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# Applying projective functors to arbitrary holonomic simple modules

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**Abstract**

We prove that applying a projective functor to a holonomic simple module over a semisimple finite-dimensional complex Lie algebra produces a module that has an essential semisimple submodule of finite length. This implies that holonomic simple supermodules over certain Lie superalgebras are quotients of modules that are induced from simple modules over the even part. We also provide some further insight into the structure of Lie algebra modules that are obtained by applying projective functors to simple modules.

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## 1 | MOTIVATION AND DESCRIPTION OF THE RESULTS

### 1.1 | Motivation from Lie superalgebras

Let $\mathfrak{g}$ be a semisimple (or reductive) finite-dimensional Lie algebra over $\mathbb{C}$. Let $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ be a finite-dimensional complex Lie superalgebra such that $\mathfrak{s}_0 \cong \mathfrak{g}$. One of the basic representation-theoretic problems for $\mathfrak{s}$ is the classification of simple $\mathfrak{s}$-supermodules, see, for example, [9, 10, 33]. A natural way to address this problem is to look for some connection between simple $\mathfrak{s}$-supermodules and simple $\mathfrak{g}$-modules. In [9, 10, 33], this approach was successfully applied to reduce the problem of classification of simple $\mathfrak{s}$-supermodules to that of simple $\mathfrak{g}$-modules. The latter problem is very difficult, the only known case in which it is "solved" (in the sense that it is reduced to the problem of classification of equivalence classes of irreducible elements over a certain noncommutative principal ideal domain) is $\mathfrak{sl}_2$, see [6].

The module categories $\mathfrak{s}$-Mod and $\mathfrak{g}$-Mod are connected by the usual induction and restriction functors. Moreover, the latter two functors are not only adjoint in the obvious way (i.e., induction is left adjoint to restriction), but they are also biadjoint, up to parity shift (i.e., induction is right adjoint to restriction, up to parity shift that depends on the parity of the dimension of $\mathfrak{s}_1$). This is equivalent to saying that induction is isomorphic to coinduction, up to parity shift. Since the universal enveloping algebra $U(\mathfrak{g})$ is noetherian and the universal enveloping algebra $U(\mathfrak{s})$ is finite over $U(\mathfrak{g})$, every simple $U(\mathfrak{s})$-supermodule $S$ has a simple quotient, say $L$, when considered as a $U(\mathfrak{g})$-module. By adjunction, it follows that $S$ is a submodule of a module that is coinduced from a simple $U(\mathfrak{g})$-module. That is a very natural fact.

Now we recall that induction and coinduction coincide, up to a parity shift. It follows that $S$ is a submodule of a module that is induced from a simple $U(\mathfrak{g})$-module. It would be more natural to expect $S$ to be a quotient of a module induced from a simple $U(\mathfrak{g})$-module. However, it seems that there is no easy argument for why that should be the case. This property is an essential ingredient in [10] where the claim is proved in type $A$ using very specific type $A$ properties established in [37, 38].

The idea is that, in order to use the correct adjunction, we need to show that $S$, when restricted to $U(\mathfrak{g})$, has a simple submodule. Note that we already know that $S$ is a submodule of an induced simple module. Therefore, it is enough to show that any $U(\mathfrak{s})$-supermodule that is induced from a simple $U(\mathfrak{g})$-module, when restricted back to $U(\mathfrak{g})$, has finite type socle, that is, it has an essential semisimple submodule of finite length.

At the level of $U(\mathfrak{g})$-modules, the composition of induction to $U(\mathfrak{s})$ followed by restriction back to $U(\mathfrak{g})$ can be described as tensoring with a finite-dimensional $U(\mathfrak{g})$-module, namely, with $\bigwedge \mathfrak{s}_1$. This naturally leads to the formulation of our main result in Theorem 22 below.

## 1.2 | Main result

Recall that a simple module over a finitely generated associative algebra of finite Gelfand–Kirillov dimension is called *holonomic*, provided that it has the minimal possible Gelfand–Kirillov dimension among all simple modules with the same annihilator.

The main result of this paper is the following statement.

**Theorem 22.** *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{C}$. Let $L$ be a holonomic simple $\mathfrak{g}$-module and let $V$ be a finite-dimensional $\mathfrak{g}$-module. Then, the $\mathfrak{g}$-module $V \otimes_{\mathbb{C}} L$ has an essential semisimple submodule of finite length.*

As an immediate corollary, it follows that any holonomic simple $\mathfrak{s}$-supermodule is, indeed, a quotient of a module that is induced from a simple $\mathfrak{g}$-module.

In type $A$, the assertion of Theorem 22 is true for all simple $\mathfrak{g}$-modules, not necessarily holonomic ones, see [10, Theorem 23]. Of course, we expect the assertion of Theorem 22 to be true for all simple $\mathfrak{g}$-modules in all types. However, at the moment, we do not see how to prove that.

## 1.3 | Structure of $V \otimes_{\mathbb{C}} L$

The main difficulty in proving the main result lies in the fact that the module $V \otimes_{\mathbb{C}} L$, in general, while being noetherian, does not have to be artinian, see [43] for an example. At the same time, in all known "natural" examples, for instance, if we assume that $L$ belongs to the BGG category $\mathcal{O}$, see [4, 19], or to the category of weight modules with finite-dimensional weight spaces, see [32], or to the category of Gelfand–Zeitlin modules, see [16, 18], the module $V \otimes_{\mathbb{C}} L$ has finite length. Therefore, in our approach, we cannot really rely on the intuition developed during the study of these classical categories of modules. We need to understand the structure of $V \otimes_{\mathbb{C}} L$ in very abstract terms and in a situation where we lack easy computable examples.

Possible subquotients of interest in $V \otimes_{\mathbb{C}} L$ split naturally into three categories:

- simple subquotients whose Gelfand–Kirillov dimension equals $\mathrm{GKdim}(L)$;
- simple subquotients whose Gelfand–Kirillov dimension is strictly smaller than $\mathrm{GKdim}(L)$;
- nonsimple subquotients that we call *strange*, see Subsection 4.5 for details, and which are defined by the property that they have Gelfand–Kirillov dimension $\mathrm{GKdim}(L)$ but they do not have any simple subquotient of Gelfand–Kirillov dimension $\mathrm{GKdim}(L)$.

Our proof of the main result essentially reduces to the statement that $V \otimes_{\mathbb{C}} L$ cannot have strange submodules.

A major part of the paper is devoted to taking a closer look at the general structure of $V \otimes_{\mathbb{C}} L$. As mentioned above, this module might fail to have finite length. However, one can define a natural Serre subquotient of the category of all $\mathfrak{g}$-module in which $V \otimes_{\mathbb{C}} L$ does have finite length, see Subsection 7.3. The structure of $V \otimes_{\mathbb{C}} L$ as an object of this Serre subquotient is similar in spirit to what was called the *rough structure* of generalized Verma modules in [38].

The correct setup for the study of modules of the form $V \otimes_{\mathbb{C}} L$ is to combine them all into a certain birepresentation of the bicategory of projective functors associated to the algebra $\mathfrak{g}$. In the case when $L$ is a simple highest weight module, these birepresentations appear frequently and were studied extensively in many papers, see [29, 34, 38] and references therein. In the general case, it is natural to expect that the corresponding birepresentations behave similarly to the case of highest weight modules. One possible direction of this expectation is formulated, in precise terms, in Conjecture 5 in Subsection 3.2. This conjecture asserts that the birepresentation in question is *simple transitive* in the terminology of [37].

In type $A$, the conjecture is proved in Subsection 4.4. In fact, in type $A$, we establish an equivalence between a birepresentation in the general case and a birepresentation in the highest weight case. We do not expect such an equivalence for other types, in general, as we know from [29] that, outside type $A$, not all simple transitive birepresentations of projective functors can be modeled naively using highest weight modules (a modeling via highest weight modules is possible, but it requires an upgrade to the level of (co)algebra 1-morphisms and the corresponding categories of (co)modules). It would be really interesting if, in general type, each simple transitive birepresentation of projective functors turned out to be constructible directly starting from some simple (but not necessarily highest weight) module. At the moment, we do not know whether this is true or not and where to look for such simple modules.

Another interesting aspect of the structure of $V \otimes_{\mathbb{C}} L$ which we analyze is the following: There is a natural preorder $\trianglerighteq$ on the set of isomorphism classes of simple $\mathfrak{g}$-modules given by $L \trianglerighteq L'$, provided that $L'$ is a quotient of $V \otimes_{\mathbb{C}} L$, for some $V$. In Conjecture 9, we predict that $\trianglerighteq$ is, in fact, an equivalence relation. Again, in type $A$, we can prove this conjecture, see Subsection 4.3. We find it very surprising that this, intuitively very natural expectation, seems to be very nontrivial in reality and even in type $A$ its proof requires quite heavy machinery. At the moment we do not know how to prove this conjecture in general.

### 1.4 | Structure of the paper

The paper is organized as follows: in Section 2, we collect all necessary preliminaries. In Sections 3, 4, and 7, we study the general structure of modules of the form $V \otimes_{\mathbb{C}} L$ in more detail. Section 3 contains all necessary preliminaries in order to formulate Conjecture 5 (in Subsection 3.2) and Conjecture 9 (in Subsection 3.3). The two conjectures are compared in Subsection 3.4. Section 4 contains proofs of Conjecture 5 and Conjecture 9 in type $A$. The main result is proved in Section 5 and is extended beyond the holonomic case to some further special cases outside type $A$ in Section 6. Section 7 is devoted to the study of strange subquotients of modules of the form $V \otimes_{\mathbb{C}} L$ as well as Serre subquotients of the category of $\mathfrak{g}$-modules in which modules of the form $V \otimes_{\mathbb{C}} L$ have finite length.

## 2 | PRELIMINARIES

### 2.1 | Category $\mathcal{O}$

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. Associated to this decomposition, we have the corresponding BGG category $\mathcal{O}$, see [4, 19]. Simple objects in $\mathcal{O}$ are the simple highest weight modules $L(\lambda)$, where $\lambda \in \mathfrak{h}^*$. The module $L(\lambda)$ is the unique simple quotient of the Verma module $\Delta(\lambda)$.

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then, $\mathcal{O}$ decomposes into a direct sum of $\mathcal{O}_\chi$, where $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is a central character of $U(\mathfrak{g})$. The category $\mathcal{O}_\chi$ is a full subcategory of $\mathcal{O}$ consisting of all modules on which the kernel of $\chi$ acts locally nilpotently. An important fact about $\mathcal{O}_\chi$ is that it is always nonzero. In other words, any character of $Z(\mathfrak{g})$ is realizable as the central character of some $L(\lambda)$, see [14, Section 7]. For a fixed $\chi$, the set of all $\lambda$ such that $L(\lambda)$ has central character $\chi$ is an orbit of the Weyl group $W$ of $(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{h}^*$ with respect to the so-called *dot-action*, which is the shift of the natural action by half the sum of all positive roots.

If $L$ is some simple $\mathfrak{g}$-module (not necessarily in category $\mathcal{O}$), then the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(L)$ of $L$ in $U(\mathfrak{g})$ is a primitive ideal and it is realizable as the annihilator of some $L(\lambda)$, see [15].

For $\lambda \in \mathfrak{h}^*$, we have the indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{O}$ and the indecomposable injective envelope $I(\lambda)$ of $L(\lambda)$ in $\mathcal{O}$.

We denote by $\mathbf{R} \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Our choice of triangular decomposition above gives rise to the decomposition of $\mathbf{R}$ into a disjoint union of positive roots $\mathbf{R}_+$ and negative roots $\mathbf{R}_-$. We denote by $\pi$ the corresponding basis of $\mathbf{R}$. We also denote by $\Xi$ the root lattice $\mathbb{Z}[\mathbf{R}]$.

By definition, a weight in $\mathfrak{h}^*$ is *integral* if it is a weight of some finite-dimensional $\mathfrak{g}$-module. We denote by $\Lambda$ the lattice of all integral weights. Note that $\Xi \subset \Lambda$ is a subgroup of finite index. This index is the determinant of the Cartan matrix for $\mathfrak{g}$, in particular, $\Xi = \Lambda$ only in types $E_8$, $F_4$, and $G_2$.

For a weight $\lambda$, we denote by $W_\lambda$ the integral Weyl group of $\lambda$, that is the subgroup of $W$ generated by all reflections $s$ for which $s \cdot \lambda - \lambda$ is an integral multiple of a root. We also denote by

$W'_\lambda$ the stabilizer of $\lambda$ in $W_\lambda$. We call $\lambda \in \mathfrak{h}^*$ *regular* if $W'_\lambda = \{e\}$. If $\lambda \in \mathfrak{h}^*$ is not regular, it is called *singular*. We call $\lambda$ *dominant* if $w \cdot \lambda \leqslant \lambda$, for all $w \in W_\lambda$.

We note that the categories $\mathcal{O}_\chi$ are usually decomposable. For $\lambda \in \mathfrak{h}^*$, we denote by $\mathcal{O}_\lambda$ the indecomposable direct summand (block) of $\mathcal{O}$ containing $L(\lambda)$. Let $\chi = \chi_\lambda$ be the central character of $L(\lambda)$. Consider $W \cdot \lambda$ and define on this finite set the equivalence relation $\equiv$ as follows: for $\mu, \nu \in W \cdot \lambda$, set $\mu \equiv \nu$ provided that $W_\mu \cdot \mu = W_\nu \cdot \nu$. If $\{\lambda_1, \dots, \lambda_k\}$ is any cross-section of equivalence classes, then $\mathcal{O}_\chi$ decomposes into the direct sum $\mathcal{O}_{\lambda_1} \oplus \cdots \oplus \mathcal{O}_{\lambda_k}$.

## 2.2 | Projective functors

In this subsection, we recall the definition and basic properties of projective functors, as introduced in [5].

Let $\mathcal{M}$ denote the category of all $\mathfrak{g}$-modules on which the action of $Z(\mathfrak{g})$ is locally finite. Note that $\mathcal{O} \subset \mathcal{M}$. Similarly to $\mathcal{O}$, the category $\mathcal{M}$ decomposes into a product of the full subcategories $\mathcal{M}_\chi$, where $\chi$ is a central character, defined as follows: $\mathcal{M}_\chi$ consists of all objects on which the kernel of $\chi$ acts locally nilpotently. Clearly, $\mathcal{O}_\chi \subset \mathcal{M}_\chi$.

For any finite-dimensional $\mathfrak{g}$-module $V$, tensoring with $V$ preserves both $\mathcal{M}$ and $\mathcal{O}$. A *projective functor* is an endofunctor of $\mathcal{M}$ (or $\mathcal{O}$) that is isomorphic to a direct summand of tensoring with some $V$. The functor $V \otimes_{\mathbb{C}} \_$ is biadjoint to $V^* \otimes_{\mathbb{C}} \_$, for any finite-dimensional $\mathfrak{g}$-module $V$, and hence, each projective functor has a biadjoint projective functor. Consequently, any projective functor is exact. Furthermore, each projective functor is isomorphic to a direct sum of indecomposable projective functors. Indecomposable projective functors are classified by their restriction to $\mathcal{O}$.

Let $\mathcal{O}'$ and $\mathcal{O}''$ be two indecomposable blocks of $\mathcal{O}$. Let $L(\lambda_1)$, $L(\lambda_2), \dots, L(\lambda_r)$ and $L(\mu_1)$, $L(\mu_2), \dots, L(\mu_s)$ be complete and irredundant lists of simples in $\mathcal{O}'$ and $\mathcal{O}''$, respectively. Assume that $\lambda_1$ and $\mu_1$ are the weights in the above lists that are dominant with respect to the corresponding integral Weyl groups (each list contains a unique such dominant weight).

Nonzero projective functors from $\mathcal{O}'$ to $\mathcal{O}''$ exist if and only if $\lambda_1 - \mu_1$ is an integral weight. Indecomposable projective functors from $\mathcal{O}'$ to $\mathcal{O}''$ are in bijection with those $\mu_i$ that are $W'_{\lambda_1}$-antidominant with respect to the dot action. In fact, for each such $\mu_i$, there is a unique indecomposable projective functor, denoted as $\theta_{\lambda_1,\mu_i}$, from $\mathcal{O}'$ to $\mathcal{O}''$ that sends $P(\lambda_1) = \Delta(\lambda_1)$ to $P(\mu_i)$.

Let now $\chi'$ and $\chi''$ be two central characters. The above can be used to classify indecomposable projective functors from $\mathcal{M}_{\chi'}$ to $\mathcal{M}_{\chi''}$. Let $\lambda$ and $\nu$ be some weights such that $L(\lambda)$ and $L(\nu)$ have the central characters $\chi'$ and $\chi''$, respectively. Without loss of generality, we may assume that $\lambda$ is dominant with respect to its integral Weyl group.

Projective functors from $\mathcal{M}_{\chi'}$ to $\mathcal{M}_{\chi''}$ exist if and only if $(W \cdot \nu) \cap (\lambda + \Lambda)$ is not empty. If this condition is satisfied, we may assume $\nu \in \lambda + \Lambda$ without loss of generality. Let

$$(W \cdot \nu) \cap (\lambda + \Lambda) = \{\nu_1, \nu_2, \dots, \nu_t\}.$$

Then, indecomposable projective functors from $\mathcal{M}_{\chi'}$ to $\mathcal{M}_{\chi''}$ are exactly the functors $\theta_{\lambda,\nu_i}$, where $\nu_i$ is $W'_\lambda$-antidominant. We note that the Serre subcategory of $\mathcal{O}$ generated by $L(\nu_1), L(\nu_2), \dots, L(\nu_t)$ does not have to be indecomposable, and hence, our indecomposable projective functors from $\mathcal{M}_{\chi'}$ to $\mathcal{M}_{\chi''}$ are not classified, in the general case, by projective functors from an indecomposable block of $\mathcal{O}$ to an indecomposable block of $\mathcal{O}$.

## 2.3 | Harish-Chandra bimodules

An alternative way to look at projective functors is using Harish-Chandra bimodules. A $\mathfrak{g}$-$\mathfrak{g}$-bimodule $B$ is called a *Harish-Chandra bimodule*, provided that it is finitely generated as a bimodule and the adjoint action of $\mathfrak{g}$ on it is locally finite with finite multiplicities. A typical example of a Harish-Chandra bimodule is the quotient of $U(\mathfrak{g})$ by the ideal generated by the kernel of some central character. The category of all Harish-Chandra bimodules is denoted as $\mathscr{H}$.

The category $\mathscr{H}$ is, naturally, a monoidal category, where the monoidal structure is given by tensoring over $U(\mathfrak{g})$. As a monoidal category, $\mathscr{H}$ is naturally $\Lambda/\Xi$-graded. Namely, for a coset $\xi \in \Lambda/\Xi$, the corresponding homogeneous component $\mathscr{H}^\xi$ consists of all bimodules $B$ such that, for any finite-dimensional simple $\mathfrak{g}$-module $V$, the fact that the multiplicity $[B^{\mathrm{ad}} : V] > 1$ implies that the support of $V$ belongs to $\xi$. For a Harish-Chandra bimodule $B$, we will denote by $B^\xi$ its projection onto $\mathscr{H}^\xi$.

Let us explain how this grading works. Given a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule $B$, the left action of $U(\mathfrak{g})$ on $B$ is given by a map $U(\mathfrak{g}) \otimes_{\mathbb{C}} B \to B$, which is a homomorphism of both left and right $\mathfrak{g}$-modules and hence also of adjoint $\mathfrak{g}$-modules. As $U(\mathfrak{g})$, considered as an adjoint $\mathfrak{g}$-module, is a direct sum of simple finite-dimensional $\mathfrak{g}$-modules whose support belongs to $\Xi$, the above action map restricts to $U(\mathfrak{g}) \otimes_{\mathbb{C}} B^\xi \to B^\xi$, for every $\xi$. Similarly, we have a restriction of the right action map, which implies that $B^\xi$ is, indeed, a $U(\mathfrak{g})$-$U(\mathfrak{g})$-subbimodule of $B$.

The above grading is motivated by the fact that the action of projective functors on category $\mathcal{O}$ behaves "slightly better" than on other natural categories of $\mathfrak{g}$-modules. As was mentioned in Subsection 2.2, projective functors are uniquely determined (up to isomorphism) by the image of dominant Verma modules in category $\mathcal{O}$. Outside category $\mathcal{O}$, it might happen that analogs of dominant Verma modules do not exist and two nonisomorphic projective functors map some module which we want to understand to isomorphic modules (see [8, Theorem B], [21, Section 2.3] and also the connection of this phenomenon to Kostant's problem, as was observed by Johan Kåhrström and explained in [27]). In such situation, one could try to consider the action of the "smaller" category $\mathscr{H}^\Xi$ and, essentially, use similar arguments as in the case of category $\mathcal{O}$. For example, this was done in [23] in the context of the study of generalized Verma modules. We are going to use this kind of trick in Section 4.

Tensoring with finite-dimensional $\mathfrak{g}$-modules both on the left and on the right preserves Harish-Chandra bimodules. Therefore, indecomposable projective functors can be viewed as summands of the Harish-Chandra bimodules $V \otimes_{\mathbb{C}} (U(\mathfrak{g})/(\ker(\chi)))$, where $\chi$ is a central character. In fact, the indecomposable projective functors with domain $\mathscr{M}_\chi$ correspond exactly to the indecomposable summands of $V \otimes_{\mathbb{C}} (U(\mathfrak{g})/(\ker(\chi)))$.

For two $\mathfrak{g}$-modules $M$ and $N$, we can consider the $\mathfrak{g}$-$\mathfrak{g}$-bimodule $\mathrm{Hom}_{\mathbb{C}}(M, N)$ and its subbimodule $\mathcal{L}(M, N)$ that consists of all elements of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ on which the adjoint action of $\mathfrak{g}$ is locally finite. If $\mathrm{Hom}_{\mathfrak{g}}(V \otimes_{\mathbb{C}} M, N)$ is finite dimensional, for any simple finite-dimensional $\mathfrak{g}$-module $V$, and both quotients $Z(\mathfrak{g})/\mathrm{Ann}_{Z(\mathfrak{g})}(M)$ and $Z(\mathfrak{g})/\mathrm{Ann}_{Z(\mathfrak{g})}(N)$ are finite dimensional, then $\mathcal{L}(M, N)$ is a Harish-Chandra bimodule, see [20, Satz 6.30]. For example, this is the case if both modules $M$ and $N$ belong to category $\mathcal{O}$.

If $M = N$, we have a natural embedding of $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M)$ into $\mathcal{L}(M, M)$. A module $M$ is called *Kostant positive*, provided that this embedding is an isomorphism. A module $M$ is called *weakly Kostant positive*, provided that the natural embedding of $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M)$ into $\mathcal{L}(M, M)^\Xi$ is an isomorphism.

## 2.4 | Soergel's combinatorial description and the integral part

Denote by $\mathcal{O}_{\mathrm{int}}$ the full subcategory of $\mathcal{O}$ that consists of all modules with integral support. It is a direct summand of $\mathcal{O}$. If a central character $\chi$ is such that $\mathcal{O}_\chi \cap \mathcal{O}_{\mathrm{int}}$ is nonzero, then $\mathcal{O}_\chi \subset \mathcal{O}_{\mathrm{int}}$ and $\mathcal{O}_\chi$ is an indecomposable block of $\mathcal{O}$. For such a $\chi$, let $\lambda$ be the unique dominant weight such that $L(\lambda) \in \mathcal{O}_\chi$. Note that $W = W_\lambda$ as $\lambda$ is integral. Soergel's combinatorial description of $\mathcal{O}$, see [40], determines $\mathcal{O}_\chi$ uniquely, up to equivalence, in terms of the algebra of $W'_\lambda$-invariants in the coinvariant algebra for $W$. Projective endofunctors between different blocks of $\mathcal{O}_{\mathrm{int}}$ can then be described in terms of induction and restriction functors between the coinvariant algebra for $W$ and its corresponding invariant subalgebras, see [41].

For a (not necessarily integral) weight $\lambda$, the category $\mathcal{O}_\lambda$ is equivalent to an integral block of $\mathcal{O}$ for a semisimple complex Lie algebra corresponding to the Weyl group $W_\lambda$. Due to Soergel's combinatorial description of projective functors mentioned above (see [41]), this equivalence is compatible with the action of those projective functors that are homogeneous of degree $\Xi$.

## 2.5 | Gelfand–Kirillov dimension

In this subsection, we recall basic facts about the Gelfand–Kirillov dimension (denoted as GKdim). We refer to [26] for all details.

To each finitely generated $\mathfrak{g}$-module $M$, we can associate its Gelfand–Kirillov dimension $\mathrm{GKdim}(M) \in \mathbb{Z}_{\geqslant 0}$ (which is the degree of the polynomial that describes the growth of $M$) and its Bernstein number $\mathrm{BN}(M) \in \mathbb{Z}_{>0}$ (which is the coefficient of the leading term of that polynomial). Contrary to the Gelfand–Kirillov dimension, the Bernstein number might depend on the choice of generators in $U(\mathfrak{g})$, so we fix such a choice for the remainder of the paper.

The algebra $U(\mathfrak{g})$ is a noetherian algebra of finite Gelfand–Kirillov dimension, namely, $\mathrm{GKdim}(U(\mathfrak{g})) = \dim(\mathfrak{g})$. Therefore, every simple $\mathfrak{g}$ module has Gelfand–Kirillov dimension at most $\dim(\mathfrak{g})$ (in reality, at most $\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}) - 1$ as $Z(\mathfrak{g})$ is a polynomial algebra in $\mathrm{rank}(\mathfrak{g})$ variables). If $\mathcal{I}$ is a primitive ideal of $U(\mathfrak{g})$ and $L$ a simple module with annihilator $\mathcal{I}$, which minimizes the Gelfand–Kirillov dimension in the class of all simple $\mathfrak{g}$-modules with annihilator $\mathcal{I}$, then $L$ is called *holonomic*. For example, all simple modules in category $\mathcal{O}$ are holonomic, see [20, Subsection 10.9]. Nonholonomic modules do certainly exist, see [13, 43]. As a matter of fact, almost all (in some sense) simple modules are nonholonomic.

For a finite-dimensional $\mathfrak{g}$-module $V$, we have

$$\mathrm{GKdim}(V) = 0, \qquad \mathrm{GKdim}(V \otimes_{\mathbb{C}} M) = \mathrm{GKdim}(M)$$

and $\mathrm{BN}(V \otimes_{\mathbb{C}} M) = \dim(V) \cdot \mathrm{BN}(M)$. If $0 \to X \to Y \to Z \to 0$ is a short exact sequence, then $\mathrm{GKdim}(Y) = \max\{\mathrm{GKdim}(X), \mathrm{GKdim}(Z)\}$. Moreover, $\mathrm{BN}(Y) = \mathrm{BN}(X) + \mathrm{BN}(Z)$ provided that $\mathrm{GKdim}(X) = \mathrm{GKdim}(Z)$. If the latter condition is not satisfied, then $\mathrm{BN}(Y)$ coincides with $\mathrm{BN}(X)$ if $\mathrm{GKdim}(X) > \mathrm{GKdim}(Z)$ and $\mathrm{BN}(Y)$ coincides with $\mathrm{BN}(Z)$ if $\mathrm{GKdim}(X) < \mathrm{GKdim}(Z)$.

**Lemma 1.** *Let $L$ and $L'$ be two simple $\mathfrak{g}$-modules such that* $\mathrm{Hom}_{\mathfrak{g}}(V \otimes_{\mathbb{C}} L, L') \neq 0$ *or* $\mathrm{Hom}_{\mathfrak{g}}(L', V \otimes_{\mathbb{C}} L) \neq 0$, *for some finite-dimensional $\mathfrak{g}$-module $V$. Then,* $\mathrm{GKdim}(L) = \mathrm{GKdim}(L')$.

*Proof.* We prove the first claim. From $\mathrm{Hom}_{\mathfrak{g}}(V \otimes_{\mathbb{C}} L, L') \neq 0$, we have $\mathrm{GKdim}(L') \leqslant \mathrm{GKdim}(L)$. At the same time, by adjunction, we have $\mathrm{Hom}_{\mathfrak{g}}(L, V^* \otimes_{\mathbb{C}} L') \neq 0$. Hence, $\mathrm{GKdim}(L) \leqslant \mathrm{GKdim}(L')$ and thus $\mathrm{GKdim}(L) = \mathrm{GKdim}(L')$. □

Due to the additivity of the Bernstein number, $V \otimes_{\mathbb{C}} L$ has a maximal semisimple quotient and this quotient has finite length (and is always nonzero). This maximal semisimple quotient is usually called the *top* of $V \otimes_{\mathbb{C}} L$. It further follows that the module $V \otimes_{\mathbb{C}} L$ has a maximal semisimple submodule and this submodule has finite length (but, potentially, may be zero). Theorem 22, in fact, shows that this maximal semisimple submodule is essential and hence is the *socle* of $V \otimes_{\mathbb{C}} L$. Note that, in general, $V \otimes_{\mathbb{C}} L$ is not of finite length, see [43]. However, we will show in Subsection 7.3 that $V \otimes_{\mathbb{C}} L$ has finite *rough length* in the sense of [38].

## 2.6 | Kazhdan–Lusztig combinatorics

To each pair $(W', S')$, where $W'$ is a Weyl group and $S'$ a fixed set of simple reflections in $W'$, we have the associated Hecke algebra $\mathbb{H} = \mathbb{H}(W', S')$, which is an algebra over $\mathbb{Z}[v, v^{-1}]$, defined by substituting the relation $(s - e)(s + e) = 0$, for $s \in S'$, in the Coxeter presentation of $W'$, by the relation $(H_s + v)(H_s - v^{-1}) = 0$ and keeping the braid relations, see, for example, [42]. It has the standard basis $\{H_w : w \in W\}$ and the Kazhdan–Lusztig basis $\{\underline{H}_w : w \in W\}$.

For $x, y \in W'$, we set $x \geqslant_L y$ provided that there is $z \in W'$ such that $\underline{H}_x$ appears with a nonzero coefficient in $\underline{H}_z \underline{H}_y$. This defines a preorder on $W'$ called the *KL-left preorder*. Equivalence classes with respect to it are called *KL-left cells*. The *KL-right preorder* $\geqslant_R$ and the corresponding *KL-right cells* are defined similarly using multiplication on the right. The *KL-two-sided preorder* $\geqslant_J$ and the corresponding *KL-two-sided cells* are defined similarly using multiplication on both sides.

If $\lambda \in \mathfrak{h}^*$ is regular and dominant, then the results of [2, 3] imply that sending $w \in W_\lambda$ to the annihilator of $L(w \cdot \lambda)$ gives rise to a bijection between the KL-left cells in $W_\lambda$ and the primitive ideals in $U(\mathfrak{g})$ containing $\mathrm{Ker}(\chi_\lambda)$.

The function that assigns to $w \in W_\lambda$ the Gelfand–Kirillov dimension of $L(w \cdot \lambda)$ is constant on KL-two-sided cell. Indeed, that this function is constant on KL-left cells follows from [20, Satz 10.9] combined with the fact mentioned above that annihilators of simple modules are constant on KL-left cells. That the function is constant on KL-right cells follows from Lemma 1 combined with the fact that simple modules inside the same KL-right cell can be obtained from each other by applying projective functors and taking subquotients.

If $W''$ is a parabolic subgroup of $W_\lambda$, $w_0''$ the longest element in $W''$ and $w_0^\lambda$ the longest element in $W_\lambda$, then $\mathrm{GKdim}(L(w_0'' w_0^\lambda \cdot \lambda))$ can be computed by a very easy formula in [20, Lemma 9.15(a)]. Namely, $\mathrm{GKdim}(L(w_0'' w_0^\lambda \cdot \lambda))$ equals the number of positive roots for $W$ (note that it is really $W$ and not $W_\lambda$) minus the number of positive roots for $W''$. In fact, if $W_\lambda$ is of type $A$, then any KL-two-sided cell of $W_\lambda$ contains some element of the form $w_0'' w_0^\lambda$ and hence the above applies. For a singular weight $\mu$, the Gelfand–Kirillov dimension of the corresponding simple highest weight module equals the Gelfand–Kirillov dimension of the simple highest weight module for a regular correspondent of $\mu$.

# 3 | STRUCTURE OF $\theta L$

## 3.1 | Quick recap of birepresentation theory

Let $\mathscr{C}$ be a finitary bicategory with involution and adjunctions (fiab bicategory), see [28]. Recall that a finitary birepresentation $\mathbf{M}$ of $\mathscr{C}$ is called *transitive*, provided that any nonzero object of $\mathbf{M}$ generates $\mathbf{M}$, as a birepresentation of $\mathscr{C}$. Furthermore, $\mathbf{M}$ is called *simple transitive* provided that it does not have any proper nontrivial $\mathscr{C}$-invariant ideals.

Recall the cell theory for bicategories, see [28, 35]. For two indecomposable 1-morphisms $\theta, \theta'$ in $\mathscr{C}$, we write $\theta \geqslant_L \theta'$ provided that there is $\theta''$ in $\mathscr{P}$ such that $\theta$ is isomorphic to a summand of $\theta'' \circ \theta'$. Equivalence classes with respect to the preorder $\geqslant_L$ are called *left cells*. Right and two-sided cells are defined similarly. This is similar to the combinatorics of KL-cells that was recalled in Subsection 2.6. Each transitive birepresentation has an *apex*, which is the maximum two-sided cell whose 1-morphisms do not annihilate this birepresentation, see [7, Subsection 3.2]. For two simple transitive birepresentations $\mathbf{M}$ and $\mathbf{N}$ of $\mathscr{C}$, we denote by $\mathrm{Dext}(\mathbf{M}, \mathbf{N})$ the set of *discrete extensions* from $\mathbf{M}$ to $\mathbf{N}$, see [7, Subsection 5.2]. The set $\mathrm{Dext}(\mathbf{M}, \mathbf{N})$ is defined as the set of all nonempty subsets $\Theta$ of the set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$, for which there exists a short exact sequence

$$0 \to \tilde{\mathbf{N}} \to \mathbf{K} \to \tilde{\mathbf{M}} \to 0 \tag{1}$$

of birepresentations of $\mathscr{C}$ (in the sense of [7, Subsection 5.2]) such that

- $\tilde{\mathbf{N}}$ is transitive with simple transitive quotient $\mathbf{N}$;
- $\tilde{\mathbf{M}}$ is transitive with simple transitive quotient $\mathbf{M}$;
- the set $\Theta$ consists of all 1-morphisms $\mathrm{F} \in \mathscr{C}$, for which there is a nonzero object $X \in \tilde{\mathbf{M}}$ such that $\mathrm{F}X$ has a nonzero summand from $\tilde{\mathbf{N}}$.

Here, the fact that (1) is a short exact sequence means that $\tilde{\mathbf{N}}$ is a full subcategory of $\mathbf{K}$ closed with respect to isomorphisms and, additionally, with respect to taking direct sums and direct summands. Furthermore, $\tilde{\mathbf{M}}$ is isomorphic to the quotient of $\mathbf{K}$ by the ideal generated by $\tilde{\mathbf{N}}$. More generally, discrete extensions between transitive representations are defined as discrete extensions between the corresponding simple transitive quotients.

Note that, in the above definition, the birepresentation $\mathbf{K}$ has exactly two weak Jordan–Hölder constituents. The fact that $\mathrm{Dext}(\mathbf{M}, \mathbf{N}) = \varnothing$ means that, in any $\mathbf{K}$ as above, the additive closure of all indecomposable objects that are not killed by projecting onto $\tilde{\mathbf{M}}$ is invariant under the action of $\mathscr{C}$. In particular, for any 1-morphism $\mathrm{F} \in \mathscr{C}$, the action of the Grothendieck class $[F]$ on the split Grothendieck group of $\mathbf{K}$ is given by a block diagonal matrix with two blocks, one corresponding to the action on the split Grothendieck group of $\tilde{\mathbf{N}}$ and the other one corresponding to the action on the split Grothendieck group of $\tilde{\mathbf{M}}$.

**Lemma 2.** *Let $\mathscr{C}$ be a fiab bicategory and $\mathbf{M}$ and $\mathbf{N}$ two transitive birepresentations of $\mathscr{C}$ with the same apex $\mathcal{J}$. Then,* $\mathrm{Dext}(\mathbf{M}, \mathbf{N}) = \varnothing$.

*Proof.* We use the idea in the proofs of [24, Corollary 20] and [7, Corollary 14]. Let $e$ be the idempotent in the real algebra $A_{\mathcal{J}}$ from [24, Subsection 9.3], whose existence was proved in

[24, Proposition 18]. Let

$$0 \to \mathbf{N} \to \mathbf{K} \to \mathbf{M} \to 0$$

be a short exact sequence of birepresentations, see [7, Subsection 5.2].

The algebra $A_{\mathcal{J}}$ acts on the split Grothendieck group of $\mathbf{K}$ with coefficients in $\mathbb{R}$. The matrix of $e$, written in the basis of indecomposable objects in $\mathbf{N}$ and $\mathbf{M}$, has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{2}$$

where $A$ and $C$ are real idempotent matrices with positive entries and $B$ has nonnegative real entries.

Recall that [17, Formula (2)] provides the following normal form for idempotent matrices with nonnegative coefficients:

$$\begin{pmatrix} J & JX & 0 & 0 \\ 0 & 0 & 0 & 0 \\ YJ & YJX & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $J$ is a block diagonal matrix with diagonal blocks $J_1, J_2, \dots, J_k$, with each $J_i$ being an idempotent matrix of rank 1 with nonnegative coefficients. In this normal form, any nonzero off-diagonal entry for which both diagonal correspondents are nonzero belongs to one of the blocks $J_i$. If we assume that $B$ has a nonzero entry, we thus obtain that the whole matrix (2) must be one block $J_i$, which contradicts the fact that $J_i$ has rank 1. Therefore, $B = 0$.

Alternatively, one can note that $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^2 = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is equivalent to $A^2 = A$, $C^2 = C$ and $AB + BC = B$. Multiplying the last equation by $A$ on the left, gives $AAB + ABC = AB$, which implies that $ABC = 0$, since $A^2 = A$. As all entries in both $A$ and $C$ are positive and in $B$ are nonnegative, the equality $ABC = 0$ is equivalent to $B = 0$. □

**Corollary 3.** *Let $\mathscr{C}$ be a fiab bicategory and $\mathbf{M}$ a finitary birepresentation of $\mathscr{C}$ such that all transitive subquotients of $\mathbf{M}$ have the same apex. Then, the objects of each transitive subquotient of $\mathbf{M}$ form a subbirepresentation.*

*Proof.* By the weak Jordan–Hölder theorem, see [37, Theorem 8], there is a short exact sequence of birepresentations

$$0 \to \mathbf{K} \to \mathbf{M} \to \mathbf{N} \to 0,$$

such that $\mathbf{N}$ is transitive and the number of transitive subquotients of $\mathbf{K}$ is one less than the number of transitive subquotients of $\mathbf{M}$. By induction, the transitive subquotients of $\mathbf{K}$ are all subbirepresentations. We need to prove that the additive closure $\mathbf{N}'$ in $\mathbf{M}$ of all indecomposables whose image in $\mathbf{N}$ is nonzero is a subbirepresentation.

Assume that this is not the case. Consider the additive closure $\mathbf{M}'$ of $\mathscr{C}\mathbf{N}'$ in $\mathbf{M}$. By our assumption, $\mathbf{M}' \neq \mathbf{N}'$. Let $\mathbf{K}'$ be the additive closure in $\mathbf{M}'$ of all indecomposable objects outside $\mathbf{N}'$. Let $\mathbf{K}''$ be some transitive quotient of $\mathbf{K}'$ and let $\mathbf{I}$ be the corresponding kernel. This gives rise to a short exact sequence of birepresentations

$$0 \to \mathbf{K}'' \to \mathbf{M}'/(\mathbf{I}) \to \mathbf{N}'/(\mathbf{I}) \to 0.$$

By construction, this gives a nontrivial discrete extension between the simple transitive quotients of $\mathbf{N}'/(\mathbf{I})$ and $\mathbf{K}''$, which contradicts Lemma 2. The claim follows. $\square$

## 3.2 | The bicategory of projective functors

Denote by $\mathscr{P}$ the locally finitary (in the sense of [30, 31]) bicategory defined as follows:

- the objects of $\mathscr{P}$ are $\mathtt{i}_\chi$, where $\chi$ is a central character of $U(\mathfrak{g})$;
- 1-morphisms in $\mathscr{P}(\mathtt{i}_\chi, \mathtt{i}_{\chi'})$ are all projective functors from $\mathcal{O}_\chi$ to $\mathcal{O}_{\chi'}$;
- 2-morphisms in $\mathscr{P}(\mathtt{i}_\chi, \mathtt{i}_{\chi'})$ are natural transformations of functors,

where all identities and compositions are defined in the obvious way.

Since projective functors can be viewed as Harish-Chandra bimodules, the bicategory $\mathscr{P}$ inherits from $\mathscr{H}$ a $\Lambda/\Xi$-grading. For $\xi \in \Lambda/\Xi$, we denote by $\mathscr{P}^\xi$ the corresponding homogeneous component. In particular, $\mathscr{P}^\Xi$ is a subbicategory of $\mathscr{P}$.

Now let $L$ be a simple $\mathfrak{g}$-module with central character $\chi$. For a central character $\chi'$, denote by $\mathbf{X}^L_{\chi'}$ the additive closure in $\mathfrak{g}$-mod of all objects of the form $\theta L$, where $\theta \in \mathscr{P}(\mathtt{i}_\chi, \mathtt{i}_{\chi'})$. Then, the collection of all these $\mathbf{X}^L_{\chi'}$ carries a natural action of $\mathscr{P}$. In other words, we get a birepresentation of $\mathscr{P}$, which we denote by $\mathbf{X}^L$. This birepresentation is locally finitary (cf. [30, 31]) in the sense that it has the properties described by the following proposition.

**Proposition 4.** *Each $\mathbf{X}^L_{\chi'}$ is an idempotent split additive category with finite-dimensional morphism spaces and finitely many isomorphism classes of indecomposable objects.*

*Proof.* Let $\theta, \theta' \in \mathscr{P}(\mathtt{i}_\chi, \mathtt{i}_{\chi'})$. Then, by adjunction,

$$\mathrm{Hom}_{\mathfrak{g}}(\theta L, \theta' L) \cong \mathrm{Hom}_{\mathfrak{g}}((\theta')^* \theta L, L),$$

where $(\theta')^*$ is the biadjoint of $\theta'$. As explained in Subsection 2.2, the right-hand side is finite dimensional. The rest now follows from the definitions. $\square$

For a fixed $\chi'$, we have the bicategory $\mathscr{P}_{\chi'} := \mathscr{P}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})$. This bicategory is finitary in the sense of [28] and $\mathbf{X}^L_{\chi'}$ is a finitary birepresentation of this bicategory. For $\chi' = \chi_0$, the central character of the trivial $\mathfrak{g}$-module, the bicategory $\mathscr{P}_{\chi_0}$ is biequivalent to the bicategory of Soergel bimodules over the coinvariant algebra of $W$, cf. [29].

Combinatorics of the action of projective functors on category $\mathcal{O}$ is governed by the Kazhdan–Lusztig basis of the Hecke algebra. Therefore, the cell structure of the latter, which was recalled in Subsection 2.6, is just a special case of the cell structure of $\mathscr{P}_{\chi_0}$. We also note that the action of projective functors on category $\mathcal{O}$ is a right action. With this in mind, the properties recalled in

Subsection 2.6 can now be reformulated in the setup of projective functors as follows: The annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(L)$ corresponds to a right cell in $\mathscr{P}$, say $\mathcal{R}$. Let $\mathcal{J}$ be the two-sided cell containing $\mathcal{R}$. Then, for any indecomposable $\theta$, the inequality $\theta L \neq 0$ implies $\theta \leqslant_J \mathcal{J}$.

For any central character $\chi'$, we denote by $\mathbf{Y}^L_{\chi'}$ the additive closure of all $\theta L$, where we take $\theta \in \mathscr{P}(\mathtt{i}_\chi, \mathtt{i}_{\chi'}) \cap \mathcal{J}$. Then, $\mathbf{Y}^L_{\chi'}$ is a full subcategory of $\mathbf{X}^L_{\chi'}$, and the collection of all these $\mathbf{Y}^L_{\chi'}$ is closed under the action of $\mathscr{P}$. We denote the corresponding birepresentation of $\mathscr{P}$ by $\mathbf{Y}^L$.

**Conjecture 5.** *The birepresentation $\mathbf{Y}^L$ is simple transitive.*

### 3.3 | A partial preorder

Consider the set $\mathrm{Irr}(\mathfrak{g})$ of isomorphism classes of simple $\mathfrak{g}$-modules. For $X, Y \in \mathrm{Irr}(\mathfrak{g})$, write $X \rhd Y$ provided that there is a finite-dimensional $\mathfrak{g}$-module $V$ such that $V \otimes_{\mathbb{C}} X \twoheadrightarrow Y$. Note that the relation $X \rhd Y$ implies the equality $\mathrm{GKdim}(X) = \mathrm{GKdim}(Y)$, see Lemma 1.

**Lemma 6.** *The relation $\rhd$ is reflexive and transitive (and hence is a partial preorder).*

*Proof.* To prove reflexivity, we can take $V$ to be the trivial module. To prove transitivity, assume that $V \otimes_{\mathbb{C}} X \twoheadrightarrow Y$ and $V' \otimes_{\mathbb{C}} Y \twoheadrightarrow Z$. Then, by exactness of projective functors,

$$(V' \otimes_{\mathbb{C}} V) \otimes_{\mathbb{C}} X \cong V' \otimes_{\mathbb{C}} (V \otimes_{\mathbb{C}} X) \twoheadrightarrow V' \otimes_{\mathbb{C}} Y \twoheadrightarrow Z.$$

This completes the proof. □

By adjunction, it follows that the relation opposite to $\rhd$ is given by the requirement that $X \hookrightarrow V^* \otimes_{\mathbb{C}} Y$.

For a central character $\chi$, let $\mathrm{Irr}(\mathfrak{g})_\chi$ denote the set of all simple $\mathfrak{g}$-modules with central character $\chi$. Then, we have

$$\mathrm{Irr}(\mathfrak{g}) = \coprod_{\chi} \mathrm{Irr}(\mathfrak{g})_\chi.$$

For a fixed $L \in \mathrm{Irr}(\mathfrak{g})$, denote by $\mathcal{Q}_L$ the set of all $L' \in \mathrm{Irr}(\mathfrak{g})$ for which there exists a finite set of elements $L = L_1, L_2, \dots, L_k = L' \in \mathrm{Irr}(\mathfrak{g})$ such that, for each $i$, we either have $L_i \rhd L_{i+1}$ or $L_{i+1} \rhd L_i$. In other words, $\mathcal{Q}_L$ is the equivalence class of $L$ in the minimal equivalence relation generated by $\rhd$.

**Proposition 7.** *Let $L \in \mathrm{Irr}(\mathfrak{g})$ and $\chi$ be a central character. Then $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is finite.*

*Proof.* Without loss of generality, we may assume $L \in \mathrm{Irr}(\mathfrak{g})_\chi$. Let $V$ be a finite-dimensional $\mathfrak{g}$-module such that all indecomposable projective endofunctors of $\mathscr{M}_\chi$ are direct summands of $V \otimes_{\mathbb{C}} \_$. In order to prove our proposition, it is enough to show that any element of $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is a subquotient of $V \otimes_{\mathbb{C}} L$, since all elements of $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ have the same Gelfand–Kirillov dimension as $L$ and there can only be finitely many of them by the additivity of the Bernstein number, see Subsection 2.5. In fact, since $V \otimes_{\mathbb{C}} \_$ already contains all projective endofunctors of

$\mathscr{M}_\chi$, it is enough to show that any element of $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is a subquotient of $V' \otimes_{\mathbb{C}} L$, for some finite-dimensional $\mathfrak{g}$-module $V'$.

Let $L' \in \mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$. Then, by definition, there exists a finite set of elements $L = L_1, L_2, \dots, L_k = L' \in \mathrm{Irr}(\mathfrak{g})$ such that, for each $i$, we either have $L_i \rhd L_{i+1}$ or $L_{i+1} \rhd L_i$. We prove the above claim by induction on $k$, with the case $k = 1$ being obvious.

For the induction step, we assume that $L_{k-1}$ is a subquotient of $V' \otimes_{\mathbb{C}} L$, for some finite-dimensional $\mathfrak{g}$-module $V'$.

Suppose $L_{k-1} \rhd L_k = L'$, that is, $V'' \otimes_{\mathbb{C}} L_{k-1} \twoheadrightarrow L_k$, for some finite-dimensional $\mathfrak{g}$-module $V''$. Then, by exactness, $L_k$ is a subquotient of $V'' \otimes_{\mathbb{C}} (V' \otimes_{\mathbb{C}} L)$ and the latter is isomorphic to $(V'' \otimes_{\mathbb{C}} V') \otimes_{\mathbb{C}} L$.

Suppose now $L' = L_k \rhd L_{k-1}$, that is, $V'' \otimes_{\mathbb{C}} L_k \twoheadrightarrow L_{k-1}$, for some finite-dimensional $\mathfrak{g}$-module $V''$. Then, by adjunction, $L_k \hookrightarrow (V'')^* \otimes_{\mathbb{C}} L_{k-1}$ and again, by exactness, $L_k$ is a subquotient of $(V'')^* \otimes_{\mathbb{C}} (V' \otimes_{\mathbb{C}} L)$, and hence of $((V'')^* \otimes_{\mathbb{C}} V') \otimes_{\mathbb{C}} L$. The claim follows. □

Note that $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is often empty. Indeed, by [25, Theorem 5.1], if $\chi'$ is the central character of $L$ and $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is not empty, then there exist dominant weights $\lambda$ and $\mu$ with the following properties: $\chi = \chi_\lambda$ and $\chi' = \chi_\mu$ such that the difference $\lambda - \mu$ is an integral weight.

**Theorem 8.** *Let $L \in \mathrm{Irr}(\mathfrak{g})$ and $\chi$ be a central character. Assume that $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$ is nonempty and the restriction of $\rhd$ to it is an equivalence relation. Then, for any central character $\chi'$, the restriction of $\rhd$ to $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_{\chi'}$ is also an equivalence relation. In fact, $\rhd$ is an equivalence relation on $\mathcal{Q}_L$.*

*Proof.* Let $\chi'$ be a central character such that $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_{\chi'}$ is not empty. Let $L_1, \dots, L_k$ be the list of all simples in $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_\chi$; in particular, they all are $\rhd$-equivalent (and hence are also equivalent with respect to the relation that is opposite to $\rhd$). Let $L' \in \mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_{\chi'}$. Then, all $L_i$ (and only they) appear both in the tops and in the socles of modules in $\mathscr{P}(\mathtt{i}_{\chi'}, \mathtt{i}_\chi)L'$. In particular, $L' \rhd L_i$, for all $i$. By adjunction, we also have $\mathrm{Hom}_{\mathfrak{g}}(L_i, \theta L') = \mathrm{Hom}_{\mathfrak{g}}(\theta^* L_i, L')$ that implies that $L_i \rhd L'$, for all $i$. The claim follows. □

**Conjecture 9.** *The relation $\rhd$ is an equivalence relation.*

*Remark* 10. It is also natural to consider the partial preorder $\to$ on $\mathrm{Irr}(\mathfrak{g})$ defined as follows: $L \to L'$, provided that $L'$ is a subquotient of $V \otimes_{\mathbb{C}} L$, for some finite-dimensional $V$. It would be interesting to understand certain properties, in particular, the equivalence classes of this preorder. For example, for simple highest weight modules in category $\mathcal{O}$, the corresponding equivalence classes are given by the KL-right cells. Also, the restriction of $\rhd$ to simple highest weight modules is an equivalence relation and the corresponding equivalence classes are given by the KL-right cells (so they coincide with the equivalence classes for the preorder $\to$).

## 3.4 | Conjecture 5 versus Conjecture 9

**Theorem 11.** *Let $L$ be a simple $\mathfrak{g}$-module such that the restriction of $\rhd$ to $\mathcal{Q}_L$ is an equivalence relation. Then, the birepresentation $\mathbf{Y}^L$ is transitive. Moreover, we have $\mathbf{Y}^L = \mathbf{Y}^{L'}$, for any $L'$ such that $L \rhd L'$.*

*Proof.* Let $\chi$ be the central character of $L$ and let $\chi'$ be some central character such that $\mathbf{Y}^L_{\chi'}$ is not zero. Denote by $\tilde{\mathscr{P}}$ the 1-full and 2-full subbicategory of $\mathscr{P}$ on the objects $\mathtt{i}_\chi$ and $\mathtt{i}_{\chi'}$. Also denote by $\tilde{\mathbf{Y}}^L$ the birepresentation of $\tilde{\mathscr{P}}$ restricted from $\mathbf{Y}^L$. To prove the first part of the theorem, it is enough to show that $\tilde{\mathbf{Y}}^L$ is transitive.

Being a finitary birepresentation of $\tilde{\mathscr{P}}$, the birepresentation $\tilde{\mathbf{Y}}^L$ has a weak Jordan–Hölder series with transitive subquotients.

Let us now assume that $\tilde{\mathbf{Y}}^L$ is not transitive and let the additive closure $\mathcal{M}$ of some $M_1, M_2, \dots, M_k$ be a transitive subbirepresentation of $\tilde{\mathbf{Y}}^L$. Let $L'$ be a simple module which appears in the top of $M_1$. Consider the corresponding $\tilde{\mathbf{Y}}^{L'}$ and let the additive closure $\mathcal{N}$ of some $N_1, N_2, \dots, N_r$ be a transitive subbirepresentation of $\tilde{\mathbf{Y}}^{L'}$. By Corollary 3, any transitive subquotient of $\tilde{\mathbf{Y}}^L$ gives, in fact, a subbirepresentation, and similarly for $\tilde{\mathbf{Y}}^{L'}$. Hence, to prove our theorem, it is enough to show that $\mathcal{M} = \mathcal{N}$.

Indeed, as $\mathcal{N}$ is arbitrary, $\mathcal{M} = \mathcal{N}$ implies that $\tilde{\mathbf{Y}}^{L'} = \mathcal{N}$ is transitive. Since the restriction of $\rhd$ to $Q_L$ is an equivalence relation, swapping the roles of $L$ and $L'$, we obtain that $\tilde{\mathbf{Y}}^L$ is transitive, a contradiction. Also, from $\mathcal{M} = \mathcal{N}$, we obtain $\tilde{\mathbf{Y}}^L = \tilde{\mathbf{Y}}^{L'}$.

The remainder of the proof is dedicated to showing that $\mathcal{M} = \mathcal{N}$. Applying projective functors to $M_1 \twoheadrightarrow L'$, we obtain that every object in $\tilde{\mathbf{Y}}^{L'}$ is a quotient of an object in $\mathcal{M}$. In particular, every object in $\mathcal{N}$ is a quotient of an object in $\mathcal{M}$.

Now recall that we have assumed that the restriction of $\rhd$ to $Q_L$ is an equivalence relation. This implies that $L$ is a quotient of some object in $\mathcal{N}$, say $N_1$. Applying projective functors to $N_1 \twoheadrightarrow L$, we obtain that every object in $\tilde{\mathbf{Y}}^L$ is a quotient of an object in $\mathcal{N}$. In particular, every object in $\mathcal{M}$ is a quotient of an object in $\mathcal{N}$.

This implies the existence of an infinite sequence of surjections

$$\cdots \twoheadrightarrow Y_2 \twoheadrightarrow X_2 \twoheadrightarrow Y_1 \twoheadrightarrow X_1 \twoheadrightarrow N_1 \twoheadrightarrow L, \tag{3}$$

where all $X_i \in \mathcal{M}$ and all $Y_j \in \mathcal{N}$. Now, in each $Y_j$, we can pick an indecomposable summand $N_{s_j}$ such that the restricted map from $N_{s_j}$ to $L$ is a surjection. Since the number of indices for $N_j$'s is finite, we can pick an infinite subsequence of the form $\cdots \to N_p \to N_p \to N_p \to L$. Again, here, at each position, the map from $N_p$ to $L$ is a surjection, in particular, all maps between all components of this sequence are nonzero.

The endomorphism algebra of $N_p$ is a local finite-dimensional algebra, see Proposition 4, and hence, its Jacobson radical is nilpotent of a fixed finite nilpotency degree. Since the above sequence is infinite and all compositions are nonzero, at least one morphism in this sequence does not belong to the Jacobson radical and hence is invertible. This means that in the original sequence (3), we have a fragment of the form $N_p \to X_i \to N_p$ such that the composition from the left to the right is invertible. Hence, $N_p$ is isomorphic to a summand of $X_i$. In other words, $\mathcal{M}$ and $\mathcal{N}$ have a nonzero intersection and thus must coincide since both carry a transitive birepresentations of $\tilde{\mathscr{P}}$. This completes the proof. $\square$

*Remark* 12. If $\mathbf{Y}^L = \mathbf{Y}^{L'}$, for any $L' \in Q_L$, then the restriction of $\rhd$ to $Q_L$ is an equivalence relation. Indeed, in this case, we claim that $L \rhd L'$ implies $L' \rhd L$. To see this, we first claim that $\mathbf{Y}^L$ contains a module with top $L$.

Consider the Duflo involution $\theta$ in the right cell that corresponds to the annihilator of $L$. This is a coalgebra 1-morphism in $\mathscr{P}$, see [29, Section 4.4], and hence, the evaluation of the counit $\theta \to \mathbb{1}$, when applied to $L$, is nonzero.

Similarly, $\mathbf{Y}^{L'}$ contains a module with top $L'$. Therefore, $\mathbf{Y}^{L} = \mathbf{Y}^{L'}$ together with $L \trianglerighteq L'$ implies $L' \trianglerighteq L$.

## 4 | PROOF OF THE TWO CONJECTURES IN TYPE *A*

In this section, we show that the statements of both Conjectures 5 and 9 are true in type $A$. So, we assume that $\mathfrak{g}$ and hence also $W$ are of type $A$.

### 4.1 | Reduction to nice blocks

Let $\lambda$ be a dominant weight and $\chi_\lambda$ the corresponding central character. We will call both $\lambda$ and $\chi_\lambda$ *nice*, provided that there is $\mu \in \lambda + \Xi$ such that $W_\mu = W'_\mu$. For example, in the set $\mathbb{Z}$ of all integral weights for $\mathfrak{sl}_2$, we have $\Xi = 2\mathbb{Z}$ and all odd weights are nice (since $-1$ is the only integral singular weight), while all even weights are not nice. Note that $W_\lambda = \{e\}$ implies that $\lambda$ is nice as we can take $\mu = \lambda$.

**Lemma 13.** *For any dominant weight $\lambda$, there is a nice dominant weight $\tilde{\lambda} \in \lambda + \Lambda$ such that*

(*a*) $W_\lambda = W_{\tilde{\lambda}}$,
(*b*) $W'_\lambda = W'_{\tilde{\lambda}}$,
(*c*) $\tilde{\lambda} - \lambda$ *is integral and dominant with respect to* $W_\lambda$.

Note that $W_\lambda = W_{\tilde{\lambda}}$ is satisfied for any $\tilde{\lambda} \in \lambda + \Lambda$, so Condition (a) above is automatic. However, the equality $W_\lambda = W_{\tilde{\lambda}}$ is necessary for Condition (b) to make sense. This is the reason why Condition (a) appears in the formulation.

*Proof.* We first prove the claim under the assumption that $\lambda$ is integral. The weight $-\rho$ is the only integral fully singular weight, so we need to look for $\tilde{\lambda}$ inside $-\rho + \Xi$. Let $D$ be the absolute value of the determinant of the Cartan matrix of $\mathfrak{g}$. Set $\tilde{\lambda} = D(\lambda + \rho) - \rho$. Since the $D$-multiples of the fundamental weights belong to $\Xi$, it follows that $\tilde{\lambda} \in -\rho + \Xi$.

Then, $\tilde{\lambda} - \lambda = (D - 1)(\lambda + \rho)$ that is dominant. Both $\lambda$ and $\tilde{\lambda}$ are integral and hence, for both of them, the integral Weyl group is just the whole Weyl group. Finally, the stabilizers of $\tilde{\lambda}$ and $\lambda$ in $W$ with respect to the dot action coincide because, after the shift by $\rho$, the dot-action becomes the usual action and this commutes with multiplication by scalars. This proves the claim for integral weights.

Take now any $\lambda$ and assume $W_\lambda \neq \{e\}$, for otherwise the claim is clear. Let $\mathbf{R}_\lambda$ be the root subsystem of $\mathbf{R}$ corresponding to $W_\lambda$. Let $\mathfrak{g}(\lambda)$ be the corresponding Lie subalgebra of $\mathfrak{g}$. Let $\Lambda(\lambda)$ be the set of all integral weights for $\mathfrak{g}(\lambda)$ and $\Xi(\lambda)$ the set of all integral linear combinations of roots for $\mathfrak{g}(\lambda)$. Choose some representatives $\mu_1 = 0, \mu_2, \dots, \mu_k$ of the cosets in $\Lambda(\lambda)/\Xi(\lambda)$. Also, denote by $\mathfrak{h}_\lambda$ the intersection of $\mathfrak{h}$ with $\mathfrak{g}(\lambda)$. Define $\mathfrak{h}_\lambda^\perp$ as the set of all $h \in \mathfrak{h}$ such that $\alpha(h) = 0$, for any root $\alpha$ of $\mathfrak{g}(\lambda)$. Then, $\mathfrak{h} = \mathfrak{h}_\lambda \oplus \mathfrak{h}_\lambda^\perp$. The inclusion $\mathfrak{h}_\lambda \hookrightarrow \mathfrak{h}$ induces the restriction map $\mathrm{Res}_\lambda : \mathfrak{h}^* \to \mathfrak{h}_\lambda^*$.

The restriction of the natural $\mathfrak{g}$-module (i.e., the module $\mathbb{C}^n$ for $\mathfrak{sl}_n$) to any simple summand $\mathfrak{a}$ of $\mathfrak{g}(\lambda)$ gives the direct sum of the natural module for $\mathfrak{a}$ with a summand on which $\mathfrak{a}$ acts trivially. Recall that the natural module generates the category of all finite-dimensional modules as an idempotent split monoidal category. This implies that

$$\mathrm{Res}_\lambda(\lambda+\rho+\Lambda) = \mathrm{Res}_\lambda(\lambda)+\Lambda(\lambda)+\rho_\lambda, \tag{4}$$

where $\rho_\lambda$ is the half of the sum of all positive roots for $\mathfrak{g}(\lambda)$. We note that both $\rho$ and $\rho_\lambda$ are integral weights, so they can be removed from (4). In particular, we can pick some representatives $\nu_1 = \lambda, \nu_2, \dots, \nu_k$ in $\lambda+\Lambda$ such that $\mathrm{Res}_\lambda(\nu_i+\rho-\lambda) = \mu_i+\rho_\lambda$.

We can now apply the already proved assertion of the lemma in the integral case to $\mathrm{Res}_\lambda(\lambda)$ to obtain the corresponding nice dominant integral weight $\widetilde{\mathrm{Res}_\lambda(\lambda)}$ for $\mathfrak{g}(\lambda)$ that satisfies (a)–(c) (with respect to $\mathrm{Res}_\lambda(\lambda+\rho)-\rho_\lambda$ for $\mathfrak{g}(\lambda)$). By (4), there is $\tilde{\lambda} \in \lambda+\Lambda$ such that $\mathrm{Res}_\lambda(\tilde{\lambda}+\rho)-\rho_\lambda = \widetilde{\mathrm{Res}_\lambda(\lambda)}$. The fact that $\tilde{\lambda}$ is nice dominant and satisfies (a)–(c) follows from the fact that $\widetilde{\mathrm{Res}_\lambda(\lambda)}$ is nice dominant and satisfies (a)–(c). □

**Example 14.** Consider $\mathfrak{g} = \mathfrak{sl}_3$ with $\mathbf{R} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$. With respect to the standard basis, we then have

$$\alpha = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \alpha+\beta = \rho = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Consider the weight $\lambda = \binom{x-1/2}{-x-1/2}$, for some irrational $x$. Then, we have $\mathbf{R}_\lambda = \{\pm(\alpha+\beta)\}$ and $\rho_\lambda = (1)$. This yields $\mathrm{Res}_\lambda(\lambda+\rho)-\rho_\lambda = 0$, which is not a nice $\mathfrak{sl}_2$-weight. Therefore, $\lambda$ is not nice.

To get a nice weight, we add to $\lambda$ the integral weight $\lambda = \binom{1}{0}$ resulting in the weight $\tilde{\lambda} = \binom{x+1/2}{-x-1/2}$. We have $\mathrm{Res}_\lambda(\tilde{\lambda}+\rho)-\rho_\lambda = 1$, which is a nice $\mathfrak{sl}_2$-weight. Clearly, the conditions (a)–(c) of Lemma 13 are satisfied for this $\tilde{\lambda}$.

One can also easily find a singular weight in $\tilde{\lambda}+\Xi$. For example, the weight $\tilde{\lambda}-(\alpha+\beta) = \binom{x-1/2}{-x-3/2}$ is singular. One checks that $\mathrm{Res}_\lambda(\_+\rho)-\rho_\lambda$ maps this weight to $-1$, namely, to the unique singular $\mathfrak{sl}_2$-weight.

*Remark* 15. It is easy to see that, if $\lambda$ is nice, then $\lambda+\Xi$ contains dominant weights of arbitrary singularity in $W_\lambda$.

If $\lambda$ and $\tilde{\lambda}$ are as above, then $\theta_{\lambda,\tilde{\lambda}}$ and $\theta_{\tilde{\lambda},\lambda}$ are mutually inverse equivalences of categories, both at the level of category $\mathcal{O}$ and at the level of category $\mathscr{M}$. In particular, these equivalences send simple objects to simple objects. Consequently, for any simple $\mathfrak{g}$-module $L$ in $\mathcal{M}_\chi$, the categories $\mathrm{add}(\mathscr{P}L)$ and $\mathrm{add}(\mathscr{P}\theta_{\lambda,\tilde{\lambda}}(L))$ coincide (in the sense that they have the same objects and morphisms).

Since both Conjectures 5 and 9 are formulated in terms of $\mathrm{add}(\mathscr{P}L)$, it follows that it is enough to prove them for simple modules with nice central characters.

## 4.2 | Reduction to singular blocks

Let $L$ be a simple $\mathfrak{g}$-module, $\chi$ be the central character of $L$, which we assume to be nice, and $\lambda \in \mathfrak{h}^*$ be some weight such that $\mathrm{Ann}_{U(\mathfrak{g})}(L) = \mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda))$. We have the bicategory $\mathscr{P}_\chi$ of all projective endofunctors of $\mathscr{M}_\chi$.

Consider the integral Weyl group $W_\lambda$ of $\lambda$. Then, $W_\lambda$ is a product of symmetric groups. Let us start by recalling special features of Kazhdan–Lusztig combinatorics in type $A$. Thanks to [22, Theorem 1.4], in type $A$, left and right cells of $\mathscr{P}^\Xi$ can be described using the Robinson–Schensted correspondence that associates to a permutation $w \in S_n$ a pair of standard Young tableaux of the same shape (which is a partition of $n$). The latter shape determines the two-sided cell. One special type $A$ feature is that each two-sided cell $\mathcal{J}$ of $\mathscr{P}^{\Xi}$ in type $A$ contains the longest element $\theta_{w_0^{W'}}$ in some parabolic subgroup $W'$ of $W_\lambda$. Another special feature is that the intersection of any left and any right cell in $\mathcal{J}$ is a singleton. This means that $\mathcal{J}$ contains the identity functor $\theta_e^{\chi'}$ on some $\mathscr{M}_{\chi'}$ (a singular block for which $W'$ is the dot-stabilizer of the dominant weight for that block) and this functor is the only projective endofunctor of $\mathscr{M}_{\chi'}$ belonging to the intersection of the cell $\mathcal{J}$ with the homogeneous component $\mathscr{P}_\chi^{\Xi}$. From Lemma 16 below, it follows that inclusion gives rise to a bijection between the left (right, two-sided) cells of $\mathscr{P}^{\Xi}$ and the corresponding cells of $\mathscr{P}$.

The elements in $\mathcal{J}$ that do not annihilate $L$ are exactly the elements of the left cell that is adjoint to the right cell that corresponds to the annihilator of $L$, see [35, Lemma 12]. Each of these left cells contains an element with target $\mathscr{M}_{\chi'}$. We choose one of those, call it $\theta$. Then $\theta L \neq 0$ and we can let $L' \in \mathscr{M}_{\chi'}$ be any simple quotient of $\theta L$. Since $\chi$ is assumed to be nice, $\theta$ is homogeneous of degree $\Xi$.

We note that, by construction, the identity projective functor $\theta_e^{\chi'}$ on $\mathscr{M}_{\chi'}$ is the only indecomposable projective endofunctor of $\mathscr{M}_{\chi'}$ that belongs to $\mathscr{P}_\chi^{\Xi}$ and does not annihilate $L'$.

Consider the annihilator $\mathbf{I} := \mathrm{Ann}_{U(\mathfrak{g})}(L')$ of $L'$ in $U(\mathfrak{g})$. Let $\mathscr{M}_{\chi'}^{\mathbf{I}}$ denote the full subcategory of $\mathscr{M}_{\chi'}$ that consists of all objects on which $\mathbf{I}$ acts locally nilpotently. Then $\theta_e^{\chi'}$ is still the identity endofunctor of $\mathscr{M}_{\chi'}^{\mathbf{I}}$ and it does not annihilate $L'$.

We want to answer the following question: What are the other projective endofunctors of $\mathscr{M}_{\chi'}^{\mathbf{I}}$ that do not annihilate $L'$?

Choose some $\lambda'$ such that $\mathrm{Ann}_{U(\mathfrak{g})}(L') = \mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda'))$, which is possible due to Duflo's theorem, see [15]. Consider $W \cdot \lambda'$ and its intersections with all $\Xi$-cosets in $\lambda' + \Lambda$. Those cosets in $\Lambda/\Xi$ for which the intersection is nontrivial form a subgroup of the cyclic group $\Lambda/\Xi$. Let $\mu_1$, $\mu_2, \ldots, \mu_k$ be the dominant weights in all the corresponding nonempty intersections (of $W \cdot \lambda'$ with the $\Xi$-cosets in $\lambda' + \Lambda$). Without loss of generality, we may assume $\lambda' \in W_{\mu_1} \cdot \mu_1$. We have, $\theta_e^{\chi'} = \theta_{\mu_1,\mu_1}$.

The integral Weyl groups $W_{\mu_i}$ are all conjugate and so are the stabilizers of the corresponding dominant weights in these $W_{\mu_i}$, see also [19, Remark 3.5]. In particular, from Soergel's combinatorial description, it follows that all the corresponding indecomposable blocks $\mathscr{O}_{\mu_i}$ (see Subsection 2.1) of category $\mathscr{O}$ are equivalent, see also [32, Lemma A.3] for an alternative argument. In particular, all $L(\mu_i)$ have the same Gelfand–Kirillov dimension, see Subsection 2.6.

Since our $\mu_i$ might be singular, the category $\mathscr{O}_{\mu_i}$ might contain some other simple highest weight modules $L(\nu)$ with the same Gelfand–Kirillov dimension as $L(\mu_i)$. In this case, we will write $\nu \sim \mu_i$.

**Lemma 16.** *Each indecomposable projective functor that does not annihilate $L'$ is of the form $\theta_{\mu_1,\nu}$, where $\nu \sim \mu_i$, for some $i$, and $\nu$ is antidominant with respect to the dot-stabilizer of $\mu_1$. Moreover, each such $\theta_{\mu_1,\nu}$ is a self-equivalence of $\mathscr{M}_{\chi'}^{\mathbf{I}}$ (but not necessarily of $\mathscr{M}_{\chi'}$).*

*Proof.* We have $\theta_{\mu_1,\nu} L' \neq 0$ if and only if $\theta_{\mu_1,\nu} L(\lambda') \neq 0$, by our choice of $\lambda'$. Since projective functors cannot increase the Gelfand–Kirillov dimension and all $L(\mu_i)$ have the same Gelfand-Kirillov dimension, $\theta_{\mu_1,\nu} L' \neq 0$ implies $\nu \sim \mu_i$, for some $i$, by our definition of $\sim$. From the classification of projective functors, we may also assume that $\nu$ is antidominant with respect to the dot-stabilizer of $\mu_1$. It remains to argue that any such $\theta_{\mu_1,\nu}$ is an equivalence.

Let $\theta^*_{\mu_1,\nu}$ denote the biadjoint of $\theta_{\mu_1,\nu}$. We have $\theta_{\mu_1,\nu}P(\mu_1) = \theta_{\mu_1,\nu}\Delta(\mu_1) = P(\nu)$, by the classification of projective functors. In particular,

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\theta_{\mu_1,\nu}\Delta(\mu_1), L(\nu)) = 1.$$

By adjunction, we thus have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu_1), \theta^*_{\mu_1,\nu}L(\nu)) = 1.$$

Note that $L(\mu_1)$ is the simple top of $\Delta(\mu_1)$ and it appears in $\Delta(\mu_1)$ with multiplicity one.

Assume that the image of a unique (up to scalar) nonzero map from $\Delta(\mu_1)$ to $\theta^*_{\mu_1,\nu}L(\nu)$ is not isomorphic to $L(\mu_1)$. Then, the socle of this image contains some $L(\nu')$, where $\nu' \sim \mu_1$ and $\nu' \neq \mu_1$. As $\theta^*_{\mu_1,\nu}L(\nu)$ is self-dual (since projective functors commute with the duality on $\mathcal{O}$), $L(\nu')$ also appears in its top. This means that the composition $\theta^*_{\mu_1,\nu}\circ\theta_{\mu_1,\nu}$ applied to $\Delta(\mu_1)$ has $P(\nu')$ as a summand. However, this is not possible as $\nu' \sim \mu_1$ and $\theta_{\mu_1,\mu_1}$ is the only projective endofunctor of $\mathcal{O}_{\mu_1}$ that does not kill $L(\mu_1)$.

From the previous paragraph, we have that the image of a unique (up to scalar) nonzero map from $\Delta(\mu_1)$ to $\theta^*_{\mu_1,\nu}L(\nu)$ is isomorphic to $L(\mu_1)$, in particular, $L(\mu_1)$ appears in the socle of $\theta^*_{\mu_1,\nu}L(\nu)$. As $\theta^*_{\mu_1,\nu}L(\nu)$ is self-dual, $L(\mu_1)$ appears in the top of $\theta^*_{\mu_1,\nu}L(\nu)$ as well. Since $\Delta(\mu_1)$ is projective and the map from it to $\theta^*_{\mu_1,\nu}L(\nu)$ is unique, the multiplicity of $L(\mu_1)$ in $\theta^*_{\mu_1,\nu}L(\nu)$ is one. Consequently, $L(\mu_1)$ is a direct summand of $\theta^*_{\mu_1,\nu}L(\nu)$. As, by the previous paragraph, no other $L(\nu')$ with $\nu' \sim \mu_1$ are allowed to appear in the top or socle of $\theta^*_{\mu_1,\nu}L(\nu)$, we have $\theta^*_{\mu_1,\nu}L(\nu) = L(\mu_1)$. By adjunction, $\theta_{\mu_1,\nu}L(\mu_1) = L(\nu)$.

This implies that

$$\theta^*_{\mu_1,\nu}\theta_{\mu_1,\nu} = \theta_{\mu_1,\nu}\theta^*_{\mu_1,\nu} = \theta_{\mu_1,\mu_1}$$

as endofunctors of $\mathscr{M}^{\mathbf{I}}_{\chi'}$, and completes the proof. □

**Example 17.** For $\mathfrak{g} = \mathfrak{sl}_2$, consider $\lambda = (-1/2)$. Then $W \cdot \lambda = \{-1/2, -3/2\}$. Both latter weights are dominant with respect to their integral Weyl group, which is trivial. However, the difference between these two weights is an integral weight. Therefore, we have two indecomposable projective functors that do not annihilate $L(-1/2)$, namely, the identity functor $\theta_{-1/2,-1/2}$ and the equivalence $\theta_{-1/2,-3/2}$ between $\mathcal{O}_{-1/2}$ and $\mathcal{O}_{-3/2}$.

## 4.3 | Proof of Conjecture 9 in type $A$

From the construction in the previous subsection, it follows that $\mathcal{Q}_L \cap \mathrm{Irr}(\mathfrak{g})_{\chi'}$ consists of modules of the form $\theta_{\mu,\mu'}(L')$, where $\theta_{\mu,\mu'}$ are equivalences. This set is, clearly, one equivalence class with respect to $\rhd$. Therefore, the claim of Conjecture 9 follows from Theorem 8.

## 4.4 | Proof of Conjecture 5 in type $A$

We now establish a crucial property of the module $L'$ constructed in Subsection 4.2, namely, its weak Kostant positivity (see Subsection 2.3):

**Lemma 18.** *The module $L'$ is weakly Kostant positive.*

*Proof.* Since we are discussing only the weak Kostant positivity of $L'$ in this lemma, in the proof below, we restrict our attention to indecomposable projective functors that are homogeneous of degree $\Xi$. Recall that $\theta_e^{\chi'}$ is the only indecomposable projective endofunctor of $\mathscr{M}_{\chi'}$ that is homogeneous of degree $\Xi$ and does not annihilate $L'$, see Section 2.

We start with the claim that $\lambda'$ may be assumed to be dominant (i.e., we may assume $\lambda' = \mu_1$). In other words, we claim that $\mathrm{Ann}_{U(\mathfrak{g})}(L') = \mathrm{Ann}_{U(\mathfrak{g})}(L(\mu_1))$. To prove this, we need to show that $L(\lambda')$ is annihilated by all indecomposable projective endofunctors of $\mathcal{O}_{\chi'}$ that are not isomorphic to the identity functor.

Let $\theta$ be an indecomposable projective endofunctor of $\mathscr{M}_{\chi'}$, homogeneous of degree $\Xi$, which is not isomorphic to the identity functor. Then, $\theta P(\mu_1) \cong P(\nu)$, for some $\nu \neq \mu_1$. Since $\theta$ is strictly bigger than $\mathcal{J}$ in the two-sided order, the Gelfand–Kirillov dimension of $L(\nu)$ is strictly greater than that of $L(\mu_1)$, see [20, Subsection 10.11]. From [20, Subsection 10.9], it then follows that $\theta L(\mu_1) = 0$. Therefore, the annihilator of $L(\mu_1)$ corresponds to a right cell inside $\mathcal{J}$ and since the right cell of $\theta_e^{\chi'}$ is the only right cell that contains a representative from projective endofunctors of $\mathcal{O}_{\chi'}$ in $\mathcal{J}$, we obtain that the annihilator of $L(\mu_1)$ corresponds to the right cell of $\theta_e^{\chi'}$. This is the same right cell that describes the annihilator of $L'$, and we obtain our claim.

Now, assuming $\lambda'$ is dominant, $L(\lambda')$ is the quotient of a projective Verma module in $\mathcal{O}_{\chi'}$. Hence, it is Kostant positive, see [20, Subsection 6.9]. For a simple finite-dimensional $V$, the multiplicity of $V$ in $\mathcal{L}(L(\lambda'), L(\lambda'))$ equals the dimension of $\mathrm{Hom}_{\mathfrak{g}}(V \otimes_{\mathbb{C}} L(\lambda'), L(\lambda'))$. We can write $V \otimes_{\mathbb{C}} \_$ as a direct sum of indecomposable projective functors. The summands which go from $\mathcal{O}_{\chi'}$ to $\mathcal{O}_{\chi'}$ are either $\theta_e^{\chi'}$ or kill $L(\lambda')$. Therefore, the above multiplicity equals the multiplicity of $\theta_e^{\chi'}$ as a summand of $V \otimes_{\mathbb{C}} \_$.

The same computation works for $L'$, under the assumption that 0 is a weight of $V$ (which is equivalent to saying that all indecomposable projective functors that appear as summands of $V \otimes_{\mathbb{C}} \_$ are homogeneous of degree $\Xi$), which is equivalent to saying that all indecomposable projective functors that are summands of $\_ \otimes_{\mathbb{C}} V$ are homogeneous of degree $\Xi$. Therefore, combining

$$U(\mathfrak{g})/\big(\mathrm{Ann}_{U(\mathfrak{g})}(L')\big) \cong U(\mathfrak{g})/\big(\mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda'))\big)$$

with

$$U(\mathfrak{g})/\big(\mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda'))\big) \cong \mathcal{L}(L(\lambda'), L(\lambda')),$$

then with

$$U(\mathfrak{g})/\big(\mathrm{Ann}_{U(\mathfrak{g})}(L')\big) \hookrightarrow \mathcal{L}(L', L')^{\Xi},$$

and, finally, with

$$[\mathcal{L}(L(\lambda'), L(\lambda')) : V] = [\mathcal{L}(L', L')^{\Xi} : V],$$

we obtain

$$U(\mathfrak{g})/\big(\mathrm{Ann}_{U(\mathfrak{g})}(L')\big) \cong \mathcal{L}(L', L')^{\Xi}.$$

This completes the proof. □

Denote by ${}^{\Xi}\mathbf{X}^{L'}$ and ${}^{\Xi}\mathbf{Y}^{L'}$ the $\mathscr{P}^{\Xi}$-analogs of $\mathbf{X}^{L'}$ and $\mathbf{Y}^{L'}$, respectively.

Since $L'$ is weakly Kostant positive and the identity projective functor $\theta_e^{\chi'}$ on $\mathscr{M}_{\chi'}$ is the only indecomposable projective endofunctor of $\mathscr{M}_{\chi'}$ that belongs to $\mathscr{P}_{\chi}^{\Xi}$ and does not annihilate $L'$, we can apply the adaptation [23, Theorem 5] of [39, Theorem 5.1] and conclude that ${}^{\Xi}\mathbf{X}^{L'}$ is equivalent to a certain category of Harish-Chandra bimodules. This equivalence is, in fact, a homomorphism of birepresentations of $\mathscr{P}^{\Xi}$.

We can also apply [23, Theorem 5] to $L(\lambda)$ and conclude that ${}^{\Xi}\mathbf{X}^{L(\lambda)}$ is equivalent to the same category of Harish-Chandra bimodules, as birepresentations of $\mathscr{P}^{\Xi}$. In other words, ${}^{\Xi}\mathbf{X}^{L'}$ and ${}^{\Xi}\mathbf{X}^{L(\lambda)}$ are equivalent as birepresentations of $\mathscr{P}^{\Xi}$. This means that such an equivalence induces an equivalence between ${}^{\Xi}\mathbf{Y}^{L'}$ and ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$.

Since we already established in Subsection 4.3 that the statement of Conjecture 9 holds for $Q_L$, Theorem 11 implies that $\mathbf{Y}^{L} = \mathbf{Y}^{L'}$. This means that ${}^{\Xi}\mathbf{Y}^{L}$ and ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$ are equivalent as birepresentations of $\mathscr{P}^{\Xi}$. It is easy to see that both $\mathbf{Y}^{L(\lambda)}$ and ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$ are simple transitive by combining [35, Theorem 22], [36, Proposition 22], and [37, Theorem 18]. Hence, ${}^{\Xi}\mathbf{Y}^{L}$ is simple transitive.

The elements of $\mathscr{P}$ inside $\mathcal{J}$ are obtained from the elements of $\mathscr{P}^{\Xi}$ inside $\mathcal{J}$ by composing with the autoequivalences given by Lemma 16. Hence, $\mathbf{Y}^{L}$ is obtained from ${}^{\Xi}\mathbf{Y}^{L}$ by applying some equivalences. Since $\mathbf{Y}^{L}$ is transitive by construction, the fact that it is simple transitive follows from the simple transitivity of ${}^{\Xi}\mathbf{Y}^{L}$. This completes the proof.

We remark that some of the arguments in this subsection, in particular, the idea of reduction to a singular block, are similar in spirit to the arguments given in the proof of [38, Theorem 67].

## 4.5 | Some corollaries

The equivalence between ${}^{\Xi}\mathbf{Y}^{L}$ and ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$ established in the previous subsection has the following consequence.

**Corollary 19.** *Let $\theta$ be an indecomposable projective functor from $\mathcal{J}$ and $M$ a subquotient of $V \otimes_{\mathbb{C}} L$ such that $\theta M \neq 0$. Then,* $\mathrm{GKdim}(M) = \mathrm{GKdim}(L)$.

*Proof.* The point is that in $\mathbf{Y}^{L(\lambda)}$ any simple subquotient $M$ of any $V \otimes_{\mathbb{C}} L(\lambda)$ satisfying $\theta M \neq 0$ appears in the top of some $V' \otimes_{\mathbb{C}} L(\lambda)$. We know this because we understand the action of projective functors on category $\mathcal{O}$ quite well. Applying to $L(\lambda)$ projective functors from $\mathcal{J}$ produces a cell birepresentation of the bicategory of projective functors. In the abelianization of this birepresentation, the action of projective functors from $\mathcal{J}$ is given by tensoring with projective bimodules that are explicitly described in [28, Proposition 4.15]. Applying such a projective module does exactly what is claimed above: applied to a summand of $V \otimes_{\mathbb{C}} L(\lambda)$ in which $M$ appears as a subquotient, it produces a module with a direct summand in which $M$ appears in the top.

Due to the equivalence between ${}^{\Xi}\mathbf{Y}^{L}$ and ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$ and the fact that $\mathbf{Y}^{L}$ is obtained from ${}^{\Xi}\mathbf{Y}^{L}$ (resp. $\mathbf{Y}^{L(\lambda)}$ from ${}^{\Xi}\mathbf{Y}^{L(\lambda)}$) using some equivalences of categories given by projective functors, it follows that $M$ has a subquotient that appears in the top of some $V'' \otimes_{\mathbb{C}} L$. This implies $\mathrm{GKdim}(M) = \mathrm{GKdim}(L)$. □

Let $L$ be a simple $\mathfrak{g}$-module and $V$ a finite-dimensional $\mathfrak{g}$-module. A subquotient $M$ of $V \otimes_{\mathbb{C}} L$ will be called *strange*, provided that the following conditions are satisfied:

- for any submodule $N \subset M$, exactly one of the modules $N$ or $M/N$ has GK-dimension $\mathrm{GKdim}(L)$,
- $M$ does not have any simple subquotient of GK-dimension $\mathrm{GKdim}(L)$.

This definition is inspired by the properties of the regular $\mathbb{C}[x]$-module. This module and all its nonzero submodules have GK-dimension 1, while any quotient of this module by a nonzero submodule has GK-dimension 0. The module itself does not have any simple submodules.

**Corollary 20.** *Let $\theta$ be an indecomposable projective functor from $\mathcal{J}$ and $M$ a strange subquotient of $V \otimes_{\mathbb{C}} L$. Then, $\theta M = 0$.*

*Proof.* As explained in the proof of Corollary 19, the assumption $\theta M \neq 0$ implies that $M$ has a subquotient that appears in the top of some $V'' \otimes_{\mathbb{C}} L$. In other words, $M$ has a simple subquotient $M'$ such that $\mathrm{GKdim}(M') = \mathrm{GKdim}(L)$. This contradicts our assumption that $M$ is strange. □

### 4.6 | The two conjectures in other type $A$ situations

Let now $\mathfrak{g}$ be of any type. Let $L$ be a simple $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$ be such that the annihilators of $L$ and $L(\lambda)$ coincide. Combining the above results with Soergel's combinatorial description, it follows that both Conjecture 5 and Conjecture 9 are true for $L$ under the assumption that $W_\lambda$ is of type $A$. More precisely, we have:

**Corollary 21.** *Assume that $W_\lambda$ is of type $A$. Then the birepresentation $\mathbf{Y}^L$ is simple transitive and the restriction of the relation $\rhd$ to $\mathcal{Q}_L$ is the full relation (i.e., any two elements are related).*

## 5 | SOCLES

### 5.1 | The main result

Let us start with repeating the formulation of the main result.

**Theorem 22.** *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{C}$. Let $L$ be a holonomic simple $\mathfrak{g}$-module and let $V$ be a finite-dimensional $\mathfrak{g}$-module. Then, the $\mathfrak{g}$-module $V \otimes_{\mathbb{C}} L$ has an essential semisimple submodule of finite length.*

### 5.2 | Reduction to a finite piece

Let $L$ be a simple $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$ be such that $\mathrm{Ann}_{U(\mathfrak{g})}(L) = \mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda))$. Consider the set $\lambda + \Lambda$ and the set $\mathcal{K}(\lambda)$ of all central characters of the form $\chi_\mu$, where $\mu \in \lambda + \Lambda$.

As we have seen in Subsection 2.4, the combinatorial datum that controls equivalences between blocks of $\mathcal{O}$ is given by triples of the form $W'_\mu \subset W_\mu \subset W$, for $\mu \in \mathfrak{h}^*$ (see [11] for an explicit classification). Therefore, it is natural to consider the finite set of all triples of the form $\tilde{\tilde{W}} \subset \tilde{W} \subset W$, where $\tilde{W}$ is the subgroup of $W$ generated by some reflections (and hence is the Weyl group of the root subsystem of $\mathbf{R}$ generated by the roots corresponding to those reflections) and $\tilde{\tilde{W}}$ is a parabolic subgroup of $\tilde{W}$ (with respect to the choice of positive roots inherited from $\mathbf{R}_+$).

Given two dominant weights $\mu$ and $\nu$ in $\lambda + \Lambda$, we have their respective integral Weyl groups $W_\mu$ and $W_\nu$ and their respective dot-stabilizers $W'_\mu$ and $W'_\nu$. If $W_\mu = W_\nu$ and $W'_\mu = W'_\nu$, then the projective functors $\theta_{\mu,\nu}$ and $\theta_{\nu,\mu}$ are mutually inverse equivalences of categories.

Now we can fix a finite set $\mu_1, \mu_2, \ldots, \mu_k$ of dominant weights in $\lambda + \Lambda$ such that $\mu_1 \in W_\lambda \cdot \lambda$ and, for any other dominant weight $\nu$ in $\lambda + \Lambda$, there is some $\mu_i$ such that we have both the equality $W_{\mu_i} = W_\nu$ of the corresponding integral Weyl groups and the equality $W'_{\mu_i} = W'_\nu$ of the corresponding dot-stabilizers. Let $\mathscr{N}$ be the direct sum of all the corresponding $\mathscr{M}_{\chi_{\mu_i}}$.

**Proposition 23.** *In order to prove Theorem 22, it is enough to prove that, for any indecomposable projective endofunctor $\theta$ of $\mathscr{N}$, the module $\theta L$ has an essential semisimple submodule of finite length.*

*Proof.* Due to our construction of $\mathscr{N}$, any nonzero projective functor $\theta'$ from $\mathscr{M}_{\chi_\lambda}$ to some $\mathscr{M}_{\chi_\nu}$ factors through some $\mathscr{M}_{\chi_{\mu_i}}$ via equivalences of categories given by projective functors. Equivalences of categories, clearly, preserve the module-theoretic property of having an essential semisimple submodule of finite length. $\square$

## 5.3 | Reduction to the maximal two-sided cell

Let $L \in \mathscr{N}$ be a simple module and $\theta$ a projective endofunctor of $\mathscr{N}$. Let $\mathcal{J}$ be the two-sided KL-cell that contains the left KL-cell corresponding to the annihilator of $L$ in $U(\mathfrak{g})$. Let $\theta_\mathcal{J}$ be a multiplicity-free direct sum of all projective endofunctors of $\mathscr{N}$ that belong to $\mathcal{J}$. Let $\tilde{\theta}$ be the Duflo element in the left KL-cell corresponding to the annihilator of $L$ in $U(\mathfrak{g})$. Then, we have a natural transformation from the identity to $\tilde{\theta}$, whose evaluation at $L$ is nonzero, see [35, Subsection 4.5]. Consequently, $L$ appears as a submodule of $\tilde{\theta}L$, and hence as a submodule of $\theta_\mathcal{J}L$, since $\tilde{\theta}$ is a summand of $\theta_\mathcal{J}$.

Applying $\theta$ to the inclusion $L \hookrightarrow \theta_\mathcal{J}L$, we get an inclusion of $\theta L$ into $\theta\theta_\mathcal{J}L$. The composition $\theta\theta_\mathcal{J}$ belongs to the additive closure of $\theta_\mathcal{J}$, modulo projective functors from strictly higher two-sided cells. The latter projective functors annihilate $L$ because of our assumption on the annihilator of $L$. This means that $\theta L$ is a submodule of $\theta' L$, for some $\theta'$ in the additive closure of $\theta_\mathcal{J}$. Therefore, if we can prove Theorem 22 for $\theta = \theta_\mathcal{J}$, it follows that Theorem 22 is true for all $\theta$.

## 5.4 | Proof of Theorem 22

Unfortunately, $\mathscr{M}$ does not have arbitrary products, which is a technical obstacle for our coming arguments that we need to deal with.

For $k \in \mathbb{Z}_{>0}$ and a central character $\chi$, denote by $\mathscr{M}^k_\chi$ the full subcategory of $\mathscr{M}_\chi$ that consists of all modules annihilated by the $k$th power of the kernel of $\chi$. Let $\mathscr{M}^k$ be the product of all $\mathscr{M}^k_\chi$. Then, by [25, Theorem 5.1], for a projective functor $\theta$ and $k \in \mathbb{Z}_{>0}$, there is $m \in \mathbb{Z}_{>0}$ such that $\theta$ maps $\mathscr{M}^k$ to $\mathscr{M}^m$. Note that $\mathscr{M}^k$ has arbitrary limits, for all $k$.

Now let $L \in \mathscr{N}$ be a simple module and $\theta$ a projective endofunctor of $\mathscr{N}$ that belongs to $\mathcal{J}$. As we only have finitely many indecomposable projective endofunctors of $\mathscr{N}$, we can fix $V$ such that all indecomposable projective endofunctors of $\mathscr{N}$ are direct summands of the projective functor $V \otimes_\mathbb{C}$ _. In particular, by the additivity of the Bernstein number with respect to short exact sequences, see Subsection 2.5, for any filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = \theta L,$$

the number of $i$ such that $\mathrm{GKdim}(M_i/M_{i-1}) = \mathrm{GKdim}(L)$ cannot be greater than $\dim(V) \cdot \mathrm{BN}(L)$.

Let $N$ be a maximal semisimple submodule of $\theta L$. We know, see Subsection 2.5, that it has finite length. Assume that it is not essential and let $K$ be a nonzero submodule of $\theta L$ such that $K \cap N = 0$. Then, $K$ has no simple submodule. From the previous paragraph, we may further assume that any quotient of $K$ by a nonzero submodule has Gelfand–Kirillov dimension strictly smaller than $\mathrm{GKdim}(L)$. Indeed, if $K$ has a nonzero submodule $K'$ such that $\mathrm{GKdim}(K/K') = \mathrm{GKdim}(L)$, we can simply replace $K$ by $K'$. After at most $\dim(V) \cdot \mathrm{BN}(L)$ replacements, we obtain a $K$ with the desired property. In particular, $K$ is a strange submodule of $\theta L$.

First of all, we note that, by adjunction,

$$0 \neq \mathrm{Hom}_{\mathfrak{g}}(K, \theta L) \cong \mathrm{Hom}_{\mathfrak{g}}(\theta^* K, L),$$

in particular, $\theta^* K \neq 0$.

On the other hand, we want to show that $\theta^* K = 0$ and in this way get a contradiction. For example, in type $A$, $\theta^* K = 0$ follows from Corollary 20. To prove $\theta^* K = 0$ in general (but, under the additional, compared to Corollary 20, assumption that $L$ is holonomic), consider the filtered diagram $\mathcal{P}$ of quotients of $K$ by nonzero submodules with respect to natural projections. The kernel of the natural map from $K$ to the limit $\lim_{\leftarrow} \mathcal{P}$ equals the intersection of all nonzero submodules of $K$. That is zero, as $K$ does not have simple submodules, in particular, it does not have a simple socle.

As $\theta^*$ has a biadjoint, we have $\theta^* \lim_{\leftarrow} \mathcal{P} \cong \lim_{\leftarrow} \theta^* \mathcal{P}$. At the same time, the Gelfand–Kirillov dimension of any $X \in \mathcal{P}$ is strictly smaller than $\mathrm{GKdim}(L)$, by our assumption on $K$. Since $\theta \in \mathcal{J}$, we have $\theta^* \in \mathcal{J}$ and hence $\theta^* X = 0$ by our assumption that $L$ is holonomic. This means that $\lim_{\leftarrow} \theta^* \mathcal{P} = 0$, which implies that $\theta^* K = 0$, a contradiction. This proves that such $K$ cannot exist and completes the proof of Theorem 22.

# 6 | BEYOND HOLONOMIC MODULES OUTSIDE TYPE $A$

## 6.1 | Results

As already mentioned in the introduction, in type $A$, the assertion of Theorem 22 is true for all simple modules $L$, not necessarily holonomic ones, see [10, Theorem 23]. The main reason why this works is the combinatorial property of type $A$ that each two-sided KL-cell contains the longest element $w_0^{\mathfrak{p}}$ of the Weyl group of some parabolic subalgebra $\mathfrak{p}$. We can generalize [10, Theorem 23] as follows.

**Theorem 24.** *Let $\mathfrak{g}$ be a semisimple classical finite-dimensional Lie algebra over $\mathbb{C}$. Let $L$ be a simple $\mathfrak{g}$-module such that the two-sided KL-cell $\mathcal{J}$ that contains the left KL-cell corresponding to the annihilator of $L$ in $U(\mathfrak{g})$ contains some $w_0^{\mathfrak{p}}$. Let $V$ be a finite-dimensional $\mathfrak{g}$-module. Then, the $\mathfrak{g}$-module $V \otimes_{\mathbb{C}} L$ has an essential semisimple submodule of finite length.*

In the same setup, we also prove both Conjectures 5 and 9.

**Theorem 25.** *Let $\mathfrak{g}$ be a semisimple classical finite-dimensional Lie algebra over $\mathbb{C}$. Let $L$ be a simple $\mathfrak{g}$-module such that the two-sided KL-cell $\mathcal{J}$ that contains the left KL-cell corresponding to the annihilator of $L$ in $U(\mathfrak{g})$ contains some $w_0^{\mathfrak{p}}$. For such $L$, the assertions of both Conjectures 5 and 9 are true.*

## 6.2 | Proof of Theorem 24

We follow the idea of the proof of [10, Theorem 23], which is also utilized, in a slightly disguised way, in Section 4. Here is a sketch of this idea.

- Due to our assumption on $\mathcal{J}$, we can translate $L$ to a singular block whose singularity corresponds to our longest element.
- The indecomposable projective endofunctors of that singular block that do not kill $L$ form a group (modulo projective functors that kill $L$), in particular, they are invertible. So, for all such projective functors, the claim of Theorem 24 is straightforward.
- The assertion of Theorem 24 is equivalent to saying that $V \otimes_{\mathbb{C}} L$ has no strange submodules. If we assume that this is wrong, then we can translate a strange submodule of $V \otimes_{\mathbb{C}} L$ to the singular block from above, which leads to a contradiction with the previous item.

The first item on the above list goes mutatis mutandis as in Subsection 4.2. We note that, outside type $A$, two-sided KL-cells do not have to contain any longest element for some parabolic subgroup. However, we explicitly assume this for our $\mathcal{J}$, which allows us to use the approach of Subsection 4.2. This approach leads to the following output: starting from $L$ with some central character $\chi$ corresponding to a dominant weight $\lambda$, we find a singular weight $\lambda'$ with the corresponding central character $\chi'$ such that the singularity $W'$ of $\lambda'$ in $W_{\lambda'}$ is a parabolic subgroup and is isomorphic to the Weyl group of our longest element in the formulation. We also find a simple module $L'$ with the same annihilator as $L(\lambda')$ and such that the additive closure of all $\theta L$, where $\theta \in \mathcal{J}$, coincides with the additive closure of all $\theta L'$, where $\theta \in \mathcal{J}$. Therefore, we can forget about $L$ and concentrate on $L'$.

For the second item on the above list, let us assume that $\lambda'$ is a singular weight with singularity $W'$ (which is a parabolic subgroup of $W_{\lambda'}$). Let $\chi'$ be the central character of $L(\lambda')$. Consider the bicategory $\mathscr{P}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})$ and the bi-ideal $\mathscr{J}_{\chi'}$ in it generated by all indecomposable objects that are not two-sided equivalent to the identity.

**Lemma 26.** *Any indecomposable object of $\mathscr{P}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})/\mathscr{J}_{\chi'}$ is invertible.*

*Proof.* Arguments similar to the ones used in the proof of Lemma 16 reduce the necessary statement to the similar statement for $\mathscr{P}^{\Xi}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})/(\mathscr{P}^{\Xi}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'}) \cap \mathscr{J}_{\chi'})$.

We can translate singular projective functors out of the wall all the way to the corresponding regular blocks. We can also translate back. Translating out and then back gives $|W'|$ copies of what we started with, with one copy in degree zero and all other copies shifted in positive degrees, see [12, Proposition 4.1]. Since the endomorphism algebra of the multiplicity-free direct sum of all indecomposable objects of the bicategory of projective functors with tops concentrated in degree zero is positively graded, see [1], it follows that $\mathscr{P}^{\Xi}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})/(\mathscr{P}^{\Xi}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'}) \cap \mathscr{J}_{\chi'})$ is biequivalent to the asymptotic category associated with the $H$-cell of $\mathcal{J}^{\Xi}$ that contains our longest element (for the definition of this asymptotic category, see, for example, [29, Subsection 3.2]).

As explained in [29, Section 8], since we assume $\mathfrak{g}$ to be classical, all the asymptotic bicategories that appear are biequivalent to the bicategory of finite-dimensional vector spaces graded by a finite group. In the latter, all indecomposable objects are invertible. The claim of the lemma follows. □

Finally, to justify the last item on the above list, let $M$ be a strange submodule of $V \otimes_{\mathbb{C}} L$. Then, for any projective functor $\theta$, the module $\theta M$ cannot have simple submodules of the same

GK-dimension as $L$. Indeed, if $\tilde{L}$ were such a submodule, then, by adjunction,

$$0 \neq \mathrm{Hom}_{\mathfrak{g}}(\tilde{L}, \theta M) \cong \mathrm{Hom}_{\mathfrak{g}}(\theta^* \tilde{L}, M)$$

and we would get a contradiction, since all quotients of $\theta^* \tilde{L}$ have the same GK-dimension as $L$ while all quotients of $M$ by nonzero submodules have strictly smaller GK-dimension.

Suppose that $M'$ is a nonzero submodule of $\theta M$. By the additivity of the Bernstein number, $M'$ can only have finitely many simple subquotients of the same GK-dimension as $L$, say $L_1, L_2, \dots, L_k$ (counted with the respective multiplicities). Let $I_i$ be the indecomposable injective hull of $L_i$. The embeddings $L_i \subset I_i$ give rise to a map from $M'$ to $I_1 \oplus I_2 \oplus \cdots \oplus I_k$. This map cannot be injective since none of the $L_i$ is a submodule of $M'$ by the previous paragraph. The kernel of this map is thus a strange submodule of $M'$.

Consequently, any nonzero submodule of $\theta M$ has a strange submodule. This means that we can translate our $M$ to our singularity $\chi'$ and obtain that $\theta' L'$ must have a strange submodule for some $\theta' \in \mathscr{P}^{\Xi}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})$. At the same time, by Lemma 26, all indecomposable summands of $\theta'$ are invertible, which implies that $\theta' L'$ is semisimple of finite length, a contradiction. This completes the proof of Theorem 24.

### 6.3 | Proof of Theorem 25

In the setup of Theorem 25, the assertion of Conjecture 9 follows from Theorem 8. Indeed, if we look at the proof of Theorem 24, the application of projective functors to $L'$ produces semisimple modules of finite length, if we restrict to the central character $\chi'$. The simple constituents of these semisimple modules, obviously, form an equivalence class with respect to $\rhd$. Hence, the assumptions of Theorem 8 are satisfied, so this theorem applies.

To prove Conjecture 5 in the setup of Theorem 25, we note that Theorem 11 already guarantees that $\mathbf{Y}^L$ is transitive. Since the underlying category $\mathbf{Y}^L(\mathtt{i}_{\chi'})$ is semisimple, $\mathbf{Y}^L(\mathtt{i}_{\chi'})$ is simple transitive as a birepresentation of $\mathscr{P}(\mathtt{i}_{\chi'}, \mathtt{i}_{\chi'})$. Since we also have $\mathbf{Y}^L = \mathbf{Y}^{L'}$ by Theorem 11, any socle constituent of any object in $\mathbf{Y}^L$ can be translated, using adjunction, back to $\mathbf{Y}^L(\mathtt{i}_{\chi'})$ in a nonzero way. If $\mathbf{Y}^L$ were not simple transitive, the kernel of the projection from $\mathbf{Y}^L$ onto its unique simple transitive quotient would kill some socle constituent of some object in $\mathbf{Y}^L$. Translating to $\mathbf{Y}^L(\mathtt{i}_{\chi'})$, we would be forced to kill a nonzero object of this category, contradicting its simple transitivity. This implies that already $\mathbf{Y}^L$ is simple transitive.

We note that the result proved in the previous paragraph can also be obtained using the results of [28, Subsection 4.8].

## 7 | STRANGE SUBQUOTIENTS, SERRE QUOTIENTS, AND ROUGH STRUCTURE

### 7.1 | Strange subquotients

Let $L$ be a simple $\mathfrak{g}$-module and $V$ a finite-dimensional $\mathfrak{g}$-module. The module $V \otimes_{\mathbb{C}} L$, clearly, does not have any strange quotients. Furthermore, Theorem 22 essentially says that $V \otimes_{\mathbb{C}} L$ does not have any strange submodules, provided that $L$ is holonomic. A priori, we cannot rule out existence of strange subquotients. However, inspired by Theorem 22, we propose the following conjecture.

**Conjecture 27.** *Strange subquotients of $V \otimes_{\mathbb{C}} L$ do not exist.*

Let $M$ be a strange subquotient of $V \otimes_{\mathbb{C}} L$. Let $N$ denote the sum of all submodules of $M$ whose GK-dimension is strictly smaller than $\mathrm{GKdim}(L)$. Since $V \otimes_{\mathbb{C}} L$ and hence also $M$ are noetherian, $N$ is finitely generated. Since each of the finitely many generators of $N$ belongs to a submodule of $M$ whose GK-dimension is strictly smaller than $\mathrm{GKdim}(L)$, it follows that $\mathrm{GKdim}(N) < \mathrm{GKdim}(L)$. Therefore, the subquotient $M/N$ is also strange and has the property that any submodule of $M/N$ has GK-dimension $\mathrm{GKdim}(L)$. We will say that $M/N$ is a strange subquotient in *normal form*. Note that strange subquotients in normal form do not have simple submodules at all.

**Proposition 28.** *Let $L$ be holonomic, $M$ a strange subquotient of $V \otimes_{\mathbb{C}} L$, and $\theta$ an indecomposable projective functor from the $\mathcal{J}$-cell corresponding to $\mathrm{Ann}_{U(\mathfrak{g})}(L)$. Then $\theta M = 0$.*

*Proof.* This is proved using the same argument as at the end of Subsection 5.4. □

## 7.2 | Serre quotients

Let $L$ be a simple $\mathfrak{g}$-module. In general, the module $V \otimes_{\mathbb{C}} L$ need not have finite length in $\mathfrak{g}$-mod. In this subsection, we introduce a natural subquotient of $\mathfrak{g}$-mod where $V \otimes_{\mathbb{C}} L$ always has finite length and a well-defined notion of composition multiplicities.

Let $\mathscr{A} = \mathscr{A}(L)$ denote the full subcategory of $\mathfrak{g}$-mod whose objects are all finitely generated $\mathfrak{g}$-modules isomorphic to subquotients of modules of the form $V \otimes_{\mathbb{C}} L$, where $V$ is a finite-dimensional $\mathfrak{g}$-module. This is an abelian subcategory of $\mathfrak{g}$-mod with the abelian structure (e.g., $\mathbb{C}$-linearity, kernels, and cokernels) being inherited from $\mathfrak{g}$-mod. Thanks to exactness of projective functors, the category $\mathscr{A}$ comes equipped with the natural action of projective functors.

Let $\mathscr{B} = \mathscr{B}(L)$ denote the full subcategory of $\mathscr{A}$ consisting of all modules of Gelfand–Kirillov dimension strictly smaller than $\mathrm{GKdim}(L)$. From Subsection 2.5, it follows that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$ as well as that $\mathscr{B}$ is stable under the action of projective functors. Therefore, $\mathscr{A}/\mathscr{B}$ is an abelian category that has a natural action of projective functors.

Let $\mathscr{C} = \mathscr{C}(L)$ denote the full subcategory of $\mathscr{A}$ consisting of all objects $M$ of $\mathscr{A}$ that have the property that $\theta M = 0$, for any indecomposable projective functor $\theta$ from the two-sided cell $\mathcal{J}$ corresponding to $\mathrm{Ann}_{U(\mathfrak{g})}(L)$. Since projective functors are exact, $\mathscr{C}$ is a Serre subcategory of $\mathscr{A}$.

**Lemma 29.** *The category $\mathscr{C}$ is stable under the action of projective functors.*

*Proof.* Let $\theta'$ be a projective functor and $\theta$ be a projective functor from $\mathcal{J}$. Then, any indecomposable summand in both $\theta\theta'$ and $\theta'\theta$ is either in $\mathcal{J}$ or annihilates $L$. This implies the claim of the lemma. □

Due to Lemma 29, the category $\mathscr{A}/\mathscr{C}$ is an abelian category that has a natural action of projective functors.

**Conjecture 30.** $\mathscr{B} = \mathscr{C}$.

Note that, if $L$ is holonomic, we have $\mathscr{B} \subset \mathscr{C}$.

## 7.3 | Rough structure of modules in $\mathscr{A}$

**Theorem 31.** *Assume that $L$ is holonomic.*

(*a*) *The category $\mathscr{A}/\mathscr{C}$ is an abelian length category.*
(*b*) *Simple objects in $\mathscr{A}/\mathscr{C}$ are in bijection with isomorphism classes of simple subquotients with GK-dimension* $\mathrm{GKdim}(L)$ *in modules of the form $V \otimes_{\mathbb{C}} L$, where $V$ is a finite-dimensional $\mathfrak{g}$-module.*
(*c*) *Every object in $\mathscr{A}/\mathscr{C}$ has well-defined composition multiplicities.*

As suggested in [38], for $X \in \mathscr{A}$, the part of the structure of $X$ that can be seen in $\mathscr{A}/\mathscr{C}$ (including the multiplicities in $X$ of simple $\mathfrak{g}$-modules with GK-dimension $\mathrm{GKdim}(L)$) is called the *rough structure* of $X$.

*Proof of Theorem* 31. A subquotient $X$ of some $V \otimes_{\mathbb{C}} L$ will be called *primitive* provided that, for any submodule $Y \subset X$, at most one of the modules $Y$ or $X/Y$ has GK-dimension $\mathrm{GKdim}(L)$.

Given a primitive subquotient $X$ of some $V \otimes_{\mathbb{C}} L$, the definitions of $\mathscr{A}$, $\mathscr{C}$ and the Serre quotient give us three options.

- The GK-dimension of $X$ is strictly smaller than $\mathrm{GKdim}(L)$. In this case, $X = 0$ in $\mathscr{A}/\mathscr{C}$.
- The module $X$ is strange. In this case, $X = 0$ in $\mathscr{A}/\mathscr{C}$.
- The module $X$ has a unique simple subquotient $X'$ of GK-dimension $\mathrm{GKdim}(L)$. In this case, $X = X'$ in $\mathscr{A}/\mathscr{C}$.

This implies Claim (b).

The category $\mathscr{A}/\mathscr{C}$ is abelian by construction. That every object in $\mathscr{A}/\mathscr{C}$ has finite length follows from the additivity of the Bernstein number, since it is a positive integer and the Bernstein number of $V \otimes_{\mathbb{C}} L$ is finite. This implies Claim (a).

Let $X$ and $Y$ be in $\mathscr{A}$ such that $Y$ is a simple $\mathfrak{g}$-module of GK-dimension $\mathrm{GKdim}(L)$. Let $I_Y$ be the injective envelope of $Y$ in $\mathfrak{g}$-Mod. Let

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

be a filtration of $X$ such that all subquotients are primitive (since the number of possible subquotient of GK-dimension $\mathrm{GKdim}(L)$ is bounded, such a filtration exists). Then $X_i/X_{i-1}$ has $Y$ as a subquotient if and only if there is homomorphism from $X_i/X_{i-1}$ to $I_Y$. As $X_i/X_{i-1}$ is assumed to be primitive, the dimension of $\mathrm{Hom}_{\mathfrak{g}}(X_i/X_{i-1}, I_Y)$ equals one (since the endomorphism algebra of $Y$ has dimension one by Dixmier–Schur's lemma). Therefore, the composition multiplicity of $Y$ in $X$ is finite and equals the dimension of $\mathrm{Hom}_{\mathfrak{g}}(X, I_Y)$. □

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