. Moreover, if $\beta < \kappa$ is a nonzero limit ordinal, then $\mathbb{P}_\beta, \Vdash_\beta \in R^*$.

Note that if $\alpha < \beta \leq \kappa$, it follows from the definition of the iteration that $\mathbb{P}_\alpha \subseteq \mathbb{P}_\beta$. Moreover, if p is a condition in $\mathbb{P}_\beta$ such that $\mathrm{dom}(F_p) \cup \mathrm{Im}(\Delta_p) \subseteq \alpha$, then p is a condition in $\mathbb{P}_\alpha$, and in fact $p = p|_\alpha$.

In the definition of $\mathbb{P}_\alpha$ we haven't made any distinctions regardless of whether α was a successor or limit ordinal, but note that if α is a nonzero limit ordinal, then a pair $p = (F_p, \Delta_p)$ is a condition in $\mathbb{P}_\alpha$ if and only if it satisfies (C0)-(C2). The next result follows from these facts.

Lemma 4.3.1. *For all $\alpha \leq \beta \leq \kappa$, $\mathbb{P}_\alpha$ is nonempty and $\mathbb{P}_\alpha \subseteq \mathbb{P}_\beta$. Moreover, $\mathbb{P}_\kappa = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$.*

Also note that for every $\alpha < \beta \leq \kappa$, if p is a condition in $\mathbb{P}_\beta$, then $\mathrm{dom}(\Delta_p) = \mathrm{dom}(\Delta_{p|_\alpha})$. In particular, since $\mathrm{dom}(\Delta_p) = \mathrm{dom}(\Delta_{p|_0})$, the domain of p is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, because $p|_0 \in \mathbb{P}_0$ by (C2).

If $p \in \mathbb{P}_\alpha$ for $\alpha \leq \kappa$, $(Q, \gamma) \in \Delta_p$ and δ is an ordinal smaller than γ, then $p^* = \big(F_p, \Delta_p \cup \{(Q, \delta)\}\big)$ is a condition in $\mathbb{P}_\alpha$. Note that Δ_p and Δ_{p^*} have the same domain, and in fact, p and p^* are clearly equivalent. This is exactly what we meant above, when we said that we only care about the largest marker of a model in a side condition.

More generally, it follows from the last two paragraphs that if $\alpha \leq \beta \leq \kappa$, $p \in \mathbb{P}_\alpha$, and $\{p_i : i < n\}$ is a finite set of conditions in $\mathbb{P}_\beta$ such that $p \leq_\alpha p_i|_\alpha$ for all $i < n$, then $\mathrm{dom}(\Delta_p) = \mathrm{dom}(\Delta_{p|_0}) \supseteq \mathrm{dom}(\Delta_{p_i}) = \mathrm{dom}(\Delta_{p_i|_\alpha})$ and the pair $\big(F_p, \Delta_p \cup \bigcup\{\Delta_{p_i|_\alpha} : i < n\}\big)$ is a condition in $\mathbb{P}_\alpha$, equivalent to p. The following lemma summarizes this.

Lemma 4.3.2. *Let $\alpha \leq \beta \leq \kappa$. If $p \in \mathbb{P}_\alpha$ and $\{p_i : i < n\}$ is a finite set of conditions in $\mathbb{P}_\beta$ such that $p \leq_\alpha p_i|_\alpha$ for all $i < n$, then $(F_p, \Delta_p \cup \bigcup\{\Delta_{p_i|_\alpha} : i < n\})$ is a condition in $\mathbb{P}_\alpha$ equivalent to p.*

There are two main conclusions that we can extract from this lemma. Let $\alpha \leq \kappa$, suppose that p is a condition in $\mathbb{P}_\alpha$, and let $(Q, \gamma) \in \Delta$. On one hand, adding the pair (Q, γ') to Δ_p, for some $\gamma' < \gamma$, results in an equivalent condition. On the other hand, if $(Q, \delta) \in \Delta_p$ for some $\delta < \gamma$, removing the pair (Q, δ) from Δ_p, also results in an equivalent condition.

Lemma 4.3.3. *Let $\alpha \leq \beta \leq \kappa$. If $q = (F_q, \Delta_q) \in \mathbb{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathbb{P}_\beta$ and $q \leq_\alpha r|_\alpha$, then*

$$r \wedge_\alpha q := \big(F_q \cup (F_r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_r\big)$$

is a condition in $\mathbb{P}_\beta$ *extending* r*.*

Proof. We show the result by induction on $\beta \geq \alpha$. If $\beta = \alpha$, then $q \leq_\alpha r$, and hence, $r \wedge_\alpha q$ is defined as $\big(F_q, \Delta_q \cup \Delta_r\big)$, which is equivalent to q. Therefore, the base case follows from lemma 4.3.2.

Assume now that $\beta = \gamma + 1$ where $\gamma \geq \alpha$. Clauses (C0) and (C1) in the definition of $\mathbb{P}_{\gamma+1}$ are clearly satisfied. On one hand, note that since $r|_\gamma \in \mathbb{P}_\gamma$, by induction hypothesis

$$r|_\gamma \wedge_\alpha q = \big(F_q \cup (F_r \restriction [\alpha, \gamma)), \Delta_q \cup \Delta_{r|\gamma}\big)$$

is a condition in $\mathbb{P}_\gamma$. On the other hand note that

$$\begin{aligned}\Delta_{(r\wedge_\alpha q)|_\gamma} &= \{(R, \min\{\delta, \gamma\}) : (R, \delta) \in \Delta_q \cup \Delta_r\} \\ &= \Delta_q \cup \Delta_{r|_\gamma}.\end{aligned}$$

Therefore, $(r \wedge_\alpha q)|_\gamma = r|_\gamma \wedge_\alpha q$, and by induction hypothesis $(r \wedge_\alpha q)|_\gamma$ is a condition in $\mathbb{P}_\gamma$ extending $r|_\gamma$. This is enough to show that clause (C2) holds for $r \wedge_\alpha q$. If $\gamma \notin \mathrm{dom}(F_r)$, then clauses (C3)-(C6) for $r \wedge_\alpha q$ follow from induction hypothesis, so we can assume that $\gamma \in \mathrm{dom}(F_r)$. Let $(R, \delta) \in \Delta_{r\wedge_\alpha q} = \Delta_q \cup \Delta_r$ for $\delta \geq \gamma + 1$ such that R is active at $\gamma + 1$. Since $q \in \mathbb{P}_\alpha$, the range of Δ_q must be contained in $\alpha + 1$, and as $\delta \leq \gamma + 1$, (R, δ) must necessarily be an element of Δ_r, by (C1) applied to q. Moreover, since $r \in \mathbb{P}_{\gamma+1}$, then $\delta \leq \gamma + 1$, and thus, the equality $\delta = \gamma + 1$ must hold. In summary, $(R, \delta) = (R, \gamma + 1) \in \Delta_r$ and, by definition of $r \wedge_\alpha q$, $F_{r\wedge_\alpha q}(\gamma) = F_r(\gamma)$. But then, clauses (C3)-(C6) follow directly from the fact that $(r \wedge_\alpha q)|_\gamma$ is an extension of $r|_\gamma$ by induction hypothesis.

If β is a nonzero limit ordinal such that $\alpha < \beta$, then clauses (C0)-(C6) follow directly from the induction hypothesis. □

The following result tells us that $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ can be seen, in a broad sense, as a forcing iteration.

Corollary 4.3.4. *For every $\alpha < \beta \leq \kappa$, every maximal antichain in $\mathbb{P}_\alpha$ is a maximal antichain in $\mathbb{P}_\beta$, and therefore $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$.*

Lemma 4.3.5. *Let $\alpha < \kappa$ and let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be conditions in $\mathbb{P}_{\alpha+1}$ such that there is a $\mathbb{P}_\alpha$-name $\dot{x} \in H(\kappa)$, a condition $r = (F_r, \Delta_r)$ in $\mathbb{P}_\alpha$, and a finite set $\{R_j : j \in n\}$ with the following properties:*

(a) *For all $j < n$,*

- $(R_j, \alpha) \in \Delta_r$,
- $\alpha + 1 \leq \sup(R_j \cap \kappa)$, *if* $R_j \in \mathcal{S} \cup \mathcal{L}$, *and*
- $\alpha + 1 \leq \sup\{\sup(N \cap \kappa) : N \in R_j\}$, *if* $R_j \in \mathcal{T}^+$.

(b) *r extends both $q_0|_\alpha$ and $q_1|_\alpha$,*

(c) *$\alpha \in \mathrm{dom}(F_0) \cap \mathrm{dom}(F_1)$ and $r \Vdash_\alpha$ "$\dot{x} \in \dot{\mathbb{Q}}_\alpha$ and $\dot{x} \leq_{\dot{\mathbb{Q}}_\alpha} F_0(\alpha), F_1(\alpha)$",*

(d) *$r \Vdash_\alpha$ "$\dot{x}$ is $(R_j[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all $j < n$ such that $R_j \in \mathcal{S}_{\alpha+1} \cup \mathcal{L}_{\alpha+1}$,*

(e) *$r \Vdash_\alpha$ "$\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all $N \in R_j \cap \mathcal{L}_{\alpha+1}$ and all $j < n$ such that $R_j \in \mathcal{T}^+$, and*

(f) *for all $j < n$, if $R_j \in \mathcal{S}_\delta$ is such that $\delta \leq \kappa$ is a limit ordinal with $cf(\delta) = \omega_1$ and $\sup(R_j \cap \delta) \leq \alpha < \delta$, then r forces that $\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic for every $N \in \dot{\mathcal{A}}^{R_j}_{\dot{G}_\alpha}$.*

Then,

$$q_2 = \big(F_r \cup \{\langle \alpha, \dot{x} \rangle\}, \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(R_j, \alpha + 1) : j \in n\}\big)$$

is a condition in $\mathbb{P}_{\alpha+1}$ extending both q_0 and q_1.

Proof. First we show that $q_2 \in \mathbb{P}_{\alpha+1}$. Showing that q_2 satisfies clause (C0) is straightforward, and (C1) follows from (a). Note that

$$q_2|_\alpha = (F_r, \Delta_r \cup \Delta_{q_0|_\alpha} \cup \Delta_{q_1|_\alpha}).$$

Hence, it follows from (b) and lemma 4.3.2 that r and $q_2|_\alpha$ are equivalent conditions in $\mathbb{P}_\alpha$, so q_2 also satisfies (C2). Clause (C3) follows from (c) and the equivalence between the conditions r and $q_2|_\alpha$. Let us check (C4) now. Let $\xi \in \mathrm{dom}(F_{q_2})$ and $S \in \mathcal{S}_{\xi+1} \cup \mathcal{L}_{\xi+1}$ such that $(S, \beta) \in \Delta_{q_2}$, for some $\beta \geq \xi + 1$. If $\xi < \alpha$, the result follows from (b) and the fact that r and $q_2|_\alpha$ are equivalent. Thus, we can assume that $\xi = \alpha$. Note that, in this case, $\beta = \alpha + 1$ and $(S, \beta) \in \Delta_0 \cup \Delta_1 \cup \{(R_j, \alpha + 1) : j \in n\}$. Hence, we have to show that r forces in $\mathbb{P}_\alpha$ that $\dot{x}$ is $(S[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic. If $(S, \alpha + 1) \in \{(R_j, \alpha + 1) : j \in n\}$, this follows from (d). If $(S, \alpha + 1) \in \Delta_i$ for some $i \in \{0, 1\}$, since $\alpha \in \mathrm{dom}(F_i)$ by (c), $q_i|_\alpha$ forces that $F_i(\alpha)$ is $(S[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic by (C4) applied to $q_i|_\alpha$. Therefore, r must force the same thing because it extends $q_i|_\alpha$, and since r forces that $\dot{x} \leq_{\dot{\mathbb{Q}}_\alpha} F_i(\alpha)$ by (c), it also forces that $\dot{x}$ is $(S[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic. Clauses (C5) and (C6) follow from a similar argument using (e) and (f), respectively, instead of (d). Lastly, note that (b), (c), and $\Delta_i \subseteq \Delta_{q_2}$ imply together that q_2 extends q_i for $i \in \{0, 1\}$. □

Lemma 4.3.6. *Let $\alpha < \kappa$ and let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be conditions in $\mathbb{P}_{\alpha+1}$, $r = (F_r, \Delta_r)$ a condition in $\mathbb{P}_\alpha$, and a finite set $\{R_j : j \in n\}$ with the following properties:*

(a) *For all $j < n$,*

- $(R_j, \alpha) \in \Delta_r$,
- $\alpha + 1 \leq \sup(R_j \cap \kappa)$, *if* $R_j \in \mathcal{S} \cup \mathcal{L}$, *and*
- $\alpha + 1 \leq \sup\{\sup(N \cap \kappa) : N \in R_j\}$, *if* $R_j \in \mathcal{T}^+$.

(b) *r extends both $q_0|_\alpha$ and $q_1|_\alpha$, and*

(c) *$\alpha \notin \mathrm{dom}(F_0) \cup \mathrm{dom}(F_1)$.*

Then,

$$q_2 = \big(F_r, \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(R_j, \alpha + 1) : j \in n\}\big)$$

is a condition in $\mathbb{P}_{\alpha+1}$ extending both q_0 and q_1.

Suppose, in addition, that $\dot{x}$ is a $\mathbb{P}_\alpha$-name in $H(\kappa)$ is such that

(d) $r \Vdash_\alpha$ "*$\dot{x}$ is $(R_j[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all $j < n$ such that $R_j \in \mathcal{S}_{\alpha+1} \cup \mathcal{L}_{\alpha+1}$,*

(e) $r \Vdash_\alpha$ "*$\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all $N \in R_j \cap \mathcal{L}_{\alpha+1}$ and all $j < n$ such that $R_j \in \mathcal{T}^+$,*

(f) *for all $j < n$, if $R_j \in \mathcal{S}_\delta$ is such that $\delta \leq \kappa$ is a limit ordinal with $cf(\delta) = \omega_1$ and $\sup(R_j \cap \delta) \leq \alpha < \delta$, then r forces that $\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic for every $N \in \dot{\mathcal{A}}^{R_j}_{\dot{G}_\alpha}$,*

(g) $r \Vdash_\alpha$ "*$\dot{x}$ is $(S[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all S such that $(S, \alpha+1) \in \Delta_0 \cup \Delta_1$ and $S \in \mathcal{S}_{\alpha+1} \cup \mathcal{L}_{\alpha+1}$,*

(h) $r \Vdash_\alpha$ "*$\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic", for all $N \in \overline{M} \cap \mathcal{L}_{\alpha+1}$ and all $\overline{M}$ such that $(\overline{M}, \alpha+1) \in \Delta_0 \cup \Delta_1$ and $\overline{M} \in \mathcal{T}^+$, and*

(i) *if $M \in \mathcal{S}_\delta$ is such that $(M, \alpha+1) \in \Delta_0 \cup \Delta_1$ and $\delta \leq \kappa$ is a limit ordinal with $cf(\delta) = \omega_1$ and $\sup(M \cap \delta) \leq \alpha < \delta$, then r forces that $\dot{x}$ is $(N[\dot{G}_\alpha], \dot{\mathbb{Q}}_\alpha)$-generic for every $N \in \dot{\mathcal{A}}^M_{\dot{G}_\alpha}$.*

Then,

$$q_2' = \big(F_r \cup \{\langle \alpha, \dot{x} \rangle\}, \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(R_j, \alpha+1) : j \in n\}\big)$$

is a condition in $\mathbb{P}_{\alpha+1}$ extending both q_0 and q_1.

Proof. Same proof as lemma 4.3.5. □

Lemma 4.3.7. *Assume that $0 \leq \gamma < \alpha \leq \kappa$. Let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be conditions in $\mathbb{P}_\alpha$ such that $\operatorname{dom}(F_0) \cup \operatorname{dom}(F_1) \subseteq \gamma$, and such that there exists a condition $r \in \mathbb{P}_\gamma$ extending both $q_0|_\gamma$ and $q_1|_\gamma$. Then,*

$$q_2 = (F_r, \Delta_r \cup \Delta_0 \cup \Delta_1)$$

is a condition in $\mathbb{P}_\alpha$ extending both q_0 and q_1.

Proof. We show by induction on β, $\gamma \leq \beta \leq \alpha$, that $q_2|_\beta$ is a condition in $\mathbb{P}_\beta$ extending both $q_0|_\beta$ and $q_1|_\beta$. The base case follows from lemma 4.3.2. The successor step follows from lemma 4.3.6. The limit step follows easily from the induction hypothesis. $\square$

Lemma 4.3.8. *Let $\beta \leq \kappa$ and let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be two conditions in $\mathbb{P}_\beta$. For every $i \in \{0, 1\}$, let*

$$A_i = \mathrm{dom}(F_{q_i}) \cup \bigcup\{R \cap \beta : R \in \mathrm{dom}(\Delta_i) \cap (\mathcal{S} \cup \mathcal{L})\}$$
$$\cup \bigcup\{N \cap \beta : N \in \bigcup(\mathrm{dom}(\Delta_i) \cap \mathcal{T}^+)\},$$

and let B_i be the set of all $\nu < \beta$ such that $(M, \nu + 1) \in \Delta_i$, for some $M \in \mathcal{S}_\delta$, where $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \nu < \delta$. Define Z_i as the union of A_i and B_i. Let $\alpha \leq \beta$ be such that $Z_0 \cap Z_1 \subseteq \alpha$, and assume that there is a condition $r = (F_r, \Delta_r)$ in $\mathbb{P}_\alpha$ extending both $q_0|_\alpha$ and $q_1|_\alpha$. Define $F_r^{0,1}$ as $F_r \cup \big(F_0 \restriction [\alpha, \beta)\big) \cup \big(F_1 \restriction [\alpha, \beta)\big)$. Then,

$$(q_0 \wedge q_1) \wedge_\alpha r := (F_r^{0,1}, \Delta_r \cup \Delta_0 \cup \Delta_1)$$

is a condition in $\mathbb{P}_\beta$ extending q_0 and q_1.

Proof. We show the result by induction on $\beta \geq \alpha$. If $\beta = \alpha$, then $(q_0 \wedge q_1) \wedge_\alpha r = (F_r, \Delta_r \cup \Delta_0 \cup \Delta_1)$, so it is equivalent to r by lemma 4.3.2.

Assume now that $\beta = \gamma + 1 > \alpha$. It is immediate to see that $(q_0 \wedge q_1) \wedge_\alpha r$ satisfies clauses (C0) and (C1). Note that

$$\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma = \big(F_r \cup (F_0 \restriction [\alpha, \gamma)) \cup (F_1 \restriction [\alpha, \gamma)), \Delta_r \cup \Delta_{q_0|_\gamma} \cup \Delta_{q_1|_\gamma}\big)$$
$$= (q_0|_\gamma \wedge q_1|_\gamma) \wedge_\alpha r.$$

Hence, by induction hypothesis $\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma$ is a condition in $\mathbb{P}_\gamma$ extending both $q_0|_\gamma$ and $q_1|_\gamma$, and this suffices to show that $(q_0 \wedge q_1) \wedge_\alpha r$ satisfies (C2). Moreover, it follows that $(q_0 \wedge q_1) \wedge_\alpha r$ satisfies clauses (C3)-(C6) for every $\xi < \gamma$

such that $\xi \in \mathrm{dom}(F_r^{0,1})$. Assume now that $\gamma \in \mathrm{dom}(F_r^{0,1})$, and let us check clauses (C3)-(C6) with respect to γ. It follows from the definition of $F_r^{0,1}$, from $\gamma \geq \alpha$, and from $Z_0 \cap Z_1 \subseteq \alpha$, that for some $i \in \{0,1\}$ and $j \in \{0,1\} \setminus \{i\}$, $\gamma \in \mathrm{dom}(F_i) \setminus \mathrm{dom}(F_j)$ and $F_r^{0,1}(\gamma) = F_i(\gamma)$. As q_i satisfies (C3), the restriction $q_i|_\gamma$ forces that $F_i(\gamma)$ is an element of $\dot{\mathbb{Q}}_\gamma$. Therefore, since $\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma$ extends $q_i|_\gamma$, it forces in $\mathbb{P}_\gamma$ that $F_r^{0,1}(\gamma)$ is an element of $\dot{\mathbb{Q}}_\gamma$. This shows that $(q_0 \wedge q_1) \wedge_\alpha r$ satisfies clause (C3). Now let $R \in \mathcal{S}_{\gamma+1} \cup \mathcal{L}_{\gamma+1}$ such that $(R, \delta) \in \Delta_r \cup \Delta_0 \cup \Delta_1$ and $\delta \geq \gamma + 1$. Since $r \in \mathbb{P}_\alpha$ and $\alpha \leq \gamma$, $(R, \delta) \notin \Delta_r$, and as $q_0, q_1 \in \mathbb{P}_{\gamma+1}$, we have that $\delta \leq \gamma + 1$, and thus, $\delta = \gamma + 1$. Since $Z_0 \cap Z_1 \subseteq \alpha$ and $\gamma \in R$, we have that $R \in \mathrm{dom}(\Delta_i)$ for the same i as above. But then, since q_i satisfies (C4), $q_i|_\gamma$ forces in $\mathbb{P}_\gamma$ that $F_i(\gamma)$ is $(R[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-generic, and hence, as $\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma$ extends $q_i|_\gamma$, it forces in $\mathbb{P}_\gamma$ that $F_r^{0,1}(\gamma)$ is $(R[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-generic. Therefore, clause (C4) holds for $(q_0 \wedge q_1) \wedge_\alpha r$. Clause (C5) is proven analogously. Let us check clause (C6) now. Let $(M, \eta) \in \Delta_r \cup \Delta_0 \cup \Delta_1$ such that $\eta \geq \gamma + 1$ for some $M \in \mathcal{S}_\delta$, where $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \gamma < \delta$. By the same reason as above, $\eta = \gamma + 1$, and hence, $(M, \gamma + 1)$ must be an element of $\Delta_0 \cup \Delta_1$. Since $B_0 \cap B_1 \subseteq Z_0 \cap Z_1 \subseteq \alpha \leq \gamma$ and $\gamma \in B_0 \cup B_1$, for the same i as above, $\gamma \in B_i$. In other words, $(M, \gamma + 1) \in \Delta_i$. But then we are done because the conclusion of clause (C6) follows from the fact that q_i satisfies (C6) and that $\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma$ extends $q_i|_\gamma$. Lastly, we have to check that $(q_0 \wedge q_1) \wedge_\alpha r$ extends q_0 and q_1. Note that $\big((q_0 \wedge q_1) \wedge_\alpha r\big)|_\gamma \leq_\gamma q_0|_\gamma, q_1|_\gamma$ implies that (D1) and (D2) hold for $\xi \in \gamma \cap \mathrm{dom}(F_r^{0,1})$. To see that (D2) holds for γ, assume that $\gamma \in \mathrm{dom}(F_r^{0,1})$ and note that it follows from the definition of $F_r^{0,1}$, from $\gamma \geq \alpha$, and from $Z_0 \cap Z_1 \subseteq \alpha$ that for some $i \in \{0,1\}$ and $j \in \{0,1\} \setminus \{i\}$, $\gamma \in \mathrm{dom}(F_i) \setminus \mathrm{dom}(F_j)$ and $F_r^{0,1}(\gamma) = F_i(\gamma)$. Finally, (D3) follows from the fact that $\Delta_0 \cup \Delta_1 \subseteq \Delta_{(q_0 \wedge q_1) \wedge_\alpha r}$.

If β is a nonzero limit ordinal greater than α, the conclusion follows directly from the induction hypothesis. □

4.4 Preservation lemmas

We will devote this section to proving the lemmas necessary to show that $\mathbb{P}_\alpha$ is $\mathcal{S}^*_\alpha$-proper, $\mathcal{L}^*_\alpha$-proper, and has the $\aleph_3$-chain condition, for every $\alpha \leq \kappa$. We will also point out (mainly in the proof of $(\mathrm{P2})_\alpha$) the reason why the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings is defined the way it is.

Definition 4.4.1. Let $\alpha \leq \kappa$ and let $p \in \mathbb{P}_\alpha$. If $Q \in \mathcal{S} \cup \mathcal{L} \cup \mathcal{T}^+$, we will say that *$p$ is $(Q, \mathbb{P}_\alpha)$-pre-generic* in case

(i) $\alpha < \kappa$ and the pair (Q, α) is in Δ_q,

(ii) $\alpha = \kappa$, $Q \in \mathcal{S} \cup \mathcal{L}$, and the pair $(Q, \sup(Q \cap \kappa))$ is in Δ_q, or else

(iii) $\alpha = \kappa$, $Q \in \mathcal{T}^+$, and the pair $(Q, \sup\{\sup(N \cap \kappa) : N \in Q\})$ is in Δ_q.

Our goal is to prove the following lemma.

Lemma 4.4.2. *For every $\alpha \leq \kappa$, the following conditions hold.*

$(P1)_\alpha$ If $Q^* \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$ and $Q = Q^* \cap H(\kappa)$, then for every $p \in \mathbb{P}_\alpha \cap Q$ there is $q \leq_\alpha p$ such that q is $(Q, \mathbb{P}_\alpha)$-pre-generic.

$(P2)_\alpha$ Let $Q^* \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$ and $Q = Q^* \cap H(\kappa)$. If $p \in \mathbb{P}_\alpha$ is a $(Q, \mathbb{P}_\alpha)$-pre-generic condition, then p is $(Q^*, \mathbb{P}_\alpha)$-generic.

$(P3)_\alpha$ Let $\overline{M} \in \mathcal{T}^+$. If $p \in \mathbb{P}_\alpha$ is an $(\overline{M}, \mathbb{P}_\alpha)$-pre-generic condition, then p is $(N^*, \mathbb{P}_\alpha)$-generic for every $N^* \in \mathcal{L}^*_\alpha$ such that $N^* \cap H(\kappa) \in \overline{M}$.

$(P4)_\alpha$ $\mathbb{P}_\alpha$ has the $\aleph_3$-chain condition.

The proof will go by induction on $\alpha \leq \kappa$, and in the precise order established in the statement of the lemma. Some of the arguments are inspired by the proof of lemma 2.22 from [12]. But before getting into the proof of lemma 4.4.2, let us show some technical results that will be useful later on.

Recall from the last section that we observed that $\mathbb{P}_\alpha \in H(\kappa^+)$. The following result is just a restatement of lemma 1.1.36.

Lemma 4.4.3. *Let $\alpha \leq \kappa$ and $Q^* \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$. If G_α is a $\mathbb{P}_\alpha$-generic filter over V, then $H(\kappa^+)^{Q^*[G_\alpha]} \preceq H(\kappa^+)^{V[G_\alpha]}$.*

The following lemmas improve some of the results from the preliminaries, exploiting the fact that the iteration will have the $\aleph_3$-chain condition.

Lemma 4.4.4. *Let $\mathbb{P} \subseteq H(\kappa)$ be a forcing notion with the μ-chain condition for some $\mu \leq \kappa$. Let $\theta > \kappa$ be a big enough cardinal so that $\mathbb{P} \in H(\theta)$. If $Q \preceq H(\theta)$ is such that $\kappa, \mathbb{P} \in Q$, and G is a $\mathbb{P}$-generic filter over V, then $(Q \cap H(\kappa)^V)[G] = Q[G] \cap H(\kappa)^V[G]$.*

Proof. The inclusion $(Q \cap H(\kappa)^V)[G] \subseteq Q[G] \cap H(\kappa)^V[G]$ is clear. Let us show the other direction. It's enough to show that $Q[G] \cap H(\kappa)^V[G] \cap OR$ is a subset of $(Q \cap H(\kappa)^V)[G] \cap OR$. First of all, note that since $\mathbb{P} \subseteq H(\kappa)$, then $H(\kappa)^V[G] \cap V = H(\kappa)^V$. Hence, $Q[G] \cap H(\kappa)^V[G] \cap OR = Q[G] \cap H(\kappa)^V \cap OR$. Let $\check{\gamma} \in Q$ be a canonical $\mathbb{P}$-name such that $\check{\gamma}_G = \gamma \in H(\kappa)^V \cap OR$. We have to show that $\check{\gamma} \in H(\kappa)^V$. We may assume that $\check{\gamma}$ is of the form $\{\langle \check{x}, q\rangle : q \in A\}$, where A is a maximal antichain and the $\check{x}$ are in $H(\kappa)^V$. Since $\mathbb{P}$ has the μ-chain condition, $|A| < \mu \leq \kappa$. Therefore, $\check{\gamma} \in H(\kappa)^V$, as we wanted. □

Lemma 4.4.5. *If $\mathbb{P} \subseteq H(\kappa)$ is a forcing notion with the μ-chain condition for some $\mu \leq \kappa$, then $\mathbb{P}$ forces that $H(\kappa)^V[\dot{G}] = H(\kappa)^{V[\dot{G}]}$. In other words, if τ is a $\mathbb{P}$-name, then $\tau \in H(\kappa)^V$ if and only if $\Vdash_{\mathbb{P}} \tau \in H(\kappa)^{V[\dot{G}]}$.*

Proof. Let τ be a $\mathbb{P}$-name. Suppose first that $\tau \in H(\kappa)^V$. Let G be a $\mathbb{P}$-generic filter over V. We will show by induction on the rank of τ that $\tau_G \in H(\kappa)^{V[G]}$. The base case is trivial. Suppose that τ has rank α, and that $\sigma_G \in H(\kappa)^{V[G]}$ for every $\mathbb{P}$-name σ in $H(\kappa)^V$ of rank $< \alpha$. The elements of τ are of the form (τ', p), where τ' is a $\mathbb{P}$-name of rank $< \alpha$ that belongs to $H(\kappa)^V$ (because $\tau \in H(\kappa)^V$), and p is a condition in $\mathbb{P}$. Therefore, by induction hypothesis, $\tau'_G \in H(\kappa)^{V[G]}$ for each $\tau' \in \mathrm{dom}(\tau)$. Hence, $\tau_G = \{\tau'_G : (\tau', p) \in \tau, p \in G\} \subseteq H(\kappa)^{V[G]}$, and as κ is preserved because $\mathbb{P}$ has the μ-chain condition, and $|\tau_G| \leq |\tau| \leq |\operatorname{trcl}(\tau)| < \kappa^V$, then $\tau_G \in H(\kappa)^V[G]$.

Assume now that $\mathbb{P}$ forces that $\tau \in H(\kappa)^{V[G]}$, i.e., that $|\operatorname{trcl}(\tau)| < \kappa^{V[\dot{G}]}$. Again, we argue by induction on the rank of τ. The base case is trivial. Suppose that τ has rank α, and that every $\mathbb{P}$-name σ of rank $< \alpha$ forced by $\mathbb{P}$ to be in $H(\kappa)^{V[\dot{G}]}$, is such that $\sigma \in H(\kappa)^V$. Note that, as $\mathbb{P}$ forces $\tau \in H(\kappa)^{V[\dot{G}]}$, every $\tau' \in \mathrm{dom}(\tau)$ is forced by $\mathbb{P}$ to be in $H(\kappa)^{V[\dot{G}]}$. Hence, by induction hypothesis $\tau' \in H(\kappa)^V$, and thus, it suffices to check that $|\tau| < \kappa^V$. But this follows from the fact that $\mathbb{P}$ forces that $|\tau| \leq |\operatorname{trcl}(\tau)| < \kappa^{V[\dot{G}]}$ and that κ is preserved thanks to the μ-chain condition of $\mathbb{P}$. □

Lemma 4.4.6. *Let $\mathbb{P} \subseteq H(\kappa)$ be a forcing notion with the μ-chain condition for some $\mu \leq \kappa$, and let $\theta > \kappa$ be a big enough cardinal so that $\mathbb{P} \in H(\theta)$. If $Q \preceq H(\theta)$ is such that $\kappa, \mathbb{P} \in Q$, and G is a $\mathbb{P}$-generic filter over V, then*

$$(Q \cap H(\kappa)^V)[G] = Q[G] \cap H(\kappa)^V[G] = Q[G] \cap H(\kappa)^{V[G]} \preceq H(\kappa)^{V[G]}.$$

Proof. Note that by lemma 1.1.36, $Q[G] \preceq H(\theta)^V[G]$, and thus, by lemma 1.1.35, $Q[G] \preceq H(\theta)^{V[G]}$. Hence, we have that $Q[G] \cap H(\kappa)^{V[G]} \preceq H(\kappa)^{V[G]}$ by proposition 1.4.8. The other two equalities follow from lemmas 4.4.4 and 4.4.5. □

4.4.1 Proof of (P1)$_\alpha$

Let $Q^* \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$ and $Q = Q^* \cap H(\kappa)$. Let $p \in \mathbb{P}_\alpha \cap Q$. We have to find an extension $q \in \mathbb{P}_\alpha$ of p such that q is $(Q, \mathbb{P}_\alpha)$-pre-generic.

⋆ **Base case**

Suppose that $\alpha = 0$.

Lemmas 2.4.19 and 2.4.18 ensure that there is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system

$\mathcal{M}$ extending $\mathrm{dom}(\Delta_p)$ and containing Q. Hence, $q = (\varnothing, \{(R, 0) : R \in \mathcal{M}\})$ is a $(Q, \mathbb{P}_0)$-pre-generic condition extending p.

⋆ **Successor case**

Suppose that $\alpha = \gamma + 1$ and that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta \leq \gamma$.

Assume first that $\gamma \in \mathrm{dom}(F_p)$. By $(\mathrm{P1})_\gamma$, since $p|_\gamma \in Q$, we may also assume, by extending $p|_\gamma$ if necessary, that $p|_\gamma$ is $(Q, \mathbb{P}_\gamma)$-pre-generic, i.e., that $(Q, \gamma) \in \Delta_{p|_\gamma}$. Let $\dot{D}$ be the $\lhd$-least $\mathbb{P}_\gamma$-name for a club D of $([H(\kappa)]^{\leq\aleph_1})^V$ in V such that D witnesses the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_\gamma$ with respect to V and $\dot{G}_\gamma$. Since $Q^* \in \mathcal{S}^*_{\gamma+1} \cup \mathcal{L}^*_{\gamma+1}$, the well-order $\lhd$ and the poset $\mathbb{P}_\gamma$ belong to Q^*, and hence, the $\mathbb{P}_\gamma$-name $\dot{D}$ belongs to Q^* as well. Note that $p|_\gamma$ forces that every element of $Q^*[\dot{G}_\gamma] \cap H(\kappa)^V$ can be covered by an element of $Q^*[\dot{G}_\gamma] \cap \dot{D}$, and thus, we can cover $Q^*[\dot{G}_\gamma] \cap H(\kappa)^V$ by an $\subseteq$-increasing sequence of length $|Q^*[G_\gamma]| \leq \aleph_1$ of elements of $Q^*[\dot{G}_\gamma] \cap \dot{D}$. Therefore, $p|_\gamma$ forces that $Q^*[\dot{G}_\gamma] \cap H(\kappa)^V \in \dot{D}$. Since $p|_\gamma$ is $(Q, \mathbb{P}_\gamma)$-pre-generic, it follows from $(\mathrm{P2})_\gamma$ that $p|_\gamma$ is $(Q^*, \mathbb{P}_\gamma)$-generic. Therefore, $p|_\gamma$ forces that $Q^*[\dot{G}_\gamma] \cap V = Q^*$, and thus, that $Q^*[\dot{G}_\gamma] \cap H(\kappa) = Q$. Hence, it forces that $Q \in \dot{D}$. Moreover, note that $p|_\gamma$ forces the following:

(1) $\{Q\} \subseteq \dot{D}$.

(2) $\{(Q, \gamma)\} \subseteq \Delta_{p|_\gamma}$ and $p|_\gamma \in \dot{G}_\gamma$.

(3) $F_p(\gamma) \in \dot{\mathbb{Q}}_\gamma$ and $\dot{\mathbb{Q}}_\gamma \in Q[\dot{G}_\gamma]$.

(4) $F_p(\gamma) \in Q[\dot{G}_\gamma]$.

The last two points follow from the fact that $\gamma, F_p(\gamma) \in Q$, since $p \in Q$ and we have (possibly) only extended $p|_\gamma$. Moreover, note that if $Q \in \mathcal{S}_\delta$ for some limit $\delta \leq \kappa$ such that $cf(\delta) = \omega_1$ and $\gamma < \delta$, then $\gamma < \sup(Q \cap \delta)$. Since $\dot{\mathbb{Q}}_\gamma$ is the $\mathbb{P}_\gamma$-name of an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notion relative to V and $\dot{G}_\gamma$, there is an extension $p^* \in \mathbb{P}_\gamma$ of $p|_\gamma$ and some $\mathbb{P}_\gamma$-name $\dot{x} \in H(\kappa)$ such that p^* forces that $\dot{x} \in \dot{\mathbb{Q}}_\gamma$ and that $\dot{x}$ is a $(Q[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-generic condition extending $F_p(\gamma)$. Let

$$q = \big(F_{p^*} \cup \{\langle \gamma, \dot{x} \rangle\}, \Delta_p \cup \Delta_{p^*} \cup \{(Q, \gamma+1)\}\big).$$

Lemma 4.3.5 ensures that q is a condition in $\mathbb{P}_{\gamma+1}$ extending p.

Now, suppose that $\gamma \notin \mathrm{dom}(F_p)$. Hence, $\mathrm{dom}(F_p) \subseteq \gamma$, as $p \in \mathbb{P}_{\gamma+1}$. Since $p \in Q$, we can find a condition $t \in \mathbb{P}_\gamma$ extending $p|_\gamma$ and such that $(Q, \gamma) \in \Delta_t$ by $(\mathrm{P1})_\gamma$. Define $q := (F_t, \Delta_t \cup \Delta_p \cup \{(Q, \gamma+1)\})$. We claim that q is a condition in $\mathbb{P}_{\gamma+1}$ extending p. Clauses (C0) and (C1) in the definition of $\mathbb{P}_{\gamma+1}$ are clearly satisfied. Note that $q|_\gamma = (F_t, \Delta_t \cup \Delta_{p|_\gamma})$, because $(Q, \gamma) \in \Delta_t$. Hence, clause (C2) also holds, because $q|_\gamma \in \mathbb{P}_\gamma$ by lemma 4.3.2. Clauses (C3)-(C6) follow from $q|_\gamma \in \mathbb{P}_\gamma$ and the fact that $\gamma \notin \mathrm{dom}(F_p)$. To see that $q \leq_{\gamma+1} p$, it's enough to note that $q|_\gamma \leq_\gamma t \leq_\gamma p|_\gamma$ and that $\Delta_p \subseteq \Delta_q$.

⋆ **<u>Limit case</u>**

Suppose that α is a nonzero limit ordinal and that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta < \alpha$.

By definition of $\mathbb{P}_\alpha$, $\mathrm{dom}(F_p) \subseteq \alpha$. Since $p \in Q$, there is some $\gamma \in Q \cap \alpha$ such that $\mathrm{dom}(F_p) \subseteq \gamma$. By $(\mathrm{P1})_\gamma$ there is a condition $t \in \mathbb{P}_\gamma$ such that $t \leq_\gamma p|_\gamma$ and $(Q, \gamma) \in \Delta_t$. Define $q = \big(F_t, \Delta_t \cup \Delta_p \cup \{(Q, \alpha \cap \sup(Q \cap \kappa))\}\big)$. It's enough to show by induction on $\xi \in [\gamma, \alpha]$ that $q|_\xi$ is a condition in $\mathbb{P}_\xi$. The base case follows from the fact that $t \leq_\gamma p|_\gamma$ and $(Q, \gamma) \in \Delta_t$. Suppose that $\xi > \gamma$. Clauses (C0) and (C1) are clearly satisfied, and clause (C2) holds by induction hypothesis. Note that since $t \in \mathbb{P}_\gamma$, $q|_\xi = \big(F_t, \Delta_t \cup \Delta_{p|_\xi} \cup \{(Q, \xi \cap \sup(Q \cap \kappa))\}\big)$ for every $\xi \in [\gamma, \alpha]$. Moreover, $\xi \notin \mathrm{dom}(F_t)$ for any $\xi \in [\gamma, \alpha]$. Therefore, clauses (C3)-(C6) follow directly.

4.4.2 Proof of $(\mathrm{P2})_\alpha$

Let $Q^* \in \mathcal{S}^*_\alpha \cup \mathcal{L}^*_\alpha$ and $Q = Q^* \cap H(\kappa)$. Let $p \in \mathbb{P}_\alpha$ be a $(Q, \mathbb{P}_\alpha)$-pre-generic condition. We will show that p is $(Q^*, \mathbb{P}_\alpha)$-generic.

⋆ **Base case**

Suppose that $\alpha = 0$.

Let E be an open dense subset of $\mathbb{P}_0$ in Q^*. We will find a condition in $E \cap Q^*$ compatible with p. We may assume, without loss of generality, that $p \in E$. Note that $p \cap Q^* = p \cap Q \in \mathbb{P}_0$, by lemmas 2.4.22 and 2.4.20. Hence, we can find a condition $q \in E \cap Q^*$ extending $p \cap Q^*$ by elementarity. Let $\mathcal{M}$ be the $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system extending both $\mathrm{dom}(\Delta_p)$ and $\mathrm{dom}(\Delta_q)$, given by lemmas 2.4.27 and 2.4.28. It follows that $r = (\varnothing, \{(R, 0) : R \in \mathcal{M}\})$ is a common extension of p and q in $\mathbb{P}_0$.

⋆ **Successor case**

Suppose that $\alpha = \gamma + 1$, that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta \leq \gamma$, and that $(\mathrm{P1})_\alpha$ holds.

Assume first that $\gamma \in \mathrm{dom}(F_p)$. Let $E \in Q^*$ be an open dense subset of $\mathbb{P}_{\gamma+1}$. We will find a condition $q \in E \cap Q^*$ compatible with p. Since E is dense open we may start by assuming that $p \in E$. Let G_γ be a $\mathbb{P}_\gamma$-generic filter over V such that $p|_\gamma \in G_\gamma$ and work in $V[G_\gamma]$. Let $\mathbb{Q}_\gamma$ be the interpretation of $\dot{\mathbb{Q}}_\gamma$ by G_γ. By $(\mathrm{P2})_\gamma$ we have that G_γ is also generic over Q^*. Recall that E/G_γ is defined as the set $\{r \in E : r|_\gamma \in G_\gamma\}$. Let $\widetilde{E}$ be the set of $\mathbb{Q}_\gamma$-conditions ν such that either

(i) there is $t \in E/G_\gamma$ such that $\gamma \in \mathrm{dom}(F_t)$ and $F_t(\gamma) = \nu$, or

(ii) there is no $t \in E/G_\gamma$ such that $\gamma \in \mathrm{dom}(F_t)$ and $F_t(\gamma) \leq_{\mathbb{Q}_\gamma} \nu$.

Clearly $\widetilde{E}$ is a dense subset of $\mathbb{Q}_\gamma$. Note that $\widetilde{E}$ is defined from γ, E, $\mathbb{P}_{\gamma+1}$, $\mathbb{Q}_\gamma$ and G_γ, and note that $E, \mathbb{P}_{\gamma+1}, \dot{\mathbb{Q}}_\gamma \in Q^*$. Therefore, since $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$

by lemma 4.4.3, we have that $\widetilde{E} \in Q^*[G_\gamma]$. Note that as $\dot{\mathbb{Q}}_\gamma \in H(\kappa)^V$, by (P4)$_\gamma$ and lemma 4.4.5 we have that $\mathbb{Q}_\gamma \in H(\kappa)^{V[G_\gamma]}$, which in particular implies that $\widetilde{E} \in H(\kappa)^{V[G_\gamma]}$. Therefore, by (P4)$_\gamma$ and lemma 4.4.4, $\widetilde{E}$ is in fact a member of $Q[G_\gamma]$.

By clause (C4) in the definition of $\mathbb{P}_{\gamma+1}$, and since $p|_\gamma \in G_\gamma$, we have that $F_p(\gamma)$ is $(Q[G_\gamma], \mathbb{Q}_\gamma)$-generic. Therefore, there is some $\nu \in Q[G_\gamma] \cap \widetilde{E}$ compatible with $F_p(\gamma)$. We claim that (i) above holds for ν. Indeed, let ν^* be a condition in $\mathbb{Q}_\gamma$ extending both $F_p(\gamma)$ and ν, and let r be a condition in G_γ extending $p|_\gamma$ and deciding ν^*. Let $p^* := (F_r \cup \{\langle \gamma, \nu^* \rangle\}, \Delta_r \cup \Delta_p)$, which is a condition in $\mathbb{P}_{\gamma+1}$ extending p, by lemma 4.3.5, and note that $p^* \in E/G_\gamma$. Moreover, p^* is a witness of the negation of (ii) for ν, so condition (i) must hold for ν. Since $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$, there must be a condition q in $Q^*[G_\gamma]$ witnessing that (i) holds for ν, i.e., $q \in E/G_\gamma \cap Q^*[G_\gamma]$ must be such that $\gamma \in \mathrm{dom}(F_q)$ and $F_q(\gamma) = \nu$. Note that as $p|_\gamma \in G_\gamma$, by (P2)$_\gamma$ we have that $Q^*[G_\gamma] \cap V = Q^*$, and hence, q must be a member of Q. It remains to see that q is compatible with p. Recall that ν and $F_p(\gamma)$ are compatible. Hence, as $p|_\gamma, q|_\gamma \in G_\gamma$, there has to be a condition $s \in G_\gamma$ extending both $p|_\gamma$ and $q|_\gamma$, and deciding some $\nu^* \in \mathbb{Q}_\gamma$ such that $\nu^* \leq_{\mathbb{Q}_\gamma} \nu, F_p(\gamma)$. In light of lemma 4.3.5, the amalgamation $(F_s \cup \{\langle \gamma, \nu^* \rangle\}, \Delta_p \cup \Delta_q \cup \Delta_s)$ is a common extension of p and q in $\mathbb{P}_{\gamma+1}$.

Assume now that $\gamma \notin \mathrm{dom}(F_p)$. Let $E \in Q^*$ be an open dense subset of $\mathbb{P}_{\gamma+1}$. We will find a condition $q \in E \cap Q^*$ compatible with p. Since E is dense open we may start by assuming, without loss of generality, that $p \in E$. Let G_γ be a $\mathbb{P}_\gamma$-generic filter such that $p|_\gamma \in G_\gamma$ and work in $V[G_\gamma]$. Note that $\gamma, E, \mathbb{P}_{\gamma+1} \in Q^*$, hence by $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$, we can find a condition $q \in Q^*[G_\gamma] \cap \mathbb{P}_{\gamma+1}$ such that

- $q \in E$,
- $q|_\gamma \in G_\gamma$, and
- $\mathrm{dom}(F_q) \subseteq \gamma$,

since the existence of q is witnessed in $V[G_\gamma]$ by p. Note that in fact $q \in Q^*$ because $Q^*[G_\gamma] \cap V = Q^*$ by the induction hypothesis $(\mathrm{P2})_\gamma$ and $p|_\gamma \in G_\gamma$. Hence, we may assume, by extending p below γ if necessary, that $p|_\gamma$ decides q, and since it forces that $q|_\gamma \in \dot{G}_\gamma$, we have that $p|_\gamma \leq_\gamma q|_\gamma$. By lemma 4.3.7, $(F_p, \Delta_p \cup \Delta_q)$ is a condition in $\mathbb{P}_{\gamma+1}$ extending both p and q.

⋆ **Limit case**

Suppose that α is a nonzero limit ordinal, that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta < \alpha$, and that $(\mathrm{P1})_\alpha$ holds.

Let $E \in Q^*$ be an open dense subset of $\mathbb{P}_\alpha$. We will find a condition $q \in E \cap Q^*$ compatible with p. Since E is dense open we may start by assuming that $p \in E$. We divide the proof in three cases.

Case 1. $cf(\alpha) = \omega$.

Note that in this case $\sup(Q \cap \alpha) = \alpha$, by lemma 1.4.11. Hence, as $\mathrm{dom}(F_p)$ is finite and contained in α, there is some $\gamma \in Q \cap \alpha$ such that $\mathrm{dom}(F_p) \subseteq \gamma$. Fix a $\mathbb{P}_\gamma$-generic filter G_γ over V such that $p|_\gamma \in G_\gamma$. Note that the parameters $\gamma, E, \mathbb{P}_\alpha$ belong to Q^*. Therefore, by $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$, we can find a condition $q \in Q^*[G_\gamma] \cap \mathbb{P}_\alpha$ such that

- $q \in E$,
- $q|_\gamma \in G_\gamma$, and
- $\mathrm{dom}(F_q) \subseteq \gamma$.

Again, the existence of q is witnessed in $V[G_\gamma]$ by p. Since $p|_\gamma \in G_\gamma$ and $p|_\gamma$ is $(Q^*, \mathbb{P}_\gamma)$-generic by induction hypothesis $(\mathrm{P2})_\gamma$, we have that $Q^*[G_\gamma] \cap V = Q^*$, and thus, $q \in Q^*$. Note that as $p|_\gamma \in G_\gamma$, we may assume by extending p below γ, that $p|_\gamma$ decides q, and moreover, since in particular it forces that $q|_\gamma \in \dot{G}_\gamma$, we have that $p|_\gamma \leq_\gamma q|_\gamma$. Lemma 4.3.7 ensures that the amalgamation $(F_p, \Delta_p \cup \Delta_q)$ is a common extension of p and q in $\mathbb{P}_\alpha$.

Case 2. $cf(\alpha) > \omega_1$.

Note that in this case, in light of lemma 1.4.11, $\sup(R \cap \alpha) < \alpha$ for very model $R \in \mathrm{dom}(\Delta_p)$ such that $\alpha \in R$. If $\mathrm{dom}(F_p) \subseteq \sup(Q \cap \alpha)$ we can argue exactly as in case 1. Hence, we may assume that $\mathrm{dom}(F_p)$ is not bounded by $\sup(Q \cap \alpha)$.

Claim 4.4.7. *For every $R \in \mathrm{dom}(\Delta_p)$ such that $\varepsilon_R < \varepsilon_Q$,*

(a) $\sup(R \cap Q \cap \alpha) < \sup(Q \cap \alpha)$, *if $R \in \mathcal{S} \cup \mathcal{L}$, and*

(b) $\sup(\bigcup R \cap Q \cap \alpha) < \sup(Q \cap \alpha)$, *if $R \in \mathcal{T}^+$.*

Proof. Assume first that $R \in \mathrm{dom}(\Delta_p) \cap (\mathcal{S} \cup \mathcal{L})$. Note that if $R \in Q[\omega_1]$, since $\alpha \in Q$, the model $Q[\omega_1]$ thinks that $R \cap \alpha$ is bounded in α. Therefore, $\sup(R \cap \alpha) \in Q[\omega_1] \cap \alpha$, and thus,

$$\sup(R \cap \alpha) < \sup(Q[\omega_1] \cap \alpha) = \sup(Q \cap \alpha).$$

Note that the equality $\sup(Q[\omega_1] \cap \alpha) = \sup(Q \cap \alpha)$ holds trivially if Q is uncountable, and if Q is countable, it holds by proposition 2.3.4. Now, if $R' \in \mathrm{dom}(\Delta_p)$ is such that $\varepsilon_{R'} < \varepsilon_Q$, since $\mathrm{dom}(\Delta_p)$ is an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system, there must be some $Q' \in \mathrm{dom}(\Delta_p)$ such that $\varepsilon_{Q'} = \varepsilon_Q$ and $R' \in Q'[\omega_1]$, and hence, $\sup\left(\Psi_{Q'[\omega_1],Q[\omega_1]}(R') \cap \alpha\right) < \sup(Q \cap \alpha)$. The symmetry of the system $\mathrm{dom}(\Delta_p)$ is needed precisely to get this result. Finally, note that if R' and Q' are as above, since $\Psi_{Q'[\omega_1],Q[\omega_1]}$ fixes $Q'[\omega_1] \cap Q[\omega_1] \cap \kappa$, it also fixes $R' \cap Q \cap \kappa$, and hence, $\sup(R' \cap Q \cap \alpha) \leq \sup\left(\Psi_{Q'[\omega_1],Q[\omega_1]}(R') \cap \alpha\right)$. Therefore, we can conclude that for every $R \in \mathrm{dom}(\Delta_p) \cap (\mathcal{S} \cup \mathcal{L})$ such that $\varepsilon_R < \varepsilon_Q$,

$$\sup(R \cap Q \cap \alpha) < \sup(Q \cap \alpha),$$

as we wanted.

Assume now that $R \in \mathcal{T}^+$. Note that if $R \in Q[\omega_1]$, then $\bigcup R \in Q[\omega_1]$, and by the same reason as above, $\sup(\bigcup R \cap \alpha) < \sup(Q \cap \alpha)$. If $R \notin Q[\omega_1]$, there is some

$Q' \in \mathrm{dom}(\Delta_p)$ such that $\varepsilon_{Q'} = \varepsilon_Q$ and $R \in Q'[\omega_1]$. Hence, $\Psi_{Q'[\omega_1],Q[\omega_1]}(R)$ is a member of $\mathrm{dom}(\Delta_p) \cap Q[\omega_1]$, and thus, $\sup(\bigcup \Psi_{Q'[\omega_1],Q[\omega_1]}(R) \cap \alpha) < \sup(Q \cap \alpha)$. Again, since $\Psi_{Q'[\omega_1],Q[\omega_1]}$ fixes $Q'[\omega_1] \cap Q[\omega_1] \cap \kappa$, it also fixes $(\bigcup R) \cap Q \cap \kappa$, so $\sup(\bigcup R \cap Q \cap \alpha) \leq \sup(\bigcup \Psi_{Q'[\omega_1],Q[\omega_1]}(R) \cap \alpha)$. Hence, we can conclude that for every $R \in \mathrm{dom}(\Delta_p) \cap \mathcal{T}^+$ such that $\varepsilon_R < \varepsilon_Q$,

$$\sup(\bigcup R \cap Q \cap \alpha) < \sup(Q \cap \alpha).$$

□

Note that for every $M \in \mathrm{dom}(\Delta_p) \cap \mathcal{S}$ such that $\varepsilon_M < \varepsilon_Q$, since M is countable and α has cofinality greater than ω_1, the same analysis from the last claim shows that $\sup(\bigcup(M \cap \mathcal{L}) \cap Q \cap \alpha) < \sup(Q \cap \alpha)$. Therefore, by claim 4.4.7, and since Δ_p and F_p are both finite, we may fix some $\gamma \in Q \cap \alpha$ such that

- $\sup(R \cap Q \cap \alpha) < \gamma$, for all $R \in \mathrm{dom}(\Delta_p) \cap (\mathcal{S} \cup \mathcal{L})$ such that $\varepsilon_R < \varepsilon_Q$,
- $\sup(\bigcup R \cap Q \cap \alpha) < \gamma$, for all $R \in \mathrm{dom}(\Delta_p) \cap \mathcal{T}^+$ such that $\varepsilon_R < \varepsilon_Q$,
- $\sup(\bigcup(M \cap \mathcal{L}) \cap Q \cap \alpha) < \gamma$, for all $M \in \mathrm{dom}(\Delta_p) \cap \mathcal{S}$ such that $\varepsilon_M < \varepsilon_Q$, and
- $\eta < \gamma$, for all $\eta \in \mathrm{dom}(F_p)$ such that $\eta < \sup(Q \cap \alpha)$.

Fix a generic filter G_γ for $\mathbb{P}_\gamma$ over V such that $p|_\gamma \in G_\gamma$. Note that the parameters $\gamma, E, \mathbb{P}_\alpha$ are members of Q^*. Therefore, by $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$, we can find a condition $q \in Q^*[G_\gamma] \cap \mathbb{P}_\alpha$ such that

- $q \in E$,
- $q|_\gamma \in G_\gamma$, and
- $\mathrm{dom}(F_q) \setminus \gamma \neq \varnothing$.

By exactly the same reasons as in case 1, $q \in Q^*$, and by extending p below γ if necessary, we may assume that $p|_\gamma$ decides q and extends $q|_\gamma$.

In order to finish the proof of $(\mathrm{P2})_\alpha$ we need to find a condition $\overline{q}$ in $\mathbb{P}_\alpha$ extending both p and q. The condition $\overline{q}$ can be built by recursion on $\mathrm{dom}(F_q)\setminus\gamma$, mimicking the proof of $(\mathrm{P1})_\beta$ for β successor. Note that $\min(\mathrm{dom}(F_p)\setminus\gamma) \geq \sup(Q\cap\alpha)$ by the choice of γ, and therefore, $\min(\mathrm{dom}(F_p)\setminus\gamma) > \max(\mathrm{dom}(F_q))$. Let $(\xi_i)_{i<r}$ be the strictly increasing enumeration of $\mathrm{dom}(F_q)\setminus\gamma$. We may assume that $r > 0$. We will build a sequence $(q_i)_{i<r}$ such that for every $i < r$, q_i is a condition in $\mathbb{P}_{\xi_i+1}$ extending both $p|_{\xi_i+1}$ and $q|_{\xi_i+1}$. The construction goes as follows.

Suppose first that $i = 0$. We want to find a condition $q_0 \in \mathbb{P}_{\xi_0+1}$ such that $q_0 \leq_{\xi_0+1} p|_{\xi_0+1}, q|_{\xi_0+1}$. We start by extending $p|_\gamma$ to a condition $\widetilde{p} \in \mathbb{P}_{\xi_0}$ such that $\widetilde{p} \leq_{\xi_0} p|_{\xi_0}, q|_{\xi_0}$. If $\xi_0 = \gamma$, we let $\widetilde{p} = p|_\gamma$, and otherwise, we let $\widetilde{p} = (F_{p|_{\xi_0}}, \Delta_{p|_{\xi_0}} \cup \Delta_{q|_{\xi_0}})$, which is a condition in $\mathbb{P}_{\xi_0}$ extending both $p|_{\xi_0}$ and $q|_{\xi_0}$ by lemma 4.3.7. Note that since $q \in Q^*$, in particular, $\mathrm{dom}(F_q) \subseteq Q^*$. Therefore, ξ_0 and $F_q(\xi_0)$ belong to Q^*, and thus, $F_q(\xi_0) \in Q = Q^* \cap H(\kappa)$. Our plan now is to find a condition $\widehat{p} \in \mathbb{P}_{\xi_0}$ and a $\mathbb{P}_{\xi_0}$-name $\dot{x} \in H(\kappa)$ such that

(i) $\widehat{p} \leq_{\xi_0} \widetilde{p}$,

(ii) $\widehat{p} \Vdash_{\xi_0}$ "$\dot{x} \leq_{\dot{\mathbb{Q}}_{\xi_0}} F_q(\xi_0)$",

(iii) for every $R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$ such that $(R, \xi_0+1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$,

$$\widehat{p} \Vdash_{\xi_0} \text{“}\dot{x} \text{ is } (R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})\text{-generic”},$$

(iv) for every $\overline{M} \in \mathcal{T}^+$ such that $(\overline{M}, \xi_0+1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$, and every $N \in \overline{M} \cap \mathcal{L}_{\xi_0+1}$,

$$\widehat{p} \Vdash_{\xi_0} \text{“}\dot{x} \text{ is } (N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})\text{-generic”},$$

and

(v) for every $M \in \mathcal{S}_\delta$ such that $(M, \xi_0+1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$, where $\delta \leq \kappa$ is a limit ordinal such that $cf(\delta) = \omega_1$ and $\sup(M\cap\delta) \leq \xi_0 < \delta$, the condition $\widehat{p}$ forces that $\dot{x}$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for every $N \in \dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}}$, so that we can define q_0 as the natural amalgamation

$$q_0 = (F_{\widehat{p}} \cup \{\langle \xi_0, \dot{x} \rangle\}, \Delta_{\widehat{p}} \cup \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}),$$

which, in light of lemma 4.3.5, is a condition in $\mathbb{P}_{\xi_0+1}$ extending both $p|_{\xi_0+1}$ and $q|_{\xi_0+1}$. Note that if $R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$ is such that $(R, \beta) \in \Delta_q$ for some $\beta \geq \xi_0+1$, then by clause (C4) of the definition of the iteration, the condition $q|_{\xi_0}$ forces that $F_q(\xi_0)$ is $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic, and thus, any $\dot{\mathbb{Q}}_{\xi_0}$-extension of $F_q(\xi_0)$ will be forced by $q|_{\xi_0}$ (and hence, by any extension of $q|_{\xi_0}$) to be $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic. Similarly for $N \in \overline{M} \cap \mathcal{L}_{\xi_0+1}$, where $\overline{M} \in \mathcal{T}^+$ is such that $(\overline{M}, \beta) \in \Delta_q$, where $\beta \geq \xi_0 + 1$, using (C5) instead of (C4). It follows from these observations that if we find a $\mathbb{P}_{\xi_0}$-name $\dot{x}$ satisfying (ii), then the set of models for which we need to check item (iii) is

$$\mathcal{M}_0 := \{R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1} : (R, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}\},$$

and the set of models for which we need to check item (iv) is

$$\mathcal{M}_1 := \{N \in \overline{M} \cap \mathcal{L}_{\xi_0+1} : \overline{M} \in \mathcal{T}^+, (\overline{M}, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}\}.$$

Note that if $R \in \mathcal{M}_0$, then $\varepsilon_R \geq \varepsilon_Q$. To see this suppose that $\varepsilon_R < \varepsilon_Q$. Since $R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$, then $\xi_0 \in R$, and hence $\xi_0 < \sup(R \cap Q \cap \alpha)$. But this is impossible by our choice of γ. Also note that if $N \in \mathcal{M}_1$, then $\xi_0 \in N$, and hence, if $\overline{M} \in \mathcal{T}^+$ is such that $(\overline{M}, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}$ and $N \in \overline{M}$, then $\xi_0 < \sup(\bigcup \overline{M} \cap Q \cap \alpha)$. Therefore, $\varepsilon_{\overline{M}} > \varepsilon_Q$ for every $\overline{M}$ as above, by our choice of γ. Moreover, note that $Q \in \mathcal{M}_0$. Indeed, since $\xi_0 \in Q$ and $Q \in \mathcal{S}_\alpha \cup \mathcal{L}_\alpha$, Q must also be a member of $\mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$, and as p is $(Q, \mathbb{P}_\alpha)$-pre-generic by assumption and $q \in Q^*$, $(Q, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}$.

A similar analysis to the one from the last paragraph shows that the models for which we need to check item (v) are those $N \in \mathcal{L}_{\xi_0+1}$ forced by $\widetilde{p}$ to be members of $\dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}}$, where $M \in \mathcal{S}_\delta$ is such that $(M, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}$, with $\delta \leq \kappa$ a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \xi_0 < \delta$. We will denote the set of models N as above, relevant for item (v), as $\mathcal{M}_2$.

Denote by $\mathcal{M}$ the set $\mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$ of all relevant models and note that $\mathcal{M} \cap Q = \varnothing$. Indeed, it's clear that $\mathcal{M}_0 \cap Q = \varnothing$ because the models in $\mathcal{M}_0$ have ω_2-height greater than or equal ε_Q. Suppose now that $N \in (\mathcal{M}_1 \cup \mathcal{M}_2) \cap Q$. Then, $N \in \mathcal{L}_{\xi_0+1}$, and thus, $\xi_0 \in N$. But this implies that $\gamma < \xi_0 < \sup(N \cap Q \cap \alpha)$, which is impossible by the choice of γ.

Recall that $\dot{\mathbb{Q}}_{\xi_0}$ is the $\mathbb{P}_{\xi_0}$-name for an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notion relative to V and $\dot{G}_{\xi_0}$. So $\widetilde{p}$ forces that the first club $D \subseteq [H(\kappa)^V]^{\leq \aleph_1}$ in V (in the well-order of $H(\kappa^+)[\dot{G}_{\xi_0}]$ induced by $\lhd$) witnessing the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_{\xi_0}$ with respect to V and $\dot{G}_{\xi_0}$ is such that $\mathcal{M}_0 \cup \mathcal{M}_1 \subseteq \dot{D}$. To see this let $R \in \mathcal{M}_0$ and let $R^* \in \mathcal{S}^*_{\xi_0+1} \cup \mathcal{L}^*_{\xi_0+1}$ such that $R = R^* \cap H(\kappa)$. Note that since $\lhd \in R^*$, R^* contains a $\mathbb{P}_{\xi_0}$-name $\dot{D}$ for the club D. This fact combined with the fact that $\widetilde{p}$ is $(R^*, \mathbb{P}_{\xi_0})$-generic by $(\mathrm{P2})_{\xi_0}$, imply that $\widetilde{p}$ forces that $R \in \dot{D}$, by the exact same argument as the one in the proof of the successor case of $(\mathrm{P1})_\alpha$. If $N \in \mathcal{M}_1$, we can argue in the exact same way, using $(\mathrm{P3})_{\xi_0}$ rather than $(\mathrm{P2})_{\xi_0}$, to show that $\widetilde{p}$ forces that $N \in \dot{D}$. The following claim shows, in particular, that $\widetilde{p}$ also forces that $\mathcal{M}_2 \subseteq \dot{D}$.

Claim 4.4.8. $\mathbb{P}_{\xi_0}$ *forces that* $\dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}} \subseteq \dot{D}$.

Proof. Let G_{ξ_0} be a $\mathbb{P}_{\xi_0}$-generic filter over V and work in $V[G_{\xi_0}]$. Let D be the interpretation of $\dot{D}$ by G_{ξ_0}. Let $N \in \mathcal{A}^M_{G_{\xi_0}}$ such that $N = N^* \cap H(\kappa)$, for some $N^* \in \mathcal{L}^*_{\xi_0+1}$. Then, $N \in M \cap \mathcal{L}_{\xi_0+1}$ and there is a condition $u \in G_{\xi_0}$ such that either $(N, \xi_0) \in \Delta_u$, or $N \in \overline{M}$ for some $\overline{M} \in \mathrm{dom}(\Delta_u) \cap \mathcal{T}^+$ such that $(\overline{M}, \xi_0) \in \Delta_u$. Hence, u is $(N^*, \mathbb{P}_{\xi_0})$-generic by $(\mathrm{P2})_{\xi_0}$, if $N \in \mathrm{dom}(\Delta_u) \cap \mathcal{L}$, or by $(\mathrm{P3})_{\xi_0}$, if $N \in \bigcup(\mathrm{dom}(\Delta_u) \cap \mathcal{T}^+)$. Since $N^* \in \mathcal{L}^*_{\xi_0+1}$, the $\mathbb{P}_{\xi_0}$-name $\dot{D}$ belongs to N^*, and as u forces that $N^*[\dot{G}_{\xi_0}] \cap H(\kappa)^V = N$, we can conclude that u forces that $N \in \dot{D}$, by the same argument of the last paragraph. The conclusion follows from the fact that u belongs to the generic G_{ξ_0}. □

Therefore, $\widetilde{p}$ forces the following:

(1) $\mathcal{M}$ is a countable subset of $\dot{D}$, which contains only finitely many small models (all of them in $\mathcal{M}_0$).

(2) $p|_{\xi_0} \in \dot{G}_{\xi_0}$ and for every $R \in \mathcal{M}$,

(2.a) $(R, \xi_0) \in \Delta_{p|_{\xi_0}}$, if $R \in \mathcal{M}_0$,

(2.b) $R \in \overline{M}$ for some $\overline{M} \in \mathcal{T}^+$ such that $(\overline{M}, \xi_0) \in \Delta_{p|_{\xi_0}}$, if $R \in \mathcal{M}_1$, and

(2.c) $R \in \dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}}$ for some $M \in \mathcal{S}_\delta$ such that $\sup(M \cap \delta) \leq \xi_0 < \delta$ and $(M, \xi_0) \in \Delta_{p|_{\xi_0}}$, where $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1, if $R \in \mathcal{M}_2$.

(3) $F_q(\xi_0) \in \dot{\mathbb{Q}}_{\xi_0}$ and $\dot{\mathbb{Q}}_{\xi_0} \in R[\dot{G}_{\xi_0}]$ for all $R \in \mathcal{M}$.

(4) $F_q(\xi_0) \in Q[\dot{G}_{\xi_0}]$ and $Q \cap \mathcal{M} = \varnothing$, where $Q \in \mathcal{M}_0$.

Therefore, it follows from the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_{\xi_0}$ relative to V and $\dot{G}_{\xi_0}$ that there are a $\mathbb{P}_{\xi_0}$-name $\dot{x} \in H(\kappa)$ and an extension $\widehat{p} \in \mathbb{P}_{\xi_0}$ of $\widetilde{p}$ that forces that $\dot{x}$ is a condition in $\dot{\mathbb{Q}}_{\xi_0}$ such that $\dot{x} \leq_{\dot{\mathbb{Q}}_{\xi_0}} F_q(\xi_0)$ and that $\dot{x}$ is $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for all $R \in \mathcal{M}$. Hence, by lemma 4.3.5 there exists a condition $q_0 \in \mathbb{P}_{\xi_0+1}$ extending both $p|_{\xi_0+1}$ and $q|_{\xi_0+1}$, as we wanted.

The definition of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notions is designed so that this proof can go through at this exact point (and others that will come later, specially in the case $cf(\alpha) = \omega_1$). This depends directly on the structure of the set of relevant models $\mathcal{M}$ that we have considered above.

For the case $i > 0$ such that $i + 1 < r$, we assume inductively that $q_i \in \mathbb{P}_{\xi_i+1}$ extends $p|_{\xi_i+1}$ and $q|_{\xi_i+1}$. We may argue exactly as in the case $i = 0$ with ξ_{i+1} instead of ξ_0 and starting with q_i rather than $p|_\gamma$, to obtain $q_{i+1} \in \mathbb{P}_{\xi_{i+1}+1}$ extending both $p|_{\xi_{i+1}+1}$ and $q|_{\xi_{i+1}+1}$.

Let $\mu = \xi_{r-1} = \max(\mathrm{dom}(F_q))$ and define

$$\overline{q} := \big(F_{q_{r-1}} \cup (F_p \upharpoonright [\mu + 1, \alpha)), \Delta_{q_{r-1}} \cup \Delta_p \cup \Delta_q\big).$$

Claim 4.4.9. *$\overline{q}$ is a condition in $\mathbb{P}_\alpha$ extending both p and q.*

Proof. We show by induction on ξ, for $\mu+1 \leq \xi \leq \alpha$, that $\overline{q}|_\xi \in \mathbb{P}_\xi$ and that $\overline{q}|_\xi$ extends both $p|_\xi$ and $q|_\xi$.

If $\xi = \mu+1$, the condition $\overline{q}|_{\mu+1}$ equals $(F_{q_{r-1}}, \Delta_{q_{r-1}} \cup \Delta_{p|_{\mu+1}} \cup \Delta_{q|_{\mu+1}})$, and as $q_{r-1} \leq_{\mu+1} p|_{\mu+1}, q|_{\mu+1}$, the conditions $\overline{q}|_{\mu+1}$ and q_{r-1} are equivalent by lemma 4.3.2. Therefore, the result follows from the fact that $q_{r-1} \in \mathbb{P}_{\mu+1}$.

Now assume that $\xi = \eta+1 > \mu+1$. Clauses (C0)-(C2) in the definition of $\mathbb{P}_{\eta+1}$ are clear by induction hypothesis. Note that if $\eta \in \mathrm{dom}(F_{\overline{q}})$ then $\eta \in \mathrm{dom}(F_p)$, and hence, as $p \in \mathbb{P}_\alpha$, clause (C3) follows from the induction hypothesis $\overline{q}|_\eta \leq_\eta p|_\eta$. Let us show clause (C4) now. Suppose that $\eta \in \mathrm{dom}(F_{\overline{q}})$, and let $(R,\beta) \in \Delta_{\overline{q}}$ be such that $\beta \geq \eta+1$ and $R \in \mathcal{S}_{\eta+1} \cup \mathcal{L}_{\eta+1}$. We need to check that $\overline{q}|_\eta$ forces that $F_{\overline{q}}(\eta)$ is $(R[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic. Note that (R,β) cannot be in $\Delta_{q_{r-1}}$, as $q_{r-1} \in \mathbb{P}_{\mu+1}$ and $\beta \geq \eta+1 > \mu+1$. Suppose that $(R,\beta) \in \Delta_{q|_{\eta+1}}$. On one hand, $\eta \in \mathrm{dom}(F_p) \setminus \gamma = \mathrm{dom}(F_p) \setminus \sup(Q \cap \alpha)$, so $\sup(Q\cap\alpha) < \eta$. On the other hand, $q \in Q^*$, so in particular, $R \in Q = Q^* \cap H(\kappa)$, and thus, $\sup(R\cap\alpha) < \sup(Q\cap\alpha)$. But note that as $R \in \mathcal{S}_{\eta+1} \cup \mathcal{L}_{\eta+1}$, then $\eta \in R$, and thus, $\eta < \sup(R\cap\alpha)$, which contradicts the fact that $\sup(Q\cap\alpha) < \eta$. Therefore, if $(R,\beta) \in \Delta_{\overline{q}}$, it must be the case that $(R,\beta) \in \Delta_{p|_{\eta+1}}$. Since $\eta \in \mathrm{dom}(F_p)$, by (C4) applied to p we have that $p|_\eta$ forces that $F_p(\eta)$ is $(R[\dot{G}_\eta], \dot{\mathbb{Q}}_\alpha)$-generic, and as $\overline{q}|_\eta \leq_\eta p|_\eta$ by induction hypothesis, and $F_{\overline{q}}(\eta) = F_p(\eta)$, clause (C4) holds for $\overline{q}|_\eta$. A very similar argument shows that (C5) holds for $\overline{q}|_{\eta+1}$. Now we check clause (C6). Suppose that $\eta \in \mathrm{dom}(F_{\overline{q}})$. Let $M \in \mathcal{S}_\delta$ such that $(M,\beta) \in \Delta_{\overline{q}}$, where $\beta \geq \eta+1$ and $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1 such that $\sup(M\cap\delta) \leq \eta < \delta$. As before, (M,β) cannot be in $\Delta_{q_{r-1}}$, as $q_{r-1} \in \mathbb{P}_{\mu+1}$ and $\beta \geq \eta+1 > \mu+1$. Suppose that $(M,\eta+1) \in \Delta_{q|_{\eta+1}}$. Then, $M \in Q$, and as $Q \in \mathcal{S} \cup \mathcal{L}$, $M \subseteq Q$. In particular, $M \cap \mathcal{L} \subseteq Q$. Hence, if $N \in M \cap \mathcal{L}_{\eta+1}$, on one hand $\eta \in N$, and thus, $\eta < \sup(N\cap\alpha)$, and on the other hand, $N \in Q$, so $\sup(N\cap\alpha) < \sup(Q\cap\alpha)$. Therefore, if $M \in \mathrm{dom}(\Delta_q)$, then $\overline{q}|_\eta$ forces that $\dot{\mathcal{A}}^M_{\dot{G}_\eta}$ is empty, and hence, clause (C6) follows vacuously. Suppose now that $(M,\eta+1) \in \Delta_{p|_{\eta+1}}$.

Then, as $F_{\overline{q}}(\eta) = F_p(\eta)$, the conclusion of clause (C6) follows from the fact that $\overline{q}|_\eta \leq_\eta p|_\eta$. Lastly, we need to check that $\overline{q}|_{\eta+1} \leq_{\eta+1} p|_{\eta+1}, q|_{\eta+1}$. Clauses (D1) and (D3) are clear by induction hypothesis and the definition of $\overline{q}$. Clause (D2) follows from induction hypothesis $\overline{q}|_\eta \leq_\eta p|_\eta, q|_\eta$, from the definition of $\overline{q}$, and from the fact that $\max(\mathrm{dom}(F_q)) = \xi_{r-1} = \mu < \eta$.

The case $\xi > \mu + 1$ limit follows from the induction hypothesis. Recall that in this case, to see that $\overline{q}|_\xi \in \mathbb{P}_\xi$ we only need to check clauses (C0)-(C2). □

Case 3. $cf(\alpha) = \omega_1$.

The proof of this case has the same structure as the one for case 2, with a few technical subtleties that require some extra work. The main issue, which has already been mentioned before, being that, while countable models $M \in \mathrm{dom}(\Delta_p)$ are bounded below α, there might be uncountable models $N \in M \cap \mathrm{dom}(\Delta_p)$ active beyond $\sup(M \cap \alpha)$ and up to α.

First of all, note that in light of lemma 1.4.11, if $R \in \mathrm{dom}(\Delta_p)$ and $\alpha \in R$, then

- $\sup(R \cap \alpha) < \alpha$, if $R \in \mathcal{S}$,
- $\sup(R \cap \alpha) = \alpha$, if $R \in \mathcal{L}$, and
- $\sup(\bigcup R \cap \alpha) = \alpha$, if $R \in \mathcal{T}^+$.

Hence, if $Q \in \mathcal{L}$ or $\mathrm{dom}(F_p) \subseteq \sup(Q \cap \alpha)$, we can argue exactly as in case 1, $cf(\alpha) = \omega$. Therefore, we may assume that $Q \in \mathcal{S}$ and that $\mathrm{dom}(F_p) \setminus \sup(Q \cap \alpha)$ is nonempty.

Let $R \in \mathrm{dom}(\Delta_p) \cap Q$. If $R \in \mathcal{L}$, then Q thinks that $R \cap \alpha$ is unbounded in α, and thus $\sup(R \cap Q \cap \alpha) = \sup(Q \cap \alpha)$. Consequently, if $R \in \mathcal{T}^+$, then $\sup(\bigcup R \cap Q \cap \alpha) = \sup(Q \cap \alpha)$. However, it follows from the symmetry of $\mathrm{dom}(\Delta_p)$ and the same argument as in the proof of claim 4.4.7, that for every $M \in \mathrm{dom}(\Delta_p) \cap \mathcal{S}$ such that $\varepsilon_M < \varepsilon_Q$, $\sup(M \cap Q \cap \alpha) < \sup(Q \cap \alpha)$.

Since Δ_p and F_p are finite, we may fix some $\gamma \in Q \cap \alpha$ such that

- $\sup(M \cap Q \cap \alpha) < \gamma$, for all $M \in \mathrm{dom}(\Delta_p) \cap \mathcal{S}$ such that $\varepsilon_M < \varepsilon_Q$, and
- $\eta < \gamma$, for all $\eta \in \mathrm{dom}(F_p)$ such that $\eta < \sup(Q \cap \alpha)$.

Fix a $\mathbb{P}_\gamma$-generic filter G_γ over V such that $p|_\gamma \in G_\gamma$. Note that if $\eta \in \mathrm{dom}(F_p)\setminus\gamma$, then $\sup(Q \cap \alpha) \leq \eta < \alpha$, where $Q \in \mathcal{S}_\alpha$ and $(Q, \alpha) \in \Delta_p$, and α is a limit ordinal of cofinality ω_1. Therefore, by (C6) we have that $p|_\eta$ forces that $F_p(\eta)$ is $(N[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic for every $N \in \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. Note that the parameters $\gamma, E, \mathbb{P}_\alpha$ are members of Q^*. Therefore, by $H(\kappa^+)^{Q^*[G_\gamma]} \preceq H(\kappa^+)^{V[G_\gamma]}$, we can find a condition $q \in Q^*[G_\gamma] \cap \mathbb{P}_\alpha$ such that

- $q \in E$,
- $q|_\gamma \in G_\gamma$,
- $\mathrm{dom}(F_q) \setminus \gamma \neq \varnothing$,
- if $\xi \in (Q \cap \alpha + 1) \setminus \gamma$, then $\Delta^{-1}_{p|_\xi}(\xi) \cap Q \subseteq \Delta^{-1}_{q|_\xi}(\xi)$, and
- for every $\xi \in \mathrm{dom}(F_q)\setminus\gamma$, we have that $q|_\xi$ forces that $F_q(\xi)$ is $(N[\dot{G}_\xi], \dot{\mathbb{Q}}_\xi)$-generic for every $N \in \dot{\mathcal{A}}^Q_{\dot{G}_\xi} \cap Q$.

The fourth item ensures that if $(R, \xi) \in \Delta_{p|_\xi} \cap Q$, then $(R, \xi) \in \Delta_{q|_\xi}$. As in the previous cases, we can argue that $q \in Q^*$, and by extending p below γ if necessary, we may assume that $p|_\gamma$ decides q and extends $q|_\gamma$.

As in the last case, the idea is to build a condition $\overline{q} \in \mathbb{P}_\alpha$ extending p and q by recursion on $\mathrm{dom}(F_q) \setminus \gamma$. Recall that $\min(\mathrm{dom}(F_p) \setminus \gamma) \geq \sup(Q \cap \alpha)$ by the choice of γ, and therefore $\min(\mathrm{dom}(F_p) \setminus \gamma) > \max(\mathrm{dom}(F_q))$. Let $(\xi_i)_{i<r}$ be the strictly increasing enumeration of $\mathrm{dom}(F_q) \setminus \gamma$. We may assume that $r > 0$. We will build a sequence $(q_i)_{i<r}$ such that for every $i < r$, q_i is a condition in $\mathbb{P}_{\xi_i+1}$ extending both $p|_{\xi_i+1}$ and $q|_{\xi_i+1}$. The construction goes exactly the same way as in the last case.

Suppose first that $i = 0$. We want to find a condition $q_0 \in \mathbb{P}_{\xi_0+1}$ such that $q_0 \leq_{\xi_0+1} p|_{\xi_0+1}, q|_{\xi_0+1}$. Extend $p|_\gamma$ to a condition $\widetilde{p} \in \mathbb{P}_{\xi_0}$ such that $\widetilde{p} \leq_{\xi_0} p|_{\xi_0}, q|_{\xi_0}$.

If $\xi_0 = \gamma$, we let $\widetilde{p} = p|_\gamma$. Otherwise, we let $\widetilde{p} = (F_{p|_{\xi_0}}, \Delta_{p|_{\xi_0}} \cup \Delta_{q|_{\xi_0}})$, which is a condition in $\mathbb{P}_{\xi_0}$ extending both $p|_{\xi_0}$ and $q|_{\xi_0}$ by lemma 4.3.7. Note that since $q \in Q^*$, in particular, $\mathrm{dom}(F_q) \subseteq Q^*$. Therefore, ξ_0 and $F_q(\xi_0)$ are elements of Q^*, and thus, $F_q(\xi_0) \in Q = Q^* \cap H(\kappa)$. As in the last case, our plan is to find a condition $\widehat{p} \in \mathbb{P}_{\xi_0}$ and a $\mathbb{P}_{\xi_0}$-name $\dot{x} \in H(\kappa)$ such that

(i) $\widehat{p} \leq_{\xi_0} \widetilde{p}$,

(ii) $\widehat{p} \Vdash_{\xi_0}$ "$\dot{x} \leq_{\dot{\mathbb{Q}}_{\xi_0}} F_q(\xi_0)$",

(iii) for every $R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$ such that $(R, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$,

$$\widehat{p} \Vdash_{\xi_0} \text{“}\dot{x} \text{ is } (R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})\text{-generic”},$$

(iv) for every $\overline{M} \in \mathcal{T}^+$ such that $(\overline{M}, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$, and every $N \in \overline{M} \cap \mathcal{L}_{\xi_0+1}$,

$$\widehat{p} \Vdash_{\xi_0} \text{“}\dot{x} \text{ is } (N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})\text{-generic”},$$

and

(v) for every $M \in \mathcal{S}_\delta$ such that $(M, \xi_0+1) \in \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}$, where $\delta \leq \kappa$ is a limit ordinal such that $cf(\delta) = \omega_1$ and $\sup(M \cap \delta) \leq \xi_0 < \delta$, the condition $\widehat{p}$ forces that $\dot{x}$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for every $N \in \dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}}$,

so that we can define q_0 as the natural amalgamation

$$q_0 = (F_{\widehat{p}} \cup \{\langle \xi_0, \dot{x} \rangle\}, \Delta_{\widehat{p}} \cup \Delta_{p|_{\xi_0+1}} \cup \Delta_{q|_{\xi_0+1}}),$$

which, in light of lemma 4.3.5, is a condition in $\mathbb{P}_{\xi_0+1}$ extending both $p|_{\xi_0+1}$ and $q|_{\xi_0+1}$. Note that if $R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1}$ is such that $(R, \beta) \in \Delta_q$ for some $\beta \geq \xi_0+1$, then by clause (C4) of the definition of the iteration, the condition $q|_{\xi_0}$ forces that $F_q(\xi_0)$ is $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic, and thus, any $\dot{\mathbb{Q}}_{\xi_0}$-extension of $F_q(\xi_0)$ will be forced by $q|_{\xi_0}$ (and hence, by any extension of $q|_{\xi_0}$) to be $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-

generic. Similarly for $N \in \overline{M} \cap \mathcal{L}_{\xi_0+1}$, where $\overline{M} \in \mathcal{T}^+$ is such that $(\overline{M}, \beta) \in \Delta_q$, where $\beta \geq \xi_0 + 1$, using (C5) instead of (C4). It follows from these observations that if we find a $\mathbb{P}_{\xi_0}$-name $\dot{x}$ satisfying (ii), then the set of models for which we need to check item (iii) is

$$\mathcal{M}_0 := \{R \in \mathcal{S}_{\xi_0+1} \cup \mathcal{L}_{\xi_0+1} : (R, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}\},$$

and the set of models for which we need to check item (iv) is

$$\mathcal{M}_1 := \{N \in \overline{M} \cap \mathcal{L}_{\xi_0+1} : \overline{M} \in \mathcal{T}^+, (\overline{M}, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}\}.$$

By the choice of γ and the same reasons as in the last case, it's easy to check that if $M \in \mathcal{M}_0 \cap \mathcal{S}$, then $\varepsilon_M \geq \varepsilon_Q$. However, we cannot argue the same for the large models in $\mathcal{M}_0 \cup \mathcal{M}_1$. This shouldn't be surprising. We have seen at the beginning of this case that large models can be active at stages beyond $\sup(Q \cap \alpha)$, unlike in the last case. But we claim that $\mathcal{M}_0 \cap Q = \varnothing$. Note that if $R \in \mathcal{M}_0 \cap Q$, then $(R, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \cap Q$ (it's clear that if $\xi_0 \in Q$, then $\xi_0 + 1 \in Q$), and hence, $(R, \xi_0 + 1) \in \Delta_{q|_{\xi_0+1}}$ by the way we have defined q. Moreover, it is still true that $Q \in \mathcal{M}_0$.

As in the last case, a similar analysis to the one above shows that the models for which we need to check item (v) are those $N \in \mathcal{L}_{\xi_0+1}$ forced by $\widetilde{p}$ to be members of $\dot{\mathcal{A}}^M_{\dot{G}_{\xi_0}}$, where $M \in \mathcal{S}_\delta$ is such that $(M, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}} \setminus \Delta_{q|_{\xi_0+1}}$, with $\delta \leq \kappa$ a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \xi_0 < \delta$. Denote the set of models N as above, relevant for item (v), as $\mathcal{M}_2$.

Denote by $\mathcal{M}$ the set $\mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$. We have shown that $\mathcal{M}_0 \cap Q = \varnothing$, but note that in general $(\mathcal{M}_1 \cup \mathcal{M}_2) \cap Q \neq \varnothing$. However, we claim that $\widetilde{p}$ forces that $F_q(\xi_0)$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for every $N \in (\mathcal{M}_1 \cup \mathcal{M}_2) \cap Q$. Suppose first that $N \in \mathcal{M}_1 \cap Q$. Then, there is some $\overline{M} \in \mathrm{dom}(\Delta_p) \cap \mathcal{T}^+$ such that $(\overline{M}, \xi_0 + 1) \in \Delta_{p|_{\xi_0+1}}$ (and hence, $(\overline{M}, \xi_0) \in \Delta_{p|_{\xi_0}}$). Therefore, since $\widetilde{p} \leq_{\xi_0} p|_{\xi_0}$, and thus, $\widetilde{p} \Vdash_{\xi_0} p|_{\xi_0} \in \dot{G}_{\xi_0}$, we have that $\widetilde{p} \Vdash_{\xi_0} N \in \dot{\mathcal{A}}^Q_{\dot{G}_{\xi_0}}$. Recall that $q|_{\xi_0}$

forces that $F_q(\xi_0)$ is $(N'[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for every $N' \in \dot{\mathcal{A}}^{Q}_{\dot{G}_{\xi_0}} \cap Q$. Therefore, as $\widetilde{p} \leq_{\xi_0} q|_{\xi_0}$, we have that $\widetilde{p}$ forces that $F_q(\xi_0)$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic, as we wanted. Suppose now that $N \in \mathcal{M}_2 \cap Q$. It's not too hard to see that $\mathbb{P}_{\xi_0}$ forces that $\dot{\mathcal{A}}^{M}_{\dot{G}_{\xi_0}} \cap Q \subseteq \dot{\mathcal{A}}^{Q}_{\dot{G}_{\xi_0}}$. Therefore, $\widetilde{p}$ forces that $N \in \dot{\mathcal{A}}^{Q}_{\dot{G}_{\xi_0}}$, and by the same argument as above, it forces that $F_q(\xi_0)$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic.

Let $\dot{D}$ be the $\lhd$-least $\mathbb{P}_{\xi_0}$-name for a club D of $([H(\kappa)]^{\leq\aleph_1})^V$ in V witnessing the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_{\xi_0}$ with respect to V and $\dot{G}_{\xi_0}$. The exact same argument from the last case shows that $\widetilde{p}$ forces that $\mathcal{M} \subseteq \dot{D}$. Therefore, we can conclude that $\widetilde{p}$ forces the following:

(1) $\mathcal{M}$ is a countable subset of $\dot{D}$, which contains only finitely many small models (all of them in $\mathcal{M}_0$).

(2) $p|_{\xi_0} \in \dot{G}_{\xi_0}$ and for every $R \in \mathcal{M}$,

  (2.a) $(R, \xi_0) \in \Delta_{p|_{\xi_0}}$, if $R \in \mathcal{M}_0$,

  (2.b) $R \in \overline{M}$ for some $\overline{M} \in \mathcal{T}^+$ such that $(\overline{M}, \xi_0) \in \Delta_{p|_{\xi_0}}$, if $R \in \mathcal{M}_1$, and

  (2.c) $R \in \dot{\mathcal{A}}^{M}_{\dot{G}_{\xi_0}}$ for some $M \in \mathcal{S}_\delta$ such that $\sup(M \cap \delta) \leq \xi_0 < \delta$ and $(M, \xi_0) \in \Delta_{p|_{\xi_0}}$, where $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1, if $R \in \mathcal{M}_2$.

(3) $F_q(\xi_0) \in \dot{\mathbb{Q}}_{\xi_0}$ and $\dot{\mathbb{Q}}_{\xi_0} \in R[\dot{G}_{\xi_0}]$ for all $R \in \mathcal{M}$.

(4) $F_q(\xi_0) \in Q[\dot{G}_{\xi_0}]$ and $F_q(\xi_0)$ is $(N[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for every $N \in Q \cap \mathcal{M}$, where $Q \in \mathcal{M}_0 \cap \mathcal{S}$ is such that $\varepsilon_Q = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$.

Therefore, it follows from the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_{\xi_0}$ relative to V and $\dot{G}_{\xi_0}$ that there are a $\mathbb{P}_{\xi_0}$-name $\dot{x} \in H(\kappa)$ and an extension $\widehat{p} \in \mathbb{P}_{\xi_0}$ of $\widetilde{p}$ that forces that $\dot{x}$ is a condition in $\dot{\mathbb{Q}}_{\xi_0}$ extending $F_q(\xi_0)$ and that $\dot{x}$ is $(R[\dot{G}_{\xi_0}], \dot{\mathbb{Q}}_{\xi_0})$-generic for all $R \in \mathcal{M}$. Hence, by lemma 4.3.5 there exists a condition $q_0 \in \mathbb{P}_{\xi_0+1}$ extending both $p|_{\xi_0+1}$ and $q|_{\xi_0+1}$, as we wanted.

This is the other point of the proof that hints us how the class of $(\mathcal{S}, \mathcal{L})$-finite proper forcings needs to be defined.

For the case $i > 0$ such that $i + 1 < r$, we assume inductively that $q_i \in \mathbb{P}_{\xi_i+1}$ extends $p|_{\xi_i+1}$ and $q|_{\xi_i+1}$. We may argue exactly as in the case $i = 0$ with ξ_{i+1} instead of ξ_0 and starting with q_i rather than $p|_\gamma$, to obtain $q_{i+1} \in \mathbb{P}_{\xi_{i+1}+1}$ extending both $p|_{\xi_{i+1}+1}$ and $q|_{\xi_{i+1}+1}$.

Let $\mu = \xi_{r-1} = \max(\mathrm{dom}(F_q))$ and define

$$\overline{q} := \big(F_{q_{r-1}} \cup (F_p \restriction [\mu+1, \alpha)), \Delta_{q_{r-1}} \cup \Delta_p \cup \Delta_q\big).$$

Claim 4.4.10. *$\overline{q}$ is a condition in $\mathbb{P}_\alpha$ extending both p and q.*

Proof. We show by induction on ξ, for $\mu + 1 \leq \xi \leq \alpha$, that $\overline{q}|_\xi \in \mathbb{P}_\xi$ and that $\overline{q}|_\xi$ extends both $p|_\xi$ and $q|_\xi$. If $\xi = \mu + 1$, the condition $\overline{q}|_{\mu+1}$ equals $(F_{q_{r-1}}, \Delta_{q_{r-1}} \cup \Delta_{p|_{\mu+1}} \cup \Delta_{q|_{\mu+1}})$, and as $q_{r-1} \leq_{\mu+1} p|_{\mu+1}, q|_{\mu+1}$, the conditions $\overline{q}|_{\mu+1}$ and q_{r-1} are equivalent by lemma 4.3.2. Therefore, the result follows from the fact that $q_{r-1} \in \mathbb{P}_{\mu+1}$.

Now assume that $\xi = \eta + 1 > \mu + 1$. Clauses (C0)-(C2) in the definition of $\mathbb{P}_{\eta+1}$ are clear by induction hypothesis. If $\eta \in \mathrm{dom}(F_{\overline{q}})$ then $\eta \in \mathrm{dom}(F_p)$, and hence, as $p \in \mathbb{P}_\alpha$, clause (C3) follows from the induction hypothesis $\overline{q}|_\eta \leq_\eta p|_\eta$. Let us show clause (C4) now. Suppose that $\eta \in \mathrm{dom}(F_{\overline{q}})$, and let $(R, \beta) \in \Delta_{\overline{q}}$ be such that $\beta \geq \eta + 1$ and $R \in \mathcal{S}_{\eta+1} \cup \mathcal{L}_{\eta+1}$. We need to check that $\overline{q}|_\eta$ forces that $F_{\overline{q}}(\eta)$ is $(R[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic. First of all, note that $F_{\overline{q}}(\eta) = F_p(\eta)$. Moreover, note that (R, β) cannot be in $\Delta_{q_{r-1}}$, as $q_{r-1} \in \mathbb{P}_{\mu+1}$ and $\beta \geq \eta+1 > \mu+1$. If (R, β) belongs to Δ_p, then we get the conclusion from the fact that $\overline{q}|_\eta \leq_\eta p|_\eta$. Hence, we may assume that $(R, \beta) \in \Delta_q$. On one hand, $\eta \in \mathrm{dom}(F_p) \setminus \gamma = \mathrm{dom}(F_p) \setminus \sup(Q \cap \alpha)$, and so $\sup(Q \cap \alpha) < \eta$. On the other hand, $q \in Q^*$, so in particular, $(R, \beta) \in Q^*$. If $R \in \mathcal{S}_{\eta+1}$, then $\eta \in R$, and hence, $\eta < \sup(R \cap \alpha) < \sup(Q \cap \alpha)$, which is impossible. However, if $R \in \mathcal{L}_{\eta+1}$, then $\sup(R \cap \alpha) = \alpha$, and moreover, as $\sup(Q \cap \alpha) < \eta < \beta$ and $\beta \in Q$, it must be the case that $\beta = \alpha$. Therefore, $(N, \eta + 1) \in \Delta_{q|_{\eta+1}}$. Recall from the beginning of this case that $p|_\eta$ forces that $F_p(\eta)$ is $(N[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic for every $N \in \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. Hence, as $\overline{q}|_\eta \leq_\eta p|_\eta$, we will be done if we show that $\overline{q}|_\eta$ forces that $R \in \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. But note that this is indeed the case. The condition $\overline{q}|_\eta$ forces that $q|_\eta \in \dot{G}_\eta$, because $\overline{q}|_\eta \leq_\eta q|_\eta$ by induction hypothesis. Hence, as $R \in Q \cap \mathcal{L}_{\eta+1}$ and $(R, \eta) \in \Delta_{q|_\eta}$, we get that $\overline{q}|_\eta$ forces that $R \in \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. In fact more is true: $\overline{q}|_\eta$ forces that $\overline{\Delta^{-1}_{q|_\eta}(\eta)} \cap \mathcal{L}_{\eta+1} \subseteq \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. We can use this, and a similar argument to the one above to show that $q|_{\eta+1}$ satisfies clause (C5). Let us show clause (C6) now. Suppose that $\eta \in \mathrm{dom}(F_{\overline{q}})$. Let $M \in \mathcal{S}_\delta$ such that $(M, \beta) \in \Delta_{\overline{q}}$, where $\beta \geq \eta+1$ and $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \eta < \delta$. As before, (M, β) cannot be in $\Delta_{q_{r-1}}$, as $q_{r-1} \in \mathbb{P}_{\mu+1}$ and $\beta \geq \eta+1 > \mu+1$. If $(M, \beta) \in \Delta_p$, then the conclusion follows from the fact that $\overline{q}|_\eta \leq p|_\eta$. Hence, we may assume that $(M, \beta) \in \Delta_q$. Note that then $M \in Q$, and as $Q \in \mathcal{S}$, $M \subseteq Q$. It's not too hard to see that $\overline{q}|_\eta$ forces that $\dot{\mathcal{A}}^M_{\dot{G}_\eta} \subseteq \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. But then we are done because $\overline{q}|_\eta \leq_\eta p|_\eta$ and $p|_\eta$ forces that $F_p(\eta)$ is $(N[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic for every $N \in \dot{\mathcal{A}}^Q_{\dot{G}_\eta}$. Lastly, we need to check that $\overline{q}|_{\eta+1} \leq_{\eta+1} p|_{\eta+1}, q|_{\eta+1}$. Clauses (D1) and (D3) are clear by induction hypothesis and the definition of $\overline{q}$. Clause (D2) follows from induction hypothesis $\overline{q}|_\eta \leq_\eta p|_\eta, q|_\eta$, from the definition of $\overline{q}$, and from the fact that $\max(\mathrm{dom}(F_q)) = \xi_{r-1} = \mu < \eta$.

The case $\xi > \mu+1$ limit follows from the induction hypothesis. Recall that in this case, to see that $\overline{q}|_\xi \in \mathbb{P}_\xi$ we only need to check clauses (C0)-(C2). □

4.4.3 Proof of (P3)$_\alpha$

Let $\overline{M} \in \mathcal{T}^+$ and let $p \in \mathbb{P}_\alpha$ be an $(\overline{M}, \mathbb{P}_\alpha)$-pre-generic condition. We will show that p is $(N^*, \mathbb{P}_\alpha)$-generic for every $N^* \in \mathcal{L}^*_\alpha$ such that $N^* \cap H(\kappa) \in \overline{M}$.

Fix some $N^* \in \mathcal{L}^*_\alpha$ such that $N^* \cap H(\kappa) \in \overline{M}$.

⋆ **Base case**

Suppose that $\alpha = 0$.

Let E be an open dense subset of $\mathbb{P}_0$ in N^*. We will find a condition in $E \cap N^*$ compatible with p. We may assume, without loss of generality, that $p \in E$. In light of lemma 2.4.29, we may assume further that $N \in \mathrm{dom}(\Delta_p)$. Now, we simply argue exactly as in the base case of $(\mathrm{P2})_\alpha$.

⋆ **Successor case**

Suppose that $\alpha = \gamma + 1$, that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta \leq \gamma$, and that $(\mathrm{P1})_\alpha$ and $(\mathrm{P2})_\alpha$ hold.

Let $E \in N^*$ be an open dense subset of $\mathbb{P}_{\gamma+1}$. It is a routine matter to check that we can argue exactly as in the proof of $(\mathrm{P2})_{\gamma+1}$, using $(\mathrm{P3})_\gamma$ instead of $(\mathrm{P2})_\gamma$, and (C5) instead of (C4), to find a condition $q \in E \cap N^*$ compatible with p.

⋆ **Limit case**

Suppose that α is a nonzero limit ordinal, that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta < \alpha$, and that $(\mathrm{P1})_\alpha$ and $(\mathrm{P2})_\alpha$ hold.

Let $E \in N^*$ be an open dense subset of $\mathbb{P}_{\gamma+1}$ and assume, without loss of generality that $p \in E$. The argument is very similar to that of $(\mathrm{P2})_\alpha$ for α limit, so we will just point out the places in which they differ. We divide the proof in three cases.

Case 1. $cf(\alpha) = \omega$.

Argue exactly as in the proof of $(\mathrm{P2})_\alpha$ for α of countable cofinality, and use $(\mathrm{P3})_\gamma$ instead of $(\mathrm{P2})_\gamma$.

Case 2. $cf(\alpha) > \omega_1$.

Note that $\sup(N \cap \alpha) < \alpha$ and assume that $\mathrm{dom}(F_p)$ is not bounded by $\sup(N \cap \alpha)$. The main difference from the proof of $(\mathrm{P2})_\alpha$ is the proof of the following claim.

Claim 4.4.11. *For every $R \in \mathrm{dom}(\Delta_p)$ such that $\varepsilon_R < \varepsilon_N$,*

(a) $\sup(R \cap N \cap \alpha) < \sup(N \cap \alpha)$, *if* $R \in \mathcal{S} \cup \mathcal{L}$, *and*

(b) $\sup(\bigcup R \cap N \cap \alpha) < \sup(N \cap \alpha)$, *if* $R \in \mathcal{T}^+$.

Proof. Assume that $R \in \mathrm{dom}(\Delta_p) \cap (\mathcal{S} \cup \mathcal{L})$. If $R \in N$, then $\sup(R \cap \alpha) < \sup(N \cap \alpha)$. Hence, assume that $R \notin N$. Then, by lemma 2.4.17, there must be an $N' \in \mathrm{dom}(\Delta_p) \cap \bigcup \mathcal{T}^+$ such that $R \in N'$ and an isomorphism $\Psi_{N',N}$ between N' and N fixing $N' \cap N$. Therefore, $\sup(\Psi_{N',N}(R) \cap \alpha) < \sup(N \cap \alpha)$. Hence, as $\Psi_{N',N}$ fixes $N' \cap N \cap \kappa$, it also fixes $R \cap N \cap \kappa$, and thus,

$$\sup(R \cap N \cap \alpha) \leq \sup(\Psi_{N',N}(R) \cap \alpha) < \sup(N \cap \alpha).$$

We can prove (b) similarly. □

The rest of the proof of this case is, word by word, a translation of the proof of $(\mathrm{P2})_\alpha$ for α limit of cofinality $> \omega_1$.

Case 3. $cf(\alpha) = \omega_1$.

Note that since $N \in \mathcal{L}$, then $\sup(N \cap \alpha) = \alpha$, and hence, $\mathrm{dom}(F_p) \subseteq \sup(N \cap \alpha)$. Therefore, we can argue exactly as in case 1, $cf(\alpha) = \omega$.

4.4.4 Proof of $(\mathrm{P4})_\alpha$

If α is a nonzero ordinal, we will assume that $(\mathrm{P1})_\beta$-$(\mathrm{P4})_\beta$ hold for every $\beta < \alpha$, and that $(\mathrm{P1})_\alpha$-$(\mathrm{P3})_\alpha$ hold.

Lemma 4.4.12. *Let G_α be a $\mathbb{P}_\alpha$-generic filter over V. Then, in $V[G_\alpha]$ the following holds:*

(1) The set $\{M \in \mathcal{S}_\alpha : (M, \alpha) \in \Delta_u$ for some $u \in G_\alpha\}$ is stationary in $[H(\kappa)]^{\aleph_0}$.

(2) The set $\{N \in \mathcal{L}_\alpha : (N, \alpha) \in \Delta_u$ for some $u \in G_\alpha\}$ is stationary in $[H(\kappa)]^{\aleph_1}$.

Proof. The proof of both (1) and (2) are exactly the same. Let us show (1). Denote the set $\{M \in \mathcal{S}_\alpha : (M, \alpha) \in \Delta_u$ for some $u \in G_\alpha\}$ by Y and let $\dot{Y}$ be a $\mathbb{P}_\alpha$-name for Y. Assume, aiming for a contradiction, that Y is non-stationary.

Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for a function from $[H(\kappa)]^{<\omega}$ into $H(\kappa)$ and let $p \in \mathbb{P}_\alpha$ be a condition forcing that $\dot{f}$ is a witness of the non-stationarity of $\dot{Y}$, i.e., p forces that, for every $M \in \dot{Y}$, M is not closed under $\dot{f}$. Since $\mathcal{S}^*_\alpha$ is a club of $[H(\theta_\alpha)]^{\aleph_0}$, there is some $M^* \in \mathcal{S}^*_\alpha$ such that $p, \dot{f} \in M^*$. Define $M := M^* \cap H(\kappa)$ and note that $p \in M$ by definition of $\mathbb{P}_\alpha$. By $(\mathrm{P1})_\alpha$ there is some $q \leq_\alpha p$ such that $(M, \alpha) \in \Delta_q$. Note that q forces that $M \in \dot{Y}$. Hence, if we show that q forces that M is closed under $\dot{f}$, we will get a contradiction. Note that it follows from $(\mathrm{P2})_\alpha$ that q is $(M^*, \mathbb{P}_\alpha)$-generic. Therefore, q forces that $M^*[\dot{G}_\alpha] \cap H(\kappa) = M$. If we combine this with the fact that $\dot{f} \in M^*$, it follows that q forces that M is closed under $\dot{f}$, as we wanted. □

Lemma 4.4.13. *Suppose that* $\alpha = \gamma + 1$. *Then,* $\mathbb{P}_\gamma$ *forces that* $\dot{\mathbb{Q}}_\gamma$ *has the* $\aleph_3$*-chain condition.*

Proof. Let G_γ be a $\mathbb{P}_\gamma$-generic filter over V and let $\mathbb{Q}_\gamma$ be the interpretation of $\dot{\mathbb{Q}}_\gamma$ by G_γ. Working in $V[G_\gamma]$, suppose that $A := \{\nu_\xi : \xi < \omega_3\}$ is a maximal antichain of $\mathbb{Q}_\gamma$. Let $\dot{A}$ and $\dot{\nu}_\xi$ be $\mathbb{P}_\gamma$-names for A and ν_ξ, respectively, for all $\xi < \omega_3$. Let D be a club of $[H(\kappa)]^{\leq \aleph_1}$ in V witnessing that $\mathbb{Q}_\gamma$ is $(\mathcal{S}, \mathcal{L})$-finitely proper relative to V and G_γ. By lemma 4.4.12, for each $\xi < \omega_3$ we can find a model $R_\xi \in \mathcal{S}_\gamma \cup \mathcal{L}_\gamma$ in D such that $\dot{\nu}_\xi, \dot{A} \in R_\xi$ and $(R_\xi, \gamma) \in \Delta_{u_\xi}$ for some $u_\xi \in G_\gamma$. Since $\varepsilon_{R_\xi} < \omega_2$ for each $\xi < \omega_3$, there are $I \in [\omega_3]^{\omega_3}$ and $\varepsilon < \omega_2$ such that $\varepsilon_{R_\xi} = \varepsilon$ for every $\xi \in I$. Since the models R_ξ have size less than or equal $\aleph_1$, we can find two different $\xi_0, \xi_1 \in I$ such that $\nu_{\xi_0} \notin R_{\xi_1}[G_\gamma]$, and as G_γ is a filter, we can find a common extension r of u_{ξ_0} and u_{ξ_1} in G_γ. Note that $\nu_{\xi_0} \in R_{\xi_0}[G_\gamma]$ and that $\{(R_{\xi_0}, \gamma), (R_{\xi_1}, \gamma)\} \subseteq \Delta_r$, where $\varepsilon_{R_{\xi_0}} = \varepsilon_{R_{\xi_1}}$. Therefore, by the $(\mathcal{S}, \mathcal{L})$-finite properness of $\mathbb{Q}_\gamma$ relative to V and G_γ, there is a $\mathbb{Q}_\gamma$-extension ν of ν_{ξ_0} that is $(R_{\xi_1}[G_\gamma], \mathbb{Q}_\gamma)$-generic. Hence, since A is a maximal antichain by assumption and it is an element of $R_{\xi_1}[G_\gamma]$, there must be a condition in $R_{\xi_1}[G_\gamma] \cap A$, different from ν_{ξ_0} and compatible with ν. Note that $R_{\xi_1}[G_\gamma] \cap A$ doesn't contain ν_{ξ_0} because ξ_0 and ξ_1 were chosen so that $\nu_{\xi_0} \notin R_{\xi_1}[G_\gamma]$, and that it is non-empty because it contains ν_{ξ_1}. Hence, this contradicts our assumption that A is a maximal antichain of $\mathbb{Q}_\gamma$. $\square$

Definition 4.4.14. Suppose that $\alpha = \gamma + 1$. We define $\dot{\mathbb{R}}_\gamma$ in $V^{\mathbb{P}_\gamma}$ as the set of all pairs $(W, \mathcal{M})$ such that W has cardinality at most 1 and

- if $W = \{\nu\}$, then there is some $r = (F_r, \Delta_r) \in \mathbb{P}_{\gamma+1}/\dot{G}_\gamma$ with $\gamma \in \operatorname{dom}(F_r)$ such that $\nu = F_r(\gamma)$ and $\mathcal{M} = \Delta_r^{-1}(\gamma + 1)$, and
- if $W = \varnothing$, then there is some $r = (F_r, \Delta_r) \in \mathbb{P}_{\gamma+1}/\dot{G}_\gamma$ with $\gamma \notin \operatorname{dom}(F_r)$ such that $\mathcal{M} = \Delta_r^{-1}(\gamma + 1)$.

We define an order on $\dot{\mathbb{R}}_\gamma$ by $(W_0, \mathcal{M}_0) \leq_{\dot{\mathbb{R}}_\gamma} (W_1, \mathcal{M}_1)$ if and only if

- $\mathcal{M}_0 \supseteq \mathcal{M}_1$, and
- if $W_1 = \{\nu_1\}$, then $W_0 = \{\nu_0\}$ for some $\nu_0 \in \dot{\mathbb{Q}}_\gamma$ which extends ν_1 in $\dot{\mathbb{Q}}_\gamma$.

Lemma 4.4.15. *Suppose that $\alpha = \gamma + 1$. Then, $\mathbb{P}_{\gamma+1}$ is isomorphic to a dense subset of $\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma$.*

Proof. Let $\widetilde{\mathbb{P}}_{\gamma+1}$ be the set of all pairs $(q|_\gamma, \check{x})$ where $x = (\{F_q(\gamma)\}, \Delta_q^{-1}(\gamma + 1))$ if $\gamma \in \operatorname{dom}(F_q)$, and $x = (\varnothing, \Delta_q^{-1}(\gamma + 1))$ if $\gamma \notin \operatorname{dom}(F_q)$, for some condition q in $\mathbb{P}_{\gamma+1}$. Note that if $(q|_\gamma, \check{x}) \in \widetilde{\mathbb{P}}_{\gamma+1}$ is as above, since $q|_\gamma$ forces that $q|_\gamma \in \dot{G}_\gamma$, it also forces that $\check{x} \in \dot{\mathbb{R}}_\gamma$. Therefore, $(q|_\gamma, \check{x}) \in \mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma$. We claim that $\widetilde{\mathbb{P}}_{\gamma+1}$ is dense in $\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma$. Let $(p, \check{x}) \in \mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma$ and assume that the first component of x is nonempty (the same argument works if the first component of x is empty). Then p forces that there is $r \in \mathbb{P}_{\gamma+1}$ such that $r|_\gamma \in \dot{G}_\gamma$ and $\check{x} = (\{F_r(\gamma)\}, \Delta_r^{-1}(\gamma+1))$. Consider the natural amalgamation $q := (F_p \cup \{\langle \gamma, F_r(\gamma)\rangle\}, \Delta_p \cup \Delta_r)$ of p and r, which is a condition in $\mathbb{P}_{\gamma+1}$ extending r by lemma 4.3.3. Let $y = (\{F_q(\gamma)\}, \Delta_q^{-1}(\gamma + 1))$. We claim that $(q|_\gamma, \check{y}) \in \widetilde{\mathbb{P}}_{\gamma+1}$ and that $(q|_\gamma, \check{y}) \leq_{\mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma} (p, \check{x})$. The first part is clear. For the second part note that as $\Delta_q = \Delta_p \cup \Delta_r$ and $p \in \mathbb{P}_\gamma$, then $\Delta_q^{-1}(\gamma+1)$ equals $\Delta_r^{-1}(\gamma + 1)$, and thus, $y = x$. Moreover, note that $q|_\gamma = (F_p, \Delta_p \cup \Delta_{r|_\gamma})$. Hence, the second part also holds. Finally, note that the map $\psi : \mathbb{P}_{\gamma+1} \to \widetilde{\mathbb{P}}_{\gamma+1}$ defined by $\psi(q) = (q|_\gamma, \check{x})$, where $x = (\{F_q(\gamma)\}, \Delta_q^{-1}(\gamma + 1))$ if $\gamma \in \operatorname{dom}(F_q)$, and

$x = (\varnothing, \Delta_q^{-1}(\gamma+1))$ if $\gamma \notin \mathrm{dom}(F_q)$, is clearly an isomorphism between $\mathbb{P}_{\gamma+1}$ and $\widetilde{\mathbb{P}}_{\gamma+1}$. □

Lemma 4.4.16. *Suppose that $\alpha = \gamma + 1$. Then, $\mathbb{P}_\gamma$ forces that $\dot{\mathbb{R}}_\gamma$ has the $\aleph_3$-chain condition.*

Proof. Let G_γ be a $\mathbb{P}_\gamma$-generic filter over V and let $\mathbb{R}_\gamma$ be the interpretation of $\dot{\mathbb{R}}_\gamma$ by G_γ. Working in $V[G_\gamma]$, suppose that $\{(W_\xi, \mathcal{M}_\xi) : \xi < \omega_3\} \subseteq \mathbb{R}_\gamma$, where for each $\xi < \omega_3$, $W_\xi = \{F_{r_\xi}(\gamma)\}$ is nonempty, $\mathcal{M}_\xi = \Delta_{r_\xi}^{-1}(\gamma+1)$, and $r_\xi = (F_{r_\xi}, \Delta_{r_\xi})$ is a condition in $\mathbb{P}_{\gamma+1}/G_\gamma$. By lemma 4.4.13, recall that $\mathbb{P}_\gamma$ forces $\dot{\mathbb{Q}}_\gamma$ to have the $\aleph_3$-chain condition. Hence, there are $\nu \in \mathbb{Q}_\gamma$, two different $\xi_0, \xi_1 \in \omega_3$, and a condition $s \in G_\gamma$, such that s forces that $\nu \leq_{\mathbb{Q}_\gamma} F_{r_{\xi_0}}(\gamma), F_{r_{\xi_1}}(\gamma)$. Since G_γ is a filter, and by extending s if necessary, we may assume that s extends both $r_{\xi_0}|_\gamma$ and $r_{\xi_1}|_\gamma$. Hence, in particular $(R, \gamma) \in \Delta_s$, for every $R \in \mathcal{M}_{\xi_0} \cup \mathcal{M}_{\xi_1}$. Therefore, the amalgamation

$$s^* = (F_s \cup \{\langle \gamma, \nu \rangle\}, \Delta_s \cup \Delta_{r_{\xi_0}} \cup \Delta_{r_{\xi_1}})$$

is a condition in $\mathbb{P}_{\gamma+1}$, extending both r_{ξ_0} and r_{ξ_1} by lemma 4.3.5. Moreover, note that since $\Delta_{s^*|_\gamma} = \Delta_s \cup \Delta_{r_{\xi_0}|_\gamma} \cup \Delta_{r_{\xi_1}|_\gamma}$ and as s extends $r_{\xi_0}|_\gamma$ and $r_{\xi_1}|_\gamma$, the conditions $s^*|_\gamma$ and s are equivalent in the forcing $\mathbb{P}_\gamma$ by lemma 4.3.2. Consequently, $s^*|_\gamma \in G_\gamma$, and hence, $(\{\nu\}, \Delta_{s^*}^{-1}(\gamma+1)) = (\{\nu\}, \mathcal{M}_{\xi_0} \cup \mathcal{M}_{\xi_1})$ is a condition in $\mathbb{R}_\gamma$ extending both $(W_{\xi_0}, \mathcal{M}_{\xi_0})$ and $(W_{\xi_1}, \mathcal{M}_{\xi_1})$.

Suppose now that, working in $V[G_\gamma]$, there is $\{(W_\xi, \mathcal{M}_\xi) : \xi \in I\} \subseteq \mathbb{R}_\gamma$, for some $I \in [\omega_3]^{\omega_3}$, such that for each $\xi \in I$, W_ξ is empty, $\mathcal{M}_\xi = \Delta_{r_\xi}^{-1}(\gamma+1)$, and $r_\xi = (F_r, \Delta_r)$ is a condition in $\mathbb{P}_{\gamma+1}/G_\gamma$. For any two different $\xi_0, \xi_1 \in I$, since G_γ is a filter, there is $s \in G_\gamma$ extending both $r_{\xi_0}|_\gamma$ and $r_{\xi_1}|_\gamma$. In this case, since $\gamma \notin \mathrm{dom}(F_{r_{\xi_0}}) \cup \mathrm{dom}(F_{\xi_1})$, instead of lemma 4.3.5, we can use the first part of lemma 4.3.6 and argue exactly as in the last case. □

Lemma 4.4.17. *$\mathbb{P}_\alpha$ has the $\aleph_3$-chain condition.*

Proof. Recall that we have assumed that GCH holds in the ground model V. Although, $2^{\aleph_1} = \aleph_2$ seems to be enough to get this result.

If $\alpha = 0$ the result follows from lemma 2.4.34. In fact, in this case $\mathbb{P}_0$ has the $\aleph_3$-Knaster condition.

If $\alpha = \gamma + 1$, the result follows from lemmas 4.4.15 and 4.4.16, together with the induction hypothesis and the fact that the $\aleph_3$-c.c. is preserved under two-step iterations (see lemma 1.1.41).

Assume now that α is a nonzero limit ordinal. Let q_ξ be a condition in $\mathbb{P}_\alpha$ for every $\xi < \omega_3$. Suppose first that $cf(\alpha) \neq \omega_3$. If $cf(\alpha) > \omega_3$, there is $\gamma < \alpha$ such that $\mathrm{dom}(F_{q_\xi}) \subseteq \gamma$ for every $\xi < \omega_3$. If $cf(\alpha) < \omega_3$, by a counting argument there must be some $\gamma < \alpha$ such that $\{\xi < \omega_3 : \mathrm{dom}(F_{q_\xi}) \subseteq \gamma\}$ has size $\aleph_3$. In both cases there are $I \in [\omega_3]^{\omega_3}$ and $\gamma < \alpha$ such that $\mathrm{dom}(F_{q_\xi}) \subseteq \gamma$ for each $\xi \in I$. By induction hypothesis there are two different $\xi_0, \xi_1 \in I$ and a condition $r \in \mathbb{P}_\gamma$ extending both $q_{\xi_0}|_\gamma$ and $q_{\xi_1}|_\gamma$. Hence, it follows from lemma 4.3.7 that q_{ξ_0} and q_{ξ_1} are compatible in $\mathbb{P}_\alpha$. Now assume that $cf(\alpha) = \omega_3$. For every $\xi < \omega_3$, define

$$\begin{aligned} A_\xi = \mathrm{dom}(F_{q_\xi}) \cup \bigcup \{R \cap \beta : R \in \mathrm{dom}(\Delta_{q_\xi}) \cap (\mathcal{S} \cup \mathcal{L})\} \\ \cup \bigcup \{N \cap \beta : N \in \bigcup (\mathrm{dom}(\Delta_{q_\xi}) \cap \mathcal{T}^+)\}, \end{aligned}$$

and let B_ξ be the set of all $\nu < \alpha$ such that $(M, \nu+1) \in \Delta_{q_\xi}$, for some $M \in \mathcal{S}_\delta$, where $\delta \leq \kappa$ is a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \nu < \delta$. Define Z_ξ as the union of A_ξ and B_ξ. By $2^{\aleph_1} = \aleph_2$ (recall that we have assumed that V is a ground model for the GCH) there is $I \in [\omega_3]^{\omega_3}$ such that $\{Z_\xi : \xi \in I\}$ forms a Δ-system with root X, by lemma 1.1.19. Since X is a set of ordinals of α of size at most $\aleph_1$ and $cf(\alpha) = \omega_3$, there must be some $\gamma < \alpha$ such that $X \subseteq \gamma$. By induction hypothesis, for any two different $\xi_0, \xi_1 \in I$ there is a condition $r \in \mathbb{P}_\gamma$ extending both $q_{\xi_0}|_\gamma$ and $q_{\xi_1}|_\gamma$. Hence, the amalgamation of r, q_{ξ_0} and q_{ξ_1} given by lemma 4.3.8 is a condition in $\mathbb{P}_\alpha$ extending q_{ξ_0} and q_{ξ_1}. □

4.5 Properness and further preservation lemmas

The next theorem follows from $(\mathrm{P1})_\alpha$-$(\mathrm{P3})_\alpha$ of lemma 4.4.2.

Theorem 4.5.1. *$\mathbb{P}_\alpha$ is $\mathcal{S}^*_\alpha$-proper and $\mathcal{L}^*_\alpha$-proper for every $\alpha \leq \kappa$.*

Together with lemma 4.4.17, we get the following.

Corollary 4.5.2. *$\mathbb{P}_\alpha$ preserves all cardinals, for all $\alpha \leq \kappa$.*

Lemma 4.5.3. *For every $\gamma < \alpha \leq \kappa$ and every $p \in \mathbb{P}_\alpha$ such that $\gamma \notin \mathrm{dom}(F_p)$ there is an extension p^* of p such that $\gamma \in \mathrm{dom}(F_{p^*})$.*

Proof. The proof is a simpler version of the proof of the limit case of $(\mathrm{P2})_\alpha$. We will divide it into two parts. Our plan is to find first an extension q of $p|_\gamma$ and a $\mathbb{P}_\gamma$-name $\dot{x} \in H(\kappa)$ so that the amalgamation

$$p' = (F_q \cup \{\langle \gamma, \dot{x} \rangle\}, \Delta_q \cup \Delta_{p|_{\gamma+1}}),$$

given by lemma 4.3.6, is a condition in $\mathbb{P}_{\gamma+1}$ extending $p|_{\gamma+1}$. Then, in case $\alpha > \gamma + 1$, we will define p^* as the natural amalgamation

$$p \wedge_{\gamma+1} p' = (F_{p'} \cup (F_p \restriction [\gamma+1, \alpha), \Delta_{p'} \cup \Delta_p),$$

which is a condition in $\mathbb{P}_\alpha$ extending p by lemma 4.3.3.

We start by defining the sets of relevant models. First, the set of relevant elementary models from $\mathrm{dom}(\Delta_p)$,

$$\mathcal{M}_0 := \{R \in \mathcal{S}_{\gamma+1} \cup \mathcal{L}_{\gamma+1} : (R, \gamma+1) \in \Delta_{p|_{\gamma+1}}\}.$$

Second, the set of relevant models that belong to non-elementary models from $\mathrm{dom}(\Delta_p)$,

$$\mathcal{M}_1 := \{N \in \overline{M} \cap \mathcal{L}_{\gamma+1} : \overline{M} \in \mathcal{T}^+, (\overline{M}, \gamma+1) \in \Delta_{p|_{\gamma+1}}\}.$$

Third, the set $\mathcal{M}_2$ of models $N \in \mathcal{L}_{\gamma+1}$ forced by $p|_\gamma$ to be members of $\dot{\mathcal{A}}^M_{\dot{G}_\gamma}$, where $M \in \mathcal{S}_\delta$ is such that $(M, \gamma+1) \in \Delta_{\gamma+1}$, with $\delta \leq \kappa$ a limit ordinal of cofinality ω_1 such that $\sup(M \cap \delta) \leq \gamma < \delta$. Denote by $\mathcal{M}$ the set $\mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$. Let $\dot{D}$ be the $\lhd$-least $\mathbb{P}_\gamma$-name for a club D of $[H(\kappa)]^{\leq \aleph_1}$ in V witnessing the $(\mathcal{S}, \mathcal{L})$-finite properness of $\dot{\mathbb{Q}}_\gamma$ with respect to V and $\dot{G}_\gamma$. We can argue as in the proof of the limit case of $(\mathrm{P2})_\alpha$ to show that $p|_\gamma$ forces that $\mathcal{M} \subseteq \dot{D}$. It's also clear that $p|_\gamma$ forces that $\dot{\mathbb{Q}}_\gamma \in R[\dot{G}_\gamma]$ for every $R \in \mathcal{M}$. Pick any $Q \in \mathcal{M}$ such that $Q \cap \mathcal{M} = \varnothing$. Since $\mathbb{1}_{\dot{\mathbb{Q}}_\gamma}$ is forced to be a member of $Q[\dot{G}_\gamma]$, there is an extension $q \in \mathbb{P}_\gamma$ of $p|_\gamma$ and a $\mathbb{P}_\gamma$-name $\dot{x} \in H(\kappa)$ such that q forces that $\dot{x}$ is a condition in $\dot{\mathbb{Q}}_\gamma$ (extending $\mathbb{1}_{\dot{\mathbb{Q}}_\gamma}$) and that $\dot{x}$ is $(R[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-generic for all $R \in \mathcal{M}$. Hence, by lemma 4.3.6 there is a condition $p' \in \mathbb{P}_{\gamma+1}$ extending $p|_{\gamma+1}$ and such that $\gamma \in \mathrm{dom}(F_{p'})$.

Now, if $\alpha = \gamma + 1$, we simply let p^* be the condition p'. Otherwise, define p^* as the amalgamation $p \wedge_{\gamma+1} p'$ given by lemma 4.3.3. □

Lemma 4.5.4. *For every $\alpha < \kappa$ and every condition $p \in \mathbb{P}_\kappa$, p forces that the set $\{y \in H(\kappa) : \exists (F, \Delta) \in \dot{G}_\kappa (F(\alpha) = y)\}$ generates a $V[\dot{G}_\alpha]$-generic filter on $\dot{\mathbb{Q}}_\alpha$.*

Proof. Let $\dot{G}^+_\alpha$ be a $\mathbb{P}_\kappa$-name for the set $\{y \in H(\kappa) : \exists (F, \Delta) \in \dot{G}_\kappa (F(\alpha) = y)\}$. It is immediate to see that $\dot{G}^+_\alpha$ is forced to be a set of pairwise compatible conditions from $\dot{\mathbb{Q}}_\alpha$. Now let $\dot{E}$ be a $\mathbb{P}_\kappa$-name for a dense subset E of $\dot{\mathbb{Q}}_\alpha$ in $V[\dot{G}_\alpha]$. We claim that the set

$$E^* := \{p \in \mathbb{P}_\kappa : \alpha \in \mathrm{dom}(F_p), p|_\alpha \Vdash_\alpha F_p(\alpha) \in \dot{E}\}$$

is a dense subset of $\mathbb{P}_\kappa$. Let $q \in \mathbb{P}_\kappa$ and find, by lemma 4.5.3, a condition $q^* \in \mathbb{P}_\kappa$ extending q and such that $\alpha \in \mathrm{dom}(F_{q^*})$. Since $\dot{E}$ is a name of a dense subset of $\dot{\mathbb{Q}}_\alpha$, there are an extension $r \in \mathbb{P}_\alpha$ of $q^*|_\alpha$ and a $\mathbb{P}_\alpha$-name $\dot{x} \in H(\kappa)$ such that $r \Vdash_\alpha$ "$\dot{x} \in \dot{E}$ and $\dot{x} \leq_{\dot{\mathbb{Q}}_\alpha} F_{q^*}(\alpha)$". By lemma 4.3.5, there is an extension $s \in \mathbb{P}_{\alpha+1}$ of $q^*|_{\alpha+1}$ with $F_s(\alpha) = \dot{x}$. The amalgamation $\overline{q} := q^* \wedge_{\alpha+1} s$, given by lemma 4.3.3, is a condition in $\mathbb{P}_\kappa$ such that $\overline{q} \leq_\kappa q^*$ and $F_{\overline{q}}(\alpha) = \dot{x}$, and, in particular, $\overline{q} \leq_\kappa q$ and $\overline{q}|_\alpha \Vdash_\alpha F_{\overline{q}}(\alpha) \in \dot{E}$. This shows that E^* is dense. Hence, for every

$\mathbb{P}_\kappa$-generic filter over V, there is some $p \in G_\kappa \cap E^*$ such that $\alpha \in \mathrm{dom}(F_p)$ and $p|_\alpha \Vdash_\alpha F_p(\alpha) \in \dot{G}^+_\alpha \cap \dot{D}$. □

Definition 4.5.5. For a $\mathbb{P}_\kappa$-name τ, a $\mathbb{P}_\kappa$*-nice name for a subset of* τ is a $\mathbb{P}_\kappa$-name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\}$, where each A_σ is an antichain of $\mathbb{P}_\kappa$[3].

Lemma 4.5.6. *$\mathbb{P}_\kappa$ forces that $\kappa^{<\kappa} = \kappa$.*

Proof. First note that $\mathbb{P}_\kappa \subseteq H(\kappa)$, so $|\mathbb{P}_\kappa| \leq |H(\kappa)| = 2^{<\kappa} = \kappa$. Moreover, since $\mathbb{P}_\kappa$ has the $\aleph_3$-c.c., there are at most $\kappa^{\aleph_2}$ many antichains. Hence, there are at most $(\kappa^{\aleph_2})^{<\kappa} = \kappa^{<\kappa} = \kappa$ many nice names for bounded subsets of κ. □

It is worth pointing out that $\mathbb{P}_\kappa$ forces that $2^{\aleph_0} = \kappa$. This follows, from example, from the fact that $\Phi(\alpha)$ is Cohen forcing for κ-many ordinals $\alpha < \kappa$, since Cohen forcing has the c.c.c., and c.c.c. forcings are included in the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings.

Lemma 4.5.7. *If $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$-name for an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing such that $\dot{\mathbb{Q}} \in H(\kappa)^V$, then $\mathbb{P}_\kappa$ forces that $\dot{\mathbb{Q}}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper relative to V and $\dot{G}_\kappa$.*

Proof. Let G_κ be a $\mathbb{P}_\kappa$-generic filter over V and let $\mathbb{Q}$ be the interpretation of $\dot{\mathbb{Q}}$ by G_κ. First, note that since $\dot{\mathbb{Q}} \in H(\kappa)^V$, then by lemma 4.4.5, $\Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{Q}} \in H(\kappa)^{V[\dot{G}_\kappa]}$. Hence, in particular, $\mathbb{Q} \in H(\kappa)^{V[G_\kappa]}$. Let $D \subseteq [H(\kappa)^{V[G_\kappa]}]^{\leq\aleph_1}$ be a club in $V[G_\kappa]$ witnessing the fact that $\mathbb{Q}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper, and let $\dot{D}$ be a $\mathbb{P}_\kappa$-name for D. Let E be a club of $[H(\kappa)]^{\leq\aleph_1}$ in V such that for every $R \in E \cap (\mathcal{S} \cup \mathcal{L})$, there is an elementary submodel R^* of a big enough $H(\theta)$ such that $\kappa, \mathbb{P}_\kappa, \dot{\mathbb{Q}}, \dot{D} \in R^*$ and $R = R^* \cap H(\kappa)$. We claim that the club E witnesses that $\dot{\mathbb{Q}}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper relative to V and $\dot{G}_\kappa$.

Let $\mathcal{M}$ be a countable subset of E such that $|\mathcal{M} \cap \mathcal{S}| < \aleph_0$ and $|\mathcal{M} \cap \mathcal{L}| \leq \aleph_0$. Suppose that $u \in G_\kappa$ is such that for every $R \in \mathcal{M}$, either

[3] See [44] for a further discussion on nice names and arguments involving counting of nice names.

- $(R, \sup(R \cap \kappa)) \in \Delta_u$, or
- $R \in \overline{M}$, where $\overline{M} \in \mathcal{T}^+$ and $(\overline{M}, \sup(\bigcup \overline{M} \cap \kappa)) \in \Delta_u$, or
- $R \in \mathcal{A}^M_{G_\kappa}$, where $M \in \mathcal{S}_\kappa$ and $(M, \sup(M \cap \kappa)) \in \Delta_u$.

Note that $\dot{\mathbb{Q}} \in R^* \cap H(\kappa)^V = R$ for every $R \in \mathcal{M}$, so $\mathbb{Q} \in R[G_\kappa]$. Moreover, note that as $\dot{D} \in R^*$ for every $R \in \mathcal{M}$, lemma 4.4.2 implies that u forces that $R \in \dot{D}$. Furthermore, suppose that u forces that $\dot{x} \in R_0[\dot{G}_\kappa] \cap \dot{\mathbb{Q}}$, where $R_0 \in \mathcal{M}$ is such that either

(a) $\mathcal{M} \cap R_0 = \varnothing$, or

(b) $R_0 \in \mathcal{S}$ and $\varepsilon_{R_0} = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$, and $\dot{x}$ is $(N[\dot{G}_\kappa], \dot{\mathbb{Q}})$-generic for every $N \in \mathcal{M} \cap R_0$.

Observe that u forces that $\varepsilon_R = \varepsilon_{R[\dot{G}_\kappa]}$ for every $R \in \mathcal{M}$, again by lemma 4.4.2. Moreover, it's not too hard to see that

$$\{R[G_\kappa] : R \in \mathcal{M} \cap R_0\} = \{N[G_\kappa] : N \in \mathcal{M}\} \cap R_0[G_\kappa].$$

Therefore, since $\dot{D}$ witnesses that $\dot{\mathbb{Q}}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper, u forces that there is an extension of $\dot{x}$ in $\dot{\mathbb{Q}}$ which is $(R[\dot{G}_\kappa], \dot{\mathbb{Q}})$-generic for every $R \in \mathcal{M}$. Hence, we can conclude that E witnesses that $\dot{\mathbb{Q}}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper relative to V and $\dot{G}_\kappa$, as we wanted. □

4.6 Consistency of forcing axioms

4.6.1 Forcing axiom for posets of restricted size

Lemma 4.6.1. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name for an $(\mathcal{S}, \mathcal{L})$-finitely proper poset of size $\mu < \kappa$ and let $(\dot{D}_i)_{i<\mu}$ be a sequence of $\mathbb{P}_\kappa$-names for dense subsets of $\dot{\mathbb{Q}}$. Then, there is a high enough $\alpha < \kappa$ such that $\dot{\mathbb{Q}}$ and each $\dot{D}_i$ are $\mathbb{P}_\alpha$-names and $\Phi(\alpha) = \dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name of an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notion with respect to V and $\dot{G}_\alpha$.*

Proof. Note that $\dot{\mathbb{Q}}$ and each $\dot{D}_i$ is decided by at most μ-many maximal antichains of $\mathbb{P}_\kappa$. By corollary 4.3.4, and since $\mathbb{P}_\kappa$ has the $\aleph_3$-chain condition, there must be some $\alpha < \kappa$ such that $\dot{\mathbb{Q}}$ and all $\dot{D}_i$ are $\mathbb{P}_\alpha$-names. Moreover, recall that the bookkeeping function Φ was chosen so that $\Phi^{-1}(\{x\})$ is unbounded in κ for each $x \in H(\kappa)$, so we may assume that $\Phi(\alpha) = \dot{\mathbb{Q}}$. In order to see that $\dot{\mathbb{Q}}$ is $(\mathcal{S}, \mathcal{L})$-finitely proper relative to V and $\dot{G}_\alpha$ we can argue exactly as in the proof of 4.5.7 by replacing all the instances of κ by α. □

Corollary 4.6.2. *$\mathbb{P}_\kappa$ forces the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings of size $< \kappa$.*

4.6.2 Forcing axiom for posets of unrestricted size

Theorem 4.6.3. *Let κ be a supercompact cardinal and suppose that Φ is a Laver function*[4]*. Then, $\mathbb{P}_\kappa$ forces the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings and $< \kappa$-many dense sets.*

Proof. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name for an $(\mathcal{S}, \mathcal{L})$-finitely proper forcing. Let $\chi < \kappa$ and let $\dot{A}_i$, for $i < \chi$, be $\mathbb{P}_\kappa$-names for maximal antichains of $\dot{\mathbb{Q}}$. Let G_κ be a $\mathbb{P}_\kappa$-generic filter over V. Let $\mathbb{Q}$ and A_i be the interpretations of $\dot{\mathbb{Q}}$ and $\dot{A}_i$ by G_κ, respectively, for all $i < \chi$. We will be done if we show, in $V[G_\kappa]$, that there is a filter on $\mathbb{Q}$ intersecting all A_i. Let $\lambda > |trcl(\dot{\mathbb{Q}})|^+$ such that $\mathbb{Q} \subseteq \lambda$. Since Φ is a Laver function, there is a (κ, λ)-supercompact embedding $j : V \to M$ such that $j(\Phi)(\kappa) = \dot{\mathbb{Q}}$ (see theorem 1.3.4). Since ${}^\lambda M \subseteq M$ and $\mathbb{P}_\kappa$ has the $\aleph_3$-c.c., we have ${}^\lambda M[G_\kappa] \subseteq M[G_\kappa]$, and hence, $\mathbb{Q} \in M[G_\kappa]$.

Note that $j(\mathbb{P}_\kappa)$ is a finite support iteration with two type symmetric systems of length $j(\kappa)$ and bookkeeping function $j(\Phi)$ in M. Moreover, as $\mathbb{P}_\kappa \subseteq H(\kappa)^V$ and $j \restriction H(\kappa)^V = id \restriction H(\kappa)^V$, we have that $\mathbb{P}_\kappa$ is the κ-th stage of the iteration $j(\mathbb{P}_\kappa)$. Therefore, $j(\Phi)(\kappa) = \dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$-name in $H(j(\kappa))^M$ of an $(j(\mathcal{S}), j(\mathcal{L}))$-finitely proper forcing notion. Hence, by lemma 4.5.7, $\mathbb{P}_\kappa = j(\mathbb{P}_\kappa)_\kappa$ forces that $\dot{\mathbb{Q}}$ is

[4] See section 1.3 for the definition of these notions.

$(j(\mathcal{S}), j(\mathcal{L}))$-finitely proper relative to M and $\dot{G}_\kappa$. Therefore, $\dot{\mathbb{Q}}$ is the κ-th iterand of the iteration $j(\mathbb{P}_\kappa)$. Let $G_{j(\kappa)}$ be a $j(\mathbb{P}_\kappa)$-generic filter over $V[G_\kappa]$. In $V[G_{j(\kappa)}]$ we can naturally extend the elementary embedding $j : V \to M$ to an elementary embedding $j^* : V[G_\kappa] \to M[G_{j(\kappa)}]$, by letting $j^*(\dot{x}_{G_\kappa}) = j(\dot{x})_{G_{j(\kappa)}}$ for every $\mathbb{P}_\kappa$-name $\dot{x}$. By lemma 4.5.4, there is a $j^*(\mathbb{Q})$-generic filter over $M[G_{j(\kappa)}]$, which has non-empty intersection with every maximal antichain in $j^*(\{A_i : i < \chi\})$. Hence, by elementarity of j^* we can conclude that there is a $\mathbb{Q}$-generic filter over $V[G_\kappa]$, which of course has non-empty intersection with all A_i, $i < \chi$. $\square$

4.7 Extensions of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings

In this section we will introduce several extensions of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcing notions, which we would like to explore further in the future. Let us recall the definition of $(\mathcal{S}, \mathcal{L})$-finite properness so that we can refer to it.

Definition 4.7.1. We say that a forcing notion $\mathbb{P} \in H(\kappa)$ is $(\mathcal{S}, \mathcal{L})$-*finitely proper* if and only if there is a club $D \subseteq [H(\kappa)]^{\leq\aleph_1}$ such that for every countable subset $\mathcal{M} \subseteq D$ such that $|\mathcal{M} \cap \mathcal{S}| < \aleph_0$ and $|\mathcal{M} \cap \mathcal{L}| \leq \aleph_0$, if $p \in \mathbb{P} \cap Q$ for some $Q \in \mathcal{M}$ such that either

(1) $\mathcal{M} \cap Q = \varnothing$, or

(2) $Q \in \mathcal{S}$ is such that $\varepsilon_Q = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$, and p is $(N, \mathbb{P})$-generic for every $N \in \mathcal{M} \cap Q$,

then there is an extension $q \leq p$ which is $(R, \mathbb{P})$-generic for every $R \in \mathcal{M}$.

There are several ways in which this class of posets can be enlarged, while preserving its main properties, namely the preservation of all cardinals and being iterable in the sense of section 4.2. The main way to do this is by giving more structure to the countable sets of models $\mathcal{M}$.

There is a clear upper bound on how much structure you can ask for $\mathcal{M}$, which has been already mentioned in section 4.1. You cannot ask for $\mathcal{M}$ to be an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system. The reason being that the symmetric system structure is not transferred, in general, from $\mathcal{M}$ to $\{Q[G_\alpha] : Q \in \mathcal{M}\}$, where G_α is a $\mathbb{P}_\alpha$-generic filter over V and $\mathbb{P}_\alpha$ is the α-th stage of a finite support iteration with two-type symmetric systems as side conditions. But we can be more precise and give a much better upper bound. Indeed, the exact amount of structure that the sets $\mathcal{M}$ admit is dictated by the sets of relevant models from the proof of the limit case of item $(\mathrm{P2})_\alpha$ of lemma 4.4.2. Therefore, by analysing that proof we can extract information about the potential ways in which the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings can be extended.

The first and most straightforward way to improve the definition of $(\mathcal{S}, \mathcal{L})$-finite properness would be to replace item (1) by the assertion that Q is a model of minimal ω_2-height among all the models in $\mathcal{M}$. In order to get this improvement we would need to prove a stronger result than that of claim 4.4.7. Namely, we would have to prove that for every $N \in \mathcal{L}$ such that $\varepsilon_N < \varepsilon_Q$ and either $N \in \overline{M}$, for some $\overline{M} \in \mathrm{dom}(\Delta_p) \cap \mathcal{T}^+$, or $N \in M$, for some $M \in \mathrm{dom}(\Delta_p) \cap \mathcal{S}$, then

$$\sup(N \cap Q \cap \alpha) < \sup(Q \cap \alpha).$$

We speculate that it should be relatively easy to prove this using the symmetry of $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric systems.

Another way in which we could extend the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings would be by considering stratified families of models, first defined in [15].

Definition 4.7.2. A subset $\mathcal{M}$ of $\mathcal{S} \cup \mathcal{L}$ is said to be *stratified* in case for all $Q_0, Q_1 \in \mathcal{M}$, if $\varepsilon_{Q_0} < \varepsilon_{Q_1}$, then in fact $\mathrm{ot}(Q_0 \cap \omega_3) < \varepsilon_{Q_1}$.

Hence, we could strengthen the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings by requiring the sets of models $\mathcal{M}$ to be stratified. Based solely on the results obtained by Asperó and Tananimit, it seems reasonable to expect that certain weakenings of square on ω_3 and failures of weak forms of Chang's Conjecture follow from the forcing axiom for the class of stratified $(\mathcal{S}, \mathcal{L})$-finitely proper forcings.

Lastly, let us describe another extension of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings with very interesting potential applications. Let $\mathcal{S}_0$ be the collection of countable elementary submodels of $(H(\omega_2); \in, T)$, let $\mathcal{L}_0$ be a collection of $\aleph_1$-sized elementary submodels of $(H(\omega_2); \in, T)$ appropriate for $\mathcal{S}_0$, and let $\mathcal{T}_0$ be the collection of all $\mathcal{L}_0$-towers.

Definition 4.7.3. Let $\mathcal{M}$ be a countable subset of $\mathcal{S} \cup \mathcal{L}$ such that $|\mathcal{M} \cap \mathcal{S}| < \aleph_0$ and $|\mathcal{M} \cap \mathcal{L}| \leq \aleph_0$. We say that $\mathcal{M}$ is $H(\omega_2)$*-chained* if for every $Q \in \mathcal{M}$ there is a set H_Q such that

(1) $H_Q \preceq H(\omega_2)^Q$,

(2) $\varepsilon_{H_Q} = \varepsilon_Q$ and $\delta_{H_Q} = \delta_Q$, and

(3) $\{H_Q : Q \in \mathcal{M}\}$ is a subset of an $(\mathcal{S}_0, \mathcal{L}_0, \mathcal{T}_0)$-chain.

The models H_Q correspond of course to the models of the form $Q[G] \cap H(\omega_2)^V = H(\omega_2)^Q$, where Q is a model in the set of relevant models from the proof of item $(\mathrm{P2})_\alpha$ of lemma 4.4.2. It's straightforward to check that these models satisfy (1)-(3) from the definition above. It's enough to note that if you take the trace of the models of an $(\mathcal{S}, \mathcal{L}, \mathcal{T}^+)$-symmetric system with $H(\omega_2)$, the resulting set forms an $(\mathcal{S}_0, \mathcal{L}_0, \mathcal{T}_0)$-chain.

Hence, we can define the following extension of the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings using the notion of $H(\omega_2)$-chainedness.

Definition 4.7.4. We say that a forcing notion $\mathbb{P} \in H(\kappa)$ is *chained* $(\mathcal{S}, \mathcal{L})$*-finitely proper* if and only if there is a club $D \subseteq [H(\kappa)]^{\leq \aleph_1}$ such that for every countable $H(\omega_2)$-chained $\mathcal{M} \subseteq D$ such that $|\mathcal{M} \cap \mathcal{S}| < \aleph_0$ and $|\mathcal{M} \cap \mathcal{L}| \leq \aleph_0$, if $p \in \mathbb{P} \cap Q$ for some $Q \in \mathcal{M}$ such that either

(1) $\mathcal{M} \cap Q = \varnothing$, or

(2) $Q \in \mathcal{S}$ is such that $\varepsilon_Q = \min\{\varepsilon_M : M \in \mathcal{M} \cap \mathcal{S}\}$, and p is $(N, \mathbb{P})$-generic for every $N \in \mathcal{M} \cap Q$,

then there is an extension $q \leq p$ which is $(R, \mathbb{P})$-generic for every $R \in \mathcal{M}$.

We believe that the forcing axiom for the class of chained $(\mathcal{S}, \mathcal{L})$-finitely proper forcings could have very interesting consequences. For instance, destroying club guessing at $S^{\omega_2}_{\omega_1}$ of any club-sequence from the ground model is one of the most straightforward applications. Forcings in this class don't seem to have the $\aleph_3$-c.c. in general, but they do have it in the specific model that we obtain after iterating them with a finite support iteration with two-type symmetric systems as side conditions (see lemma 4.4.13). So this class is still nice enough.

Although there are some limitations on how far we can extend the class of $(\mathcal{S}, \mathcal{L})$-finite proper forcings, there is still plenty of room to explore different variants of this class and find interesting applications. The main common feature of all these classes of forcing notions is that they are iterable in arbitrarily long length, while preserving all cardinals. So the consequences of these forcing axioms will be compatible with arbitrarily large values of the continuum.

Bibliography

[1] Uri Abraham. Proper forcing. In *Handbook of set theory. Vols. 1, 2, 3*, pages 333–394. Springer, Dordrecht, 2010.

[2] Uri Abraham and James Cummings. More results in polychromatic Ramsey theory. *Open Mathematics*, 10(3):1004–1016, 2012.

[3] Uri Abraham, James Cummings, and Clifford Smyth. Some results in polychromatic ramsey theory. *The Journal of Symbolic Logic*, 72(3):865–896, 2007.

[4] Uri Abraham, Matatyahu Rubin, and Saharon Shelah. On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types. *Ann. Pure Appl. Logic*, 29(2):123–206, 1985.

[5] Uri Abraham and Saharon Shelah. Forcing closed unbounded sets. *J. Symbolic Logic*, 48(3):643–657, 1983.

[6] Uri Abraham and Stevo Todorčević. Partition properties of ω_1 compatible with CH. *Fund. Math.*, 152(2):165–181, 1997.

[7] David Asperó. The consistency of a club-guessing failure at the successor of a regular cardinal. In *Infinity, computability, and metamathematics*, volume 23 of *Tributes*, pages 5–27. Coll. Publ., London, 2014.

[8] David Asperó. Adding many Baumgartner clubs. *Archive for Mathematical Logic*, 56(7-8):797–810, 2017.

[9] David Asperó and Mohammad Golshani. The special aronszajn tree property at $\aleph_2$ and *gch*. *arXiv preprint arXiv:1809.07638*, 2018.

[10] David Asperó and Mohammad Golshani. The proper forcing axiom for $\aleph_1$-sized posets, Příkrý-type proper forcing, and the size of the continuum. *arXiv preprint arXiv:2209.01395*, 2022.

[11] David Asperó and Miguel Angel Mota. Forcing consequences of PFA together with the continuum large. *Trans. Amer. Math. Soc.*, 367(9):6103–6129, 2015.

[12] David Asperó and Miguel Angel Mota. A generalization of Martin's Axiom. *Israel J. Math.*, 210(1):193–231, 2015.

[13] David Asperó and Miguel Angel Mota. Separating club-guessing principles in the presence of fat forcing axioms. *Annals of Pure and Applied Logic*, 167(3):284–308, 2016.

[14] David Asperó and Miguel Angel Mota. Few new reals. *arXiv preprint arXiv:1712.07724*, 2017.

[15] David Asperó and Nutt Tananimit. Generalizations of martin's axiom, weak square, weak chang's conjecture, and a forcing axiom failure. *arXiv preprint arXiv:2212.07324*, 2022.

[16] Joan Bagaria. Axioms of generic absoluteness. In *Logic colloquium*, volume 2, pages 28–47, 2006.

[17] James E. Baumgartner. Almost-disjoint sets, the dense set problem and the partition calculus. *Ann. Math. Logic*, 9(4):401–439, 1976.

[18] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[19] James E. Baumgartner. Applications of the Proper Forcing Axiom. In *Handbook of set-theoretic topology*, pages 913–959. North-Holland, Amsterdam, 1984.

[20] Mohamed Bekkali. *Topics in set theory*, volume 1476 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991. Lebesgue measurability, large cardinals, forcing axioms, rho-functions, Notes on lectures by Stevo Todorčević.

[21] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[22] Jörg Brendle, Andrew Brooke-Taylor, Sy-David Friedman, and Diana Carolina Montoya. Cichoń's diagram for uncountable cardinals. *Israel J. Math.*, 225(2):959–1010, 2018.

[23] Maxim R. Burke. Forcing axioms. In *Set theory (Curaçao, 1995; Barcelona, 1996)*, pages 1–21. Kluwer Acad. Publ., Dordrecht, 1998.

[24] Paul J. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences*, 50(6):1143–1148, 1963.

[25] James Cummings and Saharon Shelah. Cardinal invariants above the continuum. *Ann. Pure Appl. Logic*, 75(3):251–268, 1995.

[26] Alan Dow. An introduction to applications of elementary submodels to topology. *Topology Proc.*, 13(1):17–72, 1988.

[27] Ilijas Farah. All automorphisms of the Calkin algebra are inner. *Ann. of Math. (2)*, 173(2):619–661, 2011.

[28] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I. *Annals of Mathematics*, pages 1–47, 1988.

[29] David Fremlin. Consequences of Martin's Axiom. *Journal of Symbolic Logic*, 53(2), 1988.

[30] Sy-David Friedman. Forcing with finite conditions. In *Set theory*, Trends Math., pages 285–295. Birkhäuser, Basel, 2006.

[31] Thomas Gilton and Itay Neeman. Abraham-Rubin-Shelah open colorings and a large continuum. *J. Math. Log.*, 22(1):Paper No. 2150027, 55, 2022.

[32] Moti Gitik and Menachem Magidor. SPFA by finite conditions. *Arch. Math. Logic*, 55(5-6):649–661, 2016.

[33] Kurt Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für mathematik und physik*, 38:173–198, 1931.

[34] Kurt Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. In *Proceedings of the National Academy of Science*, volume 24, pages 556–557, 1938.

[35] Martin Goldstern. Tools for your forcing construction. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 305–360. Bar-Ilan Univ., Ramat Gan, 1993.

[36] Martin Goldstern and Saharon Shelah. The bounded proper forcing axiom. *J. Symbolic Logic*, 60(1):58–73, 1995.

[37] Tanmay C. Inamdar. On strong chains of sets and functions. *Mathematika*, 69(1):286–301, 2023.

[38] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, millennium edition, 2003.

[39] Winfried Just and Martin Weese. *Discovering modern set theory. II*, volume 18 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1997. Set-theoretic tools for every mathematician.

[40] Akihiro Kanamori. *The higher infinite.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2009. Large cardinals in set theory from their beginnings, Paperback reprint of the 2003 edition.

[41] Yurii Khomskii, Giorgio Laguzzi, Benedikt Löwe, and Ilya Sharankou.

Questions on generalised Baire spaces. *MLQ Math. Log. Q.*, 62(4-5):439–456, 2016.

[42] Piotr Koszmider. On the existence of strong chains in $\mathscr{P}(\omega_1)/\mathrm{Fin}$. *J. Symbolic Logic*, 63(3):1055–1062, 1998.

[43] Piotr Koszmider. On strong chains of uncountable functions. *Israel J. Math.*, 118:289–315, 2000.

[44] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[45] Boriša Kuzeljević and Stevo Todorčević. Forcing with matrices of countable elementary submodels. *Proc. Amer. Math. Soc.*, 145(5):2211–2222, 2017.

[46] Richard Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

[47] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel J. Math.*, 29(4):385–388, 1978.

[48] Donald A. Martin and Robert M. Solovay. Internal Cohen extensions. *Annals of Mathematical Logic*, 2(2):143–178, 1970.

[49] Heike Mildenberger and Saharon Shelah. Changing cardinal characteristics without changing ω-sequences or confinalities. *Ann. Pure Appl. Logic*, 106(1-3):207–261, 2000.

[50] William J. Mitchell. Adding closed unbounded subsets of ω_2 with finite forcing. *Notre Dame J. Formal Logic*, 46(3):357–371, 2005.

[51] Tadatoshi Miyamoto. A note on a generalized martin's axiom for the norwich posets. Unpublished note.

[52] Tadatoshi Miyamoto. A note on P^c_{ff} and $\mathrm{WTCG}(S^2_1)$. Unpublished note.

[53] Tadatoshi Miyamoto. A srongly σ-closed poset that forces a simplified $(\omega_1, 1)$-morass. Unpublished note.

[54] Tadatoshi Miyamoto. Matrices of isomorphic models and morass-like structures (reflection principles and set theory of large cardinals). *RIMS Kôkyûroku*, 1895:79–102, 2014.

[55] Tadatoshi Miyamoto. Squares by matrices with coherent sequences. *RIMS Kôkyûroku*, 1949:62–72, 2015.

[56] Tadatoshi Miyamoto. Forcing a club by a generalized fast function. *Journal of the Nanzan Academic Society Humanities and Natural Sciences*, (23):183–191, 2022.

[57] Tadatoshi Miyamoto and Teruyuki Yorioka. A fragment of Asperó-Mota's finitely proper forcing axiom and entangled sets of reals. *Fundamenta Mathematicae*, 251:35–68, 2020.

[58] Rahman Mohammadpour. New methods in forcing iteration and applications [thesis abstract]. *Bull. Symb. Log.*, 29(2):300–302, 2023.

[59] Rahman Mohammadpour. Specialising trees with small approximations I. *J. Symb. Log.*, 88(2):640–663, 2023.

[60] Justin Tatch Moore. Open colorings, the continuum and the second uncountable cardinal. *Proc. Amer. Math. Soc.*, 130(9):2753–2759, 2002.

[61] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[62] Itay Neeman. Forcing with sequences of models of two types. *Notre Dame J. Form. Log.*, 55(2):265–298, 2014.

[63] Itay Neeman. Two applications of finite side conditions at ω_2. *Arch. Math. Logic*, 56(7-8):983–1036, 2017.

[64] Neeman, Itay. Higher analog of the PFA. Slides presented at the Fields Institute, https://www.math.ucla.edu/~ineeman/, 2012.

[65] Neeman, Itay. Higher analogues of properness. Slides presented at BIRS, https://www.math.ucla.edu/~ineeman/, 2013.

[66] Andrzej Rosł anowski and Saharon Shelah. Iteration of λ-complete forcing notions not collapsing λ^+. *Int. J. Math. Math. Sci.*, 28(2):63–82, 2001.

[67] Andrzej Rosł anowski and Saharon Shelah. Reasonably complete forcing notions. In *Set theory: recent trends and applications*, volume 17 of *Quad. Mat.*, pages 195–239. Dept. Math., Seconda Univ. Napoli, Caserta, 2006.

[68] Andrzej Rosł anowski and Saharon Shelah. Lords of the iteration. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 287–330. Amer. Math. Soc., Providence, RI, 2011.

[69] Andrzej Rosł anowski and Saharon Shelah. More about λ-support iterations of $(< \lambda)$-complete forcing notions. *Arch. Math. Logic*, 52(5-6):603–629, 2013.

[70] Andrzej Rosł anowski and Saharon Shelah. The last forcing standing with diamonds. *Fund. Math.*, 246(2):109–159, 2019.

[71] Norbert W. Sauer, Robert E. Woodrow, and B. Sands, editors. *Finite and infinite combinatorics in sets and logic*, volume 411 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[72] Saharon Shelah. Infinite abelian groups, Whitehead problem and some constructions. *Israel Journal of Mathematics*, 18:243–256, 1974.

[73] Saharon Shelah. Independence results. *The Journal of Symbolic Logic*, 45(3):563–573, 1980.

[74] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[75] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

[76] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[77] Saharon Shelah. Covering of the null ideal may have countable cofinality. volume 166, pages 109–136. 2000. Saharon Shelah's anniversary issue.

[78] Saharon Shelah. The null ideal restricted to some non-null set may be $\aleph_1$-saturated. *Fund. Math.*, 179(2):97–129, 2003.

[79] Saharon Shelah. On long increasing chains modulo flat ideals. *MLQ Math. Log. Q.*, 56(4):397–399, 2010.

[80] Robert M. Solovay and Stanley Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)*, 94:201–245, 1971.

[81] Stevo Todorčević. Directed sets and cofinal types. *Transactions of the American Mathematical Society*, 290(2):711–723, 1985.

[82] Stevo Todorčević. Notes on forcing axioms, volume 26 of lecture notes series. *Institute for Mathematical Sciences. National University of Singapore. World Scientific Publishing Co., Pte. Ltd., Hackensack, NJ: Edited and with a foreword by Chong, C., Feng, Q., Yang, Y., Slaman, TA, Woodin, WH*, 2014.

[83] Stevo Todorčević. A note on the proper forcing axiom. In *Axiomatic set theory (Boulder, Colo., 1983)*, volume 31 of *Contemp. Math.*, pages 209–218. Amer. Math. Soc., Providence, RI, 1984.

[84] Stevo Todorčević. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[85] Stevo Todorčević. A dichotomy for P-ideals of countable sets. *Fund. Math.*, 166(3):251–267, 2000.

[86] Boban Veličković. Iteration of semiproper forcing revisited. *arXiv preprint arXiv:1410.5095*, 2014.

[87] Boban Veličković. Forcing axioms and stationary sets. *Adv. Math.*, 94(2):256–284, 1992.

[88] Boban Veličković and Giorgio Venturi. Proper forcing remastered. In *Appalachian set theory 2006–2012*, volume 406 of *London Math. Soc. Lecture Note Ser.*, pages 331–362. Cambridge Univ. Press, Cambridge, 2013.

[89] Matteo Viale. The proper forcing axiom and the singular cardinal hypothesis. *J. Symbolic Logic*, 71(2):473–479, 2006.

[90] Teruyuki Yorioka. Some consequences from Proper Forcing Axiom together with large continuum and the negation of martin's axiom. *Journal of the Mathematical Society of Japan*, 69(3):913–943, 2017.

[91] Jindřich Zapletal. Strongly almost disjoint functions. *Israel J. Math.*, 97:101–111, 1997.

A

Open problems and future work

In this appendix we collect a list of questions, which we think could be of interest, and propose some lines of research that we would like to follow in the future.

We believe that the technique of forcing with symmetric systems of models of two types should open the door to a plethora of consistency results of combinatorial nature at the level of ω_2. We have listed some of the most important objects known to be forceable using one-type symmetric systems at the beginning of chapter 2. We speculate that our two-type symmetric systems should allow us to improve many of these results.

The most straightforward application would be to improve our own results from chapter 3 by forcing a strong chain of functions from ω_1 to ω_1 of length ω_3. By redefining the relation $<_{\mathcal{A},\nu}$ appropriately, using some of the ideas of Veličković and Venturi [88], we should be able to define a cardinal-preserving forcing notion forcing such a strong chain. Moreover, our technique of two-type symmetric systems should be useful to force objects of size $> \aleph_2$, which are usually out of reach of Neeman's two-type side conditions, because they only grant the preservation of $\aleph_1$ and $\aleph_2$ in general.

We have already mentioned at the end of chapter 4 a few possible ways in which we could extend the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings. Exploring the different classes of posets naturally associated with finite support iterations with two-type symmetric systems and their applications is one of our main priorities. Based on our knowledge of the classes of finitely proper forcings and forcings with the $\aleph_{1.5}$-c.c., we expect our class of posets to have certain variants of Baumgartner's forcing for adding clubs of ω_2 (see [19] and [5]). This forcing, and variants thereof, add clubs not including any club from the ground model. Hence, one of their main applications is in forcing certain failures of club guessing. A test problem that we have in mind is to force the negation of the following form of club guessing on $S^{\omega_2}_{\omega_1}$: For every club-sequence $(C_\delta : \delta \in S^{\omega_2}_{\omega_1})$ there is a club D of ω_2 such that $C_\delta \setminus D$ is unbounded in δ for every δ.

Apart from exploring the class of $(\mathcal{S}, \mathcal{L})$-finitely proper forcings, we would like to know, in a broad sense, what are the possibilities of the new technique of finite support iterated forcing with symmetric systems of models of two types developed in chapter 4. Asperó and Mota's iterations with one-type symmetric systems have been an extremely fruitful source of consistency results compatible with arbitrarily large values of the continuum at the level of ω_1. Hence, we expect our own technique to have as much impact on the combinatorics of ω_2. Let us list some of the results and objects that can be obtained using Asperó and Mota's iterations, which we expect to be able to generalize to ω_2. Some of them have been already mentioned throughout the thesis.

- Very strong failures of club guessing on ω_1 together with the continuum large and with the GCH ([7], [8], [11], [12], [13], [14], [52]).
- $\mathfrak{b}(\omega_1) = \aleph_2 < \kappa = \mathfrak{d}(\omega_1) = \mathfrak{r}(\omega_1)$ ([8], [12]).
- Some weakenings of $\square_{\omega_1}$ such as $\square_{\omega_1,fin}$, $\square^{ta}_{\omega_1,\omega}$ and $\square_{\omega_1,\omega_1}$ ([15], [63]).
- Strong failures of Chang's Conjecture ([15]).
- (ω_1, ω_1)-gaps ([90]).
- Specializing functions for $\aleph_2$-Aronszajn trees, which implies Souslin's Hypothesis at $\aleph_2$, together with the GCH ([9]).
- The same negative polychromatic partition relations from [2] together with the continuum large ([13]).

- k-entangled sets of reals ([57]).

Note that there are a couple results compatible with the GCH in the list above. These results were obtained with a variant of Asperó and Mota's iterations in which the side conditions have extra structure besides being symmetric systems. In [14] they introduced the notion of *edges*, which are certain suitable graphs on symmetric systems of elementary submodels that ensure that, if your ground model satisfies CH, then the CH is preserved along the iteration. More precisely, the side conditions consist of graphs of edges $\langle (M_0, \alpha_0), (M_1, \alpha_1) \rangle$, where each (M_i, α_i) is a model with a marker, and there is the extra requirement that all information carried by the condition and contained in M_0 needs to be copied into M_1 in an appropriate way. The reason to add edges to the side conditions of the iteration is to extend the symmetry also to the working part, so that the argument of the preservation of the CH can go through at every stage of the iteration. We believe that the edge technology should be adaptable to our iterations with two-type symmetric systems, and this framework should allow us to build interesting models of high consequences of PFA together with $2^{\aleph_1} = \aleph_2$.

This is just a wild guess, but there is one extra feature of the edge technology that could be used to connect finite support iterations with symmetric systems as side conditions and the class of forcings with the $\aleph_2$-*properness isomorphism condition* (or $\aleph_2$-*p.i.c.* for short). We won't define this class here, but let us say that in the same way that Todorčević's collapse can be seen as the natural side condition used to build proper forcings (see proposition 4.1.2), symmetric systems of countable elementary submodels can be seen as the natural side condition for building forcings with the $\aleph_2$-p.i.c. Edges ensure that all the models active at some stage of the iteration form a symmetric system. Therefore, it looks like we should be able to adapt the edge technology to iterate $\aleph_2$-p.i.c. forcings with finite support. Of course, this would open the door to iterating high analogs of the class of $\aleph_2$-p.i.c. forcings using symmetric systems of models of two types, and this could lead to interesting applications such as forcing high restricted versions of combinatorial dichotomies on ω_1 such as PID($\aleph_1$) or OCA($\aleph_1$).

In a completely different direction, Asperó and Golshani [10] combined the techniques of iterated forcing with symmetric systems as side conditions and Shelah's memory iterations to force PFA restricted to posets of size $\aleph_1$ together with continuum large, answering a longstanding open question. Iterations with restricted memory appeared first in Shelah's work on the null ideal and the possible cofinalities of its covering number ([77], [78]). Later, other applications were found in a much broader context in the area of cardinal characteristics of the continuum ([49]), and recently Gilton and Neeman [31] used this technique to show that Abraham-Rubin-Shelah's Open Coloring Axiom is consistent with $2^{\aleph_0} = \aleph_3$. As a second application, a small variant of the iteration technique developed by Asperó and Golshani gives rise to a model satisfying a very useful principle of generic absoluteness. In this model the continuum is large and it satisfies a multitude of Π_2-statements that hold in models obtainable by countable support iterations of proper forcings. These include, among others, Baumgartner's Axiom for $\aleph_1$-dense sets of reals, Todorčević's Open Coloring Axiom for sets of size $\aleph_1$, Moore's Measuring principle, Todorčević's P-Ideal Dichotomy for $\aleph_1$-generated ideals on ω_1, and Baumgartner's Thinning-out Principle. It would be very interesting to combine our iterations with symmetric systems of models of two types and Shelah's memory iterations, in a similar fashion as in the work of Asperó and Golshani. If successful, we should be able to iterate, in arbitrarily long length, Neeman's class of $(\mathcal{S}, \mathcal{L})$-proper forcings restricted to partial orders of size $\aleph_2$, and hopefully also obtain a high analog of the principle of generic absoluteness mentioned above.

Lastly, we would like to know whether the assumption of a supercompact cardinal from the consistency of the forcing axiom for the class of $(\mathcal{S}, \mathcal{L})$-finitely proper posets can be weakened to, for example, just ZFC. In particular, whether some form of diamond, as in the proof of the consistency of the forcing axiom for the class of $\aleph_{1.5}$-c.c. forcings (see theorem 1.2.2), can be used to force our forcing axiom.