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A five-dimensional unemployment model with two distributed time delays

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Abstract

The purpose of this paper is to build and analyse a model of labour market slack considering unemployment along with employment in which the number of hours is limited to a level below that preferred by employees. We thus have ’underemployment’ along with unemployment. We analyse the need for potential policy action directed at the reduction of unemployment, simultaneous with an autonomous process of labour market job creation and migration. We model delays in labour market responses to past unemployment and vacancies creation and capture the effect on unemployment through a non-linear dynamic system. We observe job separation and matching but also movement into and out of underemployment. The model allows for migration in an open economy context. We analyse the stability behaviour of the resulting equilibria for our dynamic system, including Dirac and weak kernels.

KEYWORDS:
unemployment, underemployment, migration, distributed delays, Hopf bifurcation

1 | INTRODUCTION

In this paper we propose a nonlinear mathematical model of unemployment, allowing for both an extensive and an intensive margin of the labour market, generating unemployment as well as underemployment. As such, we reconsider the possibility of ’time-limited jobs’ or underemployment, as an intermediary status in the labour market - between regular employment and unemployment. Our system allows for both job market creation, and policy intervention to limit unemployment, along with the possibility that migrants take some regular jobs.

We build on the documented incidence of underemployment (for example [1]), whereas workers in either full or part-time jobs cite an interest to work a larger number of hours than those available to them in their present employment and at the going wage rate. Our paper thereby extends an earlier analysis [2] by focusing on policy effectiveness in reducing unemployment. In this context, employers have the option to draw on immigrants to fill in labour shortages, as well as to use the pool of the underemployed when the demand arises for extra hours of work. This builds on evidence in the post-2008 Great Recession period, whereby employers draw on the underemployed or their internal labour market - in particular during a recovery period. The mechanism also informs the possibility of filling in labour market gaps, such as those associated with post-pandemic labour shortages.

The adjustment in our model of unemployment dynamics could thus explain why unemployment rates changed only a little, in spite of major events such as the Great Recession or the post-pandemic recovery. Moreover, after the 2008 financial crisis, real wages stagnated during the recovery period, while underemployment stayed high relative to the level of unemployment across most European countries, and also in the USA [1]. On the other hand, the end of the COVID-19 pandemic coincided with some upward pressure on wages along with labour shortages. This happened in spite of the expectation of sharp increases in unemployment following the end of furlough schemes during the times of COVID-19 - temporarily controlling unemployment through policy intervention. Coupled with a subsequent decline in underemployment, this has also coincided with a sharp decline in international migration - showing that unemployment has been influenced by a mix of these dynamic processes, which should be modelled simultaneously. Going forward, it appears that underemployment is back to the relative levels experienced before the Great Recession [3], and time will tell whether its role in the unemployment dynamics is becoming less influential, or remains a factor to be kept in check. Our model will capture the role of underemployment in the determination of unemployment dynamics, by modelling policy intervention along with the contribution of limited-hours employment options. It further adds immigration as part of the system, which is also in line with a significant explanation of how shortages developed in the post-pandemic period along with disruptions to cross-border migrant flows.

In sum, we follow the intuition in Bell and Blanchflower [4] and reconsider the slack in the labour market beyond unemployment, to include those working fewer hours than intended. As such, underemployment needs to be added to a system of dynamic interaction of labour market processes and policy intervention to model more realistic scenarios.

We acknowledge the approach of earlier work [5,6,7,8,9,10,11,12,13,14] on unemployment control. Yet, unlike in [7] we adjust our unemployment model to allow for the possibility that workers of all skills can become subject to employment contracts that allow for fewer hours of work compared to what some of them would be happy to provide at the going wage rate.

In fact, where labour market slack is linked with underemployment measures, rather than unemployment alone, it is estimated that the share of those working part-time who would be interested in working full-time in Europe varies between 2 and almost 9 per cent of total employment over the decade leading to the Covid-19 pandemic [1]. Ultimately, we build on the fact that the availability of an underemployed workforce reduces the need for external recruitment by employers from the unemployeds. This results in similar dynamics as illustrated by a system where employers can fill in jobs from a pool of immigrants, thereby reducing the speed of unemployment reduction. Motivated by the above considerations, our paper builds on this analogy and creates a model of unemployment dynamics with both migration and underemployment, to check for the effectiveness of policy intervention and the further significance of delayed reactions.

The paper is organised as follows. In Section 2 the model for unemployment reduction considering underemployment and migration and two kernels is described. Section 3 adds a non-dimensional model. An equilibrium analysis is presented in Section 4. For different types of delay kernels a stability analysis is done in Section 5. and Section 6 for regular employment-free equilibrium. Section 7 provides the local asymptotic stability for the positive equilibrium. Numerical simulation substantiates the theoretical findings in Section 8. Finally, concluding remarks are given in Section 9.

2 | THE MATHEMATICAL MODEL

The state variables of the considered mathematical model are: the number of unemployed persons $U(t)$, the number of immigrants $M(t)$, the number of those underemployed persons or persons working limited hours $T(t)$, the number of regularly employed persons $R(t)$, and the number of available vacancies $V(t)$, at time t.

In the model description, the following assumptions are made: the separation rate of unemployed individuals and the labour market entry rate of migrants are constant at a_1 and m_1, respectively. Also, there are movements among different categories.

We model unemployment in a continuous time framework, along with a delayed reaction of markets in terms of job creation. The number of individuals claiming unemployment rises over time under the influence of external factors and we see this number diminish where a proportion of the unemployed find jobs created by recovering markets. Yet, some of those currently in regular employment or underemployed might also be dismissed, and such job losses increase unemployment. Finally, some of the unemployed can also leave or retire, thus diminishing the numbers of those in the unemployment pool. The change in unemployment is captured in Eq. $(1)_1$ below. At any time t, the number of unemployed persons, U(t) also changes by an autonomous factor a_1. The instantaneous rate of movement from unemployment to employment is jointly proportional to U(t) and the number of available vacancies V(t), where V(t) is the total number of vacancies being created by the market.

Migrants are further attracted to a particular labour market by economic opportunities, with an ongoing inflow of workers from abroad ready and able to enter the labour market of our observed economy. As they do not have access to unemployment benefits upon entry, they are not expected to add to the number of unemployed. There is an autonomous rise of migrant stocks, independent of economic conditions. Yet migrants' attraction to a labour market is a function of their employability, which depends on the available vacancies in the destination economy. By joining the labour force at destination, immigrants add to the regularly employed population. On the other hand, return migration often represents a significant proportion of the new arrivals diminishing the migrant stock along with natural attrition. Such developments are captured by Eq. $(1)_2$. As envisaged in Eq. $(1)_3$ the pool of the underemployed is enlarged by people otherwise unemployed, but who can find limited-hours employment, and diminishes where they can move to regular jobs according to job openings and the expected labour supply at going wage rates. Attrition is also occurring amongst the underemployed, due to various reasons determining labour market exit.

The number of individuals in regular employment rises through the job findings of the unemployed and by migrant workers occupying newly created jobs or vacancies and the underemployed filling in labour market gaps or temporary shortages arising in recovery periods. Total employment decreases as a consequence of a number of workers losing their regular jobs, or becoming unemployed, but also through retirement or natural loss. This is expressed formally in Eq. $(1)_4$.

Finally, we observe the creation of job opportunities by the market, as a delayed reaction to the observation of employment conditions at various points in time, with vacancy numbers also undergoing a process of decline, as modeled by Eq. $(1)_5$.

The system of differential equations, defined on the half-line ($t > 0$), is:

$$\begin{cases} \dot{U}(t) = a_1 - a_2 U(t)V(t) + a_3 R(t) - a_4 U(t) + a_5 T(t) - b_1 \int_0^\infty h_1(s)U(t-s)ds \\ \dot{M}(t) = m_1 - m_2 M(t)V(t) - b_2 M(t) \\ \dot{T}(t) = a_4 U(t) - a_5 T(t) - c_1 T(t)V(t) - b_3 T(t) \\ \dot{R}(t) = a_2 U(t)V(t) + m_2 M(t)V(t) + c_1 T(t)V(t) - a_3 R(t) - b_4 R(t) \\ \dot{V}(t) = c_2 \int_0^\infty h_2(s)R(t-s)ds - b_5 V(t) \end{cases} \tag{1}$$

where a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, m_1, m_2, c_1 and c_2 are positive constants and stand for: a_1 is the constant growth rate of unemployed persons entering the labour market, a_2 is the rate of regular hiring, a_3 is the rate of firing from regular employment, a_4 is the rate of movement from unemployment to limited hours employment, a_5 is the rate of movement from limited hours employment to unemployment; b_1 is the rate of exit from unemployment; b_2 is the rate of return or death of the migrant population; b_3 is the rate of exiting the labour market of the limited-hours employed persons; b_4 is the rate of retirement, migration or death of regularly employed persons; b_5 is the rate of vacancies decline; c_1 is the rate of upgrading from limited hours to regular employment; c_2 is the rate of vacancies creation in response to current employment conditions; m_1 represents the exogenous increase in migration and m_2 is the migrants' entry rate into regular employment.

The delay kernels $h_1, h_2 : [0,\infty) \to [0,\infty)$ are the probability density functions, assumed to be bounded, piecewise continuous and satisfying

$$\int_0^\infty h_1(s)ds = 1\ , \quad \tau_1 = \int_0^\infty s h_1(s)ds < \infty. \tag{2}$$

$$\int_0^\infty h_2(s)ds = 1\ , \quad \tau_2 = \int_0^\infty s h_2(s)ds < \infty. \tag{3}$$

Here, τ_1 is the average time delay for unemployment based on past exit from the labour market due to various factors including disillusioned workers, migration, mortality, and τ_2 represents the average time delay for available vacancies related to past regular employment levels.

3 | NON-DIMENSIONAL MODEL

The following transformations are employed with the aim of reducing the number of parameters from system (1):

$$x_1(t) = \frac{a_2 c_2}{a_5^2} U\left(\frac{t}{a_5}\right), \quad x_2(t) = \frac{a_2 c_2}{a_5^2} M\left(\frac{t}{a_5}\right), \quad x_3(t) = \frac{a_2 c_2}{a_5^2} T\left(\frac{t}{a_5}\right), \quad x_4(t) = \frac{a_2 c_2}{a_5^2} R\left(\frac{t}{a_5}\right), \quad x_5(t) = \frac{a_2}{a_5} V\left(\frac{t}{a_5}\right)$$

leading to the following equivalent non-dimensional system:

$$\begin{cases} \dot{x}_1(t) = \gamma_1 - x_1(t)x_5(t) + \alpha_3 x_4(t) - \alpha_4 x_1(t) + x_3(t) - \beta_1 \int_0^\infty k_1(s)x_1(t-s)ds, \\ \dot{x}_2(t) = \gamma_2 - \alpha_2 x_2(t)x_5(t) - \beta_2 x_2(t), \\ \dot{x}_3(t) = \alpha_4 x_1(t) - x_3(t) - \alpha_1 x_3(t)x_5(t) - \beta_3 x_3(t), \\ \dot{x}_4(t) = x_1(t)x_5(t) + \alpha_2 x_2(t)x_5(t) + \alpha_1 x_3(t)x_5(t) - \alpha_3 x_4(t) - \beta_4 x_4(t), \\ \dot{x}_5(t) = \int_0^\infty k_2(s)x_4(t-s) - \beta_5 x_5(t), \end{cases} \tag{4}$$

where the coefficients are expressed as:

$$\gamma_1 = \frac{a_1 a_2 c_2}{a_5^3}, \quad \gamma_2 = \frac{a_2 c_2 m_1}{a_5^3}$$

$$\alpha_1 = \frac{c_1}{a_2}, \alpha_2 = \frac{m_2}{a_2}, \alpha_3 = \frac{a_3}{a_5}, \alpha_4 = \frac{a_4}{a_5}$$

$$\beta_1 = \frac{b_1}{a_5}, \beta_2 = \frac{b_2}{a_5}, \beta_3 = \frac{b_3}{a_5}, \beta_4 = \frac{b_4}{a_5}, \beta_5 = \frac{b_5}{a_5}$$

and the delay kernels are: $k_1(s) = \frac{1}{a_5} h_1\left(\frac{s}{a_5}\right)$ and $k_2(s) = \frac{1}{a_5} h_2\left(\frac{s}{a_5}\right)$.

As general distributed time delays are taken into account in the mathematical model and its non-dimensional version, initial conditions for system (4) are considered of the form

$$x_i(\theta) = \varphi_i(\theta), \quad \forall\, \theta \in (-\infty, 0],\ \forall\, i = \overline{1,5},$$

where φ_i belong to the Banach space $C_{0,\mu}(\mathbb{R}_-, \mathbb{R})$ (where $\mu > 0$) of continuous real valued functions defined on $(-\infty, 0]$ such that $\lim_{t\to-\infty} e^{\mu t}\varphi(t) = 0$, considered with respect to the norm:

$$\|\varphi\|_{\infty,\mu} = \sup_{t\in(-\infty,0]} e^{\mu t}|\varphi(t)|.$$

The existence and uniqueness of solutions of the distributed delay system (4) are a consequence of the theoretical results from [15]. The positivity and boundedness of solutions of (4) can be proved by similar reasonings as in [12], and is summarized in the following Theorem:

Theorem 1. The open positive octant of $\mathbb{R}^5$ is invariant to the flow of system (4). Moreover, the set

$$\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5) : 0 \le x_1 + x_2 + x_3 + x_4 \le \frac{\gamma_1 + \gamma_2}{\beta_m},\ 0 \le x_5 \le \frac{\gamma_1 + \gamma_2}{\beta_m \beta_5} \right\},$$

where $\beta_m = \min(\beta_1, \beta_2, \beta_3, \beta_4)$ is a region of attraction for the system (1) and it attracts all the solutions initiating in the interior of the positive octant of $\mathbb{R}^5$.

Remark 1. The previous theorem states that solutions of system (4) originating from initial conditions belonging to the open positive octant of $\mathbb{R}^5$, i.e. $\varphi_i : (-\infty, 0] \to (0, \infty)$, for $i = \overline{1,5}$, remain positive for any $t > 0$. Moreover, such solutions satisfy the inequalities:

$$\limsup_{t\to\infty}[x_1(t) + x_2(t) + x_3(t) + x_4(t)] \le \frac{\gamma_1 + \gamma_2}{\beta_m} \quad \text{and} \quad \limsup_{t\to\infty} x_5(t) \le \frac{\gamma_1 + \gamma_2}{\beta_5 \beta_m}.$$

4 | EQUILIBRIA ANALYSIS

The equilibrium points are obtained by solving the system of algebraic equations

$$\begin{cases} \gamma_1 - x_1 x_5 + \alpha_3 x_4 - \alpha_4 x_1 + x_3 - \beta_1 x_1 = 0, \\ \gamma_2 - \alpha_2 x_2 x_5 - \beta_2 x_2 = 0, \\ \alpha_4 x_1 - x_3 - \alpha_1 x_3 x_5 - \beta_3 x_3 = 0, \\ x_1 x_5 + \alpha_2 x_2 x_5 + \alpha_1 x_3 x_5 - \alpha_3 x_4 - \beta_4 x_4 = 0, \\ x_4 - \beta_5 x_5 = 0. \end{cases} \tag{5}$$

From the last equation of the system (5) we get

$$x_4 = \beta_5 x_5. \tag{6}$$

We distinguish two cases:

- **Case 1:** $x_5 = 0$

Then using (6) we get $x_4 = 0$. Replacing in the second equation of the system (5) we obtain

$$x_2 = \frac{\gamma_2}{\beta_2}. \tag{7}$$

Using the first and third equations of the system (5) we get

$$x_1 = \frac{\gamma_1(1+\beta_3)}{\beta_1 + \beta_1\beta_3 + \alpha_4\beta_3} \tag{8}$$

and

$$x_3 = \frac{\alpha_4\gamma_1}{\beta_1 + \beta_1\beta_3 + \alpha_4\beta_3}. \tag{9}$$

Therefore, the first equilibrium point is denoted with S^0 and is given by

$$S^0 := \big(\delta_1(1+\beta_3), \delta_2, \delta_1\alpha_4, 0, 0\big)$$

where

$$\delta_1 = \frac{\gamma_1}{\beta_1 + \beta_1\beta_3 + \alpha_4\beta_3} \quad \text{and} \quad \delta_2 = \frac{\gamma_2}{\beta_2},$$

called regular employment-free equilibrium.

First of all, we introduce the basic reproduction number R_0 which has the role of a threshold parameter that prognosticates whether the unemployment, immigration, and underemployed problems will increase or decrease. The next generation matrix rises from the employment subsystem from the unemployment model rather than the infected subsystem in epidemic models. To find R_0 we associate the differential equations with the employed people $R(t)$ and the available vacancies $V(t)$ in model (1) as the following sub-model

$$\begin{pmatrix} \frac{dR}{dt} \\ \frac{dV}{dt} \end{pmatrix} = \begin{pmatrix} -(\alpha_3+\beta_4) & U + \alpha_2 M + \alpha_1 T \\ 1 & -\beta_5 \end{pmatrix} \begin{pmatrix} x_4(t) \\ x_5(t) \end{pmatrix}.$$

The Jacobian matrix J is evaluated at the equilibrium point S^0:

$$J(S^0) = \begin{pmatrix} -(\alpha_3+\beta_4) & \delta_1(1+\beta_3) + \alpha_2\delta_2 + \delta_1\alpha_1\alpha_4 \\ 1 & -\beta_5 \end{pmatrix} = F - W,$$

where

$$F = \begin{pmatrix} 0 & \delta_1(1+\beta_3) + \alpha_2\delta_2 + \delta_1\alpha_1\alpha_4 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad W = \begin{pmatrix} -(\alpha_3+\beta_4) & 0 \\ 1 & -\beta_5 \end{pmatrix}.$$

Therefore, the threshold R_0 is the spectral radius ρ of the next generation matrix

$$G = FW^{-1},$$

in other words

$$R_0 = \rho(G) = \frac{\delta_1(1+\beta_3+\alpha_1\alpha_4) + \delta_2\alpha_2}{(\alpha_3+\beta_4)\beta_5} = \frac{\beta_2\gamma_1 + \alpha_1\alpha_4\beta_2\gamma_1 + \beta_2\beta_3\gamma_1 + \alpha_2\beta_1\gamma_2 + \alpha_2\alpha_4\beta_3\gamma_2 + \alpha_2\beta_1\beta_3\gamma_2}{\beta_2(\beta_1 + \alpha_4\beta_3 + \beta_1\beta_3)(\alpha_3+\beta_4)\beta_5}.$$

- **Case 2:** $x_5 \neq 0$

Adding the first four equations of the system (5) we obtain

$$\gamma_1 + \gamma_2 - \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 - \beta_4 x_4 = 0. \tag{10}$$

The second equation of the system (5) leads to

$$x_2 = \frac{\gamma_2}{\beta_2 + \alpha_2 x_5}. \tag{11}$$

From the fourth equation of the system (5) we get:

$$x_1 = \alpha_3\beta_5 + \beta_4\beta_5 - \frac{\alpha_2\gamma_2}{\beta_2 + \alpha_2 x_5} - \alpha_1 x_3. \tag{12}$$

Further, we write x_1 and x_3 in terms x_5 by solving the system containing the equations (10) and (12), replacing (6) and (11)

$$\begin{cases} \beta_1 x_1 + \beta_3 x_3 = \gamma_1 + \gamma_2 - \frac{\beta_2\gamma_2}{\beta_2+\alpha_2 x_5} - \beta_4\beta_5 x_5 \\ x_1 + \alpha_1 x_3 = \alpha_3\beta_5 + \beta_4\beta_5 - \frac{\alpha_2\gamma_2}{\beta_2+\alpha_2 x_5} \end{cases} \tag{13}$$

and we have

$$x_3 = -\frac{D_0 + D_1 x_5 - \alpha_2\beta_4\beta_5 x_5^2}{(\alpha_1\beta_1 - \beta_3)(\beta_2 + \alpha_2 x_5)},$$

where

$$D_0 = \beta_2\gamma_1 + \alpha_2\beta_1\gamma_2 - \beta_1\beta_2\beta_4\beta_5 - \alpha_3\beta_1\beta_2\beta_5,$$
$$D_1 = \alpha_2\gamma_2 + \alpha_2\gamma_1 - \beta_2\beta_4\beta_5 - \alpha_2\beta_1\beta_4\beta_5 - \alpha_2\alpha_3\beta_1\beta_5.$$

We now determine x_1 in term of x_5 from the second equation of the system (13) and we get

$$x_1 = -\frac{C_0 + C_1 x_5 + \alpha_1\alpha_2\beta_4\beta_5 x_5^2}{(\alpha_1\beta_1 - \beta_3)(\beta_2 + \alpha_2 x_5)},$$

where

$$C_0 = \alpha_2\beta_2\beta_3\beta_5 + \beta_2\beta_3\beta_4\beta_5 - \alpha_1\beta_2\gamma_1 - \alpha_2\beta_3\gamma_2,$$
$$C_1 = \alpha_2\alpha_3\beta_3\beta_5 + \alpha_1\beta_2\beta_4\beta_5 + \alpha_2\beta_3\beta_4\beta_5 - \alpha_1\alpha_2\gamma_1 - \alpha_1\alpha_2\gamma_2.$$

Therefore, the second equilibrium point which is denoted by S^+ is given by

$$S^+ := \left(-\frac{Q(x_5) + \nu_1 d(x_5)}{(\beta_1 - \nu_1)(x_5 + \nu_2)}, \frac{\gamma_2}{\alpha_2(\nu_2 + x_5)}, \frac{Q(x_5) + \beta_1 d(x_5)}{\alpha_1(\beta_1 - \nu_1)(\nu_2 + x_5)}, \beta_5 x_5, x_5\right)$$

where x_5 is the solution of the following cubic equation

$$E_3 x_5^3 + E_2 x_5^2 + E_1 x_5 + E_0 = 0, \tag{14}$$

with

$$E_3 = \alpha_1\beta_4\beta_5,$$
$$E_2 = \alpha_1(\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2) + \beta_4\beta_5(1 + \beta_3 + \alpha_1\alpha_4) + \alpha_1\beta_1\mu,$$
$$E_1 = (\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2)(1 + \beta_3 + \alpha_1\alpha_4) - \nu_2\gamma_1\alpha_1 + \mu\beta_1\left(1 + \beta_3 + \alpha_1\alpha_4\frac{\nu_1}{\beta_1}\right) + (\mu\nu_2 - \gamma_2)\beta_1\alpha_1,$$
$$E_0 = \frac{\mu\gamma_1\gamma_2}{\alpha_2\delta_1\delta_2}(1 - R_0),$$

where R_0, δ_1 and δ_2 are given above and

$$\nu_1 = \frac{\beta_3}{\alpha_1}, \qquad \nu_2 = \frac{\beta_2}{\alpha_2}, \qquad \mu = (\alpha_3 + \beta_4)\beta_5,$$
$$Q(x_5) = (\beta_4\beta_5 x_5 - \gamma_1 - \gamma_2)(x_5 + \nu_2) + \gamma_2\nu_2,$$
$$d(x_5) = \mu(x_5 + \nu_2) - \gamma_2.$$

For two positive values x_1 and x_3 of the cubic equation (14) we observe

$$\begin{cases} -\frac{Q(x)+\nu_1 d(x)}{\beta_1-\nu_1} \geq 0 \\ -\frac{Q(x)+\beta_1 d(x)}{\beta_1-\nu_1} \geq 0 \end{cases}$$

which implies

$$\begin{cases} \operatorname{sign}(\beta_1 - \nu_1)[Q(x) + \nu_1 d(x)] \leq 0 \\ \operatorname{sign}(\beta_1 - \nu_1)[Q(x) + \beta_1 d(x)] \geq 0. \end{cases} \tag{15}$$

Denoting with $M = max\{\beta_1, \nu_1\}$ and $m = min\{\beta_1, \nu_1\}$ we obtain

$$\begin{cases} Q(x) + M d(x) \geq 0 \\ Q(x) + m d(x) \leq 0. \end{cases}$$

Therefore we get $(M - m)d(x) \geq 0$ then $d(x) \geq 0$. Hence $x_5 \geq \frac{\gamma_2}{\mu} - \nu_2$.

From (15) we have $-Md(x) \leq Q(x) \leq -md(x)$ then $-M \leq \frac{Q(x)}{d(x)} \leq -m$.

Moreover the components of the positive equilibrium point S^1 verify the third equation of the system (4). Replacing to obtain the cubic equation (14) we have

$$-\alpha_4 \frac{Q(x) + \nu_1 d(x)}{(\beta_1 - \nu_1)(x + \nu_2)} = \frac{Q(x) + \beta_1 d(x)}{(\beta_1 - \nu_1)(x + \nu_2)\alpha_2}(1 + \beta_3 + \alpha_1 x)$$

which implies

$$\frac{Q(x)}{d(x)} = -\frac{\beta_1(1 + \beta_3) + \alpha_4\beta_3 + \alpha_1\beta_1 x}{1 + \beta_3 + \alpha_1\alpha_4 + \alpha_1 x}$$

and after computations we get

$$\frac{Q(x)}{d(x)} = -\beta_1 + \alpha_4 \frac{\beta_1 - \nu_1}{x + \alpha_4 + \frac{1+\beta_3}{\alpha_1}} \quad \in (-M, -m).$$

Considering

$$P_3(x) = Q(x)[(1 + \beta_3 + \alpha_1 x) + \alpha_1\alpha_4] + d(x)[\beta_1(1 + \beta_3 + \alpha_1 x) + \nu_1\alpha_1\alpha_4]$$

and computing

$$Q(0) = -\gamma_1\nu_2 < 0$$
$$d(0) = \mu\nu_2 - \gamma_2$$

and

$$\begin{aligned}
P_3(0) &= -\gamma_1\nu_2(1 + \beta_3 + \alpha_1\alpha_4) + (\mu\nu_2 - \gamma_2)\frac{\gamma_1}{\delta_1} \\
&= \gamma_1 \left[-\frac{\gamma_2}{\alpha_2\delta_2}(1 + \beta_3 + \alpha_1\alpha_4) + \left(\mu \frac{\gamma_2}{\alpha_2\delta_2} - \gamma_2 \right) \frac{1}{\delta_1} \right] \\
&= \frac{\gamma_1\gamma_2}{\alpha_2} \left[-\frac{1 + \beta_3 + \alpha_1\alpha_4}{\delta_2} + \left(\mu \frac{1}{\delta_2} - \alpha_2 \right) \frac{1}{\delta_1} \right] \\
&= \frac{\gamma_1\gamma_2}{\alpha_2\delta_1\delta_2} \left[\mu - \delta_1(1 + \beta_3 + \alpha_1\alpha_4) + \alpha_2\delta_2 \right] \\
&= \frac{\mu\gamma_1\gamma_2}{\alpha_2\delta_1\delta_2}(1 - R_0),
\end{aligned}$$

it can be observed that if $P_3(0) < 0$ then $R_0 > 1$ which means we have at least one solution on $(0, +\infty)$.

In conclusion, we may either have one of the following situations:

- If $R_0 > 1$ then $E_0 < 0$ and system (4) has at least one positive equilibrum point. Moreover, if either $E_1 < 0$, or $E_1 > 0$ and $E_2 > 0$, Descartes' rule of signs guarantees the existence of a unique positive equilibrium point S^+.
- If $R_0 < 1$ then $E_0 > 0$, we may have either two or zero positive equilibrium points.

5 | LOCAL STABILITY ANALYSIS FOR S^0

We study the local stability behaviour of equilibrium S^0 by analysing the roots of the characteristic equation of (1):

$$\det \begin{bmatrix} a_{11} - \beta_1 K_1(\lambda) - \lambda & 0 & 1 & a_{14} & a_{15} \\ 0 & a_{22} - \lambda & 0 & 0 & a_{25} \\ a_{31} & 0 & a_{33} - \lambda & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} - \lambda & a_{45} \\ 0 & 0 & 0 & K_2(\lambda) & a_{55} - \lambda \end{bmatrix} = 0,$$

where: $a_{11} = -\alpha_4$, $a_{14} = \alpha_3$, $a_{15} = -\delta_1(1+\beta_3)$, $a_{22} = -\beta_2$, $a_{25} = -\delta_2\alpha_2$, $a_{31} = \alpha_4$, $a_{33} = -1-\beta_3$, $a_{35} = -\delta_1\alpha_1\alpha_4$, $a_{44} = -\alpha_3-\beta_4$, $a_{45} = \delta_1(1 + \beta_3 + \alpha_1\alpha_4) + \delta_2\alpha_2 = R_0(\alpha_3 + \beta_4)\beta_5$ and $a_{55} = -\beta_5$.

The characteristic equation is:

$$(\lambda + \beta_2)Q_1(\lambda)Q_2(\lambda) = 0 \tag{16}$$

where

$$Q_1(\lambda) = (\lambda + \alpha_4)(\lambda + \beta_3 + 1) - \alpha_4 + \beta_1(\lambda + \beta_3 + 1)K_1(\lambda),$$
$$Q_2(\lambda) = (\lambda + \alpha_3 + \beta_4)(\lambda + \beta_5) - R_0(\alpha_3 + \beta_4)\beta_5 K_2(\lambda).$$

Lemma 1. All the roots of the function $Q_2(\lambda)$ are in the open left half-plane if and only if $R_0 < 1$.

Proof. Let us first assume that $R_0 < 1$. Assuming by contradiction that there is a root λ of $Q_2(\lambda)$ such that $\Re(\lambda) \geq 0$, it follows that $|\lambda + \mu| \geq \mu$, for any $\mu > 0$. On the other hand, based on the properties of the Laplace transform, it is also easy to see that $|K_2(\lambda)| \leq 1$ and hence:

$$|(\lambda + \alpha_3 + \beta_4)(\lambda + \beta_5)| \geq (\alpha_3 + \beta_4)\beta_5 > R_0(\alpha_3 + \beta_4)\beta_5|K_2(\lambda)|.$$

Therefore $Q_2(\lambda) = 0$ cannot take place, and hence, all the roots of the function $Q_2(\lambda)$ have negative real part.

On the other hand, if we assume that $R_0 \geq 1$, we have

$$Q_2(0) = (\alpha_3 + \beta_4)\beta_5(1 - R_0) \leq 0.$$

Moreover, it is easy to see that $Q_2(\lambda) \to \infty$ as $\lambda \to \infty$, and therefore, $Q_2(\lambda)$ has at least one real root in the interval $[0, \infty)$. □

Lemma 2. If the following inequality holds:

$$\alpha_4\beta_3 - \beta_1(\beta_3 + 1) > 0 \tag{17}$$

then all the roots of the function $Q_1(\lambda)$ are in the open left half-plane, regardless of the delay kernel $k_1(t)$.

Moreover, in the absence of time delay, i.e. $k_1(t) = \delta(t)$, all the roots of the function $Q_1(\lambda)$ are in the open left half-plane.

Proof. Assuming by contradiction that there is a root λ of $Q_1(\lambda)$ such that $\Re(\lambda) \geq 0$, it follows that

$$\lambda + \alpha_4 = \frac{\alpha_4}{\lambda + \beta_3 + 1} - \beta_1 K_1(\lambda).$$

Using similar methods as in the proof of Lemma 1, based on inequality (17) we have

$$\left|\frac{\alpha_4}{\lambda + \beta_3 + 1} - \beta_1 K_1(\lambda)\right| \leq \frac{\alpha_4}{\beta_3 + 1} + \beta_1 < \alpha_4 \leq |\lambda + \alpha_4|.$$

Hence, the equality $Q_1(\lambda) = 0$ cannot take place.

Moreover, in the absence of time delay, i.e. $K_1(\lambda) = 1$, the function $Q_1(\lambda)$ is a quadratic polynomial with positive coefficients, and hence, the conclusion follows from the Routh-Hurwitz stability criterion. □

Combining the results obtained in the previous two lemmas, we can conclude the following:

Theorem 2. The following results hold for the equilibrium point S^0 of system (10):

(i) In the non-delayed case, S^0 is locally asymptotically stable if and only if $R_0 < 1$.

(ii) If $R_0 < 1$ and inequality (17) holds, the equilibrium point S^0 is locally asymptotically stable, for any choice of the delay kernels $k_1(t)$ and $k_2(t)$.

(iii) If $R_0 \geq 1$, the equilibrium S^0 is unstable, regardless of the delay kernels $k_1(t)$ and $k_2(t)$.

Proof. It is easy to see that $\lambda = -\beta_2 < 0$ is a root of the characteristic equation (16). In the absence of time delays, all the roots of $Q_1(\lambda)$ have negative real part (see Lemma 2), while the roots of $Q_2(\lambda)$ have negative real part if and only if $R_0 < 1$ (see Lemma 1). Hence, (i) is proved.

Moreover, if $R_0 < 1$ and inequality (17) holds, the asymptotic stability of the equilibrium point S^0 follows from Lemma 1 and Lemma 2, regardless of the delay kernels $k_1(t)$ and $k_2(t)$, and hence, (ii) is true.

Finally, for the proof of (iii), if $R_0 \geq 1$, Lemma 1 shows that there is at least one root of $Q_2(\lambda)$ which has positive (or null) real part, and hence the characteristic equation (16) has at least one root in the right half-plane. Therefore, the equilibrium S^0 is unstable, regardless of the delay kernels $k_1(t)$ and $k_2(t)$. □

We would like to point that in case of the presence of just the second delay kernel, as shown in [12], the equilibrium point corresponding to the state of no regular employment and no available vacancies is globally asymptotically stable. This is regardless of the considered delay kernel, if the basic reproduction number satisfies an inequality.

6 | HOPF BIFURCATION ANALYSIS FOR S^0

Let us denote by $\tau_1 > 0$ the average time delay associated to the delay kernel $k_1(t)$, and $K_1\left(\frac{\lambda}{\tau_1}\right) = \hat{K}_1(\lambda)$. Looking for pure imaginary roots of the equation $Q_1(\lambda) = 0$ of the form $\lambda = i\frac{\omega}{\tau_1}$, denoting $\hat{K}_1(i\omega) = C_1(\omega) - iS_1(\omega)$ and taking the real and imaginary parts of $Q_1(\lambda) = 0$ we have

$$\begin{cases} -\left(\frac{\omega}{\tau_1}\right)^2 + \alpha_4\beta_3 + \beta_1\left[\frac{\omega}{\tau_1}S_1(\omega) + (\beta_3+1)C_1(\omega)\right] = 0 \\ \frac{\omega}{\tau_1}(\alpha_4+\beta_3+1) + \beta_1\left[\frac{\omega}{\tau_1}C_1(\omega) - (\beta_3+1)S_1(\omega)\right] = 0 \end{cases} \tag{18}$$

with

$$C_1(\omega) = \Re[K_1(i\omega)]$$

and

$$S_1(\omega) = -\Im[K_1(i\omega)].$$

Eliminating τ_1 from system (18), we deduce:

$$\beta_1^2 S_1^2(\omega)(\beta_3+1)[\alpha_4 + \beta_1 C_1(\omega)] + [\beta_1(\beta_3+1)C_1(\omega) + \alpha_4\beta_3][\alpha_4+\beta_3+1+\beta_1 C_1(\omega)]^2 = 0. \tag{19}$$

Therefore, the following result is easily obtained:

Proposition 1. Let us assume that $R_0 < 1$ and the inequality (17) is not satisfied.

a. If for the delay kernel $K_1(\lambda)$, the equation (19) does not have any solution in $(0,\infty)$, the equilibrium point S^0 is asymptotically stable, for any average delay $\tau_1 \geq 0$.

b. If for the delay kernel $K_1(\lambda)$ the equation (19) has positive solutions ω_j, $j \in J \subset \mathbb{Z}_+$, Hopf bifurcations occur in a neighborhood of the equilibrium point S^0 at the critical values of the average delay

$$\tau_1^j = \omega_j \cdot \frac{\alpha_4+\beta_3+1+\beta_1 C_1(\omega_j)}{\beta_1(\beta_3+1)S_1(\omega_j)} \quad, j \in J,$$

provided that the above expressions are positive.

6.1 | Weak Gamma kernel

Corollary 1. If $R_0 < 1$ and the inequality (17) is not satisfied and if the weak Gamma kernel $k_1(t) = \tau_1^{-1}e^{-t/\tau_1}$ (with average delay τ_1) is considered, the equilibrium point S_0 is asymptotically stable, for any $\tau_1 \geq 0$.

Proof. Since for the weak Gamma kernel we have $C_1(\omega) = \frac{1}{1+\omega^2} > 0$, it is easy to see that equation (19) does not have any positive solutions. Hence, based on Proposition 1, the equilibrium S_0 remains asymptotically stable, for any positive value of the average delay τ_1. □

6.2 | Dirac kernel

Proposition 2. Assume that $R_0 < 1$ and the inequality (17) is not satisfied. If the Dirac kernel $k_1(t) = \delta(t-\tau_1)$ is considered, the critical values of the time delay τ_1 are:

$$\tau_1^j = \omega_j \cdot \frac{\alpha_4+\beta_3+1+\beta_1\cos(\omega_j)}{\beta_1(\beta_3+1)\sin(\omega_j)} \quad, j \in \mathbb{Z}_+,$$

where $\omega_j = \operatorname{sign}(\alpha_4+\beta_3+1-\beta_1)\arccos(u_0) + 2j\pi$, and u_0 is the unique root in the interval $[-1,0]$ of the quadratic equation

$$c_2u^2 + c_1u + c_0 = 0, \tag{20}$$

where the coefficients are:

$$\begin{aligned} c_2 &= \beta_1^2[\alpha_4 + 2\alpha_4\beta_3 + 2(1+\beta_3)^2], \\ c_1 &= \beta_1[2\alpha_4(1+\beta_3)(1+2\beta_3) + \alpha_4^2(1+3\beta_3) + (1+\beta_3)[\beta_1^2 + (1+\beta_3)^2]\}, \\ c_0 &= \alpha_4[\beta_1^2(1+\beta_3) + \beta_3(1+\alpha_4+\beta_3)^2]. \end{aligned}$$

The equilibrium S_0 is asymptotically stable if and only if $\tau_1 \in [0, \tau_1^0)$. At the critical values τ_1^j, Hopf bifurcations occur in a neighborhood of the equilibrium S_0.

Proof. When a Dirac kernel is considered, i.e. $C_1(\omega) = \cos\omega$ and $S_1(\omega) = \sin\omega$, the equation (19) reduces to the quadratic equation (20) for $u = \cos\omega$. As inequality (17) is not satisfied, it is easy to see that the quadratic equation (20) has a unique root in the interval $[-1, 0]$, denoted by u_0, and another root in the interval $(-\infty, -1)$. Hence, by means of Proposition 1, we obtain the positive critical values of the time delay τ_1.

Let us denote by $\lambda(\tau_1)$ the root of the characteristic equation $Q_1(\lambda) = 0$ such that $\lambda(\tau_1^j) = i\omega_j$, where $j \in \mathbb{Z}_+$. Hence, $\lambda(\tau)$ satisfies the equation:

$$R(\lambda) = \beta_1 e^{-\tau_1\lambda}$$

where

$$R(\lambda) = \frac{\alpha_4}{\lambda + \beta_3 + 1} - (\lambda + \alpha_4).$$

It can be easily verified that the function $\omega \mapsto |R(i\omega)|$ is strictly increasing on $(0, \infty)$, and consequently, based on [16] , we deduce that the following transversality condition holds:

$$s = \text{sign}\left(\frac{d\Re(\lambda)}{d\tau_1}\bigg|_{\tau_1=\tau_1^j}\right) = \text{sign}\left(\frac{d}{d\omega}|R(i\omega)|\bigg|_{\omega=\omega_j}\right) = 1.$$

Therefore, at each critical value τ_1^j of the time delay, a pair of complex conjugated roots of the characteristic equation cross the imaginary axis, from the left half-plane to the right half-plane. Hence, Hopf bifurcations occur in a neighborhood of S_0 at the critical values τ_1^j, and no stability switching is encountered. Based on Theorem 2, it follows that S_0 is asymptotically stable if and only if $\tau_1 \in [0, \tau_1^0)$. □

7 | LOCAL STABILITY ANALYSIS FOR S^+

We study the local stability behaviour of the positive equilibrium point S^+ by analysing the roots of the characteristic equation of (1):

$$\det\begin{bmatrix} -\alpha_4 - \beta_1 K_1(\lambda) - x_5 - \lambda & 0 & 1 & \alpha_3 & -x_1 \\ 0 & -\beta_2 - \alpha_2 x_5 - \lambda & 0 & 0 & -\alpha_2 x_2 \\ \alpha_4 & 0 & -1 - \beta_3 - \alpha_1 x_5 - \lambda & 0 & \alpha_1 x_3 \\ x_5 & \alpha_2 x_5 & \alpha_1 x_5 & -\alpha_3 - \beta_4 - \lambda & (\alpha_3 + \beta_4)\beta_5 \\ 0 & 0 & 0 & K_2(\lambda) & -\beta_5 - \lambda \end{bmatrix} = 0,$$

The characteristic equation is:

$$P_0(\lambda) + P_1(\lambda)K_1(\lambda) + P_2(\lambda)K_2(\lambda) + P_{12}(\lambda)K_1(\lambda)K_2(\lambda) = 0, \tag{21}$$

where

$$\begin{aligned} P_0(\lambda) &= -(\beta_5 + \lambda)(\beta_2 + \alpha_2 x_5 + \lambda)\{(\beta_4 + \lambda)[x_5 + \lambda + (\alpha_4 + x_5 + \lambda)(\beta_3 + \alpha_1 x_5 + \lambda)] + \\ &\quad + \alpha_3[\alpha_4(\beta_3 + \lambda) + \lambda(1 + \beta_3 + \alpha_1 x_5 + \lambda)]\}, \\ P_1(\lambda) &= -\beta_1(\alpha_3 + \beta_4 + \lambda)(\beta_5 + \lambda)(1 + \beta_3 + \alpha_1 x_5 + \lambda)(\beta_2 + \alpha_2 x_5 + \lambda), \\ P_2(\lambda) &= \alpha_2^2\alpha_4 x_2 x_5(\beta_3 + \alpha_1 x_5 + \lambda) - \alpha_2^2 x_2 x_5(1 + \beta_3 + \alpha_1 x_5 + \lambda)(\lambda + x_5) - \\ &\quad - \alpha_4\beta_5(\alpha_3 + \beta_4)(\beta_2 + \alpha_2 x_5 + \lambda) - \alpha_1\alpha_4(\beta_2 + \alpha_2 x_5 + \lambda)(x_1 x_5 + \alpha_1 x_3 x_5) + \\ &\quad + \beta_5(\alpha_3 + \beta_4)(1 + \beta_3 + \alpha_1 x_5 + \lambda)(\beta_2 + \alpha_2 x_5 + \lambda)(\alpha_4 + x_5 + \lambda) - \\ &\quad - x_5(\beta_2 + \alpha_2 x_5 + \lambda)[\alpha_1 x_3 + x_1(1 + \beta_3 + \alpha_1 x_5 + \lambda)] - \\ &\quad - \alpha_1^2 x_3 x_5(\beta_2 + \alpha_2 x_5 + \lambda)(x_5 + \lambda), \\ P_{12}(\lambda) &= \beta_1[-\alpha_2^2 x_2 x_5(1 + \beta_3 + \alpha_1 x_5 + \lambda) - \alpha_1^2 x_3 x_5(\beta_2 + \alpha_2 x_5 + \lambda) + \\ &\quad + \beta_5(\alpha_3 + \beta_4)(1 + \beta_3 + \alpha_1 x_5 + \lambda)(\beta_2 + \alpha_2 x_5 + \lambda)] \end{aligned}$$

and $K_1(\lambda)$ and $K_2(\lambda)$ are the Laplace transforms of the delays kernel $\hat{k_1}(s)$ and $\hat{k_2}(s)$.

A full theoretical analysis of this characteristic equation involving many parameters and both Laplace transforms of the delay kernels is a very tedious task, and hence, it will be omitted in this paper. However, the necessary bifurcation results can be obtained by numerical means, as described in the next section.

We add that if only the delay kernel $\hat{k_2}(s)$ is considered, i.e. $\hat{k_1}(s) = \delta(s)$, it has been shown in [12], that the positive equilibrium point is globally asymptotically stable, regardless of the delay kernel, if the rate of firing a_3 is zero.

8 | NUMERICAL SIMULATIONS

In order to exemplify the theoretical results of the model (1) we present two scenarios, the first one with basic reproduction number $R_0 < 1$, and the second one with $R_0 > 1$. It is important to note that while in the first scenario, there is only one equilibrium point S^0, in the second scenario, a regular employment-free equilibrium S^0 coexists with a positive equilibrium S^+.

For the first scenario, we consider the system parameters: $a_1 = 687.5$, $a_2 = 0.0000152588$, $a_3 = 1$, $a_4 = 0.9375$, $a_5 = 0.125$, $b_1 = 0.5$, $b_2 = 0.51$, $b_3 = 0.03125$, $b_4 = 0.5$, $b_5 = 0.5$, $c_1 = 0.000015258$, $c_2 = 0.5$, $m_1 = 25.5$, $m_2 = 0.0078125$. For this set of parameters, the only equilibrium point of system (1) is $S^0 = (1000, 50, 6000, 0, 0)$ and the computed value of the reproduction number is $R_0 = 0.331624 < 1$, and hence, based on Theorem 2 (i), the equilibrium is asymptotically stable in the non-delayed case. However, as inequality (17) does not hold, Theorem 2 does not guarantee the local asymptotic stability of the equilibrium point for any choice of the delay kernels. Indeed, in the case of Dirac kernels, the critical value for the Hopf bifurcation of the discrete-time delay τ_1 is given by Proposition 1, and is computed to be $\tau_1^0 = 29.5$. In Figure 1 , we observe the convergence of the trajectories to the asymptotically stable equilibrium point S^0 when $\tau_1 < \tau_1^0$ and the appearance of sustained oscillations for values of τ_1 exceeding the critical value.

For the second scenario, the system parameters are chosen as: $a_1 = 300$, $a_2 = 0.0589286$, $a_3 = 0.108203$, $a_4 = 0.5$, $a_5 = 0.035$, $b_1 = 0.9375$, $b_2 = 0.5$, $b_3 = 0.02625$, $b_4 = 0.003125$, $b_5 = 0.5$, $c_1 = 0.00021875$, $c_2 = 0.003125$, $m_1 = 60$, $m_2 = 0.0175$. For these parameters, the regular employment-free equilibrium $S^0 = (260.465, 120, 2126.25, 0, 0)$ is unstable, regardless of the delay kernels considered in system (1), as $R_0 = 1.0057 > 1$ (based on Theorem 2). However, there is a positive equilibrium point of system (1), namely $S^+ = (U^+, M^+, T^+, R^+, V^+) = (280, 50, 2000, 6400, 40)$.

Considering discrete time delays, Figure 2 shows the stability region of the S^+ equilibrium in the (τ_1, τ_2)-plane. The blue region under the bold blue line describes the stability region for S^+, while the region above the bold blue line represents the instability region.

For the numerical simulations from Figure 3 , the following initial condition has been considered: $U(0) = 280$, $M(0) = 50$, $T(0) = 2000$, $R(0) = 6400$, $V(0) = 40$. If we considered Dirac delay kernels and we fix $\tau_1 = 50$, the critical value of τ_2 for the Hopf bifurcation is $\tau_2^* = 39.6905$. Figure 3 displays the trajectories of the system for several values of $\tau_2 \in [0, 50]$. For lower values of the average delay τ_2, the positive equilibrium is locally asymptotically stable. On the other hand, for $\tau_2 > \tau_2^*$, the positive equilibrium becomes unstable and sustained periodic oscillations appear in its neighborhood.

9 | CONCLUSIONS

The paper has developed and analysed a model for unemployment reduction where underemployment is providing an easily available internal pool of resources for employers to temporarily draw on. Furthermore, migrant labour allows employers to fill in any shortages during a period of recovery. This model is described by a nonlinear differential system with two distributed time delays. At any time t, we have taken into account the following variables: the number of unemployed individuals, the number of regularly employed individuals, the number of underemployed individuals with limited-time jobs, the number of newly arrived immigrants, the number of total jobs on the market, and the number of vacancies created. One distributed delay relates to past exit from the labour market, reducing current unemployment due to various factors. The second distributed delay relates to past levels of regular employment that influence current vacancies.

Firstly, we used variables transformation to reduce the number of parameters and generated the non-dimensional system. We have analyzed the existence of equilibria and established two points. The first is a vacancies and regular employment free point S^0 and the second is a strictly positive one S^+.

Secondly, we have shown that the solutions of the system are positive and bounded. Then, we have undertaken the local stability analysis for S^0 and analyzed the existence of the Hopf bifurcation for various delay kernels. In the absence of delay, the equilibrium point is locally asymptotically stable under some conditions of the parameters. Additionally, we have attempted to perform a local stability analysis for S^+, however, due to the complexity of the problem, theoretical results cannot be formulated concisely. Nonetheless, we have employed computational tools in this context.

We have tested the significance of taking underemployment and migration into account when formulating policies to address unemployment. We thus have observed the evolution of unemployment, underemployment, regular employment, migration, and vacancies related to average time delays, through numerical simulations. We have established values of the average time delays by which the system becomes unstable.

In a further paper, we would consider new challenges of the labour market emerging in the economic environment following the pandemic.

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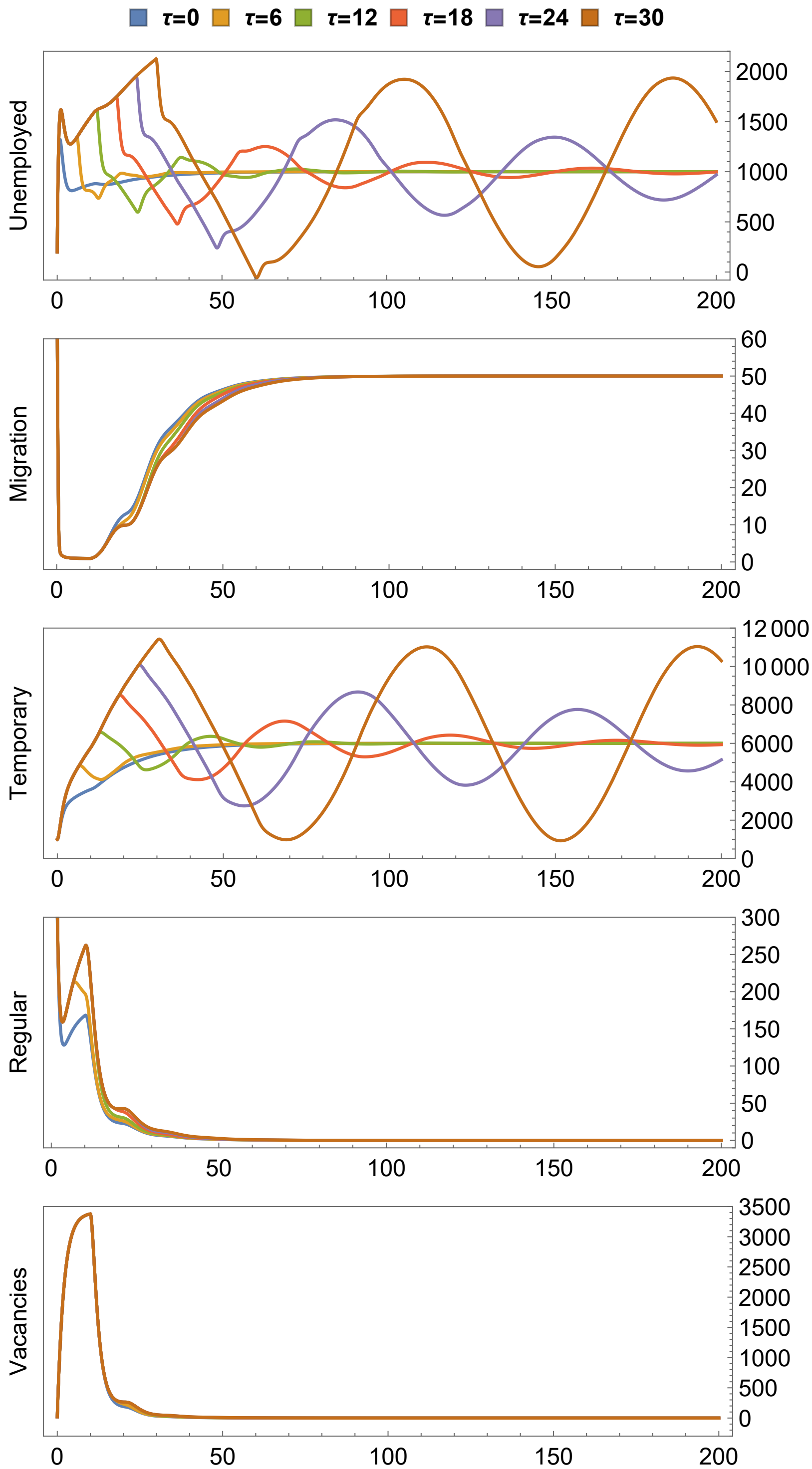

FIGURE 1 Evolution of the state variables $U(t)$, $M(t)$, $T(t)$, $R(t)$, $V(t)$ with fixed initial conditions, for fixed discrete time delays $\tau_2 = 10$ and $\tau_1 = \tau \in [0, 30]$.

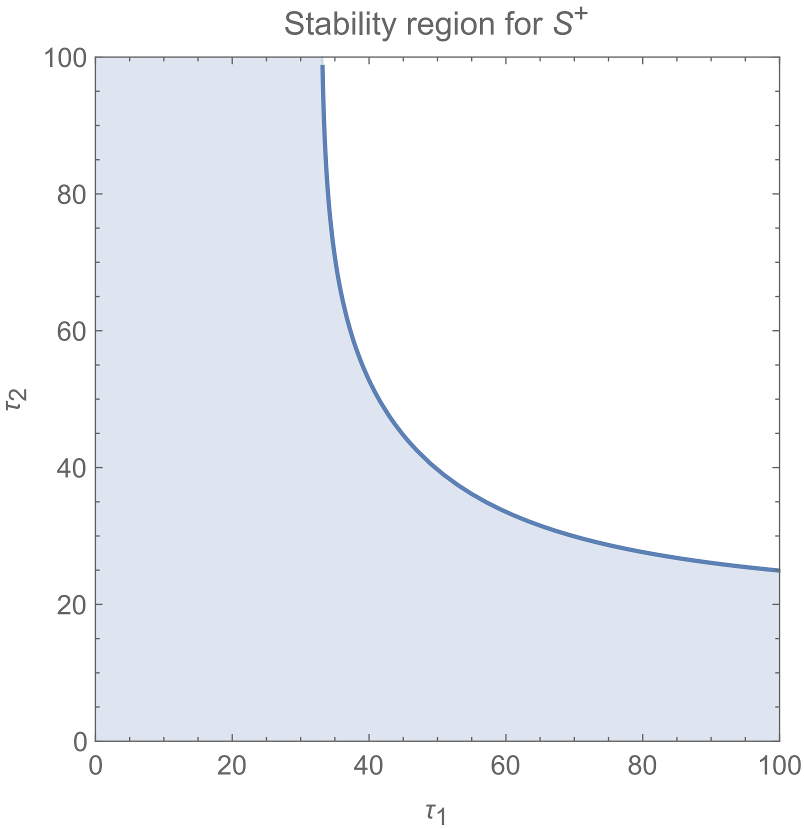

FIGURE 2 Stability region for the positive equilibrium point S^+ of system (1).

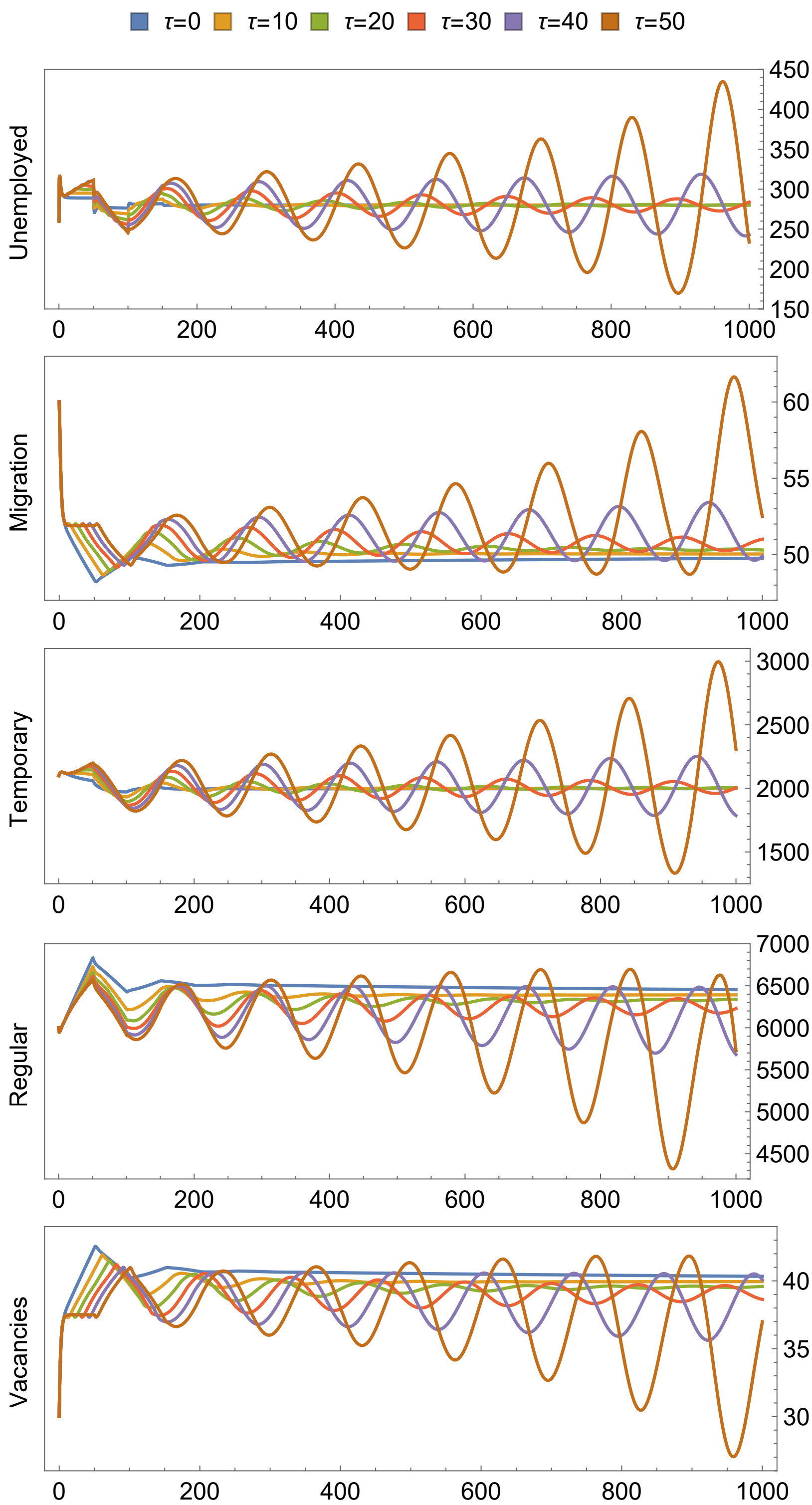

FIGURE 3 Evolution of the state variables $U(t)$, $M(t)$, $T(t)$, $R(t)$, $V(t)$ with fixed initial conditions, for fixed discrete time delays $\tau_1 = 50$ and $\tau_2 = \tau \in [0, 50]$.