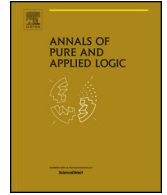Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Positive modal logic beyond distributivity

Nick Bezhanishvili [a],[*], Anna Dmitrieva [b], Jim de Groot [c], Tommaso Moraschini [d]

[a] *University of Amsterdam, the Netherlands*
[b] *University of East Anglia, United Kingdom*
[c] *The Australian National University, Canberra, Ngunnawal Country, Australia*
[d] *Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain*

A R T I C L E   I N F O

A B S T R A C T

We develop a duality for (modal) lattices that need not be distributive, and use it to study positive (modal) logic beyond distributivity, which we call weak positive (modal) logic. This duality builds on the Hofmann, Mislove and Stralka duality for meet-semilattices. We introduce the notion of Π_1-persistence and show that every weak positive modal logic is Π_1-persistent. This approach leads to a new relational semantics for weak positive modal logic, for which we prove an analogue of Sahlqvist's correspondence result.[1]

1. Introduction

Dualities between modal algebras and modal spaces on the one hand, and Heyting algebras and Esakia spaces on the other have been central to the study of modal and intermediate logics [6,8]. Indeed, many important results such as Sahlqvist canonicity and correspondence [44] can be understood through the lenses of duality techniques [45]. The duality between modal algebras and modal spaces has been extended to a duality between modal distributive lattices and modal Priestley spaces in [25,7]. This led to a Sahlqvist theory for the positive distributive modal logic introduced in [15,20,7].

---

* Corresponding author.
E-mail addresses: n.bezhanishvili@uva.nl (N. Bezhanishvili), a.dmitrieva@uea.ac.uk (A. Dmitrieva), jim.degroot@anu.edu.au (J. de Groot), tommaso.moraschini@ub.edu (T. Moraschini).
[1] This paper is partially based on the Master's thesis [14].

When the algebraic side of a duality consists of distributive lattice expansions, in the spatial side of the duality one often works with the Priestley space [41,11] of all the prime filters of a given lattice. This is no longer the case when the base lattice is non-distributive. There are many extensions of dualities for Boolean algebras and distributive lattices to the setting of all lattices, e.g. by Urquhart [46], Hartonas [27,28], Gehrke and Van Gool [22], Goldblatt [26], and Hartung [29]. Each of these uses either a ternary relation, or two-sorted frames. While these approaches have proven fruitful and interesting, they are quite different from known dualities for propositional logics such as Stone and Priestley dualities. This makes it difficult to adjust existing tools and techniques from distributive logics to non-distributive ones.

Hofmann, Mislove and Stralka (HMS) [32] developed a duality for (not necessarily distributive) meet-semilattices along the same lines of the van Kampen-Pontryagin duality for locally compact abelian groups given in [42]. This was later restricted to a duality for lattices by Jipsen and Moshier [40]. In this approach, the dual space of a lattice is based not on the prime filters, but on all proper filters. This is closely related to Holliday's possibility semantics of modal logic [34] (see also [33]) and to the choice-free duality for Boolean algebras in [4], which are also based on spaces of all proper filters. A similar approach was developed for ortholattices by Goldblatt [24] and extended later by Bimbó [5].

The aim of this paper is to investigate positive modal logic that is not necessarily distributive. We refer to this as *weak positive modal logic*. It is a logic with the same language as positive modal logic, i.e. the negation- and implication-free fragment of classical modal logic, which does not necessarily satisfy the distributivity axiom.

We study these logics via a duality that builds on HMS duality. We recall that a Priestley space is a partially ordered compact space satisfying the Priestley separation axiom

$$x \nleq y \text{ implies that there is a clopen upset } U \text{ such that } x \in U \text{ and } y \notin U.$$

These spaces provide a duality for bounded distributive lattices, which associates every Priestley space with the lattice of its clopen upsets. In HMS duality, one works with similar structures, but the role of partially ordered compact spaces is played by meet-semilattices with a compact topology, and that of clopen upsets by clopen filters. Then the HMS analogue of the Priestley separation axiom is

$$x \nleq y \text{ implies that there is a clopen filter } U \text{ such that } x \in U \text{ and } y \notin U.$$

These spaces provide a duality for bounded meet-semilattices.

Our approach is analogous to the one of Esakia duality for Heyting algebras. Recall that an Esakia space is a Priestley space where for every pair of clopen upsets U and V the Heyting implication $U \to V$ is also a clopen upset (this is sometimes formulated as the equivalent condition that $\downarrow U$ is clopen for every clopen U) [16,17]. In analogy with this, an HMS space is said to be a *lattice space* if the join in the lattice of filters of every pair U and V of clopen filters (i.e. $\{x \mid x \geq a \wedge b \text{ for some } a \in U \text{ and } b \in V\}$) is also a clopen filter.

We extend this duality to modal lattices in the signature with two unary modalities, $\Box$ and $\Diamond$. More precisely, by a modal lattice we understand a lattice with a top element $\top$ and two modalities related via Dunn's axiom $\Diamond x \wedge \Box y \leq \Diamond(x \wedge y)$ [15] and satisfying the equations $\Box\top \approx \Diamond\top \approx \top$. Furthermore, while $\Box$ will be assumed to distribute over finite meets, we require $\Diamond$ to be merely monotone. A similar phenomenon in the context of modal intuitionistic logic has been investigated in [38]. Despite the asymmetry between $\Box$ and $\Diamond$, on the dual side these modalities are interpreted via a binary relation in the standard way.

This duality allows us to define a new relational semantics for weak positive modal logics in which the analogue of a Kripke frame is a meet-semilattice with an extra relation. The meet gives rise to a partial order, so these frames can be viewed as bi-relational frames where the relations satisfy certain conditions. Formulae are interpreted as filters, disjunction is interpreted as the least filter generated by the interpretation of each disjunct, and modalities are interpreted in the standard way. This new semantics can be seen as a generalisation of the team semantics of [31] and of the modal information semantics of [3,37].

Kripke semantics for intuitionistic and modal logic is tightly related to the theory of canonical extensions [35,18,19]. This is largely due to the fact that a formula is valid in the Kripke frame associated with a Heyting or modal algebra A if and only if it is valid in the canonical extension of A. In our case, the role of canonical extensions is played by Gehrke and Priestley's Π_1-completions [21]. This is because a formula is valid in the Kripke frame associated with a modal lattice A by our duality iff it is valid in the Π_1-completion of A. Notably, the Π_1-completion of A can be described concretely as the modal lattice of all filters of the lattice of filters of A (or, equivalently, as the composition of the filter and the ideal completions of A).

Our main results are Sahlqvist-style preservation and correspondence results for weak positive modal logic with respect to this new semantics. Using a duality technique similar to that of Sambin and Vaccaro [45], we show that every sequent is preserved by Π_1-completion. Note that in the propositional setting this corresponds to the fact that every variety of lattices is closed under ideal completions and filter completions [43,2].

We also prove an analogue of the Sahlqvist correspondence result. In particular, we introduce Sahlqvist sequents in our language and show that every Sahlqvist sequent has a first-order correspondent. Furthermore, we introduce the notion of Π_1-persistence for weak positive modal logics, which is a logical analogue for the corresponding class of algebra to be closed under Π_1-completions, and we show that every weak positive modal logic is Π_1-persistent. As a result every weak positive modal logic is complete with respect to our relational semantics. We point out that an alternative approach to Sahlqvist correspondence and canonicity for non-distributive logics has been undertaken in [10], although this perspective is based on canonical extensions and is, therefore, orthogonal to the one developed in this paper.

With this paper we hope to lay a groundwork for a theory of weak positive modal logics. As discussed in the conclusion, there are many interesting directions for future research. These include the study of logics that lie between non-distributive and distributive positive (modal) logic, deriving more results for the weak modal logic presented in this paper, as well as extending weak positive logic with different types of modalities.

2. Preliminaries

We briefly recall a Stone-type duality for the category of meet-semilattices with top due to Hofmann, Mislove and Stralka [32]. We then restrict this to a duality for lattices, and show how it relates to various completions of lattices.

2.1. Dual adjunctions

2.1 Definition. By a *semilattice* we mean a meet-semilattice with top. Every semilattice $(X, \top, \wedge)$ has an underlying partial order $\leq$ given by $x \leq y$ iff $x \wedge y = x$. A *(semilattice) homomorphism* from $(X, \top, \wedge)$ to $(X', \top', \wedge')$ is a function $f : X \to X'$ such that $f(\top) = \top'$ and $f(x \wedge y) = f(x) \wedge' f(y)$ for all $x, y \in X$. We write MSL for the category of semilattices and homomorphisms.

Similarly, by a *lattice* we mean a bounded lattice, and lattice homomorphisms are assumed to preserve these bounds. We write Lat for the category of lattices and lattice homomorphisms.

If $(X, \leq)$ is a partial order (possibly coming from a semilattice $(X, \top, \wedge)$) and $a \subseteq X$ then we define the *upward closure* of a by $\uparrow a := \{y \in X \mid x \leq y \text{ for some } x \in a\}$. The set a is called *upward closed* or an *upset* if $\uparrow a = a$. If $a = \{x\}$ then we write $\uparrow x$ instead of $\uparrow\{x\}$. The *downward closure* and *downsets* are defined similarly.

2.2 Definition. A *filter* of a semilattice $(X, \top, \wedge)$ is a nonempty upset $p \subseteq X$ that is closed under meets. It is called *principal* if $p = \uparrow x$ for some $x \in X$.
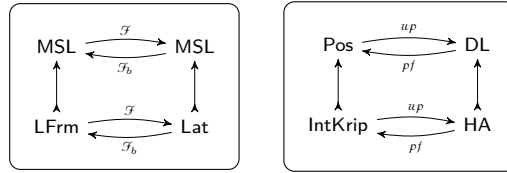
Fig. 1. Functors between categories of (semi)lattices. The upper rows are dual adjunctions. The contravariant functor *up* takes a poset to its lattice of upsets, and *pf* takes a lattice to its ordered set of prime filters.

Filters of $(X, \top, \wedge)$ correspond bijectively to homomorphisms to the two-element semilattice $2 = \{\top, *\}$: every filter p yields a characteristic map $\chi_p : X \to 2$ given by $\chi_p(x) = \top$ iff $x \in p$, and conversely for every homomorphism $f : X \to 2$, $f^{-1}(\top)$ is a filter. For every semilattice $(X, \top, \wedge)$, the collection $\mathscr{F}(X, \top, \wedge)$ of filters forms a complete semilattice ordered by subset inclusion. It is then easy to see that the filter X is the largest element in $\mathscr{F}(X, \top, \wedge)$ and that the greatest lower bound of a collection of filters is given by their intersection. Therefore it is also a (complete) lattice. The bottom element of $\mathscr{F}(X, \top, \wedge)$ is $\{\top\}$. Binary joins are given by

$$p \triangledown q = \uparrow\{x \wedge y \mid x \in p, y \in q\},$$

and the join of any set $F \subseteq \mathscr{F}(X, \top, \wedge)$ can be given by $\bigvee F = \bigcup\{p_1 \triangledown \cdots \triangledown p_n \mid p_1, \dots, p_n \in F\}$.

This assignment $\mathscr{F}$ extends to a contravariant functor $\mathscr{F} : \mathsf{MSL} \to \mathsf{MSL}$ if we define its action on a homomorphism $f : (X, \top, \wedge) \to (X', \top', \wedge')$ by $\mathscr{F}f = f^{-1} : \mathscr{F}(X', \top', \wedge') \to \mathscr{F}(X, \top, \wedge)$. For a semilattice $(X, \top, \wedge)$, let

$$\eta_{(X, \top, \wedge)} : (X, \top, \wedge) \to \mathscr{F}\mathscr{F}(X, \top, \wedge) : x \mapsto \{p \in \mathscr{F}(X, \top, \wedge) \mid x \in p\}.$$

This yield a natural transformation $\eta : id_{\mathsf{MSL}} \to \mathscr{F}\mathscr{F}$ that satisfies $\mathscr{F}\eta \circ \eta_{\mathscr{F}} = if_{\mathscr{F}}$. Therefore:

2.3 Proposition. *The functor $\mathscr{F}$ establishes a dual adjunction between* MSL *and* MSL *with both units given by η.*

In preparation for using semilattices as interpretation for weak positive logic (Section 3), and for the duality for lattices that we derive in Theorem 2.14, we restrict the functor $\mathscr{F} : \mathsf{MSL} \to \mathsf{MSL}$ to functors $\mathsf{Lat} \to \mathsf{LFrm}$ and $\mathsf{LFrm} \to \mathsf{Lat}$. One occurrence of MSL is restricted to Lat, while the other occurrence is restricted to the category of semilattices and so-called L-morphisms. The situation is analogous to that for intuitionistic logic, where the functors establishing the dual adjunction between distributive lattices and posets restrict to contravariant functors between the categories of Heyting algebras and intuitionistic Kripke frames (see Fig. 1).

2.4 Definition. An *L-morphism* between semilattices $(X, \top, \wedge)$ and $(X', \top', \wedge')$ is a semilattice homomorphism $f : (X, \top, \wedge) \to (X', \top', \wedge')$ that satisfies for all $x \in X$ and $y', z' \in X'$:

- If $f(x) = \top'$ then $x = \top$;

- If $y' \wedge z' \le f(x)$ then $\exists y, z \in X$ s.t. $y' \le' f(y)$ and $z' \le' f(z)$ and $y \wedge z \le x$. In a picture:
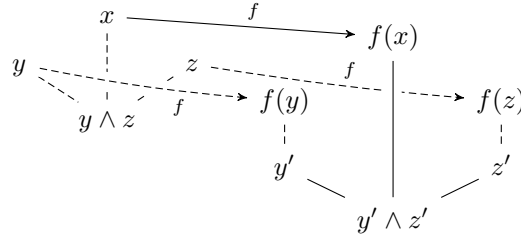
We write LFrm for the category of semilattices and L-morphisms.

The category LFrm will be used in Section 3 as frame semantics for weak positive logic.

2.5 Proposition. *If $f : (X, \top, \wedge) \to (X', \top', \wedge')$ is an L-morphism, then $f^{-1} : \mathscr{F}(X', \top', \wedge') \to \mathscr{F}(X, \top, \wedge)$ is a lattice homomorphism.*

Proof. We know that f^{-1} is a semilattice homomorphism. The map f^{-1} preserves the bottom element because $f^{-1}(\{\top'\}) = \{\top\}$. For preservation of joins, we need to show that

$$f^{-1}(a' \bigtriangledown b') = f^{-1}(a') \bigtriangledown f^{-1}(b') \tag{1}$$

for $a', b' \in \mathscr{F}(X', \top', \wedge')$. The inclusion $\supseteq$ follows from the fact that $f^{-1}(a' \bigtriangledown b')$ is a filter that contains both $f^{-1}(a')$ and $f^{-1}(b')$. Conversely, if $x \in f^{-1}(a' \bigtriangledown b')$ then $f(x) \in a' \bigtriangledown b'$ so there exist $y' \in a'$ and $z' \in b'$ such that $y' \wedge z' \le f(x)$. Since f is an L-frame morphism, we can find $y, z \in X$ such that $y' \le f(y)$ and $z' \le f(z)$ and $y \wedge z \le x$. This means that $y \in f^{-1}(a')$ and $z \in f^{-1}(b')$, and hence $x \in f^{-1}(a') \bigtriangledown f^{-1}(b')$. $\square$

It follows that $\mathscr{F}$ restricts to a contravariant functor $\mathscr{F} : \mathsf{LFrm} \to \mathsf{Lat}$.

2.6 Proposition. *Let $h : L \to L'$ be a lattice homomorphism. Then $h^{-1} : \mathscr{F}L' \to \mathscr{F}L$ is an L-morphism.*

Proof. We know that h^{-1} is a semilattice homomorphism, so we only have to show that it satisfies the additional conditions from Definition 2.4. For the first one, suppose $h^{-1}(p') = L$ (L is the top element of $\mathscr{F}L$). Then $\bot \in h^{-1}(p')$, so $\bot' = h(\bot) \in p'$, and therefore $p' = L'$.

Next, let $p' \in \mathscr{F}L'$ and $q, r \in \mathscr{F}L$ and suppose $q \cap r \subseteq h^{-1}(p')$. Let $q' := \mathord{\uparrow} h[q]$ and $r' := \mathord{\uparrow} h[r]$. Then it is easy to verify that q' and r' are filters (because q and r are), and by construction $q \subseteq h^{-1}(q')$ and $r \subseteq h^{-1}(r')$. It remains to show that $q' \cap r' \subseteq p'$. Let $a' \in L'$ be such that $a' \in q' \cap r'$. Since $a' \in q'$ there exists $a \in q$ such that $h(a) \le a'$. Since $a' \in r'$ there exists $b \in r$ such that $h(b) \le a'$. But then $a \vee b \in q \cap r$, so by assumption $h(a \vee b) \in p'$. This implies $a' \in p'$, because $h(a \vee b) = h(a) \vee h(b) \le a'$ and p' is a filter (hence up-closed). $\square$

2.2. Dual equivalences

HMS duality is obtained from the dual adjunction in Proposition 2.3 by equipping one side with a Priestley topology.

2.7 Definition. An *HMS-space* is a tuple $\mathbb{X} = (X, \top, \wedge, \tau)$ such that $(X, \top, \wedge)$ is a semilattice, (X, τ) is a compact topological space, and $\mathbb{X}$ satisfies the *HMS separation axiom*:

for all $x, y \in X$, if $x \nleq y$ then there exists a clopen filter a such that $x \in a$ and $y \notin a$.

An *HMS-morphism* is a continuous semilattice homomorphism. We write HMS for the category of HMS-spaces and HMS-morphisms.

The HMS separation axiom is a variation of the Priestley separation axiom. It immediately implies that any HMS-space is Hausdorff. Furthermore, it can be shown that every HMS-space is zero-dimensional, i.e. every open neighbourhood of a point x contains a clopen neighbourhood of x. To see this, suppose b is an open neighbourhood of a point x in an HMS-space $\mathbb{X} = (X, \top, \wedge, \tau)$. Then for each $y \in X \setminus b$ either $x \not\leq y$ or $y \not\leq x$. By the HMS separation axiom, there exist a clopen filter or a complement of a clopen filter (which is clopen as well) containing x but not y. By construction, the intersection of these clopen neighbourhoods of x is contained in b. Since b is open and $\mathbb{X}$ is compact, there exists a finite number of such neighbourhoods whose intersection is contained in b. This finite intersection is the desired clopen neighbourhood of x. Thus, (X, τ) is a Stone space.

For future reference, we prove some properties of closed sets and filters of an HMS-space.

2.8 Lemma. *Let $\mathbb{X} = (X, \top, \wedge, \tau)$ be an HMS-space and $c \subseteq X$ a filter. Then (i) c is a closed iff (ii) c is principal iff (iii) c is the intersection of clopen filters.*

Proof. The implication (ii) $\Rightarrow$ (iii) follows from the HMS separation axiom and (iii) $\Rightarrow$ (i) is obvious. For (i) $\Rightarrow$ (ii), suppose c is not principal. Then for each $x \in c$ there exists a $y \in c$ strictly below x. So for each $x \in c$, using the HMS separation axiom, we can find a clopen filter a_x such that $c \not\subseteq a_x$. Then $\{a_x \mid x \in c\}$ is an open cover of c without finite subcover. (Indeed, for every finite collection $a_{x_1}, \ldots, a_{x_n}$ we can find $y_1, \ldots, y_n$ such that $y_i \in c$ but $y_i \notin a_i$. Then $y_1 \wedge \cdots \wedge y_n$ is in c but not in any of the a_{x_i}.) So c is not compact, hence not closed. $\quad\square$

2.9 Lemma. *Let c be a closed subset of an HMS-space $\mathbb{X} = (X, \wedge, \top, \tau)$. Then $\uparrow c$ is closed as well.*

Proof. If $y \notin \uparrow c$ then for each $x \in c$ we have $x \not\leq y$, hence a clopen filter a_x containing x but not y. Then $c \subseteq \bigcup_{x \in c} a_x$, so by compactness we find a finite subcover, say, $c \subseteq a_1 \cup \cdots \cup a_n$. Since all the a_i are upward closed, we have $\uparrow c \subseteq a_1 \cup \cdots \cup a_n$. By construction, none of the a_i contain y, so $X \setminus (a_1 \cup \cdots \cup a_n)$ is an open neighbourhood of y disjoint from $\uparrow c$. $\quad\square$

The clopen filters of an HMS-space form a semilattice with the whole space as top element and intersection as meet. This gives rise to a contravariant functor

$$\mathscr{F}_{clp} : \mathsf{HMS} \to \mathsf{MSL},$$

which sends HMS-morphisms to their inverse. In the converse direction, for every semilattice $(X, \top, \wedge)$ we can equip $\mathscr{F}(X, \top, \wedge)$ with a topology to obtain an HMS space, as follows.

2.10 Definition. Let A be a semilattice. Define $\mathscr{F}_{top} A = (\mathscr{F} A, A, \cap, \tau_A)$, where τ_A is the topology generated by

$$\{\theta_A(a) \mid a \in A\} \cup \{\theta_A(a)^c \mid a \in A\}, \tag{2}$$

with $\theta_A(a) = \{p \in \mathscr{F} A \mid a \in p\}$ and $\theta_A(a)^c = \mathscr{F} A \setminus \theta_A(a)$. Defining $\mathscr{F}_{top} h = h^{-1}$ for a semilattice homomorphism h, we obtain a contravariant functor $\mathscr{F}_{top} : \mathsf{MSL} \to \mathsf{HMS}$.

We now obtain (a reformulation of) the duality of Hofmann, Mislove and Stralka [32,12,9].

2.11 Theorem. *The functors $\mathscr{F}_{clp}$ and $\mathscr{F}_{top}$ establish a dual equivalence $\mathsf{HMS} \equiv^{\mathrm{op}} \mathsf{MSL}$.*

2.12 Remark. The following alternative proof for HMS duality was pointed out by the reviewer: The forgetful functor $\mathcal{U} : \mathsf{DL} \to \mathsf{MSL}$ from distributive lattices to meet-semilattices has a left adjoint $\mathcal{F}\iota : \mathsf{MSL} \to \mathsf{DL}$. It follows that, for any semilattice L, the hom sets $\mathrm{Hom}_{\mathsf{MSL}}(L, 2)$ and $\mathrm{Hom}_{\mathsf{DL}}(\mathcal{F}\iota L, 2)$ are naturally isomorphic. Using this, it is easy to see that the Priestley space dual to $\mathcal{F}\iota L$ coincides with the HMS space dual to L. We can then derive HMS duality for semilattices by observing that the Priestley spaces dual to the free distributive lattice over a semilattice are precisely those whose underlying poset forms a semilattice. ◁

We wish to restrict this to a duality for lattices. To this end, we restrict the category HMS to L-spaces and suitable morphisms.

2.13 Definition. A *lattice space* or *L-space* is an HMS-space $\mathbb{X} = (X, \top, \wedge, \tau)$ such that $a \triangledown b$ is clopen whenever a and b are clopen filters. An L-space morphism is a continuous L-morphism. We write LSpace for the category of L-spaces and their morphisms.

2.14 Theorem. *The duality for bounded semilattices from Theorem 2.11 restricts to a duality*

$$\mathsf{LSpace} \equiv^{\mathrm{op}} \mathsf{Lat}.$$

Proof. We only have to verify that the restriction of $\mathcal{F}_{clp}$ to LSpace lands in Lat, and the restriction of $\mathcal{F}_{top}$ to Lat lands in LSpace. The former follows from the fact that the clopen filters of an L-space are closed under $\triangledown$, together with Proposition 2.5.

For the latter, suppose that L is a lattice and let $\theta_L(a)$ and $\theta_L(b)$ be two arbitrary clopen filters of $\mathcal{F}_{top}L$. Writing x, y, z for elements in $\mathcal{F}_{top}L$, we have

$$\theta_L(a) \triangledown \theta_L(b) = \uparrow\{x \cap y \mid x \in \theta_L(a), y \in \theta_L(b)\} = \theta_L(a \vee b)$$

Let us elaborate on the last equality. If $x \in \theta_L(a)$ and $y \in \theta_L(b)$ then $a \in x$ and $b \in y$, so $a \vee b \in x \cap y$. So $z \supseteq x \cap y$ implies $a \vee b \in z$, and therefore we have "⊆". Conversely, if $z \in \theta_L(a \vee b)$ then we need to find $x \in \theta_L(a)$ and $y \in \theta_L(b)$ such that $x \cap y \subseteq z$. Let $x = \uparrow a \in \theta_L(a)$ and $y = \uparrow b \in \theta_L(b)$. Then $d \in x \cap y$ implies $a \leq d$ and $b \leq d$, hence $a \vee b \leq d$. Since $z \in \theta_L(a \vee b)$ this implies $d \in z$, and therefore $x \cap y \subseteq z$. This proves "⊇". The restriction on morphisms follows from Proposition 2.6. □

2.15 Remark. In [40], Moshier and Jipsen study a spectral analogue of Hofmann, Mislove and Stralka's duality for semilattices, which they also call HMS duality. Their "HMS spaces" relate to the original ones in the same way spectral spaces relate to Priestley spaces. Moshier and Jipsen also restrict their duality to lattices, obtaining what they call "BL spaces". Likewise, these are equivalent to our L-spaces through the same change of topology. Note that, while the join on BL spaces is defined via an infinite intersection of open filters (see [40, Section 3]), it coincides with the usual join of filters considered here.

In [14], Theorem 2.14 was proven with different terminology: L-spaces and -morphisms are called "PUP spaces" and "PUP morphisms," and the category LSpace is called PUP. ◁

2.3. Completions of lattices

We relate several completions of a lattice to collections of certain filters of its dual L-space.

2.16 Definition. A *completion* of a lattice L is a pair (e, C) where C is a complete lattice and $e : L \to C$ is a lattice embedding. An element in C is called *open* if it is the join of elements in the image of e, and *closed* if it is the meet of elements in the image of e.

A completion (e, C) is called *dense* if every element of C can be written as the join of meets of elements in L, and as the meet of joins of elements in L. It is called *compact* if for any set A of closed elements of C and B of open elements of C, $\bigwedge A \leq \bigvee B$ if and only if there are finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leq \bigvee B'$.

It is well known that every lattice has a dense and compact completion which is unique up to isomorphism, see e.g. [18, Propositions 2.6 and 2.7].

2.17 Definition. Let L be a lattice.

1. The *ideal completion* of L is the collection $ie\,L$ of ideals of L ordered by inclusion, with $i : L \to ie\,L$ given by $a \mapsto \downarrow a$. Meets in $ie\,L$ are given by intersection. As a consequence, the join of a collection of ideals is the smallest ideal containing their union.
2. The *filter completion* of L is the collection $fe\,L$ of filters of L ordered by reverse inclusion, with $i : L \to fe\,L : a \mapsto \uparrow a$. Then arbitrary joins in $\mathscr{F}L$ are given by intersections, and the meet of a collection of filters in $fe\,L$ is the smallest filter of L containing their union.
3. The *canonical extension* of L is the unique dense and compact completion of L.
4. The Π_1-*completion* of a lattice L is given by the composition of the ideal and the filter completion. That is, it consists of the lattice $ie(fe\,L)$ with inclusion $a \mapsto \{p \in fe\,L \mid a \in p\}$.

The Π_1-completion was studied in [21]. Note that the ideal and filter completions are closely related. If we denote by L° the lattice L with the order reversed, then the ideals of L correspond to the filters of L° and we get $ie\,L = (fe\,L^\circ)^\circ$.

A filter p of an L-space $\mathbb{X} = (X, \top, \wedge, \tau)$ is called *saturated* if it equals the intersection of all open filters containing p. The collection of saturated filters of $\mathbb{X}$ is denoted by $\mathscr{F}_{sat}\mathbb{X}$.

2.18 Proposition. *Let L be a lattice and $\mathbb{X}_L$ its dual L-space.*

1. *The filter completion of L is isomorphic to the complete lattice $\mathscr{F}_k(\mathbb{X}_L)$ of principal filters of $\mathbb{X}_L$, with inclusion $\theta_L : L \to \mathscr{F}_k(\mathbb{X}_L) : a \mapsto \theta(a)$.*
2. *The canonical extension of L is isomorphic to the complete lattice $\mathscr{F}_{sat}(\mathbb{X}_L)$ of saturated filters of $\mathbb{X}_L$, with inclusion $\theta_L : L \to \mathscr{F}_{sat}(\mathbb{X}_L) : a \mapsto \theta(a)$.*
3. *The Π_1-completion of L is isomorphic to the complete lattice $\mathscr{F}(\mathbb{X}_L)$ of all filters of $\mathbb{X}_L$, with inclusion $\theta_L : L \to \mathscr{F}(\mathbb{X}_L) : a \mapsto \theta(a)$.*

Proof. The first item follows from the lattice of principal filters of L being isomorphic to L° and $\mathscr{F}L = (fe\,L)^\circ$. The second item is similar to [40, Theorem 4.1]. Finally, the third item follows from the connection between filter and ideal completions $ie\,L = (fe\,L^\circ)^\circ$ mentioned above, together with $\mathscr{F}L = (fe\,L)^\circ$. $\quad\square$

3. Semilattice semantics for weak positive logic

We use the duality and dual adjunction from Section 2 to give frame semantics for weak positive logic, i.e. the logic the same signature as positive logic, but with lattices as algebraic semantics. Inspired by the fact that the filters of a semilattice form a lattice, we use semilattices as frames and (principal) filters as denotations of formulae.

We start this section by giving an axiomatisation of our logic. By design the algebraic semantics is simply given by lattices. In Section 3.2 we define frames and models, give examples, and prove that the frame semantics is sound. In Section 3.3 we use the duality from Section 2 to derive completeness for weak

Table 1
Different notions of entailment.

Notation	Purpose	Location
$\varphi \vdash \psi$	Syntactic entailment	Def. 3.1
$\varphi \dashv\vdash \psi$	Algebraic entailment	Def. 3.3
$\varphi \Vdash \psi$	Semantic entailment	Def. 3.6
$\varphi \Vdash_{\mathsf{LSpace}} \psi$	Topological semantic entailment	Def. 3.17
$\varphi \models \psi$	First-order entailment	Sec. 3.4

positive logics with respect to several classes of frames. We give the standard translation into a suitable first-order logic and prove Sahlqvist correspondence in Section 3.4, where we also work out specific examples of correspondence results.

To distinguish the various notions of entailment each has their own notation, which are summarised in Table 1. We denote the interpretation of a formula φ in a lattice $\mathcal{A}$ and in a frame $\mathfrak{M}$ by $(\![\varphi]\!)_{\mathcal{A}}$ and $[\![\varphi]\!]^{\mathfrak{M}}$, respectively. Besides, we write 1 and $\curlywedge$ for the top element and meets of a semilattice when it is regarded as frame semantics, and $\preccurlyeq$ for the induced partial order.

3.1. Logic and algebraic semantics

Let $\mathbf{L}(\mathrm{Prop})$ denote the language generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi,$$

where p ranges over some arbitrary but fixed set Prop of proposition letters. If no confusion arises we omit reference to Prop and simply write $\mathbf{L}$. We define logics based on $\mathbf{L}$ as a collection of *consequence pairs*, similar to e.g. [15]. A consequence pair is an expression of the form $\varphi \trianglelefteq \psi$ where φ and ψ are formulae in $\mathbf{L}$, and intuitively means: "If φ holds, then so does ψ."

3.1 Definition. Let $\mathcal{L}$ be the smallest set of consequence pairs closed under the following axioms and rules:

$$p \trianglelefteq \top, \qquad \bot \trianglelefteq p, \qquad\qquad\qquad \textit{top and bottom}$$

$$p \trianglelefteq p, \qquad \frac{p \trianglelefteq q \quad q \trianglelefteq r}{p \trianglelefteq r}, \qquad\qquad \textit{reflexivity and transitivity}$$

$$p \wedge q \trianglelefteq p, \qquad p \wedge q \trianglelefteq q, \qquad \frac{r \trianglelefteq p \quad r \trianglelefteq q}{r \trianglelefteq p \wedge q}, \qquad\qquad \textit{conjunction rules}$$

$$p \trianglelefteq p \vee q, \qquad q \trianglelefteq p \vee q, \qquad \frac{p \trianglelefteq r \quad q \trianglelefteq r}{p \vee q \trianglelefteq r} \qquad\qquad \textit{disjunction rules}$$

If Γ is a set of consequence pairs then we let $\mathcal{L}(\Gamma)$ denote the smallest set of consequence pairs closed under uniform substitution, axioms and rules mentioned above and those in Γ. We write $\varphi \vdash_{\Gamma} \psi$ if $\varphi \trianglelefteq \psi \in \mathcal{L}(\Gamma)$ and $\varphi \dashv\vdash_{\Gamma} \psi$ if both $\varphi \vdash_{\Gamma} \psi$ and $\psi \vdash_{\Gamma} \varphi$. If Γ is the empty set then we simply write $\varphi \vdash \psi$ and $\varphi \dashv\vdash \psi$.

The algebraic semantics of the logic $\mathbf{L}$ are simply lattices. We establish this formally.

3.2 Definition. Let A be a lattice with operations $\top_A, \bot_A, \wedge_A, \vee_A$, and induced order $\leq_A$. A *lattice model* is a pair $\mathcal{A} = (A, \sigma)$ consisting of a lattice A and an assignment $\sigma : \mathrm{Prop} \to A$ of the proposition letters. The assignment σ uniquely extends to a map $(\![\cdot]\!)_{\mathcal{A}} : \mathbf{L} \to A$ by interpreting connectives with their lattice counterparts.

We say that a lattice A *validates* a consequence pair $\varphi \trianglelefteq \psi$ if $(\![\varphi]\!)_{\mathcal{A}} \leq_A (\![\psi]\!)_{\mathcal{A}}$ for all lattice models $\mathcal{A}$ based on A, notation: $A \Vdash \varphi \trianglelefteq \psi$. We also write $A \Vdash \varphi = \psi$ if A validates both $\varphi \trianglelefteq \psi$ and $\psi \trianglelefteq \varphi$. If Γ

is a set of consequence pairs then we write $\mathsf{Lat}(\Gamma)$ for the full subcategory of Lat whose objects validate all consequence pairs in Γ.

3.3 Definition. Let $\Gamma \cup \{\varphi \lhd \psi\}$ be a set of consequence pairs. Write $\varphi \vdash_\Gamma \psi$ if $(\!|\varphi|\!)_\mathcal{A} \leq_A (\!|\psi|\!)_\mathcal{A}$ for every lattice model $\mathcal{A} = (A, \sigma)$ with $A \in \mathsf{Lat}(\Gamma)$. We abbreviate $\varphi \vdash_\emptyset \psi$ to $\varphi \vdash \psi$.

Observe that $\dashv\vdash_\Gamma$ is an equivalence relation on $\mathbf{L}$. Write $L(\Gamma)$ for the set of $\dashv\vdash_\Gamma$-equivalence classes of $\mathbf{L}$, and denote by $[\varphi]$ the equivalence class of φ in $L(\Gamma)$. Then it follows from the rules in Definition 3.1 that $L(\Gamma)$ carries a lattice structure, where $\top_L = [\top]$, $\bot_L = [\bot]$, $[\varphi] \wedge_L [\psi] = [\varphi \wedge \psi]$ and $[\varphi] \wedge_L [\psi] = [\varphi \vee \psi]$. Moreover, $L(\Gamma)$ is in $\mathsf{Lat}(\Gamma)$, and setting $\sigma_L : \mathrm{Prop} \to L(\Gamma) : p \mapsto [p]$ yields lattice model $\mathscr{L}_\Gamma = (L(\Gamma), \sigma_L)$ which acts as the Lindenbaum-Tarski algebra. It follows from induction on the structure of φ that $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} = [\varphi]$ for all $\mathbf{L}$-formulae φ.

3.4 Lemma. *We have $\varphi \vdash_\Gamma \psi$ if and only if $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$.*

Proof. The "only if" holds by definition. Conversely, if $A \in \mathsf{Lat}(\Gamma)$ and $\mathcal{A} = (A, \sigma_A)$ is a lattice model, then the assignment $[p] \mapsto \sigma_A(p)$ extends to a lattice homomorphism $i : \mathscr{L}_\Gamma \to \mathcal{A}$ such that $[\varphi] = (\!|\varphi|\!)_\mathcal{A}$. (This is well defined because A validates all consequence pairs in Γ.) Then $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$ implies $[\varphi] \leq_L [\psi]$. Monotonicity of i yields $(\!|\varphi|\!)_\mathcal{A} \leq_L (\!|\psi|\!)_\mathcal{A}$, hence $\varphi \vdash_\Gamma \psi$. □

3.5 Theorem. *We have $\varphi \vdash_\Gamma \psi$ if and only if $\varphi \vdash_\Gamma \psi$.*

Proof. By Lemma 3.4 it suffices to show that $\varphi \vdash_\Gamma \psi$ if and only if $(\!|\varphi|\!)_{\mathscr{L}_\Gamma} \leq_L (\!|\psi|\!)_{\mathscr{L}_\Gamma}$. It follows from the conjunction rules, reflexivity and transitivity that $\varphi \vdash_\Gamma \psi$ if and only if $\varphi \wedge \psi \dashv\vdash_\Gamma \varphi$. Therefore we have $\varphi \vdash_\Gamma \psi$ if and only if $[\varphi \wedge \psi] = [\varphi]$ in $\mathscr{L}_\Gamma$, and since $[\varphi \wedge \psi] = [\varphi] \wedge_L [\psi]$ this holds if and only if $[\varphi] \leq_L [\psi]$. Recalling that $[\varphi] = (\!|\varphi|\!)_{\mathscr{L}_\Gamma}$ completes the proof. □

3.2. Frame semantics

The collection of filters of a semilattice forms a lattice. Therefore we can use semilattices as frame semantics of weak positive logic, with filters serving as denotations of formulae. If moreover the semilattice is a lattice, then we can also use principal filters as denotations of formulae.

3.6 Definition. A *lattice model* or *L-model* is a semilattice $(X, 1, \curlywedge)$ together with a valuation $V : \mathrm{Prop} \to \mathcal{F}(X, 1, \curlywedge)$ which assigns to each proposition letter a filter of $(X, 1, \curlywedge)$. An L-model $(X, 1, \curlywedge, V)$ is called *principal* if $(X, 1, \curlywedge)$ has a bottom element 0 and binary joins denoted by $\curlyvee$ (so it forms a lattice) and $V(p)$ is a principal filter for all $p \in \mathrm{Prop}$.

The interpretation of an $\mathbf{L}$-formula φ at a state x in a (principal) L-model $\mathfrak{M} = (X, 1, \curlywedge, V)$ is defined recursively via

$$
\begin{aligned}
\mathfrak{M}, x \Vdash \top && &\text{always} \\
\mathfrak{M}, x \Vdash \bot &&\text{iff}\quad & x = 1 \\
\mathfrak{M}, x \Vdash p &&\text{iff}\quad & x \in V(p) \\
\mathfrak{M}, x \Vdash \varphi \wedge \psi &&\text{iff}\quad & \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi \\
\mathfrak{M}, x \Vdash \varphi \vee \psi &&\text{iff}\quad & \exists y, z \in X \text{ s.t. } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi \text{ and } y \curlywedge z \preccurlyeq x
\end{aligned}
$$

We write $[\![\varphi]\!]^{\mathfrak{M}} := \{x \in X \mid \mathfrak{M}, x \Vdash \varphi\}$ for the *truth set* of φ in $\mathfrak{M}$. If the underlying (semi)lattice is fixed and we want to emphasise the role of the valuation in the interpretation, we will write $V(\varphi)$ instead of $[\![\varphi]\!]^{\mathfrak{M}}$. The *theory* of x is denoted by $\mathrm{th}_{\mathfrak{M}}(x) := \{\varphi \in \mathbf{L} \mid \mathfrak{M}, x \Vdash \varphi\}$.

Note that the (semi)lattice underlying an L-model is uniquely determined by its partial order. So we may view L-models as a type of relational semantics, where the relation is used to define a non-standard interpretation of joins. When viewed as frame semantics, we denote the top element and meet of a semilattice by 1 and $\curlywedge$ and call the semilattice itself an L-frame.

3.7 Definition. We write $\mathfrak{M}, x \Vdash \varphi \lhd \psi$ if $x \in [\![\varphi]\!]^{\mathfrak{M}}$ implies $x \in [\![\psi]\!]^{\mathfrak{M}}$, and $\mathfrak{M} \Vdash \varphi \lhd \psi$ if $[\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\psi]\!]^{\mathfrak{M}}$. If $\mathfrak{X}$ is an L-frame, we let $\mathfrak{X}, x \Vdash \varphi \lhd \psi$ if $\mathfrak{M}, x \Vdash \varphi \lhd \psi$ for all L-models $\mathfrak{M}$ based on $\mathfrak{X}$, and $\mathfrak{X} \Vdash \varphi \lhd \psi$ if $\mathfrak{X}, x \Vdash \varphi \lhd \psi$ for all states x of $\mathfrak{X}$.

We say that $\mathfrak{M}$ or $\mathfrak{X}$ *validates* $\varphi \lhd \psi$ if $\mathfrak{M} \Vdash \varphi \lhd \psi$ or $\mathfrak{X} \Vdash \varphi \lhd \psi$, respectively. If Γ is a set of consequence pairs, then we let $\mathsf{LFrm}(\Gamma)$ denote the full subcategory of LFrm whose objects validate all consequence pairs in Γ. We write $\varphi \Vdash_{\Gamma} \psi$ if $\mathfrak{X} \Vdash \varphi \lhd \psi$ for all $\mathfrak{X} \in \mathsf{LFrm}(\Gamma)$. If $\Gamma = \emptyset$ then we write $\varphi \Vdash \psi$ instead of $\varphi \Vdash_{\emptyset} \psi$.

For any L-frame $\mathfrak{F} = (X, 1, \curlywedge)$, the collection $\mathfrak{F}^* := \mathscr{F}(X, 1, \curlywedge)$ forms a lattice, called the *complex algebra* of $\mathfrak{F}$. Since valuations of $\mathfrak{F}$ correspond bijectively to assignments of $\mathfrak{F}^*$, we can define the *complex algebra* of an L-model $\mathfrak{M} = (X, 1, \curlywedge, V)$ by $\mathfrak{M}^* = (\mathscr{F}(X, 1, \curlywedge), V)$. A routine induction on the structure of φ then proves the following lemma.

3.8 Lemma. *For every L-model $\mathfrak{M}$ and $\mathbf{L}$-formula φ we have $[\![\varphi]\!]^{\mathfrak{M}} = (\!|\varphi|\!)_{\mathfrak{M}^*}$.*

The next persistence result is similar to persistence in intuitionistic logic, except we require formulae to be interpreted as (principal) filters rather than upsets.

3.9 Proposition *(Persistence). Let $\mathfrak{M} = (X, 1, \curlywedge, V)$ be a (principal) L-model. Then for each $\varphi \in \mathbf{L}$ the truth set $[\![\varphi]\!]^{\mathfrak{M}}$ of φ is a (principal) filter of $(X, 1, \curlywedge)$.*

Proof. The fact that $[\![\varphi]\!]^{\mathfrak{M}}$ is a filter for each φ follows from Lemma 3.8. Suppose $\mathfrak{M}$ is principal. Then $[\![p]\!]^{\mathfrak{M}}$ is principal by definition, as are $[\![\bot]\!]^{\mathfrak{M}} = {\uparrow}\top$ and $[\![\top]\!]^{\mathfrak{M}} = {\uparrow}0$. If $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ then we proceed by induction. We may assume that $[\![\varphi_1]\!]^{\mathfrak{M}} = {\uparrow}x_1$ and $[\![\varphi_2]\!]^{\mathfrak{M}} = {\uparrow}x_2$, so that $[\![\varphi_1 \wedge \varphi_2]\!]^{\mathfrak{M}} = {\uparrow}(x_1 \curlyvee x_2)$ and $[\![\varphi_1 \vee \varphi_2]\!]^{\mathfrak{M}} = {\uparrow}(x_1 \curlywedge x_2)$. $\square$

3.10 Theorem *(Soundness). If $\varphi \vdash_{\Gamma} \psi$ then $\varphi \Vdash_{\Gamma} \psi$.*

Proof. If $\mathfrak{M}$ is an L-model that validates all consequence pairs in Γ, then $\mathfrak{M}^* \in \mathsf{Lat}(\Gamma)$. Since $\varphi \vdash_{\Gamma} \psi$, Theorem 3.5 yields $\varphi \nvdash_{\Gamma} \psi$, and hence $(\!|\varphi|\!)_{\mathfrak{M}^*} \leq (\!|\psi|\!)_{\mathfrak{M}^*}$. Lemma 3.8 now implies $[\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\psi]\!]^{\mathfrak{M}}$, so that $\mathfrak{M}$ validates $\varphi \lhd \psi$. $\square$

We turn the collections of (principal) L-models into a category by equipping them with truth-preserving morphisms.

3.11 Definition. An *L-model morphism* from $(X, 1, \curlywedge, V)$ to $(X', 1', \curlywedge', V')$ is an L-morphism (Definition 2.4) $f : (X, 1, \curlywedge) \to (X', 1', \curlywedge')$ that satisfies $V = f^{-1} \circ V'$.

A routine induction on the structure of φ shows that L-model morphisms preserve and reflect truth of $\mathbf{L}$-formulae.

3.12 Proposition. *Let $f : \mathfrak{M} \to \mathfrak{M}'$ be an L-model morphism. Then for all states x of $\mathfrak{M}$ and all $\varphi \in \mathbf{L}$,*

$$\mathfrak{M}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}', f(x) \Vdash \varphi.$$

The remainder of this subsection is devoted to examples of L-frames and -models.

3.13 Example. Any linearly ordered set with a largest element is an L-frame. Filters in such frames are simply upsets. For example $\mathbb{N} \cup \{\infty\}$ with the usual ordering is an L-frame which is principal. The set $\mathbb{N}$ ordered by $\geq$ is also an L-frame, with top element 0. It is not principal because it lacks a bottom element.   ◁

3.14 Example. As another example, consider the collection $\mathscr{P}_\omega X$ of finite subsets of X. This forms a semi-lattice with top element $\emptyset$, and meet given by the set-theoretic union. Filters of $\mathscr{P}_\omega X$ correspond bijectively with subsets of X.   ◁

3.15 Example *(Propositional team semantics).* We briefly recall a simplified version of team semantics for propositional logics, underlying versions of modal dependence and independence logics such as the ones studied in [30,39,47,48]. Let $\mathbf{T}(\text{Prop})$ be the language be given by the grammar $\varphi ::= p \mid \neg p \mid \varphi \land \varphi \mid \varphi \lor \varphi$. Then $\mathbf{T}(\text{Prop})$-formulae can be interpreted in models consisting of a set X and a valuation $\Pi : \text{Prop} \to \mathscr{P}X$ of the proposition letters. However, rather than assigning truth of formulae to elements of X, truth is defined for *subsets* of X (the teams). Let $\mathfrak{M} = (X, \Pi)$ be such a model and $T \subseteq X$ a team, then we let

$$
\begin{aligned}
\mathfrak{M}, T \Vdash_t p \quad &\text{iff} \quad T \subseteq V(p) \\
\mathfrak{M}, T \Vdash_t \neg p \quad &\text{iff} \quad T \cap V(p) = \emptyset \\
\mathfrak{M}, T \Vdash_t \varphi \land \psi \quad &\text{iff} \quad \mathfrak{M}, T \Vdash_t \varphi \text{ and } \mathfrak{M}, T \Vdash_t \psi \\
\mathfrak{M}, T \Vdash_t \varphi \lor \psi \quad &\text{iff} \quad \exists T_1, T_2 \subseteq T \text{ s.t. } T_1 \cup T_2 = T \text{ and } \mathfrak{M}, T_1 \Vdash_t \varphi \text{ and } \mathfrak{M}, T_2 \Vdash_t \psi
\end{aligned}
$$

We can add $\top$, which is true for every team, and $\bot$ satisfying $\mathfrak{M}, T \Vdash_t \bot$ iff $T = \emptyset$.

This interpretation resembles Definition 3.6. Let us make this precise. For a set Prop of proposition letters, let $\neg\text{Prop} = \{\neg p \mid p \in \text{Prop}\}$. Then, given a team model $\mathfrak{M} = (X, \Pi)$, we can define a principal L-model $\mathfrak{M}' = (\mathscr{P}X, \emptyset, \cup, V)$, with $V(p) = \{a \in \mathscr{P}X \mid a \subseteq \Pi(p)\}$ and $V(\neg p) = \{a \in \mathscr{P}X \mid a \cap \Pi(p) = \emptyset\}$. Then for each team model $\mathfrak{M}$, team T, and formula $\varphi \in \mathbf{T}(\text{Prop})$ we have

$$\mathfrak{M}, T \Vdash_t \varphi \quad \text{iff} \quad \mathfrak{M}', T \Vdash \varphi.$$

This can be proven by induction on the structure of φ. The only non-trivial step is for joins:

$$
\begin{aligned}
\mathfrak{M}, t \Vdash_t \varphi \lor \psi \quad &\text{iff} \quad \exists T_1, T_2 \in \mathscr{P}X \text{ s.t. } T_1 \cup T_2 = T \text{ and } \mathfrak{M}, T_1 \Vdash_t \varphi \text{ and } \mathfrak{M}, T_2 \Vdash_t \psi \\
&\text{iff} \quad \exists T_1, T_2 \in \mathscr{P}X \text{ s.t. } T_1 \cup T_2 = T \text{ and } \mathfrak{M}', T_1 \Vdash \varphi \text{ and } \mathfrak{M}', T_2 \Vdash \psi \\
&\text{iff} \quad \exists T_1', T_2' \in \mathscr{P}X \text{ s.t. } T_1' \cup T_2' \supseteq T \text{ and } \mathfrak{M}', T_1' \Vdash \varphi \text{ and } \mathfrak{M}', T_2' \Vdash \psi \\
&\text{iff} \quad \mathfrak{M}', T \Vdash \varphi \lor \psi
\end{aligned}
$$

The first "iff" is the definition of $\Vdash_t$, the second follows from the induction hypothesis. The third "iff" follows from persistence and the fact that the frame is ordered by reverse inclusion, and the last "iff" hold by the definition of $\Vdash$.   ◁

3.16 Example *(Modal information logic).* Modal information logic [3] is the extension of propositional classical logic with two binary modal operators $\langle\mathsf{inf}\rangle$ and $\langle\mathsf{sup}\rangle$. These are interpreted in Kripke models $\mathfrak{M} = (X, R, V)$ where R is a pre-order on X as follows:

$$\mathfrak{M}, x \Vdash \langle\mathsf{inf}\rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z \in X \text{ s.t. } x = \inf(y, z) \text{ and } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi$$

$$\mathfrak{M}, x \Vdash \langle\mathsf{sup}\rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z \in X \text{ s.t. } x = \sup(y, z) \text{ and } \mathfrak{M}, y \Vdash \varphi \text{ and } \mathfrak{M}, z \Vdash \psi$$

Note that we need not require that every pair of states has an infimum and a supremum, nor that it is unique. The definition simply uses the fact that they might exist. Observe that we can recover the usual modal and temporal diamonds via $\Diamond\varphi = \langle\mathsf{inf}\rangle(\varphi, \top)$ and $\blacklozenge\varphi = \langle\mathsf{sup}\rangle(\varphi, \top)$.

Clearly, every L-model is a model for modal information logic. Interestingly, the interpretation of $\langle\mathsf{inf}\rangle$ is closely aligned to our interpretation of joins; the only difference is that the infimum is allowed to be *below* the state under consideration. Taking this into account, our interpretation of joins in an L-model $\mathfrak{M} = (X, 1, \curlywedge, V)$ coincides with

$$\varphi \vee \psi = \blacklozenge(\langle\mathsf{inf}\rangle(\varphi, \psi)),$$

where $\vee$ is the non-classical join of weak positive logic. ◁

3.3. Descriptive frames and completeness

We have already seen a duality for lattices by means of L-spaces. Since every L-space is based on a complete semilattice, L-spaces can be viewed as topologised (principal) L-frames. In this subsection we define clopen valuations for L-spaces and show how this gives rise to completeness results. We denote L-spaces and L-spaces with a valuation by $\mathbb{X}$ and $\mathbb{M}$. If $\mathbb{X}$ is an L-space, then we write $\kappa\mathbb{X}$ for its underlying (principal) L-frame.

3.17 Definition. A *clopen valuation* for an L-space $\mathbb{X}$ is an assignment $V : \mathrm{Prop} \to \mathscr{F}_{clp}\mathbb{X}$, which assigns to each proposition letter a clopen filter of $\mathbb{X}$. We call a pair $\mathbb{M} = (\mathbb{X}, V)$ of an L-space and a clopen valuation an *L-space model*. The interpretation $[\![\varphi]\!]^{\mathbb{M}}$ of an **L**-formula φ in an L-space model $\mathbb{M} = (\mathbb{X}, V)$ is defined as in the underlying L-model $(\kappa\mathbb{X}, V)$.

An L-space model $\mathbb{M}$ validates a consequence pair $\varphi \lhd \psi$ if $[\![\varphi]\!]^{\mathbb{M}} \subseteq [\![\psi]\!]^{\mathbb{M}}$, notation: $\mathbb{M} \Vdash \varphi \lhd \psi$. We say that an L-space $\mathbb{X}$ validates $\varphi \lhd \psi$ if every L-space model based on it validates $\varphi \lhd \psi$. Finally, we write $\varphi \Vdash_{\mathsf{LSpace}} \psi$ if every L-space validates $\varphi \lhd \psi$.

3.18 Lemma. *Let $\mathbb{X}$ be an L-space, A its dual lattice, and $\varphi, \psi \in \mathbf{L}$. Then*

$$\mathbb{X} \Vdash \varphi \text{ iff } A \vdash\!\!\!\dashv \varphi \quad \text{and} \quad \mathbb{X} \Vdash \varphi \lhd \psi \text{ iff } A \vdash\!\!\!\dashv \varphi \lhd \psi.$$

Proof. The first "iff" follows from the fact that clopen valuations of $\mathbb{X}$ correspond bijectively to assignments of the proposition letters for A, together with a routine induction on the structure of φ. The second "iff" follows immediately from the first. □

3.19 Remark. We can alternatively describe L-spaces as *descriptive L-frames*. This is similar to the perspective of Esakia spaces as descriptive intuitionistic Kripke frames, see [17, Chapter 3] and [8, Section 8]. We briefly sketch this alternative perspective.

A *general L-frame* is a tuple $(X, 1, \curlywedge, A)$ such that $(X, 1, \curlywedge)$ is an L-frame and A is a collection of filters of $(X, 1, \curlywedge)$ containing X and $\emptyset$, and closed under $\cap$ and $\triangledown$. Let $-A = \{X \setminus a \mid a \in A\}$. A *descriptive L-frame* is a general L-frame $(X, 1, \curlywedge, A)$ that is

- *refined*: for all $x, y \in X$ such that $x \not\preccurlyeq y$ there exists an $a \in A$ such that $x \in a$ and $y \notin a$;
- *compact*: if $C \subseteq A \cup -A$ has the finite intersection property then $\bigcap C \neq \emptyset$.

A general L-morphism from $(X, 1, \curlywedge, A)$ to $(X', 1', \curlywedge', A')$ is an L-morphism $f : (X, 1, \curlywedge) \to (X', 1', \curlywedge')$ such that $f^{-1}(a') \in A$ for all $a' \in A'$. Write D-LFrm for the category of descriptive L-frames and general L-morphisms. Then we have D-LFrm $\cong$ LSpace.   ◁

Next, we use the notion of Π_1-preservation to prove a general completeness result.

3.20 Definition. A consequence pair $\varphi \trianglelefteq \psi$ is called Π_1-*persistent* if for every L-space $\mathbb{X}$,

$$\mathbb{X} \Vdash \varphi \trianglelefteq \psi \quad \text{implies} \quad \kappa\mathbb{X} \Vdash \varphi \trianglelefteq \psi.$$

It is well known that filter and ideal completions preserve all (in)equalities (see e.g. [43]). Combining this with Lemmas 3.8 and 3.18 we find:

3.21 Lemma. *Any consequence pair $\psi \trianglelefteq \chi$ of **L**-formulae is Π_1-persistent.*

3.22 Theorem. *Let Γ be a set of consequence pairs. Then the logic $\mathcal{L}(\Gamma)$ is sound and complete with respect to the following classes of frames:*

- D-LFrm(Γ) (*descriptive L-frames validating Γ*);
- PLFrm(Γ) (*principal L-frames validating Γ*);
- LFrm(Γ) (*L-frames validating Γ*).

Proof. Soundness holds by definition, so we prove completeness. Suppose $\varphi \nvdash_\Gamma \psi$. Then by Theorem 3.5 we can find a lattice A validating all consequence pairs in Γ, but not $\varphi \trianglelefteq \psi$. As a consequence of Lemma 3.18 the L-space $\mathbb{X}$ dual to A validates all consequence pairs in Γ but does not validate $\varphi \trianglelefteq \psi$, thus we find completeness with respect to D-LFrm(Γ).

Since $\mathbb{X} \nVdash \varphi \trianglelefteq \psi$ there must exist a clopen valuation V such that $(\mathbb{X}, V) \nVdash \varphi \trianglelefteq \psi$. Therefore $(\kappa\mathbb{X}, V) \nVdash \varphi \trianglelefteq \psi$. Besides, Lemma 3.21 implies that $\kappa\mathbb{X}$ validates all consequence pairs in Γ. This implies completeness with respect to LFrm(Γ). Lastly, we note that $\mathbb{X}$ is a principal L-frame and since V is clopen it assigns to each proposition letter a principal filter. Thus $\mathbb{X}$ is a principal L-frame validating Γ but not $\varphi \trianglelefteq \psi$, proving completeness with respect to PLFrm(Γ).   □

3.23 Remark. Another way to prove Theorem 3.22 is via a Sahlqvist-style argument and the correspondence proved in Section 3.4. This resembles the approach taken by Sambin and Vaccaro [45]. For the detailed order-topological proof we refer to [14, Section 4.2]. The basic idea is as follows: if a sequent is refuted on some L-frame, then it is refuted on this frame via a minimal valuation which is closed. An analogue of the so-called intersection lemma entails that the value of a positive formula on a closed valuation is the intersection of the values of this formula on a clopen valuation. This produces a clopen valuation refuting the sequent.   ◁

3.24 Remark. The Π_1-persistence discussed here allows us to move from valuations that interpret proposition letters as clopen filters to valuations that assign to each proposition letter an arbitrary filter. It is analogous to d-persistence in intuitionistic and modal logics [6,8]. In the classical setting d-persistence allows one to move from clopen valuations to arbitrary valuations, and in the intuitionistic case from valuations into clopen upsets to valuations into all upsets. We point out that, while in the distributive setting this corresponds algebraically to canonical extensions, in our setting the corresponding algebraic structure is the Π_1-completion. ◁

3.4. The first-order translation and Sahlqvist correspondence

In this section we define the standard translation of **L** into a suitable first-order logic. We use this to derive a Sahlqvist correspondence result. We prove that for every consequence pair $\psi \lessdot \chi$, the collection of L-frames validating $\psi \lessdot \chi$ is first-order definable. Our proof of the correspondence result follows a standard proof from normal modal logic, such as found in [6, Section 3.6]. Thus, it showcases how our duality for lattices allows us to transfer classical techniques to the positive non-distributive setting. However, it is complicated (or rather, made more interesting) by the non-standard interpretation of disjunctions.

3.25 Definition. Let **FOL** be the single-sorted first-order language which has a unary predicate P_p for every proposition letter p, and a binary relation symbol R.

Intuitively, the relation symbol of our first-order language accounts of the poset structure of L-frames. It is used in the translation of disjunctions.

If x, y and z are first-order variables, then we can express that x is above every lower bound of y and z in the ordering induced by the relation symbol R using a first-order sentence. In order to streamline notation we abbreviate this as follows:

$$\mathsf{abovemeet}(x; y, z) := \forall w((wRy \wedge wRz) \rightarrow wRx).$$

If $x, y_1, \ldots, y_n$ is a finite set of variables and $n \geq 1$ then we define $\mathsf{abovemeet}(x; y_1, \ldots, y_n)$ in the obvious way. In particular, $\mathsf{abovemeet}(x; y)$ is simply yRx.

We are now ready to define the standard translation.

3.26 Definition. Let x be a first-order variable. Define the standard translation $\mathrm{st}_x : \mathbf{L} \rightarrow \mathbf{FOL}$ recursively via

$$\mathrm{st}_x(p) = P_p x$$
$$\mathrm{st}_x(\top) = (x = x)$$
$$\mathrm{st}_x(\bot) = \forall y(yRx)$$
$$\mathrm{st}_x(\varphi \wedge \psi) = \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi)$$
$$\mathrm{st}_x(\varphi \vee \psi) = \exists y \exists z(\mathsf{abovemeet}(x; y, z) \wedge \mathrm{st}_y(\varphi) \wedge \mathrm{st}_z(\psi))$$

Furthermore, we define the standard translation of a consequence pair $\varphi \lessdot \psi$ as

$$\mathrm{st}_x(\varphi \lessdot \psi) = \mathrm{st}_x(\varphi) \rightarrow \mathrm{st}_x(\psi).$$

Every L-model $\mathfrak{M} = (X, 1, \curlywedge, V)$ gives rise to a first-order structure $\mathfrak{M}^\circ = (X, \preccurlyeq, V)$ for **FOL**: $\preccurlyeq$ accounts for the interpretation of the binary relation symbol, and the interpretation of the unary predicates is given via the valuations of the proposition letters. We write $\mathfrak{M}^\circ$ for the L-model $\mathfrak{M}$ conceived of as a first-order structure for **FOL**.

3.27 Proposition. *For every L-model $\mathfrak{M}$ and state w of $\mathfrak{M}$ we have*

1. $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi)[w]$;
2. $\mathfrak{M}, w \Vdash \varphi \lessdot \psi$ iff $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi \lessdot \psi)[w]$.

Proof. The first item follows immediately from the definition of the standard translation, and the second item follows from the first one. □

In order to obtain similar results as in Proposition 3.27 for frames, we need to quantify the unary predicates in **FOL** corresponding to the proposition letters. We can do so in a second-order language, say, **SOL**. However, getting a second-order correspondent for a consequence pair $\varphi \lhd \psi$ that is satisfied in a frame if and only if $\varphi \lhd \psi$ is, is not as easy as simply quantifying over all possible interpretations of the unary predicates. That is, we cannot simply add $\forall P_1 \cdots \forall P_n$ in front of $\mathrm{st}_x(\varphi \lhd \psi)$. Indeed, we wish to only take those interpretations into account that arise from a valuation of the proposition letters *as filters*.

Thus we wish to quantify over interpretations of the unary predicates corresponding to filters in the underlying frame. We can force this by adding conditions that ensure that the P's are interpreted as filters in the antecedent of the implication $\mathrm{st}_x(\varphi) \rightarrow \mathrm{st}_x(\psi)$. Then the implication is vacuously true for "illegal" interpretations of the unary predicates. This intuition motivates the following definition of the *second-order translation* of a consequence pair.

3.28 Definition. Let $p_1, \ldots, p_n$ be the proposition letters occurring in ψ and χ, and let $P_1, \ldots, P_n$ denote their corresponding unary predicates. For each P_i, abbreviate

$$\mathsf{isfil}(P_i) = \exists w P_i w \wedge \forall x \forall y \forall z ((P_i y \wedge P_i z \wedge \mathsf{abovemeet}(x; y, z)) \rightarrow P_i x).$$

Using this abbreviation, we define the *second order translation* of the consequence pair $\psi \lhd \chi$ by

$$\mathrm{so}(\psi \lhd \chi) = \forall P_1 \cdots \forall P_n \big((\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n) \wedge \mathrm{st}_x(\psi)) \rightarrow \mathrm{st}_x(\chi)\big). \tag{3}$$

To disburden notation, we will often abbreviate $\mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n)$ as ISFIL.

Since all unary predicates in $\mathrm{so}(\psi \lhd \chi)$ are in the scope of a quantifier, the formula $\mathrm{so}(\psi \lhd \chi)$ can be interpreted in a first-order structure with a single relation. Therefore, every L-model $\mathfrak{X}$ gives rise to a structure $\mathfrak{X}^\circ$ for **SOL** in which we can interpret second order translations.

3.29 Lemma. *For all L-frames $\mathfrak{X} = (X, 1, \curlywedge)$ and all consequence pairs $\psi \lhd \chi$ we have*

$$\mathfrak{X}, w \Vdash \psi \lhd \chi \quad \textit{iff} \quad \mathfrak{X}^\circ \models \mathrm{so}(\psi \lhd \chi)[w].$$

Proof. If $\mathfrak{X}, w \Vdash \psi \lhd \chi$ then $w \in V(\psi)$ implies $w \in V(\chi)$ for every valuation V for $\mathfrak{X}$. If any of the P_i is interpreted as a subset of X that is not a filter, then the implication inside the quantifiers in (3) is automatically true, because the antecedent is false. If all P_i are interpreted as filters, then the implication holds because of the assumption. The converse is similar. □

Next, we show how one can use the second-order translation to obtain local correspondence results. We first define what we mean by local correspondence.

3.30 Definition. Let $\varphi \lhd \psi$ be a consequence pair and $\alpha(x)$ a first-order formula with free variable x. Then we say that $\varphi \lhd \psi$ and $\alpha(x)$ are *local frame correspondents* if for any L-frame $\mathfrak{X}$ and any state w we have

$$\mathfrak{X}, w \Vdash \varphi \lhd \psi \quad \text{iff} \quad \mathfrak{X}^\circ \models \alpha(x)[w].$$

Since our language is positive, every formula is upward monotone. That is, extending the valuation increases the truth set of formulae.

3.31 Lemma. *Let $\mathfrak{X}$ be an L-frame and let V and V' be valuations for $\mathfrak{X}$ such that $V(p) \subseteq V'(p)$ for all $p \in \mathrm{Prop}$. Then for all $\varphi \in \mathbf{L}$ we have $V(\varphi) \subseteq V'(\varphi)$.*

We now prove that every consequence pair has a local correspondent.

3.32 Theorem. *Any consequence pair $\psi \lhd \chi$ of **L**-formulae locally corresponds to a first-order formula with one free variable.*

Proof. We know that $\mathfrak{X}, w \Vdash \psi \lhd \chi$ if and only if $\mathfrak{X}^\circ \models \mathrm{so}(\psi \lhd \chi)[w]$. Our strategy for obtaining a first-order correspondent is to remove all second-order quantifiers from the second-order translation. We assume that no two quantifiers bind the same variable.

If ψ is equivalent to $\top$ then as a consequence of Lemma 3.31 $\psi \lhd \chi$ is equivalent to $\top \lhd \chi'$, where χ' is obtained from χ by replacing all proposition letters with $\bot$. This, in turn, implies that $\mathrm{so}(\top \lhd \chi')$ is a first-order correspondent of $\top \lhd \chi$, since the lack of proposition letters in χ' implies that there are no second-order quantifiers in $\mathrm{so}(\top \lhd \chi')$. If ψ is equivalent to $\bot$ then $\psi \lhd \chi$ is vacuously valid on all L-frames. So we may assume that the antecedent does not use $\top$ or $\bot$.

Let $p_1, \ldots, p_n$ be the propositional variables occurring in ψ, and write $P_1, \ldots, P_n$ for their corresponding unary predicates. We assume that every proposition letter that occurs in χ also occurs is ψ, for otherwise we may replace it by $\bot$ to obtain a formula which is equivalent in terms of validity on frames.

Step 1. Use equivalences of the form

$$(\exists w(\alpha(w)) \wedge \beta) \leftrightarrow \exists w(\alpha(w) \wedge \beta), \qquad (\exists w(\alpha(w)) \to \beta) \leftrightarrow \forall w(\alpha(w) \to \beta)$$

to pull out all quantifiers that arise in $\mathrm{st}_x(\psi)$. Let $Y := \{y_1, \ldots, y_m\}$ denote the set of (bound) variables that occur in the antecedent of the implication from the second-order translation. We end up with a formula of the form

$$\forall P_1 \cdots \forall P_n \forall y_1 \cdots \forall y_m \big((\mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}) \to \mathrm{st}_x(\chi)\big). \tag{4}$$

Here $\mathrm{ISFIL} = \mathsf{isfil}(P_1) \wedge \cdots \wedge \mathsf{isfil}(P_n)$, AT is a conjunction of formulae of the form $P_i z$ and REL is a conjunction of formulae of the form $\mathsf{abovemeet}(z; z', z'')$, where $z, z', z'' \in Y \cup \{x\}$.

Step 2. Next we read off minimal instances of the P_i making the antecedent true. Intuitively, these correspond to the smallest valuations for the p_i making the antecedent true. For each proposition letter P_i, let $P_i y_{i_1}, \ldots, P_i y_{i_k}$ be the occurrences of P_i in AT in the antecedent of (4). We define the valuation of p_i to be the filter generated by the (interpretations of) $y_{i_1}, \ldots, y_{i_k}$. Formally,

$$\sigma(P_i) := \lambda u.\mathsf{abovemeet}(u; y_{i_1}, \ldots, y_{i_k}).$$

(If $k = 0$, i.e. there is no variable y with $P_i y$, then we let $\sigma(P_i) = \lambda u.(u \neq u)$.) Then for each L-model $\mathfrak{M}$ and states $x', y'_1, \ldots, y'_m$ in $\mathfrak{M}$ we have

$$\mathfrak{M}^\circ \models \mathrm{AT} \wedge \mathrm{REL}[x, y'_1, \ldots, y'_m] \quad \text{implies} \quad \mathfrak{M}^\circ \models \forall y(\sigma(P_i)y \to P_i y).$$

If we replace each unary predicate P_i in (4) with $\sigma(P_i)$, then all conjoints in ISFIL and AT become true. Writing $[\sigma(P)/P]\,\mathrm{st}_x(\chi)$ for the formula obtained from $\mathrm{st}_x(\chi)$ by replacing each instance of a unary predicate P_i with $\sigma(P_i)$, we arrive at the first-order formula

$$\forall y_1 \cdots \forall y_m \big(\mathrm{REL} \to [\sigma(P_i)/P_i]\,\mathrm{st}_x(\chi)\big) \tag{5}$$

Step 3. Finally, we claim that for every L-frame $\mathfrak{X}$, $\mathfrak{X}^\circ$ validates (4) if and only if it validates (5). The implication from left to right is simply an instantiation of the quantifiers as filters. For the converse, assume that $\mathfrak{M}$ is some model based on $\mathfrak{X}$, so that $\mathfrak{M}^\circ$ is an extension of $\mathfrak{X}^\circ$ giving the interpretations of the unary predicates as filters. We may disregard the case where any of them is not a filter as that would make the antecedent in (4) false, hence the whole implication true. Let $x', y'_1, \ldots, y'_m$ be states in $\mathfrak{M}$ and assume that

$$\mathfrak{M}^\circ \models \text{ISFIL} \wedge \text{AT} \wedge \text{REL}[x', y'_1, \ldots, y'_m]. \tag{6}$$

We need to show that $\mathfrak{M}^\circ \models \text{st}_x(\chi)[x', y'_1, \cdots, y'_m]$. It follows from the assumption that (5) holds that $\mathfrak{M}^\circ \models [\sigma(P)/P]\,\text{st}_x(\chi)[x', y'_1, \cdots, y'_m]$. Moreover, as a consequence of (6) we have $\mathfrak{M}^\circ \models \forall y(\sigma(P)(y) \to Py)$ for all $P \in \{P_1, \ldots, P_n\}$. Using Lemma 3.31 it follows that $\mathfrak{M}^\circ \models \text{st}_x(\chi)[x', y'_1, \cdots, y'_m]$, as desired. $\quad\square$

Let us work through some explicit examples so we can see the proof of the theorem in action. Recall that $\mathsf{abovemeet}(x;y)$ is simply yRx.

3.33 Example. Consider the formula $p \wedge (q \vee q') \trianglelefteq (p \wedge q) \vee (p \wedge q')$. This corresponds to distributivity; the reverse consequence pair is always valid. We temporarily abbreviate $\chi := (p \wedge q) \vee (p \wedge q')$. The second-order translation of this formula is

$$\text{so}(p \wedge (q \vee q') \trianglelefteq \chi) = \forall P \forall Q \forall Q' \big(\big[\mathsf{isfil}(P) \wedge \mathsf{isfil}(Q) \wedge \mathsf{isfil}(Q')$$
$$\wedge\, Px \wedge \exists y \exists y' (\mathsf{abovemeet}(x;y,y') \wedge Qy \wedge Q'y') \big] \to \text{st}_x(\chi) \big)$$

As per instructions, we rewrite this to

$$\forall P \forall Q \forall Q' \forall y \forall y' \big((\text{ISFIL} \wedge Px \wedge \mathsf{abovemeet}(x;y,y') \wedge Qy \wedge Q'y') \to \text{st}_x(\chi) \big). \tag{7}$$

Next, we obtain $\sigma(P) = \lambda u.\mathsf{abovemeet}(u;x)$, which is simply xRu. Similarly we find $\sigma(Q) = \lambda u.yRu$ and $\sigma(Q') = \lambda u.y'Ru$. Plugging these into the antecedent yields

$$[\sigma(P_i)/P_i](\text{st}_x(\chi)) = \exists z \exists z' \big(\mathsf{abovemeet}(x;z,z') \wedge (xRz) \wedge (yRz) \wedge (xRz') \wedge (y'Rz') \big).$$

Thus we find the following first-order correspondent:

$$\forall y \forall y' \big(\mathsf{abovemeet}(x;y,y') \to \big[\exists z \exists z' (\mathsf{abovemeet}(x;z,z') \wedge (xRz) \wedge (yRz)$$
$$\wedge\, (xRz') \wedge (y'Rz')) \big] \big).$$

Recall that the predicate R is interpreted as the partial order $\preccurlyeq$ underlying an L-frame. Furthermore, since $z \wedge z' Rx$ and xRz and xRz', we find that $z \wedge z' = x$. Thus, a state w in an L-frame $(X, 1, \curlywedge)$ with partial order $\preccurlyeq$, satisfies distributivity if and only if for all $w \in X$ we have

$$\forall y \forall y' \big((y \curlywedge y' \preccurlyeq w) \to \exists z \exists z'((w = z \curlywedge z') \wedge (y \preccurlyeq z) \wedge (y' \preccurlyeq z')) \big).$$

Therefore, an L-frame validates distributivity if it is a distributive semilattice. Note that here $\curlywedge$ and $\preccurlyeq$ are L-frame operators while the quantifiers and $\to, \wedge$ are connectives from the first-order language used to reason about L-frames. $\quad\triangleleft$

3.34 Example. Next consider the modularity axiom

$$((p_1 \wedge p_3) \vee p_2) \wedge p_3 \trianglelefteq (p_1 \wedge p_3) \vee (p_2 \wedge p_3).$$

Writing χ for the right hand side of the consequence pair, after applying Step 1 of the proof of Theorem 3.32 we have

$$\forall P_1 \forall P_2 \forall P_3 \forall y \forall y' \big((\mathsf{ISFIL} \wedge P_1 y \wedge P_3 y \wedge P_2 y' \wedge P_3 x \wedge \mathsf{abovemeet}(x; y, y')) \to \mathsf{st}_x(\chi)\big) \tag{8}$$

We then get $\sigma(P_1) = \lambda u.yRu$, $\sigma(P_2) = \lambda u.y'Ru$ and $\sigma(P_3) = \lambda u.\mathsf{abovemeet}(u; x, y)$. Substituting these and leaving out ISFIL and AT yields

$$\forall y \forall y' \big[\big(\mathsf{abovemeet}(x; y, y') \wedge \mathsf{abovemeet}(y; x, y) \wedge \mathsf{abovemeet}(x; x, y)\big)$$
$$\to \exists z \exists z' (\mathsf{abovemeet}(x; z, z') \wedge (yRz) \wedge \mathsf{abovemeet}(z; x, y) \wedge (y'Rz') \wedge \mathsf{abovemeet}(z'; x, y))\big].$$

Leaving out everything that is trivially true, this yields the following condition. A world w in an L-frame $(X, 1, \curlywedge)$ with partial order $\preccurlyeq$ satisfies the modularity axiom if and only if

$$\forall y \forall y' ((y \curlywedge y' \preccurlyeq w) \to \exists z \exists z' ((z \curlywedge z' \preccurlyeq w) \wedge (y \preccurlyeq z) \wedge (y' \preccurlyeq z') \wedge (w \curlywedge y \preccurlyeq z'))).$$

In yet other words, w satisfies modularity if for all $y, y' \in X$ such that $y \curlywedge y' \preccurlyeq w$ we can find z, z' above y, y', respectively, such that $z \curlywedge z' \preccurlyeq w$ and $w \curlywedge y \preccurlyeq z'$. ◁

4. Normal modal extension

We investigate the extension of weak positive logic with two modal operators, $\square$ and $\diamond$, interpreted via a relation in the usual way (see e.g. [6, Definition 1.20]). As our point of departure we take L-frames with an additional relation. We stipulate sufficient conditions on the relation to ensure persistence, but we do not enforce any axioms on the modalities. It then turns out that $\square$ preserves finite conjunctions, while, as a consequence of the non-standard interpretation of disjunctions, $\diamond$ only satisfies monotonicity. This is reminiscent of the modal extension of intuitionistic logic investigated by Kojima [38]. The interaction axioms relating $\square$ and $\diamond$ are closely aligned to Dunn's axioms for positive modal logic [15], see Remark 4.8.

After investigating the modal logic from a semantic point of view, we use our newly developed intuition to give a syntactic definition of the logic and its algebraic semantics in Section 4.2, and a duality in Section 4.3. We then extend the definition of the filter- and Π_1-completion to the modal setting and prove completeness for weak positive modal logic in Section 4.4.

Finally, in Section 4.5 we extend the Sahlqvist correspondence result for weak positive logic to the modal setting. It is no longer the case that any consequence pair is Sahlqvist, and we identify as Sahlqvist consequence pairs precisely the negation-free Sahlqvist formulae from normal modal logic [6, Definition 3.51], where the implication is replaced by ◁.

One may wonder whether it would be more natural to insist that $\diamond$ be normal as well. We do not, because the additional conditions required to ensure that $\diamond$ is normal complicate the presentation of the semantics and duality. Moreover, in order to make $\diamond$ normal we only need to extend our basic system with the consequence pair

$$\diamond(p \vee q) \mathrel{◁} \diamond p \vee \diamond q,$$

which is Sahlqvist! We compute its local correspondent in Example 4.42.

4.1. Relational meet-frames

Let $\mathbf{L}_{\square\diamond}$ be the language generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \square\varphi \mid \diamond\varphi,$$
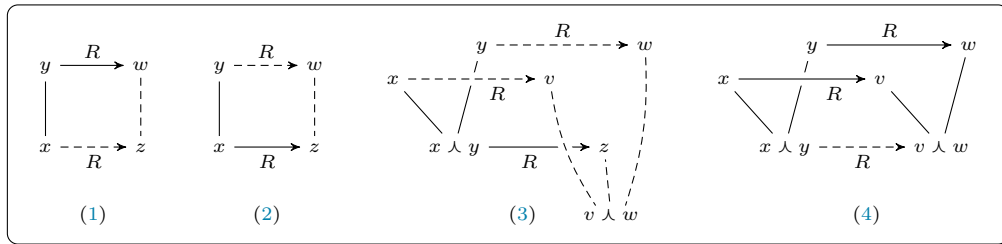
Fig. 2. The four conditions of a modal L-frame. Lines denote the poset order, with high nodes being bigger. Arrows denote the relation R.

where p ranges over some set Prop of proposition letters. A *modal consequence pair* is an expression of the form $\varphi \lhd \psi$, where $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$. We derive an appropriate notion of modal L-frame, such that the truth set of each formula is guaranteed to be a filter.

4.1 Definition. A *modal L-frame* is a tuple $(X, 1, \curlywedge, R)$ where $(X, 1, \curlywedge)$ is an L-frame with underlying partial order $\preccurlyeq$, and R is a binary relation on X such that:

1. If $x \preccurlyeq y$ and yRz then there exists a $w \in X$ such that xRw and $w \preccurlyeq y$;
2. If $x \preccurlyeq y$ and xRw then there exists a $z \in X$ such that yRz and $w \preccurlyeq z$;
3. If $(x \curlywedge y)Rz$ then there exist $v, w \in X$ such that xRv and yRw and $v \curlywedge w \preccurlyeq z$;
4. If xRv and yRw then $(x \curlywedge y)R(v \curlywedge w)$;
5. For all x, $1Rx$ if and only if $x = 1$.

A *modal L-model* is a modal L-frame together with a valuation V that assigns to each proposition letter a filter of $(X, 1, \curlywedge)$.

Just like in the propositional case in Section 3.2, we can identify a class of frames where formulae can be interpreted exclusively as *principal* filters. To this end, we define a *principal modal L-frame* as a modal L-frame $(X, 1, \curlywedge, R)$ that additionally satisfies:

0. $(X, 1, \curlywedge)$ has binary joins and all non-empty meets;
3'. If $(\curlywedge x_i)Rz$, where the i range over some index set I, then there exist a z_i such that $x_i R z_i$ for all $i \in I$ and $\curlywedge z_i \preccurlyeq z$;
4'. If $x_i R y_i$, where i ranges over some index set I, then $(\curlywedge x_i)R(\curlywedge y_i)$;

Clearly, items (3') and (4') subsume (3) and (4). A *principal modal L-model* is a principal modal L-frame with a valuation that assigns to each proposition letter a principal filter.

4.2 Definition. The interpretation of $\mathbf{L}_{\Box\Diamond}$-formulae in a (principal) modal L-model $\mathfrak{M}$ is defined via the clauses from Definition 3.6 and

$$\mathfrak{M}, x \Vdash \Box\varphi \quad \text{iff} \quad \forall y \in X, xRy \text{ implies } \mathfrak{M}, y \Vdash \varphi$$

$$\mathfrak{M}, x \Vdash \Diamond\varphi \quad \text{iff} \quad \exists y \in X \text{ such that } xRy \text{ and } \mathfrak{M}, y \Vdash \varphi$$

Satisfaction and validity of formulae and modal consequence pairs are defined as expected. In particular, if $\mathcal{K}$ is a class of modal L-frames and $\varphi \lhd \psi$ is a modal consequence pair, then we write $\varphi \Vdash_{\mathcal{K}} \psi$ if the consequence pair $\varphi \lhd \psi$ is valid on all frames in $\mathcal{K}$.

The first four conditions of a modal L-frame are depicted in Fig. 2. Observe that (1) and (5) together imply seriality, i.e. every state has an R-successor. If $(X, 1, \curlywedge, R)$ is a modal L-frame, then we define for $x \in X$ and filters $a \subseteq X$:

$$R[x] := \{y \in X \mid xRy\}, \qquad [R]a = \{x \in X \mid R[x] \subseteq a\}, \qquad \langle R\rangle a = \{x \in X \mid R[x] \cap a \neq \emptyset\}.$$

Definition 4.1(1) and (2) together say that $x \preccurlyeq y$ implies $R[x] \sqsubseteq R[y]$, where $\sqsubseteq$ denotes the Egli-Milner order on $\mathscr{P}X$ [1, Definition 6.2.2]. Furthermore, if $\mathfrak{M}$ is a modal L-model then

$$\llbracket \Box\varphi \rrbracket^{\mathfrak{M}} = [R]\llbracket \varphi \rrbracket^{\mathfrak{M}}, \qquad \llbracket \Diamond\varphi \rrbracket^{\mathfrak{M}} = \langle R\rangle\llbracket \varphi \rrbracket^{\mathfrak{M}}.$$

Next we prove persistence.

4.3 Proposition. *Let $\mathfrak{M} = (X, 1, \curlywedge, R, V)$ be a (principal) modal L-model. Then for each $\varphi \in \mathbf{L}_{\Box\Diamond}$ the set $\llbracket \varphi \rrbracket^{\mathfrak{M}}$ is a (principal) filter in $(X, 1, \curlywedge)$.*

Proof. We assume that $\mathfrak{M}$ is not principal; the case for principal modal L-models is similar. The proof proceeds by induction on the structure of φ. The only non-trivial cases are the modal cases. We prove the statement for $\varphi = \Diamond\psi$; the case $\varphi = \Box\psi$ is similar.

Suppose $\varphi = \Diamond\psi$, $\mathfrak{M}, x \Vdash \Diamond\psi$ and $x \preccurlyeq y$. Then there exists an R-successor z of x satisfying ψ, and by (2) we can find an R-successor w of y such that $z \preccurlyeq y$. By the induction hypothesis we then find $\mathfrak{M}, w \Vdash \psi$ and therefore $\mathfrak{M}, y \Vdash \Diamond\psi$. Next, suppose that both x and y satisfy $\Diamond\psi$. Then there exist $v \in R[x]$ and $w \in R[y]$ satisfying ψ. By (4) we have $(x \curlywedge y)R(v \curlywedge w)$ and by the induction hypothesis $\mathfrak{M}, v \curlywedge w \Vdash \psi$. Therefore $\mathfrak{M}, x \curlywedge y \Vdash \Diamond\psi$. Lastly, (5) implies $\mathfrak{M}, 1 \Vdash \Diamond\psi$. We conclude that $\llbracket \Diamond\psi \rrbracket^{\mathfrak{M}}$ is a filter in $(X, 1, \curlywedge)$. $\square$

4.4 Remark. We could have slightly weakened condition (4) by requiring the existence of some $(x \curlywedge y)$-successor above $v \curlywedge w$. We use the current formulation because it aligns more closely to the notion of a modal L-space. $\triangleleft$

Morphisms between modal L-frames and -models are a combination of L-morphisms and an adaptation of p-morphisms for positive modal logic [7].

4.5 Definition. A *bounded L-morphism* from $(X, 1, \curlywedge, R)$ to $(X', 1', \curlywedge', R')$ is an L-morphism $f : (X, 1, \curlywedge) \to (X', 1', \curlywedge')$ such that for all $x, y \in X$ and $z' \in X'$:

1. If xRy then $f(x)R'f(y)$;
2. If $f(x)R'z'$ then there exists a $z \in X$ such that xRz and $f(z) \preccurlyeq' z'$;
3. If $f(x)R'z'$ then there exists a $w \in X$ such that xRw and $z' \preccurlyeq' f(w)$.

(See also Fig. 3.) A bounded L-morphism between models is bounded L-morphism between the underlying frames that preserves and reflects truth of proposition letters.

4.6 Proposition. *Let $\mathfrak{M} = (X, 1, \curlywedge, R, V)$ and $\mathfrak{M}' = (X', 1', \curlywedge', R', V')$ be two (principal) modal L-models. If $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded L-morphism, $x \in X$ and $\varphi \in \mathbf{L}_{\Box\Diamond}$, then*

$$\mathfrak{M}, x \Vdash \varphi \quad \text{iff} \quad \mathfrak{M}', f(x) \Vdash \varphi.$$

Proof. This follows from a routine induction on the structure of φ. We showcase the modal cases of the proof. Suppose $\varphi = \Box\psi$. It follows immediately from Definition 4.5(1) that $\mathfrak{M}', f(x) \Vdash \Box\psi$ implies $\mathfrak{M}, x \Vdash \Box\psi$.
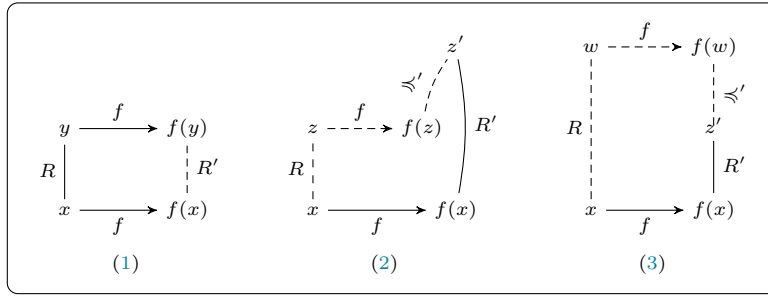
Fig. 3. The conditions of a bounded L-morphism.

So suppose $\mathfrak{M}, x \Vdash \Box\psi$. If y' is an R'-successor of $f(x)$, then there exists some $z \in X$ such that xRz and $f(z) \preccurlyeq' y'$. This implies $\mathfrak{M}, z \Vdash \psi$ and by induction $\mathfrak{M}', f(z) \Vdash \psi$. Persistence then yields $\mathfrak{M}', y' \Vdash \psi$. Therefore $\mathfrak{M}', f(x) \Vdash \Box\psi$.

If $\varphi = \Diamond\psi$ Then the preservation from left to right follows from Definition 4.5(1). Conversely, if $\mathfrak{M}', f(x) \Vdash \Diamond\psi$, then there exists a $y' \in X'$ such that $f(x)R'y'$ and $\mathfrak{M}', y' \Vdash \psi$. By (3) we can find some $w \in X$ such that xRw and $y' \preccurlyeq' f(w)$. Persistence implies $\mathfrak{M}', f(w) \Vdash \psi$ and induction yields $\mathfrak{M}, w \Vdash \psi$. Therefore $\mathfrak{M}, x \Vdash \Diamond\psi$. $\square$

We give a number of modal consequence pairs that are valid in every modal L-frame. These motivate the definition of a modal lattice in Section 4.2.

4.7 Lemma. *Let $(X, 1, \curlywedge, R)$ be a modal L-frame. The following consequence pairs are valid:*

$$\top \vartriangleleft \Box\top \qquad\qquad \top \vartriangleleft \Diamond\top \qquad\qquad\qquad (modal\ top)$$

$$\Box(\varphi \wedge \psi) \vartriangleleft \Box\varphi \wedge \Box\psi \qquad\qquad \Diamond\varphi \vartriangleleft \Diamond(\varphi \vee \psi) \qquad\qquad (monotonicity)$$

$$\Box\varphi \wedge \Box\psi \vartriangleleft \Box(\varphi \wedge \psi) \qquad\qquad \Diamond\varphi \wedge \Box\psi \vartriangleleft \Diamond(\varphi \wedge \psi) \qquad (normality\ and\ duality)$$

Proof. All of these besides $\top \vartriangleleft \Diamond\top$ follow immediately from the definition of the interpretation of $\Box$ and $\Diamond$. In particular, they do not rely on any of the conditions from Definition 4.1. The validity of $\top \vartriangleleft \Diamond\top$ follows from Definition 4.1(5) and (1). $\square$

Observe that the consequence pair $\top \vartriangleleft \Diamond\top$ corresponds to seriality, i.e. the frame condition that every state has an R-successor. In presence of $\top \vartriangleleft \Box\top$ and the duality axiom it is equivalent to $\Box\varphi \vartriangleleft \Diamond\varphi$.

4.8 Remark. The duality axiom in Lemma 4.7 corresponds to one of Dunn's duality axioms for positive modal logic [15]. It seems that the non-standard interpretation of joins makes Dunn's other duality axiom, $\Box(\varphi \vee \psi) \vartriangleleft \Box\varphi \vee \Diamond\psi$, unsuitable in our context. On the other hand, we have $\Box\varphi \vartriangleleft \Diamond\varphi$, which is not assumed by Dunn. We flag investigation of the connection between the various axioms relating $\Box$ and $\Diamond$ as an interesting direction for further research. $\vartriangleleft$

4.2. Logic and modal lattices

Guided by the validities from Lemma 4.7, we define the logic $\mathcal{L}_{\Box\Diamond}$ as follows.

4.9 Definition. Let $\mathcal{L}_{\Box\Diamond}$ be the smallest set of modal consequence pairs closed under uniform substitution, the axioms and rules from Definition 3.1, and under:

$$\top \trianglelefteq \Box\top \qquad\qquad \top \trianglelefteq \Diamond\top \qquad\qquad\qquad (\textit{modal top})$$

$$\frac{\varphi \trianglelefteq \psi}{\Box\varphi \trianglelefteq \Box\psi} \qquad\qquad \frac{\varphi \trianglelefteq \psi}{\Diamond\varphi \trianglelefteq \Diamond\psi} \qquad\qquad (\textit{Becker's rules})$$

$$\Box\varphi \wedge \Box\psi \trianglelefteq \Box(\varphi \wedge \psi) \quad \Diamond\varphi \wedge \Box\psi \trianglelefteq \Diamond(\varphi \wedge \psi) \qquad (\textit{linearity} \text{ and } \textit{duality})$$

If Γ is a set of modal consequence pairs then $\mathcal{L}_{\Box\Diamond}(\Gamma)$ denotes the smallest set of modal consequence pairs closed under the axioms and rules above and those in Γ. We write $\varphi \vdash_\Gamma \psi$ if $\varphi \trianglelefteq \psi \in \mathcal{L}_{\Box\Diamond}(\Gamma)$ and $\varphi \dashv\vdash_\Gamma \psi$ if both $\varphi \vdash_\Gamma \psi$ and $\psi \vdash_\Gamma \varphi$, omitting Γ if it is empty.

Observe that Becker's rule together with linearity for $\Box$ implies that $\Box$ is a normal modal operator. The algebraic semantics of the logic is given by modal lattices.

4.10 Definition. A *modal lattice* is a tuple $(A, \Box, \Diamond)$ consisting of a lattice A and two maps $\Box, \Diamond : A \to A$ satisfying for all $a, b \in A$:

$$\top = \Box\top \qquad\qquad \top = \Diamond\top$$

$$\Diamond a \le \Diamond(a \vee b) \qquad \Box(a \wedge b) = \Box a \wedge \Box b \qquad \Diamond a \wedge \Box b \le \Diamond(a \wedge b)$$

A *modal lattice homomorphism* from $(A, \Box, \Diamond)$ to $(A', \Box', \Diamond')$ is a lattice homomorphism $h : A \to A'$ such that $h(\Box a) = \Box' h(a)$ and $h(\Diamond a) = \Diamond' h(a)$ for all $a \in A$. We write MLat for the category of modal lattice and modal lattice homomorphisms.

4.11 Example. Let $\mathfrak{X} = (X, 1, \curlywedge, R)$ be a modal L-frame. Then $\mathfrak{X}^* := (\mathscr{F}(X, 1, \curlywedge), [R], \langle R \rangle)$ is a modal lattice. If $\mathfrak{X}$ is principal then $\mathfrak{X}^\ddagger := (\mathscr{F}_p(X, 1, \curlywedge), [R], \langle R \rangle)$ is a modal lattice. $\quad \triangleleft$

Formulae $\varphi \in \mathbf{L}_{\Box\Diamond}$ can be interpreted in a modal lattice $\mathscr{A} = (A, \Box, \Diamond)$ with an assignment $\sigma : \mathrm{Prop} \to A$. Analogous to Section 3.1, the interpretation of proposition letters is given by the assignment, and the connectives and modalities as interpreted via their counterparts in $\mathscr{A}$. This gives rise to validity of formulae and modal consequence pairs in a modal lattice $\mathscr{A}$.

If $\mathfrak{M} = (\mathfrak{X}, V)$ is a modal L-model then V is an assignment for $\mathfrak{X}^*$ and we write $\mathfrak{M}^* = (\mathfrak{X}^*, V)$. If $\mathfrak{M} = (\mathfrak{X}, V)$ is a principal modal L-model then V is an assignment for $\mathfrak{X}^\ddagger$ and we let $\mathfrak{M}^\ddagger = (\mathfrak{X}^\ddagger, V)$. We obtain the following counterpart of Lemma 3.8.

4.12 Lemma. *Let $\mathfrak{M}$ be a modal L-model, $\mathfrak{N}$ a principal modal L-model, and $\varphi \in \mathbf{L}_{\Box\Diamond}$. Then*

$$[\![\varphi]\!]^{\mathfrak{M}} = (\!|\varphi|\!)_{\mathfrak{M}^*} \quad \textit{and} \quad [\![\varphi]\!]^{\mathfrak{N}} = (\!|\varphi|\!)_{\mathfrak{N}^\ddagger}.$$

For a modal lattice A we define $A \Vdash \varphi \trianglelefteq \psi$ and $A \Vdash \varphi = \psi$ the same way as in Definition 3.2. We also write $\varphi \Vdash_\Gamma \psi$ if any modal lattice that validates all consequence pairs in Γ also validates $\varphi \trianglelefteq \psi$. Then we can prove the next theorem in the same way as in Section 3.1.

4.13 Theorem. *Let $\Gamma \cup \{\varphi \trianglelefteq \psi\}$ be a set of modal consequence pairs. Then $\varphi \vdash_\Gamma \psi$ iff $\varphi \Vdash_\Gamma \psi$.*

4.3. Modal L-spaces and duality

We define the modal counterpart of L-spaces as follows.

4.14 Definition. A *modal L-space* is a tuple $(X, 1, \curlywedge, \tau, R)$ such that

1. $(X, 1, \curlywedge, \tau)$ is an L-space;
2. R is a binary relation on X such that $1Rx$ iff $x = 1$ for all $x \in X$;
3. If a is a clopen filter, then so are $[R]a$ and $\langle R \rangle a$;
4. For all $x, y \in X$ we have xRy if and only if for all $a \in \mathcal{F}_{clp}\mathbb{X}$:

   - If $x \in [R]a$ then $y \in a$;
   - If $y \in a$ then $x \in \langle R \rangle a$.

Truth and validity in modal L-spaces is defined as usual, using clopen valuations.

The third item is a condition often seen in the definition of general frames. Item (4) is our counterpart of the tightness condition, and has previously been used in [7, Section 2]. Next, we prove that each modal L-space has an underlying (principal) modal L-frame.

4.15 Lemma. *Let $\mathbb{X} = (X, 1, \curlywedge, \tau, R)$ be a modal L-space. Then $R[x]$ is closed for all $x \in X$.*

Proof. Suppose $y \notin R[x]$. Then there exists a clopen filter a such that either $x \in [R]a$ and $y \notin a$, or $y \in a$ and $x \notin \langle R \rangle a$. In the first case $X \setminus a$ is a clopen neighbourhood of y disjoint from $R[x]$. In the second case a is a clopen neighbourhood of y disjoint from $R[x]$. $\square$

4.16 Proposition. *Let $\mathbb{X} = (X, 1, \curlywedge, \tau, R)$ be a modal L-space. Then $(X, 1, \curlywedge, R)$ is a principal modal L-frame.*

Proof. We know that L-spaces have all non-empty meets, and hence also binary joins, so (0) is satisfied. Furthermore, (5) is satisfied by definition. We verify the other conditions from Definition 4.1, starting with (4′).

Condition 4′. Suppose $x_i R y_i$, where i ranges over some index set I. If $I = \emptyset$ then this condition states $1R1$ which holds by definition, so assume that $I \neq \emptyset$. By the tightness condition of modal L-spaces, in order to prove $(\curlywedge x_i)R(\curlywedge y_i)$ it suffices to show that for all clopen filters a, $\curlywedge x_i \in [R]a$ implies $\curlywedge y_i \in a$ and $\curlywedge y_i \in a$ implies $\curlywedge x_i \in \langle R \rangle a$.

First assume $\curlywedge x_i \in [R]a$. Since $[R]a$ is a clopen filter and $\curlywedge x_i \preccurlyeq x_j$ for each $j \in I$ we have $x_j \in [R]a$, so $R[x_j] \subseteq a$. By assumption $x_j R y_j$, so $y_j \in a$ for all $j \in I$. Since a is a clopen filter, it is principal, hence $\curlywedge y_i \in a$. Second, suppose $\curlywedge y_i \in a$. Then $y_j \in a$ for all $j \in I$, which implies $x_j \in \langle R \rangle a$ for all $j \in I$. Since $\langle R \rangle a$ is a clopen filter, hence principal, we find $\curlywedge x_i \in \langle R \rangle a$.

Condition 1. Let $x \preccurlyeq y$ and yRz. Suppose towards a contradiction that there exists no $w \in X$ such that xRw and $w \preccurlyeq z$. Let $x' = \curlywedge R[x]$ be the minimal element in $R[x]$ (which is an R-successor of x by (4′)). Then $x' \not\preccurlyeq z$, so we can find a clopen filter a containing x' such that $z \notin a$. This implies $R[x] \subseteq a$, so that $x \in [R]a$, but $y \notin [R]a$ because yRz and $z \notin a$. As $x \preccurlyeq y$ this violates the fact that $[R]a$ is a filter.

Condition 2. Let $x \preccurlyeq y$ and xRw. Suppose towards a contradiction that there exists no $z \in X$ such that yRz and $w \preccurlyeq z$. Then $R[y] \cap {\uparrow}w = \emptyset$. Both $R[y]$ and ${\uparrow}w$ are closed, as a consequence of Lemmas 4.15 and 2.9 and the fact that singletons in Stone spaces are always closed. Therefore we can find a clopen filter a containing ${\uparrow}z$ which is disjoint from $R[y]$. This implies that $x \in \langle R \rangle a$ while $y \notin \langle R \rangle a$. Since $x \preccurlyeq y$ this contradicts the fact that $\langle R \rangle a$ is a filter.

Condition 3′. Finally, let $\{x_i \mid i \in I\}$ be some collection of elements of X and suppose $(\curlywedge x_i)Rz$. If I is empty then $\curlywedge x_i = 1$ and Definition 4.14(2) implies $z = 1$, and the empty collection witnesses truth of (3′). So assume $I \neq \emptyset$. Since $\curlywedge x_i \preccurlyeq x_j$ for all $j \in I$, condition (2) implies that $R[x_j] \neq \emptyset$ for all $j \in I$. As a consequence of (4′) there is a smallest element $z_j := \curlywedge R[x_j]$ in each $R[x_j]$. We claim that $\curlywedge z_j \preccurlyeq z$. Suppose

not, then there is a clopen filter a containing $\curlywedge z_j$ such that $z \notin a$. This implies $x_j \in [R]a$ for all $j \in I$, but $\curlywedge x_i \notin [R]a$ because $(\curlywedge x_i)Rz$ and $z \notin a$. But this contradicts the fact that $[R]a$ is principal filter. $\quad \square$

This proposition motivates the following definition.

4.17 Definition. Let $\mathbb{X} = (X, 1, \curlywedge, \tau, R)$ be a modal L-space. Then we write $\pi\mathbb{X} = (X, 1, \curlywedge, R)$ for the underlying principal modal L-frame, and $\kappa\mathbb{X} = (X, 1, \curlywedge, R)$ for the underlying principal modal L-frame regarded as a modal L-frame.

While they may appear the same, the difference between $\pi\mathbb{X}$ and $\kappa\mathbb{X}$ lies in the valuations they allow for. While valuations of $\pi\mathbb{X}$ necessarily interpret proposition letters as principal filters, a valuation for $\kappa\mathbb{X}$ can assign any filter to a proposition letter. As a consequence, the frames differ in terms of *validity*.

For future reference, we prove the following lemma about modal L-spaces. It states that the action of $[R]$ and $\langle R \rangle$ on any filter is determined by their action on clopen filters.

4.18 Lemma. *Let $\mathbb{X} = (X, 1, \curlywedge, \tau, R)$ be a modal L-space. Then for every closed filter $c \in \mathscr{F}_k(\mathbb{X})$,*

$$[R]c = \bigcap \big\{ [R]a \mid a \in \mathscr{F}_{clp}(\mathbb{X}), c \subseteq a \big\} \quad and \quad \langle R \rangle c = \bigcap \big\{ \langle R \rangle a \mid a \in \mathscr{F}_{clp}(\mathbb{X}), c \subseteq a \big\}. \tag{9}$$

Furthermore, for every filter $d \in \mathscr{F}(\mathbb{X})$ we have

$$[R]d = \bigvee \big\{ [R]c \mid c \in \mathscr{F}_k(\mathbb{X}), c \subseteq d \big\} \quad and \quad \langle R \rangle d = \bigvee \big\{ \langle R \rangle c \mid c \in \mathscr{F}_k(\mathbb{X}), c \subseteq d \big\}. \tag{10}$$

Proof. The left-to-right inclusion of the first equality follows from the fact that $c \subseteq a$ implies $[R]c \subseteq [R]a$. For the converse, suppose $x \in \mathbb{X}$ is such that $x \notin [R]c$. Then $R[x] \nsubseteq c$, so there is some $y \in R[x]$ such that $y \notin c$. Since c is a closed filter, it is the intersection of all clopen filters that contain it. A routine compactness argument yields a clopen filter b containing c such that $y \notin b$. Therefore $R[x] \nsubseteq b$, so $x \notin [R]b$ and hence x is not in the right hand side of the equality.

For the second equality, it suffices to prove the right-to-left inclusion as well. Suppose $x \notin \langle R \rangle c$. Then $R[x] \cap c = \emptyset$. Since $R[x]$ is closed, we can use compactness to find some clopen filter a containing c disjoint from $R[x]$, and the argument proceeds as above.

Next, consider (10). We know that $c \subseteq d$ implies $[R]c \subseteq [R]d$, and since the latter is a filter containing $[R]c$ for all closed filters $c \subseteq d$, it contains the filter generated by them. This proves $\supseteq$. Conversely, suppose $x \in [R]d$. Then $R[x] \subseteq d$. Since d is up-closed we also have $\uparrow R[x] \subseteq d$. The set $\uparrow R[x]$ is closed by Lemmas 4.15 and 2.9, and it is a filter because $R[x]$ is closed under $\curlywedge$. Thus we have found a closed filter $c := \uparrow R[x]$ contained in d such that $x \in [R]c$. This proves $\subseteq$.

The right-to-left inclusion of the second equality in (10) is easy again. For $\subseteq$, suppose $x \in \langle R \rangle d$. Then $R[x] \cap d \neq \emptyset$, so we can find some $y \in R[x] \cap d$. Now $\uparrow y$ is a closed filter contained in d such that $x \in \langle R \rangle(\uparrow y)$, witnessing $\subseteq$. $\quad \square$

Since we know that each modal L-space has an underlying modal L-frame, we can conveniently define the morphisms between them as follows.

4.19 Definition. A *modal L-space morphism* from $(X, 1, \curlywedge, \tau, R)$ to $(X', 1', \curlywedge', \tau', R')$ is a function $f : X \to X'$ such that $f : (X, 1, \curlywedge, \tau) \to (X', 1', \curlywedge', \tau')$ is an L-space morphism and $f : (X, 1, \curlywedge, R) \to (X', 1', \curlywedge', R')$ is a bounded L-morphism. We denote the resulting category by $\mathsf{MLSpace}$.

We work towards a duality between modal lattices and modal L-spaces.

4.20 Proposition. *If $\mathbb{X} = (X, 1, \curlywedge, \tau, R)$ is a modal L-space then $(\mathscr{F}_{clp}\mathbb{X}, [R], \langle R \rangle)$ is a modal lattice. Moreover, if $f : \mathbb{X} \to \mathbb{X}'$ is a modal L-space morphism, then*

$$\mathscr{F}_{clp}f = f^{-1} : (\mathscr{F}_{clp}\mathbb{X}', [R'], \langle R' \rangle) \to (\mathscr{F}_{clp}\mathbb{X}, [R], \langle R \rangle)$$

is a modal lattice homomorphism.

Proof. The maps $[R], \langle R \rangle$ are functions on $\mathscr{F}_{clp}\mathbb{X}$ by definition. It follows from Proposition 4.16 and Lemma 4.7 that they satisfy the conditions from Definition 4.10.

If f is a modal L-space morphism then in particular it is an L-space morphism, so $\mathscr{F}_{clp}f$ is a lattice homomorphism from $\mathscr{F}_{clp}\mathbb{X}'$ to $\mathscr{F}_{clp}\mathbb{X}$. So we only have to show that f^{-1} preserves the modalities. This can be proven in the same way as in Proposition 4.6. $\square$

We now show how to turn a modal lattice into a modal L-space.

4.21 Definition. Let $\mathscr{A} = (A, \square, \Diamond)$ be a modal lattice. Then we define the binary relation R_A on $\mathscr{F}A$ by

$$p R_A q \quad \text{iff} \quad \square^{-1}(p) \subseteq q \subseteq \Diamond^{-1}(p).$$

4.22 Lemma. *Let $(A, \square, \Diamond)$ be a modal lattice and $p \in \mathscr{F}A$. Then $\square^{-1}(p)$ is a filter in $\mathscr{F}A$ and $p R_A \square^{-1}(p)$.*

Proof. The set $\square^{-1}(p)$ is a filter because $\square : A \to A$ preserves conjunctions and the top element. To show $p R_A \square^{-1}(p)$ we need to prove $\square^{-1}(p) \subseteq \square^{-1}(p) \subseteq \Diamond^{-1}(p)$. The left inclusion is trivial. For the right one, suppose $a \in \square^{-1}(p)$. Then $\square a \in p$. Besides, $\Diamond\top = \top \in p$. So the duality axiom implies $\Diamond\top \wedge \square a \leq \Diamond(\top \wedge a) = \Diamond a \in p$, hence $a \in \Diamond^{-1}(p)$. $\square$

4.23 Lemma. *Let $(A, \square, \Diamond)$ be a modal lattice. Then for each $a \in A$ we have*

$$[R_A]\theta_A(a) = \theta_A(\square a) \quad and \quad \langle R_A \rangle \theta_A(a) = \theta_A(\Diamond a).$$

Proof. Suppose $p \in [R_A]\theta_A(a)$. By Lemma 4.22 we have $p R_A \square^{-1}(p)$ and by assumption $a \in \square^{-1}(p)$. This implies that $\square a \in p$ and therefore $p \in \theta_A(\square a)$. For the reverse inclusion, suppose $p \in \theta_A(\square a)$. Then $a \in \square^{-1}(p)$, so $p R_A q$ implies $a \in q$, and hence $p \in [R_A]\theta_A(a)$.

Next, suppose $p \in \langle R_A \rangle \theta_A(a)$. Then there exists a filter q such that $p R_A q$ and $a \in q$. By definition of R_A this implies $a \in \Diamond^{-1}(p)$ and hence $\Diamond a \in p$, so $p \in \theta_A(\Diamond a)$. Conversely, suppose $p \in \theta_A(\Diamond a)$. Let q be the filter generated by $\square^{-1}(p)$ and a. We claim that $c \in q$ implies $\Diamond c \in p$. To see this, note that for each $c \in q$ there exists some $d \in \square^{-1}(p)$ such that $d \wedge a \leq c$. We then have $\square d \in p$, and $\Diamond a \in p$ by assumption, so $\square d \wedge \Diamond a \in p$. Since $\square d \wedge \Diamond a \leq \Diamond(d \wedge a) \leq \Diamond c$ we find $\Diamond c \in p$. The filter q is nonempty because it contains $\top$. Furthermore, we have $\square^{-1}(p) \subseteq q$ by definition of q and we just showed that $c \in q$ implies $\Diamond c \in p$, so that $q \subseteq \Diamond^{-1}(p)$. This proves $p R_A q$. By design $a \in q$ so q witnesses the fact that $p \in \langle R_A \rangle \theta_A(a)$. $\square$

4.24 Lemma. *If $\mathscr{A} = (A, \square, \Diamond)$ is a modal lattice, then $\mathscr{A}_* := (\mathscr{F}A, A, \cap, \tau_A, R_A)$ is a modal L-space.*

Proof. We verify the conditions from Definition 4.14. Item (1) follows from Theorem 2.14. Item (2) follows from the definition of R_A. Item (3) follows from Lemma 4.23. Item (4) follows from the definition of R_A and the fact that each clopen filter is of the form $\theta_A(a)$. $\square$

4.25 Lemma. *Let $h : \mathscr{A} \to \mathscr{A}'$ be a modal lattice homomorphism. Then $\mathscr{F}h = h^{-1} : \mathscr{A}'_* \to \mathscr{A}_*$ is a bounded L-space morphism.*

Proof. It follows from the duality between lattices and L-spaces that h^{-1} is an L-space morphism, so we only have to verify the three conditions from Definition 4.5. We write R' and R for the relations from $\mathscr{A}'_*$ and $\mathscr{A}_*$.

1. Let p' and q' be filters of $\mathscr{A}'$ (elements of $\mathscr{A}'_*$) such that $p'R'q'$. In order to prove that $h^{-1}(p')Rh^{-1}(q')$ we have to show that $\square^{-1}(h^{-1}(p')) \subseteq h^{-1}(q') \subseteq \diamond^{-1}(h^{-1}(p'))$. Let $a \in \square^{-1}(h^{-1}(p'))$. Then $\square a \in h^{-1}(p')$ so $\square'(h(a)) = h(\square a) \in p'$. Therefore $h(a) \in (\square')^{-1}(p')$, and since $p'R'q'$ this implies $h(a) \in q'$, so that $a \in h^{-1}(q')$. Next, if $a \in h^{-1}(q')$ then $h(a) \in q'$, so $h(\diamond a) = \diamond'h(a) \in p'$. Therefore $\diamond a \in h^{-1}(p')$ so that $a \in \diamond^{-1}(h^{-1}(p'))$.

2. Suppose $h^{-1}(p')Rq$. Lemma 4.22 then implies that $p'R'(\square')^{-1}(p')$. So it suffices to show that $h^{-1}((\square')^{-1}(p')) \subseteq q$. To this end, suppose $a \in h^{-1}((\square')^{-1}(p'))$. Then $h(\square a) = \square'h(a) \in p'$, so $\square a \in h^{-1}(p')$. Since $h^{-1}(p')Rq$ this implies $a \in q$, as desired.

3. Suppose $h^{-1}(p')Rq$. Then $\uparrow h[q]$ is a filter (since h is a lattice homomorphism it is non-empty and closed under meets). Let q' be the filter generated by $\uparrow h[q]$ and $(\square')^{-1}(p')$. Then $q \subseteq h^{-1}(q')$ by construction, so it suffices to show that $p'R'q'$. We have $(\square')^{-1}(p') \subseteq q'$ by definition, so we only have to show that $q' \subseteq (\diamond')^{-1}(p')$. Let $a' \in q'$. Then there are $b' \in (\square')^{-1}(p')$ and $c \in q$ such that $b' \wedge' h(c) \leq a'$. We find

$$\square'b' \wedge h(\diamond c) = \square'b' \wedge \diamond'h(c) \leq \diamond a'.$$

By construction we have $\square'b' \in p'$. Furthermore, $c \in q$ implies $\diamond c \in h^{-1}(p')$ and hence $h(\diamond c) \in p'$. Therefore $\diamond a' \in p'$, and consequently $p'R'q'$. $\square$

Using the above lemmas we now establish a duality for modal lattices.

4.26 Theorem. *The duality between* Lat *and* LSpace *lifts to a duality*

$$\mathsf{MLat} \equiv^{\mathrm{op}} \mathsf{MLSpace}.$$

Proof. Let $\mathbb{X} = (X, \leq, \tau, R)$ be a modal L-space. This gives rise to the modal lattice $\mathbb{X}^* = (\mathscr{F}_{clp}\mathbb{X}, [R], \langle R \rangle)$, which in turn yields a modal L-space $(\mathbb{X}^*)_* = (\mathscr{F}\mathscr{F}_{clp}\mathbb{X}, \mathscr{F}_{clp}\mathbb{X}, \cap, \tau_{\mathscr{F}_{clp}\mathbb{X}}, R_{\mathscr{F}_{clp}\mathbb{X}})$. As a consequence of the duality for lattices (Theorem 2.14) we know that $(X, 1, \curlywedge, \tau)$ is isomorphic to $(\mathscr{F}\mathscr{F}_{clp}\mathbb{X}, A, \cap, \tau_{\mathscr{F}_{clp}\mathbb{X}})$ via $x \mapsto \eta_{\mathbb{X}}(x) = \{a \in \mathscr{F}_{clp}\mathbb{X} \mid x \in a\}$. So we only have to show that R and $R_{\mathscr{F}_{clp}\mathbb{X}}$ coincide. This can be seen as follows:

$$
\begin{aligned}
xRy \quad &\text{iff} \quad \forall a \in \mathscr{F}_{clp}\mathbb{X}\big(x \in [R]a \text{ implies } y \in a \text{ and } y \in a \text{ implies } \langle R \rangle a\big) \\
&\text{iff} \quad \forall a \in \mathscr{F}_{clp}\mathbb{X}\big([R]a \in \eta_{\mathbb{X}}(x) \text{ implies } a \in \eta_{\mathbb{X}}(y) \text{ and } a \in \eta_{\mathbb{X}}(y) \text{ implies } \langle R \rangle a \in \eta_{\mathbb{X}}(x)\big) \\
&\text{iff} \quad \forall a \in \mathscr{F}_{clp}\mathbb{X}\big([R]^{-1}(\eta_{\mathbb{X}}(x)) \subseteq \eta_{\mathbb{X}}(y) \subseteq \langle R \rangle^{-1}(\eta_{\mathbb{X}}(x))\big) \\
&\text{iff} \quad \eta_{\mathbb{X}}(x)R_{\mathscr{F}_{clp}\mathbb{X}}\eta_{\mathbb{X}}(y)
\end{aligned}
$$

Next, let $\mathscr{A} = (A, \square, \diamond)$ be a modal lattice, $\mathscr{A}_* = (\mathscr{F}_{top}A, R_A)$ and $(\mathscr{A}_*)^* = (\mathscr{F}_{clp}\mathscr{F}_{top}A, [R_A], \langle R_A \rangle)$. Then the duality for lattices from Theorem 2.14 tells us that A and $\mathscr{F}_{clp}\mathscr{F}_{top}A$ are isomorphic via $a \mapsto \theta_A(a) = \{p \in \mathscr{F}_{top}A \mid a \in p\}$, so we just have to show that $\square$ coincides with $[R_A]$ and $\diamond$ coincides with $\langle R_A \rangle$. That is, we have to show that $\theta_A(\square a) = [R_A]\theta_A(a)$ and $\theta_A(\diamond a) = \langle R_A \rangle \theta_A(a)$. But we have already proven that in Lemma 4.23.

The two paragraphs above establish the duality on objects. The duality for morphisms follows immediately from Theorem 2.14 and the fact that all our categories are concrete. $\square$

4.4. Completions of modal lattices

We extend the Π_1-completion to modal lattices. Lemma 4.18 suggests the following definition of completions of a modal lattice.

4.27 Definition. Let $\mathscr{A} = (A, \Box, \Diamond)$ be a modal lattice. Let $i : A \to fe(A)$ be the filter completion of A, and $j : A \to ie(A)$ the ideal completion.

1. The *filter completion* of $\mathscr{A}$ is the modal lattice $fe(\mathscr{A}) = (fe(A), \Box^{fe}, \Diamond^{fe})$ where

$$\star^{fe} c = \bigwedge \{i(\star a) \mid a \in A \text{ and } c \leq i(a)\} \qquad \text{for } \star \in \{\Box, \Diamond\}.$$

2. The *ideal completion* of $\mathscr{A}$ is the modal lattice $ie(\mathscr{A}) = (ie(A), \Box^{ie}, \Diamond^{ie})$ where

$$\star^{ie} c = \bigvee \{j(\star a) \mid a \in A \text{ and } j(a) \leq c\} \qquad \text{for } \star \in \{\Box, \Diamond\}.$$

3. The Π_1-*completion* of $\mathscr{A}$ is $ie(fe(\mathscr{A}))$, i.e. the composition of the filter and the ideal completion.

Concretely, if we view $fe(A)$ as sitting inside $\Pi_1(A)$, then the Π_1-*completion* of $\mathscr{A}$ is the modal lattice $\Pi_1(\mathscr{A}) = (\Pi_1(A), \Box^{\Pi_1}, \Diamond^{\Pi_1})$ where

$$\star^{\Pi_1} d = \bigvee \{\star^{fe} c \mid c \in fe(A) \text{ and } c \leq d\} \qquad \text{for } \star \in \{\Box, \Diamond\}. \tag{11}$$

Recall that the modal L-frame underlying a modal L-space is principal, so both the collection of closed filters as well as the collection of all filters form modal lattices. We already proved (for the duality result) that the function $\theta : \mathscr{A} \to (\mathscr{A}_*)^*$ satisfies $\theta(\Box a) = [R_A]\theta(a)$ and $\theta(\Diamond a) = \langle R_A \rangle \theta(a)$, so we can use that.

4.28 Proposition. *Let $\mathscr{A} = (A, \Box, \Diamond)$ be a modal lattice and $\mathbb{X}$ its dual modal L-space. Then:*

1. *$fe(\mathscr{A})$ is isomorphic to the modal lattice of closed filters of $\mathbb{X}$, i.e. to $(\pi\mathbb{X})^{\ddagger}$;*
2. *$\Pi_1(\mathscr{A})$ is isomorphic to the modal lattice of filters of $\mathbb{X}$, i.e. to $(\kappa\mathbb{X})^*$.*

Proof. Let $\mathbb{X} = (X, 1, \curlywedge, \tau, R_A)$ be the modal L-space dual to $\mathscr{A}$. By Proposition 2.18 we can identify the filter extension of A with the lattice of closed filters of $\mathbb{X}$ with inclusion $\theta : A \to \mathscr{F}_k(\mathbb{X})$. In $\mathscr{F}_k(\mathbb{X})$, the top, bottom and meet are given by X, $\{1\}$ and intersection, and the join of a collection of filters is the smallest filter containing their union. So we only have to verify that for all $c \in \mathscr{F}_k(\mathbb{X})$ we have $\Box^{fe} c = [R_A]c$ and $\Diamond^{fe} c = \langle R_A \rangle c$. Thus compute

$$\begin{aligned}
\Box^{fe} c &= \bigcap \{\theta(\Box a) \mid a \in A, c \subseteq \theta(a)\} && \text{(Definition of } \Box^{fe}) \\
&= \bigcap \{[R_A]\theta(a) \mid a \in A, c \subseteq \theta(a)\} && \text{(Lemma 4.23)} \\
&= \bigcap \{[R_A]b \mid b \in \mathscr{F}_{clp}(\mathbb{X}), c \subseteq b\} && (\theta \text{ is an iso from } A \text{ to } \mathscr{F}_{clp}(\mathbb{X})) \\
&= [R_A]c. && \text{(Lemma 4.18)}
\end{aligned}$$

Similarly we find $\Diamond^{fe} c = \langle R_A \rangle c$ for all $c \in \mathscr{F}_k(\mathbb{X})$.

For the second item we adopt a similar strategy. Using Proposition 2.18, we identify the Π_1-completion of A with $\mathscr{F}(\mathbb{X})$, the filters of $\mathbb{X}$ with operators as in (11), and inclusion $\theta : A \to \mathscr{F}(\mathbb{X})$. We view the filter

completion of A as sitting inside the Π_1-completion, just like $\mathscr{F}_k(\mathbb{X}) \subseteq \mathscr{F}(\mathbb{X})$. Then we already know that $\square^{fe}c = [R_A]c$ for all closed filters, therefore

$$\square^{\Pi_1}d = \bigvee \{\square^{fe}c \mid c \in fe(A), c \subseteq d\} = \bigvee \{[R_A]c \mid c \in \mathscr{F}_k(\mathbb{X}), c \subseteq d\} = [R]d$$

for all filters d. We use Lemma 4.18 for the last equality. Similarly we find $\lozenge^{\Pi_1}d = \langle R_A \rangle d$. $\square$

Proposition 4.28 immediately implies:

4.29 Corollary. *The filter and Π_1-completion of a modal lattice are modal lattices themselves.*

Next, we work towards a preservation theorem.

4.30 Theorem. *Let $\mathscr{A} = (A, \square, \lozenge)$ be a modal lattice and $\varphi, \psi \in \mathbf{L}_{\square\lozenge}$. Then*

$$\mathscr{A} \Vdash \varphi \lessdot \psi \quad \textit{iff} \quad fe(\mathscr{A}) \Vdash \varphi \lessdot \psi.$$

The proof of the theorem relies on Lemma 4.31. We use following definition: if θ_1 and θ_2 are valuations for $\mathscr{A}$, then we define the valuation $\theta_1 \wedge \theta_2$ by $(\theta_1 \wedge \theta_2)(p) := \theta_1(p) \wedge \theta_2(p)$.

4.31 Lemma. *Let $\mathscr{A} = (A, \square, \lozenge)$ be a modal lattice, $fe(\mathscr{A})$ its filter completion, and σ a valuation of the proposition letters for $fe(\mathscr{A})$.*

1. *If $a \in A$ and $\varphi \in \mathbf{L}_{\square\lozenge}$ are such that $a \in \sigma(\varphi)$, then there exists a valuation θ for $\mathscr{A}$ such that $\theta(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$ and $\theta(\varphi) \leq a$.*
2. *If θ is a valuation for $\mathscr{A}$ such that $\theta(p) \in \sigma(p)$ for all p, then $\theta(\varphi) \in \sigma(\varphi)$ for all $\varphi \in \mathbf{L}_{\square\lozenge}$.*

Proof. We start with the first statement. We proceed by induction on the structure of φ. If $\varphi = p \in \mathrm{Prop}$ then we set $\theta(p) = a$ and $\theta(q) = \top$ for all other $q \in \mathrm{Prop}$. If $\varphi = \top$ then $\theta(\top) = \top \in \{\top\} = \sigma(\top)$, and if $\varphi = \bot$ then $\theta(\bot) = \bot \in A = \sigma(\bot)$.

Suppose $\varphi = \varphi_1 \wedge \varphi_2$. If $a \in \sigma(\varphi_1 \wedge \varphi_2) = \sigma(\varphi_1) \triangledown \sigma(\varphi_2)$ then there are $a_1 \in \sigma(\varphi_1)$ and $a_2 \in \sigma(\varphi_2)$ such that $a_1 \wedge a_2 \leq a$. The induction hypothesis then gives valuations θ_1, θ_2 for $\mathscr{A}$ such that $\theta_1(\varphi_1) \leq a_1$ and $\theta_2(\varphi_2) \leq a_2$, and such that $\theta_1(p) \in \sigma(p)$ and $\theta_2(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$. Let $\theta = \theta_1 \wedge \theta_2$. Since $\theta_i(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$, and $\sigma(p)$ is a filter, we have $\theta(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$. Then $\theta(\varphi_i) \leq \theta_i(\varphi_i)$ for $i \in \{1, 2\}$, and we find $\theta(\varphi_1 \wedge \varphi_2) = \theta(\varphi_1) \wedge \theta(\varphi_2) \leq \theta_1(\varphi_1) \wedge \theta_2(\varphi_2) \leq a_1 \wedge a_2 \leq a$.

If $\varphi = \varphi_1 \vee \varphi_2$, then $a \in \sigma(\varphi_1 \vee \varphi_2) = \sigma(\varphi_1) \cap \sigma(\varphi_2)$ so there are valuations θ_1, θ_2 such that $\theta_1(\varphi_1) \leq a$ and $\theta_2(\varphi_2) \leq a$, and such that $\theta_1(p) \in \sigma(p)$ and $\theta_2(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$. Let again $\theta = \theta_1 \wedge \theta_2$. Then just as above $\theta(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$. Moreover, $\theta(\varphi_1 \vee \varphi_2) = \theta(\varphi_1) \vee \theta(\varphi_2) \leq \theta_1(\varphi_1) \vee \theta_2(\varphi_2) \leq a \vee a = a$.

Finally, let $\star \in \{\square, \lozenge\}$ and $\varphi = \star\varphi_1$. Suppose $a \in \sigma(\star\varphi_1)$. Then

$$a \in \sigma(\star\varphi_1) = \star^{fe}\sigma(\varphi_1) = \bigwedge\{\uparrow(\star b) \mid b \in A, \sigma(\varphi_1) \leq \uparrow b\} = \bigwedge\{\uparrow(\star b) \mid b \in A, b \in \sigma(\varphi_1)\}.$$

The last equality follows from the fact that filters are ordered by reverse inclusion, so that $\sigma(\varphi_1) \leq \uparrow b$ iff $\sigma(\varphi_1) \supseteq \uparrow b$ iff $b \in \sigma(\varphi_1)$. Since $\bigwedge$ in $fe(\mathscr{A})$ is defined as $\bigvee$, there exist $b_1, \ldots, b_n \in A$ such that $b_i \in \sigma(\varphi_1)$ for all $i \in \{1, \ldots, n\}$ and $\star b_1 \wedge \cdots \wedge \star b_n \leq a$. The induction hypothesis implies that there exist valuations $\theta_1, \ldots, \theta_n$ for $\mathscr{A}$ such that $\theta_i(\varphi_1) \leq b_i$ and $\theta_i(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$ and $i \in \{1, \ldots, n\}$. Since $\star$ is monotone we find $\theta_i(\star\varphi_1) = \star\theta_i(\varphi_1) \leq \star b_i$ for all i. Letting $\theta = \theta_1 \wedge \cdots \wedge \theta_n$ gives $\theta(\star\varphi_1) \leq \theta_i(\star\varphi_1) \leq \star b_i$ for all i, so that $\theta(\star\varphi_1) \leq \star b_1 \wedge \cdots \wedge \star b_n \leq a$, as desired. Since $\sigma(p)$ is a filter and $\theta_i(p) \in \sigma(p)$ for all $i \in \{1, \ldots, n\}$ we find $\theta(p) \in \sigma(p)$ for all $p \in \mathrm{Prop}$.

We prove the second item by induction on the structure of φ as well. The base cases $\varphi = \top, \bot, p \in \mathrm{Prop}$ are routine. Suppose $\varphi = \varphi_1 \wedge \varphi_2$. By induction we have $\theta(\varphi_1) \in \sigma(\varphi_1)$ and $\theta(\varphi_2) \in \sigma(\varphi_2)$. So $\theta(\varphi_1 \wedge \varphi_2) = \theta(\varphi_1) \wedge \theta(\varphi_2) \in \sigma(\varphi_1) \triangledown \sigma(\varphi_2) = \sigma(\varphi_1 \wedge \varphi_2)$. If $\varphi = \varphi_1 \vee \varphi_2$, then with the same induction hypothesis we find $\theta(\varphi_i) \leq \theta(\varphi_1 \vee \varphi_2)$ so $\theta(\varphi_1 \vee \varphi_2) \in \sigma(\varphi_i)$ for $i \in \{1,2\}$. Therefore $\theta(\varphi_1 \vee \varphi_2) \in \sigma(\varphi_1) \cap \sigma(\varphi_2) = \sigma(\varphi_1 \vee \varphi_2)$. Lastly suppose $\star \in \{\Box, \Diamond\}$ and $\varphi = \star\varphi_1$. By the induction hypothesis we have $\theta(\varphi_1) \in \sigma(\varphi_1)$, so $\sigma(\varphi_1) \supseteq \uparrow\theta(\varphi_1)$ and since $fe(\mathcal{A})$ is ordered by reverse inclusion, it follows from the definition of $\star^{fe}$ that $\star^{fe}\sigma(\varphi_1) \supseteq \uparrow\star\theta(\varphi_1)$, hence

$$\theta(\star\varphi_1) = \star\theta(\varphi_1) \in \star^{fe}\sigma(\varphi_1) = \sigma(\star\varphi_1)$$

as desired. $\square$

We can now prove the theorem.

Proof of Theorem 4.30. We have $\mathcal{A} \Vdash \varphi \lessdot \psi$ iff $\mathcal{A} \Vdash \varphi \wedge \psi = \varphi$, so it suffices to focus on equalities. The proposition letters play the role of variables, and every valuation of the proposition letters to elements in $\mathcal{A}$ extends to valuations for φ and ψ in the obvious way.

So suppose $\mathcal{A} \Vdash \varphi = \psi$. Let σ be any valuation for $fe(\mathcal{A})$ and $a \in A$. We aim to prove that $a \in \sigma(\varphi)$ if and only if $a \in \sigma(\psi)$. So suppose $a \in \sigma(\varphi)$. Then by Lemma 4.31(1) we can find a valuation θ for $\mathcal{A}$ such that $\theta(\varphi) \leq a$. By assumption $\mathcal{A} \Vdash \varphi = \psi$, so $\theta(\psi) = \theta(\varphi) \leq a$, hence by Lemma 4.31(2) we find $a \in \sigma(\psi)$. Similarly we can prove that $a \in \sigma(\psi)$ implies $a \in \sigma(\varphi)$ so that $\sigma(\varphi) = \sigma(\psi)$. Since σ is arbitrary, we conclude $fe(\mathcal{A}) \Vdash \varphi = \psi$. $\square$

We note that the proof of Lemma 4.31 only relies on monotonicity of the modalities. Thus it yields an analogue of Theorem 4.30 for lattices with monotone operators.

The arguments from Lemma 4.31 and Theorem 4.30 can be dualised to similar results for ideal completions. The ideal completion of a modal lattice need not give rise to a modal lattice, but it does give rise to a lattice with two monotone operators, and the modal cases from Lemma 4.31 and Theorem 4.30 only rely on monotonicity of the modalities. Thus we obtain similar results for the ideal completion and hence for the Π_1-completion of a modal lattice.

4.32 Theorem. *Let $\mathcal{A} = (A, \Box, \Diamond)$ be a modal lattice and $\varphi, \psi \in \mathbf{L}_{\Box\Diamond}$. Then*

$$\mathcal{A} \Vdash \varphi \lessdot \psi \quad \textit{iff} \quad fe(\mathcal{A}) \Vdash \varphi \lessdot \psi \quad \textit{iff} \quad \Pi_1(\mathcal{A}) \Vdash \varphi \lessdot \psi.$$

We can use this algebraic result to obtain completeness for weak positive modal logics. Theorem 4.32 yields the following analogue of Lemma 3.21.

4.33 Lemma. *Any consequence pair $\psi \lessdot \chi$ of $\mathbf{L}_{\Box\Diamond}$-formulae is Π_1-persistent.*

4.34 Theorem. *Let Γ be a set of consequence pairs. Then the logic $\mathcal{L}_{\Box\Diamond}(\Gamma)$ is sound and complete with respect to the following classes of frames:*

- *Modal L-spaces validating Γ;*
- *Principal modal L-frames validating Γ;*
- *Modal L-frames validating Γ.*

Proof. Similar to the proof of Theorem 3.22. $\square$

4.5. Sahlqvist correspondence

We extend the results from Section 3.4 to obtain Sahlqvist correspondence for modal L-frames. Our definition of a Sahlqvist consequence pair is closely aligned to Sahlqvist formulae from normal modal logic (see e.g. [6, Definition 3.51]). To account for the additional relation in the definition of a modal L-model, we work with a first-order logic with an extra binary relation symbol (compared to Section 3.4). That is, we let **FOL$_2$** be the first-order logic with a unary predicate for each proposition letter and two binary predicates S (corresponding to the partial order) and R (corresponding to the modal relation). We write **SOL$_2$** for the second-order logic with the same predicates where we allow quantification over unary predicates. Every modal L-model $\mathfrak{M}$ gives rise to a first-order structure $\mathfrak{M}^\circ$ for **FOL$_2$** in the obvious way, and similarly every modal L-frame $\mathfrak{X}$ yields a structure $\mathfrak{X}^\circ$ where we can interpret **SOL$_2$**-formulae with no free predicates. We extend the standard translation from Definition 3.26 to a translation $\mathrm{st}_x : \mathbf{L}_{\square\diamond} \to \mathbf{FOL_2}$ by adding the clauses

$$\mathrm{st}_x(\square\varphi) = \forall y(xRy \to \mathrm{st}_y(\varphi)), \qquad \mathrm{st}_x(\diamond\varphi) = \exists y(xRy \wedge \mathrm{st}_y(\varphi)).$$

We then have the following counterpart of Proposition 3.27.

4.35 Proposition. *Let $\mathfrak{M}$ be a modal L-model, w a state in $\mathfrak{M}$ and φ an $\mathbf{L}_{\square\diamond}$-formula. Then*

1. $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi)[w]$;
2. $\mathfrak{M}, w \Vdash \varphi \lessdot \psi$ iff $\mathfrak{M}^\circ \models \mathrm{st}_x(\varphi \lessdot \psi)[w]$.

Defining the second-order translation of a modal consequence pair $\varphi \lessdot \psi$ as in Definition 3.28, we can extend Lemma 3.29 to the next lemma:

4.36 Lemma. *For all modal L-frames $\mathfrak{X} = (X, 1, \curlywedge, R)$ and modal consequence pairs $\psi \lessdot \chi$,*

$$\mathfrak{X}, w \Vdash \psi \lessdot \chi \quad \textit{iff} \quad \mathfrak{X}^\circ \models \mathrm{so}(\psi \lessdot \chi)[w].$$

Finally, we still have monotonicity of all connectives of $\mathbf{L}_{\square\diamond}$, so the following analogue of Lemma 3.31 goes through without problems.

4.37 Lemma. *Let $\mathfrak{X}$ be a modal L-frame and let V and V' be valuations for $\mathfrak{X}$ such that $V(p) \subseteq V'(p)$ for all $p \in \mathrm{Prop}$. Then for all $\varphi \in \mathbf{L}_{\square\diamond}$ we have $V(\varphi) \subseteq V'(\varphi)$.*

We are now ready to define Sahlqvist consequence pairs and prove a correspondence result. We make use of the following notion of a boxed atom.

4.38 Definition. A *boxed atom* is a formula of the form

$$\square^n p := \underbrace{\square \cdots \square}_{n \text{ times}} p,$$

where p is a proposition letter. A *Sahlqvist antecedent* is a formula made from boxed atoms, $\top$ and $\bot$ by freely using $\wedge$, $\vee$ and $\diamond$. A modal consequence pair $\varphi \lessdot \psi$ is called *Sahlqvist* if φ is a Sahlqvist antecedent (and ψ is any formula).

If R is a relation, then we write R^n for the n-fold composition of R. That is, $xR^n y$ if there exist $x_0, \ldots, x_n$ such that $x = x_0$, $y = x_n$ and $x_i R x_{i+1}$ for all $i \in \{0, \ldots, n-1\}$. With this definition, truth of $\Box^n p$ in a modal L-model $\mathfrak{M} = (X, \leq, R, V)$ can be given as

$$\mathfrak{M}, x \Vdash \Box^n p \quad \text{iff} \quad \forall y \in X, xR^n y \text{ implies } \mathfrak{M}, y \Vdash p.$$

We legislate $xR^0 y$ iff $x = y$. Then the interpretation of $\Box^0 p$ simply coincides with p.

4.39 Theorem. *Any Sahlqvist modal consequence pair $\psi \lessdot \chi$ locally corresponds to a first-order formula with one free variable.*

Proof. By Lemma 4.36 we have $\mathfrak{X}, w \Vdash \psi \lessdot \chi$ if and only if $\mathfrak{X}^\circ \models \mathrm{so}(\psi \lessdot \chi)[w]$. As in Theorem 3.32, our strategy for obtaining a first-order formula is to remove all second-order quantifiers from $\mathrm{so}(\psi \lessdot \chi)[w]$. We assume that no two quantifiers bind the same variable. The case where $\psi = \top$ or $\bot$ can be handled as in Theorem 3.32. Let $p_1, \ldots, p_n$ be the propositional variables occurring in ψ, and write $P_1, \ldots, P_n$ for their corresponding unary predicates. We assume that every proposition letter that occurs in χ also occurs is ψ, for otherwise we may replace it by $\bot$ to obtain a formula which is equivalent in terms of validity on frames.

Step 1. Use equivalences of the form

$$(\exists w(\alpha(w)) \wedge \beta) \leftrightarrow \exists w(\alpha(w) \wedge \beta), \qquad (\exists w(\alpha(w)) \vee \beta) \leftrightarrow \exists w(\alpha(w) \vee \beta),$$

and

$$(\exists w(\alpha(w)) \rightarrow \beta) \leftrightarrow \forall w(\alpha(w) \rightarrow \beta)$$

to pull out all existential quantifiers that arise in $\mathrm{st}_x(\psi)$. Let $Y := \{y_1, \ldots, y_m\}$ denote the set of (bound) variables that arise in the antecedent of the implication from the second-order translation. We end up with a formula of the form

$$\forall P_1 \cdots \forall P_n \forall y_1 \cdots \forall y_m \big((\mathrm{ISFIL} \wedge \mathrm{AT} \wedge \mathrm{REL}) \rightarrow \mathrm{st}_x(\chi) \big) \tag{12}$$

where

- ISFIL is a conjunction of formulae of the form $\mathsf{isfil}(P_i)$;
- BOX-AT is a conjunction of standard translations of boxed proposition letters, i.e. formulae of the form $\forall z(z' R^n z \rightarrow P_i z)$ (here $P_i z$ is viewed as $\forall z(z' R^0 z \rightarrow P_i z)$);
- REL is a conjunction of formulae of the form zRz' and $\mathsf{abovemeet}(z; , z', z'')$.

Step 2. Next we read off minimal instances of the P_i making the antecedent true. For each proposition letter P_i, let $\forall y_{i_1}(z_{i_1} R^{n_1} y_{i_1} \rightarrow P_i y_{i_1}), \ldots, \forall y_{i_k}(z_{i_k} R^{n_k} y_{i_k} \rightarrow P_i y_{i_k})$ be the occurrences of P_i in BOX-AT in the antecedent of (12). Intuitively, we define the valuation of p_i to be the filter generated by the (interpretations of) $y_{i_1}, \ldots, y_{i_k}$. Formally,

$$\sigma(P_i) := \bigvee \Big\{ \exists w_{i_1} \cdots \exists w_{i_k} \big(z_{i_1} R^{n_1} w_{i_1} \wedge \cdots \wedge z_{i_k} R^{n_k} w_{i_k} \\ \wedge \mathsf{abovemeet}(u; w_{i_1}, \ldots, w_{i_k}) \big) \mid \{i_1, \ldots, i_k\} \Big\}.$$

(If $k = 0$, i.e. there are no boxed atoms involving P_i in the formula, then we take the empty meet to be falsum, i.e. $\sigma(P_i) = \lambda u.(u \neq u)$.)

The remainder of the proof is analogous to the proof of Theorem 3.32.   □

In the next example we apply the algorithm of the proof of Theorem 4.39 to two simple consequence pairs, $p \lhd \Diamond p$ and $\Box p \lhd p$. This shows the mechanism of the proof in action. Moreover, it demonstrates that the duality between $\Box$ and $\Diamond$ is weaker than in the classical case, because the formulae locally correspond to different frame conditions.

In the correspondents, we write R for the modal relation and $\preccurlyeq$ for the poset order. Technically this should be S, which is then interpreted as $\preccurlyeq$. Besides, note that $\mathsf{abovemeet}(u; z)$ is the same as zSu, which we denote by $z \preccurlyeq u$. Similarly, $\mathsf{abovemeet}(u; z, z')$ means $z \curlywedge z' \preccurlyeq u$.

4.40 Example. The second-order translation of $p \lhd \Diamond p$ is

$$\forall P(\mathsf{isfil}(P) \wedge Px \rightarrow \exists y(xRy \wedge Py))$$

This is already of the desired shape, so we proceed to Step 2. We find $\sigma(P) = \lambda u.x \preccurlyeq u$. Substituting this gives the first-order formula $\forall x(\mathsf{isfil}(P) \wedge (x \preccurlyeq x) \rightarrow \exists y(xRy \wedge (x \preccurlyeq y)))$. The antecedent of the formula is always true, so the (simplified) local correspondent of $p \lhd \Diamond p$ is

$$\exists y(xRy \wedge x \preccurlyeq y).$$

Thus, a frame satisfies $p \lhd \Diamond p$ if $\forall x \exists y(xRy \wedge x \preccurlyeq y)$.   ◁

4.41 Example. Next consider $\Box p \lhd p$. The second-order translation is

$$\forall P(\mathsf{isfil}(P) \wedge \forall y(xRy \rightarrow Py) \rightarrow Px).$$

Then $\sigma(P) = \lambda u.\exists y(xRy \wedge y \preccurlyeq u)$. Instantiating this makes the antecedent of the outer implication vacuously true, so that we get the local correspondent $\exists y(xRy \wedge y \preccurlyeq x)$. Validity of $\Box p \lhd p$ on a frame then corresponds to $\forall x \exists y(xRy \wedge y \preccurlyeq x)$.   ◁

Next, we use Theorem 4.39 to enforce normality for the diamond operator. It follows from Lemma 4.7 that $\Diamond p \vee \Diamond q \lhd \Diamond (p \vee q)$ is valid in every modal L-frame, so we focus on its converse.

4.42 Example. If we want $\Diamond$ to preserve binary joins we need to add

$$\Diamond(p \vee q) \lhd \Diamond p \vee \Diamond q \tag{13}$$

to our logic. This is Sahlqvist, so we can use the algorithm from Theorem 4.39 to obtain a first-order frame condition ensuring its validity. The second-order translation is

$$\forall P \forall Q \big(\mathsf{isfil}(P) \wedge \mathsf{isfil}(Q) \wedge \exists y(xRy \wedge (\exists z \exists z'(\mathsf{abovemeet}(y; z, z') \wedge Pz \wedge Qz'))) \rightarrow \mathrm{st}_x(\Diamond p \vee \Diamond q)\big).$$

Processing the formula, we obtain the following second-order translation:

$$\forall P \forall Q \forall y \forall z \forall z' (\mathrm{ISFIL} \wedge xRy \wedge \mathsf{abovemeet}(y; , z, z') \wedge Pz \wedge Qz' \rightarrow \mathrm{st}_x(\Diamond p \vee \Diamond q)). \tag{14}$$
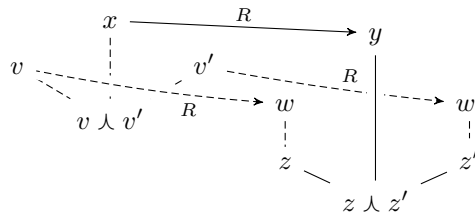
This gives $\sigma(P) = \lambda u.z \preccurlyeq u$, $\sigma(Q) = \lambda u.z' \preccurlyeq u$. The standard translation of the antecedent is

$$\mathrm{st}_x(\chi) = \exists v \exists v'(\mathsf{abovemeet}(x; v, v') \wedge \exists w(vRw \wedge Pw) \wedge \exists w'(v'Rw' \wedge Qw')).$$

Substituting P and Q and omitting trivial terms we obtain the following local correspondent:

$$\forall y \forall z \forall z' \big((xRy \wedge (z \curlywedge z' \preccurlyeq y)) \to \exists v \exists v'((v \curlywedge v' \preccurlyeq x) \wedge \exists w(vRw \wedge z \preccurlyeq w) \wedge \exists w'(v'Rw' \wedge z' \preccurlyeq w'))\big).$$

In a picture:

A modal L-frame satisfies normality of $\diamond$ if this holds for all states x. It is a relational analogue of the second condition from Definition 2.4 that ensures preservation of joins. ◁

5. Conclusion

We have given a new duality for bounded (not necessarily distributive) lattices which resembles Stone-type dualities. It builds on a known duality for the category of meet-semilattices with top given by Hofmann, Mislove and Stralka [32]. The relation between our duality and the duality by Hofmann, Mislove and Stralka is similar to the relation between Esakia duality and Priestley duality. It can also be seen as a Stone-type analogue of Jipsen and Moshier's spectral duality for lattices [40].

We also extended the duality to one for a modal extension of weak positive logic with □ and ◇. While these are interpreted using a relation in the same way as in normal modal logic over a classical base, the non-standard interpretation of joins causes ◇ to no longer be join-preserving. This interesting phenomenon has also been observed in the context of modal intuitionistic logic [38].

As the dualities presented in this paper resemble known dualities used in modal logic, they allow us to use similar tools and techniques. To showcase this, we proved Π_1-persistence and Sahlqvist correspondence results along the lines of [6].

There are many intriguing avenues for further research, some of which we list below.

Finite model property. While it is easy to derive the finite model property for weak positive logic, the same result for the modal extension presented in this paper appears to be non-trivial.

Relation to ortho(modular) lattices. Ortholattices and orthomodular lattices provide other interesting classes of lattices with operators. However, in ortholattices the orthocomplement turns joins into meets. Duality for these structures has been discussed by Goldblatt [23,24] and Bimbo [5]. In [14, Chapter 6] the duality for lattices is extended to account for modal operators that turn joins into meets. Recently, modal ortholattices have been studied in [33]. We leave it as an interesting open problem to see whether the preservation and correspondence results of this paper can be extended to this setting. It is also open whether this technique could be extended to orthomodular lattices [36]. This is especially interesting as orthomodalur lattices provide algebraic structures of quantum logic [13], so these methods could be relevant in the study of quantum logic.

Different modalities. Yet another question is what other modal extensions of weak positive logic we can define. In particular, it would be interesting to define a form of neighbourhood semantics based on the L-frames used in this paper and investigate the behaviour of the resulting modalities.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

References

[1] S. Abramsky, A. Jung, Domain Theory, vol. 3, Clarendon Press, Oxford, 1994, pp. 1–168.
[2] K.A. Baker, A.W. Hales, From a lattice to its ideal lattice, Algebra Univers. 4 (1974) 250–258.
[3] J. van Benthem, Relational patterns, partiality, and set lifting in modal semantics, in: Saul Kripke on Modality, 2022.
[4] N. Bezhanishvili, W. Holliday, Choice-free stone duality, J. Symb. Log. 85 (1) (2020) 109–148.
[5] K. Bimbó, Functorial duality for ortholattices and de Morgan lattices, Log. Univers. 1 (2) (2007) 311–333.
[6] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, Cambridge, 2001.
[7] S.A. Celani, R. Jansana, Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic, Log. J. IGPL 7 (1999) 683–715.
[8] A. Chagrov, M. Zakharyaschev, Modal Logic, Clarendon Press, 1997.
[9] D.M. Clark, B.A. Davey, Natural Dualities for the Working Algebraist, Cambridge Studies in Advanced Mathematics, vol. 57, Cambridge University Press, Cambridge, UK, 1998.
[10] W. Conradie, A. Palmigiano, Algorithmic correspondence and canonicity for non-distributive logics, Ann. Pure Appl. Log. 170 (9) (2019) 923–974.
[11] B. Davey, H. Priestley, Introduction to Lattices and Order, second edition, Cambridge University Press, 2002.
[12] B.A. Davey, H. Werner, Dualities and equivalences for varieties of algebras, in: Contributions to Lattice Theory, Szeged, 1980, 1983.
[13] C. De Ronde, G. Domenech, H. Freytes, Quantum logic in historical and philosophical perspective, in: The Internet Encyclopedia of Philosophy, 2014.
[14] A. Dmitrieva, Positive modal logic beyond distributivity: duality, preservation and completeness, Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2021.
[15] J.M. Dunn, Positive modal logic, Stud. Log. 55 (1995) 301–317.
[16] L. Esakia, Topological Kripke models, Sov. Math. Dokl. 15 (1974) 147–151.
[17] L. Esakia, Heyting Algebras, Springer, 2019. English translation of the original 1985 book.
[18] M. Gehrke, J. Harding, Bounded lattice expansions, J. Algebra 238 (2001) 345–371.
[19] M. Gehrke, B. Jónsson, Bounded distributive lattices with operators, Math. Jpn. 40 (2) (1994) 207–215.
[20] M. Gehrke, H. Nagahashi, Y. Venema, A Sahlqvist theorem for distributive modal logic, Ann. Pure Appl. Log. 131 (1–3) (2005) 65–102.
[21] M. Gehrke, H. Priestley, Canonical extensions and completions of posets and lattices, Rep. Math. Log. 43 (2008) 133–152.
[22] M. Gehrke, S. Van Gool, Distributive envelopes and topological duality for lattices via canonical extensions, Order 31 (3) (2014) 435–461.
[23] R. Goldblatt, Semantic analysis of orthologic, J. Philos. Log. 3 (1–2) (1974) 19–35.
[24] R. Goldblatt, The Stone space of an ortholattice, Bull. Lond. Math. Soc. 7 (1) (1975) 45–48.
[25] R. Goldblatt, Varieties of complex algebras, Ann. Pure Appl. Log. 44 (3) (1989) 173–242.
[26] R. Goldblatt, Morphisms and duality for polarities and lattices with operators, J. Appl. Log. 7 (6) (2020) 1019–1072.
[27] C. Hartonas, Discrete duality for lattices with modal operators, J. Log. Comput. 29 (1) (2019) 71–89.
[28] C. Hartonas, E. Orlowska, Representation of lattices with modal operators in two-sorted frames, Fundam. Inform. 166 (1) (2019) 29–56.
[29] G. Hartung, A topological representation of lattices, Algebra Univers. 29 (1992) 273–299.
[30] L. Hella, K. Luosto, K. Sano, J. Virtema, The expressive power of modal dependence logic, in: R. Goré, B.P. Kooi, A. Kurucz (Eds.), Proc. AIML 2014, 2014, pp. 294–312.
[31] W. Hodges, Compositional semantics for a language of imperfect information, Log. J. IGPL 5 (4) (1997) 539–563.
[32] K.H. Hofmann, M. Mislove, A. Stralka, The Pontryagin Duality of Compact 0-Dimensional Semi-Lattices and Its Applications, Springer, Berlin, New York, 1974.
[33] W. Holliday, M. Mandelkern, The orthologic of epistemic modals, http://128.84.4.18/abs/2203.02872, 2022.

[34] W.H. Holliday, Possibility semantics, in: M. Fitting (Ed.), Selected Topics from Contemporary Logics, in: Landscapes in Logic, vol. 2, College Publications, London, 2021.
[35] B. Jónsson, A. Tarski, Boolean algebras with operators. Part I, Am. J. Math. 73 (4) (1951) 891–939.
[36] G. Kalmbach, Orthomodular Lattices, Academic Press, 1983.
[37] S. Brinck Knudstorp, Modal information logic, Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2022.
[38] K. Kojima, Relational and neighborhood semantics for intuitionistic modal logic, Rep. Math. Log. (2012) 87–113.
[39] J. Kontinen, J.S. Müller, H. Schnoor, H. Vollmer, Modal independence logic, in: R. Goré, B.P. Kooi, A. Kurucz (Eds.), Proc. AIML 2014, 2014, pp. 353–372.
[40] M. Moshier, P. Jipsen, Topological duality and lattice expansions, I: a topological construction of canonical extensions, Algebra Univers. (2014) 109–126.
[41] H. Priestley, Representation of distributive lattices by means of ordered stone spaces, Bull. Lond. Math. Soc. 2 (2) (1970) 186–190.
[42] D.W. Roeder, Category theory applied to Pontryagin duality, Pac. J. Math. 52 (2) (1974) 519–527.
[43] D. Sachs, Identities in finite partition lattices, Proc. Am. Math. Soc. 12 (1961) 944–945.
[44] H. Sahlqvist, Completeness and correspondence in the first and second order semantics for modal logic, Stud. Logic Found. Math. 82 (1975) 110–143.
[45] G. Sambin, V. Vaccaro, A new proof of Sahlqvist's theorem on modal definability and completeness, J. Symb. Log. 54 (3) (1989) 992–999.
[46] A. Urquhart, A topological representation theory for lattices, Algebra Univers. 8 (1) (1978) 45–58.
[47] F. Yang, Modal dependence logics: axiomatizations and model-theoretic properties, Log. J. IGPL 25 (5) (October 2017) 773–805.
[48] F. Yang, J. Väänänen, Propositional team logics, Ann. Pure Appl. Log. 168 (7) (July 2017) 1406–1441.