EVALUATION BIREPRESENTATIONS OF AFFINE TYPE A SOERGEL BIMODULES

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ABSTRACT. In this paper, we use Soergel calculus to define a monoidal functor, called the evaluation functor, from extended affine type A Soergel bimodules to the homotopy category of bounded complexes in finite type A Soergel bimodules. This functor categorifies the well-known evaluation homomorphism from the extended affine type A Hecke algebra to the finite type A Hecke algebra. Through it, one can pull back the triangulated birepresentation induced by any finitary birepresentation of finite type A Soergel bimodules to obtain a triangulated birepresentation of extended affine type A Soergel bimodules. We show that if the initial finitary birepresentation in finite type A is a cell birepresentation, the evaluation birepresentation in extended affine type A has a finitary cover, which we illustrate by working out the case of cell birepresentations with subregular apex in detail.

CONTENTS

1. Introduction

Finitary birepresentation theory of finite type Soergel bimodules in characteristic zero has been a topic of intensive study, with many interesting results, in the last couple of years [KMMZ2019, MM2017, MMMTZ2019, MT2019, Zimm2017]. In this paper, we initiate the study of a class of finitary and triangulated birepresentations of affine type A Soergel bimodules. The bicategories of these Soergel bimodules are no longer finitary and, therefore, new phenomena show up in their birepresentation theory. For example, there are no known interesting triangulated birepresentations in finite type, whereas we do give examples of such birepresentations in affine type A.

To describe these, let us briefly recall the decategorified setting first. In type A, as is well-known, there are evaluation maps from the affine Hecke algebra to the finite type Hecke algebra. These are homomorphisms of algebras, so any representation of the latter algebra can be pulled back to a representation of the former algebra through such a map. These so-called *evaluation representations* form an important and well-studied class of finite-dimensional representations of affine type A Hecke algebras, see e.g. [CP1996, DF2016, LNT2003] and references therein.

Several authors ([MT2017, Introduction] and [E2018, Section 1.6]) have conjectured that these evaluation maps can be categorified by monoidal *evaluation functors* (i.e., pseudofunctors between one-object bicategories) from affine type A Soergel bimodules to the homotopy category of bounded complexes in finite type A Soergel bimodules. In this paper, we indeed define such functors and use them to categorify the aforementioned evaluation representations in the form of triangulated birepresentations, obtained by pulling back the triangulated birepresentations induced by finitary birepresentations of finite type A Soergel bimodules through these functors. Moreover, in case the original finitary birepresentation is simple transitive, we show that the evaluation birepresentation admits a *finitary cover*, i.e., a finitary birepresentation together with an essentially surjective and epimorphic morphism of additive birepresentations from that cover to the evaluation birepresentation. This categorifies the well-known fact that the corresponding evaluation representations are quotients of certain cell representations defined by Graham and Lehrer [GL1998].

Let us finish this introduction with a disclaimer. We do not present a theory of triangulated birepresentations in this paper. First of all, it is not yet clear whether our evaluation functors can be extended to triangulated functors between the homotopy category of bounded complexes in affine type A Soergel bimodules and its counterpart in finite type A. Proving the existence of such an extension is a non-trivial exercise in obstruction theory, which will have to be addressed in the future. This extension problem was first mentioned in [E2018, Section 16], where it is conjectured to be solvable, and a similar problem will have to be solved in order to prove [AL-ELR, Conjecture 1.2] for the categorification of the internal braid group action on quantum groups. Secondly, some ingredients for a theory of triangulated birepresentations can already be found in the literature, e.g. [E2018, EH2018, Hog2017, LM2022, Stev2011], but many foundational results are still missing. In general, it is not clear which parts of finitary birepresentation theory, e.g. the notion of simple transitive birepresentation, the categorical (weak) Jordan-Hölder theorem, the relation with (co)algebra 1-morphisms, the double-centralizer theorem (see [MMMTZ2020] and references therein), generalize to the triangulated setting and/or in which form exactly. These questions need to be answered first, before one can even think of categorifying the induction product of evaluation representations from [LNT2003, Section 2.5]. Finally, all of this is just for affine type A. Hecke algebras of other affine Coxeter types also have interesting finite-dimensional representations, but there are no evaluation morphisms in those cases, so other ideas will be needed to categorify those representations. In other words, the results in this paper are (hopefully) just the tip of a (tricky) triangulated iceberg.

Plan of the paper. In Section 2, we recall the basics of extended and non-extended affine Hecke algebras of affine type A, the evaluation maps, the Graham-Lehrer cell modules and the evaluation representations. Everything in this section is well-documented in the literature and we only recall the details that are needed in the rest of this paper.

In Section 3, we briefly recall Soergel calculus in finite and affine type A, the latter both in the non-extended and the extended version. Again, nothing new is presented, so the specialists can skip this section and move on to the next one. Of course, in the remainder we often refer to the diagrammatic equations in this section, which is exactly why we recall them.

In Section 4, we first recall some basic results on Rouquier complexes in finite type A and then focus on a special type of Rouquier complex, which is fundamental for the definition of the evaluation functors in the next section. In particular, we develop a mixed diagrammatic calculus for morphisms between products of Bott-Samelson bimodules and these special Rouquier complexes, all in finite type A. To the best of our knowledge, this extension of the usual Soergel calculus is new.

In Section 5, we define the evaluation functors by assigning a bounded complex of finite type A Soergel bimodules (or, more precisely, of finite type A Bott-Samelson bimodules) to each extended affine type A Bott-Samelson bimodule and a map between such complexes to each generating extended affine type A Soergel calculus diagram. The main result of this section, and of this paper, is that this assigment is well-defined up to homotopy equivalence.

In Section 6, we first introduce the notion of a triangulated birepresentation of an additive bicategory and define evaluation birepresentations of Soergel bimodules in extended affine type A, which are important examples. We then prove that each evaluation birepresentation has a (possibly non-unique) finitary cover. Finally, we study in detail the simplest non-trivial evaluation birepresentations, which are the ones induced by cell birepresentations of finite type A with subregular apex. As we show, these admit a simple transitive finitary cover whose underlying algebra is a signed version of the zigzag algebra of affine type A.

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2. The decategorified story

From now on, fix $d \in \mathbb{N}_{\geq 3}$ and let $\widehat{I} := \mathbb{Z}/d\mathbb{Z}$ and $I := \{1, \dots, d-1\}$. By a slight abuse of notation, we will often identify $\widehat{I}$ with the set of representatives $\{0, 1, \dots, d-1\}$ and consider I as a subset of $\widehat{I}$.

Let $\widehat{\mathfrak{S}}_d$ be the affine Weyl group of type $\widehat{A}_{d-1}$. It is generated by $s_i,\ i \in \widehat{I}$, subject to relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } |i-j| > 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

for $i \in \widehat{I}$. The *extended* affine Weyl group $\widehat{\mathfrak{S}}_d^{\mathrm{ext}}$ is the semidirect product

$$\langle \rho \rangle \ltimes \widehat{\mathfrak{S}}_d,$$

where $\langle \rho \rangle$ is an infinite cyclic group generated by ρ and

$$\rho s_i \rho^{-1} = s_{i+1},$$

for $i \in \widehat{I}$. The finite Weyl group of type A_{d-1} is the symmetric group on d letters, $\mathfrak{S}_d$, corresponding to the subgroup of $\widehat{\mathfrak{S}}_d$ generated by $s_i,\ i \in I$.

Remark 2.1. In some papers, the name *extended affine Weyl group* of type $\widehat{A}_{d-1}$ is used for the quotient of $\widehat{\mathfrak{S}}_d$ by the ideal generated by ρ^d. However, there are no evaluation maps from the extended affine Hecke algebra corresponding to that quotient to the finite type Hecke algebra, so we will not consider it in this paper.

2.1. **Hecke algebras.** Let $\Bbbk = \mathbb{C}(q)$, where q is a formal parameter. The *extended affine Hecke algebra* $\widehat{H}_d^{ext}$ is the $\Bbbk$-algebra generated by $T_i,\ i \in \widehat{I}$, and $\rho^{\pm 1}$, with relations

$$(T_i + q)(T_i - q^{-1}) = 0, \qquad T_i T_j = T_j T_i \text{ if } |i-j| > 1, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{1}$$

$$\rho \rho^{-1} = 1 = \rho^{-1} \rho, \qquad \rho T_i \rho^{-1} = T_{i+1}, \tag{2}$$

for $i, j \in \widehat{I}$. Note that T_i is invertible for every $i \in \widehat{I}$ with

$$T_i^{-1} = T_i + q - q^{-1}.$$

As is well-known, $\widehat{H}_d^{ext}$ is a q-deformation of the group algebra $\mathbb{C}[\widehat{\mathfrak{S}}_d^{\mathrm{ext}}]$ with basis (the *regular basis*) given by $\{\rho^m T_w \mid m \in \mathbb{Z}, w \in \widehat{\mathfrak{S}}_d\}$, where $T_w := T_{i_1} \cdots T_{i_\ell}$ for any *reduced expression* (rex) $s_{i_1} \cdots s_{i_\ell}$ of w.

Another presentation is given in terms of the *Kazhdan–Lusztig generators* $b_i := T_i + q$, for $i \in \widehat{I}$, and $\rho^{\pm 1}$, subject to relations

$$b_i^2 = [2] b_i, \qquad b_i b_j = b_j b_i \text{ if } |i-j| > 1, \qquad b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i, \tag{3}$$

$$\rho \rho^{-1} = 1 = \rho^{-1} \rho, \qquad \rho b_i \rho^{-1} = b_{i+1}, \tag{4}$$

for $i \in \widehat{I}$, where $[2] := q + q^{-1}$. Note that $T_i = b_i - q$ and $T_i^{-1} = b_i - q^{-1}$, for every $i \in \widehat{I}$. The *Kazhdan–Lusztig basis* is given by $\{\rho^m b_w \mid m \in \mathbb{Z}, w \in \widehat{\mathfrak{S}}_d\}$, where b_w is defined for an arbitrary rex of w (and is independent of that choice).

The (non-extended) *affine Hecke algebra* $\widehat{H}_d$ is the subalgebra of $\widehat{H}_d^{ext}$ generated by either $T_i,\ i \in \widehat{I}$, subject to relations (1), or $b_i,\ i \in \widehat{I}$, subject to relations (3).

The *finite Hecke algebra* H_d is the $\Bbbk$-subalgebra of $\widehat{H}_d$ generated by either $T_i,\ i \in I$, subject to relations (1), or $b_i,\ i \in I$ subject to relations (3).

2.2. **Evaluation maps.**

Definition 2.2. For any $a \in \Bbbk^{\times}$, there are two *evaluation maps* $\mathrm{ev}_a, \mathrm{ev}'_a \colon \widehat{H}_d^{ext} \to H_d$. These are defined as the homomorphisms of $\Bbbk$-algebras determined by

$$\mathrm{ev}_a(T_i) = T_i, \quad \text{for } i \in I, \tag{5}$$

$$\mathrm{ev}_a(\rho) = aT_1^{-1} \cdots T_{d-1}^{-1} \tag{6}$$

and

$$\mathrm{ev}'_a(T_i) = T_i, \quad \text{for } i \in I, \tag{7}$$

$$\mathrm{ev}'_a(\rho) = aT_1 \cdots T_{d-1}, \tag{8}$$

respectively.

The definition implies that

$$\mathrm{ev}_a(T_0) = \mathrm{ev}_a(\rho^{-1}T_1\rho) = T_{d-1} \cdots T_2 T_1 T_2^{-1} \cdots T_{d-1}^{-1} \tag{9}$$

and

$$\mathrm{ev}'_a(T_0) = \mathrm{ev}'_a(\rho^{-1}T_1\rho) = T_{d-1}^{-1} \cdots T_2^{-1} T_1 T_2 \cdots T_{d-1}, \tag{10}$$

so the restrictions of ev_a and ev'_a to $\widehat{H}_d$ do not depend on a.

In terms of the Kazhdan–Lusztig generators we have

$$\mathrm{ev}_a(b_i) = b_i, \quad \text{for } i \in I, \tag{11}$$

$$\mathrm{ev}_a(b_0) = \mathrm{ev}_a(\rho^{-1}b_1\rho) = (b_{d-1} - q) \cdots (b_1 - q) b_1 (b_1 - q^{-1}) \cdots (b_{d-1} - q^{-1}) \tag{12}$$

and

$$\mathrm{ev}'_a(b_i) = b_i, \quad \text{for } i \in I, \tag{13}$$

$$\mathrm{ev}'_a(b_0) = \mathrm{ev}'_a(\rho^{-1}b_1\rho) = (b_{d-1} - q^{-1}) \cdots (b_1 - q^{-1}) b_1 (b_1 - q) \cdots (b_{d-1} - q). \tag{14}$$

Another way of saying this is that the evaluation maps do not preserve the bar involution, but rather satisfy

$$\overline{\mathrm{ev}_a(x)} = \mathrm{ev}'_{\overline{a}}(\overline{x}), \tag{15}$$

for any $x \in \widehat{H}_d^{ext}$ and $a = a(q) \in \Bbbk^{\times}$.

One can also define ev_a and ev'_a using a third presentation of $\widehat{H}_d^{ext}$, called the *Bernstein presentation*. In that presentation, $\widehat{H}_d^{ext}$ is defined as some sort of semidirect product of H_d and

$\Bbbk[Y_1^{\pm 1}\dots,Y_d^{\pm 1}]$. However, there are several possible choices for the algebra of Laurent polynomials. In [E2018], two such choices are given with different variables: $y_1,\dots,y_d$ and $y_1^*,\dots,y_d^*$ respectively. The interaction of H_d and these polynomial algebras is defined by

$$T_i^{-1}y_iT_i^{-1}=y_{i+1} \tag{16}$$

and

$$T_iy_i^*T_i=y_{i+1}^*, \tag{17}$$

respectively, for $i\in I$.

The relation between these two Bernstein presentations and our first presentation of $\widehat{H}_d^{ext}$ is given by

$$y_1=\rho T_{d-1}\cdots T_2T_1, \tag{18}$$

$$y_i=T_{i-1}^{-1}\cdots T_2^{-1}T_1^{-1}\rho T_{d-1}\cdots T_{i+1}T_i,\quad i=2,\dots,d-1, \tag{19}$$

resp.

$$y_1^*=\rho T_{d-1}^{-1}\cdots T_2^{-1}T_1^{-1}, \tag{20}$$

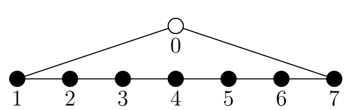
$$y_i^*=T_{i-1}\cdots T_2T_1\rho T_{d-1}^{-1}\cdots T_{i+1}^{-1}T_i^{-1},\quad i=2,\dots,d-1. \tag{21}$$

It follows that the evaluation map $\mathrm{ev}_a\colon \widehat{H}_d^{ext}\to H_d$ is the unique homomorphism of algebras sending T_i to T_i, for $i\in I$, and y_1 to a, while $\mathrm{ev}_a'\colon \widehat{H}_d^{ext}\to H_d$ is the unique homomorphism of algebras sending T_i to T_i, for $i\in I$, and y_1^* to a. The latter coincides with the flattening map $\flat$ in [E2018, §2.6] for $a=1$.

We will categorify the evaluation map ev_a in Section 5.1. The categorification of ev_a' is very similar and the relation between the two evaluation maps in (15) also categorifies, since the categorification of the bar-involution is given by flipping diagrams upside-down, inverting the orientation of the differentials in complexes and changing the sign of homological and grading shifts.

Remark 2.3. Some remarks about the various conventions in the literature are in order. We try to follow conventions close to those in [E2018]. Our presentation of the extended affine Hecke algebra in Section 2.1 agrees with [E2018], as does the relation between the standard generators and the Kazhdan–Lusztig generators. Some authors use the inverse of ρ in (2). Our choice of conventions implies the absence of certain powers of q in the definition of the evaluation maps, in comparison with some of the sources in the literature. For more information on evaluation maps, see e.g. [CP1996, §5.1] and [DF2016, (5.0.2)]. There are more possible evaluation maps, but we only consider these two in this paper.

2.3. **Graham-Lehrer cell modules.** Consider the $\widehat{A}_{d-1}$ Coxeter diagram $\widehat{\Gamma}_{d-1}$ with its vertices ordered counterclockwise and top vertex numbered 0, e.g.



for $d = 8$. For any $z \in \Bbbk^\times$, the *Graham-Lehrer cell module* $\widehat{M}_z$ of $\widehat{H}_d$ corresponding to z and the partition $(d-1,1)$ has underlying vector space

$$\widehat{M}_z := \mathrm{Span}_{\Bbbk}\left\{m_i \mid i \in \widehat{I}\right\} \tag{22}$$

and the action of $\widehat{H}_d$ on $\widehat{M}_z$ is given by

$$b_i m_j = \begin{cases} [2]m_i, & \text{if } j \equiv i \bmod d; \\ zm_1, & \text{if } i-1 \equiv 0 \equiv j \bmod d; \\ z^{-1}m_0, & \text{if } i \equiv 0 \equiv j-1 \bmod d; \\ m_j, & \text{if } i \equiv j \pm 1 \bmod d, \text{ but none of the above;} \\ 0, & \text{else.} \end{cases} \tag{23}$$

It is easy to see that $\widehat{M}_z$ is isomorphic to $W_{d-2,\pm\sqrt{z}}(d)$ in [GL1998, Definition 2.6], where m_i is identified with the cup diagram on a cylinder with $d-2$ straight lines and only one cup, whose endpoints are i and $i+1$. When $i \neq 0$, the whole diagram corresponding to m_i lives on the front part of the cylinder, but when $i = 0$, the cup of m_0 goes around the back of the cylinder. Note that we have used $\delta = [2]$, rather than $\delta = -[2]$. As remarked in [GL1998, text above Corollary 2.9.1], $W_{d-2,\sqrt{z}}(d)$ and $W_{d-2,-\sqrt{z}}(d)$ are isomorphic, which is clear from the fact that both are isomorphic to $\widehat{M}_z$.

The Graham-Lehrer cell module $\widehat{M}_z$ can be made into an $\widehat{H}_d^{ext}$-module, but not in a unique way. As a matter of fact, for each $\lambda \in \Bbbk^\times$, we can define

$$\rho\, m_j = \lambda z^{\delta_{j,0}} m_{j+1}, \tag{24}$$

for $j \in \widehat{I}$. It is easy to verify that this gives a well-defined action and we denote the corresponding Graham-Lehrer cell module of $\widehat{H}_d^{ext}$ by $\widehat{M}_{z,\lambda}$. Note that the restriction of $\widehat{M}_{z,\lambda}$ to $\widehat{H}_d$ is equal to $\widehat{M}_z$, for all $\lambda \in \Bbbk^\times$, and that the action of ρ^d on $\widehat{M}_{z,\lambda}$ is simply multiplication by $\lambda^d z$.

Graham and Lehrer [GL1998, Theorem 2.8] defined a $\Bbbk$-bilinear form

$$\langle \cdot, \cdot \rangle \colon \widehat{M}_z \otimes \widehat{M}_{z^{-1}} \to \Bbbk, \tag{25}$$

which in our notation is determined by

$$\langle m_i, m_j \rangle = \begin{cases} [2], & \text{if } j \equiv i \bmod d; \\ z, & \text{if } i \equiv 0 \equiv j-1 \bmod d; \\ z^{-1}, & \text{if } i-1 \equiv 0 \equiv j \bmod d; \\ 1, & \text{if } i \equiv j \pm 1 \bmod d, \text{ but none of the above;} \\ 0, & \text{else.} \end{cases} \tag{26}$$

This induces a $\Bbbk$-bilinear form on $\widehat{M}_{z,\lambda} \otimes \widehat{M}_{z^{-1},\lambda^{-1}}$, satisfying $\langle \rho^n b_w m_j, m_k \rangle = \langle m_j, b_w^\star \rho^{-n} m_k \rangle$, for any $w \in \widehat{W}$, $n \in \mathbb{Z}$ and $j, k \in \widehat{I}$, where $b_w^\star = b_{w^{-1}}$ is the dual Kazhdan-Lusztig basis element. Therefore, the radical of the bilinear form

$$\mathrm{rad}(\langle \cdot, \cdot \rangle) = \left\{ m \in \widehat{M}_{z,\lambda} \mid \langle m, m' \rangle = 0,\ \forall m' \in \widehat{M}_{z^{-1},\lambda^{-1}} \right\}$$

is an $\widehat{H}_d^{ext}$-submodule of $\widehat{M}_{z,\lambda}$. Graham and Lehrer [GL1998, Theorem 2.8] proved that the quotient module $\widehat{M}_z/\mathrm{rad}(\langle\cdot,\cdot\rangle)$ of $\widehat{H}_d$ is simple, and the same holds for the quotient module $\widehat{M}_{z,\lambda}/\mathrm{rad}(\langle\cdot,\cdot\rangle)$ of $\widehat{H}_d^{ext}$, of course. A straightforward calculation shows that the radical of the bilinear form on $\widehat{M}_{z,\lambda}$ is zero unless $z = (-q)^{\pm d}$ (independently of λ), in which case it has dimension one and is generated by

$$n_\pm := \sum_{k=1}^{d} (-q)^{\mp k} m_k. \tag{27}$$

Note that, when $z = (-q)^{\pm d}$, we have $\rho\, n_\pm = \lambda(-q)^{\pm 1} n_\pm$ and $b_i n_\pm = 0$ for all $i \in \widehat{I}$.

When $z = (-q)^{\pm d}$, put $\widehat{M}^{\pm}_{d,\lambda} := \widehat{M}_{(-q)^{\pm d},\lambda^{\pm 1}}$ and let

$$\widehat{L}^{\pm}_{d,\lambda} := \widehat{M}^{\pm}_{d,\lambda}/\langle n_\pm \rangle \tag{28}$$

be the simple quotient $\widehat{H}_d^{ext}$-modules of dimension $d-1$. Finally, denote the restriction of these simple modules to $\widehat{H}_d$ by

$$\widehat{L}^{\pm}_{d} := \widehat{M}^{\pm}_{d}/\langle n_\pm \rangle. \tag{29}$$

As explained above, these restrictions do not depend on $\lambda \in \Bbbk^\times$.

2.4. **Evaluation modules.** Let M be a finite-dimensional H_d-module (over $\Bbbk$). Recall that, for any $a \in \Bbbk^\times$, there are two evaluation maps $\mathrm{ev}_a, \mathrm{ev}'_a \colon \widehat{H}_d^{ext} \to H_d$ (see Definition 2.2).

Definition 2.4. For any $a \in \Bbbk^\times$, the *evaluation modules* M^{ev_a} and $M^{\mathrm{ev}'_a}$ of $\widehat{H}_d^{ext}$ are the pull-backs of M through ev_a and ev'_a, respectively.

The actions of $\widehat{H}_d^{ext}$ on M^{ev_a} and $M^{\mathrm{ev}'_a}$ can be computed using the explicit formulas in Definition 2.2 and below. In this paper, we only consider the case when $M := M_d$ is the simple H_d-module corresponding to the partition $(d-1,1)$. There are several ways to define M_d explicitly and the definition we choose here is tailor-made for categorification. Take $M_d := \mathrm{span}_\Bbbk\{m_i \mid i \in I\}$, with the action of H_d being given by

$$b_i m_j = \begin{cases} [2]m_i, & \text{if } j = i; \\ m_i, & \text{if } j = i \pm 1; \\ 0, & \text{else,} \end{cases} \tag{30}$$

for $i, j \in I$. It is easy to show that M_d is simple, but this is well-known so we leave it as an exercise to the reader. The action of the $T_i^{\pm 1} = b_i - q^{\pm 1}$ is also easy to give explicitly:

$$T_i^{\pm 1} m_j = \begin{cases} q^{\mp 1} m_i, & \text{if } j = i; \\ m_i - q^{\pm 1} m_j, & \text{if } j = i \pm 1; \\ -q^{\pm 1} m_j, & \text{else.} \end{cases} \tag{31}$$

Note that, as vector spaces, $M_d^{\mathrm{ev}_a} = M_d^{\mathrm{ev}'_a} = M_d$, and the action of $b_i \in \widehat{H}_d^{ext}$, for $i \in I$, is the same as above because $\mathrm{ev}_a(b_i) = b_i$. A simple calculation now shows that

$$\mathrm{ev}_a(\rho)m_j = aT_1^{-1}\cdots T_{d-1}^{-1}m_j = \begin{cases} a(-q)^{2-d}m_{j+1}, & \text{if } j = 1,\ldots,d-2; \\ aq\sum_{k=1}^{d-1}(-q)^{1-k}m_k, & \text{if } j = d-1, \end{cases} \tag{32}$$

and

$$\mathrm{ev}'_a(\rho)m_j = aT_1\cdots T_{d-1}m_j = \begin{cases} a(-q)^{d-2}m_{j+1}, & \text{if } j = 1,\ldots,d-2; \\ aq^{-1}\sum_{k=1}^{d-1}(-q)^{k-1}m_k, & \text{if } j = d-1. \end{cases} \tag{33}$$

The actions of b_0 can then be computed using the equation $b_0 = \rho^{-1}b_1\rho$, but we omit the calculation because will not need the result.

Recall the simple quotients $\widehat{L}_{d,\lambda}^{\pm}$ of the Graham-Lehrer cell modules $\widehat{M}_{d,\lambda}^{\pm}$, defined in (28).

Theorem 2.5. *Let $a = \lambda(-q)^{d-2}$. There are two isomorphisms of $\widehat{H}_d^{ext}$-modules*

$$\widehat{L}_{d,\lambda}^{+} \cong M_d^{\mathrm{ev}_a};$$
$$\widehat{L}_{d,\lambda}^{-} \cong M_d^{\mathrm{ev}'_{a^{-1}}}.$$

Moreover, there is a perfect pairing of $\widehat{H}_d^{ext}$-modules

$$M_d^{\mathrm{ev}_a} \otimes M_d^{\mathrm{ev}'_{a^{-1}}} \to \Bbbk.$$

Proof. To show the first part, it suffices to compute the action of ρ on $\widehat{L}_{d,\lambda}^{+}$ and compare it to (32). Let $\overline{m}_k$ be the image of m_k under the projection $\widehat{M}_{d,\lambda}^{+} \to \widehat{L}_{d,\lambda}^{+}$, for $k \in \widehat{I}$. Then $\{\overline{m}_1,\ldots,\overline{m}_{d-1}\}$ is a basis of $\widehat{L}_{d,\lambda}^{+}$, because $\overline{m}_0 = -\sum_{k=1}^{d-1}(-q)^k\overline{m}_{d-k}$. This implies that in $\widehat{L}_{d,\lambda}^{+}$ we have

$$\rho\,\overline{m}_j = \begin{cases} \lambda\overline{m}_{j+1}, & \text{if } j = 1,\ldots,d-2; \\ -\lambda\sum_{k=1}^{d-1}(-q)^k\overline{m}_{d-k}, & \text{if } j = d-1. \end{cases}$$

This is indeed the same as in (32) because $aq = \lambda(-q)^{d-2}q = -\lambda(-q)^{d-1}$.

Similarly, $\widehat{L}_{d,\lambda}^{-} \cong M_d^{\mathrm{ev}'_{a^{-1}}}$, as in $\widehat{L}_{d,\lambda}^{-}$ we have $\overline{m}_0 = -\sum_{k=1}^{d-1}(-q)^{-k}\overline{m}_{d-k}$, so

$$\rho\,\overline{m}_j = \begin{cases} \lambda^{-1}\overline{m}_{j+1}, & \text{if } j = 1,\ldots,d-2; \\ -\lambda^{-1}\sum_{k=1}^{d-1}(-q)^{-k}\overline{m}_{d-k}, & \text{if } j = d-1, \end{cases}$$

which is the same as in (33) because $a^{-1}q^{-1} = \lambda^{-1}(-q)^{2-d}q^{-1} = -\lambda^{-1}(-q)^{1-d}$.

For the second part, note that the two $\widehat{H}_d^{ext}$-modules $\widehat{L}_{d,\lambda}^{+}$ and $\widehat{L}_{d,\lambda}^{-}$ are *dual* to each other, because we could also consider the radical defined by

$$\mathrm{rad}'(\langle\cdot,\cdot\rangle) = \left\{m' \in \widehat{M}_{z^{-1},\lambda^{-1}} \mid \langle m, m'\rangle = 0,\ \forall m \in \widehat{M}_{z,\lambda}\right\},$$

which is an $\widehat{H}_d^{ext}$-submodule of $\widehat{M}_{z^{-1},\lambda^{-1}}$. As before, this radical is zero unless $z = (-q)^{\pm d}$. For these two values of z and any value of $\lambda \in \Bbbk^{\times}$, the two simple quotients of $\widehat{M}_{z^{-1},\lambda^{-1}}$ are isomorphic to $\widehat{L}^{\mp}_{d,\lambda}$ and the bilinear form descends to a perfect pairing

$$\widehat{L}^{+}_{d,\lambda} \otimes \widehat{L}^{-}_{d,\lambda} \to \Bbbk.$$

By the first part, this is equivalent to a perfect pairing

$$M_d^{\mathrm{ev}_a} \otimes M_d^{\mathrm{ev}'_{a^{-1}}} \to \Bbbk,$$

for $a = \lambda(-q)^{d-2}$. □

Remark 2.6. We claim no originality w.r.t. Theorem 2.5, but we do not know of any reference in the literature where one can find it explicitly, which is why we have proved it here.

3. Reminders on Soergel categories

In this section we briefly recall the definition of the diagrammatic Soergel category of non-extended and extended affine type A and finite type A, but before we do that we start with a brief section on graded categories and categories with shift.

3.1. **Graded categories and categories with shift.** All categories in this paper are assumed to be essentially small, meaning that they are equivalent to small categories, so set-theoretic questions play no role.

We call a $\mathbb{C}$-linear category $\mathcal{A}$ *graded* if it is enriched over the category of $\mathbb{Z}$-graded vector spaces, and we call a $\mathbb{C}$-linear functor between such graded categories *degree-preserving* if it preserves the degrees of homogeneous morphisms.

We say that a $\mathbb{C}$-linear category $\mathcal{A}$ has a *shift* (or, alternatively, that it is a *category with shift*) if there is a $\mathbb{C}$-linear automorphism $\langle 1\rangle$ of $\mathcal{A}$. If such a shift exists, we define $\langle r\rangle$ as the composite of r copies of $\langle 1\rangle$ for any $r \in \mathbb{Z}_{\geq 0}$, and $-r$ copies of the inverse of $\langle 1\rangle$ for any $r \in \mathbb{Z}_{\leq 0}$. By definition, therefore, we have $\langle r+s\rangle = \langle r\rangle \circ \langle s\rangle$, for all $r, s \in \mathbb{Z}$, and $\langle 0\rangle = \mathrm{Id}_{\mathcal{A}}$.

Given a graded category $\mathcal{A}$, let $\mathcal{A}^{\mathrm{sh}}$ be the associated $\mathbb{C}$-linear category with shift, whose objects are formal integer shifts of objects in $\mathcal{A}$ and whose hom-spaces are defined by

$$\mathcal{A}^{\mathrm{sh}}\left(X\langle r\rangle, Y\langle s\rangle\right) := \mathcal{A}\left(X, Y\right)_{s-r}$$

for every $X, Y \in \mathcal{A}$ and $r, s \in \mathbb{Z}$. Note that $\mathcal{A}^{\mathrm{sh}}$ is no longer a graded category. If the Hom-spaces of $\mathcal{A}$ are finite-dimensional in every degree, then the hom-spaces of $\mathcal{A}^{\mathrm{sh}}$ are finite-dimensional.

Given two graded categories $\mathcal{A}$ and $\mathcal{B}$, any degree-preserving, $\mathbb{C}$-linear functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a unique $\mathbb{C}$-linear functor $F\colon \mathcal{A}^{\mathrm{sh}} \to \mathcal{B}^{\mathrm{sh}}$, denoted by the same symbol, which commutes with the shifts.

Conversely, given any $\mathbb{C}$-linear category $\mathcal{A}$ with shift, let $\mathcal{A}^{\mathrm{gr}}$ be the associated graded category with shift, whose objects are those of $\mathcal{A}$ and whose graded Hom-spaces are defined by

$$\mathcal{A}^{\mathrm{gr}}\left(X, Y\right) := \bigoplus_{s\in\mathbb{Z}} \mathcal{A}\left(X, Y\langle s\rangle\right),$$

for any $X, Y \in \mathcal{A}$.

Given two $\mathbb{C}$-linear categories $\mathcal{A}$ and $\mathcal{B}$ with shifts, any $\mathbb{C}$-linear functor $F\colon \mathcal{A} \to \mathcal{B}$ commuting with the shifts induces a unique degree-preserving, $\mathbb{C}$-linear functor $F\colon \mathcal{A}^{\mathrm{gr}} \to \mathcal{B}^{\mathrm{gr}}$, denoted by the same symbol.

Thus $(-)^{\mathrm{sh}}$ and $(-)^{\mathrm{gr}}$ define a pair of 2-functors between the 2-category of graded categories and the 2-category of $\mathbb{C}$-linear categories with shift. It is not hard to show, see e.g. [EMTW2020, Proposition 11.9], that $(-)^{\mathrm{sh}}$ is left adjoint to $(-)^{\mathrm{gr}}$, i.e., that there is a functorial isomorphism

$$\mathrm{Fun}\left(\mathcal{A}^{\mathrm{sh}}, \mathcal{B}\right) \cong \mathrm{Fun}\left(\mathcal{A}, \mathcal{B}^{\mathrm{gr}}\right)$$

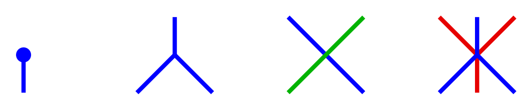
for $\mathcal{A}$ a graded category and $\mathcal{B}$ a $\mathbb{C}$-linear category with shift. Here the first functor category is between categories with shift and the second between graded categories.

For more details on graded categories and categories with shift, and also on additive closures and idempotent completions (a.k.a. Karoubi closures/envelopes), see e.g. [EMTW2020, Sections 11.2.1-11.2.4].

3.2. **Soergel calculus in finite and non-extended affine type A.** The finite type A diagrammatic Soergel calculus was introduced by Elias–Khovanov [EKh2010] and generalized to all Coxeter types by Elias–Williamson [EW2016]. The extended affine Soergel calculus was first defined in [MT2017] and studied more systematically in [E2018]. We refer to the latter two papers for more details. For the specialists, we remark that we use the so-called *root span realization* of the Cartan datum of finite and affine type A below.

Denote by $S = \{s_i \mid i \in \widehat{I}\}$ the set of simple reflections of $\widehat{\mathfrak{S}}_d$. The *diagrammatic Bott-Samelson category* of type $\widehat{A}_{d-1}$, denoted $\widehat{\mathcal{BS}}_d$, is the $\mathbb{Z}$-graded, $\mathbb{C}$-linear, additive, monoidal category whose objects are formal finite direct sums of finite words in the alphabet S, and whose graded vector spaces of morphisms are defined below in terms of homogeneous generating diagrams and relations. In general, we can write the objects as vectors of words and morphisms as matrices of equivalence classes of diagrams.

As usual, we will color the strands to facilitate the reading of the diagrams. These colors correspond to the elements of $\widehat{I}$, so henceforth we will also refer to those elements as colors. When there are too many different colors in a diagram, the colors are sometimes indicated by labels next to the strands. We say that two colors $i, j \in \widehat{I}$ are *adjacent* if $i \equiv j \pm 1 \bmod d$ and that they are *distant* otherwise. The generating diagrams are

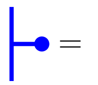


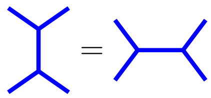
Degree $\quad 1 \quad -1 \quad 0 \quad 0$

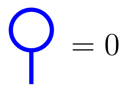
and the diagrams obtained from these by a rotation of 180 degrees (which have the same degrees). The colors of the 4-valent vertices are assumed to be distant, whereas those of the 6-valent vertices are assumed to be adjacent.

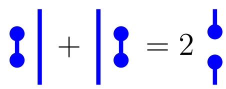
Diagrams can be stacked vertically (composition of morphisms) and juxtaposed horizontally (monoidal product of morphisms), while adding the degrees, and are subject to the relations below. We denote by Id_X the identity morphism of X and write fg for the monoidal product of morphisms f and g (or, equivalently, horizontal composition when considering the monoidal category as a one-object bicategory). We also assume isotopy invariance and cyclicity, meaning that closed parts of the diagrams can be moved around freely in the plane as long as they do not cross any other strands and the boundary is fixed, and all diagrams can be bent and rotated and the bent and rotated versions of the relations also hold.

- Relations involving one color:

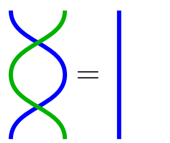
(34) 

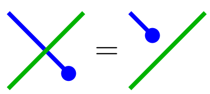
(35) 

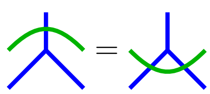
(36) 

(37) 

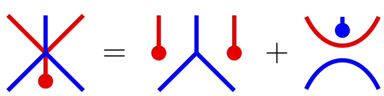
- Relations involving two distant colors:

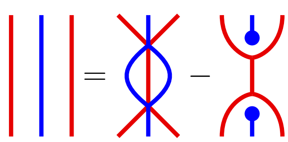
(38) 

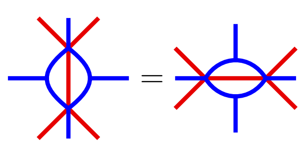
(39) 

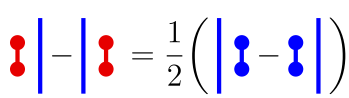
(40) 

- Relations involving two adjacent colors:

(41) 

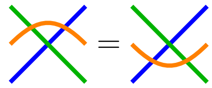
(42) 

(43) 

(44) 

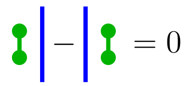
- Relation involving three distant colors:

(45)



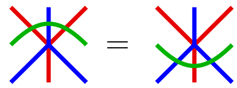
- Relation involving distant dumbbells:

(46)



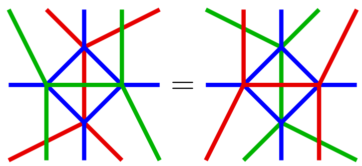
- Relation involving two adjacent colors and one distant from the other two:

(47)



- Relation involving three colors such that one of them is adjacent to the other two:

(48)



Note that the empty word is the identity object in $\widehat{\mathcal{BS}}_d$ and its endomorphisms are the closed diagrams, which by the relations above are equal to polynomials in the colored dumbbells

As each dumbbell has degree 2, the degree of any polynomial in these dumbbells, as a morphism in $\widehat{\mathcal{S}_{BS}}$, is twice its polynomial degree. From now on, we denote this polynomial algebra by R.

Note further that, by relations (37), (44) and (46), the morphism

(49) $$\sum_{i=0}^{d-1}$$ 

is central, in the sense that it can be slid through all diagrams (i.e. it commutes horizontally with all morphisms). Note that this morphism is equal to $\boxed{\mathrm{y}}$ (up to sign, depending on conventions) in [MT2017], because it is equal to the sum of all simple roots.

Let $\widehat{\mathcal{BS}}_d^{\mathrm{sh}}$ be the category with shift associated to $\widehat{\mathcal{BS}}_d$, see Section 3.1.

Definition 3.1. The *diagrammatic Soergel category* $\widehat{\mathcal{S}}_d$ is the idempotent completion of the diagrammatic Bott-Samelson category with shift $\widehat{\mathcal{BS}}_d^{\mathrm{sh}}$.

Remark 3.2. In the following sections, we sometimes state and prove diagrammatic equations in $\widehat{\mathcal{BS}}_d$, in which case there are no shifts for the source and target objects, instead of $\widehat{\mathcal{S}}_d$, in which case the source and target objects are carefully shifted. This is just to simplify notation and makes no essential difference in our case. As long as the equations in $\widehat{\mathcal{BS}}_d$ are between homogeneous diagrams of the same degree, they give rise to an equality between morphisms in $\widehat{\mathcal{S}}_d$, which is the key point.

The diagrammatic Bott-Samelson category $\widehat{\mathcal{BS}}_d$ is equivalent to the algebraic category of Bott-Samelson bimodules and bimodule maps and the diagrammatic Soergel category $\widehat{\mathcal{S}}_d$ is equivalent to the algebraic category of Soergel bimodules and degree-preserving bimodule maps, see [EW2016, Theorem 6.28]. For convenience, we will therefore denote the objects of $\widehat{\mathcal{BS}}_d$ by $\mathrm{B}_{\underline{w}} = \mathrm{B}_{s_{i_1}} \cdots \mathrm{B}_{s_{i_\ell}}$, where $\underline{w} = s_{i_1} \cdots s_{i_\ell}$ is a finite word in the alphabet S. In particular, the monoidal product is given by $\mathrm{B}_{\underline{u}}\mathrm{B}_{\underline{v}} = \mathrm{B}_{\underline{uv}}$, where $\underline{uv}$ is the concatenation of the words $\underline{u}$ and $\underline{v}$.

Let us also recall the so-called *Categorification Theorem*, due to Soergel in finite type A, to Härterich [Har1999] in affine type A and to Elias–Williamson [EW2014, EW2016] in general Coxeter type.

Theorem 3.3. *For any $w \in \widehat{\mathcal{S}}_d$ and rex $\underline{w} = s_{i_1} \cdots s_{i_\ell}$ of w, there is an indecomposable object $\mathrm{B}_w \in \widehat{\mathcal{S}}_d$, independent of the choice of rex, such that*

$$\mathrm{B}_{\underline{w}} \cong \mathrm{B}_w \oplus \bigoplus_{u \prec w} \mathrm{B}_u^{\oplus h_{w,u}},$$

where $\prec$ is the Bruhat order in $\widehat{\mathcal{S}}_d$ and $h_{w,u} \in \mathbb{N}[q, q^{-1}]$ is the graded multiplicity of B_u in the decomposition of $\mathrm{B}_{\underline{w}}$.

Moreover, the $\mathbb{Z}[q, q^{-1}]$-linear map

$$\widehat{H}_d^{\mathbb{Z}[q,q^{-1}]} \to [\widehat{\mathcal{S}}_d]_{\oplus}$$
$$b_w \mapsto \mathrm{B}_w, \quad w \in \widehat{\mathcal{S}}_d$$

is an isomorphism of algebras, where $\widehat{H}_d^{\mathbb{Z}[q,q^{-1}]}$ is the integral form of $\widehat{H}_d$.

Let $\widehat{\mathcal{S}}_d^{\mathrm{gr}}$ be the graded monoidal category associated to $\widehat{\mathcal{S}}_d$, see Section 3.1. For every $u, v \in \widehat{\mathcal{S}}$, the graded Hom-space

$$\widehat{\mathcal{S}}_d^{\mathrm{gr}}(\mathrm{B}_u, \mathrm{B}_v) = \bigoplus_{t \in \mathbb{Z}} \widehat{\mathcal{S}}_d(\mathrm{B}_u, \mathrm{B}_v\langle t \rangle)$$

is a free left (or right) graded R-module of finite graded rank, given by *Soergel's Hom-formula*:

$$\mathrm{grk}_R\left(\widehat{\mathcal{S}}_d^{\mathrm{gr}}(\mathrm{B}_u, \mathrm{B}_v)\right) = (b_u, b_v), \tag{50}$$

where $(-,-)$ is the well-known sesquilinear form on $\widehat{H}_d$, see e.g. [EW2016, Section 2.4 and Theorem 3.15].

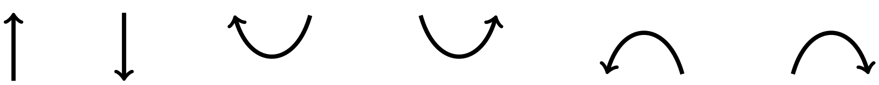
Definition 3.4. The diagrammatic Bott-Samelson category and the diagrammatic Soergel category of finite type A_{d-1}, denoted $\mathcal{BS}_d$ and $\mathcal{S}_d$ respectively, are defined as $\widehat{\mathcal{BS}}_d$ and $\widehat{\mathcal{S}}_d$ but only using the colors I.

Note that $\mathcal{BS}_d$ and $\mathcal{S}_d$ are monoidal subcategories of $\widehat{\mathcal{BS}}_d$ and $\widehat{\mathcal{S}}_d$, respectively, but that the natural embeddings are not full because e.g. the 0-colored dumbbell is not a morphism in $\mathcal{BS}_d$ and $\mathcal{S}_d$.

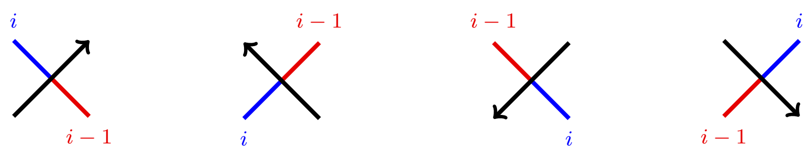
3.3. **Soergel calculus in extended affine type A.** In this subsection we briefly sketch how to enhance $\widehat{\mathcal{BS}}_d$ and $\widehat{\mathcal{S}}_d$ to get the extended diagrammatic Soergel category of type $\widehat{A}_{d-1}$, denoted $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ and $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$, which were introduced in [MT2017] and further studied in [E2018]. We refer to those two papers for more details.

The objects of $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ are formal direct sums of words in the alphabet $S \cup \{\rho, \rho^{-1}\}$. Because of the link with algebraic bimodules, we write B_ρ^n for ρ^n, for any $n \in \mathbb{Z}$.

There are also new generating diagrams, all of degree zero, involving oriented strands. The generators involving only oriented strands are

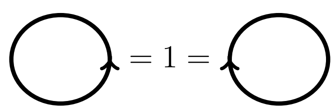
(51) 

and the generating diagrams involving oriented strands and adjacent colored strands are

(52) 

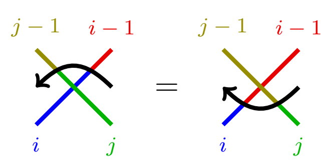
The new morphisms satisfy the following relations, where we again assume isotopy invariance and cyclicity.

- Relations involving only oriented strands:

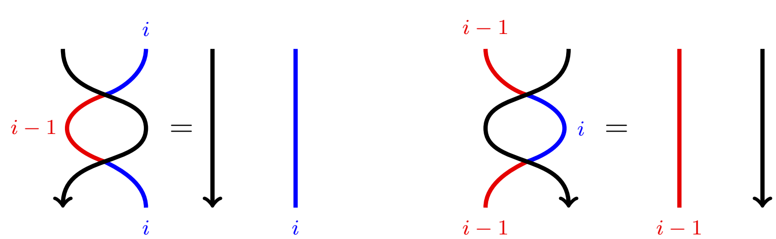
(53) 

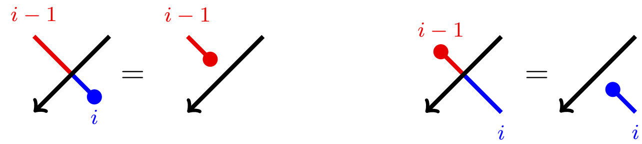
(54) 

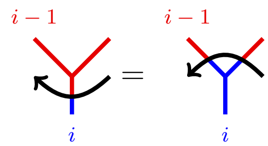
- Relation involving oriented strands and distant colored strands:

(55) 

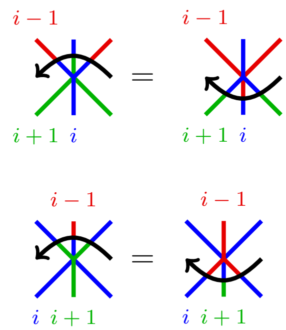
- Relations involving oriented strands and two adjacent colored strands:

(56) 

(57)


(58)


- Relations involving oriented strands and three adjacent colored strands:

(59)


By relations (57), the sum of all colored dumbbells in (49) also commutes with oriented strands, so the corresponding morphism is also central in $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$.

In general, any object in $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ is isomorphic to a direct sum of objects of the form $\mathrm{B}_\rho^n \mathrm{B}_{\underline{w}}$, for some $n \in \mathbb{Z}$ and word $\underline{w}$ in S. By the relations in (53), there is an isomorphism of vector spaces (and of algebras)

$$\left(\widehat{\mathcal{BS}}_d^{\mathrm{ext}}\right)^0 \left(\mathrm{B}_\rho^m, \mathrm{B}_\rho^n\right) \cong \begin{cases} \mathbb{C}\mathrm{id}_{\mathrm{B}_\rho^m}, & \text{if } m = n; \\ \{0\}, & \text{else.} \end{cases}$$

Recall that $R = \widehat{\mathcal{BS}}(\varnothing, \varnothing)$ is the polynomial algebra in the colored dumbbells. Then the isomorphism above generalizes to an isomorphism of graded R-R-bimodules

$$\widehat{\mathcal{BS}}_d^{\mathrm{ext}} \left(\mathrm{B}_\rho^m, \mathrm{B}_\rho^n\right) \cong \begin{cases} R^{\tau^m}, & \text{if } m = n; \\ \{0\}, & \text{else,} \end{cases}$$

where τ is the automorphism of R which sends the i-colored dumbbell to the $i+1$-colored dumbbell, for any $i \in \widehat{I}$, and R^{τ^m} is the free rank-one R-R-bimodule with the normal left R-action and the right R-action twisted by τ^m.

Moreover, the black oriented part and the non-oriented colored part of any diagram can be separated by the above relations, resulting in an isomorphism of graded R-R-bimodules

$$\widehat{\mathcal{BS}}_d^{\mathrm{ext}} \left(\mathrm{B}_\rho^m \mathrm{B}_{\underline{u}}, \mathrm{B}_\rho^m \mathrm{B}_{\underline{v}}\right) \cong \begin{cases} R^{\tau^m} \otimes_R \widehat{\mathcal{BS}}_d \left(\mathrm{B}_{\underline{u}}, \mathrm{B}_{\underline{v}}\right), & \text{if } m = n; \\ \{0\}, & \text{else.} \end{cases}$$

In particular, this implies that the natural embedding $\widehat{\mathcal{BS}}_d \hookrightarrow \widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ is full. For the proofs of these results, see [E2018, Section 3.3].

Definition 3.5. The *extended diagrammatic Soergel category* $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$ is the idempotent completion of $\left(\widehat{\mathcal{BS}}_d^{\mathrm{ext}}\right)^{\mathrm{sh}}$.

The above results on the Hom-spaces in $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ and Theorem 3.3 imply the following generalization to the extended case, see [MT2017, Theorem 2.5].

Theorem 3.6. *For any $n \in \mathbb{Z}$ and $w \in \widehat{\mathcal{S}}_d$, the object $\mathrm{B}_\rho^n \mathrm{B}_w \in \widehat{\mathcal{S}}_d^{\mathrm{ext}}$ is indecomposable. Moreover, the $\mathbb{Z}[q, q^{-1}]$-linear map*

$$\left(\widehat{H}_d^{ext}\right)^{\mathbb{Z}[q,q^{-1}]} \to [\widehat{\mathcal{S}}_d^{\mathrm{ext}}]_{\oplus}$$
$$\rho^n b_w \mapsto \mathrm{B}_\rho^n \mathrm{B}_w, \quad n \in \mathbb{Z}, w \in \widehat{\mathcal{S}}_d$$

is an isomorphism of algebras.

4. Rouquier complexes

For $\mathcal{A}$ a $\mathbb{C}$-linear, additive category, we write $\mathcal{K}^b(\mathcal{A})$ for the homotopy category of bounded complexes in $\mathcal{A}$. If $\mathcal{A}$ is monoidal, then the usual monoidal product of chain complexes equips $\mathcal{K}^b(\mathcal{A})$ with a monoidal structure as well. If $\mathcal{A}$ is graded, then $\mathcal{K}^b(\mathcal{A})$ is bigraded and we denote the shift inherited from $\mathcal{A}$ by $\langle \cdot \rangle$ and the homological shift by $[\cdot]$.

Remark 4.1. Throughout this section, we sometimes state and prove diagrammatic equations in $\mathcal{K}^b(\mathcal{BS}_d)$, instead of $\mathcal{K}^b(\mathcal{S}_d)$. This makes no real difference in our case, as the differentials of the complexes in $\mathcal{K}^b(\mathcal{BS}_d)$ below are always given by matrices of homogeneous diagrams of the same degree, so they always give rise to objects in $\mathcal{K}^b(\mathcal{S}_d)$. See also Remark 3.2.

Let $\mathcal{C} = \mathcal{S}_d$. For the simple reflection $s_i \in W$ the *Rouquier complex* $\mathrm{T}_i := \mathrm{T}_{s_i} \in \mathcal{K}^b(\mathcal{S}_d)$ is defined by

$$\mathrm{T}_i := \underline{\mathrm{B}_i} \longrightarrow R\langle 1 \rangle, \tag{60}$$

with B_i placed in homological degree zero (we always underline terms in homological degree zero). This complex is invertible in $\mathcal{K}^b(\mathcal{S}_d)$, with inverse given by

$$\mathrm{T}_i^{-1} := R\langle -1 \rangle \longrightarrow \underline{\mathrm{B}_i}, \tag{61}$$

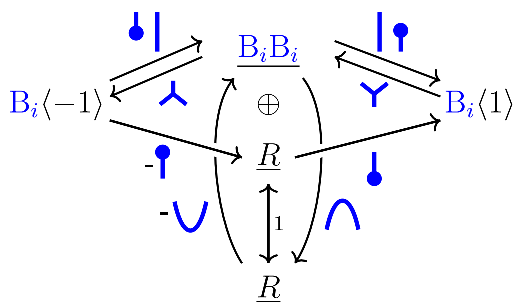
as follows from the homotopy equivalences which we recall below. These complexes were introduced in [Rou2006] and categorify the usual generators of the braid group, in particular, they satisfy the braid relations up to homotopy equivalence [Rou2006, Theorem 3.2]. By Matsumoto's theorem, this implies that, for any $w \in S_n$, the complex T_w can be defined as

$$\mathrm{T}_w := \mathrm{T}_{i_1} \cdots \mathrm{T}_{i_\ell}, \tag{62}$$

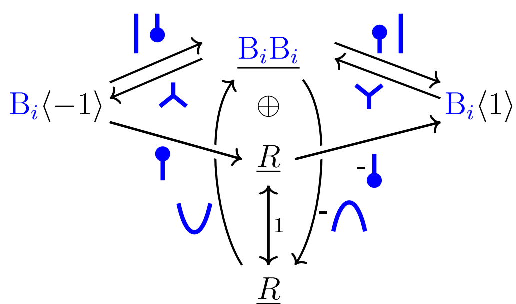
where $\underline{w} = s_{i_1} \cdots s_{i_\ell}$ is any rex of w (i.e., up to homotopy equivalence, the complex does not depend on the choice of rex).

In subsection 4.1, we briefly recall the results on Rouquier complexes that are relevant for the definition of the evaluation functor. For more details, see [Rou2006], [EKr2010, §3] and [EMTW2020, Chapter 19]. In Subsection 4.2, we introduce a special Rouquier complex, denoted T_ρ, and develop a diagrammatic calculus for morphisms in $\mathcal{K}^b(\mathcal{S}_d)$ whose source and/or target contain tensor powers of T_ρ and T_ρ^{-1}. To the best of our knowledge, this extension of Soergel calculus has not appeared in the literature before.

4.1. **Some diagrammatic shortcuts I: general Rouquier complexes.** For $i \in I$, let $\phi_i \colon \mathrm{T}_i^{-1}\mathrm{T}_i \to R$ denote the homotopy equivalence (where 1 stands for the identity map)

(63)


and $\psi_i \colon \mathrm{T}_i\mathrm{T}_i^{-1} \to R$ the analogous homotopy equivalence

(64)


in $\mathcal{K}^b(\mathcal{S}_d)$. These maps are well-known, see e.g. [EKr2010, §3].

Remark 4.2. The backward arrows (from right to left) in (63) and (64) indicate the homotopies which prove that the composites of the downward arrows followed by the upward arrows are homotopic to the identity on $\mathrm{T}_i^{-1}\mathrm{T}_i$ and $\mathrm{T}_i\mathrm{T}_i^{-1}$, respectively. Throughout the paper, we will use backward arrows to indicate homotopies.

Let further $\eta_{i,\pm} \colon \mathrm{T}_i^{\pm 1}\mathrm{T}_i^{\mp 1} \to \mathrm{T}_i^{\pm 1} R \mathrm{T}_i^{\mp 1}$ be the canonical isomorphisms $\mathcal{K}^b(\mathcal{S}_d)$, for any $i \in I$, both given by $ab \mapsto a1b$. To simplify notation, we write 1^m for $1 \cdots 1$ (m times) in the sequel.

The following is the Movie Move MM2 in [EKr2010, §3].

Lemma 4.3. *For any $i \in I$, the composite maps*

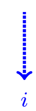
$$\mathrm{T}_i \xrightarrow{a\mapsto 1a} R\mathrm{T}_i \xrightarrow{\psi_i^{-1}\,\mathrm{Id}_{\mathrm{T}_i}} \mathrm{T}_i\mathrm{T}_i^{-1}\mathrm{T}_i \xrightarrow{\mathrm{Id}_{\mathrm{T}_i}\,\phi_i} \mathrm{T}_iR \xrightarrow{ab\mapsto ab} \mathrm{T}_i,$$
$$\mathrm{T}_i \xrightarrow{a\mapsto a1} \mathrm{T}_iR \xrightarrow{\mathrm{Id}_{\mathrm{T}_i}\,\phi_i^{-1}} \mathrm{T}_i\mathrm{T}_i^{-1}\mathrm{T}_i \xrightarrow{\psi_i\,\mathrm{Id}_{\mathrm{T}_i}} R\mathrm{T}_i \xrightarrow{ba\mapsto ba} \mathrm{T}_i,$$

are both equal to $\mathrm{Id}_{\mathrm{T}_i}$ in $\mathcal{K}^b(\mathcal{S}_d)$, and the composite maps

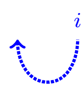
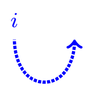
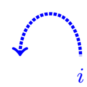
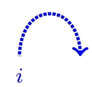
$$\mathrm{T}_i^{-1} \xrightarrow{a\mapsto 1a} R\mathrm{T}_i^{-1} \xrightarrow{\phi_i^{-1}\,\mathrm{Id}_{\mathrm{T}_i}^{-1}} \mathrm{T}_i^{-1}\mathrm{T}_i\mathrm{T}_i^{-1} \xrightarrow{\mathrm{Id}_{\mathrm{T}_i}^{-1}\,\psi_i} \mathrm{T}_i^{-1}R \xrightarrow{ab\mapsto ab} \mathrm{T}_i^{-1},$$
$$\mathrm{T}_i^{-1} \xrightarrow{a\mapsto a1} \mathrm{T}_i^{-1}R \xrightarrow{\mathrm{Id}_{\mathrm{T}_i}^{-1}\,\psi_i^{-1}} \mathrm{T}_i^{-1}\mathrm{T}_i\mathrm{T}_i^{-1} \xrightarrow{\phi_i\,\mathrm{Id}_{\mathrm{T}_i}^{-1}} R\mathrm{T}_i^{-1} \xrightarrow{ba\mapsto ba} \mathrm{T}_i^{-1},$$

are both equal to $\mathrm{Id}_{\mathrm{T}_i^{-1}}$ in $\mathcal{K}^b(\mathcal{S}_d)$.

We now introduce the diagrammatics for the maps involving $\mathrm{T}_i^{\pm 1}$ that will be needed in the sequel. For any $i \in I$, we depict the identity morphisms of $\mathrm{T}_i^{\pm 1}$ as

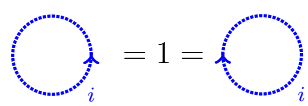
$\mathrm{Id}_{\mathrm{T}_i} :=$  and $\mathrm{Id}_{\mathrm{T}_i^{-1}} :=$ 

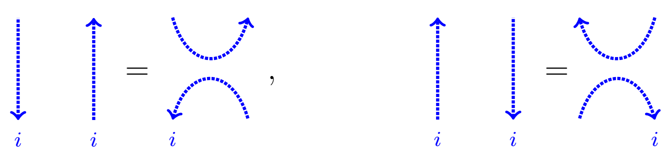
The degree zero homotopy equivalences in (63) and (64) (which are the units and counits of left and right adjunction of T_i and T_i^{-1}) are then depicted as

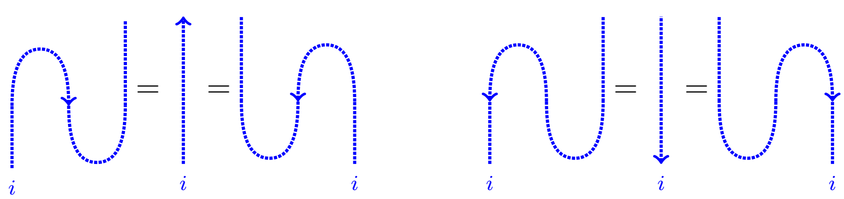
 ,  ,  and 

and the above remarks translate into the following diagrammatic relations.

Lemma 4.4. *For any $i \in I$, we have the following relations between morphisms of $\mathcal{K}^b(\mathcal{S}_d)$:*

(65) 

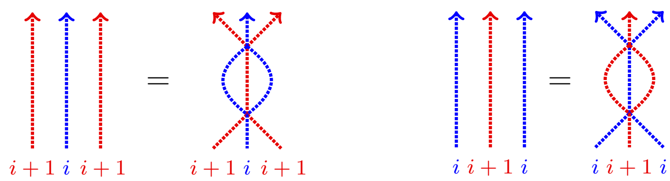
(66) 

(67) 

Remark 4.5. Just for the record, we give some further results about Rouquier complexes, all well-known to experts.

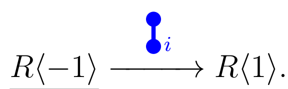
- For any $i \in \{1, \ldots, d-2\}$, the isomorphism between $\mathrm{T}_i\mathrm{T}_{i+1}\mathrm{T}_i$ and $\mathrm{T}_{i+1}\mathrm{T}_i\mathrm{T}_{i+1}$ in $\mathcal{K}^b(\mathcal{S}_d)$ (see [EKr2010, §3] for the maps) can be represented by the degree zero diagrams

satisfying the relations



There are similar diagrams and relations for braid moves involving the inverses of Rouquier complexes, see e.g. [EW2017, §5]. In Remark 4.13 below, we introduce some new diagrams.

- For any $i \in I$, the cone of the map $f\colon \mathrm{T}_i \to \mathrm{T}_i^{-1}$, which is the identity on B_i and zero everywhere else, is isomorphic to



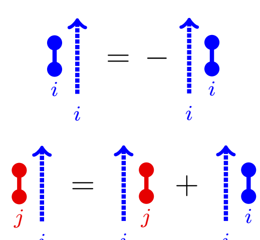
in $\mathcal{K}^b(\mathcal{S}_d)$. The distinguished triangle

$$\mathrm{T}_i^{-1} \to \mathsf{Cone}(f) \to \mathrm{T}_i \to \mathrm{T}_i[1]$$

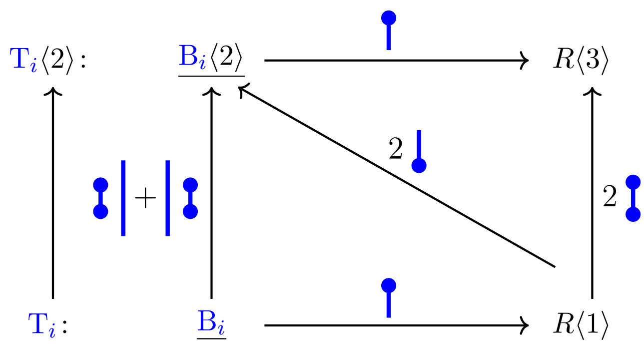
categorifies the quadratic relation in the Hecke algebra H_d.

The remaining lemmas of this subsection are all known to experts and not hard to derive. Some of them can be found in the literature (see e.g. [GH2022]). We give all relevant homotopy equivalences explicitly for completeness. Further, to keep the notation as simple as possible, we state some equations in $\mathcal{K}^b(\mathcal{BS}_d)$. Being homogeneous, they also give rise to equations between morphisms in $\mathcal{K}^b(\mathcal{S}_d)$, as explained in Remark 4.1.

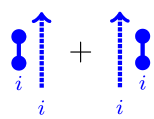
Lemma 4.6. *For any $i, j \in I$ such that $j = i \pm 1$, the following* dumbbell-slide *relations hold in $\mathcal{K}^b(\mathcal{BS}_d)$:*



Proof. We are actually going to prove the equations in $\mathcal{K}^b(\mathcal{S}_d)$, fixing the shifts of the objects. For the first equation, consider

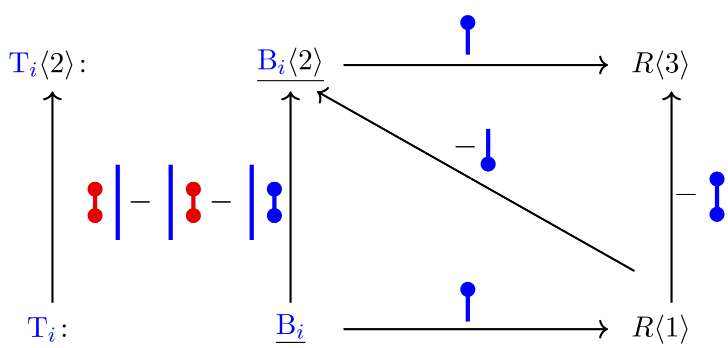


The vertical arrows correspond to the map of complexes represented by

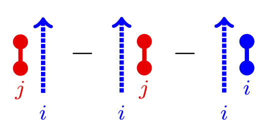


and the diagonal arrow is a homotopy. Using (37), we see that the map of complexes is null-homotopic.

For the second equation, consider



The vertical arrows correspond to the map of complexes represented by



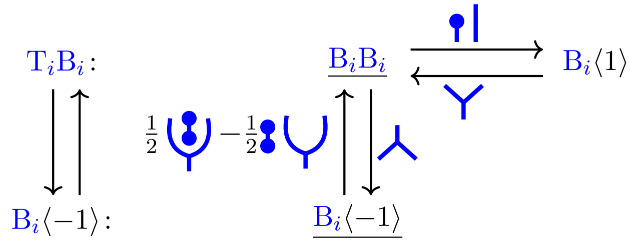
and the diagonal arrow is a homotopy. Using (44), we see that the map of complexes is null-homotopic. ☐

Lemma 4.7. *There is an isomorphism*

$$\mathrm{T}_i^{\pm 1}\mathrm{B}_i\mathrm{T}_i^{\mp 1} \cong \mathrm{B}_i$$

in $\mathcal{K}^b(\mathcal{S}_d)$.

Proof. Recall that $B_iB_i \cong B_i\langle 1\rangle \oplus B_i\langle -1\rangle$ in $\mathcal{S}_d$. Using that isomorphism, it is easy to see that $T_iB_i \cong B_i\langle -1\rangle$ in $\mathcal{K}^b(\mathcal{S}_d)$, with the homotopy equivalence between the complexes being given by



An analogous homotopy equivalence shows that $B_iT_i \cong B_i\langle -1\rangle$ in $\mathcal{K}^b(\mathcal{S}_d)$ and thus that $T_iB_i \cong B_iT_i$ in $\mathcal{K}^b(\mathcal{S}_d)$.

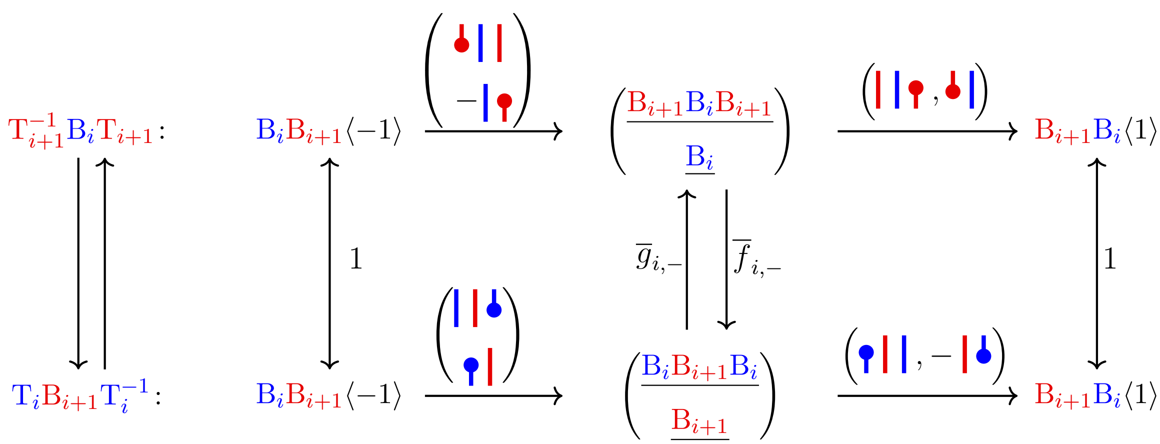
Of course, the above also implies that $B_iT_i^{-1} \cong B_i\langle 1\rangle \cong T_i^{-1}B_i$ in $\mathcal{K}^b(\mathcal{S}_d)$. □

Lemma 4.8. *For each $1 \leq i \leq d-2$, there are isomorphisms*

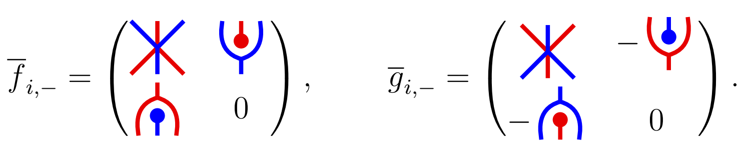
$$f_{i,\pm}\colon T_{i+1}^{\pm 1}B_iT_{i+1}^{\mp 1} \to T_i^{\mp 1}B_{i+1}T_i^{\pm 1}$$

in $\mathcal{K}^b(\mathcal{S}_d)$.

Proof. In this case, the complexes are actually isomorphic, not just homotopy equivalent. In the following figure, we exhibit the isomorphism $f_{i,-}\colon T_{i+1}^{-1}B_iT_{i+1} \to T_iB_{i+1}T_i^{-1}$ and its inverse $g_{i,-}$ (to avoid cluttering, we do not write labels in diagrams if they are clear from context):



Here $\overline{f}_{i,-}$ and $\overline{g}_{i,-}$ are, respectively,



The maps $f_{i,-} = (1, \overline{f}_{i,-}, 1)$ and $g_{i,-} = (1, \overline{g}_{i,-}, 1)$ are mutual inverses and a pleasant exercise, using the relation in (41), shows that both of them are chain maps. The complexes $T_{i+1}B_iT_{i+1}^{-1}$

and $\mathrm{T}_i^{-1}\mathrm{B}_{i+1}\mathrm{T}_i$ are isomorphic too, as they are adjoint to $\mathrm{T}_{i+1}^{-1}\mathrm{B}_i\mathrm{T}_{i+1}$ and $\mathrm{T}_i\mathrm{B}_{i+1}\mathrm{T}_i^{-1}$, respectively. Similarly, we obtain the isomorphism $f_{i,+}\colon \mathrm{T}_{i+1}\mathrm{B}_i\mathrm{T}_{i+1}^{-1} \to \mathrm{T}_i^{-1}\mathrm{B}_{i+1}\mathrm{T}_i$ and its inverse $g_{i,+}$. □

Recall the homotopy equivalences $\phi_i\colon \mathrm{T}_i^{-1}\mathrm{T}_i \to R$ and $\psi_i\colon \mathrm{T}_i\mathrm{T}_i^{-1} \to R$ and put $\delta_{i,+} := \phi_i^{-1} \circ \psi_{i+1}$ and $\delta_{i,-} := \psi_i^{-1} \circ \phi_{i+1}$ (we suppress the maps $\eta_{i,\pm}$ whenever we use the diagrams $\overset{|}{\bullet}$ and $\underset{|}{\bullet}$). Below, we keep the notation from Lemma 4.8.

Lemma 4.9. *For each $1 \leq i \leq d-2$, the following maps are equal to zero in $\mathcal{K}^b(\mathcal{S}_d)$:*

(68) $$f_{i,\pm} \circ \big(\mathrm{Id}_{\mathrm{T}_{i+1}^{\pm1}} \overset{|}{\underset{i}{\bullet}} \mathrm{Id}_{\mathrm{T}_{i+1}^{\mp1}}\big) - \big(\mathrm{Id}_{\mathrm{T}_i^{\mp1}} \overset{|}{\underset{i+1}{\bullet}} \mathrm{Id}_{\mathrm{T}_i^{\pm1}}\big) \circ \delta_{i,\pm}\colon \mathrm{T}_{i+1}^{\pm1}\mathrm{T}_{i+1}^{\mp1} \to \mathrm{T}_i^{\mp1}\mathrm{B}_{i+1}\mathrm{T}_i^{\pm1}\langle 1\rangle,$$

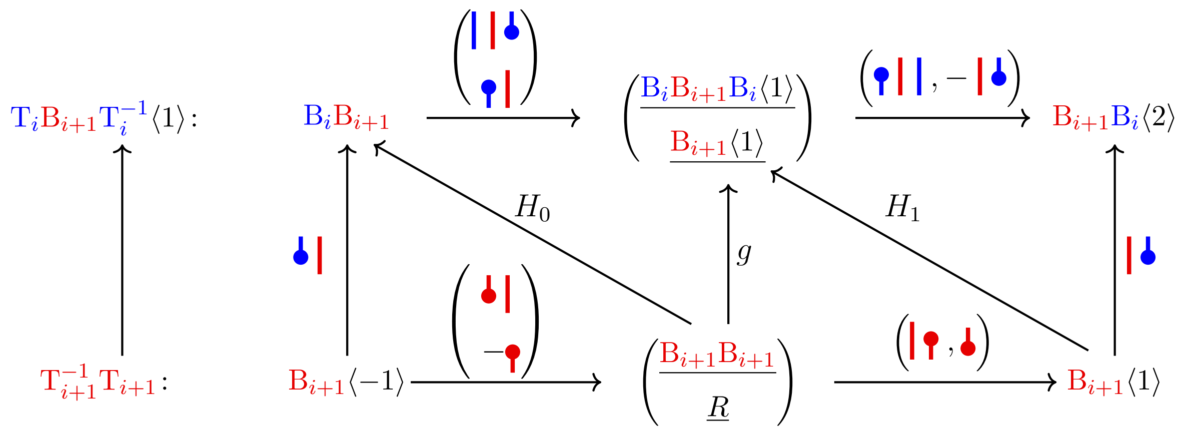
(69) $$\big(\mathrm{Id}_{\mathrm{T}_i^{\mp1}} \underset{i+1}{\overset{\bullet}{|}} \mathrm{Id}_{\mathrm{T}_i^{\pm1}}\big) \circ f_{i,\pm} - \delta_{i,\pm} \circ \big(\mathrm{Id}_{\mathrm{T}_{i+1}^{\pm1}} \underset{i}{\overset{\bullet}{|}} \mathrm{Id}_{\mathrm{T}_{i+1}^{\mp1}}\big)\colon \mathrm{T}_{i+1}^{\pm1}\mathrm{B}_i\mathrm{T}_{i+1}^{\mp1}\langle -1\rangle \to \mathrm{T}_i^{\mp1}\mathrm{T}_i^{\pm1},$$

(70) $$f_{i,\pm}^{-1} \circ \big(\mathrm{Id}_{\mathrm{T}_i^{\mp1}} \overset{|}{\underset{i+1}{\bullet}} \mathrm{Id}_{\mathrm{T}_i^{\pm1}}\big) - \big(\mathrm{Id}_{\mathrm{T}_{i+1}^{\pm1}} \overset{|}{\underset{i}{\bullet}} \mathrm{Id}_{\mathrm{T}_{i+1}^{\mp1}}\big) \circ \delta_{i,\pm}^{-1}\colon \mathrm{T}_i^{\mp1}\mathrm{T}_i^{\pm1} \to \mathrm{T}_i^{\pm1}\mathrm{B}_i\mathrm{T}_i^{\mp1}\langle 1\rangle,$$

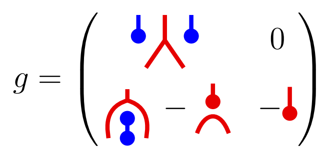
(71) $$\big(\mathrm{Id}_{\mathrm{T}_{i+1}^{\pm1}} \underset{i}{\overset{\bullet}{|}} \mathrm{Id}_{\mathrm{T}_{i+1}^{\mp1}}\big) \circ f_{i,\pm}^{-1} - \delta_{i,\pm}^{-1} \circ \big(\mathrm{Id}_{\mathrm{T}_i^{\mp1}} \underset{i+1}{\overset{\bullet}{|}} \mathrm{Id}_{\mathrm{T}_i^{\pm1}}\big)\colon \mathrm{T}_i^{\mp1}\mathrm{B}_{i+1}\mathrm{T}_i^{\pm1}\langle -1\rangle \to \mathrm{T}_{i+1}^{\pm1}\mathrm{T}_{i+1}^{\mp1}.$$

Proof. We only need to prove that the maps in (68) and (69) are null-homotopic for $f_{i,-}$ and $\delta_{i,-}$. Pre- and post-composing those two maps with the appropriate isomorphisms proves the analogous statement for the maps in (70) and (71) as well. Note further that $f_{i,+}^{-1}$ and $f_{i,-}$ become equal after switching i and $i+1$, and so do $\delta_{i,+}^{-1}$ and $\delta_{i,-}$. Since the two-color Soergel calculus relations are invariant under switching the two colors, the relations in this lemma hold for the pairs $(f_{i,-}^{\pm1}, \delta_{i,-}^{\pm1})$ if and only if they hold for the pairs $(f_{i,+}^{\mp1}, \delta_{i,+}^{\mp1})$.

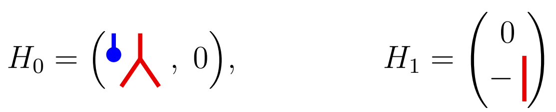
Let us, therefore, prove the first two equations for $f_{i,-}$ and $\delta_{i,-}$. It is not hard to compute that the map of complexes in (68) is given by the vertical arrows in the diagram below:



where

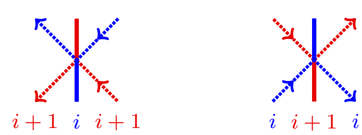


It is also easy to check that this map is null-homotopic, with homotopies

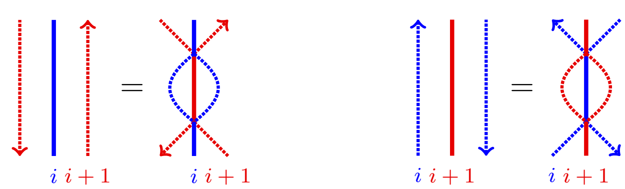


This establishes (68). The proof of (69) can be obtained by a vertical reflexion of the diagrams above and exchanging the labels i and $i+1$. ☐

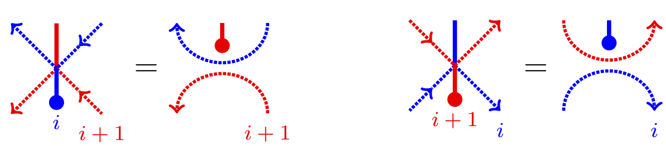
Remark 4.10. The isomorphisms in Lemma 4.8 have a diagrammatic interpretation in terms of degree zero generators in $\mathcal{K}^b(\mathcal{BS}_d)$

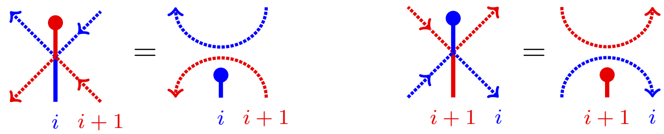


and relations



Using these diagrams, Lemma 4.9 translates into





There are also analogous diagrams and relations with reversed orientation of the oriented strands.

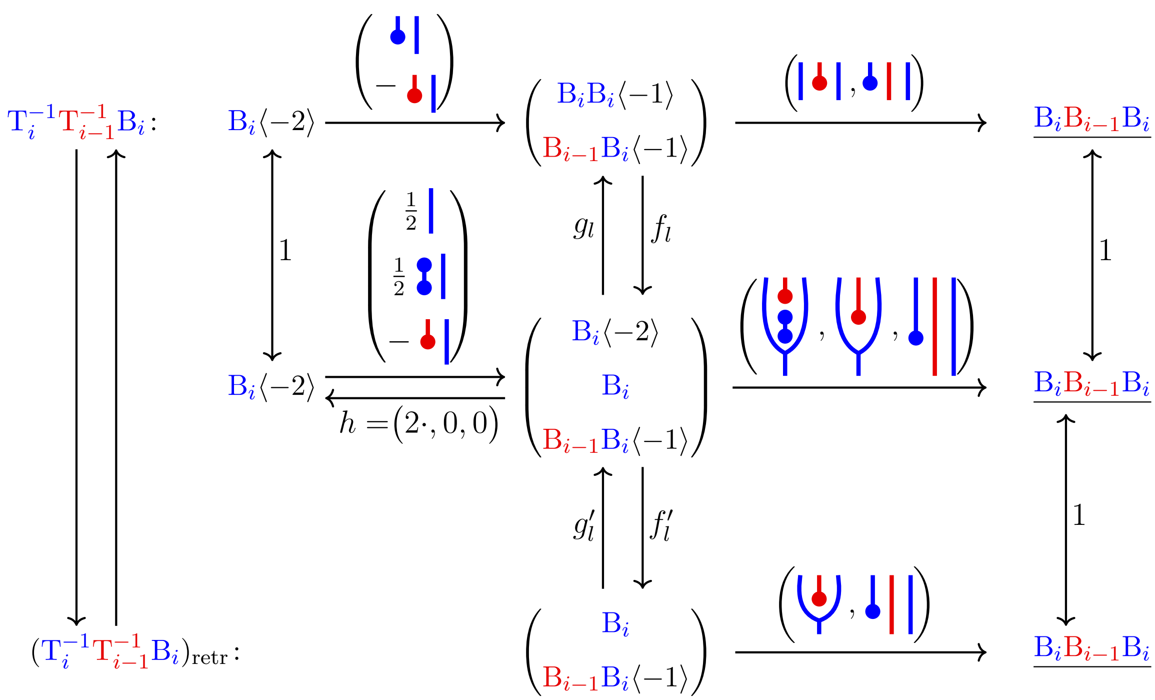
Lemma 4.11. *There exist the following isomorphisms in* $\mathcal{K}^b(\mathcal{S}_d)$*:*

(72) $\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}\mathrm{B}_i \cong \mathrm{B}_{i-1}\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}, \qquad i \in I,$

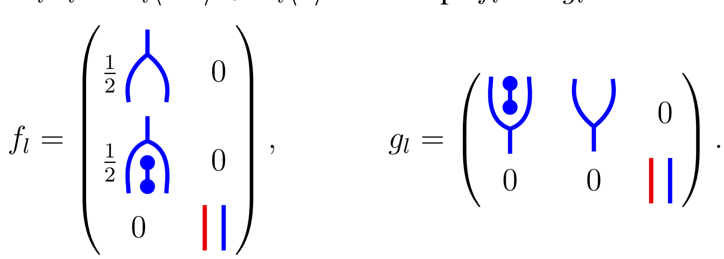
(73) $\mathrm{T}_{i-1}\mathrm{T}_i\mathrm{B}_{i-1} \cong \mathrm{B}_i\mathrm{T}_{i-1}\mathrm{T}_i, \qquad i \in I,$

(74) $\mathrm{T}_i^{\pm 1}\mathrm{T}_{i-1}^{\pm 1}\mathrm{B}_j \cong \mathrm{B}_j\mathrm{T}_i^{\pm 1}\mathrm{T}_{i-1}^{\pm 1}, \qquad j \notin \{i-2, i-1, i, i+1\}.$

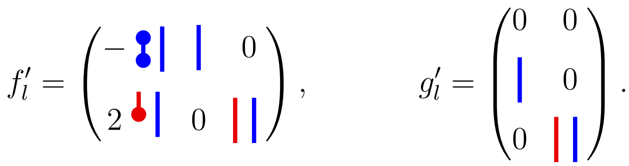
Proof. We start with (72), proving that both chain complexes have retractions that are homotopy equivalent to each other. Here is a homotopy equivalence between the complex $\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}\mathrm{B}_i$ and its retraction $(\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}\mathrm{B}_i)_{\mathrm{retr}}$:



The upper vertical arrows correspond to the mutually inverse maps $(1, f_l, 1)$ and $(1, g_l, 1)$ induced by the isomorphism $\mathrm{B}_i\mathrm{B}_i \cong \mathrm{B}_i\langle -1\rangle \oplus \mathrm{B}_i\langle 1\rangle$. The maps f_l and g_l are

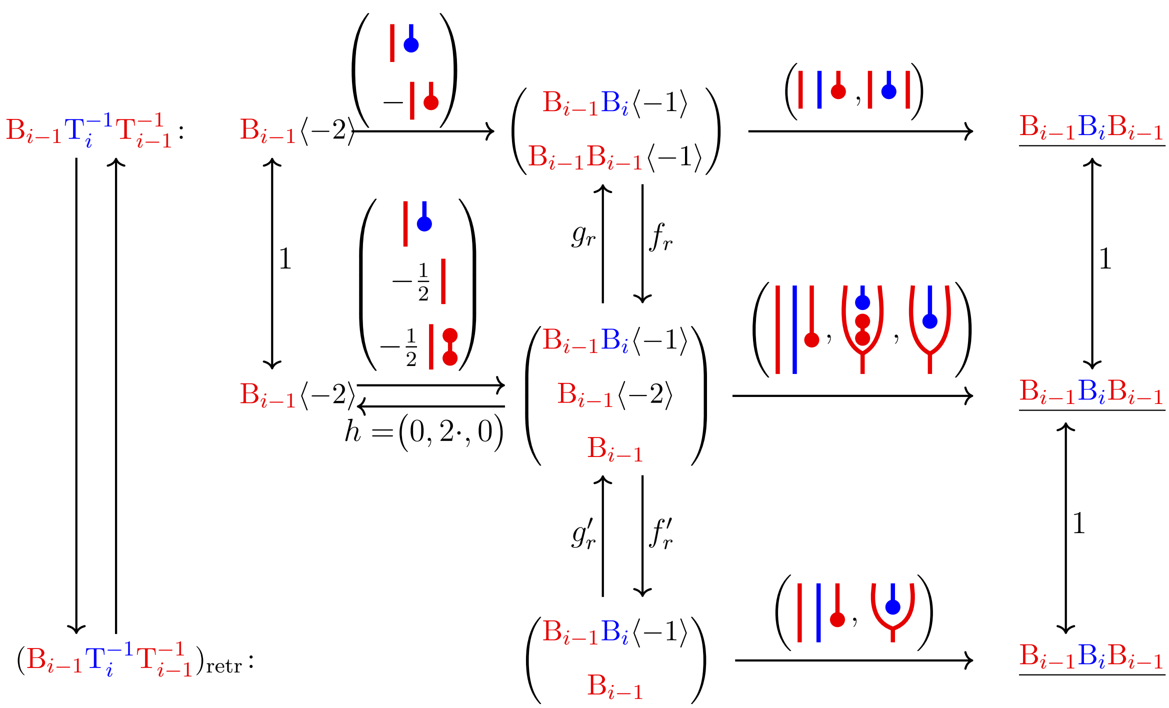


The lower vertical maps are the mutually up-to-homotopy inverse maps $(f'_l, 1)$ and $(g'_l, 1)$ given below.

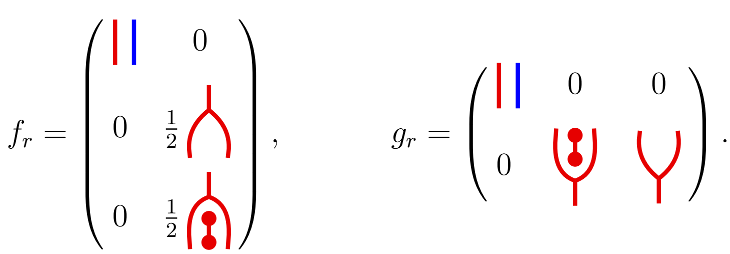


The fact that they define a homotopy equivalence uses the homotopy h (whose only non-zero entry is multiplication by 2) in the complex in the middle. We leave the details to the reader.

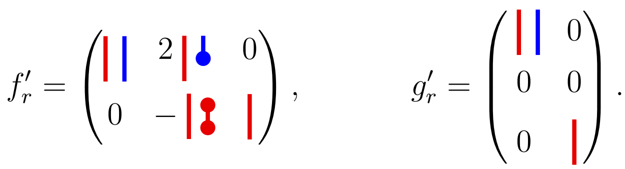
The following diagram gives a homotopy equivalence between the complex $\mathrm{B}_{i-1}\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}$ and its retraction $(\mathrm{B}_{i-1}\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1})_{\mathrm{retr}}$:



The upper vertical arrows correspond to the mutually inverse maps $(1, f_r, 1)$ and $(1, g_r, 1)$, with f_r and g_r being

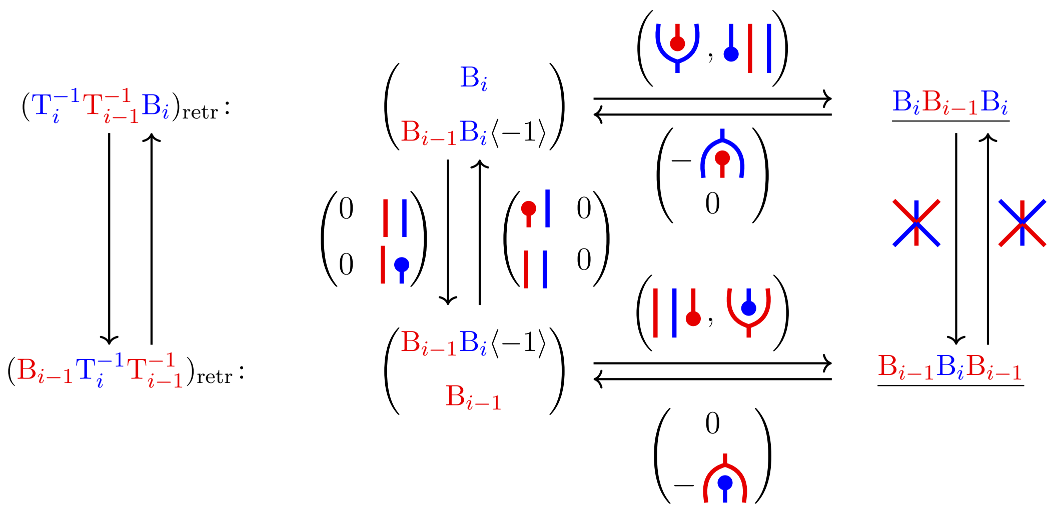


The lower vertical arrows correspond to the mutually up-to-homotopy inverse maps $(f'_r, 1)$ and $(g'_r, 1)$ given below.



We leave the details to the reader.

The diagram below shows that the complexes $(\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1}\mathrm{B}_i)_{\mathrm{retr}}$ and $(\mathrm{B}_{i-1}\mathrm{T}_i^{-1}\mathrm{T}_{i-1}^{-1})_{\mathrm{retr}}$ are homotopy equivalent.

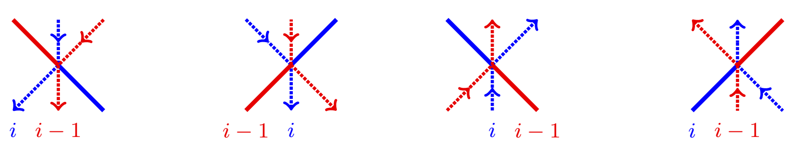


This finishes the proof of the existence of the isomorphism in (72).

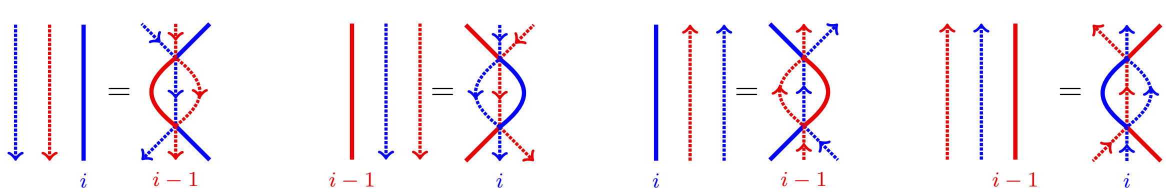
Tensoring both complexes in (72) with $\mathrm{T}_{i-1}\mathrm{T}_i$ on the left and on the right yields the isomorphism in (73).

The equivalence in (74) is clear, because the two complexes are canonically isomorphic. □

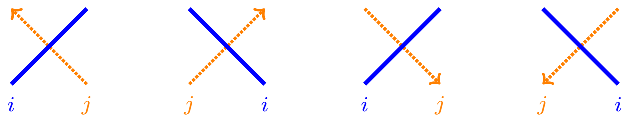
Remark 4.12. The isomorphisms in Lemma 4.11 also have a diagrammatic interpretation in terms of degree zero generators in $\mathcal{K}^b(\mathcal{BS}_d)$



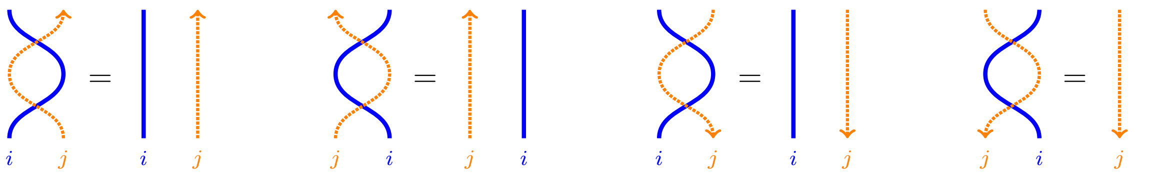
and relations



Remark 4.13. The following will not be used in the sequel. The canonical isomorphisms $\mathrm{B}_i\mathrm{T}_j^{\pm 1} \cong \mathrm{T}_j^{\pm 1}\mathrm{B}_i$, for distant i and j, translate into the generators

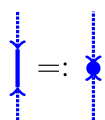


satisfying the relations



There are also maps $\mathrm{B}_i \to \mathrm{T}_i^{-1}$, $\mathrm{T}_i \to \mathrm{B}_i$, $R \to \mathrm{T}_i$ and $\mathrm{T}_i^{-1} \to R$ of non-zero degree in $\mathcal{K}^b(\mathcal{BS}_d)$, depicted respectively as

and satisfying certain diagrammatic relations, which are easy to deduce. Note also that the composite



is the map $\mathrm{T}_i^{-1} \to \mathrm{T}_i$ mentioned in Remark 4.5.

4.2. **Some diagrammatic shortcuts II: special Rouquier complexes $\mathrm{T}_\rho^{\pm 1}$.** In this subsection, we introduce and study a special Rouquier complex, denoted T_ρ, which will play an important role in the definition of the evaluation functors.

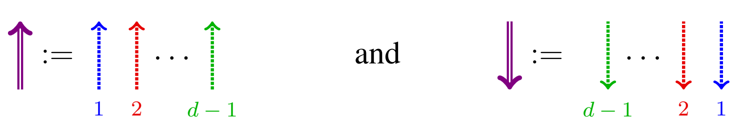
Definition 4.14. Define

$$\mathrm{T}_\rho := \mathrm{T}_1 \cdots \mathrm{T}_{d-1} \quad \text{and} \quad \mathrm{T}_\rho^{-1} := \mathrm{T}_{d-1}^{-1} \cdots \mathrm{T}_1^{-1}$$

in $\mathcal{K}^b(\mathcal{S}_d)$.

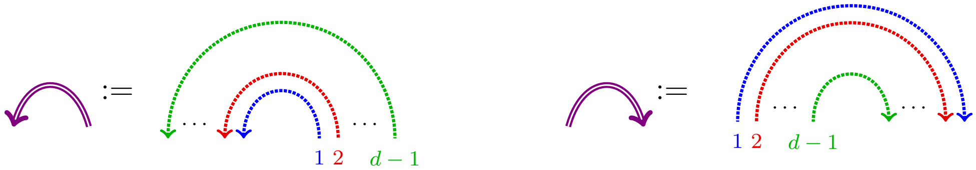
In order to develop a diagrammatic calculus for these special Rouquier complexes, we first picture the *identity morphisms* of T_ρ and T_ρ^{-1} as upward and downward oriented arrows, respectively:

(75)



Further, we introduce *oriented cups and caps*

(76) 

(77) 

These correspond to the units and counits of left and right adjunction for T_ρ and T_ρ^{-1} in $\mathcal{K}^b(\mathcal{S}_d)$. Algebraically, they can be expressed in terms of the maps given in Section 4.1:

(78)  $:= (1^{d-2}\psi_{d-1}^{-1}1^{d-2}) \circ (1^{d-3}\eta_{d-2,+}\psi_{d-2}^{-1}1^{d-3}) \circ \cdots \circ (1\eta_{2,+}\psi_2^{-1}1)$

$\circ\, (\eta_{1,+}\psi_1^{-1})\colon R \to \mathrm{T}_\rho\mathrm{T}_\rho^{-1},$

(79)  $:= (1^{d-2}\phi_1^{-1}1^{d-2}) \circ (1^{d-3}\eta_{2,-}\phi_2^{-1}1^{d-3}) \circ \cdots \circ (1\eta_{d-2,-}\phi_{d-2}^{-1}1)$

$\circ\, (\eta_{d-1,-}\phi_{d-1}^{-1})\colon R \to \mathrm{T}_\rho^{-1}\mathrm{T}_\rho,$

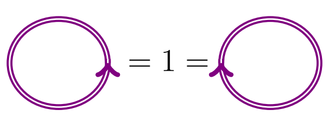
(80)  $:= (\phi_1\eta_{1,-}^{-1}) \circ (1\phi_2\eta_{2,-}^{-1}1) \circ \cdots \circ (1^{d-3}\phi_2\eta_{2,-}^{-1}1^{d-3})$

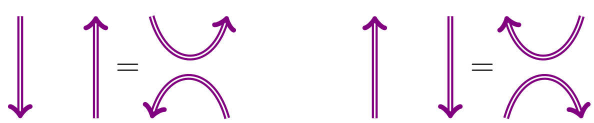
$\circ\, (1^{d-2}\phi_1 1^{d-2})\colon \mathrm{T}_\rho^{-1}\mathrm{T}_\rho \to R,$

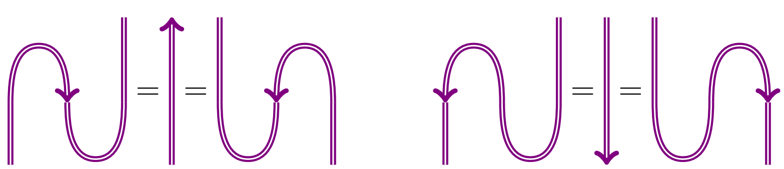
(81)  $:= (\psi_{d-1}\eta_{d-1,+}^{-1}) \circ (1\psi_{d-2}\eta_{d-2,+}^{-1}1) \circ \cdots \circ (1^{d-3}\psi_2\eta_{2,+}^{-1}1^{d-3})$

$\circ\, (1^{d-2}\psi_{d-1}1^{d-2})\colon \mathrm{T}_\rho\mathrm{T}_\rho^{-1} \to R.$

Lemma 4.15. *The oriented cups and caps satisfy the following relations in* $\mathcal{K}^b(\mathcal{S}_d)$

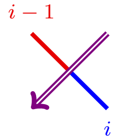
(82) 

(83) 

(84) 

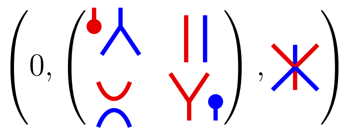
Proof. The relations in (84) are a consequence of Lemma 4.3. The other relations are immediate. □

The next diagrammatic generators involving oriented strands are the *mixed crossings*, which correspond to the following degree-zero isomorphisms in in $\mathcal{K}^b(\mathcal{S}_d)$, for $i \in I$:

(85) 

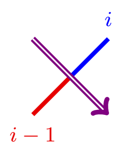
$= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}} F_{i,r}\, \mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}} \colon \mathrm{T}_\rho^{-1}\mathrm{B}_i \to \mathrm{B}_{i-1}\mathrm{T}_\rho^{-1},$

where in homological degrees -2, -1 and 0, respectively, we define

(86) $F_{i,r} :=$ 

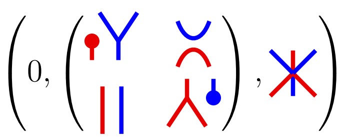
This is the map obtained from the homotopy equivalence in Lemma 4.11 by tensoring on the left with the identity morphism of $\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}$ and on the right with the identity morphism of $\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}$, and using when necessary the permutation isomorphism between $\mathrm{T}_i^{-1}\mathrm{B}_j$ and $\mathrm{B}_j\mathrm{T}_i^{-1}$ if $|i-j| \neq 1$.

Analogously,

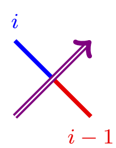


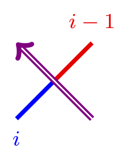
$= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}} G_{i,r}\, \mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}} \colon : \mathrm{B}_{i-1}\mathrm{T}_\rho^{-1} \to \mathrm{T}_\rho^{-1}\mathrm{B}_i,$

with

(87) $G_{i,r} =$ 

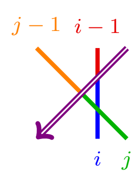
Of course, there are also mixed crossings involving T_ρ, which are depicted as

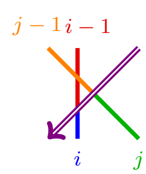


$: \mathrm{T}_\rho\mathrm{B}_{i-1} \to \mathrm{B}_i\mathrm{T}_\rho$ and 

$: \mathrm{B}_i\mathrm{T}_\rho \to \mathrm{T}_\rho\mathrm{B}_{i-1}.$

Lemma 4.16. *For distant colors $i, j \in I$, we have*

(88) 

$=$ 

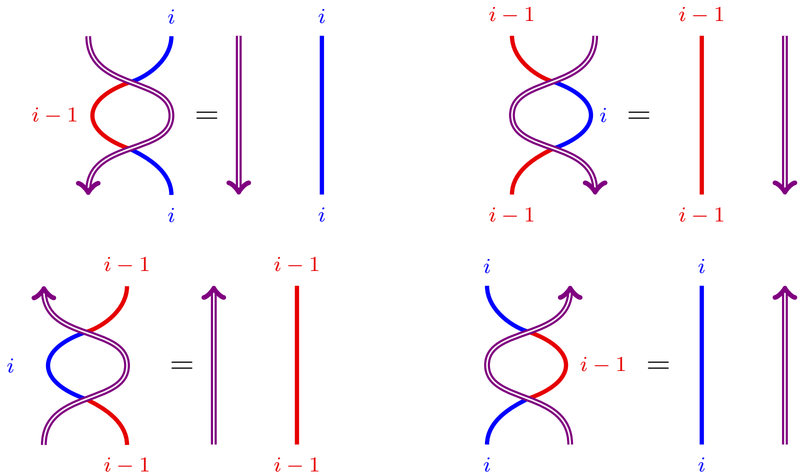
in $\mathcal{K}^b(\mathcal{S}_d)$.

Proof. It is clear that the map in (85) commutes with the 4-valent crossing for distant colors. □

The proof of the following lemma is immediate and, therefore, omitted.

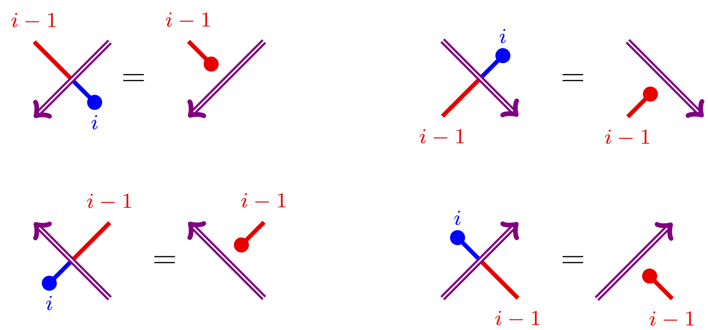
Lemma 4.17. *The mixed crossings in $\mathcal{K}^b(\mathcal{S}_d)$ satisfy the relations*

(89)

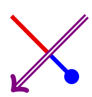


Lemma 4.18. *The following diagrammatic relations hold in in $\mathcal{K}^b(\mathcal{BS}_d)$:*

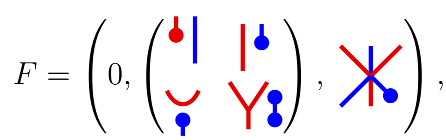
(90)



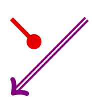
Proof. We prove the first relation in (90), the proof of the others being similar. By (86), the maps of complexes corresponding to the two sides of (90) are

 $= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}}\, F\, \mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}},$

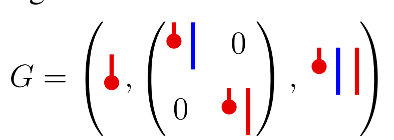
where in homological degrees $(-2,-1,0)$, respectively, we have



and

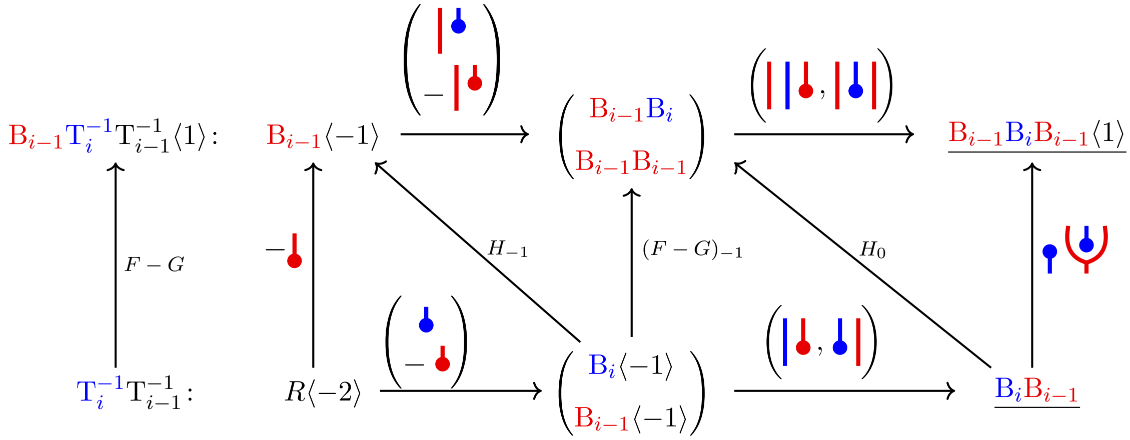
 $= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}}\, G\, \mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}},$

where in the same homological degrees we have

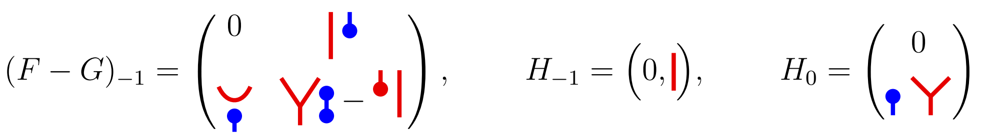


The diagram below shows that $F - G$ is zero in $\mathcal{K}^b(\mathcal{S}_d)$:

(91)



with

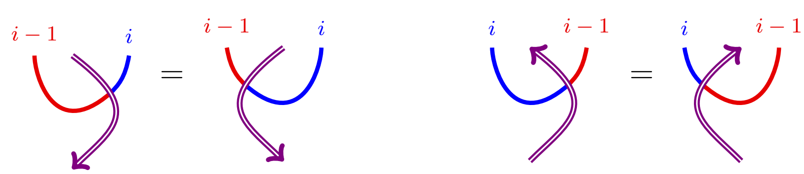


This finishes the proof. ☐

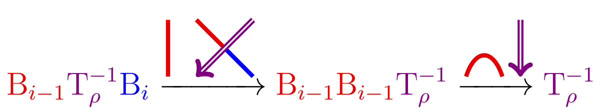
Lemma 4.19. *The following diagrammatic equalities hold in* $\mathcal{K}^b(\mathcal{S}_d)$*:*

(92)

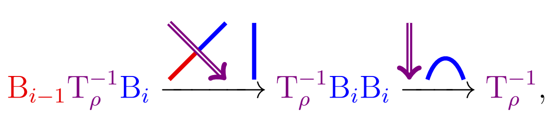
(93)



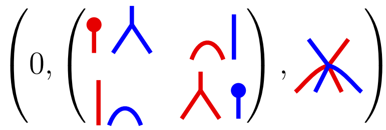
Proof. We prove the first relation in (92), as the other can be proved in a similar way. The proof is a consequence of the fact that the composites



and

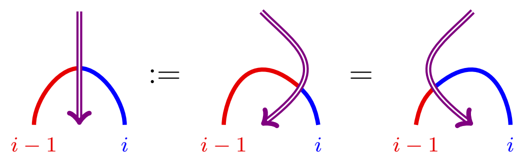


are both given by

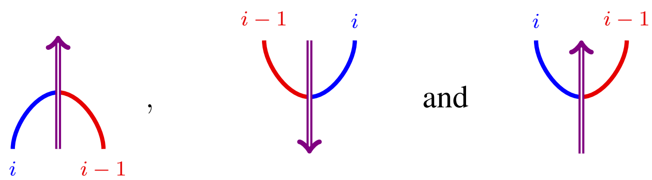
(94) $$\mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots \mathrm{T}_{i+1}^{-1}}$$  $$\mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots \mathrm{T}_{1}^{-1}}\,.$$

This computation is straightforward and uses (86) and (87). □

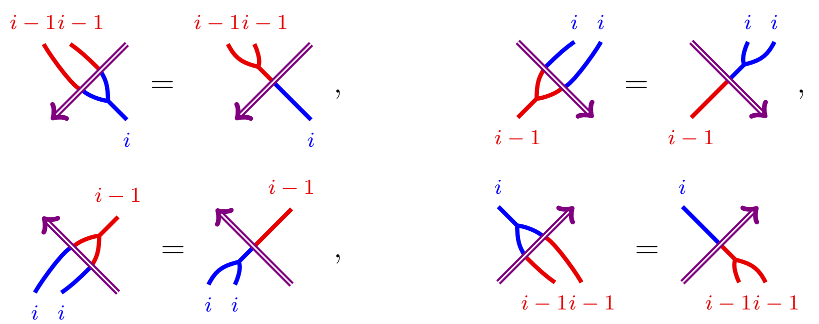
Remark 4.20. By Lemma 4.19, we can define



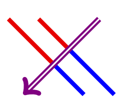
and similarly



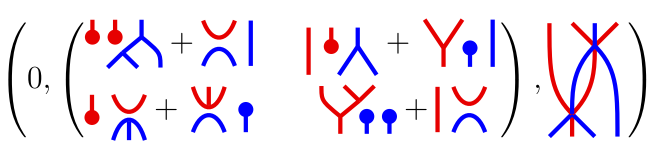
Lemma 4.21. *The following* pitchfork *relations hold in* $\mathcal{K}^b(\mathcal{BS}_d)$*:*

(95)


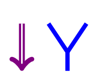
Proof. We only prove the first relation in (95), as the others can be proved in a similar way. Relations (85) and (86) imply that

 $$= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots \mathrm{T}_{i+1}^{-1}}\, F \,\mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots \mathrm{T}_{1}^{-1}} : \mathrm{T}_\rho^{-1}\mathrm{B}_i\mathrm{B}_i \to \mathrm{B}_{i-1}\mathrm{B}_{i-1}\mathrm{T}_\rho^{-1},$$

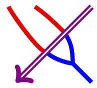
where F in homological degrees -2, -1 and 0, respectively, is given by

(96) $F =$ 

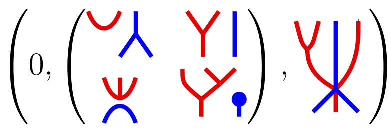
Pre-composing with



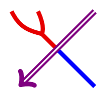
results in

 $= \mathrm{Id}_{\mathrm{T}_{d-1}^{-1}\cdots\mathrm{T}_{i+1}^{-1}}\, F_{\text{pitchfork}}\, \mathrm{Id}_{\mathrm{T}_{i-2}^{-1}\cdots\mathrm{T}_{1}^{-1}},$

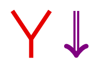
where

(97) $F_{\text{pitchfork}} =$ 

In homological degree zero we have used (43) with a blue dot on the leftmost blue endpoint. The proof is now completed by the observation that

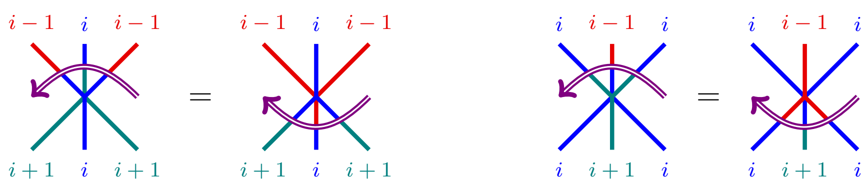


is given by exactly the same map, which can be seen immediately by post-composing the mixed crossing in (85) with



and using (86). □

Lemma 4.22. *The following diagrammatic equalities hold in $\mathcal{K}^b(\mathcal{S}_d)$, for any adjacent triple $i-1, i, i+1 \in I$:*

(98) 

Proof. Both diagrams in the first equality represent morphisms between $\mathrm{T}_\rho^{-1}\mathrm{B}_{i+1}\mathrm{B}_i\mathrm{B}_{i+1}\mathrm{T}_\rho$ and $\mathrm{B}_{i-1}\mathrm{B}_i\mathrm{B}_{i-1}$. By (56), there is an isomorphism $\mathrm{T}_\rho^{-1}\mathrm{B}_{i+1}\mathrm{B}_i\mathrm{B}_{i+1}\mathrm{T}_\rho \cong \mathrm{B}_i\mathrm{B}_{i-1}\mathrm{B}_i$, so both diagrams correspond to morphisms in

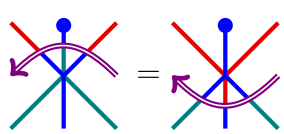
$$\mathcal{S}_d\left(\mathrm{B}_i\mathrm{B}_{i-1}\mathrm{B}_i, \mathrm{B}_{i-1}\mathrm{B}_i\mathrm{B}_{i-1}\right).$$

Recall that $\mathrm{B}_i\mathrm{B}_{i-1}\mathrm{B}_i \cong \mathrm{B}_{i(i-1)i} \oplus \mathrm{B}_i$ and $\mathrm{B}_{i-1}\mathrm{B}_i\mathrm{B}_{i-1} \cong \mathrm{B}_{i(i-1)i} \oplus \mathrm{B}_{i-1}$, which implies that

$$\mathcal{S}_d\left(\mathrm{B}_i\mathrm{B}_{i-1}\mathrm{B}_i, \mathrm{B}_{i-1}\mathrm{B}_i\mathrm{B}_{i-1}\right) \cong \mathcal{S}_d\left(\mathrm{B}_{i(i-1)i}, \mathrm{B}_{i(i-1)i}\right) \cong \mathbb{C}$$

by Soergel's Hom-formula in (50).

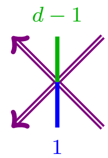
In particular, this implies that the two diagrams in the first equality are multiples of each other. To check that they are actually equal, one can attach a dot at an appropriate place. For example, one can easily check that



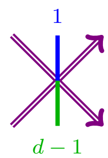
in $\mathcal{K}^b(\mathcal{BS}_d)$ by using relations (41) and (90), followed by (89) and (95). The second equality of the statement is proved in the same way. □

The *mixed 6-valent vertices* represent the following isomorphisms in $\mathcal{K}^b(\mathcal{S}_d)$, obtained by recursive application of Lemma 4.8:

(99)

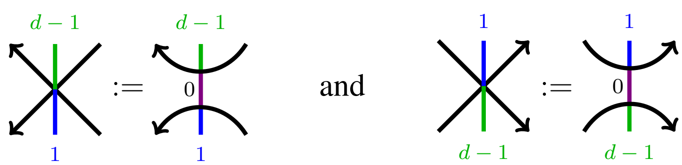


$$: \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho \to \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}$$



$$: \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1} \to \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho$$

Remark 4.23. To understand why we have introduced the mixed 6-valent vertices above, recall that the evaluation functors are (yet to be defined) functors from $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$ to $\mathcal{K}^b(\mathcal{S}_d)$, and that in $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$ there are mutually inverse isomorphisms



Lemma 4.24. *The mixed 6-valent vertices satisfy*

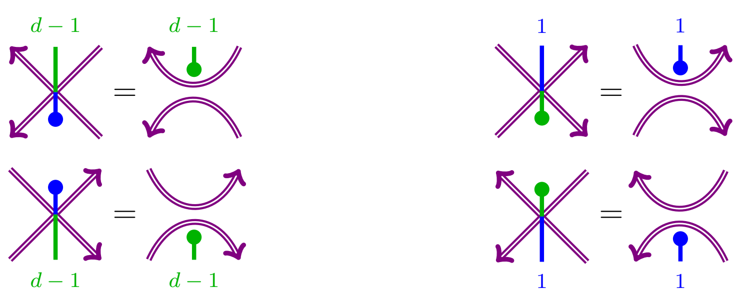
(100)

in $\mathcal{K}^b(\mathcal{S}_d)$.

Lemma 4.25. *The mixed 6-valent vertices also satisfy the following* dot relations *in* $\mathcal{K}^b(\mathcal{BS}_d)$*:*

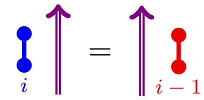
(101)

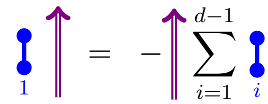
(102)

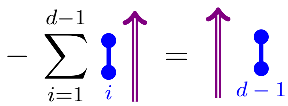


Proof. Apply Lemma 4.9 recursively. □

Lemma 4.26. *The following* mixed dumbbell-slide *relation holds in* $\mathcal{K}^b(\mathcal{BS}_d)$*:*

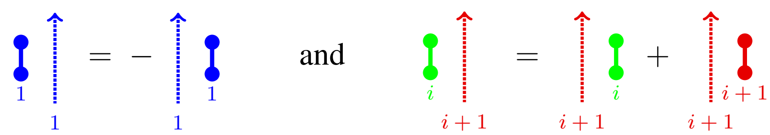
(103)  $i = 2, \ldots, d-1$

(104) 

(105) 

Proof. The equality in (103) is an immediate consequence of (90).

For (104) apply the non-oriented dumbbell-slides from Lemma 4.6

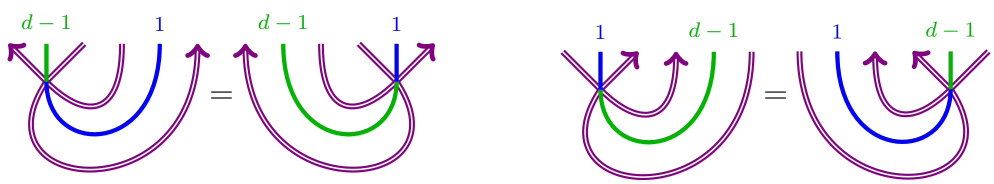


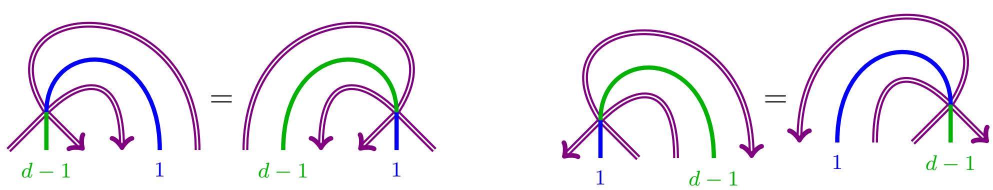
recursively.

Finally, for (105) use the same non-oriented dumbbell-slides as above but with the colors i and $i+1$ swapped. □

To prove Lemmas 4.27 to 4.30 below, we use the same strategy as in the proof of Lemma 4.22: we first check that a certain hom-space is one-dimensional and then conclude that two morphisms in that hom-space are equal by attaching dots to the corresponding diagrams.

Lemma 4.27. *The mixed 6-valent vertices satisfy the following* cyclicity *relations in* $\mathcal{K}^b(\mathcal{S}_d)$*:*

(106) 

(107) 

Proof. We only prove the first relation in (106), as the remaining ones are proved in the same way. We claim that the two morphisms in (106) are multiples of one another. To see this, note that $\mathrm{T}_\rho \mathrm{B}_{d-1} \mathrm{T}_\rho^{-1} \cong \mathrm{T}_\rho^{-1} \mathrm{B}_1 \mathrm{T}_\rho$ and $\mathrm{B}_1\mathrm{B}_1 \cong \mathrm{B}_1\langle -1\rangle \oplus \mathrm{B}_1\langle 1\rangle$, whence

$$\begin{aligned}\mathcal{K}^b(\mathcal{S}_d)\left(R, \mathrm{T}_\rho \mathrm{B}_{d-1} \mathrm{T}_\rho^{-1} \mathrm{T}_\rho^{-1} \mathrm{B}_1 \mathrm{T}_\rho\right) &\cong \mathcal{K}^b(\mathcal{S}_d)\left(R, \mathrm{T}_\rho \mathrm{B}_1 \mathrm{B}_1 \mathrm{T}_\rho\right) \\ &\cong \mathcal{S}_d\left(R, \mathrm{B}_1\mathrm{B}_1\right)\end{aligned}$$

$$\cong \mathcal{S}_d\left(R, \mathrm{B}_1\langle -1\rangle \oplus \mathrm{B}_1\langle 1\rangle\right),$$

where we have used the biadjointness of T_ρ and its inverse, and the fullness of the natural embedding of $\mathcal{S}_d$ in $\mathcal{K}^b(\mathcal{S}_d)$, for the second isomorphism. By Soergel's Hom-formula in (50), we know that

$$\dim_{\mathbb{C}}\left(\mathcal{S}_d\left(R, \mathrm{B}_1\langle -1\rangle\right)\right) = 0 \quad \text{and} \quad \dim_{\mathbb{C}}\left(\mathcal{S}_d\left(R, \mathrm{B}_1\langle 1\rangle\right)\right) = 1$$

and hence

$$\dim_{\mathcal{C}}\left(\mathcal{K}^b(\mathcal{S}_d)\left(R, \mathrm{T}_\rho \mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{T}_\rho^{-1}\mathrm{B}_i\mathrm{T}_\rho\right)\right) = 1.$$

Attaching a dot to one of the colored strands (say with 1) on both sides of (106) and using the relations in Lemma 4.25 and certain isotopies shows that both morphisms are equal in $\mathcal{K}^b(\mathcal{BS}_d)$.

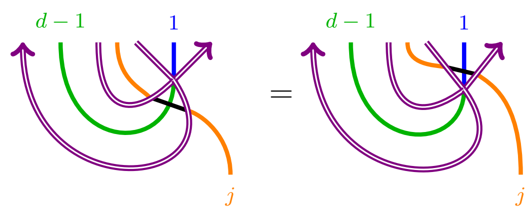
□

Lemma 4.28. *For each $j \in I$ distant from 1 and $d-1$, the following equalities hold in $\mathcal{K}^b(\mathcal{S}_d)$:*

(108)

Proof. We only prove the first equality, as the other can be proved in the same way. By adjointness, proving the first equaltiy in (108) is equivalent to proving the equality

(109)



in $\mathcal{K}^b(\mathcal{S}_d)$.

For any $j \in I$ distant from 1 and $d-1$, the same arguments as before (and the fact that B_1 and B_{j+1} commute) prove the following isomorphisms of hom-spaces:

$$\begin{aligned}
\mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{B}_j, \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_j\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\right) &\cong \mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{B}_j, \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_j\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\right)\\
&\cong \mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{T}_\rho\mathrm{B}_j\mathrm{T}_\rho^{-1}, \mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_j\mathrm{T}_\rho^{-1}\mathrm{B}_1\right)\\
&\cong \mathcal{S}_d\left(\mathrm{B}_{j+1}, \mathrm{B}_1\mathrm{B}_{j+1}\mathrm{B}_1\right)\\
&\cong \mathcal{S}_d\left(\mathrm{B}_{j+1}, \mathrm{B}_{j+1}\mathrm{B}_1\mathrm{B}_1\right)\\
&\cong \mathcal{S}_d\left(\mathrm{B}_{j+1}, \mathrm{B}_{j+1}\mathrm{B}_1\langle -1\rangle \oplus \mathrm{B}_{j+1}\mathrm{B}_1\langle 1\rangle\right)
\end{aligned}$$

By Soergel's Hom-formula in (50), we know that

$$\dim_{\mathbb{C}}\left(\mathcal{S}_d\left(\mathrm{B}_{j+1}, \mathrm{B}_{j+1}\mathrm{B}_1\langle -1\rangle\right)\right) = 0 \quad \text{and} \quad \dim_{\mathbb{C}}\left(\mathcal{S}_d\left(\mathrm{B}_{j+1}, \mathrm{B}_{j+1}\mathrm{B}_1\langle 1\rangle\right)\right) = 1$$

whence

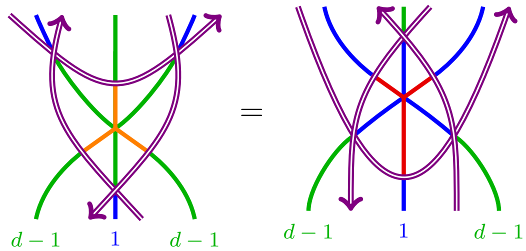
$$\dim_{\mathbb{C}}\left(\mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{B}_j, \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_j\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\right)\right) = 1$$

and the equality in (109) can be proved by attaching dots to these diagrams at appropriate places.

□

Lemma 4.29. *The following equalities are true in* $\mathcal{K}^b(\mathcal{S}_d)$*:*

(110)



Proof. We first note that

$$\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho \cong \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_{d-1}\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1} \cong \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{B}_{d-2}\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1},$$

and

$$\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1} \cong \mathrm{B}_{d-1}\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_{d-1} \cong \mathrm{T}_\rho\mathrm{B}_{d-2}\mathrm{B}_{d-1}\mathrm{B}_{d-2}\mathrm{T}_\rho^{-1},$$

and therefore

$$\mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1}, \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\right) \cong \mathcal{S}_d\left(\mathrm{B}_{d-1}\mathrm{B}_{d-2}\mathrm{B}_{d-1}, \mathrm{B}_{d-2}\mathrm{B}_{d-1}\mathrm{B}_{d-2}\right)$$

By the decompositions

$$\mathrm{B}_{d-1}\mathrm{B}_{d-2}\mathrm{B}_{d-1} \cong \mathrm{B}_{(d-1)(d-2)(d-1)} \oplus \mathrm{B}_{d-1} \quad \text{and} \quad \mathrm{B}_{d-2}\mathrm{B}_{d-1}\mathrm{B}_{d-2} \cong \mathrm{B}_{(d-1)(d-2)(d-1)} \oplus \mathrm{B}_{d-2}$$

and Soergel's Hom-formula in (50), we conclude that

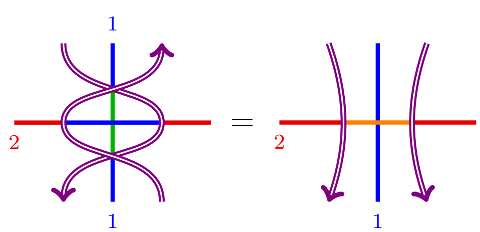
$$\dim_{\mathbb{C}}\left(\mathcal{K}^b(\mathcal{S}_d)\left(\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1}, \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho\right)\right) = 1.$$

Thus the two diagrams in (110) are scalar multiples of each other and the equality now follows by attaching dots to these diagrams at appropriate places. □

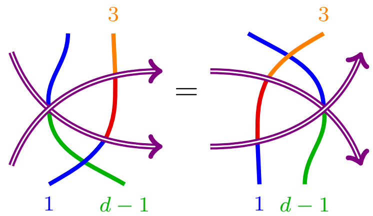
The proof of the following lemma uses exactly the same arguments as above and is left as an exercise to the reader.

Lemma 4.30. *The following equalities hold in* $\mathcal{K}^b(\mathcal{S}_d)$*:*

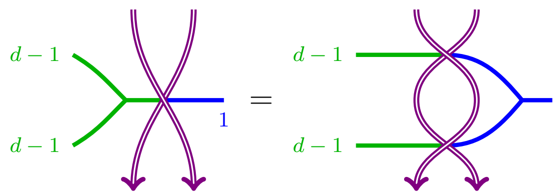
(111)



(112)



(113)



5. Evaluation functors

In this section, we finally define the evaluation functors $\mathcal{E}v_{r,s}\colon \widehat{\mathcal{S}}_d^{\mathrm{ext}} \to \mathcal{K}^b(\mathcal{S}_d)$, for $r,s \in \mathbb{Z}$, which categorify the evaluation maps ev_a from Definition 2.2, for $a = (-1)^s q^r$ with $r,s \in \mathbb{Z}$. The other evaluation maps in that definition, denoted ev'_a, can be categorified likewise, but we don't work out the details here.

Remark 5.1. To be really precise, we actually define a degree-preserving functor from $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ to $\mathcal{K}^b((\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}})$ which uniquely determines $\mathcal{E}v_{r,s}$, see Remarks 3.2 and 4.1. Note that $(\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}}$ is a graded category with shift, and that $X\langle t\rangle \cong X$ for every $X \in (\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}}$ and $t \in \mathbb{Z}$. The natural, degree-preserving embedding of $\mathcal{BS}_d$ into $(\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}}$ is therefore fully faithful and essentially surjective, although it is not an equivalence of graded categories because its inverse is not degree-preserving. However, for our purposes all that matters is that the monoidal subcategory of degree-zero morphisms $((\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}})^0$ is isomorphic with $\mathcal{BS}_d^{\mathrm{sh}}$, which implies that the idempotent completion of both is $\mathcal{S}_d$. This might sound a bit complicated, but we can not simply define a functor from $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ to $\mathcal{K}^b(\mathcal{BS}_d)$ because the image of B_ρ requires non-trivial internal shifts when $r \neq 0$.

5.1. **Definition.** Let $r,s \in \mathbb{Z}$ be arbitrary but fixed for the remainder of this section.
The *evaluation functor* is the monoidal, $\mathbb{C}$-linear functor

$$\mathcal{E}v_{r,s}\colon \widehat{\mathcal{S}}_d^{\mathrm{ext}} \to \mathcal{K}^b(\mathcal{S}_d) \tag{114}$$

commuting with shifts which is uniquely determined (see Remark 5.1) by the monoidal, degree-preserving, $\mathbb{C}$-linear functor

$$\mathcal{E}v_{r,s}\colon \widehat{\mathcal{BS}}_d^{\mathrm{ext}} \to \mathcal{K}^b((\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}}) \tag{115}$$

defined below. Note that we use the same notation for both functors.

- On the (non-full) subcategory $\mathcal{BS}_d$ of $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$, the evaluation functor $\mathcal{E}v_{r,s}$ is the identity. More specifically, this means that $\mathcal{E}v_{r,s}(\mathrm{B}_i) := \mathrm{B}_i$ for every $i \in I$ and that $\mathcal{E}v_{r,s}$ sends any diagram without unoriented 0-colored strands and oriented strands to itself.

On other objects of $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$, it is defined as

$$\mathcal{E}v_{r,s}(\mathrm{B}_0) := \mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho, \tag{116}$$

$$\mathcal{E}v_{r,s}(\mathrm{B}_\rho^{\pm 1}) := \mathrm{T}_\rho^{\pm 1}\langle \pm r\rangle[\pm s]. \tag{117}$$

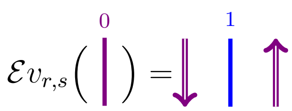
On other morphisms it is defined as follows.

- On *oriented and* 0*-colored* generators:

(118) 

(119) 

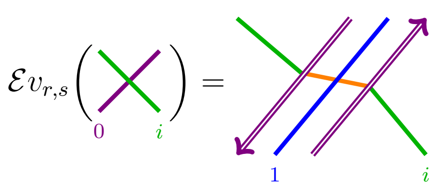
(120) 

(121) 

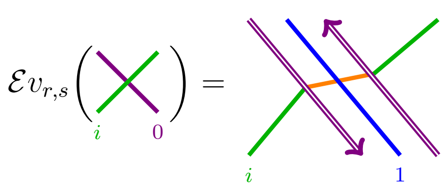
(122) 

(123) 

- On generators including strands with *distant colors*:

(124) 

for $i \neq 1, d-1$

(125) 

for $i \neq 1, d-1$

- On generators including strands with *adjacent colors*:

(126) 

(127)

if $i \neq 0, 1$, while

(128)

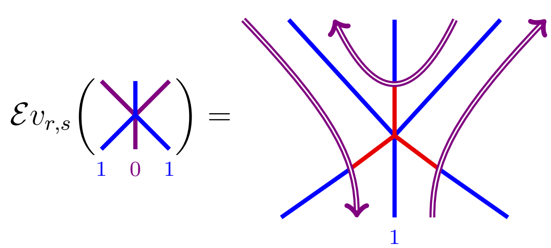
(129)

(130)

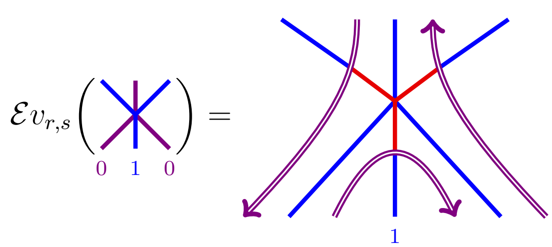
(131)

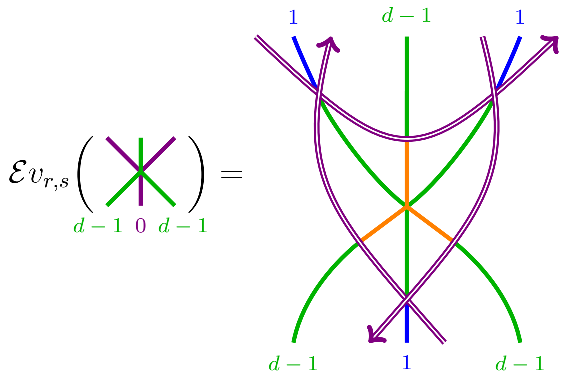
and

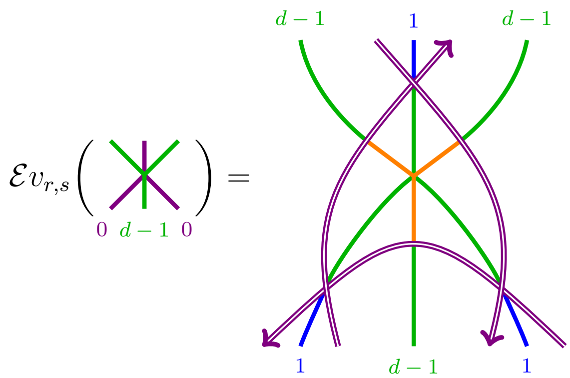
(132)



(133)



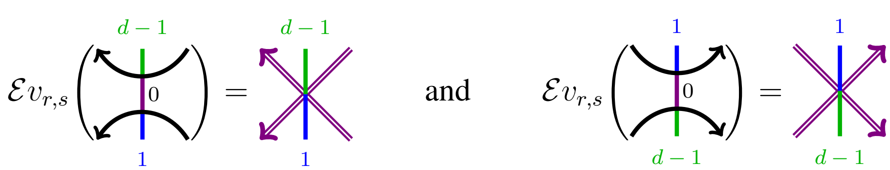
(134) 

(135) 

This ends the definition of $\mathcal{E}v_{r,s}$.

Remark 5.2. Since $\mathrm{T}_\rho^{-1}\mathrm{B}_1\mathrm{T}_\rho \cong \mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}$ in $\mathcal{K}^b((\mathcal{BS}_d^{\mathrm{sh}})^{\mathrm{gr}})$, we could have defined $\mathcal{E}v_{r,s}(\mathrm{B}_0)$ as $\mathrm{T}_\rho\mathrm{B}_{d-1}\mathrm{T}_\rho^{-1}$. These two choices result in naturally isomorphic evaluation functors, the isomorphim being induced by the 6-valent vertices (99), as can be checked by straightforward diagrammatic calculations.

Remark 5.3. The apparent lack of symmetry between the image via $\mathcal{E}v_{r,s}$ of the mixed 4-vertices involving strands colored 0 and 1, and the corresponding image of the mixed 4-vertices involving colored 0 and $d-1$ ((128) to (131)) is explained by Remark 5.2. Note also that

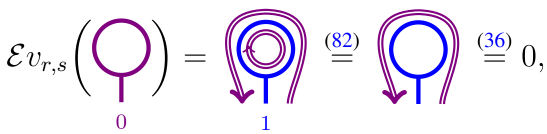


5.2. **Proof of well-definedness.**

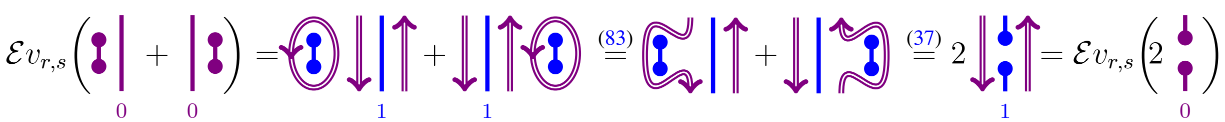
Theorem 5.4. *The monoidal functor $\mathcal{E}v_{r,s}$ is well-defined.*

Proof. The fact that $\mathcal{E}v_{r,s}$ preserves *isotopy invariance* follows from Lemma 4.15, Lemma 4.19 and Lemma 4.27, together with isotopy invariance of the usual (non-oriented) Soergel calculus.

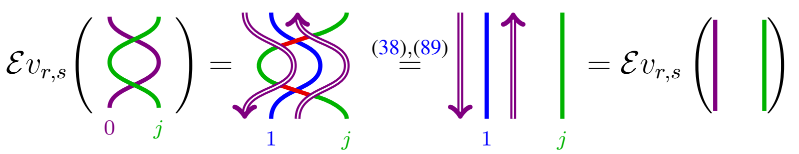
- Relations involving only *one color*. We only need to check for color 0. Relations (34) and (35) are clear. For the remaining one-color relations we have

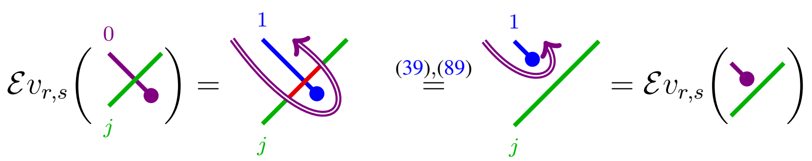


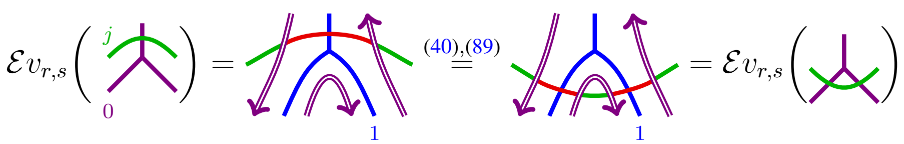
and



- Relations involving *two distant colors*. Here $j \neq 1, d-1$.

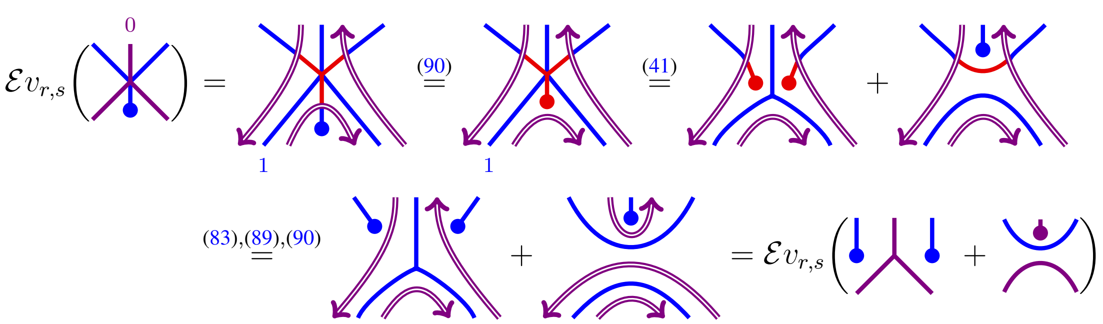


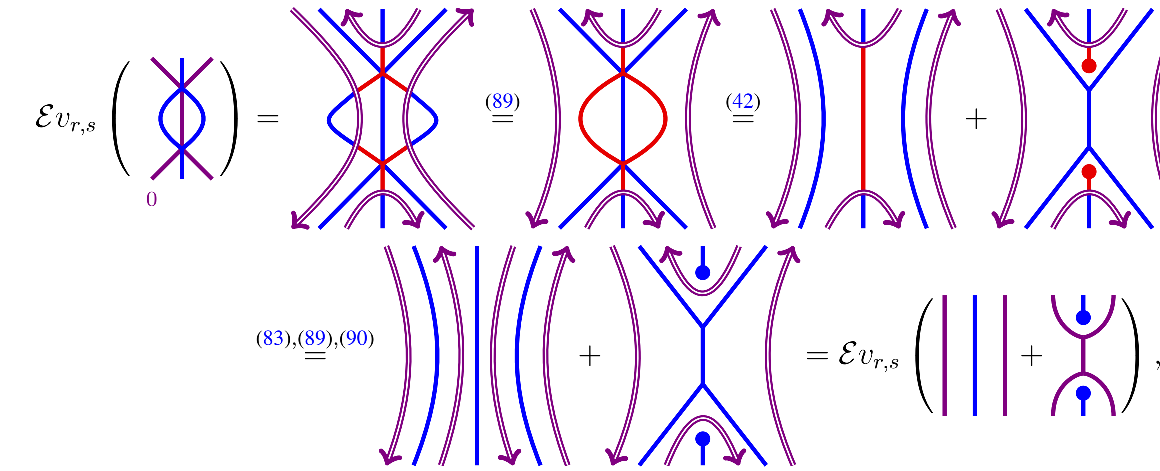


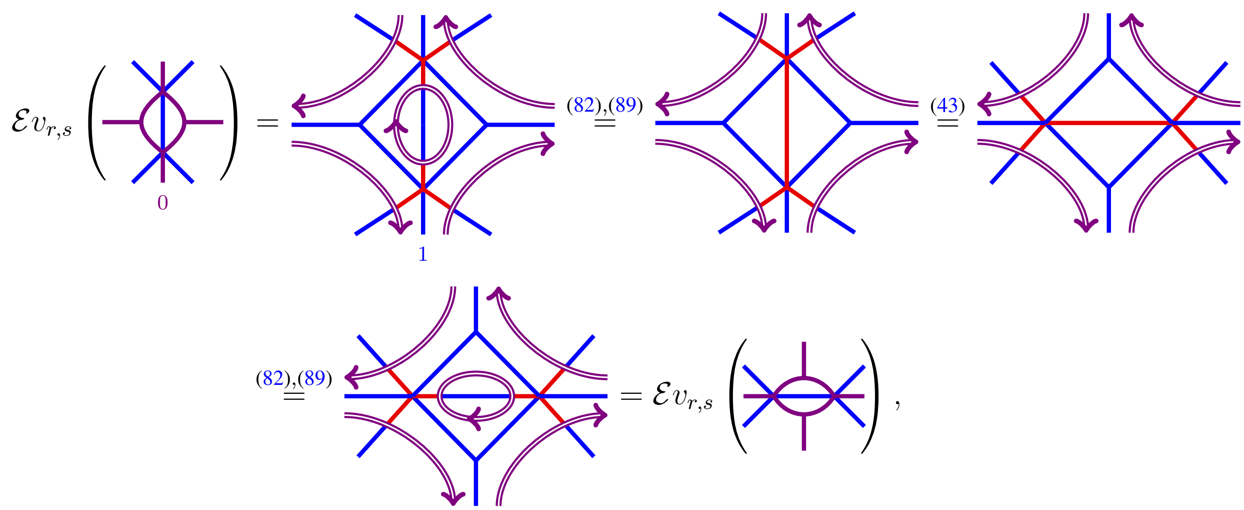


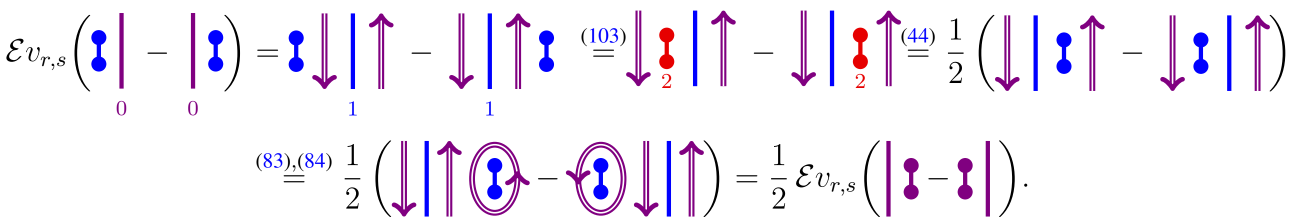
The corresponding relations with the colors 0 and j switched are proved in the same way.

- Relations involving *two adjacent colors*. We have to check the cases involving either the pair $(0,1)$ or the pair $(0,d-1)$. For the pair $(0,1)$ we compute:





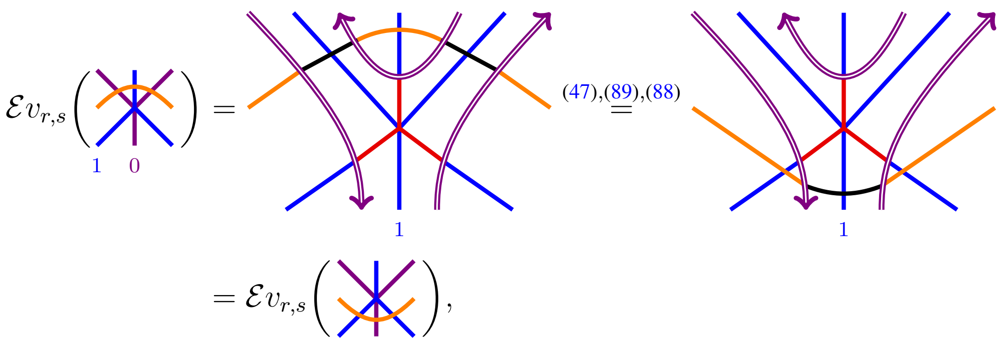




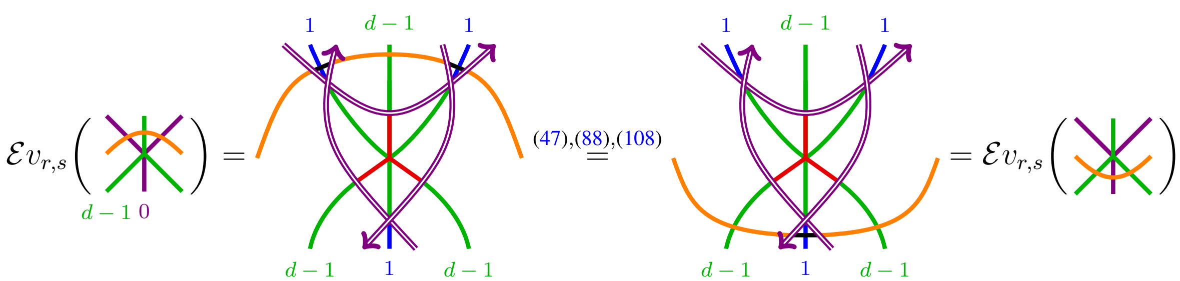
The relations with the colors 0 and 1 switched are proved in the same way. The relations for the pair $(0, d-1)$ can be proved similarly, using the image of the corresponding mixed 6-valent vertex, of course.

- The relation involving *three distant colors* is straightforward and follows from the observation that the case involving colors 0, i and j, with $i, j \in I$ and distant implies checking a relation involving the colors 1, $i+1$ and $j+1$, which are still distant.

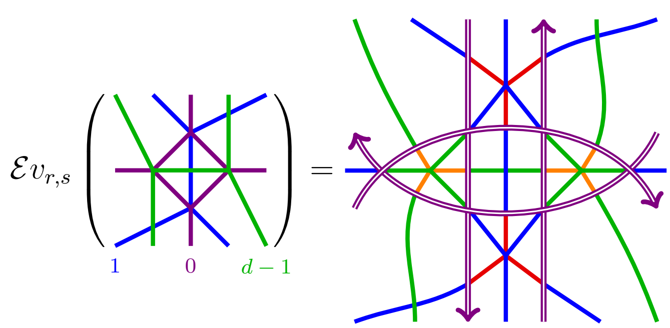
- The relation involving a *distant dumbbell* colored $i \in \{2, \dots, d-2\}$ and a straight line colored 0 is straightforward, because (103) implies that it reduces to the same relation involving a distant dumbbell with color $i+1$ and a straight line colored 1. Similarly, the relation involving a *distant dumbbell* colored 0 and a straight line colored $i \in \{2, \dots, d-2\}$ reduces to the relation involving a distant dumbbell colored 1 and a straight line colored $i+1$, thanks to (89).
- Relation involving *two adjacent colors and one distant from the other two*. If the distant color in (47) is 0, the proof is straightforward. Otherwise, we compute



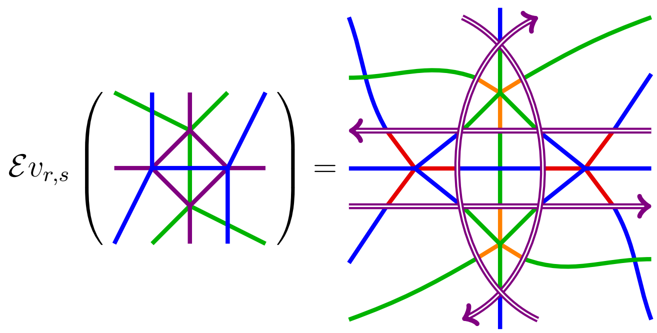
  and



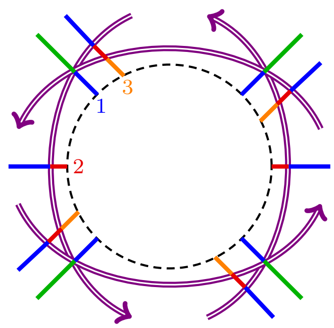
  The relations with the adjacent colors exchanged are proved in the same way.
- Relation involving *three adjacent colors*. We need to check the cases of three adjacent colors belonging to $\{d-2, d-1, 0, 1, 2\}$. Starting with the case of $(0, 1, d-1)$, we have



and

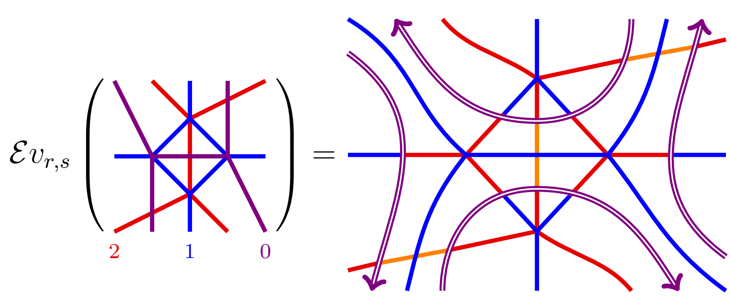


To prove that these are equal, first use the relations in Lemma 4.29 and Lemma 4.30 to write them in the form



Then observe that the parts of the diagrams inside the dashed circle are exactly as the two sides of (48) with colors $(1, 2, 3)$, which completes the proof of this case.

The remaining cases can be proved in similar ways, but they are actually a bit easier. For example, for the colors $(0, 1, 2)$ we have



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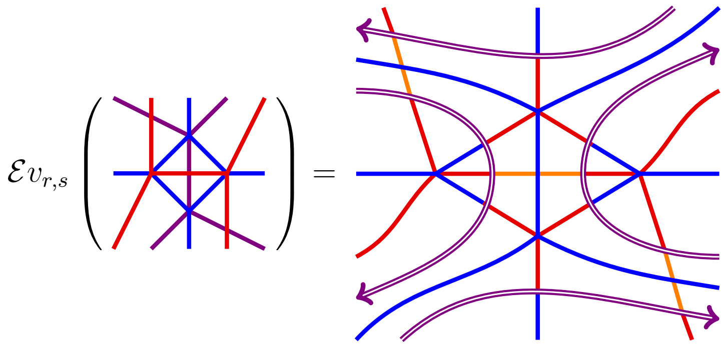
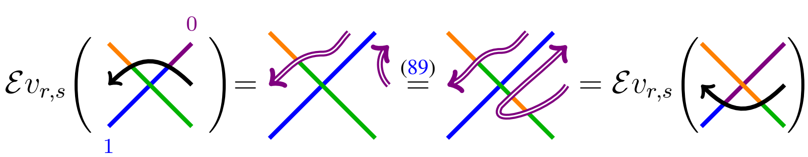


Proceeding as in the previous case, but using the relations in Lemma 4.22 and Lemma 4.17, results in two diagrams which differ only by parts that are equal to the two sides of (48) with colors $(1, 2, 3)$ again.

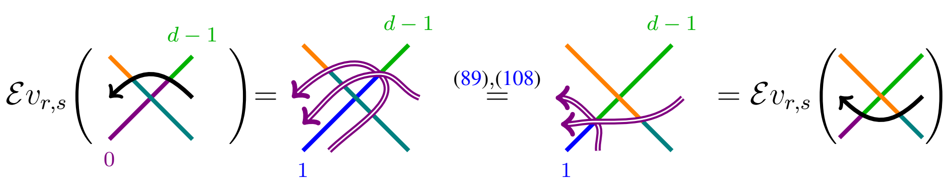
- Relations involving *oriented strands*. Relations (53) and (54) translate under $\mathcal{E}v_{r,s}$ into relations (82) and (83), respectively. The remaining relations (55) to (59) translate into relations (88), (89), (90), (95) and (98) (together with some obvious relations in the usual (non-oriented) Soergel calculus), respectively, if they don't involve the color 0.

  However, if one of the strands is colored 0, then there is something to check. For each relation, we prove one case involving the colors 0 and 1 and one case involving the colors 0 and $d-1$, the other cases being similar.

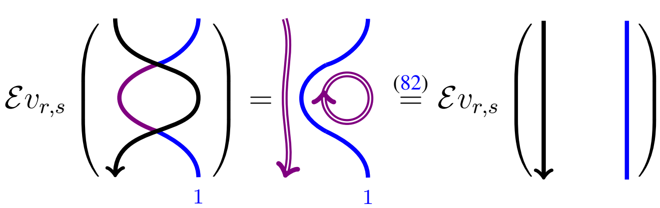
  – For relation (55), we have



  and



  – For relation (56), we have



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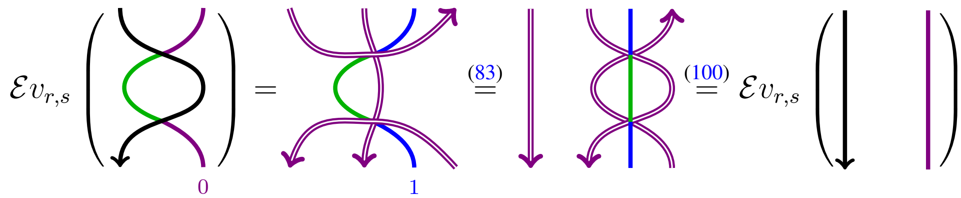
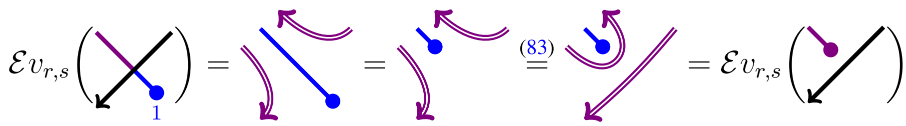


– for relation (57), we have



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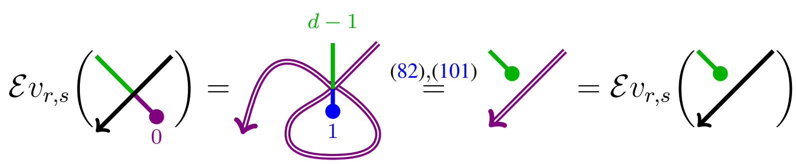
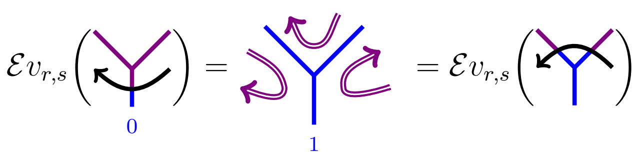


– For relation (58), we have



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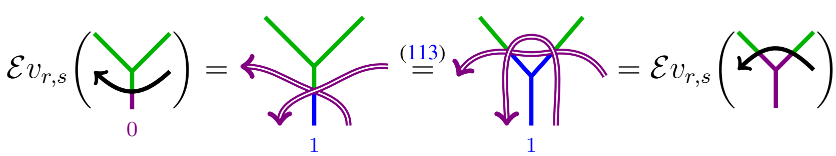
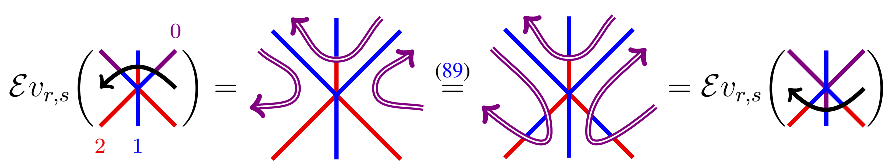
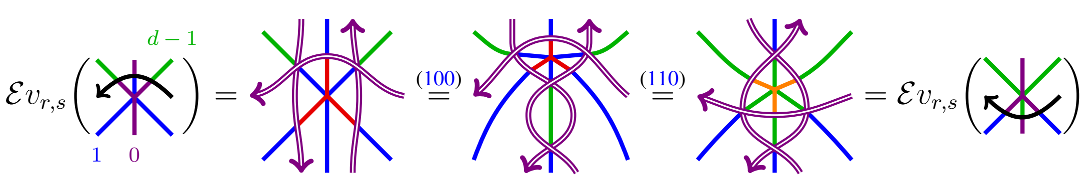


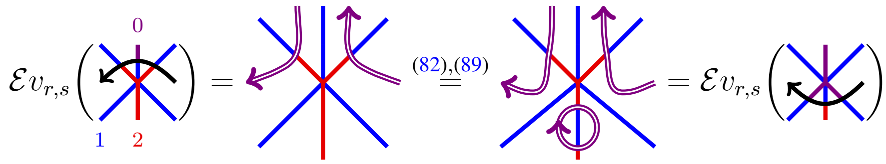
– Relation (59) actually consists of two (similar) relations. For the first of them, we have



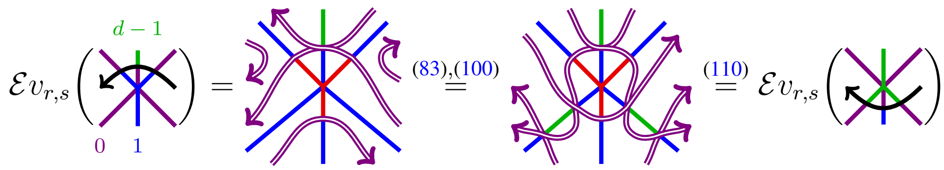
and



To check this relation with colors $(d-2, d-1, 0)$, use (89) and (100).
For the second relation in (59), we have



and



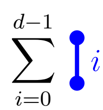
Checking the relation with colors $(d-2, d-1, 0)$ uses (100), (89) and (83).
This ends the proof of Theorem 5.4. □

Remark 5.5. (1) The functor $\mathcal{E}v_{r,s}$ is not full: For example, the special Rouquier complex $\mathcal{E}v_{r,s}(\mathrm{B}_\rho^{-1}) = \mathrm{T}_\rho^{-1}\langle -r\rangle[-s]$ has the form

$$\cdots \to \mathrm{B}_{d-1}\cdots\mathrm{B}_1\langle -r\rangle[-s] \to 0.$$

Therefore, there is an obvious (non-null-homotopic) map from $\mathrm{B}_{d-1}\cdots\mathrm{B}_1\langle -r\rangle[-s]$ to $\mathrm{T}_\rho^{-1}\langle -r\rangle[-s]$, which is the identity on $\mathrm{B}_{d-1}\cdots\mathrm{B}_1\langle -r\rangle[-s]$ and zero elsewhere, but this map is not in the image of $\mathcal{E}v_{r,s}$.

(2) By (51) and (104), the evaluation functor $\mathcal{E}v_{r,s}$ maps the central morphism



to zero. We could have defined $\widehat{\mathcal{BS}}_d^{\mathrm{ext}}$ over the polynomial ring $\mathbb{C}[y, x_1, \dots, x_{d-1}]$ as in [MT2017] and extended $\mathcal{E}v_{r,s}$ to that "base ring". In that case, the central morphism $\boxed{\mathrm{y}}$ (which is equal to the above dumbbell sum, as already remarked) would be sent to zero by the evaluation functor, which makes perfect sense as the extended base ring of $\mathcal{BS}_d$ would be $\mathbb{C}[x_1, \dots, x_d]$.

6. Evaluation birepresentations and finitary covers

6.1. **Recollections on birepresentation theory.** In the following, we will work with graded (finitary or triangulated) birepresentations of graded, additive bicategories. The particular bicategory we are interested in is, of course, $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$, which we view as a bicategory with one object in the usual way.

We call a graded, $\mathbb{C}$-linear, additive category $\mathcal{A}$ *graded-finitary* if $\mathcal{A}^{\mathrm{sh}}$ is idempotent complete, morphism spaces between indecomposables are finite-dimensional and there are only finitely many isomorphism classes of indecomposables up to isomorphism and grading shift. Note that $\mathcal{A}$ need not be finitary, because the Hom-spaces might be infinite-dimensional, although they are finite-dimensional in each degree. This is why we write *graded-finitary* and not *graded, finitary*. We denote the 2-category of graded, resp. graded-finitary, $\mathbb{C}$-linear, additive categories, degree-preserving $\mathbb{C}$-linear functors and natural transformations by $\mathfrak{A}_{\mathbb{C}}^{g}$, resp. $\mathfrak{A}_{\mathbb{C}}^{gf}$. A *(locally) graded, additive bicategory* $\mathcal{C}$ is one whose morphism categories are enriched over $\mathfrak{A}_{\mathbb{C}}^{g}$ and a *(locally) graded-finitary bicategory* $\mathcal{C}$ is one whose morphism categories are enriched over $\mathfrak{A}_{\mathbb{C}}^{gf}$ and whose identity 1-morphisms are indecomposable. Note that, to shorten the string of adjectives, we drop the adjective $\mathbb{C}$-linear, even though it is implicit in the enrichment. A *graded, additive* (resp. *graded-finitary*) *birepresentation* is a degree-preserving pseudofunctor from $\mathcal{C}$ to $\mathfrak{A}_{\mathbb{C}}^{g}$ (resp. $\mathfrak{A}_{\mathbb{C}}^{gf}$).

Since we are mainly interested in $\widehat{\mathbb{S}}_{d}^{\mathrm{ext}}$, we will also abuse notation and call additive (bi)categories of the form $\mathcal{A}^{\mathrm{sh}}$ graded-finitary provided $\mathcal{A}$ is. Similarly, given a graded-finitary birepresentation $\mathbf{M}$ of a graded, additive bicategory $\mathcal{C}$, we will also call the birepresentation $\mathbf{M}^{\mathrm{sh}}$ of $\mathcal{C}^{\mathrm{sh}}$ (which acts on categories $\mathbf{M}(\mathtt{i})^{\mathrm{sh}}$, for objects $\mathtt{i}$, via functors which commute with shifts) graded-finitary. For more detail on these constructions, we refer to [MMMTZ2019, Section 2.6].

We will also be considering triangulated birepresentations of graded, additive bicategories. Denote by $\mathfrak{T}_{\mathbb{C}}$ the bicategory of triangulated, $\mathbb{C}$-linear categories, ($\mathbb{C}$-linear) triangulated functors and natural transformations. A *triangulated birepresentation* of a $\mathbb{C}$-linear, additive bicategory $\mathcal{C}$ is a ($\mathbb{C}$-linear) pseudofunctor from $\mathcal{C}$ to $\mathfrak{T}_{\mathbb{C}}^{gf}$. In order to consider graded versions, we restrict ourselves to the 2-full subbicategory $\mathfrak{T}_{\mathbb{C}}^{g}$ of $\mathfrak{T}_{\mathbb{C}}$ whose objects are triangulated categories of the form $\mathcal{K}^{b}(\mathcal{A}^{\mathrm{sh}})$ for a graded, $\mathbb{C}$-linear, additive category $\mathcal{A}$, and whose functors are degree-preserving triangulated functors. A *graded-triangulated birepresentation* of an additive, graded bicategory $\mathcal{C}$ is then a degree-preserving ($\mathbb{C}$-linear) pseudofunctor from $\mathcal{C}$ to $\mathfrak{T}_{\mathbb{C}}^{g}$.

Similarly to the finitary case above, we will call a birepresentation graded, triangulated if a bicategory of the form $\mathcal{C}^{\mathrm{sh}}$ acts on triangulated categories of the form $\mathcal{T}^{\mathrm{sh}}$ via triangulated functors commuting with shifts. These are birepresentations obtained by taking a graded birepresentation of $\mathcal{C}$ acting on $\mathcal{T}$, closing under shifts, and then restricting to morphisms of degree zero.

In some cases, graded-finitary birepresentations will have an additional shift functor (coming from the homological shift in a triangulated birepresentation), with respect to which morphisms in the underlying categories will have degree zero. We call such birepresentations bigraded-finitary.

Given a (locally) additive, graded bicategory, the set of isomorphism classes of indecomposable 1-morphisms up to grading shift can be given three natural partial preorders: the *left* preorder ($[\mathrm{F}] \leq_{L} [\mathrm{G}]$ if and only if $[\mathrm{G}]$ appears as a direct summand of $[\mathrm{HF}]$ for some 1-morphism H), the *right* preorder ($[\mathrm{F}] \leq_{R} [\mathrm{G}]$ if and only if $[\mathrm{G}]$ appears as a direct summand of $[\mathrm{FH}]$ for some 1-morphism H) and the *two-sided* preorder ($[\mathrm{F}] \leq_{J} [\mathrm{G}]$ if and only if $[\mathrm{G}]$ appears as a direct summand of $[\mathrm{H}_1\mathrm{F}\mathrm{H}_2]$ for some 1-morphisms $\mathrm{H}_1, \mathrm{H}_2$), and the corresponding equivalence classes are called *left, right* and *two-sided cells*, respectively.

If $\mathcal{C}$ is graded-finitary, we can associate to any left cell a so-called *graded cell 2-representation*, which is the quotient of the left 2-ideal in $\mathcal{C}$ generated by the identities on the 1-morphisms in the cell, by the unique maximal ideal of the resulting birepresentation (i.e. the unique maximal ideal of the underlying categories which is stable under the action of $\mathcal{C}$). For more details (in the ungraded case, but the graded one is analogous), see e.g. [MM2016, Section 3.3].

6.2. **Finitary covers of evaluation cell birepresentations.** Let $\mathbf{M}$ be a graded-finitary birepresentation of $\mathcal{S}_d$, for any $d \in \mathbb{N}_{\geq 2}$. Then $\mathcal{K}^b(\mathbf{M})$, as a graded, triangulated birepresentation of $\mathcal{K}^b(\mathcal{S}_d)$, induces a graded, triangulated birepresentation of $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$, the *evaluation birepresentation* $\mathbf{M}^{\mathcal{E}v_{r,s}}$, resp. $\mathbf{M}^{\mathcal{E}v'_{r,s}}$, by pull-back through the evaluation functors $\mathcal{E}v_{r,s}$, resp. $\mathcal{E}v'_{r,s}$, for any $r, s \in \mathbb{Z}$.

In this subsection we show that, if $\mathbf{M}$ is a cell birepresentation of $\mathcal{S}_d$, then $\mathbf{M}^{\mathcal{E}v_{r,s}}$ has a bigraded-finitary cover in the following sense.

Definition 6.1. A *bigraded-finitary cover* of a graded, triangulated birepresentation $\mathbf{N}$ of a graded, additive bicategory $\mathcal{C}$ is a bigraded-finitary birepresentation $\mathbf{L}$ of $\mathcal{C}$ together with a faithful morphism of linear additive bigraded birepresentations $\Phi \colon \mathbf{L} \to \mathbf{N}$ whose essential image generates $\mathbf{N}$ as a graded triangulated category.

Proposition 6.2. *Let* $\mathbf{M}$ *be the graded cell birepresentation associated to some left cell* $\mathcal{L}$ *of* $\mathcal{S}_d$. *Then* $\mathbf{M}^{\mathcal{E}v_{r,s}}$ *has a bigraded-finitary cover.*

Proof. By [EH2018, Proposition 4.31], T_ρ^d acts as $\mathrm{Id}\langle x\rangle[y]$ on $\mathbf{M}^{\mathcal{E}v_{r,s}}$, for some $x, y \in \mathbb{Z}$. Let $\mathbf{L}$ be the closure under isomorphisms, direct sums, direct summands, grading and homological shifts of the $\mathcal{E}v_{r,s}(\mathrm{T}_\rho^i)\mathrm{B}_w$, for $i \in \widehat{I}$ and $w \in \mathcal{L}$. Relation (56) implies that $\mathbf{L}$ is a bigraded-finitary birepresentation of $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$.

The inclusion functor $\mathbf{L} \hookrightarrow \mathbf{M}^{\mathcal{E}v_{r,s}}$ is a morphism of linear additive bigraded birepresentations and its essential image generates $\mathcal{K}^b(\mathbf{M})$, which is the underlying graded triangulated category of $\mathbf{M}^{\mathcal{E}v_{r,s}}$. □

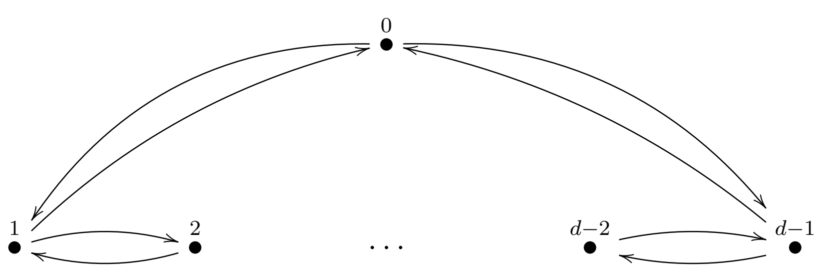
We refer to Corollary 6.5 for an example demonstrating that Φ is not full in general.

Remark 6.3. It is easy to see that $\mathbf{L}$ is transitive, and it looks likely that calculations, using the explicit descriptions of the representing bimodules for the B_w given in [MMMTZ2019, Section 4.3], one can verify that it is indeed simple transitive.

6.3. **The zigzag algebras.** Let us first recall the *affine zigzag algebra* $\widehat{Z}_d$ over $\mathbb{C}$ associated to the $\widehat{A}_{d-1}$ Dynkin diagram. As is well-known, there are two isomorphism classes of affine zigzag algebras with invertible integer coefficients, and we use a specific representative of either one or the other depending on the parity of d.

Let $e_i,\ i \in \widehat{I}$, denote the orthogonal idempotents associated to the vertices of the zigzag quiver

(136)



and $i_1|i_2|\ldots|i_k$ the path in the quiver from i_k to i_1 via $i_{k-1}, \ldots, i_2$. The relations in $\widehat{Z}_d$ are

$$i|i+1|i+2 = 0 = i|i-1|i-2, \quad i \in \widehat{I}; \tag{137}$$

(138) $$i|i+1|i = i|i-1|i, \quad i \in I;$$

(139) $$0|1|0 = (-1)^d(0|d-1|0).$$

For convenience, we also use the notation

$$\ell_i := i|i+1|i,$$

for any $i \in \widehat{I}$. This algebra has dimension $4d$, it is positively graded by putting the degree of every path equal to its length, and it is a graded Frobenius algebra with non-degenerate trace defined by

(140) $$\mathrm{tr}(\ell_i) = 1 \text{ for every } i \in \widehat{I}; \quad \mathrm{tr}(a) = 0 \text{ when } \deg(a) \neq 2.$$

This means that $\widehat{Z}_d^\star \cong \widehat{Z}_d\langle 2\rangle$ as graded left, resp. right, $\widehat{Z}_d$-modules. Define the non-degenerate bilinear pairing $\langle .,. \rangle \colon \widehat{Z}_d \otimes \widehat{Z}_d \to \mathbb{C}$ as usual

(141) $$\langle a,b\rangle := \mathrm{tr}(ab), \ a,b \in \widehat{Z}_d,$$

and recall that two bases of $\widehat{Z}_d$, say $\{a_i \mid i = 1\ldots,4d\}$ and $\{a_i^\star \mid 1,\ldots,4d\}$, are called *dual* to each other if they satisfy

$$\langle a_i, a_j^\star\rangle = \delta_{i,j}, \ i,j = 1,\ldots,4d,$$

where $\delta_{i,j}$ is the Kronecker delta. With respect to the bilinear form on $\widehat{Z}_d$, there is a natural pair of dual bases $\{e_i, \ell_i, (i|i\pm 1)\}_{i\in\widehat{I}}$ and $\{e_i^\star, \ell_i^\star, (i|i\pm 1)^\star\}_{i\in\widehat{I}}$, such that

(142) $$e_i^\star = \ell_i, \ \ell_i^\star = e_i, \quad i \in \widehat{I};$$

(143) $$(i|(i\pm 1))^\star = (i\pm 1)|i, \quad i \in I;$$

(144) $$(0|(d-1))^\star = (-1)^d((d-1)|0).$$

Note that $\widehat{Z}_d$ is symmetric when d is even and only weakly symmetric when d is odd.

Let $\widehat{Z}_d-\mathrm{fgproj}$, resp. $\mathrm{fgproj}-\widehat{Z}_d$, be the category of finite-dimensional, graded, projective left, resp. right, $\widehat{Z}_d$-modules and degree-preserving module maps. The indecomposable objects in these categories are isomorphic to $\widehat{Z}_d e_i\langle t\rangle$, resp. $e_i\widehat{Z}_d\langle t\rangle$, for some $i \in \widehat{I}$ and $t \in \mathbb{Z}$.

Finally, let $\widehat{Z}_d-\mathrm{fgbiproj}-\widehat{Z}_d$ be the monoidal category of all finite-dimensional, graded, biprojective $\widehat{Z}_d-\widehat{Z}_d$-bimodules and degree-preserving bimodule maps. A bimodule is called biprojective if it is projective as a graded left module and as a graded right module, but not necessarily as a graded bimodule. Every indecomposable projective object in this category is isomorphic to

$$\widehat{Z}_d e_i \otimes e_j \widehat{Z}_d\langle t\rangle,$$

for some $i,j \in \widehat{I}$ and $t \in \mathbb{Z}$. The monoidal structure of $\widehat{Z}_d-\mathrm{fgbiproj}-\widehat{Z}_d$ is given by tensoring over $\widehat{Z}_d$ and the unit object is $\widehat{Z}_d$, which is biprojective but not projective as a bimodule over itself. Recall that any exact, graded endofunctor of $\widehat{Z}_d-\mathrm{fgproj}$ is naturally isomorphic to $B \otimes_{\widehat{Z}_d} -$, for some $B \in \widehat{Z}_d-\mathrm{fgbiproj}-\widehat{Z}_d$. Natural transformations between exact, graded endofunctors correspond to $\widehat{Z}_d-\widehat{Z}_d$-bimodule maps and the composition of endofunctors corresponds to the tensor product of the corresponding bimodules over $\widehat{Z}_d$.

Let τ be the degree-preserving algebra automorphism of $\widehat{Z}_d$ induced by the counterclockwise rotation of the Dynkin diagram defined by

$$e_i \mapsto e_{i+1}, \quad 0|(d-1) \mapsto (-1)^d(1|0), \quad i|j \mapsto (i+1)|(j+1), \tag{145}$$

for $i, j \in \widehat{I}$, such that $j = i \pm 1$ but $(i,j) \neq (0, d-1)$. Note that $\tau^d = \mathrm{id}$ when d is even, and $(\tau)^{2d} = \mathrm{id}$ when d is odd. By definition, the *twisted bimodule*

$$\widehat{Z}_n^\tau \in \widehat{Z}_d\text{−fgbiproj−}\widehat{Z}_d \tag{146}$$

has underlying vector space $\widehat{Z}_d$, while the left and right $\widehat{Z}_d$-actions are defined by

$$a \cdot_L b \cdot_R c := ab\tau(c), \tag{147}$$

for $a, b, c \in \widehat{Z}_d$. It is clear that $\widehat{Z}_d^\tau \cong \widehat{Z}_d$ as left and as right $\widehat{Z}_d$-modules, but not as $\widehat{Z}_d$-$\widehat{Z}_d$-bimodules. As a consequence, $\widehat{Z}_d^\tau$ is biprojective. It is, however, not projective as a $\widehat{Z}_d$-$\widehat{Z}_d$-bimodule. We record the existence of an isomorphism

$$\widehat{Z}_d^{\tau^k} \otimes_{\widehat{Z}_d} \widehat{Z}_d^{\tau^m} \cong \widehat{Z}_d^{\tau^{k+m}} \tag{148}$$

in $\widehat{Z}_d$−fgbiproj−$\widehat{Z}_d$, for every pair $k, m \in \mathbb{Z}$.

Note further that there exist isomorphisms of left, resp. right, $\widehat{Z}_d$-modules

$$\widehat{Z}_d^\tau \otimes_{\widehat{Z}_d} \widehat{Z}_d e_i \cong \widehat{Z}_d e_{i+1} \quad \text{and} \quad e_i \widehat{Z}_d \otimes_{\widehat{Z}_d} \widehat{Z}_d^\tau \cong e_{i-1}\widehat{Z}_d \tag{149}$$

and, therefore, an isomorphism of $\widehat{Z}_d$-$\widehat{Z}_d$-bimodules

$$\widehat{Z}_d^\tau \otimes_{\widehat{Z}_d} \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \cong \widehat{Z}_d e_{i+1} \otimes e_{i+1} \widehat{Z}_d \otimes_{\widehat{Z}_d} \widehat{Z}_d^\tau \tag{150}$$

for every $i \in \widehat{I}$.

The *zigzag algebra* Z_d of finite type A_{d-1} is by definition the idempotent subalgebra

$$(e_1 + \cdots + e_{d-1})\widehat{Z}_d(e_1 + \cdots + e_{d-1}). \tag{151}$$

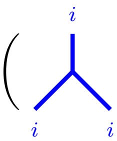
6.4. **The birepresentations.** Let $Z = Z_d$ denote the zigzag algebra of finite type A_{d-1}. Recall the finitary birepresentation $\mathbf{M}_d$ of $\mathcal{S}_d$ acting on Z-gproj, the finitary category of finite-dimensional, graded projective Z-modules, by graded, biprojective Z-Z-bimodules. Under this birepresentation, $\mathbb{1} = R$ acts by tensoring (over Z) with Z and each B_i acts by tensoring (over Z) with $Ze_i \otimes e_i Z\langle 1\rangle$, for $i \in I$. The image of the generating Soergel diagrams is given by

(152) $\mathbf{M}_d\Big($  $\Big)$

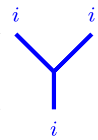
$$: \quad Ze_i \otimes e_i Z\langle 1\rangle \to Z, \qquad ae_i \otimes e_i b \mapsto ae_i b,$$

(153) $\mathbf{M}_d\Big($  $\Big)$

$$: \quad Z \to Ze_i \otimes e_i Z\langle 1\rangle, \qquad e_j \mapsto \begin{cases} (-1)^i\left(\ell_i \otimes e_i + e_i \otimes \ell_i\right), & j = i; \\ (-1)^i\left(j|i \otimes i|j\right), & j \pm 1 = i, \end{cases}$$

(154) $\mathbf{M}_d\Big($ 

$\Big)$ $$: \quad Ze_i \otimes e_i Ze_i \otimes e_i Z\langle 2\rangle \;\to\; Ze_i \otimes e_i Z\langle 1\rangle$$
$$e_i \otimes e_i a e_i \otimes e_i \;\mapsto\; (-1)^i \mathrm{tr}(e_i a e_i) e_i \otimes e_i$$

(155) $\mathbf{M}_d\Big($ 

$\Big)$ $$: \quad Ze_i \otimes e_i Z\langle 1\rangle \;\to\; Ze_i \otimes e_i Ze_i \otimes e_i Z\langle 2\rangle$$
$$e_i \otimes e_i \;\mapsto\; e_i \otimes e_i \otimes e_i,$$

while all other generating Soergel diagrams are sent to zero. The proof that this is well-defined is a straightforward computation and similar to the proof of [MT2019, Theorem I]. It is easy to see that this birepresentation decategorifies to the representation M_d of H_d, given in (30).

Now, consider the triangulated birepresentations $\mathbf{M}^{\mathcal{E}v_{r,s}}$ and $\mathbf{M}^{\mathcal{E}v'_{-r,-s}}$ of $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$, for $r, s \in \mathbb{Z}$, obtained by pulling $\mathcal{K}^b(\mathbf{M})$ back through the evaluation functors $\mathcal{E}v_{r,s}$ and $\mathcal{E}v'_{-r,-s}$. These decategorify to M^{ev_a} and $M^{\mathrm{ev}_{a^{-1}}}$ defined in (32) and (33), respectively, where $a = (-1)^s q^r$. The case $(r, s) = (d-2, 2-d)$ is somewhat special, as it corresponds to the so-called *Tate twist*, but the general case can easily be derived from this one by shifting the bigrading in all arguments below. To keep the notation simple, we therefore consider $\mathbf{M}^{\mathcal{E}v_{r,s}}$ for the fixed choice $(r, s) = (d-2, 2-d)$ first.

Define the complex

(156) $$X_0 := \big(\underline{Ze_{d-1}\langle 1\rangle} \to Ze_{d-2}\langle 2\rangle \to \cdots \to Ze_1\langle d-1\rangle\big)$$

where the term $Ze_{d-1}\langle 1\rangle$ is in homological degree 0 and the differential in position i is given by right multiplication by $d-i-1|d-i-2$. We further set $X_i := Ze_i$, for $i \in I$.

In Proposition 6.2, the rank of the bigraded-finitary cover $\mathbf{L}$ of an evaluation cell birepresentation is not necessarily minimal. In the following proposition, we give a minimal finitary cover for $\mathbf{M}^{\mathcal{E}v_{r,s}}$.

Proposition 6.4. *The bigraded-finitary subcategory*
$$\widehat{\mathbf{M}}_{d-2,2-d} := \mathrm{add}\,\{(X_0 \oplus X_1 \oplus \cdots \oplus X_{d-1})\langle i\rangle[j] \mid i, j \in \mathbb{Z}\}$$
is stable under the action of $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$*, and hence carries the structure of a finitary birepresentation of* $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$*, which we denote by the same symbol.*

Proof. We need to check stability under $\mathrm{B}_1 \dots, \mathrm{B}_{d-1}$ and T_ρ. The action of $\mathrm{B}_1 \dots, \mathrm{B}_{d-1}$ stabilises $\mathrm{add}\,\{X_1 \oplus \cdots \oplus X_{d-1}\langle i\rangle[j] \mid i, j \in \mathbb{Z}\}$ since this is just the finitary birepresentation of $\mathcal{S}_d$ described above. We therefore first compute $\mathrm{B}_i(X_0)$ for $i \in I$ and then verify stability of $\mathrm{add}\,\{X_1 \oplus \cdots \oplus X_{d-1}\langle i\rangle[j] \mid i, j \in \mathbb{Z}\}$ under T_ρ.

Notice that, for $i \in \{2, \cdots d-2\}$, $\mathrm{B}_i(X_0)$ is given by the complex
$$Ze_i \otimes \Big(e_i Ze_{i+1}\langle d-i\rangle \to e_i Ze_i\langle d-i+1\rangle \to e_i Ze_{i-1}\langle d-i+2\rangle\Big).$$

Notice that, as a Z-module, this is just Ze_i tensored with a complex of vector spaces, so it suffices to argue that said complex of vector spaces is null-homotopic. This is indeed the case since the first map embeds a one-dimensional space into a two-dimensional space, and the second map is a surjection onto another one-dimensional space. It follows that the whole complex is null-homotopic.

Further, $\mathrm{B}_1(X_0)$ is given by

$$Ze_1 \otimes \Big(e_1 Ze_2\langle d-1\rangle \to e_1 Ze_1 \langle d\rangle \Big)$$

with map $1|2 \mapsto \ell_1$, which is injective, hence the summand surviving Gaussian elimination is $Ze_1 \otimes e_1\langle d\rangle$ in homological degree $d-2$. Thus the result is homotopy equivalent to $Ze_1\langle d\rangle[2-d] = X_1\langle d\rangle[2-d]$.

On the other extreme, $\mathrm{B}_{d-1}(X_0)$ is given by

$$Ze_{d-1} \otimes \Big(\underline{e_{d-1}Ze_{d-1}\langle 2\rangle} \to e_{d-1}Ze_{d-2}\langle 3\rangle \Big)$$

where the map is right multiplication by $d-1|d-2$, which is surjective. The kernel is thus $Ze_{d-1}\otimes \ell_{d-1}\langle 2\rangle$ and the result is homotopy equivalent to $Ze_{d-1} = X_{d-1}$ without any shifts.

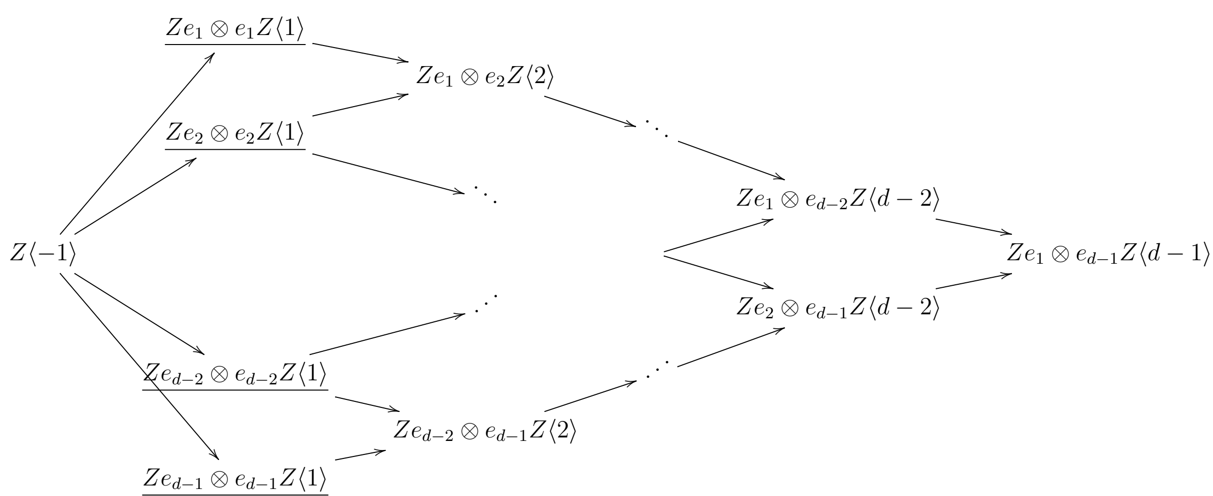
Thus $\mathrm{add}\,\{X_0 \oplus X_1 \oplus \cdots \oplus X_{d-1}\langle i\rangle[j] \mid i,j\in\mathbb{Z}\}$ is stable under the action of $\mathrm{B}_1,\ldots,\mathrm{B}_{d-1}$.

It remains to show that $\widehat{\mathbf{M}}$ is stable under the action of T_ρ. Recall from Section 5.1 that $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho) = \mathrm{T}_1^{-1}\cdots\mathrm{T}_{d-1}^{-1}\langle d-2\rangle[2-d]$ and

$$\mathrm{T}_i^{-1} = R\langle -1\rangle \xrightarrow{\ \bullet\ } \underline{\mathrm{B}_i}.$$

Using the definition of $\mathbf{M}_d$ above, it is easy to see that the complex representing $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)$ is

(157)

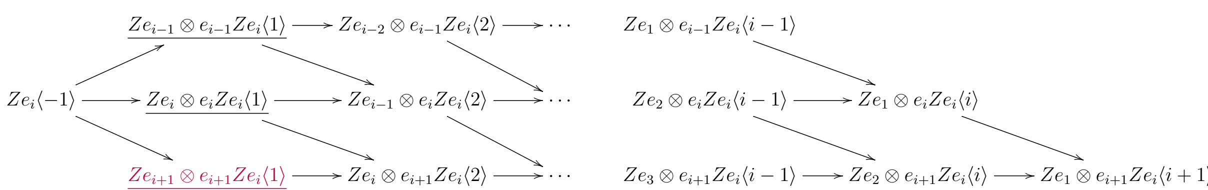


Here, in the first differential, whose source is $Z\langle -1\rangle$, the component mapping to $Ze_i \otimes e_i Z\langle 1\rangle$ is given by

$$e_j \mapsto \begin{cases} \ell_i \otimes e_i + e_i \otimes \ell_i, & \text{if } i = j; \\ j|i \otimes i|j, & \text{if } i \neq j. \end{cases}$$

The other differentials are all vectors of Z-Z-bimodule maps which are equal to the tensor product of $\pm\mathrm{id}$ on one tensor factor and $i|i+1$, for some $i = 1, \dots, d-2$, on the other tensor factor. For our arguments below, the signs of these maps are not important.

We are first going to prove that $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)(X_i) \simeq X_{i+1}$, for any $i = 1, \dots, d-2$. Since $e_j Z e_i = \{0\}$ when $|i-j| > 1$, the non-zero part of the complex corresponding to $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)(X_i)$ is



By Gaussian elimination, one can then see that this is homotopy equivalent to the purple $Ze_{i+1} \otimes e_{i+1}Ze_i\langle 1\rangle$ in homological degree zero, which is isomorphic to X_{i+1}. To explain this, we identity each vertex of the diagram above by its pair of coordinates (row number, column number), where we number the rows of the complex by 1,2,3 from top to bottom and the columns by their homological degree. As in the diagram above, we omit the signs of all maps below, since they are not important for our argument. Using these conventions, first note that the the part of the complex $(2,-1) \to (2,0) \to (3,1)$ is given by

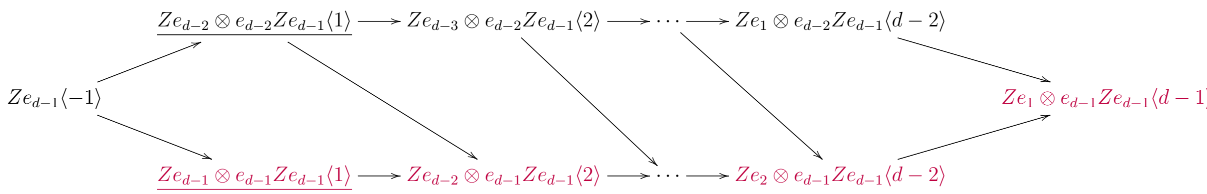
$$Ze_i \otimes \left(\Bbbk\langle -1\rangle \to \underline{e_i Z e_i\langle 1\rangle} \to e_{i+1}Ze_i\langle 2\rangle \right)$$

where the complex of vector spaces is split by the same arguments as above and hence null-homotopic. Thus these three terms cancel in the Gaussian elimination procedure. Similarly, every part of the complex of the form $(1,j) \to (2,j+1) \to (3,j+2)$, for $j = 0, \dots, i-2$, is given by

$$Ze_{i-j-1} \otimes \left(e_{i-1}Ze_i\langle j+1\rangle \to e_iZe_i\langle j+2\rangle \to e_{i+1}Ze_i\langle j+3\rangle \right)$$

is split and hence null-homotopic. Hence all these triples of terms cancel in the Gaussian elimination procedure, which in the end only leaves the purple one, proving the desired homotopy equivalence.

The next homotopy equivalence we are going to prove is $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)(X_{d-1}) \simeq X_0$. The non-zero part of the complex $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)(X_{d-1})$ is



where the differentials are as above. Note that again all maps pointing to the south-east in the complex are given by the tensor product of the identity of some Ze_{d-j} with an injective map of vector spaces hence split. Thus, by Gaussian elimination, this complex is homotopy equivalent to the direct summand of the purple subcomplex for which the right tensor factor is restricted to multiples of e_{d-1}. This direct summand is indeed isomorphic to X_0.

The remaining case of the action of $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)$ on X_0 can be replaced by considering the action of $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)^{-1}$ on X_1, which is analogous to the action of $\mathcal{E}v_{d-2,2-d}(\mathrm{T}_\rho)$ on X_{d-2}. □

Similarly, we can define an additive birepresentation $\widehat{\mathbf{M}}_{r,s}$ of $\widehat{\mathbb{S}}_d^{\mathrm{ext}}$, for any $r, s \in \mathbb{Z}$.

Corollary 6.5. *For any $r, s \in \mathbb{Z}$, there is a morphism of additive $\widehat{\mathbb{S}}_d^{\mathrm{ext}}$-birepresentations $\Phi\colon \widehat{\mathbf{M}}_{r,s} \to \mathbf{M}^{\mathcal{E}v_{r,s}}$, induced by the embedding from Proposition 6.4. This makes $\widehat{\mathbf{M}}_{r,s}$ into a finitary cover of $\mathbf{M}^{\mathcal{E}v_{r,s}}$.*

Proof. All assertions follow immediately from Proposition 6.4. □

Remark 6.6. Note that $\widehat{\mathbf{M}}_{r,s}$ decategorifies to the Graham–Lehrer cell module $\widehat{M}_{d,\lambda}$ with $\lambda = (-1)^{s-(2-d)}q^{r-(d-2)}$, as can be easily seen by comparing the action of the generators on the X_i with the decategorified action in (23) and (24). Moreover, Φ decategorifies to the projection of $\widehat{M}_{r,s}$ onto $L^+_{d,(-1)^{s-(2-d)}q^{r-(d-2)}}$.

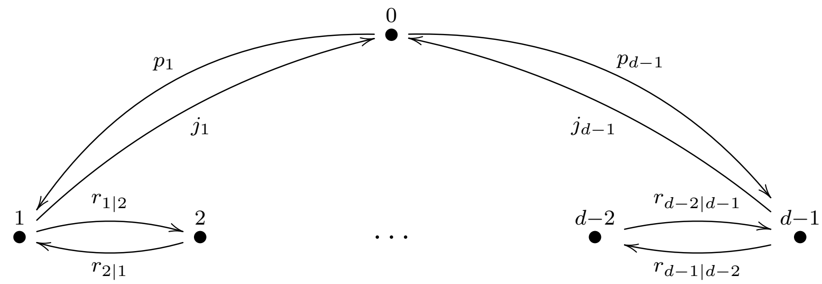
Proposition 6.7. *For any $r, s \in \mathbb{Z}$, there is an isomorphism of ungraded algebras*

$$\mathrm{End}_{\mathbf{M}^{\mathcal{E}v_{r,s}}}(X_0 \oplus \cdots \oplus X_{d-1}) \cong \widehat{Z}.$$

Proof. Without loss of generality, we assume that $(r, s) = (d-2, 2-d)$, as before. Denote by $p_{d-1}\colon X_0 \to X_{d-1}$ the projection onto the component in homological degree 0 and by $j_d\colon X_{d-1} \to X_0$ the map induced by multiplication with ℓ_{d-1}. Similarly, denote by $j_1\colon X_1[2-d] \to X_0$ the inclusion of the component in homological degree $d-2$ and by $p_1\colon X_0 \to X_1[2-d]$ the map induced by multiplication with ℓ_1. We remark that $p_{d-1}, j_{d-1}, j_1, p_1$ have degrees $1, 1, 1-d, d+1$, respectively. Moreover, we denote the maps $Ze_i \to Ze_{i\pm1}$ given by right multiplication by $i|i\pm1$ by $r_{i|i\pm1}$. Then it is a straightforward calculation to verify that

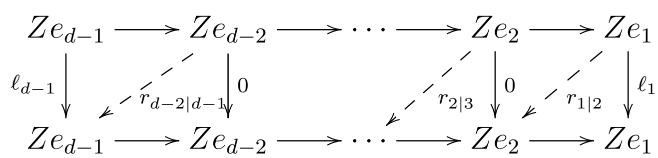
$\mathrm{End}_{\mathbf{M}^{\varepsilon_{v_a}}}(X_0 \oplus \cdots \oplus X_{d-1})$ is given by the path algebra of the quiver

(158)



modulo the relations defining $\widehat{Z}$ under the isomorphism sending $r_{i|i\pm1}$ to $i \pm 1|i$, p_i to $i|0$ and j_i to $0|i$ for $i \in \{1, d-1\}$. To verify the sign in the relation involving 0 we observe that the endomorphism of X_0 given by $j_1p_1 + (-1)^{d-1}j_{d-1}p_{d-1}$ is (omitting shifts for readability) given by the solid arrows in the diagram

(159)



and is null-homotopic via the homotopy indicated by the dashed arrows. $\square$

Remark 6.8. The natural bigrading of $\mathrm{End}_{\mathbf{M}^{\varepsilon_{v_{r,s}}}}(X_0 \oplus \cdots \oplus X_{d-1})$ induces a bigrading on $\widehat{Z}$ via the isomorphism in Proposition 6.7. Note that it does not depend on $(r, s) \in \mathbb{Z}^2$, as long as we keep the gradings of the X_i fixed. Assuming that $(r, s) = (d-2, 2-d)$, we see that it is given by

$$\deg(i|i+1) = (1,0), \quad i \in I; \tag{160}$$

$$\deg(i|i-1) = (1,0), \quad i \in \widehat{I}\backslash\{1\}; \tag{161}$$

$$\deg(0|1) = (d+1, 2-d); \tag{162}$$

$$\deg(1|0) = (1-d, d-2). \tag{163}$$

Note that the first entry of this bigrading is compatible with the above grading of $\widehat{Z}$ except for the degrees of the arrows between 0 and 1.

The explicit 2-action of $\widehat{\mathcal{S}}_d^{\mathrm{ext}}$ on $\widehat{\mathbf{M}}_{r,s}$ is given

- on 1-morphisms by

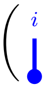
$$F(i) := \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d\langle 1\rangle, \quad i \in \widehat{I}; \tag{164}$$

$$F(\pm) := \widehat{Z}_d^{\tau^{\pm1}}\langle r\rangle[s], \tag{165}$$

- on 2-morphisms by

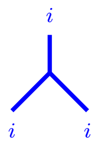
$$F\left(\underset{i}{\bullet\!\!|}\right) : \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d\langle 1\rangle \to \widehat{Z}_d, \qquad ae_i \otimes e_i b \mapsto ae_i b, \tag{166}$$

(167)

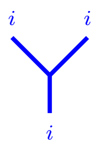
F() $: \widehat{Z}_d \to \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \langle 1 \rangle$

$$e_j \mapsto \begin{cases} (-1)^i (\ell_i \otimes e_i + e_i \otimes \ell_i), & j = i; \\ (-1)^i (j|i \otimes i|j), & j \pm 1 = i \neq 0; \\ 1|0 \otimes 0|1, & j = 1, i = 0; \\ (-1)^d (d-1|0 \otimes 0|d-1), & j = d-1, i = 0, \end{cases}$$

(168)

F() $: \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \langle 2 \rangle \to \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \langle 1 \rangle$

$$e_i \otimes e_i a e_i \otimes e_i \mapsto (-1)^i \mathrm{tr}(e_i a e_i) e_i \otimes e_i$$

F() $: \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \langle 1 \rangle \to \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d e_i \otimes e_i \widehat{Z}_d \langle 2 \rangle$

$$e_i \otimes e_i \mapsto e_i \otimes e_i \otimes e_i.$$

The generating 2-morphisms involving an oriented black strand in (51) and (52) are sent to the isomorphisms in (148) and (150), respectively, and all other generating 2-morphisms are sent to zero.

Remark 6.9. We could alternatively have used the evaluation functor $\mathcal{E}v'_{-r,-s}$ to obtain another evaluation birepresentation and its finitary cover.

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