Up with Categories, Down with Sets; Out with Categories, In with Sets!

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Abstract

Practical approaches to the notions of subsets and extension sets are compared, coming from broadly set-theoretic and category-theoretic traditions of mathematics. I argue that the set-theoretic approach is the most practical for "looking down" or "in" at subsets and the category-theoretic approach is the most practical for "looking up" or "out" at extensions, and suggest some guiding principles for using these approaches without recourse to either category theory or axiomatic set theory.

1 Introduction

In mathematics we often write $A \subseteq B$ to mean that A is a subset of B, or equivalently that B is an extension (superset) of A. Often in mathematical discourse we do not get A and B simultaneously. Either we first have A, and are then considering B as an extension of A ("going up"), or we first have B and we are then considering A as a subset of it ("going down"). In this note, I discuss two approaches from mathematical practice to thinking and writing about these notions. These two approaches come broadly from the set-theoretic and category-theoretic traditions of mathematics. The set-theoretic approach will be familiar to most practising mathematicians, while the category-theoretic approach to sets is more recent and not as universally well-known. I suggest some principles drawn from these approaches which can be used in mathematical work without any knowledge of category theory at all, nor of set theory beyond the very basics. I argue that the approach from the set-theoretic tradition is more useful when going down, and the approach from the category-theoretic tradition is more useful when going up, hence the first slogan in the title. Towards the end of the note, I broaden the argument slightly and change the metaphor to replace "up" with "out" and "down" with "in", hence the second slogan.

These arguments are about the pragmatic issues which arise when doing and writing mathematics. My intention is not to say anything about the philosophical foundations of mathematics, but I hope this presentation also has some novelty for those who are familiar with that subject. I will assume some basic algebra of fields for the main running example, and the subject matter requires some familiarity with the interplay between isomorphism and equality.

It seems common to associate the set-theoretic and category-theoretic traditions with different branches of mathematics. For example, the set-theoretic tradition is often considered to be followed in analysis (at least real analysis) and much of mathematical logic, including model theory and (unsurprisingly!) axiomatic set theory. The category-theoretic tradition is more common in algebra and geometry (at least at the research level, although less so at undergraduate level). Indeed, Maddy [Mad19] writes:

> No reasonable observer would suggest that an algebraic geometer or algebraic topologist would do better to think in set-theoretic rather than category-theoretic terms. But it seems equally unreasonable to suggest that an analyst, or for that matter a set theorist, would do better to think in category-theoretic terms.

She also quotes Mac Lane, albeit writing in 1986 and so perhaps outdated, in support of this view.

However, I suggest that this view is not the whole story. Complex analysis is perhaps a good example of a branch of mathematics with a foot in each camp. The more geometric areas of complex analysis use category-theoretic methods such as sheaves, but there are also areas much closer to real analysis where set-theoretic methods are central. In my research I find myself using ideas and tools from algebra, algebraic geometry, real and complex analysis and model theory, sometimes separately and sometimes together, and I find that it is not always the traditional approach to thinking in that branch of mathematics which is the most fruitful, nor always the easiest in which to write precisely and clearly. Rather, I find that the more set-theoretic approach is better suited for dealing with subsets of some fixed set (*looking down*) and a more category-theory inspired approach is better for dealing with extensions of a fixed set (*looking up*). Moreover, I find that the flexibility to use either or both of the approaches as the situation dictates is better than using either approach rigidly.

Maddy's interest in [Mad19] is to address the desiderata of philosophical foundations for mathematics. While I am not addressing these foundational issues, one point of overlap is Maddy's desideratum of the *Essential Guidance* that an approach to mathematics provides. She writes in [Mad19]:

> . . . a foundation that would guide mathematicians toward the important structures and characterize them strictly in terms of their mathematically essential features. Such a foundation would actually be useful to mainstream mathematicians in their day-to-day work, [not remote, largely irrelevant, like set theory]; it would provide **Essential Guidance.**

I have put brackets around the comment that set theory is remote and largely irrelevant to mainstream mathematicians, because in context it refers to the particular way that mathematical structures are coded in axiomatic set theory, not to set theory as a subject, nor to the more basic set theory which is the subject of this note. (Furthermore, this is a simplified viewpoint which Maddy is presenting, not defending.) I would maintain that, irrespective of their foundational status, both the set-theoretic and category-theoretic viewpoints provide essential guidance to practising mathematicians.

While I am using the terms *set-theoretic* and *category-theoretic* traditions, really what I am describing are two approaches to sets, not to categories, which Shulman [Shu19] calls the *material set theory* and the *structural set theory* approaches. As formal theories, the material approach is characterised by placing the element relation centrally, while the structural approach centres on functions. Shulman's paper shows (following much older work from the 1970s) that the foundational theories from these two approaches are mutually interpretable, hence equiconsistent, so a choice between the approaches is really one of language, akin to a choice of computer programming language, not a difference in the underlying mathematics which can be done. Structuralism as a philosophy of mathematics is discussed in detail in [Sha97].

Through the rest of this note, I explain the material (set-theoretic) and then structural (category-inspired) approaches to subsets, and then to extensions, and compare them. I then finish with some comments on non-injective functions and some final discussion.

I would like to thank my students and two anonymous referees for useful comments on earlier drafts.

2 Down with Sets

Suppose we want to work with subfields of a given field F. (We will take fields as a running example, but the same ideas apply to many other sort of mathematical objects, algebraic and otherwise.) The easy way to work with them is with material set-theoretic conventions. We regard F as a set, equipped with the field operations of addition and multiplication, and a subfield A is a subset of F, equipped with and closed under the restrictions of the field operations. The set F is composed of elements, and a subset is determined by which of those elements it contains. The relevant principle is the Extensionality principle from Set Theory, or more specifically the Subset Extensionality principle.

Subset Extensionality Given a set X, two subsets A and B of X are equal if and only if they contain exactly the same elements of X.

To preempt any confusion that might arise, I should point out that the meaning of the word *extensionality* in the Extensionality principle is somewhat different from the meaning of extension as the counterpart to a subset: if A is a subset of B then B is an extension (or superset) of A. One can reconcile the meanings by thinking of Subset Extensionality as stating that a subset of X is determined by its *extent*, or in informal terms, how much of X it covers, while in contrast, if B is an extension of A then it *extends* A, that is, increases its extent. (The original purpose of the term *extensionality* is to contrast with an *intensional* definition.)

This material approach to subsets works well for fields and similar algebraic objects. It can be adapted for things like graphs, if we regard a graph G as two sets: the set $V(G)$ of vertices and the set $E(G)$ of edges (together with the information assigning the endpoints to each edge), where a subgraph H is then given by (compatible) subsets of both $V(G)$ and of $E(G)$. The same applies to categories: we regard a category as given by its sets of objects and of arrows (together with source and target assignments, and the composition assignment for compatible arrows), and then a subcategory is given by subsets of both. In practice, this set-theoretic convention is a useful and easy way to think about subgraphs and subcategories and to work with them.

In the subject of Axiomatic Set Theory, it is customary to consider a universe U of all possible elements, and then a set is always a "subset" of U. The inverted commas are included because usually not all "subsets" of U are considered as sets, for example U itself is usually not considered as a set, but as a proper class, to avoid inconsistencies arising from Russell's paradox. Usually but not always, the elements of U are themselves all sets, and then often U is the cumulative hierarchy, denoted by V. This approach gives us the principle of Set Extensionality, which is one of the axioms of ZF set theory:

Set Extensionality Two sets A and B are equal if and only if they contain exactly the same elements.

This principle makes sense without having an explicit universe U of all possible elements, but it does require that given any possible sets A and B and any elements $a \in A$, $b \in B$, we can make sense of the question of whether or not $a = b$, and it should have a definite answer.

In model theory, when studying fields we often adopt the following convention.

Monster Model Convention (for fields) There is a universal (monster) field U (of a fixed characteristic), and by *field* (of that characteristic) we will mean *subfield of* U.

Of course there is no claim that every field which could be considered is a subfield of U, rather that we are only considering those fields which are.

The Monster Model Convention is often applied to a complete first-order theory T, and then every model of T considered is an elementary submodel of the monster model U. However, the convention is equally applicable to other contexts such as for inductive theories, which includes the case of fields here. (An inductive theory is one axiomatised by $\forall\exists$-sentences, or equivalently one whose category of models and their embeddings is closed under unions of chains.) In many situations the Monster Model Convention is useful because it simplifies notation, and the loss of generality is often not relevant to the discussion at hand.

There are technical requirements for a monster model, that it be sufficiently universal and saturated, but there is no harm in assuming that such monster models exist. There is a brief discussion on approaches to the related foundational issues in Chapter 27 of [Kir19] and more in [HK21].

We can view the use of a universe U of all sets and their elements, as a Monster Model Convention for Sets. Historically, I imagine that this way of doing Set Theory inspired the Monster Model Convention in Model Theory. A similar convention was used in Algebraic Geometry by Weil in the 1950s, under the name *universal domain* [Wei62] (although that set-theoretic approach is no longer much used in algebraic geometry).

3 Down with Categories

An alternative viewpoint on subfields, or subsets, comes from category theory. Given a field F, we can consider other fields A and B together with field embeddings $\theta : A \hookrightarrow F$ and $\varphi : B \hookrightarrow F$. We say that the pairs (A, θ) and (B, φ) are *equivalent* if there is an isomorphism $\psi : A \to B$ such that $\varphi \circ \psi = \theta$. A subfield of F is then considered to be an equivalence class of pairs (A, θ). In practice we work with a particular representative of an equivalence class, or if we use more than one representative then we also have to be careful to fix a choice of isomorphism between the representatives.

It is easy to see that this *structural* or *function-based* notion of subfield is theoretically equivalent to the *material* set-theoretic approach. However, it is obviously clunkier and more awkward to use in practice if we are working with subfields of a fixed F. In practice, I have seen many instances of algebraists and geometers working with these conventions, but then dropping the notation for the inclusion arrows, without any accompanying text explaining that we are now to treat the field A as a subfield of F via the (now notationally absent) embedding θ. Often this remains unambiguous, but not always.

The structural approach is also very useful in some situations, for example in algebraic topology where we might want to identify homotopy-equivalent subspaces, or in geometry, where we might want the notion of subobject to capture a little more information than the element-based approach naturally does. For example, in algebraic geometry, we might want to distinguish the subvariety of $\mathbb{C}$ given by $(z-1)^2 = 0$ from that given by $z - 1 = 0$, to capture the multiplicity of the zero at $z = 1$. In complex analysis, it is important to distinguish the two paths $\theta, \gamma : [0, 1] \to \mathbb{C}$ given by $\theta(t) = e^{2\pi i t}$ and $\gamma(t) = e^{-4\pi i t}$ even though they have the same image, the unit circle, for example because $\int_\theta \frac{\mathrm{d}z}{z} = 2\pi i$ and $\int_\gamma \frac{\mathrm{d}z}{z} = -4\pi i$.

In these examples, the structural approach gives an easy way to consider subvarieties or paths as subobjects, whereas the material approach does not, precisely because the equivalence relation between representatives is not (at least not naturally) expressible in terms of elements in the image of a function. However, when we *can* use the material approach based on the Extension Principle for Subsets, it seems to be the easiest way to think about subfields, subsets, or other subobjects, and it gives rise to the easiest notation.

One lesson that I like to take away from the structural approach, however, is the tremendously useful idea that the concept of a subfield (or subset) is different from the concept of a field (or set).

Structural Principle for Subsets A subset A of a set X is given by a set A together with an embedding of A into X. The embedding is not recoverable from the abstract sets A and X.

In practice, applying this principle often means that I find myself correcting my writing and that of students and coauthors by replacing the words "set" and "field" by "subset" and "subfield" wherever the chosen embedding matters. From the material point of view, this is usually just changing a word without altering the meaning, but I find the extra precision of this Structural Principle for Subsets often helpful for clarity of thought.

If we adopt the Monster Model Convention for Fields, or Set Extensionality, and A and F are fields, then the statement that A is a subfield of F is either true or false, depending on whether every element of A is in F, which makes sense because all these elements are elements of U. (The addition and multiplication must also agree, but if we are assuming the Monster Model Convention this is automatic.)

Without these conventions, we have to be more careful. The Structural Principle for Subsets explains exactly how we should be careful. Given A and F just as fields, either we have a specific chosen field embedding $A \hookrightarrow F$ or we do not. If we do, we can use Subset Extensionality to switch to the material approach inside F. But if not, the statement that A is a subfield of F is neither true nor false, but makes no sense.

4 Up with Sets

Now we start with a field F, and want to consider an extension field K of it. What does it mean? From the material set-theoretic perspective, we should distinguish the concept of a field K with a field embedding $\theta : F \hookrightarrow K$ from the concept of a a field K such that F is a subfield of K in the sense of Section 2 above. Both notions are known as extensions. If we have the Monster Model Convention for Fields, or just Set Extensionality, then it is clear what this second notion means.

However, in practice, this strict material notion of extension is almost never how we consider abstract field extensions. For example, if F is the rational field $\mathbb{Q}$ and K is given by adjoining $\sqrt{-1}$, we usually construct K as the quotient ring $\mathbb{Q}[X]/I$ where I is the ideal $I = \langle X^2 + 1 \rangle$ as a subset of the polynomial ring $\mathbb{Q}[X]$. Then the element 1 in K is the coset $1 + I$, which is not the same element of the set-theoretic universe U as the element 1 of $\mathbb{Q}$.

An alternative construction of $\mathbb{Q}(\sqrt{-1})$ is as the set of ordered pairs of rational numbers $(a, b) \in \mathbb{Q}^2$, written as $a + ib$, with addition and multiplication given by the familiar rules. The notation $a + ib$ allows us to consider elements of $\mathbb{Q}$ as those elements of $\mathbb{Q}(\sqrt{-1})$ such that $b = 0$, but if we also regard $\mathbb{Q}(\sqrt{-1})$ as $\mathbb{Q}^2$ (let alone as $\mathbb{Q}[X]/I$) this is incompatible with the Monster Model Convention for Fields, and even with any reasonable notion of Set

Extensionality: we cannot reasonably have an element 1 equal to an ordered pair $(1, 0)$, and equal to a set $1 + I$ of polynomials within a universe U of elements of sets.

The convenient approach is to construct the extension $\mathbb{Q}(\sqrt{-1})$ of $\mathbb{Q}$, or more generally the extension K of F, by whatever means convenient, and then to regard K as a set and to redefine F to be the subset of K, so the elements of F are now considered to be the elements of K. For example, when dealing with complex numbers, we consider real numbers to be those complex numbers whose imaginary part is 0. But we defined the complex numbers originally as ordered pairs of real numbers, so we have to accept this as a change of viewpoint.

What about number fields generally, that is, finite extensions of $\mathbb{Q}$, which are the main subject matter of algebraic number theory? Although there is no universal number field, we can construct the universal algebraic extension $\mathbb{Q}^{\mathrm{alg}}$ of $\mathbb{Q}$, and regard number fields as those subfields of $\mathbb{Q}^{\mathrm{alg}}$ which are finite-dimensional over $\mathbb{Q}$.

For fields considered just as algebraic structures (that is, considered up to field isomorphism rather than equipped with a topology or norm or other structure) this is not too much of a problem in principle, because any set of extensions of a field can be *amalgamated*. That is, given a field F_0 and any set of extensions $f_i : F_0 \hookrightarrow F_i$, for $i \in I$, we can find a field K and field embeddings $g_i : F_i \hookrightarrow K$ such that for all $i, j \in I$ and all $a \in F_0$ we have $g_i(f_i(a)) = g_j(f_j(a))$. (It may be that we have some $a \in (F_i \cap F_j) \smallsetminus F_0$ such that $g_i(f_i(a)) \neq g_j(f_j(a))$.) So given F_0 and the extensions F_i, we can regard them all as subfields of some K, so the f_i become inclusion maps, provided we do not mind suspending Set Extensionality for the initial renaming of the elements. If we continue to consider new extensions, then the repeated renaming of elements rather challenges Set Extensionality.

This approach is not so much *Up with Sets* as *Up by another approach and then Down with Sets*.

However, by and large, number theorists do not regard number fields as subfields of $\mathbb{Q}^{\mathrm{alg}}$, or as subfields of $\mathbb{C}$. One reason for this is that they consider number fields not just as algebraic structures, but also as equipped with various norms: the Euclidean norm and the p-adic norms. Then they want to consider a number field F embedded in the completions in $\mathbb{R}$ or $\mathbb{C}$ and in $\mathbb{Q}_p$ for different primes p. The product formula for norms requires us to consider all these embeddings simultaneously. But these extensions of F do not amalgamate as normed fields. And even if we want to regard them just as algebraic structures, without the norms, there is another problem of having to make arbitrary choices in an amalgam. For example, there are three cube roots of 2 in $\mathbb{Q}^{\mathrm{alg}}$. In $\mathbb{R}$ there is just one, with decimal expansion $\alpha = 1.2599210\ldots$. In the 5-adic numbers there is also just one cube root of 2, say β, with 5-adic expansion $\beta = \ldots 204132203$, that is,

$$\beta = \cdots + 2 \times 5^8 + 0 \times 5^7 + 4 \times 5^6 + 1 \times 5^5 + 3 \times 5^4 + 2 \times 5^3 + 2 \times 5^2 + 0 \times 5^1 + 3 \times 5^0.$$

If we amalgamate $\mathbb{R}$ and $\mathbb{Q}_5$ over $\mathbb{Q}$, or over any number field not containing cube roots of 2, we have to decide whether α and β should be equal or not. But this choice is completely arbitrary and, if we forget that, we can be misled that it is meaningful to say that α is or is not equal to β in $\mathbb{Q}^{\mathrm{alg}}$. Koblitz [Kob84, p83] gives various fallacious proofs arising from similar such identifications.

So it is often unhelpful to think of a number field purely as a subfield of $\mathbb{Q}^{\mathrm{alg}}$ or of another algebraically closed field, and indeed in number theory or algebra, extension fields K of F should generally be considered as embeddings $\theta : F \hookrightarrow K$, not as in the material sense.

5 Up with Categories

In the structural approach, we define an extension field K of F to be a field K together with an embedding $\theta : F \hookrightarrow K$. We do not require θ to be an inclusion, and indeed the distinction between inclusions and other embeddings does not makes sense without Set Extensionality. We are often interested in an extension K only up to isomorphism over F, as with the extensions $\mathbb{Q}(\sqrt{-1})$ discussed above, in which case we are free to use any of the constructions of it as convenient. If we wish, we can subsequently identify F with its image as a subfield of K. However it is good practice to be explicit (at least to ourselves, and generally in writing as well unless it is clear) when we are doing that.

Sometimes this approach seems more complicated, especially at first, because we have to consider embeddings which are not inclusion maps, which is extra notation, writing $\theta(a)$ for an element of K rather than just a. However, the structural approach suggests that we *only* consider embeddings, not inclusion maps, which is then a simplification in situations where we might otherwise be considering both. And then, if we like, we can simply declare any (compatible) embeddings to be inclusions for the purposes of notation, but we have the freedom to do this how we like, when we like. There is a different perspective to that of material set theory. The mathematical object we are considering at a given point in a mathematical discourse is a configuration of fields and field embeddings, rather than a collection of subfields of a monster field U together with some field embeddings which may not be inclusions. However, this structural perspective is more flexible, and gives the freedom to choose notation for elements in the fields which is convenient for whatever you are doing at a given point in the discourse.

This structural approach could be described as the opposite of Set Extensionality: instead of having a universe of all possible elements of sets, we have no such universe. While each set comes with its elements, which are used via Subset Extensionality to describe its subsets, we do not have a way of comparing the elements of different sets X and Y unless we explicitly identify both these sets as subsets of a common extension Z (which may be X or Y or something else) via an inclusion map. I find it useful to capture this way of thinking via the following:

Principle of No Shared Elements It does not makes sense to say that elements of different sets are equal to each other or not.

The phrasing of this Principle is deliberately designed to exclude the possibility of identifying the concepts of set and subset, and hence to challenge the habits of someone who is accustomed to using the terminology of "set" from the material approach without thought as to whether they mean "subset". To repeat, of course we can compare elements of different subsets A and B of a set X, as subsets of X, but not as sets, because if we think of A and B just as sets rather than as subsets of X, we have exactly forgotten the means of comparing their elements.

This Principle encapsulates a more category-theoretic approach to mathematics which is common and, I think, acknowledged in (research level) algebra. This approach is perhaps not suited to all areas of mathematics, but Leinster's series of blog posts on Large Sets [Lei21] is an interesting example of this approach being used in axiomatic set theory, specifically an exposition of large cardinals.

6 Out with Categories

We can generalise from embeddings to homomorphisms which are not necessarily injective, so going *out* rather than necessarily *up*, although we have to move slightly away from the example of fields, as all field homomorphisms are embeddings.

Given a field F and a subfield C of F, we can form the F-vector space $\Omega(F/C)$ of Kähler differentials, which is generated by symbols $\{da \mid a \in F\}$ subject to the relations

$$d(a+b) = da + db, \qquad d(ab) = adb + bda, \quad \text{and} \quad dc = 0$$

for all $a, b \in F$ and all $c \in C$. This also gives us a canonical C-linear map $d : F \to \Omega(F/C)$. See [Eis95, p386] for a detailed explanation.

If we embed F in a larger field F', there is an obvious associated embedding of vector spaces $\Omega(F/C) \hookrightarrow \Omega(F'/C)$. Typically, we use the same notation da for the element either in $\Omega(F/C)$ or in $\Omega(F'/C)$. This is not very surprising, since we can regard the embedding as an inclusion of sets.

We can also consider an intermediate field $C \subseteq C' \subseteq F$, and then we get a quotient map $\Omega(F/C) \twoheadrightarrow \Omega(F/C')$. In Commutative Algebra, the same notation da is often used for an element of $\Omega(F/C)$ and for its image in $\Omega(F/C')$. Of course one can also distinguish them as $d_{F/C}a$ and $d_{F/C'}a$ if one needs to, but the point is that often this notation is not needed to avoid ambiguity. One sees similar notational conventions in other areas of mathematics where cohomological ideas occur.

These two elements written da cannot consistently be equal to each other in any material sense because the quotient map is not injective. For example, if the characteristic is 0 and $a \in C'$ is transcendental over C then $da \neq 0$ in $\Omega(F/C)$, but $da = 0$ in $\Omega(F/C')$. But freed from the constraints of a universe of elements, we can use notation for elements locally within each set, and then we can allow notation for elements to be continued not just along injective functions which we choose to treat as inclusions of sets, but also along non-injective functions as in this case where we so choose, provided we are careful. Using the same notation for two different things is not uncommon in mathematics, and is often called *abuse of notation*. In computer programming essentially the same thing is known as *overloading*. Both terms are somewhat pejorative, but really this is usually done for good reason: a judgement is made that it is better to simplify the notation because the difference between the two notions is not important, or can be deduced from context when needed. For example, in both the material and structural approaches to sets it would be normal to identify the 0 in the rational field with the 0 in the real field, and not write them as $0_{\mathbb{Q}}$ and $0_{\mathbb{R}}$, even though from a material set theory perspective, $0_{\mathbb{R}}$ is perhaps a Dedekind cut, so an uncountable set of rationals, not equal to a single rational. Likewise, we write $+$ for both the addition in $\mathbb{Q}$ and in $\mathbb{R}$. There is a unique field embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$, hence a canonical inclusion of sets, so the shift in viewpoint mentioned earlier, to regard $\mathbb{Q}$ as a subset of $\mathbb{R}$, makes this unproblematic. However, the material viewpoint does not have any such way to make sense of identifying elements along a non-injective function, whereas there is no such problem from the structural viewpoint. The elements $d_{F/C}a$ and $d_{F/C'}a$ lie in different sets, so according to the Principle of No Shared Elements there is no a priori sense in which they can be equal or unequal, so no problem in using the same notation da for them.

7 Final Comments

Summary

Using Set-theoretic (material) conventions for dealing with subsets, via Subset Extensionality, is fairly universal in mathematical practice. If we consider a universe U of possible elements of sets, we can extend the idea to Set Extensionality, but this principle seems much less universal in mathematical practice, and much less clearly useful. In practice, we might want to build a sufficiently universal domain for working in, such as an algebraically closed field, a Riemann surface in complex analysis, an abstract variety in algebraic geometry, or a Fraïssé limit or monster model in model theory, and then work with subsets of it via Subset Extensionality. However the "building" process often involves "going up" in ways that Set Extensionality is not helpful for. Indeed for field extensions we have seen that taking it too seriously can be genuinely problematic. The Principle of No Shared Elements is an alternative which simplifies matters in an opposite way, and I often find this much more useful for the building process.

Closely related to this Principle of No Shared Elements is the Structural Principle for Subsets. I have often found that whether using set-theoretic or category-theoretic approaches, ambiguities in writing can often be spotted, and then removed, by applying this principle. The Monster Model Convention does not technically clash with this Structural Principle for Subsets because one explicitly only considers sets which are subsets of some U, but it is somewhat opposite in perspective, and in the use of language, because it rules out making a distinction between set/subset (or field/subfield etc).

In any case, I encourage my students to take seriously the idea of working with both the

Subset Extensionality principle, and the Principle of No Shared Elements, whilst also using the Monster Model Convention where it really helps to simplify concepts or notation, but not to use it uncritically.

Analysis and logic versus algebra and geometry

In the introduction, I noted that the set-theoretic tradition is stronger in real analysis, model theory, and set theory, while the category-theoretic tradition is stronger in algebra and in geometry. Perhaps the bulk of mathematical practice in analysis and much of mathematical logic is concerned with going down, that is, working inside a prebuilt object (the real field, a Banach space, a monster model, ...) whereas in algebra and geometry there is much more going up and out, to build new objects. I thank an anonymous referee for explaining that much of functional analysis consists of extending classical function spaces "up" to more sophisticated generalized function spaces, using category-theoretic methods, and pointing out a relevant discussion in answers to a MathOverflow question [Wik].

Slogans

"Up with Categories, Down with Sets" is a simple slogan which I hope is memorable. Of course, it is somewhat provocative given the negative connotations of the downward direction! If we incorporate the discussion from Section 6, and also view a subset as being inside a set rather than down from it, we get the alternative slogan "Out with Categories, In with Sets" with much the same meaning, but the opposite connotations. I chose these two slogans to form the title of this note as I think they summarise well the basic ideas I want to get across. However, as I warned at the end of the introduction, the subject matter is not Category Theory, nor even Set Theory, but the material and structural approaches to using sets in mathematical practice. Hence a more accurate but less catchy slogan would be "Out with Structure, In with Material", or perhaps, "Out with Functions, In with Elements".

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