Consistent and Inconsistent Forcing Axioms

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Abstract

This thesis explores the relationship between forcing axioms and square principles. While classical forcing axioms, at the level of ω_1, are incompatible with square principles, the situation is different for forcing axioms at ω_2; in fact, sufficiently strong generalizations of MA_{ω_2} actually imply square principles at ω_2. Specifically, we prove that the forcing axiom $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies the weak square $\square_{\omega_1,\omega_1}$. Using this result, we prove the inconsistency of the forcing axiom $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.). Moreover, we also prove that $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies weak Chang's Conjecture and the existence of a locally compact scattered (LCS) space of height ω_2 and width ω.

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Chapter 1

Introduction

Since the dawn of set theory, when set-theorists started adopting the axiomatic approach, we have been searching for a set of axioms that would be best at formulating our field of study. While the usual axiomatic system, namely Zermelo-Fraenkel Set Theory with the Axiom of Choice, ZFC, may be more than enough for the average mathematician in terms of the strength of implications, it fails to answer many questions naturally asked by set-theorists. Among the first to be considered is the question of the size of the continuum. In fact, Georg Cantor famously spent the last years of his life at the turn of 20th century trying to prove or disprove the Continuum Hypothesis, CH ($2^{\aleph_0} = \aleph_1$), in the framework of ZF. As is well-known, his efforts were spent in vain, as CH was to be shown to be independent of ZF by the famous work of Kurt Gödel in around 1938 [12] (who proved that CH is consistent with ZFC), and Paul Cohen, in 1963 [6], who through the invention of the forcing method showed that $\neg\,\mathsf{CH}$ is also consistent together with ZFC. Thus began the search for natural axioms which, when supplemented to ZFC, would decide this, and other questions about the set-theoretic universe left open by ZFC.

One such axiom is Gödel's Axiom of Constructibility, $V = L$. This axiom says that the set-theoretic universe is precisely L, namely the smallest possible (under inclusion) inner model of ZF containing all ordinals. This axiom is very powerful in that it decides essentially all questions occurring naturally in set theory. Among its consequences one can mention CH, and in fact the Generalized Continuum Hypothesis (GCH), Jensen's $\diamondsuit_\kappa$ principle for all infinite regular cardinals κ, or Jensen's $\Box_\kappa$ principle for every infinite cardinal κ ([15]). It is worth pointing out that $\diamondsuit_{\omega_1}$ implies the existence of a Souslin tree on ω_1, and that $\Box_{\omega_1} + \mathsf{CH}$ implies the existence of Souslin tree on ω_2 ([13]). Hence, all these tree existence axioms are true under $V = L$. However, despite its successes, $V = L$ is usually seen as an undesirable axiom due to its limitative nature. One instantiation of this is the fact that $V = L$ is incompatible with most large cardinal axioms considered in set theory; in fact it is with measurable cardinals and anything above that ([25]). The family of axioms we will consider next are of an opposite character, they are 'maximality' principles, and are consistent with all large cardinal axioms.

We are referring to forcing axioms. These are axioms postulating some suitable saturation of the universe with respect to forcing extensions. The guiding idea is that "many of the things that one can force are already true in the universe". Formally, these are axioms asserting the existence, given any member $\mathbb{P}$ of some given nice class Σ of forcing axioms, and any family $\mathcal{D}$ of relatively small size (typically of size $\aleph_1$) consisting of dense subsets of $\mathbb{P}$, of a filter of $\mathbb{P}$ meeting all members of $\mathcal{D}$. The best known forcing axioms include Martin's Axiom, MA,[1] the Proper Forcing Axiom, PFA, and Martin's Maximum, MM. It is straightforward to show that MA implies the failure of CH. What is perhaps more surprising is that sufficiently strong (classical) forcing axioms, like PFA, imply an exact value of $2^{\aleph_0}$, with this value being $\aleph_2$ (this is due to Todorčvić and Veličković, s. [30]). PFA has also other remarkable consequences; for example that any two $\aleph_1$-dense set of reals are isomorphic ([4]), or the failure of the combinatorial principle square $\Box_{\omega_1}$ ([29]). This last implication will be one focal point of this thesis.

In this thesis, we will explore forcing axioms above $\aleph_1$ (i.e., forcing axioms for meeting all members of collections $\mathcal{D}$ of dense sets for $|\mathcal{D}| > \aleph_1$). These are high analogues of 'classical' forcing a forcing axioms (where $\mathcal{D}$ has size $\aleph_1$). In particular, we will consider (consistent) extensions of Martin's Axiom and (inconsistent) extensions of weak forms of Martin's Maximum. We will focus on the consistency, or otherwise, of these principles

[1] This was in fact the first forcing axiom to be isolated.

and on their consequences.

For basic resources and definitions, one can refer to Jech's [14] and Kunen's [18] textbooks.

Chapter 2

Forcing Axioms, Infinite Combinatorial Principles and Their Relations

2.1 Square principles

The combinatorial principle $\square_\kappa$ was introduced by Jensen [15] when he was investigating the consequences of the axiom of constructibility $V = L$.

Definition 2.1.1. Let κ be an uncountable cardinal. A $\square_\kappa$-*sequence* is a sequence $(C_\alpha : \alpha \in \lim(\kappa^+))$ such that for every $\alpha \in \lim(\kappa^+)$

1. C_α is closed and unbounded[1] in α.

2. If $\mathrm{cf}(\alpha) < \kappa$, then $\mathrm{ot}(C_\alpha) < \kappa$.

3. For all $\beta \in \lim(C_\alpha), C_\beta = C_\alpha \cap \beta$.

We say $\square_\kappa$ *holds* if there exists a $\square_\kappa$-sequence.

[1] From here on we shall use the word "club" to abbreviate "closed and unbounded set".

In the paper [15], Jensen found that if $V = L$, then $\square_\kappa$ holds for every $\kappa \geq \omega_1$. Jensen used a $\square_\kappa$-sequences to construct κ^+-Souslin trees in L which motivated others to use such sequences to construct other objects of size κ^+.

Let κ be a regular uncountable cardinal. A set $S \subseteq \kappa$ is stationary if $S \cap C \neq \emptyset$ for every club subset C of κ.

Definition 2.1.2. Let $\kappa \geq \omega_1$ and let S be a stationary subset of κ.

1. S *reflects* at α if $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.

2. $\mathrm{Refl}(S)$ *holds* if every subset of S which is stationary in κ reflects at some α.

3. S is *non-reflecting* if S does not reflect at any α.

Stationary reflection and the square principle are inherently connected concepts in infinite combinatorics. This result is due to Solovay.

Fact 2.1.3. *$\square_\kappa$ implies that for every stationary subset S of κ^+ there is a subset T of S which is stationary in κ^+ and such that $\mathrm{Refl}(S')$ fails.*

Proof. Let $(D_\alpha : \alpha \in \lim(\kappa^+))$ be a $\square_\kappa$-sequence and S a stationary subset of κ^+. Let $T \subseteq S$ be stationary and such that, for some $\beta_0 < \kappa$, $\mathrm{ot}(D_\alpha) = \beta_0$ for all $\alpha \in T$ (T and β_0 exist since $\{\alpha \in S : \mathrm{ot}(D_\alpha) = \beta\}$, for $\beta \leq \kappa$, partitions S into κ-many pieces). Now suppose, towards a contradiction, that S' reflects at $\alpha_0 < \kappa^+$. Let D be the set of limit points of D_{α_0}. Then $D \cap T$ is stationary in α_0. Now, if $\alpha \in D \cap T$, then $D_\alpha = D_{\alpha_0} \cap \alpha$ (by coherence), and therefore $\mathrm{ot}(D_\alpha) = \mathrm{ot}(D_{\alpha_0} \cap \alpha) = \beta_0$. But of course there is at most one α such that $\mathrm{ot}(D_{\alpha_0} \cap \alpha) = \beta_0$. Hence, $D \cap T$ has at most one element, and so it cannot be stationary in α_0. $\square$

Fact 2.1.3 implies that there exists a non-reflecting stationary subset of S^2_0,[2] which Gregory [13] used to show that GCH and $\square_{\omega_1}$ implies the existence of an ω_2-Souslin tree.

Corollary 2.1.4. *$\square_\kappa$ implies that for every infinite reglar cardinal $\lambda \leq \kappa$ there is a subset S of $S^{\kappa^+}_\lambda$ which is stationary in κ^+ and which does not reflect.*

[2]Here, $S^2_0 = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.

In that same paper, Jensen [15] introduced the *weak square*, a weaker version of the square principle presented above. The motivation was to attempt to weaken the assumption of Fact 2.1.3.

Definition 2.1.5. Let κ be an infinite cardinal. A $\square^*_\kappa$*-sequence* is a sequence $(\mathcal{C}_\alpha : \alpha \in \lim(\kappa^+))$ such that

1. $\mathcal{C}_\alpha$ is a nonempty collection of clubs of α and $|\mathcal{C}_\alpha| \leq \kappa$..

2. If $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{ot}(C) < \kappa$ for every $c \in \mathcal{C}_\alpha$

3. For every $C \in \mathcal{C}_\alpha$ and every $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

We say $\square^*_\kappa$ *holds* if there exists a $\square^*_\kappa$-sequence.

The weak square sequence $\square^*_\kappa$ is an object of interest when κ is singular due to the fact that $\kappa^{<\kappa} = \kappa$ implies $\square^*_\kappa$. Jensen [15] showed that the weak square $\square^*_\kappa$ is equivalent to the existence of a special κ-Aronszajn tree. On the other hand, the weak square principle does not imply failure of stationary reflection like the the square principle does. A counterexample can be found in Theorem 21 of the paper of Cummings et al.[7].

The next definition is due to Schimmerling [24], *the weak square with λ clubs on each level.*

Definition 2.1.6. Let κ be an infinite cardinal and a cardinal $\lambda \leq \kappa^+$. A $\square_{\kappa,<\lambda}$*-sequence* is a sequence $(\mathcal{C}_\alpha : \alpha \in \lim(\kappa^+))$ such that

1. $\mathcal{C}_\alpha$ is a nonempty collection of clubs of α and $|\mathcal{C}_\alpha| < \lambda$.

2. If $\mathrm{cf}(\alpha) < \kappa$, then $\mathrm{ot}(C) < \kappa$ for every $c \in \mathcal{C}_\alpha$

3. For every $C \in \mathcal{C}_\alpha$ and every $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

We say that $\square_{\kappa,<\lambda}$ *holds* if there exists a $\square_{\kappa,<\lambda}$-sequence and $\square_{\kappa,\lambda}$ *holds* if $\square_{\kappa,<\lambda^+}$ holds.

Clearly, $\square^*_\kappa = \square_{\kappa,\kappa}$, $\square_\kappa = \square_{\kappa,1}$, and $\square_{\kappa,\lambda}$ is weaker as λ increases. Schimmerling [24] discovered that for sufficiently small λ, we can construct non-reflecting stationary sets by using $\square_{\kappa,<\lambda}$. Here we note that $\kappa^{<\lambda} = \kappa$ implies $\lambda \leq \mathrm{cf}(\kappa)$.

Fact 2.1.7. *[24] Let λ and κ be regular cardinals such that $\lambda \leq \kappa^+$ and suppose $\square_{\kappa,<\lambda}$ holds. If $S \subseteq \kappa^+$ is stationary, then there exists a stationary set $T \subseteq S$ such that T does not reflect at any α with $\mathrm{cf}(\alpha) \geq \lambda$.*

The research regarding square principles branches off in many ways. One of the most important among them is the square principles when κ is a singular cardinal along with some form of large cardinal. Shelah's PCF theory also shares some connection with the square principles, namely the fact that $\square_{\kappa,\lambda}$ implies a very good scale for singular cardinal κ [7].

2.2 Forcing Axioms

Forcing axioms are a class of axioms that assert the existence of a filter that is sufficiently generic[3] for a partial order in some class of partial orders Γ. The axioms are useful in that they imply that the universe is rich enough relative to the generic extensions of the forcing notions in Γ in the sense that some of the information present in the forcing extensions is already there in the base universe and there is no need for a proper generic extension of one.[4]

Let us recall some forcing axioms.

Notation 2.2.1. Martin's Axiom, MA. If $\mathbb{P}$ is a partial order which has the ccc[5] and $\mathcal{D}$ is a collection of fewer than $2^{\aleph_0}$ dense subsets of $\mathbb{P}$, then there is a subset G of $\mathbb{P}$ such that

- if p and q are elements of G, then there is some r in G such that $r \leq p$ and $r \leq q$,
- if $p \geq q$ and $q \in G$, then $p \in G$, and
- $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

We say that G is a $\mathbb{P}$-generic filter. $\mathsf{MA}(\kappa)$ is the statement of MA replacing the clause "$\mathcal{D}$ is a collection of fewer than $2^{\aleph_0}$ dense subsets of $\mathbb{P}$" with "$\mathcal{D}$ is a collection of κ dense subsets of $\mathbb{P}$".

Given a set X, $[X]^\omega$ denotes the collection of all countable subsets of X.

We make a note here that when we say a stationary subset S of $[A]^\omega$ for some uncountable set A we mean the subset S meets every club under $\subseteq$ on $[A]^\omega$. More precisely, $S \subseteq [A]^\omega$ is stationary if for every $F : [A]^{<\omega} \to A, S$ contains a closure point of F.

Definition 2.2.2. A forcing notion is *proper* if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

[3]Some authors call this a "pseudo generic" filter

[4]Here, we use the terms "partial order" and "forcing notion" interchangeably.

[5]ccc stands for the 'countable chain condition'. $\mathbb{P}$ having the ccc means that $\mathbb{P}$ does not have uncountable antichains.

When we work with a proper forcing notion, we will be working with models which satisfy fractions of ZFC. The models will satisfy the theory ZFC minus the power set axiom. Consider the collection $H(\theta)$ of all sets of hereditary cardinality less than θ. Then $H(\theta)$ satisfies this theory, and so does every countable elementary submodel M of $(H(\theta), \in)$. We say that a cardinal θ is large enough for an object X to mean that $\mathcal{P}(X) \in H(\theta)$, and we say that θ is large enough if θ is large enough for every object of interest.

Fact 2.2.3. *[14](Theorem 31.7) A forcing notion $\mathbb{P}$ is proper if and only if for every large enough θ there is a club E of $[H(\theta)]^\omega$ such that for every $N \in E$ and every $p \in N \cap \mathbb{P}$ there is some $q \leq_{\mathbb{P}} p$ which is $(N, \mathbb{P})$-generic (i.e., given any dense set $D \subseteq \mathbb{P}$, $D \in N$, every $q' \leq_{\mathbb{P}} q$ is $\leq_{\mathbb{P}}$-compatible with some $r \in D \cap N$).*

Notation 2.2.4. The proper forcing axiom, PFA, is the statement of MA_{ω_1} replacing "$\mathbb{P}$ is a partial order which has the ccc" with "$\mathbb{P}$ is a proper forcing notion"

Notation 2.2.5. Martin's Maximum, MM, is the statement of PFA replacing "$\mathbb{P}$ is a proper partial order" with "$\mathbb{P}$ is a forcing notion preserving stationary subsets of ω_1". We will sometimes just say that "$\mathbb{P}$ preserves stationary sets".

In order to streamline our notations of axioms, we shall use the following notation scheme.

Notation 2.2.6. Given a class Γ of forcing notions and a cardinal κ,

$$\mathrm{FA}_\kappa(\Gamma)$$

is the statement that for every $\mathbb{P} \in \Gamma$ and every collection $\mathcal{D}$ of κ dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Remark 2.2.7. The forcing axioms presented above can be written in the following way.

- MA is $\mathrm{FA}_{<2^\omega}(\mathrm{ccc})$.
- MA_κ is $\mathrm{FA}_\kappa(\mathrm{ccc})$.
- PFA is $\mathrm{FA}_{\omega_1}(\mathrm{proper})$.

- MM is FA_{ω_1}(preserves stationary sets).

Fact 2.2.8. *The following diagram illustrates the relationships among the properties of a forcing notion $\mathbb{P}$ (s. [14]).*

$$\begin{gathered}\mathbb{P} \textit{ has the ccc}\\ \Downarrow\\ \mathbb{P} \textit{ is proper}\\ \Downarrow\\ \mathbb{P} \textit{ preserves stationary subsets of } \omega_1\\ \Downarrow\\ \mathbb{P} \textit{ preserves } \omega_1\end{gathered}$$

The relationships between the forcing axioms presented thus far is the following.

$$\mathsf{MM} \Rightarrow \mathsf{PFA} \Rightarrow \mathsf{MA}.$$

A forcing iteration $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \beta < \lambda, \alpha \leq \lambda \rangle$ is said to have countable supports if for every $\alpha \leq \lambda$, $\mathcal{P}_\alpha$ consists of α-sequences p with $p \restriction \beta \Vdash_{\mathcal{P}_\beta} p(\beta) \in \dot{\mathcal{Q}}_\beta$ for all $\beta < \alpha$ and such that $|\{\beta < \alpha : p(\beta) \neq 1_{\dot{\mathcal{Q}}_\beta}\}| \leq \aleph_0$, where $1_{\dot{\mathcal{Q}}_\beta}$ denotes the maximum (weakest) condition in $\dot{\mathcal{Q}}_\beta$.

Fact 2.2.9 (Shelah)**.** *[14](Theorem 31.15) Properness is preserved by countable support iteration.*

Hence, it is possible to iterate proper forcing in any length while preserving properness, so in particular preserving ω_1. This provides a way to build models with interesting combinatorics for objects of size $\aleph_1$.

A significant consequence of this fact is the countable support iteration of proper forcing preserves $\aleph_1$.

Now we shall present some well known consequences of PFA.

Fact 2.2.10. *[30] PFA implies $2^{\aleph_0} = \aleph_2$.*

Fact 2.2.11. *[29] PFA implies the failure of $\Box_\lambda$ for all regular $\lambda \geq \omega_2$.*

Now from Fact 2.2.9, if κ is a supercompact cardinal, then there is a countable support iteration $(\mathbb{P}_\alpha : \alpha < \kappa)$, $\mathbb{P}_\kappa \subseteq V_\kappa$, such that $\mathbb{P}_k$ forces PFA (Baumgartner and Shelah). All V-cardinals λ such that $\omega_1 < \lambda < \kappa$ are collapsed to ω_1 along the iteration. Hence, $2^{\aleph_0} = \aleph_2$ in the final extension.

In fact, if there exists a supercompact cardinal κ, then square fails above κ.

Fact 2.2.12. *[28] If κ is supercompact, then $\Box_\lambda$ fails for all $\lambda \geq \kappa$.*

Under MM, the square principle and its weak forms fail in general.

Fact 2.2.13. *[8] Suppose MM holds and λ is an uncountable cardinal. Then:*

(1) If $\mathrm{cf}(\lambda) = \omega$, then $\Box^*_\lambda$ fails.

(2) If $\mathrm{cf}(\lambda) = \omega_1$, then $\Box_{\lambda,\mu}$ fails for every $\mu < \lambda$.

(3) If $\mathrm{cf}(\lambda) \geq \omega_2$, then $\Box_{\lambda,\mu}$ fails for every $\mu < \mathrm{cf}(\lambda)$.

This is due to the fact that MM implies the following stationary reflection principles studied in [11]:

(i) if λ is an uncountable cardinal and S is a stationary subset of $[\lambda^+]^{\aleph_0}$, then there is $X \subseteq \lambda^+$ such that $|X| = \aleph_1$ and $S \cap [X]^{\aleph_0}$ is stationary.

(ii) if κ is a regular cardinal with $\kappa > \omega_1$ and $(S_i : i < \omega_1)$ is a sequence of stationary subsets of κ with $\mathrm{cf}(\omega)$, then there are stationarily many $\alpha \in \kappa$ with $\mathrm{cf}(\omega)$ such that $S_i \cap \alpha$ is stationary for every $i < \omega_1$.

By Theorems 7, 8 in [7], stationary reflection in the form of (i) is incompatible with weak square $\Box^*_\lambda$ when $\mathrm{cf}(\lambda) = \omega$. Thus, it can be shown that when MM holds, $\Box^*_\lambda$ fails for every singular λ where $\mathrm{cf}(\lambda) = \omega$.

Similarly, the other two claims in Fact 2.2.13 can be proved by using the stationary reflection principle (ii) and the fact that if λ is uncountable and $\Box_{\lambda,\mu}$, then for every stationary subset of λ^+ there is a stationary subset which does not reflect at any point of uncountable cofinality.

The weak square $\square^*_\kappa$ is not, however, the weakest form of square principles. In fact, Foreman and Magidor [10] introduced the *Very Weak Square principle*, denoted VWS_κ, which is considerably weaker than the weak square but still offers some applications to algebra and topology. However, this notion will not be used in this thesis, so to the readers who are interested, we refer to the reference we cited.

Interestingly, Martin's Maximum also has some influence on very weak squares.

Fact 2.2.14. *[19] MM implies the failure of VWS_κ where $\mathrm{cf}(\kappa) = \omega$.*

Chapter 3

A Consistent Forcing Axiom, Weak Square and An Inconsistent Forcing Axiom

In this chapter, we shall first briefly explore the forcing axioms in the form of $\mathsf{MA}^{1.5}_{\kappa}$ extending MA_{κ} (i.e., Martin's Axiom at κ), we will define a class of *stratified* forcing notions, and will establish basic facts. Then, we shall present the effect of the forcing axiom $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified), that is, the usual forcing axiom $\mathsf{MA}^{1.5}_{\kappa}$ but restricted to the class of stratified partial orders, on the weak square $\square_{\omega_1,\omega_1}$. And lastly, we shall observe the inconsistency which arises when we strengthen the forcing axiom to $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$.

3.1 Notions and basic facts

The forcing axioms in the form of $\mathsf{MA}^{1.5}_{\kappa}$ are the forcing axioms $\mathrm{FA}_{\kappa}(\Gamma)$, where Γ is the class of those partial order which satisfy what we shall call the $\aleph_{1.5}$-*chain condition.* The property is indeed named in such a way to emphasize the fact that a partial order with the countable chain condition satisfies $\aleph_{1.5}$-chain condition, and likewise, every partial order with the $\aleph_{1.5}$-chain condition class satisfies the $\aleph_2$-chain condition.

Definition 3.1.1. A partial order $\mathbb{P}$ has the $\aleph_{1.5}$-*chain condition*, or is $\aleph_{1.5}$-*c.c* for short, if for every large enough cardinal θ (i.e., every cardinal θ such that $\mathbb{P} \in H(\theta)$) there is a club E of $N \in [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq E$ and every $N_0 \in \mathcal{N}$, if N_0 has minimum height within $\mathcal{N}$, then for every $p_0 \in N_0 \cap \mathbb{P}$ there is some extension $p \in \mathbb{P}$ of p_0 such that p is $(N, \mathbb{P})$-generic for all $N \in \mathcal{N}$.

Indeed, we shall define the strengthening $\mathsf{MA}^{1.5}_\kappa$ of MA_κ for any cardinal κ using the notation scheme introduced in the previous chapter.

Definition 3.1.2. Let κ be a cardinal. We write $\mathsf{MA}^{1.5}_\kappa$ to denote

$$\mathrm{FA}_\kappa(\{\mathbb{P} : \mathbb{P} \text{ has the } \aleph_{1.5}\text{-c.c.}\}),$$

that is, $\mathsf{MA}^{1.5}_\kappa$ is the following statement: For every $\aleph_{1.5}$-c.c. partial order $\mathbb{P}$ and every collection $\mathcal{D}$ of size κ consisting of dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

This particular forcing axiom was introduced in the paper [2] and in the same paper (Theorem 2.1) it was proved that if CH holds and $\lambda \geq \omega_2$ is a regular cardinal which is closed enough, that is, $\mu^{\aleph_0} < \lambda$ for any $\mu < \lambda$, and, another combinatorial principle similar to square, the diamond principle $\diamondsuit(\{\alpha < \lambda : \mathrm{cf}(\alpha) \geq \omega_1\})$ holds, then there is a proper cardinal-preserving forcing notion $\mathbb{P}$ of size λ such that $\mathsf{MA}^{1.5}_\lambda$ holds in the generic extension by $\mathbb{P}$. We will not make use of diamond principle in this thesis. Interested readers can refer to general texts on set theory such as [14].

The forcing axiom which we shall be working on is a mild strengthening of $\mathsf{MA}^{1.5}_\kappa$, which will be called $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified), defined by restricting the finite family $\mathcal{N} \subseteq E$ to have a certain nice structural property. First, we shall define the notion of a *stratified* collection of models.

Definition 3.1.3. A collection $\mathcal{N}$ of countable elementary submodels of $H(\theta)$, for some infinite cardinal θ, is *stratified* if for all $N_0, N_1 \in \mathcal{N}$, if $N_0 \cap \omega_1 < N_1 \cap \omega_1$, then $\mathrm{ot}(N_0 \cap \omega_2) < N_1 \cap \omega_1$.

Indeed, since the elementary submodels are countable, there can only be so many ordinals below ω_1 for such a model N, certainly not uncountably many of them when viewed from outside of the model. We will now define a natural notion of height of a (countable) model, and will say what it means for a model to be closed under a given sequence of bijections between ordinals in ω_2 and their cardinality.

Definition 3.1.4. 1. Given a set N such that $N \cap \omega_1 \in \omega_1$, we denote $N \cap \omega_1$ by δ_N, *the height of N*.

2. Given a sequence $\vec{e} = (e_\alpha : \alpha \in \omega_2)$, where $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_2$, we say that a set N is *closed under $\vec{e}$* if

   (a) $e_\alpha \restriction \xi + 1 \in N$ whenever $\alpha \in \omega_2 \cap N$ and $\xi \in |\alpha| \cap N$, and

   (b) $e_\alpha^{-1}(\xi) \in N$ whenever $\alpha, \xi \in N$

We will be using the following well-known fact repeatedly, sometimes without mention, and for the sake of completeness we shall give its simple proof.

Fact 3.1.5. *Suppose $\vec{e} = (e_\alpha : \alpha \in \omega_2)$, where $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_2$, and suppose N_0 and N_1 are countable submodels of $H(\omega_2)$ closed under $\vec{e}$ such that $\delta_{N_0} \leq \delta_{N_1}$. Then $N_0 \cap \alpha \subseteq N_1$ for every $\alpha \in N_0 \cap N_1 \cap \omega_2$.*

Proof. Given any $\bar{\alpha} \in N_0 \cap \alpha$, $\xi = e_\alpha^{-1}(\bar{\alpha}) \in N_0 \cap |\alpha|$. But since α and ξ are both members of N_1 as $|\alpha| \leq \omega_1$, we also have that $\bar{\alpha} = e_\alpha(\xi) \in N_1$. $\square$

Corollary 3.1.6. *Suppose $\vec{e} = (e_\alpha : \alpha \in \omega_2)$, where $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_2$, and suppose N_0 and N_1 are countable submodels of $H(\omega_2)$ closed under $\vec{e}$ of the same height. Then $N_0 \cap N_1 \cap \omega_2$ is an initial segment of both $N_0 \cap \omega_2$ and $N_1 \cap \omega_2$.*

Now, back to the topic at hand of defining the forcing axiom $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified).

Definition 3.1.7. We say that a forcing notion $\mathbb{P}$ *has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models* iff for every infinite cardinal θ such that $\mathbb{P} \in H(\theta)$ there is a club $E \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite stratified $\mathcal{N} \subseteq E$, if $p_0 \in N_0 \cap \mathbb{P}$, where

$N_0 \in \mathcal{N}$ is of minimal height within $\mathcal{N}$, then there is an extension p of p_0 in $\mathbb{P}$ such that p is $(N, \mathbb{P})$-generic for every $N \in \mathcal{N}$.

Clearly, every partial order with the $\aleph_{1.5}$-c.c. also has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models.

Definition 3.1.8. Given a cardinal κ, we write $\mathsf{MA}^{1.5}_{\kappa}$(stratified) to denote

$$\mathrm{FA}_{\kappa}(\mathcal{K}),$$

where $\mathcal{K}$ is the class of partial orders $\mathbb{P}$ such that $\mathbb{P}$ has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models.

The following proposition extends the aforementioned fact that every forcing with the $\aleph_{1.5}$-c.c. is proper and has the $\aleph_2$-c.c.

Proposition 3.1.9. *If a forcing notion has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models, then it is proper and has the $\aleph_2$-c.c.*

Proof. Suppose $\mathbb{P}$ is a forcing notion with the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models. Let θ be a cardinal such that $\mathbb{P} \in H(\theta)$ and let $E \subseteq [H(\theta)]^{\aleph_0}$ be a club witnessing, for $H(\theta)$, that $\mathbb{P}$ has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models. Given any $N \in E$, $\{N\}$ is trivially stratified, and therefore for every $p \in \mathbb{P} \cap N$ there is an $(N, \mathbb{P})$-generic extension of p. This shows that $\mathbb{P}$ is proper.

To prove that $\mathbb{P}$ has the $\aleph_2$-chain condition let us assume, towards a contradiction, that there is a maximal antichain A of $\mathbb{P}$ such that $|A| \geq \aleph_2$, and let $(p_i : i < \lambda)$ be a one-to-one enumeration of A, for some $\lambda \geq \omega_2$. Let M be an elementary submodel of some large enough $H(\chi)$ such that

1. E, $A, (p_i : i < \lambda) \in M$ and
2. $|M| = \aleph_1$ and $\aleph_1 \subseteq M$.

Let $i_0 \in \omega_2 \setminus M$ and let $N \preccurlyeq H(\chi)$ be countable and such that p_{i_0}, E, $M \in N$. Let $\tau = M \cap \omega_2$ and $\beta = \sup(N \cap \tau)$.

Now, by correctness of M we may find $i_1 \in \omega_2 \cap M$ for which there is some $N' \in E \cap M$ such that $\delta_{N'} = \delta_N$ and $p_{i_1} \in N'$. Indeed, the existence of such an N' is expressed by a true sentence, as witnessed by i_0 and N, with δ_N, E and $(p_i : i < \lambda)$ as parameters.

We note that $\mathcal{N} = \{N, N'\}$ is a stratified family of members of E as $\delta_N = \delta_{N'}$. It follows, since $p_{i_0} \in N$ and $\delta_N = \delta_{N'}$, that we may find an $(N', \mathbb{P})$-generic condition p extending p_{i_0}. Then there must be condition p' extending p and extending some $\bar{p} \in A \cap N'$. But that is impossible since A is an antichain and $\bar{p} \neq p_{i_0}$ as $N' \subseteq M$. $\square$

We will be using essentially the same forcing construction from [2], which shows the consistency of $\mathsf{MA}^{1.5}_{<\kappa}$, for any given closed enough κ. This construction can be used to prove the following theorem.

Theorem 3.1.10. *(CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$-chain condition such that the following statements hold in the generic extension by $\mathcal{P}$.*

(1) $2^{\aleph_0} = \kappa$

(2) For every $\lambda < \kappa$, $\mathsf{MA}^{1.5}_{\lambda}$(stratified)

The reason why, we claim, the proof works here is because, coincidentally, all relevant collections of models appearing in the proof from [2] are themselves stratified. In particular, Theorem 3.1.10 will show that the forcing axioms of the form $\mathsf{MA}^{1.5}_{<\kappa}$(stratified) with a suitable κ are relatively consistent with ZFC. However, we will not verify these assertions in this thesis.

3.2 $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\square_{\omega_1,\omega_1}$

The main goal in this section is the following.

Theorem 3.2.1. *$\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\square_{\omega_1,\omega_1}$.*

The strategy we shall employ to prove this theorem is by constructing a partial order $\mathcal{P}$ of finite approximations to a weak square such that $\mathcal{P}$ satisfies the conditions of $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified). Thus, the forcing axiom will imply the existence of a weak square. The partial order is defined as follows.

Let $\vec{e} = (e_\alpha : \alpha < \omega_2)$ be such that $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_2$. We define the following forcing notion $\mathcal{P}$.

Conditions in $\mathcal{P}$ are triples

$$p = (h^p, i^p, \mathcal{N}_p)$$

with the following properties.

1. h^p is a function such that $\mathrm{dom}(h^p) \in [\mathrm{Lim}(\omega_2) \times \omega_1 \times \mathrm{Lim}(\omega_1 + 1)]^{<\omega}$ and such that for each $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$, $h^p(\alpha, \nu, \tau) \subseteq \tau \times \alpha$ is a finite function which can be extended to a strictly increasing and continuous function $f : \tau \longrightarrow \alpha$ with range cofinal in α.

2. For every $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$:

   (a) if $\mathrm{cf}(\alpha) = \omega_1$, then $\tau = \omega_1$ and $\nu = 0$;

   (b) if $\mathrm{cf}(\alpha) = \omega$, then $\tau \in \mathrm{Lim}(\omega_1)$.

3. For every α and ν there is at most one τ such that $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$.

4. i^p is a function whose domain is the set of triples $(\alpha, \nu, \bar{\tau})$ such that $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ for some τ and $\bar{\tau} \in \mathrm{dom}(h^p(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$, and $i^p(\alpha, \nu, \bar{\tau}) \in \omega_1$ for each $(\alpha, \nu, \bar{\tau}) \in \mathrm{dom}(i^p)$.

5. For every $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ and every limit ordinal $\bar{\tau} \in \mathrm{dom}(h^p(\alpha, \nu, \tau))$,

$$(h^p(\alpha, \nu, \tau)(\bar{\tau}), i^p(\alpha, \nu, \bar{\tau}), \bar{\tau}) \in \mathrm{dom}(h^p)$$

   and

$$h^p(h^p(\alpha, \nu, \tau)(\bar{\tau}), i^p(\alpha, \nu, \bar{\tau}), \bar{\tau}) = h^p(\alpha, \nu, \tau) \restriction \bar{\tau}.$$

6. $\mathcal{N}_p$ is a finite stratified collection of countable elementary submodels of $(H(\omega_2); \in)$ closed under $\vec{e}$.

7. For every $N \in \mathcal{N}_p$ and every $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ such that α, $\nu \in N$:

   (a) $\tau \in N$;

   (b) $h^p(\alpha, \nu, \tau) \restriction N \subseteq N$;

   (c) If $\mathrm{cf}(\alpha) = \omega_1$, then
$$\delta_N \in \mathrm{dom}(h^p(\alpha, \nu, \omega_1))$$
   and
$$h^p(\alpha, \nu, \omega_1)(\delta_N) = \sup(N \cap \alpha)$$

   (d) For every $\bar{\tau} \in \mathrm{dom}(h^p(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$, $i^p(\alpha, \nu, \bar{\tau}) \in N$.

Given $\mathcal{P}$-conditions p_0 and p_1, p_1 extends p_0 if and only if:

1. $\mathrm{dom}(h^{p_0}) \subseteq \mathrm{dom}(h^{p_1})$;

2. for every $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p_0})$,

   (a) $h^{p_0}(\alpha, \nu, \tau) \subseteq h^{p_1}(\alpha, \nu, \tau)$, and

   (b) $i^{p_1}(\alpha, \nu, \bar{\tau}) = i^{p_0}(\alpha, \nu, \bar{\tau})$ for each $\bar{\tau} \in \mathrm{dom}(h^{p_0}(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$.

3. $\mathcal{N}_{p_0} \subseteq \mathcal{N}_{p_1}$.

Given $p \in \mathcal{P}$, we denote $\{\alpha \in S^2_1 : (\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ for some ν, $\tau\}$ by X_p.[1]

Here each $h^p(\alpha, \nu, \tau)$ serves as a finite sequence which we will extend, using the generic filter of $\mathcal{P}$, to a sequence $C \in \mathcal{C}_\alpha$ as in the definition of the weak square. The function $i^p(\alpha, \nu, \tau)$ is the index function which keeps track of the correct subsequences of $h^p(\alpha, \nu, \tau)$. It is necessary for tracing back the correct sequence of C; without such an index function we would not have enough information to construct $\mathcal{C}_\alpha$.

[1]For the sake of simplicity in the notation, we shall write S^2_1 to denote the set of limit ordinals with cofinality ω_1 in ω_2. This is also denoted $S^{\omega_2}_{\omega_1}$ in the literature.

Now, the first step in the proof of Theorem 3.2.1 is to show that the partial order $\mathcal{P}$ indeed satisfies the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models.

Lemma 3.2.2. *$\mathcal{P}$ has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models.*

Proof. Let θ be such that $\mathcal{P} \in H(\theta)$ and let $\mathcal{N}^*$ be a finite stratified collection of countable elementary submodels of $H(\theta)$ containing $\vec{e}$. We will assume that for each $N^* \in \mathcal{N}^*$, $N^* = \bigcup_{\nu<\delta_{N^*}} N^*_\nu$, where $(N^*_\nu)_{\nu<\delta_{N^*}}$ is a continuous $\in$-chain of countable elementary submodels of $H(\theta)$ containing $\vec{e}$. In fact, if there is any cardinal $\chi > \theta$ such that N^* is of the form $N^{**} \cap H(\theta)$ for a countable $N^{**} \preccurlyeq H(\chi)$ with $\vec{e}$, $\theta \in N^{**}$, then N^* is a continuous $\in$-chain of countable elementary submodels of $H(\theta)$ as above.[2] This is because if we let $(x_n)_{n<\omega}$ be an enumeration of N^* and let $(\beta_n)_{n<\omega}$ be a strictly increasing sequence converging to δ_{N^*}, then by correctness of N^{**} we may build a sequence $(\vec{N}^*_n)_{n<\omega}$ of members of N^{**} such that

1. for each n, $\vec{N}^*_n = (N^*_i)_{i\leq\beta_n}$ is a continuous $\in$-chain of length $\beta_n + 1$ consisting of countable elementary submodels of $H(\theta)$ containing $\vec{e}$ and x_n;

2. $\vec{N}^*_{n+1}$ end-extends $\vec{N}^*_n$.

$\bigcup_{n<\omega} \vec{N}^*_n$ is then as desired.

Let $\mathcal{N} = \{N^* \cap H(\omega_2) \,:\, N^* \in \mathcal{N}^*\}$. Let also $N_0 \in \mathcal{N}$ be of minimal height and let $p_0 \in N_0$ be a $\mathcal{P}$-condition. Given any $\alpha \in X_{p_0}$, let $(\alpha(k) \,:\, k < m_\alpha)$ be the strictly increasing enumeration of

$$\{\sup(N \cap \alpha) \,:\, N \in \mathcal{N},\, \alpha \in N\}$$

and, for every $k < m_\alpha$, let $\delta^\alpha_k = \delta_N$ for any $N \in \mathcal{N}$ such that $\alpha \in N$ and $\sup(N \cap \alpha) = \alpha(k)$. Note that by Fact 3.1.5, δ^α_k is well-defined for every $k < m_\alpha$ (i.e., δ^α_κ is independent from the choice of N as long as $\alpha \in N$ and $\sup(N \cap \alpha) = \alpha(k)$) as in fact $N \cap \alpha = N' \cap \alpha$ whenever N, $N' \in \mathcal{N}$ are such that $\alpha \in N \cap N'$ and $\delta_N = \delta_{N'}$. For each $\alpha \in X_{p_0}$ and

[2] It is enough to assume the above since for all sets $X \subseteq Y$ and for every club C of $[Y]^\omega$, the collection of set of the form $X \cap N$ where $N \in C$ contains a club of $[X]^\omega$.

$k < m_\alpha$, let $i(\alpha, k) \in \delta_k^\alpha \setminus \mathrm{range}(i^{p_0})$ be such that $\delta_N < i(\alpha, k)$ for each $N \in \mathcal{N}$ with $\delta_N < \delta_k^\alpha$.

In order to prove the lemma, it suffices to show that

$$p^* = (h^{p^*}, i^{p^*}, \mathcal{N}_{p_0} \cup \mathcal{N})$$

is an $(N^*, \mathcal{P})$-generic condition for each $N^* \in \mathcal{N}^*$, where

$$\mathrm{dom}(h^{p^*}) = \mathrm{dom}(h^{p_0}) \cup \{(\alpha(k), i(\alpha, k), \delta_k^\alpha) : \alpha \in X_{p_0},\, k < m_\alpha\}$$

and where for each $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p^*})$:

1. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p_0})$ and $\mathrm{cf}(\alpha) = \omega$, then
   (a) $h^{p^*}(\alpha, \nu, \tau) = h^{p_0}(\alpha, \nu, \tau)$ and
   (b) $i^{p^*}(\alpha, \nu, \bar{\tau}) = i^{p_0}(\alpha, \nu, \bar{\tau})$ for each $\bar{\tau} \in \mathrm{dom}(h^{p_0}(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$;
2. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p_0})$ and $\mathrm{cf}(\alpha) = \omega_1$, then
   (a) $h^{p^*}(\alpha, \nu, \tau) = h^{p_0}(\alpha, \nu, \tau) \cup \{(\delta_k^\alpha, \alpha(k)) : k < m_\alpha\}$,
   (b) $i^{p^*}(\alpha, \nu, \bar{\tau}) = i^{p_0}(\alpha, \nu, \bar{\tau})$ for every $\bar{\tau} \in \mathrm{dom}(h^{p_0}(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$, and
   (c) $i^{p^*}(\alpha, \nu, \delta_k^\alpha) = i(\alpha, k)$ for each $k < m_\alpha$;
3. if $\alpha \in X_{p_0}$ and $k < m_\alpha$, then
   (a) $h^{p^*}(\alpha(k), i(\alpha, k), \delta_k^\alpha) = \{(\delta_{k'}^\alpha, \alpha(k')) : k' < k\}$ and
   (b) $i^{p^*}(\alpha(k), i(\alpha, k), \delta_{k'}^\alpha) = i(\alpha, k')$ for each $k' < k$.

Claim 3.2.3. *If $\alpha_0 < \alpha_1$ are such that α_0, $\alpha_1 \in X_{p_0}$, then $\alpha_0(k_0) < \alpha_0 < \alpha_1(k_1)$ for all $k_0 < m_{\alpha_0}$ and $k_1 < m_{\alpha_1}$.*

Proof. The inequality $\alpha_0(k_0) < \alpha_0$ is immediate given that $\alpha_0(k_0) = \sup(M \cap \alpha_0)$ for some countable M. Also, we note that if $N \in \mathcal{N}$ is such that $\alpha_1 \in N$ and $\alpha_1(k_1) = N \cap \alpha_1$, then $\alpha_0 \in N$ by Fact 3.1.5 since $\alpha \in N_0$ and $\delta_{N_0} \leq \delta_N$. Hence $\alpha_0 < \sup(N \cap \alpha_1) = \alpha_1(k_1)$. □

Claim 3.2.4. *For every $N \in \mathcal{N}$, $\alpha \in X_{p_0}$ and $k < m_\alpha$, if $\alpha(k)$, $i(\alpha, k) \in N$ and $k' < k$, then $\alpha(k') \in N$.*

Proof. Since $i(\alpha, k) \in N$, we have that $\delta_N \geq \delta^\alpha_k$. But $\alpha(k') \in M \cap \alpha(k)$ for every $M \in \mathcal{N}$ such that $\alpha \in M$ and $\delta_M = \delta^\alpha_k$, and $M \cap \alpha(k) \subseteq N \cap \alpha(\kappa)$, where the inclusion follows from Fact 3.1.5 since $\delta_N \geq \delta^\alpha_k$. □

The proof of the following claim is essentially the same.

Claim 3.2.5. *For all $N \in \mathcal{N}$ and $\alpha \in X_{p_0}$, if $\alpha \in N$, then $h^{p_0}(\alpha, 0, \omega_1) \in N$ and $i^{p_0}(\alpha, 0, \bar{\tau}) \in N$ for every $\bar{\tau} \in \mathrm{dom}(h^{p_0}(\alpha, 0, \omega_1)) \cap \mathrm{Lim}(\omega_1)$.*

Using the above two claims together with Fact 3.1.5, one can easily verify that p^* is a $\mathcal{P}$-condition, and it obviously extends p_0. Let now $N^* \in \mathcal{N}^*$ and let us prove that p^* is $(N^*, \mathcal{P})$-generic. For this, let $D \in N^*$ be an open and dense subset of $\mathcal{P}$ and let $p \in D$ extend p^*. We will prove that there is a condition $r \in D \cap N^*$ compatible with p.

Let $(N^*_\nu)_{\nu < \delta_{N^*}}$ be a continuous $\in$-chain of countable elementary submodels of $H(\theta)$ containing $\vec{e}$ such that $N^* = \bigcup_{\nu < \delta_{N^*}} N^*_\nu$. Since $\mathcal{N}_p$ is stratified and $N^* \cap H(\omega_2) \in \mathcal{N}_p$, we may find some $\nu_0 < \delta_{N^*}$ such that

1. $(h^p \cup i^p) \cap N^* \subseteq N^*_{\nu_0}$,

2. there is some $\eta \in N^*_{\nu_0} \cap \omega_2$ such that $[\eta, \omega_2) \cap N^*_{\nu_0} \cap N = \emptyset$ for every $N \in \mathcal{N}_p$ with $\delta_N < \delta_{N^*_{\nu_0}}$, and

3. for every $\alpha \notin N^*_{\nu_0}$ such that $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ for some ν, τ, and such that $\alpha^* = \min((N^*_{\nu_0} \cap \omega_2) \setminus \alpha)$ exists, there is some $\eta_\alpha \in N^*_{\nu_0} \cap \alpha^*$ with $[\eta, \alpha) \cap N^*_{\nu_0} \cap N = \emptyset$ for every $N \in \mathcal{N}_p$ such that $\delta_N < \delta_{N^*_{\nu_0}}$.

Given a $\mathcal{P}$-condition q, let $\mathbb{M}(q)$ be a structure with universe

$$\mathcal{U}_q := \{(\alpha, \nu, \tau, \xi, \beta) \,:\, (\alpha, \nu, \tau) \in \mathrm{dom}(h^q),\ (\xi, \beta) \in h^q(\alpha, \nu, \tau)\} \cup i^q$$

coding h^q and i^q in some fixed canonical way.

Let us denote $N^*_{\nu_0}$ by N^+. Let $R = \mathcal{U}_p \cap N^+$. Working in N^+ we may find a condition $r \in D$ such that $\mathcal{U}_p \cap N^+ \subseteq \mathcal{U}_r$ and for which there is an isomorphism

$$\pi : \mathbb{M}(p) \longrightarrow \mathbb{M}(r)$$

which is the identity on $\mathcal{U}_p \cap \mathcal{U}_r$ and is such that the following holds for each $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$:

1. if $\alpha \geq \sup(N^+ \cap \omega_2)$, then $\pi(\alpha) > \eta$;
2. if $\alpha \notin N^+$ and $\alpha^* = \min((N^+ \cap \omega_2) \setminus \alpha)$ exists, then $\pi(\alpha) > \eta_\alpha$;
3. if $\alpha \in N^+$ but $\nu \notin N^+$, then $\pi(\nu) > \delta_N$ for each $N \in \mathcal{N}_p$ such that $\delta_N < \delta_{N^+}$;
4. $\mathcal{N}_p \cup \mathcal{N}_r$ is stratified.

Such an r can indeed be found in N^+ since the existence of a condition with the properties above is a true statement, as witnessed by p, which can be expressed over $H(\theta)$ by a sentence with parameters in N^+.

In order to finish the proof it suffices to show that p and r can be amalgamated into a condition $p' \in \mathcal{P}$. This condition p' can be obtained as $p' = (h^{p'}, i^{p'}, \mathcal{N}_p \cup \mathcal{N}_r)$ by the following construction, very similar to that of p^* from p_0.

For every $\alpha \in X_r$, let $(\alpha(k) \,:\, k < m_\alpha)$ be the strictly increasing enumeration of

$$\{\sup(N \cap \alpha) \,:\, N \in \mathcal{N}_p,\, \alpha \in N\}$$

and, for every $k < m_\alpha$, let $\delta^\alpha_k = \delta_N$ for any $N \in \mathcal{N}$ such that $\alpha \in N$ and $\sup(N \cap \alpha) = \alpha(k)$. As in the construction of p^* from p_0, each δ^α_k is well-defined. For each $\alpha \in X_r$ and $k < m_\alpha$, let $i(\alpha, k) \in \delta^\alpha_k \setminus \mathrm{range}(i^p)$ be such that $\delta_N < i(\delta, k)$ for each $N \in \mathcal{N}_p$ with $\delta_N < \delta^\alpha_k$.

We define $h^{p'}$ and $i^{p'}$ by letting $h^{p'}$ be a function with

$$\mathrm{dom}(h^{p'}) = \mathrm{dom}(h^p) \cup \mathrm{dom}(h^r) \cup \{(\alpha(k), i(\alpha, k), \delta^\alpha_k) \,:\, \alpha \in X_r,\, k < m_\alpha\}$$

and making the following definitions for each $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p'})$ (where, given a condition $t \in \mathcal{P}$ and a tuple $(\alpha, \nu, \tau) \notin \mathrm{dom}(h^t)$, we define $h^t(\alpha, \nu, \tau) = \emptyset$ if $(\alpha, \nu, \tau) \notin \mathrm{dom}(h^t)$, and similarly with i^t in place of h^t):

1. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p) \cup \mathrm{dom}(h^r)$ and $\mathrm{cf}(\alpha) = \omega$, then
   (a) $h^{p'}(\alpha, \nu, \tau) = h^p(\alpha, \nu, \tau) \cup h^r(\alpha, \nu, \tau)$ and
   (b) for each $\bar{\tau}$ in $(\mathrm{dom}(h^p(\alpha, \nu, \tau)) \cup \mathrm{dom}(h^r(\alpha, \nu, \tau))) \cap \mathrm{Lim}(\omega_1)$, $i^{p'}(\alpha, \nu, \bar{\tau}) = i^p(\alpha, \nu, \bar{\tau}) \cup i^r(\alpha, \nu, \bar{\tau})$;
2. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$ and $\mathrm{cf}(\alpha) = \omega_1$, then
   (a) $h^{p'}(\alpha, \nu, \tau) = h^p(\alpha, \nu, \tau)$,
   (b) $i^{p'}(\alpha, \nu, \bar{\tau}) = i^p(\alpha, \nu, \bar{\tau})$ for every $\bar{\tau} \in \mathrm{dom}(h^p(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$ and
   (c) $i^{p'}(\alpha, \nu, \bar{\tau}) = i^r(\alpha, \nu, \bar{\tau})$ for every $\bar{\tau} \in \mathrm{dom}(h^r(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$;
3. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^r)$ and $\mathrm{cf}(\alpha) = \omega_1$, then
   (a) $h^{p'}(\alpha, \nu, \tau) = h^r(\alpha, \nu, \tau) \cup \{(\delta^\alpha_k, \alpha(k)) \,:\, k < m_\alpha\}$,
   (b) $i^{p'}(\alpha, \nu, \bar{\tau}) = i^r(\alpha, \nu, \bar{\tau})$ for every $\bar{\tau} \in \mathrm{dom}(h^r(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$ and
   (c) $i^{p'}(\alpha, \nu, \delta^\alpha_k) = i(\alpha, k)$ for each $k < m_\alpha$;
4. if $\alpha \in X_r$ and $k < m_\alpha$, then
   (a) $h^{p'}(\alpha(k), i(\alpha, k), \delta^\alpha_k) = \{(\delta^\alpha_{k'}, \alpha(k')) \,:\, k' < k\}$,
   (b) $i^{p'}(\alpha(k), i(\alpha, k), \bar{\tau}) = i^r(\alpha, 0, \bar{\tau})$ for every limit ordinal $\bar{\tau} \in \mathrm{dom}(h^r(\alpha, 0, \omega_1))$ and
   (c) $i^{p'}(\alpha(k), i(\alpha, k), \delta^\alpha_{k'}) = i(\alpha, k')$ for each $k' < k$.

The choice of η and of η_α, for $\xi \in X_p \backslash N^+$ such that $\min((N^+ \cap \omega_2) \backslash \alpha)$ exists, together with the way r has been fixed, immediately yields the following.

Claim 3.2.6. *For every $N \in \mathcal{N}_p$ and every $\alpha \in X_r \backslash X_p$, if $\alpha \in N$, then $\delta_N \geq \delta_{N^+}$.*

Using Claim 3.2.6, we can prove the following versions of Claims 3.2.3 and 3.2.5.

Claim 3.2.7. *If $\alpha_0 < \alpha_1$ are such that α_0, $\alpha_1 \in X_r$, then $\alpha_0(k_0) < \alpha_0 < \alpha_1(k_1)$ for all $k_0 < m_{\alpha_0}$ and $k_1 < m_{\alpha_1}$.*

Claim 3.2.8. *For all $N \in \mathcal{N}_p$ and $\alpha \in X_r$, if $\alpha \in N$, then $h^r(\alpha, 0, \omega_1) \in N$ and $i^r(\alpha, 0, \bar{\tau}) \in N$ for every $\bar{\tau} \in \mathrm{dom}(h^r(\alpha, 0, \omega_1)) \cap \mathrm{Lim}(\omega_1)$.*

We also have the following counterpart of Claim 3.2.4, proved in exactly the same way.

Claim 3.2.9. *For every $N \in \mathcal{N}_p$, $\alpha \in X_r$ and $k < m_\alpha$, if $\alpha(k)$, $i(\alpha, k) \in N$ and $j < k$, then $\alpha(j) \in N$.*

Using the corresponding forms of Claims 3.2.3 and 3.2.4, together with Fact 3.1.5 and the particular choice of r together with Claims 3.2.7, 3.2.8 and 3.2.9, we can then verify that p' is a condition in $\mathcal{P}$, which finishes the proof of the lemma since then p' of course extends both p and r. □

We will need the following four density lemmas to verify that the function h^p can be extended in order to form a weak square sequence.

Lemma 3.2.10. *For every $\alpha < \omega_1$ of countable cofinality and every $p \in \mathcal{P}$ there is a condition $p' \in \mathcal{P}$ extending p and such that $(\alpha, 0, \omega) \in \mathrm{dom}(h^{p'})$.*

Proof. We simply let $p' = (h^p \cup \{((\alpha, 0, \omega), \emptyset)\}, i^p, \mathcal{N}_p)$. □

Lemma 3.2.11. *For every $\alpha \in S^2_1$ and every $p \in \mathcal{P}$ there is a condition $p' \in \mathcal{P}$ extending p and such that $\alpha \in X_{p'}$.*

Proof. We may obviously assume $\alpha \notin X_p$. We may also assume that $\alpha \in N$ for some $N \in \mathcal{N}_p$ as the conclusion in the other case is immediate. Let $(\alpha(k) : k < m_\alpha)$ be the strictly increasing enumeration of

$$\{\sup(N \cap \alpha) : N \in \mathcal{N}_p,\, \alpha \in N\}$$

and, for every $k < m_\alpha$, let $\delta^\alpha_k = \delta_N$ for any $N \in \mathcal{N}$ such that $\alpha \in N$ and $\sup(N \cap \alpha) = \alpha(k)$. As usual, using Fact 3.1.5 we have that each δ^α_k is well-defined. For each $\alpha \in X_r$ and $k < m_\alpha$, let $i(\alpha, k) \in \delta^\alpha_k \setminus \mathrm{range}(i^p)$ be such that $\delta_N < i(\alpha, k)$ for each $N \in \mathcal{N}_p$ with $\delta_N < \delta^\alpha_k$.

We can now easily verify that the following is a condition $p' \in \mathcal{P}$ as required: $p' = (h^{p'}, i^{p'}, \mathcal{N}_p)$, where

$$\mathrm{dom}(h^{p'}) = \mathrm{dom}(h^p) \cup \{(\alpha, 0, \omega_1)\} \cup \{(\alpha(k), i(\alpha, k), \delta^\alpha_k) \,:\, k < m_\alpha\}$$

and where for each $(\alpha, \nu, \tau) \in \mathrm{dom}(h^{p'})$:

1. if $(\alpha, \nu, \tau) \in \mathrm{dom}(h^p)$, then
   (a) $h^{p'}(\alpha, \nu, \tau) = h^p(\alpha, \nu, \tau)$ and
   (b) $i^{p'}(\alpha, \nu, \bar{\tau}) = i^p(\alpha, \nu, \bar{\tau})$ for each $\bar{\tau} \in \mathrm{dom}(h^p(\alpha, \nu, \tau)) \cap \mathrm{Lim}(\omega_1)$;
2. $h^{p'}(\alpha, 0, \omega_1) = \{(\delta^\alpha_k, \alpha(i)) \,:\, i < m_\alpha\}$ and $i^{p'}(\alpha, 0, \delta^\alpha_k) = i(\alpha, k)$ for each $k < m_\alpha$;
3. for each $k < m_\alpha$,
   (a) $h^{p'}(\alpha(k), i(\alpha, k), \delta^\alpha_k) = \{\delta^\alpha_{k'} \,:\, k' < k\}$ and
   (b) $i^{p'}(\alpha(k), i(\alpha, k), \delta^\alpha_{k'}) = i(\alpha, k')$ for each $k' < k$.

$\square$

Lemmas 3.2.12 and 3.2.13 are also easy.

Lemma 3.2.12. *For every $p \in \mathcal{P}$, $\alpha \in X_p$, and every $\nu < \omega_1$ there is a condition $p' \in \mathcal{P}$ extending p and such that $\nu \in \mathrm{dom}(h^{p'}(\alpha, 0, \omega_1))$.*

Lemma 3.2.13. *For every $p \in \mathcal{P}$, $\alpha \in X_p$, every nonzero limit ordinal $\delta \in \mathrm{dom}(h^p(\alpha, 0, \omega_1))$, and every $\eta < h^p(\alpha, 0, \omega_1)(\delta)$ there is a condition $p' \in \mathcal{P}$ extending p together with some limit ordinal $\mu \in \mathrm{dom}(h^{p'}(\alpha, 0, \omega_1)) \cap \delta$ such that $h^{p'}(\alpha, 0, \omega_1)(\mu) > \eta$.*

Proof. We can choose a countable elementary submodel N of $H(\omega_2)$ closed under $\vec{e}$ such that $\delta_{N_0} < \delta_N < \delta_{N_1}$, where $\sup(N_0 \cap \alpha) \geq \eta$ and $\sup(N_1 \cap \alpha) = \delta$. By Fact 3.1.5 and using the construction of p' with $\mathcal{N}_{p'} = \mathcal{N}_p \cup \{N\}$ in the proof of Lemma 3.2.11 yields then the result. $\square$

Given a $\mathcal{P}$-generic filter G, a limit ordinal $\alpha < \omega_2$, and $\nu < \omega_1$, we define $C^G_{\alpha,\nu}$ as

$$\bigcup\{\text{range}(h^p(\alpha,\nu,\tau)) \,:\, p \in G,\, (\alpha,\nu,\tau) \in \text{dom}(h^p) \text{ for some } \tau\}$$

Let also

$$\mathcal{C}^G_\alpha = \{C^G_{\alpha,\nu} \,:\, \nu < \omega_1,\, C^G_{\alpha,\nu} \neq \emptyset\}$$

We immediately obtain the following corollary from Lemmas 3.2.2, the density lemmas 3.2.11–3.2.13, and the definition of $\mathcal{P}$.

Corollary 3.2.14. *If G is a $\mathcal{P}$-generic filter over V, then*

$$(\mathcal{C}^G_\alpha \,:\, \alpha \in \text{Lim}(\omega_2))$$

is a $\square_{\omega_1,\omega_1}$-sequence.

Corollary 3.2.14 yields the following.

Corollary 3.2.15. *$\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\square_{\omega_1,\omega_1}$.*

The original goal of our work was to show that the forcing axiom $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies the square $\square_{\omega_1,\omega}$ but it became clear very soon that out proof strategy just didn't work when ν ranges over ω; there is simply too little space to accommodate all the sequences $C^G_{\alpha,\nu}$. This limitation of our approach naturally yields the following question.

Question 3.2.16. *Does $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) imply $\square_{\omega_1,\omega}$?*

It is proved in [21] that $\mathsf{MA}^{1.5}_\kappa$, for any given κ, is consistent with $\neg\square_{\omega_1,\omega}$.

Question 3.2.17. *Does $\mathsf{MA}^{1.5}_{\aleph_2}$ imply $\square_{\omega_1,\omega_1}$?*

Finally, the following corollary is an immediate consequence of Corollary 3.2.14.

Corollary 3.2.18. ZFC *proves that there is a poset* $\mathcal{P}$ *such that*

1. $\mathcal{P}$ *is proper,*

2. $\mathcal{P}$ *has the* $\aleph_2$*-c.c., and*

3. $\mathcal{P}$ *forces weak square.*

Similar results have been obtained by Neeman. One of the results in [21] is that a forcing axiom $\mathsf{MA}^{1.5}_{\omega_2}(U)$, whose definition involves a certain parameter $U \subseteq [\omega_2]^{\aleph_0}$, implies both $\square_{\omega_1,<\omega}$ and the following strengthening $\square^{\mathrm{ta}}_{\omega_1,\omega}$ of $\square_{\omega_1,\omega}$:[3] Given cardinals $\lambda \leq \kappa$ such that $\kappa \geq \omega_1$, $\square^{\mathrm{ta}}_{\kappa,\lambda}$ holds if and only if there is a $\square_{\kappa,\lambda}$-sequence $(\mathcal{C}_\alpha \, : \, \alpha \in \mathrm{Lim}(\kappa^+))$ such that for every $\alpha \in \mathrm{Lim}(\kappa^+)$ and for all C, $C' \in \mathcal{C}_\alpha$, C and C' agree on a tail, i.e., there is some $\beta < \alpha$ such that $C \setminus \beta = C' \setminus \beta$.[4]

Neeman also points out in [21] that both $\square_{\omega_1,\omega}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$ follow from some of his strong high analogues of PFA. On a related vein,[5] Sakai shows in [23] that Martin's Maximum proves that $\square^{\mathrm{p}}_{\omega_1}$ (i.e, partial square at ω_2) holds and that this is not the case for PFA.

Let us write $\mathsf{PFA}_{\aleph_2}(\aleph_2\text{-c.c.})$ to denote $\mathrm{FA}_{\aleph_2}(\mathcal{K})$, where $\mathcal{K}$ is the class of proper forcing notions with the $\aleph_2$-chain condition.

Theorem 3.1.10, as well as other similar strengthenings of the main result from [2], motivate the following question which was the original goal of this project.[6]

Question 3.2.19. *Is* $\mathsf{PFA}_{\aleph_2}(\aleph_2\text{-}c.c.)$ *consistent?*

We are not able to answer this question. However, in Section 3.4 we will show that $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$, a natural strengthening of $\mathsf{PFA}_{\aleph_2}(\aleph_2\text{-c.c.})$, is in fact inconsistent.

[3] On the other hand, the definition of $\mathsf{MA}^{1.5}_{\omega_2}$(stratified) is parameter-free.

[4] The superscript ta stands for 'tail agreement'.

[5] One important aspect in which Sakai's result is different from both Neeman's results and the first main result in the present paper is that Sakai's theorem involves strong forcing axiom at ω_1, whereas the others are implications from forcing axioms at ω_2.

[6] This question is also motivated by the main result from [1], to the effect that it is consistent, for arbitrary choice of κ, that $\mathrm{FA}_\kappa(\{\mathbb{P} \, : \, \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\})$ holds. It is worth pointing out that the proof of this theorem is very different from the proof of the main result from [2].

$\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ is $\mathrm{FA}_{\aleph_2}(\mathcal{K}^*)$, where $\mathcal{K}^*$ is the class of forcing notions that both preserve stationary subsets of ω_1 and have the $\aleph_2$-chain condition.

One of the ingredients of this proof will be the fact that $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ implies $\square_{\omega_1,\omega_1}$ (since $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ extends $\mathsf{MA}^{1.5}_{\aleph_2}(\text{stratified})$).

3.3 Uniformization property for $\aleph_2$

This section contains the necessary facts which we shall employ in Section 3.4 to prove another result in this thesis.

Given a set S of ordinals and a set X, let us denote by $\mathsf{Unif}_{S,X}$ the statement that for every sequence $(f_\alpha : \alpha \in S)$ of colourings f_α with colour set X such that $\mathrm{dom}(f_\alpha)$ is a club of α there is a function $H : \bigcup S \longrightarrow X$ such that for every $\alpha \in S$,

$$\{\xi \in \mathrm{dom}(f_\alpha) \,:\, f_\alpha(\xi) = H(\xi)\}$$

contains a club of α.

Shelah proves the following theorem in [26], Appendix, Chapter 3. A more accessible alternative proof can be found in Rinot's blog [22].

Theorem 3.3.1. *(Shelah) $\mathsf{Unif}_{S^2_1,2}$ is false.*

We can also define a natural weakening $\mathsf{Unif}^{\mathrm{c}}_{S,X}$ of $\mathsf{Unif}_{S,X}$ by restricting to sequences $(f_\alpha : \alpha \in S)$ of constant colourings (i.e., for every $\alpha \in S$, f_α is a constant function).[7] It is immediate to see that for any $S \subseteq \mathsf{Ord}$ and any set X, $\mathsf{Unif}^{\mathrm{c}}_{S,X}$ can be equivalently stated as the assertion that for every function $F : S \longrightarrow X$ there is a function $H : \bigcup S \longrightarrow X$ with the property that for every $\alpha \in S$ there is a club $C \subseteq \alpha$ of α such that $H(\xi) = F(\alpha)$ for every $\xi \in C$. We will say that H *uniformizes F mod. clubs.*

The following is implicit in [26], Appendix, Chapter 3.

Theorem 3.3.2. *(Shelah) If $S \subseteq S^2_1$ is stationary and $\mathsf{Unif}^{\mathrm{c}}_{S,2}$ holds, then CH holds as well.*

[7] The superscript c is for 'constant'.

Proof. $\mathsf{Unif}^{\mathrm{c}}_{S,2}$ clearly implies $\mathsf{Unif}^{\mathrm{c}}_{S,\mathbb{R}}$: Given $F : S \longrightarrow {}^{\omega}2$, let $F_n : S \longrightarrow 2$ be defined by $F_n(\alpha) = (F(\alpha))(n)$ (for each $n < \omega$). Applying $\mathsf{Unif}^{\mathrm{c}}_{S,2}$ to each F_n we obtain functions $H_n : S \longrightarrow 2$ and clubs $D^n_\alpha \subseteq \alpha$, for $\alpha \in S$ and $n < \omega$, such that $H_n(\xi) = F_n(\alpha)$ for all $\xi \in D^n_\alpha$. But then, if we define $H : S \longrightarrow {}^{\omega}2$ by letting $H(\xi) = (H_n(\xi) : n < \omega)$, it follows that H uniformizes F mod. clubs as witnessed by the clubs D_α, for $\alpha \in S$, where $D_\alpha = \bigcap_n D^n_\alpha$.

Thus, if $2^{\aleph_0} \geq \aleph_2$ and $\mathsf{Unif}^{\mathrm{c}}_{S,2}$ holds, then $\mathsf{Unif}^{\mathrm{c}}_{S,\omega_2}$ holds as well. Now suppose $\mathsf{Unif}^{\mathrm{c}}_{S,2}$ holds and $2^{\aleph_0} \geq \aleph_2$. Letting F be the identity function on S, we apply $\mathsf{Unif}^{\mathrm{c}}_{S,\omega_2}$ to F and get a corresponding uniformizing function $H : \omega_2 \longrightarrow \omega_2$ and clubs $D_\alpha \subseteq \alpha$ for $\alpha \in S$. Since S is stationary, we may find $\alpha \in S$ closed under H. But now we reach a contradiction since there is obviously no club $D \subseteq \alpha$ such that $H(\xi) = F(\alpha) = \alpha$ for all $\xi \in D$.[8] $\square$

We can straightforwardly prove the above assertion with the full statement of $\mathsf{Unif}_{S,2}$ to show that CH holds with the same main idea of the proof. The full proof can be found in [22].

Remark 3.3.3. If $S \subseteq S^2_1$ is stationary and $\mathsf{Unif}_{S,2}$ holds, then CH holds.

Remark 3.3.4. Given a club-sequence $\vec{C} = (C_\alpha : \alpha \in S)$ such that $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$ for each $\alpha \in S$, we can define the following strengthening $\mathsf{Unif}^{\mathrm{c},\vec{C},\mathrm{cbd}}_{S,2}$ of $\mathsf{Unif}^{\mathrm{c}}_{S,2}$: $\mathsf{Unif}^{\mathrm{c},\vec{C},\mathrm{cbd}}_{S,2}$ is the statement that for every function $F : S \longrightarrow 2$ there is a function $H : \sup(S) \longrightarrow 2$ such that for every $\alpha \in S$,

$$\{\xi \in C_\alpha : H(\xi) = F(\alpha)\}$$

is co-bounded in α.[9]

If CH holds and $\vec{C} = (C_\alpha : \alpha \in S)$ is as above, $\mathsf{Unif}^{\mathrm{c},\vec{C},\mathrm{cbd}}_{S^2_1,2}$ can be forced by a σ-closed and $\aleph_2$-c.c. forcing, obtained as the direct limit of a long enough countable support iteration of σ-closed forcing notions with the $\aleph_2$-c.c. At any given stage of the iteration, the corresponding iterand is the forcing $\mathcal{Q}_{\vec{C},F}$ for adding a uniformizing function on $\vec{C}$ mod. co-bounded sets, for some given colouring $F : S \longrightarrow 2$: A condition in $\mathcal{Q}_{\vec{C},F}$ is a

[8]There is obviously not even any nonempty $D \subseteq \alpha$ like that.

[9]We say that H uniformizes F on $\vec{C}$ modulo co-bounded sets.

function $q = (b^q_\alpha \, : \, \alpha \in Z_q)$, for some countable $Z_q \subseteq S^2_1$, such that $b^q_\alpha < \alpha$ for each $\alpha \in Z_q$, and such that $\sup(\mathrm{dom}(C_{\alpha'}) \cap \alpha) < b^q_{\alpha'}$ for all $\alpha < \alpha'$ in Z_q with $F(\alpha) \neq F(\alpha')$. The extension relation is reverse inclusion.

Question 3.3.5. *Is the dependence on a fixed club-sequence in the consistency proof in Remark 3.3.4 necessary? In other words, is the following strengthening of* $\mathsf{Unif}^{c,\vec{C}_*,\,cbd}_{S,2}$*, for a fixed club-sequence* $\vec{C}_* = (C_\alpha, \colon \alpha \in S^2_1)$ *with* $\mathrm{ot}(C_\alpha) = \omega_1$ *for each* α*, consistent? Suppose* $\vec{C} = (C_\alpha \, : \, \alpha \in S)$ *is a club-sequence such that* $\mathrm{ot}(C_\alpha) = \omega_1$ *for each* $\alpha \in S$*. Then for every function* $F : S \longrightarrow 2$ *there is a function* $H : \sup(S) \longrightarrow 2$ *such that for every* $\alpha \in S$*,*

$$\{\xi \in C_\alpha \, : \, H(\xi) = F(\alpha)\}$$

is co-bounded in α*.*

Recall the definition of generalized diamond principle $\diamondsuit$. For any cardinal number κ and a stationary set $S \subseteq \kappa$, $\diamondsuit(S)$ is the statement there exists a $\diamondsuit$-sequence $(A_\alpha \subseteq \alpha : \alpha \subseteq S)$ such that for every $A \subseteq \kappa$, $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in κ.

Remark 3.3.6. The statement that $\mathsf{Unif}^{c}_{S,2}$ holds for every stationary $S \subseteq S^2_1$ is not equivalent to CH as, for example, the assumption that $\diamondsuit(S)$ holds for every stationary $S \subseteq S^2_1$ implies $\neg\,\mathsf{Unif}^{c}_{S,2}$ for every such S: Suppose $(A_\alpha \, : \, \alpha \in S)$ is a $\diamondsuit$-sequence and let $F : S \longrightarrow 2$ be such that for every $\alpha \in S$, $F(\alpha) = 1 - i$ if A_α codes a function $H_\alpha : \alpha \longrightarrow 2$ and there are club-many $\xi \in \alpha$ such that $H_\alpha(\xi) = i$. It is easy to see that no function $H : \omega_2 \longrightarrow 2$ can uniformize F mod. clubs.

Given a class $\mathcal{K}$ of countable models, let us say that a proper forcing $\mathbb{P}$ is *proper with respect to* $\mathcal{K}$ in case for every cardinal θ such that $\mathbb{P} \in H(\theta)$ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every $N \in D \cap \mathcal{K}$ and every condition $p \in \mathbb{P} \cap N$ there is an extension $p^* \in \mathbb{P}$ of p which is $(N, \mathbb{P})$-generic.

Given a cardinal θ, a set $\mathcal{S} \subseteq [H(\theta)]^{\aleph_0}$ is a *projective stationary subset of* $H(\theta)$ in case for every stationary $S \subseteq \omega_1$ and every club D of $[H(\theta)]^{\aleph_0}$ there is some $N \in \mathcal{S} \cap D$ such that $\delta_N \in S$. The following proposition is standard.

Proposition 3.3.7. *Let $\mathcal{K}$ be a class of models such that $\mathcal{K} \cap [H(\theta)]^{\aleph_0}$ is a projective stationary subset of $[H(\theta)]^{\aleph_0}$ for every cardinal $\theta > \omega_1$ such that $\mathbb{P} \in H(\theta)$. Let $\mathbb{P}$ be a forcing notion which is proper with respect to $\mathcal{K}$. Then $\mathbb{P}$ preserves stationary subsets of ω_1.*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for a club of ω_1^V, let $S \subseteq \omega_1$ be stationary, and let $p \in \mathbb{P}$. Let θ be large enough and, using the projective stationarity of $\mathcal{K} \cap H(\theta)$, let $N \prec H(\theta)$ be countable and such that $\mathbb{P}$, $\dot{C}$, $p \in N$ and $\delta_N \in S$. Let p^* be an $(N, \mathbb{P})$-generic condition stronger than p. Then p^* forces that $\delta_N \in S$ is a limit of ordinals in $\dot{C}$ and therefore, since $\dot{C}$ is a $\mathbb{P}$-name for a closed set, that $\delta_N \in \dot{C}$. $\square$

3.4 $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.$)$ is false

This section is one of the main results of this thesis. As has already been presented, the square principles are not compatible with the forcing axioms which are modelled by collapsing a supercompact cardinal to ω_2 such as PFA and MM. Weaker forcing axioms such as Martin's Axiom are, however, compatible with square principles.

In the previous section, we have shown that one of these weaker axioms, more precisely, $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified), not only is compatible with square principles but also outright implies one of them.

We define $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.$)$ as follows. Given a cardinal κ, $\mathsf{MM}_\kappa(\aleph_2$-c.c.$)$ denotes $\mathrm{FA}_\kappa(\Gamma)$, where Γ is the class of all posets $\mathbb{P}$ such that

- $\mathbb{P}$ preserves stationary subsets of ω_1 and
- $\mathbb{P}$ has the $\aleph_2$-c.c.

Theorem 3.4.1. *$\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.$)$ is false.*

Let us assume, towards a contradiction, that $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.$)$ holds. In particular $\mathrm{FA}_{\aleph_2}({}^{<\omega}2)$ holds and therefore CH fails.[10] Let $S = S^2_1$. It follows, by Theorem 3.3.2,

[10] In fact $2^{\aleph_0} \geq \aleph_3$.

that there is a function $F : S \longrightarrow 2$ for which there is no function $H : \omega_2 \longrightarrow 2$ uniformizing F mod. clubs.

Since $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) also holds, we may fix a $\square_{\omega_1,\omega_1}$-sequence $\vec{\mathcal{C}} = (\mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\omega_2))$ (by Theorem 3.2.1). Let also $\vec{e} = (e_\alpha : \alpha < \omega_2)$ be such that $e_\alpha : |\alpha| \longrightarrow \alpha$ is a bijection for each $\alpha < \omega_2$.

Let $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ be the class of countable models N such that $N \cap \omega_2 = \bigcup_{\gamma \in C} e_\gamma``\delta_N$ for some $C \in \mathcal{C}_\alpha$, where $\alpha = \sup(N \cap \omega_2)$.

The following is quite standard.

Claim 3.4.2. *For every cardinal $\theta > \omega_1$, $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}} \cap H(\theta)$ is a projective stationary subset of $[H(\theta)]^{\aleph_0}$.*

Proof. Suppose D is a club of $[H(\theta)]^{\aleph_0}$ and $S \subseteq \omega_1$ is stationary. Let $f : {}^{<\omega}\omega_2 \longrightarrow \omega_2$ be a finitary function such that for every $X \in [\omega_2]^{\aleph_0}$, if $f``[X]^{<\omega} \subseteq X$, then $X = N \cap \omega_2$ for some $N \in D$. Let $\alpha \in S^2_0$ be such that $\omega_1 < \alpha$ and $f``[\alpha]^{<\omega} \subseteq \alpha$ and let $C \in \mathcal{C}_\alpha$. But now, since $E = \{M \cap \alpha \,:\, M \prec (H(\omega_2); \in, \vec{e}, C)\}$ contains a club of $[\alpha]^{\aleph_0}$, we may pick some $X \in E$ closed under f and such that $\delta_X \in S$, and if $N \in D$ is such that $N \cap \omega_2 = X$, then N will be a member of $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ such that $\delta_N \in S$.[11] $\square$

We will show that there is a forcing notion $\mathcal{Q}$ which is proper with respect to $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$, has the $\aleph_2$-c.c., and forces the existence of a function $H : \omega_2 \longrightarrow 2$ uniformizing F mod. clubs. This will yield a contradiction since then $\mathcal{Q}$ will preserve stationary subsets of ω_1 by Proposition 3.3.7 and Claim 3.4.2, and so the existence of such a function H will follow from an application of $\mathrm{FA}_{\aleph_2}(\{\mathcal{Q}\})$.

Remark 3.4.3. One can prove directly that $\square_{\omega_1,\omega_1}$, which we know follows $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$, implies that for every sequence $\vec{f} = (f_\alpha : \alpha \in S^2_1)$ of colourings as in the definition of $\mathsf{Unif}_{S^2_1,2}$ there is a proper forcing notion $\mathcal{Q}_{\vec{f}}$ which is proper with respect to $\mathcal{K}^*$, has the $\aleph_2$-c.c., and forces the existence of a function $H : \omega_2 \longrightarrow 2$ such that $\{\xi \in \mathrm{dom}(f_\alpha) \,:\, f_\alpha(\xi) = H(\xi)\}$ contain a club for every $\alpha \in S^2_1$. It then follows that $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ is inconsistent

[11] The choice of α being of countable cofinality is inessential. We could have taken α of cofinality ω_1, considered $E = \{M \cap \alpha \,:\, M \prec (H(\omega_2); \in, \vec{e}, \vec{\mathcal{C}})\}$, and continued the argument using the coherence of $\vec{\mathcal{C}}$.

since it implies $\mathsf{Unif}_{S^2_1,2}$, which always fails by Theorem 3.3.1. However, proceeding as we are doing here has the advantage of producing a self-contained proof, not relying on Theorem 3.3.1.

Notation 3.4.4. For each $\alpha \in \mathrm{Lim}(\omega_2)$, let us fix an enumeration $(C_{\alpha,\nu} : \nu < \omega_1)$ of $\mathcal{C}_\alpha$. Also, given a set X, we will write $\mathrm{cl}(X)$ to denote $X \cup \overline{X \cap \mathsf{Ord}}$, where $\overline{X \cap \mathsf{Ord}}$ denotes the closure of X in the order topology.[12]

Definition 3.4.5. Let us say that a family $\mathcal{N}$ of countable models is $\vec{\mathcal{C}}$*-stratified* in case the following holds.

1. $\mathcal{N} \subseteq \mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$
2. For all N_0, $N_1 \in \mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$ but $N_0 \cap \omega_2 \neq N_1 \cap \omega_2$, then
   (a) $\alpha_i := \min((N_i \cap \omega_2) \setminus N_{1-i})$ exists for each $i \in 2$,
   (b) $\mathrm{cf}(\alpha_0) = \mathrm{cf}(\alpha_1) = \omega_1$, and
   (c) there is no ordinal α above $\sup(N_0 \cap N_1 \cap \omega_2)$ such that $\alpha \in \mathrm{cl}(N_0 \cap \omega_2) \cap \mathrm{cl}(N_1 \cap \omega_2)$.
3. For all N_0, $N_1 \in \mathcal{N}$, if $\delta_{N_0} < \delta_{N_1}$, then
   $$\alpha := \max(\mathrm{cl}(N_0 \cap \omega_2) \cap \mathrm{cl}(N_1 \cap \omega_2))$$
   exists, $\alpha \in N_1$, and there is some $\nu < \delta_{N_1}$ such that
   $$N_0 \cap \alpha = \bigcup_{\gamma \in C_{\alpha,\nu}} e_\gamma``\delta_{N_0}.$$

We note that every $\vec{\mathcal{C}}$-stratified family of models is stratified.

The following simple remark will be quite useful.

[12] We stress that X need not be a set of ordinals.

Remark 3.4.6. Suppose $\mathcal{N}$ is a $\vec{\mathcal{C}}$-stratified family of models, $\bar{\alpha} < \omega_2$, N_0, $N_1 \in \mathcal{N}$, and $\alpha_0 \in N_0 \cap S$ and $\alpha_1 \in N_1 \cap S$ are such that

$$\sup(N_0 \cap \alpha_0) = \sup(N_1 \cap \alpha_1) = \bar{\alpha}$$

Then $\delta_{N_0} = \delta_{N_1}$. Hence, if $\alpha_0 \neq \alpha_1$, then $\alpha_0 = \min((N_0 \cap \omega_2) \setminus N_1)$ and $\alpha_1 = \min((N_1 \cap \omega_2) \setminus N_0)$.

Definition 3.4.7. Let us say that a $\vec{\mathcal{C}}$-stratified family $\mathcal{N}$ of models is *compatible with* F in case for all N_0, $N_1 \in \mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$, $N_0 \cap \omega_2 \neq N_1 \cap \omega_2$, and $\alpha_i = \min((N_i \cap \omega_2) \setminus N_{1-i})$ for each $i \in 2$, then $F(\alpha_0) = F(\alpha_1)$.

Definition 3.4.8. We define $\mathcal{Q}$ to be the forcing notion consisting of ordered pairs

$$q = ((\mathcal{I}^q_\alpha \,:\, \alpha \in X_q), \mathcal{N}_q)$$

with the following properties.

1. $X_q \in [S]^{<\omega}$
2. For every $\alpha \in X_q$, $\mathcal{I}^q_\alpha$ is a finite collection of pairwise disjoint intervals of the form $[\gamma_0, \gamma_1)$ with $\gamma_0 < \gamma_1 < \alpha$.
3. For all α_0, $\alpha_1 \in X_q$, if $F(\alpha_0) \neq F(\alpha_1)$, then $\min(I) \neq \min(I')$ for all $I \in \mathcal{I}^q_{\alpha_0}$ and $I' \in \mathcal{I}^q_{\alpha_1}$.
4. $\mathcal{N}_q$ is a finite family of countable elementary submodels of the structure $(H(\omega_2); \in, \vec{e}, \vec{\mathcal{C}})$ which is $\vec{\mathcal{C}}$-stratified and compatible with F.
5. The following are equivalent for every $\alpha \in X_q$ and every $\beta < \alpha$.
   (a) $\beta = \min(I)$ for some $I \in \mathcal{I}^q_\alpha$.
   (b) $\beta = \sup(N \cap \alpha)$ for some $N \in \mathcal{N}_q$ such that $\alpha \in N$.

Given conditions q_0, $q_1 \in \mathcal{Q}$, q_1 extends q_0 iff

1. $X_{q_0} \subseteq X_{q_1}$,

2. for every $\alpha \in X_{q_0}$ and every $I \in \mathcal{I}^{q_0}_\alpha$ there is some (necessarily unique) $I' \in \mathcal{I}^{q_1}_\alpha$ such that $\min(I') = \min(I)$ and $\sup(I') \geq \sup(I)$, and

3. $\mathcal{N}_{q_0} \subseteq \mathcal{N}_{q_1}$

We will use the two following density lemmas.

Lemma 3.4.9. *For every $\mathcal{Q}$-condition q and every $\alpha \in S$ there is some $q^* \in \mathcal{Q}$ extending q and such that $\alpha \in X_{q^*}$.*

Proof. If $\alpha \in q$ then we are already done. Now, suppose $\alpha \notin X_q$. Let

$$\mathcal{I} = \{\{\sup(N \cap \alpha)\} \,:\, N \in \mathcal{N}_q,\, \alpha \in N\}$$

. We are going to show that

$$q^* := ((\mathcal{I}^q_\beta \,:\, \beta \in X_q) \cup \{(\alpha, \mathcal{I})\}, \mathcal{N}_q)$$

is the extension of q in $\mathcal{Q}$ as desired. To see this, let $\bar{\alpha} = \sup(N \cap \alpha)$ for some $N \in \mathcal{N}_q$ with $\alpha \in N$. For q^* to fail to be a condition in $\mathcal{Q}$, there must be some $\alpha' \in X_q$ and some $N' \in \mathcal{N}_q$ such that $\alpha' \in N'$, $\sup(N \cap \alpha') = \bar{\alpha}$, and $F(\alpha') \neq F(\alpha)$. In fact, this is the only way for q^* to fail to be a condition since $\mathcal{I}^q_\beta \subseteq \mathcal{I}^{q^*}_\beta$. By $\vec{\mathcal{C}}$-stratification of $\mathcal{N}_q$ and Remark 3.4.6 we have that $\delta_N = \delta_{N'}$, $\alpha = \min((N \cap \omega_2) \setminus N')$, and $\alpha' = \min((N' \cap \omega_2) \setminus N)$. But then $F(\alpha) = F(\alpha')$ since $\mathcal{N}_q$ is compatible with F, which is a contradiction. □

Lemma 3.4.10. *For all $q \in \mathcal{Q}$, $\alpha \in X_q$, and $\eta < \alpha$ there is some extension $q^* \in \mathcal{Q}$ together with some $I \in \mathcal{I}^{q^*}_\alpha$ such that* $\min(I) > \eta$.

Proof. Let N be a sufficiently correct elementary submodel of $H(\omega_2)$ containing q and η, i.e., the chosen N allows us to construct q^* as

$$q^* = ((\mathcal{I}^{q^*}_\beta \,:\, \beta \in X_q), \mathcal{N}_q \cup \{N\}),$$

where

$$\mathcal{I}^{q^*}_\beta = \mathcal{I}^q_\beta \cup \{\{\sup(N \cap \beta)\}\}$$

for each $\beta \in X_q$.

It is clear that q^* is condition in $\mathcal{Q}$ stronger than q. Also, $\eta < \sup(N \cap \alpha)$ since $\eta \in N$. $\square$

Another crucial part of the proof is to show that $\mathcal{Q}$ is proper for the relevant models. The following is the relevant properness lemma.

Lemma 3.4.11. *Let θ be a cardinal such that $\mathcal{Q} \in H(\theta)$ and let M^0 and M^1 be countable elementary submodels of $H(\theta)$ of the same height such that F, $\vec{C}$, $\vec{e} \in M^0 \cap M^1$ and $\{M^0, M^1\}$ is a $\vec{\mathcal{C}}$-stratified family compatible with F. Then for every $q_0 \in \mathcal{Q} \cap M^0$ there is an extension $q^* \in \mathcal{Q}$ of q_0 such that q^* is $(M^i, \mathcal{Q})$-generic for $i = 0, 1$.*

Proof. Let

$$\mathcal{N} = \{M^0 \cap H(\omega_2), M^1 \cap H(\omega_2)\}$$

and for every $\alpha \in X_{q_0}$ let $\rho_\alpha = \sup(M^0 \cap \alpha)$.

The proof will be complete once we show that

$$q^* = ((\mathcal{I}^{q^*}_\alpha : \alpha \in X_{q_0}), \mathcal{N}_{q_0} \cup \mathcal{N})$$

is an $(M^i, \mathcal{Q})$-generic condition for $i = 0$, 1,

where $\mathcal{I}^{q^*}_\alpha = \mathcal{I}^q_\alpha \cup \{\{\rho_\alpha\}\}$.[13]

We start by noting that $\mathcal{N}_{q_0} \cup \mathcal{N}$ is $\vec{\mathcal{C}}$-stratified. This comes from the hypothesis that $\mathcal{N}$ is $\vec{\mathcal{C}}$-stratified and compatible with F, and since $\mathcal{N}_{q_0} \in M^0$ and $\delta_{M^0} = \delta_{M^1}$. The fact that $\mathcal{N}_{q_0} \cup \mathcal{N}$ is $\vec{\mathcal{C}}$-stratified is immediate. The condition q^* is indeed a condition in $\mathcal{Q}$ since for every $\alpha \in X_{q_0}$ and every $I \in \mathcal{I}^{q_0}_\alpha$, $\min(I) < \rho_\alpha$ and $\rho_\alpha \notin M^0$, and for all $\alpha < \alpha'$ in X_{q_0}, $\rho_\alpha < \alpha < \rho_{\alpha'}$. Now, since the condition q^* is defined to be an extension of q_0, it suffices to prove that q^* is $(M^i, \mathcal{Q})$-generic for each $i = 0$, 1. To do this, suppose $D \in M^i$

[13] Here, we shall use $\{\rho_\alpha\}$ in place of $[\rho_\alpha, \rho_\alpha + 1)$ for the sake of simplicity.

is an open and dense subset of $\mathcal{Q}$ and q is an extension of q^* in D. We will find a condition in $D \cap M^i$ compatible with q.

Let $\Delta = \{\delta_N : N \in \mathcal{N}_q\} \cap \delta_{M^i}$. Let us note that, by $\vec{\mathcal{C}}$-stratification of $\mathcal{N}_q$ and $M^i \in \mathcal{N}_q$,

$$R_q = \{N \cap \omega_2 \cap M^i : N \in \mathcal{N}_q, \, \delta_N \in \Delta\} \in M^i.$$

Using this, and by a reflection argument as in the proof of Lemma 3.2.2, we may find in M^i a condition $r \in D$ such that

$$q' := ((\mathcal{I}^q_\alpha \oplus \mathcal{I}^r_\alpha : \alpha \in X_q \cup X_r), \mathcal{N}_q \cup \mathcal{N}_r) \in \mathcal{Q},$$

where $\mathcal{I}^q_\alpha \oplus \mathcal{I}^r_\alpha$ is defined as follows for each $\alpha \in X_q \cup X_r$.

1. If $\alpha \in X_q \setminus X_r$, then $\mathcal{I}^q_\alpha \oplus \mathcal{I}^r_\alpha = \mathcal{I}^q_\alpha$.

2. If $\alpha \in X_r \setminus X_q$, then

$$\mathcal{I}^q_\alpha \oplus \mathcal{I}^r_\alpha = \mathcal{I}^r_\alpha \cup \{\{\sup(N \cap \alpha)\} : N \in \mathcal{N}_q, \, \alpha \in N\}$$

3. If $\alpha \in X_q \cap X_r$, then $\mathcal{I}^q_\alpha \oplus \mathcal{I}^r_\alpha$ is the unique set $\mathcal{I}$ of pairwise disjoint intervals with

$$\{\min(I) : I \in \mathcal{I}\} = \{\min(I) : I \in \mathcal{I}^q_\alpha \cup \mathcal{I}^r_\alpha\}$$

   such that for every $I \in \mathcal{I}$, if $\gamma_0 = \min(I)$, then

   (a) $\sup(I) = \sup(I_0)$ in case $I_0 \in \mathcal{I}^q_\alpha$, $\min(I_0) = \gamma_0$, and there is no $J \in \mathcal{I}^r_\alpha$ such that $\min(J) = \gamma_0$;

   (b) $\sup(I) = \sup(I_1)$ in case $I_1 \in \mathcal{I}^r_\alpha$, $\min(I_1) = \gamma_0$, and there is no $J \in \mathcal{I}^q_\alpha$ such that $\min(J) = \gamma_0$;

   (c) $\sup(I) = \max\{\sup(I_0), \sup(I_1)\}$ in case $I_0 \in \mathcal{I}^q_\alpha$, $I_1 \in \mathcal{I}^r_\alpha$, $\min(I_0) = \gamma_0$, and $\min(I_1) = \gamma_0$.

More specifically, we find $r \in D \cap M^i$ with the following properties.

1. For all $\alpha \in X_r$, $I \in \mathcal{I}^r_\alpha$, $\alpha' \in X_q$, and $I' \in \mathcal{I}^q_{\alpha'}$, if $\min(I) = \min(I')$, then $F(\alpha) = F(\alpha')$.

2. For every $N \in \mathcal{N}_r$ such that $\delta_N \in \Delta$ there is some $X \in R_q$ such that X is an initial segment of $N \cap \omega_2$. Moreover, X is a proper initial segment of $N \cap \omega_2$ if and only if there if there is some $N' \in \mathcal{N}_q$ such that X is a proper initial segment of $N' \cap \omega_2$, in which case, for every such N', if $\alpha_0 = \min((N \cap \omega_2) \setminus X)$ and $\alpha_1 = \min((N' \cap \omega_2) \setminus X)$, then $F(\alpha_0) = F(\alpha_1)$.

We can indeed find such an $r \in M^i$, by correctness of M^i, since the existence of an r with the above properties is a true fact, as witnessed by q, which can be expressed by a sentence with parameters in M^i. And given $r \in M^i$ as above, the amalgamation q' of r and q described earlier is a condition in $\mathcal{Q}$. This finishes the proof of the lemma since q' extends both q and r. □

The following is an immediate corollary from Lemma 3.4.11.

Corollary 3.4.12. *$\mathcal{Q}$ is proper with respect to $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$.*

Lemma 3.4.13. *$\mathcal{Q}$ has the $\aleph_2$-c.c.*

Proof. Suppose, towards a contradiction, that $(q_i\,;\, i < \lambda)$, for some cardinal $\lambda \geq \omega_2$, is a one-to-one enumeration of a maximal antichain A of $\mathcal{Q}$. Let θ be a large enough cardinal. For every $i < \omega_2$ let M_i be a countable elementary submodel of $H(\theta)$ belonging to $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ and such that p_i, F, $\vec{\mathcal{C}}$, $\vec{e}$, $A \in M_i$.

Let P be an elementary submodel of some higher $H(\chi)$ such that $|P| = \aleph_1$ and $\vec{\mathcal{C}}$, $((q_i, M_i) \,:\, i < \lambda) \in P$. Since all q_i are distinct and $\lambda \geq \omega_2$, we may find i_0 such that $q_{i_0} \notin P$. Now, working in P and since $M_{i_0} \cap P \in P$ as $M_{i_0} \in \mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$, we may find $i_1 \in P \cap \lambda$ such that $\delta_{M_{i_0}} = \delta_{M_{i_1}}$ and $\{M_{i_0}, M_{i_1}\}$ is $\vec{\mathcal{C}}$-stratified. By Lemma 3.4.11 there is a condition $q^* \in \mathcal{Q}$ extending q_{i_0} and such that q^* is $(M_{i_1}, \mathcal{Q})$-generic. But now, since $A \in M_{i_1}$ is a maximal antichain of $\mathcal{Q}$, we can find a common extension q' of q^* and some $q_{i_2} \in A \cap M_{i_1}$, which is a contradiction since $q_{i_2} \neq q_{i_0}$ yet q' extends both q_{i_0} and q_{i_2}. $\square$

Let now G be a $\mathcal{Q}$-generic filter. Given any $\alpha \in S$, let

$$D^G_\alpha = \{\min(I) : I \in \mathcal{I}^q_\alpha,\, q \in G,\, \alpha \in X_q\}$$

By Lemma 3.4.9, D^G_α is an unbounded subset of α.

Lemma 3.4.14. *For every $\alpha \in S$, D^G_α is closed in α.*

Proof. Let $\delta < \alpha$ be a limit ordinal forced by some $q \in \mathcal{Q}$ with $\alpha \in X_q$ to be a limit point of $D^{\dot{G}}_\alpha$ and suppose, towards a contradiction, that $\delta \neq \min(I)$ for any $I \in \mathcal{I}^q_\alpha$. By the choice of q we may assume that there is some $I \in \mathcal{I}^q_\alpha$ such that $\min(I) < \delta$. Letting I_0 be the unique such I with $\min(I)$ maximal within $\mathcal{I}^q_\alpha$ we may now extend q to a condition q' such that $[\min(I_0), \delta + 1) \in \mathcal{I}^{q'}_\alpha$. But q' forces that $D^{\dot{G}}_\alpha \cap \delta \subseteq \min(I_0) + 1 < \delta$, which contradicts the assumption that q forced $D^{\dot{G}}_\alpha$ to be cofinal in δ. $\square$

It follows from Lemmas 3.4.9, 3.4.10, and 3.4.14 together that if we aim to define $H : \omega_2 \longrightarrow 2$ by letting $H(\eta) = F(\alpha)$ for any $\alpha \in S$ such that $\eta \in D^G_\alpha$ (and $H(\eta) = 0$ if there is no α as above), then H is a well-defined function with uniformizes F mod. clubs, as witnessed by D^G_α for $\alpha \in S$. This concludes the proof of Theorem 3.4.1.

There are other results in the literature dealing with failures of forcing axioms at $\aleph_2$ or above. In this respect we single out the following theorem of Shelah [27], extended by the main result in this section that there is no naive higher analogue of MM, that is, For every regular cardinal $\kappa \geq \omega_2, \mathrm{FA}_\kappa(\Gamma_\kappa)$ fails where Γ_k is the class of forcing notions preserving stationary sets of μ for every uncountable regular $\mu \leq \kappa$.[14]

Theorem 3.4.15. *(Shelah) Given any regular cardinal $\lambda > \omega_1$, $\mathrm{FA}_\lambda(\mathcal{K}_\lambda)$ is false, where $\mathcal{K}_\lambda$ is the class of forcing notions preserving all stationary subsets of μ for every uncountable regular cardinal $\mu \leq \lambda$.*

[14]Here, $\mathsf{MM}_{\aleph_2}(\aleph_2\text{−c.c.})$ is $\mathrm{FA}_{\aleph_2}(\Gamma_{\aleph_1})$.

Remark 3.4.16. We point out that the inconsistency proof of the forcing axiom $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ we have given shows the impossibility of having $\mathrm{FA}_{\aleph_2}(\Gamma)$ for the class Γ of posets which have the $\aleph_{1.5}$-c.c. with respect to families of models which are simultaneously $\vec{\mathcal{C}}$-stratified, for a fixed $\square_{\omega_1,\omega_1}$-sequence $\vec{\mathcal{C}}$, and F-compatible for arbitrarily fixed choices of F. On the other hand, the methods of [2] allow us to build models of the forcing axiom $\mathrm{FA}_{\aleph_2}(\Gamma_0)$, where Γ_0 is the class of partial orders with the $\aleph_{1.5}$-c.c. with respect to families which are $\vec{\mathcal{C}}$-stratified, for a fixed $\square_{\omega_1,\omega_1}$-sequence $\vec{\mathcal{C}}$.

Chapter 4

More on the consequences of $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) and another proof of inconsistency of $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$

In this chapter we shall present additional implications of $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified). These include a negation of the weak Chang's conjecture and the existence of an LCS partial order on $\omega \times \omega_2$. Furthermore, we will also present another proof of the failure of the forcing axiom $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ using the other implication of $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified), the negation of the weak Chang's conjecture.

4.1 $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\neg\,$wCC.

The weak Chang's conjecture, denoted wCC, is the statement that there is no function from ω_1 into ω_1 bounding all canonical functions (modulo club). In other words, that for every function f from ω_1 to ω_1 there is some $\alpha < \omega_2$ such that $\{\nu < \omega_1 : f(\nu) < g_\alpha(\nu)\}$ is stationary, where g_α is the *canonical function* for α defined as follows.

Definition 4.1.1. Let $\alpha < \omega_2$ be a nonzero ordinal and $\pi : \omega_1 \to \alpha$ a surjection. The function $g_\alpha : \omega_1 \to \omega_1$ defined by letting $g_\alpha(\nu) = \mathrm{ot}(\pi``\nu)$ is called a *canonical function for*

α.

A canonical function for an ordinal α is clearly increasing. The degree of uniqueness of canonical functions for a given α is given by the following obvious observation, which in particular implies that g_α is uniquely determined modulo clubs.

Fact 4.1.2. *Given $\alpha < \omega_2$ and given surjections $\pi_0, \pi_1 : \omega_1 \longrightarrow \alpha$ there is a club of $\nu < \omega_1$ such that $\pi_0``\nu = \pi_1``\nu$.*

Another standard fact about canonical functions for α is that they represents the ordinal α in every generic ultrapower of V obtained by forcing with $\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1}$. In other words, if g is a canonical function for α, then $\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1}$ forces that, letting $M = ((^{\omega_1^V} V) \cap V)/\dot{G}$ be the generic ultrapower of V obtained from $\dot{G}$, the set of M-ordinals below the class $[g]_{\dot{G}}$ of g in M is well-ordered in order type α.

A natural weakening of CB is *weak Chang's Conjecture*, wCC [9]. Martin's Maximum implies the saturation of NS_{ω_1} [11], and hence also CB. On the other hand, it is not difficult to see that not even wCC follows from PFA (s. e.g. [2] for strong forms of this non-implication).

Notation 4.1.3. *Club-bounding by canonical functions*, CB, is the statement that every function $f : \omega_1 \longrightarrow \omega_1$ is bounded on a club by the canonical function of some nonzero $\alpha < \omega_2$. CB is a weakening of NS_{ω_1} being saturated.

The goal of this section is to prove the following theorem.

Theorem 4.1.4. *$\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\neg\,$wCC.*

As usual we fix a sequence $\vec{e} = (e_\alpha \,:\, 0 < \alpha < \omega_2)$, where $e_\alpha : \omega_1 \longrightarrow \alpha$ is a surjection for each α.

We consider the following forcing notion $\mathcal{R}$. A condition in $\mathcal{R}$ is a triple $p = (f_p, (h^p_\alpha : \alpha \in X_p), \mathcal{N}_p)$, where:

1. $f_p \subseteq \omega_1 \times \omega_1$ is a finite function.

2. $X_p \in [\omega_2 \setminus \{0\}]^{<\omega}$

3. For each $\alpha \in X_p$,

   (a) $h^p_\alpha \subseteq \omega_1 \times \omega_1$ is a finite function which can be extended to a continuous strictly increasing function $h : \omega_1 \longrightarrow \omega_1$, and

   (b) for each $\nu \in \mathrm{dom}(h^p_\alpha)$, $\nu \in \mathrm{dom}(f_p)$ and $\mathrm{ot}(e_\alpha``h^p_\alpha(\nu)) < f_p(h^p_\alpha(\nu))$.

4. $\mathcal{N}_p$ is a finite stratified family of countable elementary submodels of $(H(\omega_2); \in, \vec{e}, \vec{f})$.

5. The following holds for each $N \in \mathcal{N}_p$.

   (a) $f_p \restriction \delta_N \subseteq N$;

   (b) $\delta_N \in \mathrm{dom}(f_p)$ and $f_p(\delta_N) \geq \mathrm{ot}(N \cap \omega_2)$;

   (c) for every $\alpha \in X_p \cap N$,

      i. $h^p_\alpha \restriction \delta_N \subseteq N$,

      ii. $\delta_N \in \mathrm{dom}(h^p_\alpha)$, and

      iii. $h^p_\alpha(\delta_N) = \delta_N$.

Given $\mathcal{R}$-conditions p_0 and p_1, p_1 extends p_0 iff

1. $f_{p_0} \subseteq f_{p_1}$,

2. $X_{p_0} \subseteq X_{p_1}$, and

3. for every $\alpha \in X_{p_0}$, $h^{p_0}_\alpha \subseteq h^{p_1}_\alpha$. .

The following density lemmas are easy.

Lemma 4.1.5. *For every $p \in \mathcal{R}$ and every nonzero $\beta < \omega_2$ there is a $\mathcal{R}$-condition p^* extending p and such that $\beta \in X_{p^*}$.*

Proof. We may of course assume that $\beta \notin X_p$. It then suffices to set

$$p^* = (f_p, (h^p_\alpha \,:\, \alpha \in X_p) \cup \{(\beta, \{(\delta_N, \delta_N) \,:\, N \in \mathcal{N}_p,\, \beta \in N\})\}, \mathcal{N}_p)$$

To see that this is a condition in $\mathcal{R}$ it suffices to note that if $N \in \mathcal{N}_p$ is such that $\beta \in N$, then $\delta_N \in \mathrm{dom}(f_p)$ and

$$\mathrm{ot}(e_\beta``\delta_N) = \mathrm{ot}(N \cap \beta) < \mathrm{ot}(N \cap \omega_2) \leq f_p(\delta_N),$$

and that, since $\mathcal{N}_p$ is stratified, $\mathrm{ot}(N \cap \omega_2) < \delta_{N'}$ for every $N' \in \mathcal{N}_p$ such that $\delta_N < \delta_{N'}$. $\square$

Lemmas 4.1.6, 4.1.7 and 4.1.8 are straightforward.

Lemma 4.1.6. *For every $p \in \mathcal{R}$, $\alpha \in X_p$ and $\nu < \omega_1$ there is some $p^* \in \mathcal{R}$ extending p and such that $\nu \in \mathrm{dom}(h_\alpha^{p^*})$.*

Lemma 4.1.7. *For every $p \in \mathcal{R}$, $\alpha \in X_p$, every nonzero limit ordinal $\delta \in \mathrm{dom}(h_\alpha^p)$, and every $\eta < h_\alpha^p(\delta)$ there is a condition $p^* \in \mathcal{R}$ extending p together with some $\nu \in \mathrm{dom}(h_\alpha^{p^*}) \cap \delta$ such that $h_\alpha^{p^*}(\nu) > \eta$.*

Lemma 4.1.8. *For every $p \in \mathcal{R}$ and every $\nu < \omega_1$ there is a condition $p^* \in \mathcal{R}$ extending p and such that $\nu \in \mathrm{dom}(f_{p^*})$.*

It follows from Lemmas 4.1.5–4.1.8 together that if G is $\mathcal{R}$-generic and we set

$$f^G = \bigcup \{f_p \,:\, p \in G\}$$

and

$$C_\alpha^G = \bigcup \{\mathrm{range}(h_\alpha^p) \,:\, p \in G,\, \alpha \in X_p\}$$

for each nonzero $\alpha < \omega_2$, then $f^G : \omega_1^V \longrightarrow \omega_1^V$ is a function, each C_α^G is a club of ω_1^V, and $\mathrm{ot}(e_\alpha``\nu) < f^G(\nu)$ for each α and $\nu \in C_\alpha^G$. Hence, if we can show that $\mathcal{R}$ has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models, an application of $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) to $\mathcal{R}$ will show that $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\neg\,\mathrm{wCC}$.

Lemma 4.1.9. *$\mathcal{R}$ has the $\aleph_{1.5}$-c.c. with respect to finite stratified families of models.*

Proof. Let θ be a large enough cardinal, let $\mathcal{N}^*$ be a finite stratified family of countable elementary submodels of $H(\theta)$ containing $\vec{e}$ and $\vec{f}$, and let $p_0 \in \mathcal{R} \cap N_0^*$, where N_0^* is of minimum height within $\mathcal{N}^*$. We will prove that there is a condition $p^* \in \mathcal{R}$ stronger than p_0 such that p^* is $(N^*, \mathcal{R})$-generic for each $N^* \in \mathcal{N}^*$.

Let $\mathcal{N} = \{N^* \cap H(\omega_2) : N^* \in \mathcal{N}^*\}$ and for every $\delta \in \{\delta_N : N \in \mathcal{N}\}$ let

$$\mu(\delta) = \max\{\mathrm{ot}(N \cap \omega_2) : N \in \mathcal{N},\, \delta_N = \delta\}$$

Let

$$p^* = (f_{p_0} \cup \{(\delta_N\, \mu(\delta_N)) : N \in \mathcal{N}\}, (h^{p^*}_\alpha : \alpha \in X_{p_0}), \mathcal{N}_{p_0} \cup \mathcal{N}),$$

where

$$h^{p^*}_\alpha = h^{p_0}_\alpha \cup \{(\delta_N, \delta_N) : N \in \mathcal{N},\, \alpha \in N\}$$

for each $\alpha \in X_{p_0}$. It is easy to check that p^* is a condition in $\mathcal{R}$, and it of course extends p_0 by construction. Hence, it will be enough to show that p^* is $(N^*, \mathcal{R})$-generic for every $N^* \in \mathcal{N}^*$. Let $D \in N^*$ be a dense and open subset of $\mathcal{R}$ and let p be an extension of p^* in D. We will show that there is a condition in $D \cap N^*$ compatible with p.

As in the proof of Lemma 3.2.2, we may assume that $N^* = \bigcup_{\nu < \delta_{N^*}} N^*_\nu$, where $(N^*_\nu)_{\nu < \delta_{N^*}}$ is a $\subseteq$-continuous $\in$-chain of models. By moving to a suitable $N^*_{\nu_0}$ and arguing there as in the proof of Lemma 3.2.2 using the stratification of $\mathcal{N}_p$, we may find a condition $r \in D \cap N^*_{\nu_0}$ such that

1. for every $\alpha \in X_p \cap X_r$, $h^p_\alpha \cup h^r_\alpha$ can be extended to a strictly increasing and continuous function $h : \omega_1 \longrightarrow \omega_1$,

2. for every $\alpha \in X_r \cap X_q$ and every $N \in \mathcal{N}_p$ such that $\delta_N < \delta_{N^*}$, $\alpha \notin N$, and

3. $\mathcal{N}_p \cup \mathcal{N}_r$ is stratified.

Let now

$$p' = (f_r \cup (f_p \restriction (\omega_1 \setminus \delta_{N^*})), (h^{p'}_\alpha : \alpha \in X_p \cup X_r), \mathcal{N}_p \cup \mathcal{N}_r),$$

where

1. for every $\alpha \in X_p \cap X_r$, $h^{p'}_\alpha = h^p_\alpha \cup h^r_\alpha$,

2. for every $\alpha \in X_p \setminus X_r$, $h_\alpha^{p'} = h_\alpha^p$, and

3. for every $\alpha \in X_r \setminus X_p$, $h_\alpha^{p'} = h_\alpha^r \cup \{(\delta_N, \delta_N) : N \in \mathcal{N}_p, \alpha \in N\}$.

Then p' is a condition in $\mathcal{R}$, which finishes the proof of the lemma since p' is of course stronger than both p and r. $\square$

The above lemma concludes the proof of Theorem 4.1.4.

4.2 Another proof of the inconsistency of $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.)

In this section of the thesis we will give another proof of Theorem 3.4.1 in Section 3.4. Our argument is essentially due to Shelah showing that in some circumstances stationary forcing argument cannot be iterated without collapsing ω_1.

Let us assume that $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.) holds. Hence, $\mathsf{MA}_{1.5}$(stratified) holds as well and so, by Theorem 4.1.4, there is a function $f : \omega_1 \longrightarrow \omega_1$ such that $\{\nu < \omega_1 : g(\nu) < f(\nu)\}$ contains a club for every nonzero $\alpha < \omega_2$ and every canonical function g for α. We will build a sequence $(f_n)_{n<\omega}$ of functions from ω_1 to ω_1, together with clubs C_n of ω_1, such that for every n and every $\nu \in C_n$, $f_{n+1}(\nu) < f_n(\nu)$. This of course will yield a contradiction since then, if $\nu \in \bigcap_n C_n$, then $f_{n+1}(\nu) < f_n(\nu)$ for all n, which is impossible.

We will make sure that the construction can keep going by arranging, for every $n < \omega$ and every nonzero $\alpha < \omega_2$, that f_n dominates every canonical function for α on a club. We start our construction by letting $f_0 = f$.

Given $n < \omega$ and assuming f_n has been constructed, we will find f_{n+1} by an application of $\mathsf{MM}_{\aleph_2}(\aleph_2$-c.c.) to the following slight variant $\mathcal{R}_{f_n}$ of the poset $\mathcal{R}$ in the proof of Theorem 4.1.4.

Let $\mathcal{K}^{f_n}$ be the collection of countable $N \preccurlyeq (H(\omega_2); \in, \vec{e}, \vec{f}, f_n)$ such that $\mathrm{ot}(N \cap \omega_2) < f_n(\delta_N)$.

A condition in $\mathcal{R}_{f_n}$ is a tuple $p = (f_p, d_p, (h_\alpha^p : \alpha \in X_p), \mathcal{N}_p)$ with the following properties.

1. $f_p \subseteq \omega_1 \times \omega_1$ is a finite function.

2. $d_p \subseteq \omega_1 \times \omega_1$ is a finite function which can be extended to a continuous strictly increasing function $d : \omega_1 \longrightarrow \omega_1$.

3. For every $\nu \in \mathrm{dom}(d_p)$, $f_p(d_p(\nu)) < f_n(d_p(\nu))$.

4. $X_p \in [\omega_2 \setminus \{0\}]^{<\omega}$

5. For each $\alpha \in X_p$,

   (a) $h^p_\alpha \subseteq \omega_1 \times \omega_1$ is a finite function which can be extended to a continuous strictly increasing function $h : \omega_1 \longrightarrow \omega_1$, and

   (b) for each $\nu \in \mathrm{dom}(h^p_\alpha)$, $\nu \in \mathrm{dom}(f_p)$ and $\mathrm{ot}(e_\alpha``h^p_\alpha(\nu)) < f_p(h^p_\alpha(\nu))$.

6. $\mathcal{N}_p$ is a finite stratified family of members of $\mathcal{K}^{f_n}$.

7. The following holds for each $N \in \mathcal{N}_p$.

   (a) $f_p \restriction \delta_N \subseteq N$;

   (b) $\delta_N \in \mathrm{dom}(f_p)$ and $f_p(\delta_N) \geq \mathrm{ot}(N \cap \omega_2)$;

   (c) $d_p \restriction \delta_N \subseteq N$, $\delta_N \in \mathrm{dom}(d_p)$, and $d_p(\delta_N) = \delta_N$.

   (d) for every $\alpha \in X_p \cap N$,

      i. $h^p_\alpha \restriction \delta_N \subseteq N$,

      ii. $\delta_N \in \mathrm{dom}(h^p_\alpha)$, and

      iii. $h^p_\alpha(\delta_N) = \delta_N$.

Given $\mathcal{R}_{f_n}$-conditions p_0 and p_1, p_1 extends p_0 iff

1. $f_{p_0} \subseteq f_{p_1}$,

2. $d_{p_0} \subseteq d_{p_1}$,

3. $X_{p_0} \subseteq X_{p_1}$, and

4. for every $\alpha \in X_{p_0}$, $h^{p_0}_\alpha \subseteq h^{p_1}_\alpha$.

We now have the following density lemmas. Lemma 4.2.1 is proved by the same argument as in the proof of Lemma 4.1.5 using the fact that all models of $\mathcal{N}_p$ are in $\mathcal{K}^{f_n}$, and Lemmas 4.2.2–4.2.5 are straightforward.

Lemma 4.2.1. *For every $p \in \mathcal{R}_{f_n}$ and every nonzero $\beta < \omega_2$ there is a $\mathcal{R}_{f_n}$-condition p^* extending p and such that $\beta \in X_{p^*}$.*

Lemma 4.2.2. *For every $p \in \mathcal{R}_{f_n}$, $\alpha \in X_p$ and $\nu < \omega_1$ there is some $p^* \in \mathcal{R}_{f_n}$ extending p and such that $\nu \in \mathrm{dom}(h_\alpha^{p^*})$.*

Lemma 4.2.3. *For every $p \in \mathcal{R}_{f_n}$, $\alpha \in X_p$, every nonzero limit ordinal $\delta \in \mathrm{dom}(h_\alpha^p)$, and every $\eta < h_\alpha^p(\delta)$ there is a condition $p^* \in \mathcal{R}_{f_n}$ extending p together with some $\nu \in \mathrm{dom}(h_\alpha^{p^*}) \cap \delta$ such that $h_\alpha^{p^*}(\nu) > \eta$.*

Lemma 4.2.4. *For every $p \in \mathcal{R}_{f_n}$ and every $\nu < \omega_1$ there is a condition $p^* \in \mathcal{R}_{f_n}$ extending p and such that $\nu \in \mathrm{dom}(f_{p^*})$.*

Lemma 4.2.5. *For every $p \in \mathcal{R}_{f_n}$ and every $\nu < \omega_1$ there is a condition $p^* \in \mathcal{R}_{f_n}$ extending p and such that $\nu \in \mathrm{dom}(d_{p^*})$.*

It follows from the above density lemmas that if G is $\mathcal{R}_{f_n}$-generic and we let

$$D^G = \bigcup\{\mathrm{range}(d_p) \,:\, p \in G\},$$

$$f^G = \bigcup\{f_p \,:\, p \in G\},$$

and

$$C_\alpha^G = \bigcup\{\mathrm{range}(h_\alpha^p) \,:\, p \in G,\, \alpha \in X_p\}$$

for each nonzero $\alpha < \omega_2$, then D^G is a club of ω_1^V, $f^G : \omega_1^V \longrightarrow \omega_1^V$ is a function, each C_α^G is a club of ω_1^V, $\mathrm{ot}(e_\alpha``\nu) < f^G(\nu)$ for each α and $\nu \in C_\alpha^G$, and $f^G(\nu) < f_n(\nu)$ for each $\nu \in D^G$. It follows that an application of $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ to $\mathcal{R}_{f_n}$ will provide us with f_{n+1}. Hence, we just need to prove that $\mathcal{R}_{f_n}$ preserves stationary subsets of ω_1 and has the $\aleph_2$-c.c. This we will prove by means of the following version of Lemma 3.4.11 in Section 3.4.

Lemma 4.2.6. *Let θ be a cardinal such that $\mathcal{R}_{f_n} \in H(\theta)$ and let M^0, $M^1 \prec H(\theta)$ be countable models of the same height and such that $\mathcal{R}_{f_n} \in M^0 \cap M^1$ and $M^0 \cap H(\omega_2)$, $M^1 \cap H(\omega_2) \in \mathcal{K}^{f_n}$. Then for every $p_0 \in \mathcal{R}_{f_n} \cap M^0$ there is an extension $p^* \in \mathcal{Q}$ of p_0 such that q^* is $(M^i, \mathcal{R}_{f_n})$-generic for $i = 0$, 1.*

Proof. Let $\mu = \max\{\mathrm{ot}(M^0 \cap \omega_2), \mathrm{ot}(M^1 \cap \omega_2)\}$ and

$$p^* = (f_{p^*}, d_{p^*}, (h_\alpha^{p^*} : \alpha \in X_{p_0}), \mathcal{N}_{p_0} \cup \{M^0 \cap H(\omega_2), M^1 \cap H(\omega_2)\}),$$

where

1. $f_{p^*} = f_{p_0} \cup \{(\delta_{M^0}, \mu)\}$,
2. $d_{p^*} = d_{p_0} \cup \{(\delta_{M^0}, \delta_{M^0})\}$, and
3. $h_\alpha^{p^*} = h_\alpha^{p_0} \cup \{(\delta_{M^0}, \delta_{M^0})\}$ for each $\alpha \in X_{p_0}$.

Using the fact that $M^0 \cap H(\omega_2)$, $M^1 \cap H(\omega_2) \in \mathcal{K}^{f_n}$, it is immediate to check that $p^* \in \mathcal{R}_{f_n}$, and it of course extends p_0. It will thus suffice to show that p^* is $(M^i, \mathcal{R}_{f_n})$-generic for $i = 0$, 1. For this, let $D \in M^i$ be a dense and open subset of $\mathcal{R}_{f_n}$ and let $p \in D$ be an extension of p^*. We will show that there is a condition $r \in D \cap M^i$ compatible with p.

We can find r by arguing, in M^i, in the same way as in the reflection argument in the proof of Lemma 4.1.9. More specifically, and exactly as in that proof, we may assume that $M^i = \bigcup_{\nu < \delta_{M^i}} M^i_\nu$, where $(M^i_\nu)_{\nu < \delta_{M^i}}$ is a $\subseteq$-continuous $\in$-chain of models. Then, by moving to a suitable $M^i_{\nu_0}$ and arguing there as in the proof of Lemma 3.2.2 using the stratification of $\mathcal{N}_p$, we may find a condition $r \in D \cap M^i_{\nu_0}$ such that

1. for every $\alpha \in X_p \cap X_r$, $h_\alpha^p \cup h_\alpha^r$ can be extended to a strictly increasing and continuous function $h : \omega_1 \longrightarrow \omega_1$,
2. for every $\alpha \in X_r \cap X_q$ and every $N \in \mathcal{N}_p$ such that $\delta_N < \delta_{M^i}$, $\alpha \notin N$, and
3. $\mathcal{N}_p \cup \mathcal{N}_r$ is stratified.

Let us define $h^{p'}_\alpha$, for $\alpha \in X_p \cup X_r$, as follows:

1. for every $\alpha \in X_p \cap X_r$, $h^{p'}_\alpha = h^p_\alpha \cup h^r_\alpha$;

2. for every $\alpha \in X_p \setminus X_r$, $h^{p'}_\alpha = h^p_\alpha$;

3. for every $\alpha \in X_r \setminus X_p$, $h^{p'}_\alpha = h^r_\alpha \cup \{(\delta_N, \delta_N) : N \in \mathcal{N}_p, \alpha \in N\}$.

We then have that

$$p' = (f_r \cup (f_p \restriction (\omega_1 \setminus \delta_{M^i})), d_p \cup d_r, (h^{p'}_\alpha : \alpha \in X_p \cup X_r), \mathcal{N}_p \cup \mathcal{N}_r)$$

is a common extension in $\mathcal{R}_{f_n}$ of p and r, which finishes the proof of the lemma. □

Lemma 4.2.7. *$\mathcal{K}^{f_n}$ is projective stationary.*

Proof. Given a cardinal $\theta \geq \omega_2$, a function $F : [\omega_2]^{<\omega} \longrightarrow \omega_2$, and a stationary set $S \subseteq \omega_1$, it is enough to show that there is a countable $X \subseteq \omega_2$ closed under F such that $\delta := X \cap \omega_1 \in S$ and such that $f_n(\delta) > \mathrm{ot}(X)$.

In order to find such an X we first pick $\alpha \in S^2_0$ above ω_1 such that $F``[\alpha]^{<\omega} \subseteq \alpha$. We then let $N$ be a countable elementary submodel of some larger $H(\chi)$ containing $F$, $\alpha$, $f_n$, and $\vec{f}$ and such that $\delta_N \in S$, and let $X = N \cap \alpha$. Then $F``[X]^{<\omega} \subseteq X$, $X \cap \omega_1 = \delta_N \in S$, and $f_n(\delta_N) \geq \mathrm{ot}(N \cap \omega_2) > \mathrm{ot}(N \cap \alpha) = \mathrm{ot}(X)$. □

We now have the following corollary from Lemmas 4.2.6 and 4.2.7.

Corollary 4.2.8. *$\mathcal{R}_{f_n}$ is proper with respect to $\mathcal{K}^{f_n}$ and therefore it preserves stationary subsets of ω_1.*

We also have the following corollary from Lemma 4.2.6.

Lemma 4.2.9. *$\mathcal{R}_{f_n}$ has the $\aleph_2$-c.c.*

Proof. This is similar to the proof of Lemma 3.4.13. Suppose $(p_i\,;\, i < \lambda)$, for some cardinal $\lambda \geq \omega_2$, is a one-to-one enumeration of a maximal antichain A of $\mathcal{R}_{f_n}$. Let θ be a large enough cardinal and for every $i < \omega_2$ let M_i be a countable elementary submodel of $H(\theta)$ such that p_i, $A \in M_i$ and such that $M_i \cap H(\omega_2) \in \mathcal{K}^{f_n}$.

Let P be an elementary submodel of some higher $H(\chi)$ such that $|P| = \aleph_1$ and $((p_i, M_i) : i < \lambda) \in P$. We may then find i_0 such that $p_{i_0} \notin R_{f_n}$. Working in P, we may find $i_1 \in P \cap \lambda$ such that $\delta_{M_{i_0}} = \delta_{M_{i_1}}$. By Lemma 4.2.6 there is a condition $p^* \in \mathcal{Q}$ extending p_{i_0} and such that p^* is $(M_{i_1}, \mathcal{R}_{f_n})$-generic. Since $A \in M_{i_1}$ is a maximal antichain of $\mathcal{R}_{f_n}$, we can find a common extension p' of p^* and some $p_{i_2} \in A \cap M_{i_1}$, which is a contradiction since $p_{i_2} \neq p_{i_0}$ yet p' extends both p_{i_0} and p_{i_2}. $\square$

Lemmas 4.2.8 and 4.2.9 complete our second proof of Theorem 3.4.1.

4.3 $\mathsf{MA}^{1.5}_{\aleph_2}$ (stratified) and LCS partial orders

A LCS space is a Hausdorff, locally compact and scattered topological space constructed using the α-Cantor-Bendixson derivative of a topological space X. The notion of cardinal sequences is relevant in the study of LCS spaces. Juhász and Weiss proved the result that an infinite cardinal sequence $(\kappa_\alpha)_{\alpha<\omega_1}$ is a sequence of an LCS space if and only if $\kappa_\beta < \kappa_\alpha^\omega$ for every $\alpha < \beta < \omega_1$ [17]. Some attempts at characterizing cardinal sequences have been made. See [16] and [20] for more on this. It is relatively consistent with ZFC that there exists a LCS space with length ω_2 and width ω, as shown in [5]. For more on LCS spaces, we refer the reader to the survey paper by Joan Bagaria [3], In this last section, we will prove that $\mathsf{MA}^{1.5}_{\aleph_2}$ (stratified) implies the existence of a LCS space with length ω_2 and width ω.

Definition 4.3.1. A partial order $\leq$ is said to be *LCS* partial order on $\omega \times \omega_2$ if

(1) $\leq$ is a subset of $\omega \times \omega_2$,

(2) $(n_0, \alpha_0) < (n_1, \alpha_1) \Rightarrow \alpha_0 < \alpha_1$,

(3) $\forall (n, \alpha), \forall \beta < \alpha, \exists$ infinitely many m such that $(m, \beta) < (n, \alpha)$ and

(4) $\forall x_0, x_1 \in \omega \times \omega_2, \exists$ finite $b(\{x_0, x_1\}) \subseteq \{x : x \leq x_0, x_1\}$ and $\forall z \leq x_0, x_1, \exists x \in b(\{x_0, x_1\})$ such that $z \leq x$.

Let $LCS(\omega, \omega_2)$ be the statement that there exists a LCS partial order on $\omega \times \omega_2$.

$LCS(\omega, \omega_2)$ implies the existence of a LCS space which is thin-tall since the height is greater than the width.

Theorem 4.3.2. *$\mathsf{MA}^{1.5}_{\aleph_2}$ (stratified) implies $LCS(\omega, \omega_2)$.*

Let $\mathbb{P}$ be a forcing notion whose conditions are the following:

$$p = (\leq_p, b_p, \mathcal{N}_p, \mathcal{A}_p)$$

where

(1) $\leq$ is a finite subset of $\omega \times \omega_2$ such that $(n_0, \alpha_0) < (n_1, \alpha_1) \Rightarrow \alpha_0 < \alpha_1$,

(2) $b_p : [\mathrm{dom}(\leq_p)]^2 \to [\mathrm{dom}(\leq_p)]^{<\omega}$ such that $\forall \{x_0, x_1\} \in [\mathrm{dom}(\leq_p)]^2$,

  (i) $b_p(\{x_0, x_1\}) \subseteq \{x \in \mathrm{dom}(\leq_p) : x \leq_p x_0, x_1\}$ and

  (ii) $\forall x_0, x_1 \in \mathrm{dom}(\leq_p), z \leq_p x_0, x_1 \Rightarrow \exists x \in b(\{x_0, x_1\})$,

(3) $\mathcal{N}_p$ finite stratified family of countable $N \preccurlyeq (H(\omega_2); \in, \vec{e})$,

(4) $\mathcal{A}_p \subseteq \mathcal{N}_p$ and

(5) $\forall N \in \mathcal{A}_p, \forall x_0 \neq x_1$ in $\mathrm{dom}(\leq_p) \cap N, b_p(\{x_o, x_1\}) \in N$.

and if $p_0, p_1 \in \mathbb{P}$, $p_1 \leq p_0$ iff

(1) $\leq_{p_0} \subseteq \leq_{p_1}$,

(2) $\forall x_0, x_1 \in \mathrm{dom}(\leq_{p_0}), b_{p_0}(\{x_o, x_1\}) = b_{p_1}(\{x_o, x_1\})$,

(3) $\mathcal{N}_{p_0} \subseteq \mathcal{N}_{p_1}$,

(4) $\mathcal{A}_{p_0} \subseteq \mathcal{A}_{p_1}$.

We will need the following density lemmas.

Lemma 4.3.3. *For every $p \in \mathbb{P}$, and every $(n, \alpha) \in \omega \times \omega_2$, there exists $p^* \leq p$ such that $(n, \alpha) \in \operatorname{dom}(\leq_{p^*})$.*

Proof. In case if $\alpha > \delta_N$ for all $N \in \mathcal{N}_p$, then we can find a sufficiently correct elementary submodel of $H(\omega_2)$ containing (n, α). We construct p^* as follows. Let

$$p^* = (\leq_{p^*}, b_{p^*}, \mathcal{N}_p \cup \{N\}, \{N\})$$

So, for every $x \neq (n, \alpha)$ in $\operatorname{dom}(\leq_{p^*})$, $b(\{x, (n, \alpha)\}) = b(\{x, (n^*, \alpha^*)\}) \cup \{(n^*, \alpha^*)\}$ for some $(n^*, \alpha^*) < (n, \alpha)$. Thus, we can easily check that p^* is a condition in $\mathbb{P}$. □

Similarly, we have the following.

Lemma 4.3.4. *For every $p \in \mathbb{P}, (n, \alpha) \in \operatorname{dom}(\leq_p), \beta \in \alpha,$ and every $n \in \omega, \exists n^* > n$, there exists $p^* \leq p, (n^*, \beta) \leq (n, \alpha)$.*

Therefore, the forcing axiom with this particular forcing notion yields an LCS partial order on $\omega \times \omega_2$.

Corollary 4.3.5. $\mathrm{FA}_{\aleph_2}(\{\mathbb{P}\})$ *implies $LCS(\omega, \omega_2)$.*

To finish proving Theorem 4.3.2, we now need to show that the forcing notion in question has the $\aleph_{1.5}$-c.c. with respect to stratified families.

Lemma 4.3.6. $\mathbb{P}$ *has the $\aleph_{1.5}$-c.c. with respect to stratified families.*

Proof. Let $\mathcal{N}^*$ be a finite stratified family of countable submodels $N^* \preccurlyeq H(\theta)$ such that θ is a large enough cardinal number,[1] $\vec{e} \in N^*$ and $N^* = \bigcup_{\nu \in \delta_{N^*}} N^*_\nu$ where $(N^*_\nu)_{\nu < \delta_{N^*}}$ is an $\in$-increasing, $\subseteq$-continuous chain such that $\vec{e} \in N^*_\nu$ and $N^*_\nu \subseteq H(\theta)$ for all ν.

Let $p = (\leq_p, b_p, \mathcal{N}_p, \mathcal{A}_p) \in N^*_0 \cap \mathbb{P}$ where N^*_0 is a model of minimal height in $\mathcal{N}^*$ and

$$p^* = (\leq_p, b_p, \mathcal{N}_p \cup \{N^* \cap H(\omega_2) : N^* \in \mathcal{N}^*\}, \mathcal{A}_p \cup \{N^* \cap H(\omega_2) : N^* \in \mathcal{N}^*\}).$$

[1]In this particular case, θ can be $(2^{\aleph_1})^+$.

Claim 4.3.7. *$p^* \in \mathbb{P}$ and $p^* \leq_{\mathbb{P}} p$.*

Now we need to show that p^* is $(N^*, \mathbb{P})$-generic for every $N^* \in \mathcal{N}^*$. Let $p' \leq_{\mathbb{P}} p^*, N^* \in \mathcal{N}^*, D \in N^*$ is a dense subset of $\mathbb{P}$. Suppose $p' \in D$. We want to find $r \in D \cap N^*$ such that there exists $q \in \mathbb{P}$ where $q \leq r, p'$. Let $\nu \in \delta_{N^*}$ such that $D \in N^*_\nu$ and for all $(n, \alpha) \in \mathrm{dom}(\leq_{p'}) \setminus N^*_\nu$ there is $\eta_\alpha \in N^*_\nu \cap \alpha$ such that for every $M \in \mathcal{A}_p$,

$$M \cup [\eta_\alpha, \sup(N^*_\nu \cap \alpha)) \cap N^*_\nu = \emptyset \text{ whenever } \delta_M < \delta_{N^*}.$$

Working in N^*_ν find $r \in D$ such that

(1) $\leq_{p'} \cap N^*_\nu \subseteq \leq_r$,

(2) $b_{p'} \restriction [N^*_\nu]^2 \subseteq b_r$,

(3) there exists $\pi : (\leq_{p'}, b_{p'}) \longrightarrow (\leq_r, b_r)$ where π is the identity on $\mathrm{dom}(\leq_{p'}) \cap N^*_\nu$,

(4) for every $(n, \alpha) \in \mathrm{dom}(\leq_{p'}) \setminus N^*_\nu, \eta_\alpha \leq \mathrm{lv}(\pi(n, \alpha)) < \alpha$[2] and

(5) $\mathcal{N}_{p'} \cup \mathcal{N}_r$ is stratified.

Let $q = (\leq, b, \mathcal{N}_{p'} \cup \mathcal{N}_r, \mathcal{A}_{p'} \cup \mathcal{A}_r)$ where

(1) $\leq$ is the transitivization of $\leq_{p'} \cup \leq_r$[3],

(2) $b = b_{p'} \cup b_r \cup \{b(\{x_0, \pi(x_1)\} : x_0, x_1 \in \mathrm{dom}(\leq_{p'}) \setminus N^*_\nu$ where

$$b(\{x_0, \pi(x_1)\}) = \bigcup \{b_{p'}(\{x_0, x\}) : x \in \mathrm{dom} \leq_{p'} \cap N^*_\nu, x ||_{\leq_{p'}} x_0, x \leq_r \pi(x_1)\} \cup$$
$$\bigcup \{b_{p'}(\{x, \pi(x_1)\}) : x \in \mathrm{dom} \leq_{p'} \cap N^*_\nu, x \leq_{p'} x_0, x ||_r \pi(x_1)\}.$$

Claim 4.3.8. $q \in \mathbb{P}$

All conditions hold trivially except clause (5). Let $x \neq y \in \mathrm{dom}(\leq_{p'}) \cup \mathrm{dom}(\leq_r)) \cap N$ where $N \in \mathcal{A}_{p'} \cup \mathcal{A}_r$. Then we have the following cases:

(1) $N \in \mathcal{A}_{p'}, x, y \in \mathrm{dom}(\leq_{p'})$;

[2] $\mathrm{lv}(n, \alpha) = \alpha$.

[3] that is, $\leq$ is the smallest LCS partial order on $\omega \times \omega_2$ such that $\leq_{p'} \cup \leq_r \subseteq \leq$

(2) $N \in \mathcal{A}_r, x, y \in \mathrm{dom}(\leq_r)$.

Both these cases are trivial;

(3) $N \in \mathcal{A}_{p'}$,

(3.1) $\delta_N < \delta_{N^*_\nu}$. Then $x, y \in \mathrm{dom}(\leq_{p'})$, so $b(\{x,y\}) = b_{p'}(\{x,y\}) \in N$;

(3.2) $\delta_{N^*_\nu} \leq \delta_N$. WLOG $y \in \mathrm{dom}(\leq_r) \backslash \mathrm{dom}(\leq_{p'})$.

(3.2.1) Suppose $x \in \mathrm{dom}(\leq_r) \backslash \mathrm{dom}(\leq_{p'})$ then $\mathrm{lv}(x), \mathrm{lv}(y) \in N, \delta_N \geq \delta_{N^*_\nu}, \omega \times (N^*_\nu \cap \min\{\mathrm{lv}(x), \mathrm{lv}(y)\}) \subseteq N$ where $b_r(\{x,y\}) \subseteq N^*_\nu \cap \min\{\mathrm{lv}(x), \mathrm{lv}(y)\}$;

(3.2.2) $x \in \mathrm{dom}(\leq_{p'}) \backslash N^*_\nu$. Then for every z such that $x ||_{\leq_{p'}} z, z \subseteq_r y \Rightarrow \mathrm{lv}(z) \in N$ since $\delta_N \geq \delta_{N^*_n}$ and $\mathrm{lv}(y) \in N^*_\nu$. Hence $b_{p'}(\{x,z\}) \in N$. Also for every $z \in \mathrm{dom}(\leq_r), b_r(\{y,z\}) \in N$ again since $\mathrm{lv}(y) \in N$ and $\delta_N \geq \delta_{N^*_\nu}, b(\{x,y\}) \in N$.

□

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