Beyond the Mathematics of the Moment: Exploring Teacher Discourse at the Mathematical Horizon

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Στη μαμά και τον μπαμπά μου

(To my mum and dad)

Abstract

The thesis aims to explore secondary mathematics teachers' communicational actions beyond the mathematics of the moment, namely beyond the boundaries of the curriculum. Building on the Theory of Commognition (Sfard, 2008) and the construct of Discourse at the Mathematical Horizon (Cooper, 2016), I explore teachers' patterns of communication about mathematical ideas and practices that are not predicated by curriculum guidelines. The study takes place in England with nine secondary mathematics teachers, three groups of 11 to 14-year-old students and three teacher educators. The data include sixteen lesson observations, thirteen interviews with teachers and teacher educators, and one focus group discussion with four teachers around a pilot vignette-based activity (Biza et al., 2018).

The analysis revealed taken and potential opportunities for discussions beyond the mathematics of the moment in a mathematics classroom, including: ideas and practices that run across the mathematics curriculum or which students might encounter in the future; mathematical conventions; and, applications of mathematics. Effective communication beyond the mathematics of the moment is achieved through intersubjective discursive elements between teacher and student discourses (e.g., bespoke word use and visual mediators). The findings suggest that Discourse at the Mathematical Horizon is an amalgamation of advanced mathematical and pedagogical discourses where patterns of mathematical communication are attributed pedagogical meaning. Individual teachers' discourses are influenced by educational, personal and professional experiences, and by narratives about students' 'abilities' and engagement. Thus, Discourse at the Mathematical Horizon is refined as a meta-discourse of advanced mathematics enriched with intersubjective discursive elements, atypical in advanced mathematics but relevant to the current or future experiences of the students. Finally, findings from the focus group indicate shifts in participants' discourses. I use these findings to propose a methodological approach for creating resources for professional development and research activities to bring up discourses at the Mathematical Horizon.

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Contents

List of Abbreviations

AST	Advanced Skills Teacher
BSRLM	British Society for Research into Learning Mathematics
CERME	Congress of the European Society for Research in Mathematics Education
CPD	continuous professional development
DfE	Department for Education
GCSE:	General Certificate of Secondary Education
ITT	Initial Teacher Training
KS	Key Stage within the National Curriculum
NCETM	National Centre for Excellence in the Teaching of Mathematics
OFSTED	Office for Standards in Education
PGCE	Postgraduate Certificate in Education
PME	Psychology of Mathematics Education
QTS	Qualified Teacher Status
RME	Research group in Mathematics Education (University of East Anglia)
SCITT	School-Cantered Initial Teacher Training
STEM	Science, Technology, Engineering and Mathematics
TSST	Teacher Subject Specialism Training
UEA	University of East Anglia

List of figures

List of tables

1 Introduction

> A curriculum is an attempt to communicate the essential principles and features of an educational proposal in such a form that it is open to critical scrutiny and capable of effective translation into practice.
>
> (Stenhouse, 1975, p. 4)

> The curriculum is conceived both as a tool for classroom work and as a contract or transaction between what society expects schools to offer and what their authorities accept it should offer in terms of contents to be taught and institutional experiences to be provided.
>
> (Braslavsky, 2002, p. 10)

The thesis aims to explore secondary mathematics teachers' communicational actions beyond the mathematics of the moment, i.e., beyond the boundaries of the curriculum. My interest in the topic arose during my final year of undergraduate studies in Mathematics. At the time, I was attending an optional module in Mathematics Education which included school visits and weekly seminars to discuss seminal research in Mathematics Education, teaching practices and our experiences from the school visits. In one of the first seminars, we were looking at different representations used for teaching fractions. The lecturer asked us to find connections between the notion of fractions as taught in school and mathematics we had learnt during our undergraduate studies. The question puzzled me, the room fell quiet for a moment until one of my peers mentioned equivalence classes. At that time, I experienced my very own 'aha' moment. Upon reflection, I started recognising more connections among the mathematical topics I had encountered throughout the years, and I realised that some of my teachers had hinted at connections I could not fully grasp at the time. Although I did not think much about my teachers' comments for a while, their words remained in my memory and developed new meanings for me years later.

The following year, I started tutoring and enrolled in the Masters programme "Mathematics for Education". I started experimenting with my practice and building routines that would allow me to initiate discussions beyond the mathematics of the moment with my students. I was watchful for opportunities to discuss ideas beyond what it is explicitly included in the school curriculum because I thought it might help my students realise the world around them differently in the future. During my Masters, I came across the concept of Horizon Content Knowledge (Ball & Bass, 2009) which prompted me to look for suggestions in existing research in Mathematics Education. Despite my enthusiasm, my dedication to the cause and the hours I spent studying, I was not always able to find the right words to talk about ideas and practices beyond the mathematics of the moment with my students. My experiences and my desire to better understand a key trait of my identity as a mathematics teacher influenced my decision to explore the matter more deeply by pursuing a PhD. The rest of this work is an attempt to present what I learnt during my journey as a doctoral student in a linear fashion.

Embarking on a journey to explore discussions beyond the mathematics of the moment, I first need to elaborate how curriculum is seen in the context of this study. My understanding of curriculum is shaped by the works of Lawrence Stenhouse and Cecilia Braslavsky. The opening excerpts encapsulate qualities I view as essential in any curriculum. I view the curriculum as an act of communication, a contract, between policy makers and education professionals. This contract should be flexible, open to scrutiny, and pragmatic.

Mathematics curricula, in particular, need to meet the educational demands of the 21st century which include connections with mathematical practices in students' future careers and life (Wake, 2014). Mathematics is intra- and inter-connected. One can find applications of mathematics in almost every field and profession. In many instances, the mathematics required goes beyond basic skills (e.g., the importance of prime numbers in network security or Arrow's impossibility theorem in political science[1]). On the other hand, mathematics is a discipline with many domains and many connections across them. Theories that give rise to new mathematical ideas often bring together seemingly unrelated parts of mathematics. By its nature, mathematics is a complex and growing web of ideas and applications. Mathematics taught in schools, even if it might be seen as elementary, it is still part of this complex web.

[1] General information about Arrow's impossibility theorem can be found here: https://plato.stanford.edu/entries/arrows-theorem/

> [M]athematics curricula need to reposition mathematics as a discipline that builds connections; indeed, that it should be at the nexus of our interaction as individuals and communities with each other and with a range of situations and contexts. It is important, therefore, to design curricula in ways that ensure that mathematics is valued by learners as they attempt to make sense of, and with, mathematics in ways that facilitate their being able to engage in practice (doing) and developing their identity (becoming)(Wake, 2014, p. 288).

Wake's (2014) proposal indicates a shift of focus from ensuring which topics should be taught towards fostering opportunities for learners to engage with mathematics in meaningful ways and help them develop their mathematical identities. In presence of such curricula, governments, schools, and teachers share responsibilities. Seen as such, curricula are not exhaustive lists of topics but rather they include provisions of what should be learnt. Therefore, although a teacher is expected to follow a curriculum and is responsible for its implementation in her[2] classroom, there is space for divergence.

We live in a society where needs are fast-changing. Mathematics and its applications appear to have a central role in understanding and adapting to the uncertainties that lie before us. However, there is evidence to suggest variations in the operationalisation of real-world applications in mathematics curricula around the world (Smith & Morgan, 2016). Adapting the content of the curriculum every few years to meet these changes is neither realistic nor sustainable. Therefore, a teacher should be prepared to recognise and facilitate opportunities for making connections that would benefit the learner's future. She should, also, be prepared to adapt to changes even in cases where the curriculum does not explicitly account for them. An understanding of mathematics beyond the curriculum could benefit the teacher in supporting her students to get a glimpse into the world of mathematics and its applications while keeping the conversation relevant to students' needs and grounded in ideas that the students already know.

Despite reform efforts, curricula are not always open to interpretation and flexibility is often misunderstood as absence of density (Braslavsky, 2002). For example, in England, there have

[2] I use the feminine pronouns (she/her/hers) to signify when an argument might be also influenced by my positionality.

been revisions in the National Curriculum. However, vagueness remains a central issue[3]. In theory, the guidelines are open to different interpretations and offer flexibility. In practice, the vagueness could lead schools to focus more on exam-like questions to balance the uncertainty, thus narrowing down the variety of practices. For instance, the National Curriculum for Key stage 4 suggests briefly that students "should also apply their mathematical knowledge wherever relevant in other subjects and in financial contexts" (Department for Education, 2014a, p. 3) providing limited information or exemplary ideas to be explored by teachers. In practice, the interdisciplinary content is determined by the additional guidelines of the exam boards. The guidelines of the exam boards converge to a list of basic topics, e.g., interpreting distance-time graphs, using the formula of density, or calculating compound interest. Schools often focus on the list of topics proposed by the boards to prepare students for exams, thus, limiting the flexibility to explore other interdisciplinary connections, for example, connections with computer science or sports. Educational research and policy makers are working towards understanding and delineating the issue. In the meantime, schools and teachers are faced with a challenge to balance the suggestions of the existing curricular resources. In this context, a teacher should be flexible and prepared to adapt the content she is teaching to reflect the priorities of the school where she currently works. An understanding of the mathematical ideas and practices beyond the curriculum a teacher is expected to teach in a particular school could aid in navigating her teaching career in a fast-changing world.

A question stemming from the arguments presented thus far is how can a teacher develop an understanding of mathematics beyond the curriculum? A probable answer is through initial teacher education. Yet, the form and structure of initial mathematics teacher education varies around the world (Dahl, 2005; Ingvarson & Rowley, 2017; Tatto & Hordern, 2017). One difference is the tertiary degrees future teachers are required to obtain before or concurrently with their initial teacher training. For example, about half of mathematics teachers in England have a degree in a discipline other than mathematics (Allen & Sims, 2018) which makes their mathematical backgrounds diverse. In other words, their experiences with

[3] In the academic year 2021-22, the government published additional non-statutory guidance for Key Stage 3 based on NCETM secondary mastery framework, as an effort to address the impact of COVID-19. https://www.gov.uk/government/publications/teaching-mathematics-at-key-stage-3

the mathematical content vary depending on their discipline and they possibly bring different mathematical discourses into their teaching practices. This diversity has an advantage. A teacher might have some experience with the mathematical content in other disciplines as well as in mathematics that she could bring in the classroom and share with students. However, looking into initial teacher education is not sufficient in answering the question.

This study is about teaching practices beyond the mathematics of the moment. My aim is to investigate secondary mathematics teachers' practices that are not grounded in the curriculum, what are the characteristics which lead to effective communication with their students and how teachers develop such practices. I use the Theory of Commognition (Sfard, 2008) as a lens to study the discourses of teachers and the communication between the teacher and the students beyond the mathematics of the moment. Specifically, I build on and operationalise Discourse at the Mathematical Horizon (Cooper, 2016) to explore teachers' patterns of communication about mathematical ideas and practices that are not predicated by curriculum guidelines. This study also explores the idea of transforming empirical data into vignette-based research and professional development resources. In the context of this study, the proposed design draws on the MathTASK design principles (Biza et al., 2018, 2021).

Specifically, my study addresses the following research questions and sub-questions:

RQ 1 What are the characteristics of discussions *beyond the mathematics of the moment*?

- **1a.** What taken and potential opportunities for discussion beyond the mathematics of the moment can be identified in everyday teaching practice?
- **1b.** How could teachers and students communicate effectively when opportunities for discussion beyond the mathematics of the moment are taken?
- **1c.** What experiences shape individual discourses at the Mathematical Horizon?

RQ 2 How can empirical evidence be used to create practice-based resources for developing teachers' [d]Discourse[s] at the Mathematical Horizon?

Following this chapter, Chapter 2 focuses on the core literature around teachers' knowledge, teacher education, and teaching practice which informed the theoretical and conceptual approach I adopt in this study. I present an overview of different models of teachers' knowledge and discuss their conceptual and practical suitability for exploring the idea of teachers navigating beyond the mathematics of the moment. Next, I focus on the literature about initial and continued teacher education to look at what is proposed in relation to teachers' professional learning for teaching beyond the mathematics of the moment. Finally, I shift my attention to and critique the dichotomy between knowing and teaching and identify the gap in the literature which this study aims to address.

Chapter 3 aims to provide the reader with insights regarding the conceptualisation of Discourse at the Mathematical Horizon in my study. Firstly, I present an overview of the theory of Commognition which takes a sociocultural and discursive perspective on teaching and learning mathematics. Then, I describe the main tenets of the theory that are relevant to the study. Having outlined the various models that describe teachers' knowledge in Chapter 2, I proceed with a review of the literature about Horizon Content Knowledge, looking into the proposed conceptualisations and identifying similarities, differences, and gaps among different research narratives about mathematical horizon. Finally, I revisit Cooper's (2016) original definition of Discourse at the Mathematical Horizon and propose a refined working definition to be operationalised in subsequent chapters.

Chapter 4 discusses the research design and methodology of the study. I begin by describing the research design and research questions. Next, I offer an overview of the context and the demographics of the study, followed by a summary of the methods used. I comment on my role as a researcher. Then, I discuss the data analysis, I outline my actions to ensure the trustworthiness of the study, I reflect on ethical considerations and explain how I addressed methodological issues. Finally, I reflect on my experience doing research during the COVID-19 pandemic.

Then, the three empirical chapters follow. Chapters 5 and 6 focus on exploring the characteristics of discussions beyond the mathematics of the moment (RQ1) based on empirical data collected through lesson observations and semi-structured interviews. Chapter 5 addresses specifically the research questions RQ 1a and RQ 1b. In this chapter I discuss the patterns of communication in taken and potential opportunities to go beyond the mathematics of the moment in sections that correspond to four emerging themes: *across the mathematics curriculum*; *what comes next*; *conventions*; and *applications*. At the end of the chapter, I summarise and discuss the key findings on opportunities to go beyond the mathematics of the moment in relation to the aims of the study.

In Chapter 6, I explore the participants' reported experiences which influence individual discourses at the Mathematical Horizon and address the research question RQ 1c. In this chapter, I present and discuss the views of participants regarding aspects of their education, personal and professional life that influence their discourses at the Mathematical Horizon. The emerging findings include experience with mathematics and other disciplines in *tertiary education*; *professional experiences* in previous jobs; *reading books* and *sourcing information on the internet*; *interactions within the school environment*; *interpretations of the curriculum* and *narratives about students' 'abilities' and engagement*. Then, I summarise and discuss how the key findings contribute to addressing the aims of the study.

Chapter 7 is the final empirical chapter which focuses on the idea of creating and using professional development resources to explore and develop Discourse at the Mathematical Horizon using data and empirical findings of the study (RQ 2). I base my design on the MathTASK principles (Biza et al., 2018, 2021). The first part of the chapter concerns the design, implementation, and evaluation of a pilot MathTASK activity (mathtask). I provide an a-priori analysis of the pilot mathtask. Then, I discuss and evaluate the implementation of the activity based on empirical findings in Chapters 5 and 6. The second part of the chapter concerns the refinement of the design principles of the resources to bring up discourses at the Mathematical Horizon by utilising the collected data from lesson observations. To discuss my proposal, I draw on the MathTASK design principles, the empirical findings in Chapters 5, 6 and the implementation of the pilot mathtask during the focus group.

In the concluding chapter, Chapter 8, I first synthesise the results presented in Chapters 5,6 and 7 and discuss the substantive, theoretical and methodological contribution of my study. Then, I discuss implication for policy and practice. I report the limitations of the study and discuss ideas for further research. Finally, I provide a reflective account of how conducting this study contributes to my journey as an early career researcher.

2 Setting the scene: Knowledge for teaching, teacher education and teaching practice

The aim of this chapter is to position the current work in relation to the relevant literature and identify the gap which my study will address. In the following sections, I am looking into the literature that contributes to exploring teaching practices beyond the content covered explicitly in the curriculum documentations, i.e., beyond the mathematics of the moment. Starting from research aiming to model professional knowledge and continuing with relevant literature on professional development, I conclude on how these two aspects are linked with teaching beyond the mathematics of the moment.

2.1. Knowing for teaching mathematics

As a starting point of my enquiry, I explore the literature around mathematical knowledge for teaching. My aim is to identify how understandings of mathematical content and practices beyond the school curriculum are conceptualised and linked with teaching practices.

The nature of the mathematical knowledge for teaching has been studied even before mathematics education was established as a separate domain of educational research. Felix Klein in his innovative work "Elementary Mathematics from an Advanced Standpoint" (Klein, 1908/2016) highlighted key mathematical ideas in arithmetic, algebra, analysis and geometry that were connected to school mathematics of his era. The aim of the work was partly to address an issue faced by prospective mathematics teachers[4], the issue of "double discontinuity". Double discontinuity, as Klein (1908/2016) called it, can be described as the challenge of an individual aiming to get into teaching to identify connections between school mathematics, university mathematics and teaching practices. Later, Pólya (1962/2004) suggested that in order for mathematics teachers to develop learners' experience and rational thinking, they need a kind of knowledge not obtained through mathematics teacher

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[4] At the time when Klein's work was first published, mathematics teachers were males who have studied mathematics at university.

education at the time. Since then, our insights into the nature and kinds of knowing for teaching have been broadened.

In educational studies, an important contribution was that of Shulman (1986) which set the basis for later work by highlighting the complexity of teaching. Here, I focus on the three areas of curriculum knowledge he identified: (1) knowledge of the “curriculum materials for a given subject or topic within a grade” (ibid, p. 10) which, in the context of the present study, can be seen as curricular knowledge about mathematics of the moment; (2) lateral curriculum knowledge and (3) vertical curriculum knowledge. Of relevance to the aims of the study are the notions of lateral and vertical curriculum knowledge. Lateral curriculum knowledge refers to familiarity with the curriculum in other subjects within the same grade level while vertical curriculum knowledge refers to familiarity with the curriculum of the subject in previous and later years of schooling (Shulman, 1986). Both vertical and lateral curriculum knowledge are seen as underlying ideas motivating communications beyond the mathematics of the moment.

During a period of teachers’ evaluation reform in the United States (US), Shulman’s seminal work challenged deficit assumptions about teachers and teaching, questioned the relationship between teaching practice and policy making and proposed a conceptualisation of knowledge for teaching that encompasses subject matter, pedagogical and curriculum content knowledge (Shulman, 1986). At the time, his work aimed to set the foundations for teaching reforms in the USA and argued for the implementation of educational research in policy (Shulman, 1987). The seminal works of Shulman and Klein have shaped the conceptualisation of professional knowledge of mathematics teachers (Ball et al., 2008; Carrillo-Yañez et al., 2018; Dreher et al., 2018; Rowland et al., 2005; Stockton & Wasserman, 2017; Zazkis & Leikin, 2010).

The Mathematical Knowledge for Teaching model (Ball et al., 2008) is frequently cited in research on teachers’ mathematical knowledge. Ball and her colleagues at the University of Michigan (e.g., Ball et al., 2008; Ball & Bass, 2002; Hill et al., 2004) proposed the model for conceptualising the knowledge required in teaching mathematics. Focusing on primary school teachers in the US, they developed an inquiry-based model proposing a refinement of Shulman’s (1986) ideas of Subject-Matter Knowledge and Pedagogical Content Knowledge in an attempt to understand and measure Mathematical Knowledge for Teaching. They proposed a categorisation of Subject-Matter and Pedagogical Content Knowledge comprising three sub-domains for each idea. Under the umbrella of Subject Matter Knowledge, Common Content Knowledge is knowledge of mathematical content also applicable in situations other than teaching (e.g., how to solve a quadratic equation), Specialised Content Knowledge includes mathematical knowledge and skills applicable in teaching only (e.g., evaluating non-standard approaches) and Horizon Content Knowledge which is described as "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Pedagogical Content Knowledge includes Knowledge of Content and Students, Knowledge of Content and Teaching and finally knowledge of Curriculum that combine knowledge of content in relation to students (e.g., anticipated students' ways of thinking) to teaching (e.g., choice of appropriate examples) and curriculum materials (e.g., familiarity with curriculum guidelines). Figure 2.1 depicts a visualisation of the sub-domains, the representation is sometimes referred to as "the egg" due to its oval shape.

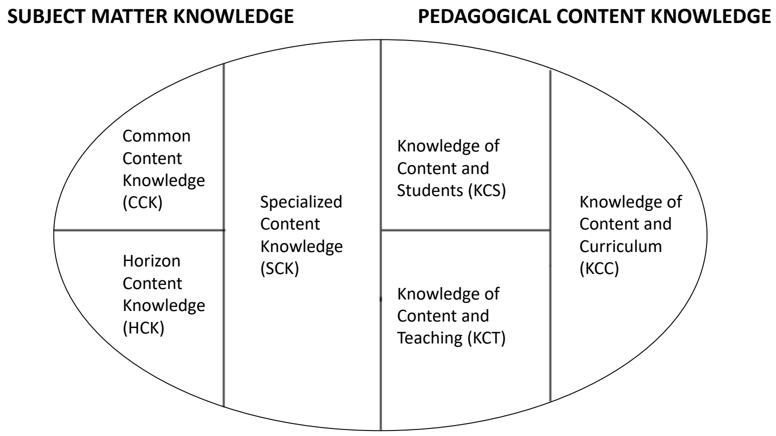


Figure 2.1 'The egg' representation. Recreated based on Ball et al. (2008, p. 403)

Endeavouring to describe teachers' actions at the boundaries of the curriculum and beyond, I was inspired by Ball and colleagues' idea of the Horizon Content Knowledge that was further described as:

> an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (Ball & Bass, 2009, para. 17)

Ball and her colleagues suggest that Horizon Content Knowledge is a notion complementary to Klein's idea of an advanced perspective in elementary mathematics and theorised that it influences, among others, teaching practices related to noticing and evaluating mathematical significance in what the students are saying, foreseeing and making connections across educational levels and disciplines and evaluating opportunities (Ball & Bass, 2009). However, the boundaries of the curriculum, and more so going beyond them, are not always identifiable. So far, several studies drew on the idea of Horizon Content Knowledge with variations in the use and the narratives about the 'horizon' metaphor (see Section 3.2). Evidence of the role of Horizon Content Knowledge in the quality of teaching relies on the different uses and narratives of the 'horizon' metaphor (Jakobsen et al., 2012; Mosvold & Fauskanger, 2013; Papadaki, 2019). Horizon Content Knowledge, among the sub-domains of the Mathematical Knowledge for Teaching model, requires more study in terms of its nature, its relation to teaching and ways of development. My study aims to address a gap in the literature regarding teachers' actions that are not explicitly included in the curriculum. Exploring the literature around Horizon Content Knowledge provides a starting point for this endeavour (see also Section 3.2).

Since its first conceptualisation, the Mathematical Knowledge for Teaching model has been refined (e.g., Hurrell, 2013) and adapted (Carrillo-Yañez et al., 2018; Cooper, 2016). For example, the Mathematics Teacher's Specialised Knowledge model (Carrillo-Yañez et al., 2018), is an adaptation of Ball's et al. (2008) model that considers only the mathematical and pedagogical knowledge that is unique to teaching mathematics. While, the Mathematical Discourse for Teaching model (Cooper, 2016; Mosvold, 2015) is an adaptation that provides a basis to study teaching as a discursive practice.

The Mathematical knowledge for Teaching model was developed primarily to address a demand for evaluating and developing teachers' knowledge in the US and became widely popular at the time. However, the model is by no means perfect, researchers have highlighted limitations in the distinguishability and organisation of the domains (e.g., Charalambous et al., 2019; Fernández et al., 2011; Koponen et al., 2019 etc.) and its potential to account for the knowledge of mathematics teachers in secondary and tertiary level (e.g., Speer et al., 2015). Next, I consider works and alternative models for conceptualising knowing for teaching mathematics. The remainder of the section reflects my thought process in terms of how the consideration of various theoretical constructs adds to the conceptualisation of teaching practices beyond the mathematics of the moment for the purposes of my study.

Another widely used framework worldwile is the Knowledge Quartet (Rowland et al., 2005). The Knowledge Quartet was proposed by researchers in the United Kingdom (UK) and is widely used to analyse teachers' professional knowledge and beliefs. The Knowledge Quartet is a framework introduced initially to locate ways in which preservice primary teachers draw on their mathematical and pedagogical knowledge during practice. The framework was later applied to account for knowledge of mathematics teachers in secondary and tertiary levels of education (e.g., Breen et al., 2018; Kayali, 2019; Rowland, 2010). Rowland et al. (2005) used a grounded approach to describe the four dimensions of the Knowledge Quartet: Foundation, Transformation, Connection and Contigency. The Foundation dimension includes academic mathematical and pedagogical knowledge while the other three dimensions – Transformation, Connection and Contigency – include knowledge that is described 'in-action'.

Both the Mathematical Knowledge for Teaching model and the Knowledge Quartet are influenced by Shulman's work balancing the scale between mathematical and pedagogical knowledge. In contrast to the Mathematical Knowledge for Teaching model, in the Knowledge Quartet, mathematical and pedagogical knowledge are not separated, mathematical and pedagogical aspects of teaching are included in each dimension. The Knowledge Quartet can be used to reflect on teachers' practices as well as their knowledge (Neubrand, 2018)[5]. Aspects of knowledge and actions of teachers beyond the curriculum are reflected in the codes of the Knowledge Quartet that belong in different dimensions. For example, the codes "making connections between procedures" (Rowland et al., 2005, p. 265) and "making connections between concepts" (ibid, p. 265), which belong to the connection dimension, and "deviation from agenda" (ibid, p. 266), which belongs to the contingency dimension, are both relevant in navigating across and beyond the curriculum. Considering teaching practices beyond the mathematics of the moment under this approach highlights the multi-dimensionality of knowledge and practices in relation to teachers' actions that are not explicitly included in delivering curriculum content. Utilising the tenets of Knowledge Quartet, Fernández et al. (2011) proposed a refinement of the Mathematical Knowledge for Teaching model. Horizon Content Knowledge has a central role in the proposed refinement.

[5] A practical advantage of using the Knowledge Quartet as a frame for my study could be that it emerged to address challenges in teacher training in England (Rowland, 2013). Being developed in the same context could potentially require less adaptations to fit the context of the current study.

Specifically, Horizon Content Knowledge is not viewed as one of the sub-domains of the model, rather, it is seen as the mathematical knowledge that brings together the domains of the model “from a continuous mathematical education point of view” (Fernández et al., 2011, p. 2645) (see also Sections 3.2 and 3.3).

The works discussed so far were initially developed for primary school teaching and then adapted and applied to mathematics teaching for more advanced educational levels. The main differences between teaching mathematics in primary and secondary school are the complexity of the topics and the specialisation of the teachers. Primary school teachers typically teach most of the subjects to their students, including mathematics and they are not expected to have a specialisation in mathematics in order to teach. On the other hand, secondary school teachers specialise in teaching mathematics, although the types and the levels of expected specialisations differ among countries (see Section 2.2). Now, I turn to literature on secondary school mathematics teaching. Specifically, I focus on the debate as to whether knowing advanced mathematics influences teaching in secondary school. The following works are influenced by the work of Klein and the notion of elementary mathematics viewed from an advanced perspective.

A model of secondary school mathematics teacher’s knowledge to consider is School Related Content Knowledge (Dreher et al., 2018). Influenced by Klein’s work, Dreher et al.(2018) aim to bridge the gap of double discontinuity (Klein, 1908/2016) between school and university mathematics. They consider the non-trivial relationship between school and university mathematics and propose a model that represents this relationship from two perspectives:

> This construct includes on the one hand knowledge about the curricular structure of school mathematics as well as the legitimation of this structure from a (meta-) mathematical perspective and, on the other hand, knowledge about interrelations between school and academic mathematics at the level of specific contents in both top-down and bottom-up directions (Dreher et al., 2018, p. 337)

Made evident by the aims of the authors, the model considers only knowledge of mathematical content and not pedagogical factors in teaching mathematics. The idea of bridging the gap between academic and school mathematics is very interesting. The model was developed in Germany where mathematics teachers are required to study the subject at tertiary level before obtaining a qualification to teach. However, this conceptualisation would be problematic if applied in contexts where mathematics teacher education does not require the teacher to obtain a degree in Mathematics. In other countries, including England where my study takes place, mathematics teachers might come from different backgrounds and have diverse mathematical experiences. Therefore, university mathematics might be difficult to be considered as common ground.

Zazkis and Leikin (2010) define as Advanced Mathematical Knowledge the "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (p. 264) and examine the views of teachers on whether Advanced Mathematical Knowledge is useful in their teaching practice. In Zazkis and Leikin's (2010) study, teachers indicated reasons why this kind of knowledge could be useful in their practice. However, the majority of the participants failed to provide specific examples (Zazkis & Leikin, 2010). In more recent studies (Mamolo & Taylor, 2018), the gap between Advanced Mathematical Knowledge per se and teaching beyond the curriculum is still evident in the findings. Moreover, there is evidence to suggest that Advanced Mathematical Knowledge could be beneficial but also limiting (Hallman-Thrasher et al., 2019). Particularly, Hallman-Thrasher et al. (2019) conclude:

> content knowledge alone was insufficient to develop robust *knowledge of student understanding*. In fact, confidence in *discipline knowledge* seemed to obscure the need for our participants to explore the content they would be teaching with a critical lens in order to understand students' ways of thinking about content and difficulties students might encounter (p. 270, emphasis in the original).

The lack of consistent evidence to suggest that secondary mathematics teachers benefit from traditional university mathematics courses calls for re-visiting the role of university mathematics courses in teacher education.

Zazkis and Mamolo (2011) proposed a reconceptualisation of Horizon Content Knowledge as an application of Advanced Mathematical Knowledge in teaching (see Section 3.2). A contextual limitation of this approach is that it does not account for diversity in the mathematical backgrounds of secondary teachers. As mentioned earlier, in some countries (e.g., Australia, England, or Switzerland), there are different paths one can follow to become a secondary mathematics teacher. Thus, not every mathematics teacher has to go through an undergraduate course in Mathematics and conceptualising advanced mathematics as a body of knowledge for mathematics teachers would include incompatible narratives about the nature and the scope of mathematics as a discipline. For example, an individual who has encountered advanced calculus applied in physics might have a different experience with calculus compared to someone who studied pure mathematics (Hitier & González-Martín, 2022). Another issue stemming from focusing solely on advanced mathematical knowledge is the underlying assumption that the development of advanced mathematical understanding is attributed to university studies. The institutionalisation of knowledge goes against the idea of acting beyond the curriculum, as the agency of what there is to know is taken away from the individual. Specifically, advanced subject knowledge in this case is confined by the university curricula. Thus, what the teacher knows is seen as limited by what is provided in formal educational contexts. Furthermore, such assumptions amplify the gap between knowledge and practice. Under certain circumstances, university programmes provide opportunities for advanced mathematical learning useful for teaching (e.g., Even, 2011; Pinto & Cooper, 2022). However, other opportunities for learning and development through life and professional experience and participation should be taken into serious consideration (e.g., participating in discussions with colleagues or reading a book).

Other conceptions to consider in relation to knowledge beyond the curriculum can be found in the works pertaining to research of local and nonlocal mathematics (Stockton & Wasserman, 2017; Wasserman, 2015, 2018). Wasserman (2015) proposes mapping mathematical concepts in relation to their proximity of the mathematical content being taught to aid in studying the (mental) actions of ‘transforming’ mathematical knowledge into teaching. The idea of locality is refined:

> “Close” in this sense entailed both the degree to which mathematical ideas were closely connected–i.e., the degree of interdependence between them–but also temporally close in relation to when mathematical ideas were typically developed. In other words, this is a topological description about the landscape of mathematical ideas, defining two regions: the local mathematical neighborhood of the mathematics being taught, and the nonlocal mathematical neighborhood, which consists of ideas that are further away. Notably, the definitions of these two mathematical regions are

> in relation to the content that a teacher teaches–the content of the regions' changes depending on what a particular teacher is responsible for teaching. (Wasserman, 2018, p. 118)

Knowledge of nonlocal mathematics include understanding of both school and advanced mathematics and was considered in relation to literature about Horizon Content Knowledge (Stockton & Wasserman, 2017; Wasserman, 2015, 2018). The study of local and nonlocal mathematics takes into account both mathematical (connections) and pedagogical (timing) considerations. The proximity is defined by how concepts are connected in mathematics and in the curriculum. Yet, the two measures of figurative 'distances' are not always compatible. Take triangles and circles, for example, mathematically the two concepts are directly related (e.g., every triangle can be inscribed in a circle). However, in primary and early secondary school this idea is not explored, therefore, the two concepts are further away in consideration of the curriculum guidelines. Another issue arises when, instead of mathematical concepts as the centre of the model, one considers mathematical practices. In contrast to concepts, mathematical practices (e.g., proving, abstracting, or modelling) can be seen as local in terms of connections with the content being taught but nonlocal in terms of timing. For example, in secondary school mathematics a formal geometric proof might not be part of the requirements of the curriculum, therefore from a temporal perspective proving would be a nonlocal practice. However, students are expected to reason mathematically based on known geometric theorems, and reasoning as a practice is essential in proving. Despite the limitations, the idea of local and nonlocal neighbourhoods seem to provide a direction that bridges the gap between knowledge and practice by proposing a dynamic mapping of mathematical concepts ad hoc (see also Sections 3.2 and 3.3).

To conclude, this section presents an overview of the literature around teacher knowledge focusing on mathematics not predicated by the content covered in the curriculum documentations. The main ideas presented in this section will be revisited and elaborated further in Section 3.2.

2.2. Initial Teacher Education and Professional Development

The previous section provides an overview of different theoretical constructs which describe what researchers have found to be important for mathematics teachers to know. The constructs were considered in relation to the idea of teachers navigating beyond the mathematics of the moment. However, how these models are utilised in teachers' professional learning and development depends on the context and the choice of a specific model (Goldsmith et al., 2014). I now turn to literature on professional learning during tertiary education and throughout one's teaching career and explore suggestions related to the aims of my study. I begin with remarks on initial teacher education and professional development in different educational contexts. Then, I focus on evidence and suggestions for teaching mathematics beyond what is explicitly included in secondary school curricula. I also consider how those suggestions fit in the broader educational practices in mathematics teacher education.

The form and structure of initial mathematics teacher education varies around the world (Dahl, 2005; Ingvarson & Rowley, 2017; Tatto & Hordern, 2017). For instance, Ingvarson and Rowley (2017) implemented a comparative, quantitative study among mathematics teachers in 17 countries[6], and found that quality in mathematics teacher education programmes varies considerably across countries. National policies on quality of teacher education programmes in each of the countries was associated with teachers' scores in mathematical and pedagogical tests and the achievements of students in international assessments (Ingvarson & Rowley, 2017).

One difference in teacher education across different countries can be found in the tertiary degrees future teachers are required to obtain before or concurrently with their initial teacher training. For example, in Australia the qualification of secondary school mathematics teachers includes a degree in Secondary Education with a minor in two subjects (e.g., mathematics and physics) followed by a postgraduate initial teacher training course. On the other hand, in many European countries, secondary school mathematics teachers are required to obtain a degree in Mathematics before they enter a postgraduate teacher training programme (Caena, 2014). Moreover, in some countries mathematics teachers are eligible to teach across secondary school levels (e.g., in Brazil). While in countries where secondary schools are distinguished into lower and upper secondary level (e.g., in Norway), future mathematics teachers are required to undertake different training routes depending on the

[6] The UK is not included in the 17 countries.

levels they wish to teach. Finally, differences lie in the educational institutions who are responsible for delivering teacher training. In some cases, for example in Turkey, the process is centralised under university programmes and the involvement of schools varies depending on the context. However, in England, for example, there are three main routes to becoming a qualified teacher[7]: through a university-led course studying towards a Postgraduate Certificate in Education (PGCE), through School-Centred Initial Teacher Training (SCITT) or through School Direct Initial Teacher Training (School Direct ITT). It is also important to note that, partly due to issues in recruitment and retention (Allen & Sims, 2018; Royal Society, 2018), mathematics teachers in England do not always have a degree in mathematics.

Throughout their career mathematics teachers are expected to take part in professional development activities. The structure, intensity and variety of professional development opportunities vary across countries. Similar to initial teacher training routes, there is a variety of continuous professional development activities offered to mathematics teachers by universities, schools or networks of schools (e.g., regional associations), and professional development centres.

The comparison of teacher education and professional development across countries highlights the complexity of adopting and adapting research suggestions based on what works in different educational contexts. Next, I turn to research which explores teaching interventions and make suggestions in relation to professional learning beyond the mathematics of the moment. Large scale quantitative studies can provide evidence of similarities and differences across contexts, while qualitative studies and design experiments illustrate innovative approaches which could inspire practice and further research. The literature is discussed in two strands: encounters with advanced mathematics and encounters with applications of mathematics in other disciplines and professions.

Typically, mathematics teachers encounter advanced mathematics during their undergraduate degrees. Drawing on Rowland et al. (2005), Advanced Mathematical Knowledge is seen as foundation knowledge, part of teachers' "overt subject knowledge" (p. 265) alongside "theoretical underpinning of pedagogy; use of terminology" (p. 265) among

[7] The different routes are presented in detail in section 4.2.

others. Depending on the nature of the course, Advanced Mathematical Knowledge might be embeded in modules targeted specifically at teachers or as part of a general mathematics course at university, in other words traditional mathematics courses.

Evidence from large-scale studies suggests that participation in traditional courses on advanced mathematics do not relate to the quality of teaching or students' achievements (Begle, 1972; Monk, 1994). Similar results are also evident in more recent qualitative studies where mathematics teachers are asked to indicate connections between abstract mathematics courses attended and their teaching practices (Goulding et al., 2003; Zazkis & Leikin, 2010). On the other hand, advanced mathematics courses designed specifically for prospective or in-service teachers indicate promising results in terms of contributing to professional knowledge and developing teaching practices (Even, 2011; Larsen et al., 2018; Pinto & Cooper, 2022; Wasserman et al., 2019). Research on Advanced Mathematical Knowledge focuses on secondary mathematics teachers who have acquired or are pursuing a degree in mathematics. As stated earlier, acquiring a mathematics degree is not always a necessary qualification to become a mathematics teacher. Out-of-field teachers, i.e., teachers who have studied and taught other subjects, make up a considerable part of the teaching community. In this case, courses that bridge mathematical knowledge and teaching practice for out-of-field professionals have been found to be effective not only in enriching teachers' mathematical knowledge and practices in and beyond the curriculum (Adler et al., 2014; Vale et al., 2011) but also in building identities as mathematics teachers (Vale et al., 2011).

Design studies aiming to enrich teachers' Advanced Mathematical Knowledge and Horizon Content Knowledge highlight the need to invest in making connections between elementary and advanced mathematics grounded in practice. Larsen et al. (2018) suggest a collaboration among researchers with focus on preparing teachers with regards to mathematical content and those with focus on pedagogy to design modules for teachers. Whilst, other studies (Wasserman et al., 2018, 2019; Weber et al., 2020) propose a circular instructional model that incorporates the discussion of teaching situations. Using these situations prospective teachers participating in the study discussed mathematical and pedagogical actions and secondary school mathematics topics, learnt about advanced mathematics connected to the topic and then engaged with the situation again, reflecting on the mathematical and pedagogical actions in light of the advanced mathematics (Wasserman et al., 2018, 2019; Weber et al., 2020).

Research has shown that learning with and from others is an important aspect of professional development opportunities (Matos et al., 2009). Professional development activities which are found to have promising outcomes are structured in such ways as to facilitate interactions among the teachers (e.g., Vale et al., 2011; Vermunt et al., 2019) but also across communities, e.g., between educators, mathematicians and mathematics teachers (Cooper & Karsenty, 2018; Grau et al., 2017; Pinto & Cooper, 2022), teachers of different grade levels and different subjects (Nelson & Slavit, 2007). Regarding opportunities for learning beyond the curriculum, Suh et al. (2019) showcase how a Lesson Study approach, which includes collaboration between communities of primary and secondary school teachers and a teacher educator, provides opportunities for professional learning at the Horizon. Taking a slightly different approach, Pinto and Cooper (2022) suggest that a design which facilitates collaborative learning between mathematicians and secondary mathematics teachers could bridge the gap between Advanced Mathematics per se and teaching practice.

The literature of Horizon Content Knowledge and Advanced Mathematical Knowledge, presented in Section 2.1, principally focuses on the idea of understanding mathematics as a discipline. Making interdisciplinary connections is understated in the models which focus on understanding mathematics as a discipline. Despite that, connecting mathematics learnt in school with professions and everyday life is another area in which mathematics beyond the curriculum is involved. Even though some curricula include provisions for such connections to be made, the teachers are required to have an awareness of the applications of mathematics in specific contexts and an understanding of other disciplines beyond the curriculum they are expected to teach.

Preparing teachers to teach mathematics for a changing world is a priority for researchers and policy makers alike all around the world. Maas and Engeln (2019) suggest and implement a professional development course for mathematics and science teachers aiming to prepare teachers for making connections with the "world of work"(p. 697). The design principles include:

- design tasks showing the connection between the subjects and the WoW [world of work] in cooperation between educators and industry in order to overcome 'language' barriers, and find authentic contextual questions in which the subjects are used;
- give teachers insight into how mathematics and science are used in work places to give them meta-knowledge on these subjects and prepare them to develop their own tasks. (Maass & Engeln, 2019, p. 969)

Their analysis of data from 13 countries suggest that the program was successful in aiding teachers to incorporate connections into their teaching. However, the influence of the institutional context is evident in those results. Specifically, teachers who felt more supported by policy (e.g., the curriculum, resources) include more connections in their practice. Apart from policy support, other contextual factors that were found to be related include teachers' beliefs, classroom management and availability of time (Maass & Engeln, 2019). The review of the literature indicates a need for further research on preparing mathematics teachers for implementing cross-curricular connections and life applications in teaching.

In brief, professional development courses aiming to develop an understanding of mathematics beyond the curriculum in ways meaningful to teaching have a common characteristic: they aim to bridge the gap between subject matter knowledge and practice. Bridging the gap is sometimes achieved by bringing different communities or researchers together during the design phase (Larsen et al., 2018), or during the implementation (Maass & Engeln, 2019; Nelson & Slavit, 2007; Pinto & Cooper, 2022) or by grounding the discussions in secondary school teaching practices (Wasserman et al., 2018, 2019; Weber et al., 2020). However, the choice of activities in each case is different, sometimes focusing on a mathematical problem while in other cases building the mathematical content around practice-based resources.

In recent years, practice-based activities are frequently used in initial and continued teacher education (Grossman, 2018). Practice-based activities can take many forms, for example, teaching rehearsals (Averill et al., 2016) and role-plays (Mamolo, 2021), video clubs discussing excerpts of their own and others' teaching practices (Beisiegel et al., 2018; Gamoran Sherin & van Es, 2009), or vignettes (Beilstein et al., 2017; Biza et al., 2018; Friesen & Kuntze, 2021).

Practice-based resources facilitate discussions and provide opportunities for teachers to collaborate and form communities (Grossman, 2018).

Of particular interest in the context of this study is the latter type of practice-based activities, vignettes. Vignettes are short open-ended stories based on research or real-life situations. Vignette-based resources have been used in mathematics education for research, initial, and continued education purposes (e.g., Beilstein et al., 2017; Biza et al., 2018; Friesen & Kuntze, 2021; Herbst et al., 2011; Skilling & Stylianides, 2020). They can take different formats e.g., text, comics, animation or video each of which have both advantages and shortcomings (Chazan & Herbst, 2012; Herbst et al., 2011). The interest in vignette-based resources stems from personal experiences as an undergraduate and my engagement with the MathTASK project[8](Biza et al., 2007, 2018). MathTASK is a research and development program that brings together mathematics teachers, teacher educators and researchers in the UK, Greece and Brazil. The programme focuses primarily on secondary mathematics teachers' mathematical and pedagogical discourses and transforming their aspirations into teaching practices (Biza et al., 2007, 2015, 2018). Part of the aims of the programme is to design vignette-based resources for teachers. Engaging with MathTASK activities, mathematics teachers are invited to discuss with each other and with researchers and teacher educators. The MathTASK principles make provisions to foster Horizon Content Knowledge to surface (Biza et al., 2018, 2021). However, the claim has not yet been sufficiently elaborated.

Following the idea of bridging the gap between knowledge and practice, I side with the view that activities should be grounded in practice. I propose that vignette-based activities could be used for professional development courses and sessions. The use of vignettes in professional development settings could offer opportunities to bridge the gap between knowledge and practice both by bringing diverge communities together and through grounding discussions about mathematics beyond the curriculum in teaching practice.

2.3. Teaching: The dichotomy between knowledge and practice

In the literature reviewed so far it seems that there is evidence to support the idea that an understanding of mathematics beyond the boundaries of the curriculum is part of teachers'

[8] https://www.uea.ac.uk/groups-and-centres/a-z/mathtask

knowledge. Moreover, there are studies aiming to explore how such understanding could be achieved. However, there is a gap between knowledge and practice that persists in the various conceptualisations of teacher's knowledge (Neubrand, 2018).

Ball, referring to the gap, proposes the need to shift the focus of research from knowledge to "knowing and doing" (p. 14) in teaching. She also notes:

> Research was not capturing the dynamic of what teachers actually do when they listen to students, make decisions about what to say next, move around the room, and decide on the next example. Scholars were studying classrooms and analyzing discourse, tasks, and interactions, but were not unpacking what is involved for the teacher in doing those things. The measurement work also led scholars to break up teaching into compartments, which is not the way teaching is enacted in practice. (Ball, 2017, p. 14)

The shift from knowing to doing is evident in more recent works of the Michigan group which draws attention to the ways teachers use language to communicate with students, listen and react to issues of equity (e.g., Hoover et al., 2016). In a similar vein, the Mathematical Discourse for Teaching Model (Cooper, 2016; Mosvold, 2015) shifts the attention to discursive elements of mathematics and pedagogy abandoning the metaphor of acquisition/transfer of knowledge. Within the model, Discourse at the Mathematical Horizon could be a starting point allowing further study about how understanding of mathematics beyond the curriculum is involved in teaching. It is worth noticing here that changing the metaphor of knowledge from acquisition to participation in communities of discourse does not necessarily bridge the gap between what teachers know and what teachers do during teaching. Nonetheless, the barrier that was created by considering 'knowing' as a mental activity while 'doing' as a physical/ communicational activity is removed.

In a nutshell, a review of the literature on professional knowledge and teacher education provides evidence that understanding and communicating about mathematics that are beyond the school curriculum are seen as desired qualities for mathematics teachers. Encounters with advanced mathematics has a central role in secondary teachers' initial education. However, there is a debate as to how understanding advanced mathematics is related to teaching. Moreover, in response to educational demands in the 21st century, there is a need for teachers to make cross-curricular connections and connections with the world of work. However, these connections are not always supported by the curriculum. Therefore, more research is needed into teaching practices beyond the curriculum and how teachers learn these practices, through professional development, their practice and/or personal engagement with the discipline(s).

Endorsing a discursive view of teaching, I propose shifting from knowledge acquisition to discursive activity, in particular, Discourse at the Mathematical Horizon to investigate aspects of teaching practices beyond the curriculum that remain hidden under the knowledge-practice dichotomy. Shifting from knowledge to discourse is not merely a matter of changing focus. The theory of Commognition (Sfard, 2008), where the Mathematical Discourse for Teaching model (Cooper, 2016) is based, involves re-conceptualisation of knowing, learning and teaching as forms of communication. In the following chapter, I provide an account of how learning and teaching mathematics is conceptualised under the umbrella of the theory of Commognition. Then, I bring in some ideas briefly mentioned in this chapter, including the literature on Horizon Content Knowledge, Advanced Mathematical Knowledge and nonlocal mathematics to identify key ideas that are shared across the different conceptualisations. Finally, I refine the definition of Discourse at the Mathematical Horizon to be operationalised in the rest of the study.

3 Conceptualising Discourse at the Mathematical Horizon

This chapter aims to provide the reader with insights regarding the conceptualisation of Discourse at the Mathematical Horizon in my study. Firstly, I present an overview of the theory of Commognition, the theoretical lens used in this work, and provide definitions for the key concepts which will be used to explore Discourse at the Mathematical Horizon. Then, I proceed with a review of the literature about Horizon Content Knowledge, looking into the proposed conceptualisations and identifying similarities, differences and gaps between the views of different research narratives. Finally, I propose a refined working definition of Discourse at the Mathematical Horizon, to be operationalised in subsequent chapters.

3.1. Discourse

Sociocultural perspectives have had a strong presence in mathematics education in recent years. Sociocultural perspectives see the individual "as a participant in established historically evolving cultural practices" (Cobb, 2006, p. 151). The gradual shift from seeing learning as an individual cognitive effort to seeing it as a social interaction resulted in a variety of sociocultural theories. Discursive perspectives focus on language use and communication, viewing learning as initiation in a specific discourse community. In mathematics education research, discourse has been conceptualised and used by many researchers in different ways and with various degrees of specificity (Ryve, 2011).

Discursive perspectives fit with the aim of the study which is to explore secondary mathematics teachers' communicational actions beyond the mathematics of the moment. In particular, this work adopts and builds upon the theory of Commognition, a discursive perspective influenced by the works of Wittgenstein (1953/2009), Vygotsky (1978) and Lave and Wenger (1991). At the early stages of developing a research plan, I considered various sociocultural perspectives and their potential as a theoretical lens for the study[9]. The choice

[9] During my undergraduate and postgraduate studies, I had engaged predominately with cognitive frameworks (somewhat driven by their popularity among the communities I was part of at the time). However, I turned to sociocultural perspectives as felt that the conceptualisation of learning as a cognitive activity only – outside the context of the social environment in which this learning is taking place – is in conflict with how I perceive my own learning.

of commognition as a theoretical lens for the study is driven by my ontological and epistemological views. I found the proposed conceptualisations of learning and mathematics consistent with my views about the world and my experiences as a learner, teacher, and young researcher (see also Section 4.5). The theory provides a rigorous basis and methodological tools and at the same time it is relatively new and malleable. Therefore, I can see myself learning more about the world by participating in a community of commognitive researchers and eventually contributing to the theory. In the following section of the chapter, I present the basic tenets of the theory.

3.1.1. *An overview of the theory of Commognition*

Commognition is a portmanteau word stemming from the words: *communication* and *cognition*. According to Sfard (2008) cognition and communication are inseparable. Therefore, thinking can be viewed as "an individualised version of interpersonal communication" (ibid, p. 81). In Commognition, the theory developed under the above assumption, learning is no longer viewed as an acquisition of knowledge but rather as participation in discourse.

The social world can be seen as consisting of partly overlapping discourses where each individual can participate in various communities of discourse. Discourses are "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by the participants" (ibid, p. 93). Whether, one is looking into a community or individual participation, a first indicator that a communication is part of a particular discourse is identifying the objects involved in the narratives. However, identifying the objects does not provide sufficient information to characterise and study the discourse. Take, for example, the following excerpt:

> The oddly shaped object that came whizzing past the Sun and Earth in 2017 on a trajectory from outside our Solar System prompted wild speculation. Most scientists think the cigar-shaped visitor, less than 1 kilometer long, was a comet or asteroid from a nearby star or some other cosmic flotsam. But theoretical astrophysicist Avi Loeb of Harvard University argued that 'Oumuamua' Hawaiian for "scout," was an alien creation—a light sail, antenna, or even a spaceship. Today he announced a plan to

> look for more such objects: a philanthropy-backed effort called the Galileo Project. (Clery, 2021)

The object discussed in the excerpt is an extra-terrestrial object that entered the solar system in 2017. With just this information in mind, many communities would be interested into talking about it (e.g., astronomers, space enthusiasts, journalists, science fiction writers, or conspiracy theorists). I would like to encourage the reader to pause and ponder for a moment about where this quotation was taken from and what makes them think so.

The excerpt is taken from an article on the Science Magazine website. The article and the author represent a community of public discourses about science. Paying more attention to the writing style, the quotation could not have been published within a scientific journal, as there are almost no references for the claims and the terminology used is not scientific (e.g., “whizzing”, “cigar-shaped visitor”). The journalist tries to create a mystery around this object not only with words but also with a picture labelled as “artistic representation” of the object. Moreover, looking at the patterns the author follows throughout the article we can gain more information about the discourse in which they engage.

According to Sfard (2008), there are two complementary types of rules of a discourse, object-level rules and metarules. *Object-level rules* are “narratives about regularities in the behaviour of objects of the discourse” (ibid, p. 201), e.g., “[it] was a comet or asteroid from a nearby star” (Clery, 2021). On the other hand, *metarules* “define patterns in the activity of the discussants trying to produce and substantiate object-level narratives” (Sfard, 2008, p. 201), e.g. the journalist uses attractive words and strong statements to substantiate his argument possibly in an attempt to catch the attention of the reader. This example illustrates how one needs to consider the mediators used (written or verbal) and the rules of communication which the interlocutors follow. The object in this example is a physical object, however, similar observations can be made for discursive objects, i.e., abstract ideas talked about, such as education or knowledge etc.

In the theory of Commognition, discourses have four elements: *word use*, *visual mediators*, *narratives* and *routines*. Specifically, *word use* includes terms or colloquial words used for the purpose of communication. According to Sfard (2008), v*isual mediators* are “visible objects that are operated upon as a part of the process of communication” (ibid, p. 133). *Narratives* are “any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to endorsement or rejection” (ibid, p. 133) and a *routine* is a “set of metarules defining a discursive pattern that repeats itself in certain types of situations” (ibid, p. 301). Routines were later refined and operationalised as the pair of a task and the procedure followed to achieve the task (Lavie et al., 2019). These elements will be revisited in the next sections in the context of mathematics and teaching.

Given that discourses are “partly overlapping” (Sfard, 2008, p. 91), the limits of a discourse are not always clear. However, a useful first distinction could be between colloquial and scientific discourse. Colloquial discourses are types of communication anyone uses for their day-to-day communication such as with a cashier at the supermarket or a discussion with friends. Colloquial talk is about everyday objects (e.g., a bag of chips or an upcoming trip) and the rules one has to follow usually depend on the relation of the interlocutors (e.g., customer-cashier or best friends). Scientific discourses, on the other hand, are types of communication about scientific subjects, their mediators and rules are partly specified by the particular subject (e.g., physics, biology, sociology, etc.).

There are two ways to approach discourses, looking at the community at large or focusing on individual discourses within a community. Henceforth, to distinguish between the two, I adopt Gee’s (1996) notation. I will be referring to the first case as Discourses (with a capital D) and the latter, which focuses on the individualised discursive activity, as discourses (with a lower-case d). For this study, the purpose of the distinction is to provide a communicational signal to the reader every time I am zooming in to an individual or I am zooming out to the community.

My study is focused on teaching mathematics. This endeavour requires looking at both Mathematics and Pedagogy as Discourses relevant to the aims of the study. Therefore, I would like to mention one further distinction among scientific discourses, ontic and deontic discourses (Sfard, 2019). An *ontic* discourse is concerned with how the physical world is constructed and how it works, “characterised by the strict subject–object dichotomy” (Sfard, 2019, p. 93) and the endorsement of narratives are based on strict procedures (e.g., proving, replicability). On the other end, a *deontic* discourse is concerned with issues of the social world (e.g., what is right and what is wrong). A characteristic of deontic discourses is that the distinction between the subject and the object is blurred, therefore one cannot completely separate themselves from the objects of the discourse (others). In this case, the adoption of a narrative "is an act of joining a community and has more to do with people's identity work" (Sfard, 2019, p. 94). For example, mathematics is the discourse about mathematical objects. To communicate about these objects, interlocutors use words with mathematical meaning and symbols to produce narratives with mathematical meaning that can be endorsed or rejected by the community of the discourse based on rigorous procedures accepted by the community of interlocutors (e.g., proofs). Another example of scientific discourse is philosophy, the discourse which seeks to tackle fundamental questions of our reality. In contrast to mathematics, the endorsement of philosophical narratives depends on the authority of the source (e.g., Plato, Wittgenstein, etc.). These two latest examples make clear that scientific discourses have significant differences not only in the objects but also in the metarules.

Research is "a particular, well-defined kind of discourse producing cogent narratives with which other human practices can be mediated, modified, and gradually improved in their effectiveness and productivity" (Sfard, 2019, p. 35). Research in Mathematics Education, in particular, is the discourse that includes communication about mathematics and about teaching, thus, it inherits discursive elements from Mathematics, Social Science and Psychology while being a well-established, standalone discourse. The objects of Research in Mathematics Education include mathematical objects and human interactions and behaviours. In other words, narratives about mathematics, social constructs and behaviours are subsumed under Mathematics Education.

Both the distinction between colloquial and scientific discourses and the distinction between ontic and deontic discourses are meant to be seen as opposite ends of spectrums. These distinctions are helpful in navigating complicated networks of discourses but do not constitute 'solid' boundaries between discourses. Mathematics classroom discourse, for example, includes both elements of colloquial and mathematical discourse. In social sciences, "the researchers must alternate between being insiders and outsiders to the social relations under study, entering respectively either deontic or ontic mode" (Sfard, 2019, p. 93). Similarly, mathematics teachers are acting considering both mathematics and their students.

In the following sections, I give an overview of mathematical discourse, and the mathematical and pedagogical discourses in mathematics teaching.

3.1.2. *Mathematical Discourse*

Following an overview of the main tenets of the theory of Commognition, a closer look at Mathematical Discourse from a commognitive perspective is essential in establishing the discourse of the current study. Mathematics is the Discourse about mathematical objects. Mathematical objects are abstract discursive objects building upon other discursive objects constituting mathematics as "a multi-layered recursive structure of discourses about discourse" (Sfard, 2008, p. 161). Take functions, for example, this mathematical object was historically developed by mathematicians for the purpose of algebraically describing objects related to a curve such as its coordinates or its slope. Over the years what mathematicians called 'function', changed to encompass new ideas and realisations enriching the object-level discourse about functions, Calculus. The concept of a function, which was used by mathematicians to communicate about the characteristics of a curve, led to the development of a new mathematical field, Analysis. Calculus was subsumed in Analysis. In Sfard (2008) the term 'meta-discourse' is used to describe the relation between two discourses where one is subsumed by the other[10]. In this example, Analysis is a meta-discourse (i.e., discourse about another discourse) about Calculus. In other words, Analysis can explain the discursive elements of Calculus (e.g., Calculus narratives or routines etc).

Looking at the historical development of mathematical objects such as the function, we should "consider not only mathematical aspects, but also social and cultural contexts in which these are situated" (Moustapha-Corrêa et al., 2021, p. 2). Moreover, the development of mathematics is not linear but spiral in nature: "some familiarity with the objects of the discourse seems a precondition for participation, but at the same time participation in the discourse is a precondition for gaining this familiarity" (Sfard, 2008, p. 161). However, mathematics learners and teachers are able to communicate about functions overcoming this paradox created when we focus on mathematical objects per se. According to the theory of Commognition, learning is conceptualised as change in learners' discourse (Sfard, 2008). The change can be at object-level or meta-level. For example, students usually first come across

[10]In the Glossary of the book, Sfard formally defines 'meta-level discourse' as "discourse about another (object-level) discourse" (Sfard, 2008, p. 299). Subsequent publications in commognitive literature, use both the terms 'meta-discourse' and 'metalevel discourse' to refer to the same idea. Here, the term 'meta-discourse' is preferred for economy of space.

cosine as the ratio of the adjacent side of a right triangle to the hypotenuse. Object-level learning is the enrichment of students' discourse about cosine with new vocabulary, notation and new – to the students – narratives about cosine. An instance of meta-level learning would be a change in the meta-rules of the discourse about cosine, thus, a new (meta-)discourse, e.g., studying cosine as a trigonometric function. The two discourses described above are incommensurable, i.e., "discourses that differ in their use of words and mediators or in their routines" (Sfard, 2008, p. 299), and contain seemingly conflicting narratives. For example, considering cosine as the ratio of sides in the right triangle. The ratio can be calculated only for angles $0 < \vartheta < 90°$. However, the domain of the cosine function includes any real number. I refer to the changes mentioned above as discursive shifts (Viirman & Nardi, 2018). Thus, any change in the word use, visual mediators, narratives and/or routines of an individual or a community is considered as a change of their discursive practices and thus a discursive shift. Based on this definition, discursive shifts are indicators of object-level and meta-level learning.

Meta-level learning occurs through explicit discussions of commognitive conflicts (Nachlieli & Heyd-Metzuyanim, 2022; Sfard, 2008). Commognitive conflict is:

> the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules (Sfard, 2008, p. 161).

In this case, the discourses of the interlocutors are initially incommensurable. By accepting the metarules that govern the teacher's discourse the student enters a new discourse.

Studying the teaching and learning of mathematics relies on looking at the elements of mathematical discourse. Mathematical discourse is, therefore, defined as the type of communication signified by four elements: keywords and how they are used (e.g., the uses of the words: function, variable, unknown, equal, etc), visual mediators (drawn shapes and graphs, mathematical symbols, etc), narratives (e.g., theorems and definitions), and routines (e.g., discursive patterns of solving equations or proving, differentiation etc). Routines are paired with task situations:

> "a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task" (Lavie et al., 2019, p. 161).

The terms "exploration" and "ritual" are used to characterise routines based on their goal: the endorsement of a narrative or pleasing others and social approval respectively (Sfard, 2008). Both rituals and explorations are present in mathematics learning, mathematical routines might be performed by a student first as rituals, to please the teacher, and then transformed into explorations as the student progressively becomes a more independent participant in a new discourse.

Identifying and studying the discursive elements in a written or verbal mathematical communication can provide insights about the mathematics discourse of the interlocutor(s)[11]. For example, the following excerpt and accompanying Figure 3.1 are taken from the data of this study to illustrate the use of the elements:

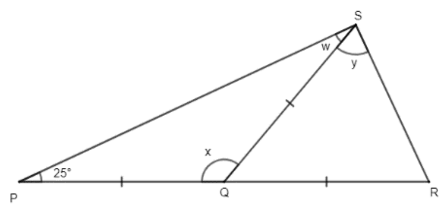


Figure 3.1 A recreation of the shape Liz and her student discuss.

Student: If y is 40, x is fif-, no, SQR is 50, that means that the other corner [RSP] would have to be 90.

Liz: How did you work out what y is?

Student: Because I think that, that, that was a right angle.

Liz: How did you work out that that was a right angle? Remember, this isn't drawn to scale?

Student: Yeah [Liz moves on].

The teacher, Liz, and her 11-year-old student communicate about the shape in Figure 3.1, used as a visual mediator for the discussion. The mathematical object talked about is the angle RSP. Throughout the dialogue, the interlocutors use words to describe the object and attribute values (e.g., "the other corner", "that", "right angle", "90"). The student attempts

[11] In the case of written communication, the person uses a medium (e.g., a piece of paper) to communicate her ideas asynchronously (e.g., to the teacher, an examiner, a colleague, or her future self). In this case, one might have access to the mathematical discourse of only one person, while at the same time, be herself part of the communication as the reader of the work.

to produce a narrative about the angle RSP. The narrative is a conjecture that the angle RSP is a right angle. Liz attempts to challenge the conjecture of the student prompting him to use angle facts to "work out" the angles as the shape "isn't drawn to scale". In this dialogue there are two routines talked about. The first is the routine the student used to determine that the angle is 90^o based on how it looks. The second is the routine of using angle facts to calculate the angle, proposed by Liz.

As an observer of the conversation and the lesson, I thought of solving the task using yet another routine, using the property that if the median is equal to one-half of the side the corresponding angle is a right angle. I learnt the property as part of my secondary school mathematics education. However, this property is not taught to students of this age in England, and although Liz could be aware of the property, she would not suggest to her student to use something that he has not learnt. This example illustrates how a task can be perceived differently by people. The difference stems from precedent events (Lavie et al., 2019), i.e., events that preceded the task situation and the person recognises them as relevant in the current task.

As the teacher, Liz has the role of the expert in the conversation. Liz's contribution in the dialogue is in the form of questioning and making suggestions on the mathematical task. In order to study why and how she acted, we also need to consider pedagogical influences and their relation to teaching mathematics. For example, why did Liz choose to make a suggestion rather than correcting the student? In the following section, I focus on the conceptualisations of teaching and discourses for teaching within the theory of Commognition.

3.1.3. *Discourses for teaching mathematics*

At the early stages of the development of the theory, Commognition has been predominantly used to approach mathematical learning. Commognitive researchers usually focus on the elements of the learner's mathematical discourse and identities and how a learner's discourse could change as she becomes a more experienced participant in the community of the classroom (e.g., Heyd-Metzuyanim & Graven, 2016; Newton, 2012; Sfard, 2007, 2008; Sinclair & Heyd-Metzuyanim, 2014). More recently, research in teaching – an activity that could gear students' opportunities to learn – and teachers´ discourses has started growing (Cooper, 2014; Mosvold, 2015; Nachlieli & Elbaum-Cohen, 2021; Nachlieli & Tabach, 2019; Tabach & Nachlieli, 2016; Viirman, 2015).

The role of the teacher as an expert mediating change in the student's discourse is indispensable. While object-level learning could be achieved by oneself through ritual-enabling and exploration-requiring opportunities to learn (Nachlieli & Tabach, 2019), "any substantial change in individual discourse, one that involves a modification in metarules or introduction of whole new mathematical objects, must be mediated by experienced interlocutors" (Sfard, 2008, p. 254). Yet, for teaching to become an object of study under the umbrella of the theory, it requires a definition to be operationalised. Tabach and Nachlieli (2016) define teaching as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (p. 303). In the case of school mathematics, the canonic discourse is the mathematical discourse of the classroom. The limits of any discourse are not easily drawn, however, for the purpose of this study mathematical discourse of the classroom depends on the age of the students and is bounded by what the students are expected to learn as outlined in the national curriculum and the teaching instructions.

A first step to enhance the potential of the theory to be used to study teachers´ discourses, was the development and use of the Mathematical Discourse for Teaching model (Cooper, 2016, 2014; Cooper & Karsenty, 2018; Mosvold, 2015). Cooper, in his PhD, explores primary school teachers' mathematical and pedagogical discourses during a professional development course facilitated by a researcher mathematician. During the course the teachers had the role of the learners. The study highlights the particularities of the communications between the teachers, experts in the mathematical discourse of the classroom in primary school; and the researcher mathematician, expert in advanced mathematical discourse[12] (Cooper, 2016, 2014; Cooper & Karsenty, 2018). For his study, Cooper developed an adaptation of the Mathematical Knowledge for Teaching model (Ball et al., 2008) by drawing on the theory of Commognition. The mathematical Discourse for Teaching model (Cooper, 2016) draws the attention to discursive elements of mathematics

[12] In this work, I differentiate between University Mathematics Discourse and Advanced Mathematical Discourse. In contrast to Advanced Mathematical Knowledge (Zazkis & Leikin, 2010), Advanced Mathematical Discourse should not be solely associated with University Mathematics.

and pedagogy that are considered relevant to mathematics teaching. These include:

> “keywords of teachers' Discourse and how these words are used; rules by which mathematical and pedagogical narratives are endorsed or rejected; goals of participation in mathematical discourse (explorative or ritualistic); the kinds of mathematical activities that are valued, etc.” (p. 21).

Analogous to Subject Matter Knowledge and Pedagogical Content Knowledge (Ball et al., 2008), Cooper’s model consists of two Discourses, Mathematical Discourse and Pedagogical Content[13] Discourse. Each of these two Discourses are then categorised into three sub-Discourses following the original idea proposed by Ball and her colleagues (e.g., Ball et al., 2008). Recalling the ‘egg’ representation of the Mathematical Knowledge for Teaching model (Figure 2.1) which depicted the domains as separate cells, the ‘egg’ might create an impression that the six domains are completely separable. However, the six sub-Discourses are explicitly described as overlapping (Cooper, 2016; Mosvold, 2015), highlighting that one should not try to draw a clear line between them. In Table 3.1, I illustrate the adaptation focusing on the descriptions given for each of the six Discourses in parallel to the original six domains of knowledge of the Mathematical Knowledge for Teaching model as described in the works of the Michigan group (Ball et al., 2008; Ball & Bass, 2009; Hill et al., 2008).

The Mathematical Discourse for Teaching provides the ground to study teachers’ discourses and claims to bridge the gap between knowledge and practice that the Mathematical Knowledge for Teaching model could not, by framing “communication (e.g. discussions) and action (e.g. teaching) […] as aspects of a single entity (Discourse)” (Cooper, 2016, p. 24).

[13] The word ‘content’ is used to signify the boundaries of the Discourse. While Pedagogical Discourse includes elements of pedagogy in general, Pedagogical Content Discourse includes pedagogical elements that are in relation to the content being taught, in this case Mathematics.

Table 3.1. A side-by-side overview of the Mathematical Discourse for Teaching and Mathematical Knowledge for Teaching models

	Mathematical Discourse for Teaching	Mathematical Knowledge for Teaching	
Mathematical Discourse	***Common Content Discourse*** "is concerned with the ways in which educated adults participate in mathematical discourse." (Cooper, 2016, p. 22)	***Common Content Knowledge*** is "the mathematical knowledge and skill used in settings other than teaching." (Ball et al., 2008, p. 399)	*Mathematical Content Knowledge*
	Specialised Content Discourse is the "mathematical discourse that is typical of teachers of mathematics" (Cooper & Karsenty, 2018, p. 242)	***Specialised Content Knowledge*** "is the mathematical knowledge and skill unique to teaching." (Ball et al., 2008, p. 400)	
	Discourse at the Mathematical Horizon are the "patterns of mathematical communication that are appropriate in a higher grade level" (Cooper & Karsenty, 2018, p. 242).	***Horizon Content Knowledge*** is "an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated." (Ball & Bass, 2009, para. 17)	
Pedagogical Content Discourse	***Discourse of Content and Students*** "is concerned with teachers' practices regarding content and students, including metarules of their discourse, such as routines of listening to students and building instruction on their ideas, and narratives on the importance of students' ideas and errors in planning instructions." (Cooper, 2016, p. 23)	***Knowledge of Content and Students*** is "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content"(Hill et al., 2008, p. 375)	*Pedagogical Content Knowledge*
	Discourse of Content and Teaching "attends not only to teachers' narratives about teaching, but also to their practices, and to the metarules of their discourse, such as teaching routines (professed or practiced), and rules for endorsing new teaching practices – based on "expert" teacher education, past experience, or peers." (Cooper, 2016, pp. 23–24)	***Knowledge of Content and Teaching*** "combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. […] Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning." (Ball et al., 2008, p. 401)	
	Discourse of Curriculum "focuses attention on teachers' interaction with their curricular "knowledge" – the metarules that influence their communication and actions regarding curricular material – which curricular materials do they value, which do they use, what is the basis for their decision, etc." (Cooper, 2016, p. 24)	***Knowledge of Content and Curriculum*** is knowledge about "educational goals, standards, state assessments, grade levels where particular topics are typically taught, etc." (Ball & Bass, 2009, para. 15)	

At the time when the Mathematical Discourse for Teaching model (Cooper, 2016) was developed, teaching was not yet clearly defined in commognitive terms and neither were the objects of the Pedagogical Discourses. Building on Cooper's (2016) conceptualisation, I revisit the Mathematical Discourse for Teaching model utilising more recent developments in commognitive research and argue for a nuanced view of the model and its operationalisation in different contexts (e.g. in a secondary school mathematics classroom). A first step to this direction is the elaboration of Pedagogical Discourses:

> Pedagogical Discourses shape and orient teachers towards what to teach students, how to teach them, why certain teaching actions are more effective than others and, often not talked about but still very important, who can learn (or not learn) (Heyd-Metzuyanim & Shabtay, 2019, p. 543).

Pedagogical Discourses are conceptualised as a spectrum between *Delivery Pedagogical Discourse*, "which values teacher "delivering" knowledge and students "acquiring" it" (Nachlieli & Heyd-Metzuyanim, 2022, p. 348), and at the other end *Exploration Pedagogical Discourse*, "which values students' explorations" (Nachlieli & Heyd-Metzuyanim, 2022, p. 348). The concept of Pedagogical Discourse was originally operationalised to account for teachers' statements about what, who and how to teach (Heyd-Metzuyanim & Shabtay, 2019). Nachlieli and Elbaum-Cohen (2021) expand this operationalisation and propose that Pedagogical Discourses "include any communication about teaching and learning mathematics" (p. 8), encompassing any representation of lessons, for example audio and video recordings or transcripts of lessons. A teaching practice is "the task as seen by the performing teacher together with the procedure she executed to perform the task" (Nachlieli & Elbaum-Cohen, 2021, p. 7). Thereby, mathematics teaching practices – namely, routines performed by the teacher in the mathematics classroom such as explanation routines, motivation routines and question posing routines (Viirman, 2015) – could also be considered as objects in pedagogical discourse. Returning to the example from the data presented in Section 3.1.2, Liz's performed routines of questioning the student and suggesting alternative approaches echo her pedagogical narratives.

Considering the Mathematical Discourse for Teaching model in light of the works on pedagogical discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Elbaum-Cohen, 2021; Nachlieli & Heyd-Metzuyanim, 2022; Viirman, 2015), I argue that the sub-Discourses include characteristics of ontic and deontic discourses. The objects of Mathematical Discourse for Teaching are both mathematics and humans. The two main sub-Discourses could be viewed as a way to partially distinguish between mathematical narratives situated in the social context of the classroom (Mathematical Discourse) and social/pedagogical narratives situated in the context of teaching mathematics (Pedagogical Content Discourse). Despite both Discourses still having characteristics of a deontic and ontic Discourse respectively, Mathematical Discourse gravitates more towards to an ontic Discourse and Pedagogical Content Discourse towards a deontic Discourse.

Until now, the Mathematical Discourse for Teaching model has been operationalised in the context of professional development activities (Cooper, 2016, 2014; Cooper & Karsenty, 2018; Mosvold, 2015). In these studies, the teachers' discourses were studied while interacting with other teachers and teacher educators. Following Nachlieli and Elbaum-Cohen (2021) proposal of expanding the view of Pedagogical Discourses to include classroom talk with emphasis on teaching practices, I argue that the Mathematical Discourse for Teaching model can be operationalised to study teaching practices in the mathematics classroom using data from lesson observations.

My doctoral project focuses on the study of Discourse at the Mathematical Horizon aiming to examine and expand its conceptualisation within the Mathematical Discourse for Teaching model. In chapter 2, I presented an overview of the relevant literature and briefly mentioned Horizon Content Knowledge as a starting point for exploring teaching practices beyond the content covered explicitly in the curriculum documentations. In the following section, I take a closer look at the literature about Horizon Content Knowledge using commognition as a critical lens to highlight variations in the different conceptualisations of the concept. Then, I focus on Discourse at the Mathematical Horizon, refining its conceptualisation and proposing a way of operationalising it in the context of the mathematics classroom.

3.2. Horizon Content Knowledge: A literature Review

Before I proceed to look into the concept of Discourse at the Mathematical Horizon, I revisit the concept of Horizon Content Knowledge, the domain in the Mathematical Knowledge for Teaching model, with a commognitive lens. I should begin the section by disclosing that I see myself as a commognitive researcher. As such, I use the theory of Commognition not only as a lens to study mathematical and pedagogical discourses but also as an aid in systematically reviewing the literature about Horizon Content Knowledge and Discourse at the Mathematical Horizon. I argue that a commognitive approach may bring to light the different conceptualisations of 'horizon' and contribute theoretically to the investigation of Discourse at the Mathematical Horizon. Part of the review that follows on the different conceptualisations and operationalisations of knowledge at the mathematical Horizon in research was presented in my report for the BSRLM Spring 2019 Conference (Papadaki, 2019). The section includes arguments that have been published in the conference proceedings as well as arguments that were developed later, and thus, were not included in the original report.

The domain is more commonly referred to as "Horizon (Content) Knowledge" (Ball et al., 2008; Ball & Bass, 2009; Jakobsen et al., 2012). However, the phrase "Knowledge at the Mathematical Horizon" is also used to refer to a variation of the concept (e.g. Zazkis & Mamolo, 2011). Going forward, I will be using 'Horizon Content Knowledge' to refer to the construct and its different conceptualisations in the literature, for the purpose of simplicity.

Horizon Content Knowledge was first described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). A year later, Ball and Bass (2009), in an attempt to clarify the concept, gave the following definition:

> We define horizon knowledge as an **awareness** [emphasis added] – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (para. 17)

According to the Cambridge dictionary, awareness means "knowledge that something exists or an understanding of a situation or subject at the present time based on information or experience" (https://dictionary.cambridge.org/dictionary/english/awareness). All the other domains of the Mathematical Knowledge for Teaching model are defined using the words 'knowledge' or 'knowing about', the choice of word here might hint a difference in the quality of Horizon Content Knowledge in comparison to the other subdomains. The words 'knowledge' and 'awareness' signify a variation in the level of engagement required. It is unlikely, that the words are chosen at random by the researchers in the Michigan Group, as their use is consistent throughout their work (Ball et al., 2008; Ball & Bass, 2009; Hill et al., 2004, 2008). Using the word ‘awareness’ to describe Horizon Content Knowledge could indicate that the focus is not on knowing specific characteristics of concepts but rather knowing about mathematics as a discipline, understanding its nature and acknowledging that the ideas communicated during a lesson are part of a larger discourse. In contrast to the word ‘knowledge’ which is often connected with the development of skills.

It could be possible for a teacher to follow instructions (tour guide) on how to make connections across mathematical ideas proposed by the curriculum or a syllabus without being aware of their importance in the mathematical theory (appreciative tourist). The researchers claim that such awareness mainly aids the teachers in identifying and taking into account the mathematical significance of students’ sayings, noticing and value opportunities to highlight key points, connections or “possible precursors to later mathematical confusion or misrepresentation” (Ball & Bass, 2009, para. 17). Based on empirical evidence, Horizon Content Knowledge might include three categories: insights of subject matter, mathematical connections and meta-mathematics of elementary mathematics (Guberman & Gorev, 2015, p. 179). In an attempt to clarify the boundaries of this category, Ball and Bass (2009) identified four elements:

- A sense of the mathematical environment surrounding the current “location” in instruction
- Major disciplinary ideas and structures
- Key mathematical practices
- Core mathematical values and sensibilities

(Ball & Bass, 2009, para. 18)

After its first introduction, Horizon Content Knowledge has been used and elaborated in research, leading to diverse discourses, challenging its conceptualisation and use in research. Nonetheless, Horizon Content Knowledge is still considered to be a grey area, with different interpretations and meaning, compared to the other sub-domains of Mathematical Knowledge for Teaching model. Next, I will go through the most cited different conceptualisations of Horizon Content Knowledge proposed over the years as adaptations of the original idea (Fernández et al., 2011; Jakobsen et al., 2012; Zazkis & Mamolo, 2011).

Jakobsen et al. (2012) developed a working definition of Horizon Content Knowledge based on Ball and Bass' (2009) description:

> Horizon Content Knowledge (HCK) is an **orientation** to and **familiarity** [emphasis added] with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes **awareness** [emphasis added] of core disciplinary orientations and values, and of major structures of the discipline. (Jakobsen et al., 2012, p. 4642)

In their definition, the word 'awareness' is connected to mathematics as a discipline and not to specific concepts; it specifically refers to the core disciplinary values and orientations. Moreover, the words 'orientation' and 'familiarity' could be interpreted as 'knowing about' mathematics but on a deeper level than 'being aware'. The phrase 'explicit knowledge of the ways of and tools for knowing in the discipline' supports the view of the expectation of understanding the academic routines which might be part of a teacher's tertiary education, but they do not explicitly suggest engagement with the academic discourse through institutionalised education.

Zazkis and Mamolo (2011) attempted to extend the idea of Horizon Content Knowledge:

> We consider **application of advanced mathematical knowledge** [emphasis added] in a teaching situation as an instantiation of teachers' knowledge at the mathematical horizon. More explicitly, a teacher's use of the mathematical subject matter knowledge acquired in undergraduate studies is recognized as an instantiation of knowledge at the mathematical horizon when such knowledge is applied to a teaching situation. (p. 9)

Zazkis and Mamolo (2011) refer to 'inner' and 'outer' horizon and argue the importance of learning advanced mathematics at university level. According to their perspective, inner horizon includes characteristics of a mathematical object that might not be at the centre of the teaching attention while the outer horizon includes "connections between different disciplinary strands and contexts in which the object may exist" (ibid, p. 9).

Advanced mathematical knowledge, in Zazkis and Mamolo's narrative, is seen as knowing university mathematics (Zazkis & Leikin, 2010). The researchers propose an institutionalised view of 'horizon'. Nonetheless, the nature of such knowledge and its benefits to teaching were challenged (Figueiras et al., 2011; Foster, 2011). Although, there are claims that advanced mathematical knowledge could aid teacher talk to students about mathematics beyond the curriculum, the evidence is scarce and mostly limited to discussions with gifted students or during maths clubs (Yan et al., 2021). At university level, the students have opportunities to consider connections across mathematical ideas and structures, but these opportunities might be constrained by the requirements of the curriculum and the objectives of the modules. Thus, the institutionalisation of the definition of Horizon Content Knowledge might be limiting (see also Section 2.1).

Despite the initial scrutiny that the proposed definition by Zaskis and Mamolo (2011) received (Figueiras et al., 2011; Foster, 2011; Zazkis & Mamolo, 2012), it generated further important research and discussions that provided the ground to uncover overlapping ideas that could be related to the concept. One such idea is that of 'location' in respect to advanced and elementary mathematics. Specifically, Zaskis and Mamolo (2011) proposed the notions of "inner" and "outer" horizon of a mathematical object to locate where the horizon 'meets' elementary mathematics. In response to this idea, Foster (2011) proposed using the notion of "peripheral mathematical knowledge" that is "useful pedagogically but is not knowledge which the learner might be expected to learn now, or even […] in the future" (Foster, 2011, p. 24) to sieve the Advanced Mathematical Knowledge that is relevant to teaching. However, the description of peripheral mathematical knowledge leads to questioning whether Horizon Content Knowledge – as described thus far – is related to peripheral mathematical knowledge. For "inner and outer horizon" the 'location' is determined based on the mathematical object in question, while for "peripheral mathematical knowledge" the 'location' is determined compared to the proximity of the topic to the content being taught. Both descriptions, however, relate only to the mathematical object and not to what the students are currently learning or what the teacher is teaching.

Wasserman and Stockton (2013), taking into account the various descriptions of Horizon Content Knowledge that were present at the time, talked about “curricular mathematical horizon” which includes school mathematics beyond what the teacher is currently teaching and “advanced mathematical horizon” which includes more advanced mathematical topics. This description brings to the picture the teacher and what she is expected to teach. In an attempt to distance himself from the ‘horizon’ metaphor, Wasserman (2018), proposed the ideas of “local” and “nonlocal mathematical neighbourhoods” to map the ‘distance’ and connection of two mathematical ideas in relation to “the degree of interdependence between them – but also temporally close in relation to when mathematical ideas were typically developed” (Wasserman, 2018, p. 118). Similarly, as with the notions of curricular and advanced horizon, the descriptors local and nonlocal refer to both the content and the expected development of the students that might include age, the orientation of the educational system and is linked to what a teacher is expected to teach. Despite that these descriptions were developed within a cognitive framework, there is enough information to indicate that the relation between two neighborhoods is not static, rather:

> [They] are in relation to the content that a teacher teaches–the content of the regions' changes depending on what a particular teacher is responsible for teaching. (Wasserman, 2018, p. 118)

Wasserman (2018) highlights the importance of looking at advanced mathematical knowledge not only in relation to the content of school mathematics but also to the teaching of school mathematics. Figure 3.2 demonstrates how the author visualises the interactions between local and nonlocal mathematical knowledge and teaching. Wasserman (2018) notes:

> a “hole” has been created within the mathematical landscape to make room for ideas connected not just to understanding local content, but to teaching local content. In this approach, advanced mathematics must serve as a mathematically powerful understanding […]–something that fundamentally transforms (i.e., a new color) one's own understanding of the local mathematical neighborhood (p. 125).

It is not clear, however, whether the ‘fundamental transformation’ is thought of in respect to teaching of local mathematics or if the learning of local mathematics is also expected to be transformed through the interactions.

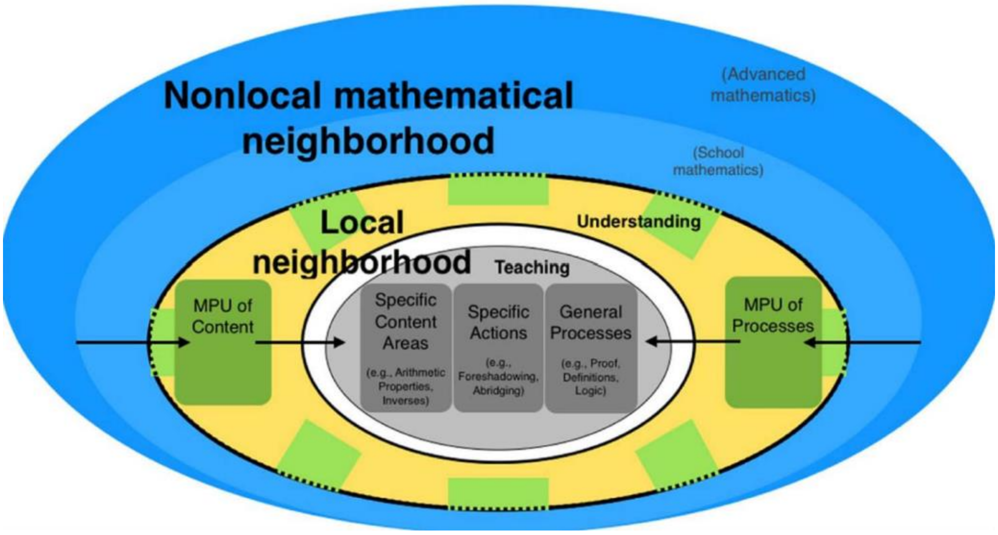


Figure 3.2 Nonlocal mathematical knowledge interacting with local teaching. MPU stands for mathematically powerful understanding. Reprinted from Wasserman (2018, p. 125).

Table 3.2. Standpoints and metaphors used in Ball and Bass (2009), Zazkis and Mamolo (2011), and Jakobsen et al. (2012)

Standpoint	**Metaphors used**
Elementary perspective on advanced mathematics	• "peripheral vision" (Ball & Bass, 2009, para. 1 and 17) • "a view of the large mathematical landscape" (Ball & Bass, 2009, para. 17) • "mathematical environment surrounding the current 'location'" (Ball & Bass, 2009, para. 18) • "an orientation" (Jakobsen et al., 2012, p. 4642)
Advanced perspective on elementary mathematics	• "where the land appears to meet the sky" (Zazkis & Mamolo, 2011, p. 9) • "the higher one stands, the farther away the horizon is and the more it encompasses." (Zazkis & Mamolo, 2011, p. 10)

The word horizon is used figuratively, possibly to indicate the idea of the connection between mathematics in general and mathematics taught in school. The extensive use of metaphors to describe Horizon Content Knowledge has been stated in the past (Jakobsen et al., 2013). Wasserman (2018) distances himself from the use of the metaphor and the Mathematical Knowledge for Teaching model, maintaining, however, the idea of navigating the map of mathematics by creating a conceptual map. In most instances (Jakobsen et al., 2012; Zazkis &

Mamolo, 2011), the researchers position their conceptualisation of Horizon Content Knowledge comparative to Klein's idea of knowing elementary mathematics from an advanced standpoint (Klein, 1908/2016). The metaphors the researchers use to describe Horizon Content Knowledge seem to line up with the different perspectives found in the papers. Table 3.2 summarises my reading of the literature in relation to the standpoints and the metaphors used in seminal work on Horizon Content Knowledge.

Ball and Bass (2009) and Jakobsen et al. (2012) adopt a standpoint complementary to Klein's. For them, Horizon Content Knowledge is a kind of elementary perspective on advanced mathematics. The researchers' discourse includes analogies between the literal horizon in a landscape and the mathematical horizon. The word 'orientation' that Jakobsen et al. (2012) use in their definition can be interpreted as "the position of something in relation to its surroundings" (according to the Cambridge dictionary) which might indicate a hidden metaphor. In all these metaphors, there is an underlying assumption that the person is fixed in a 'location' (i.e., elementary mathematics) looking to the horizon (i.e., advanced mathematics).

On the other hand, Zazkis and Mamolo (2011) visualise Horizon Content Knowledge as being able to approach elementary mathematics from an advanced perspective. To support their ideas about advanced mathematics, they use a physical property, that the higher above sea level one stands (i.e., advanced mathematics) the horizon (i.e., elementary mathematics) seems to be further away. Comparing the two perspectives, for Ball and Bass (2009) as well as for Jakobsen et al. (2012) the horizon seems to include the connections spanning across mathematics, whereas, for Zazkis and Mamolo (2011) the horizon is the limit of what the teacher knows.

Researchers who adopt one or the other standpoint, usually – but not always – signify their choice also by the ways of referring to the concept. Using the name 'Horizon Content Knowledge' researchers often side with Ball and Bass (2009) conceptualisation or 'Knowledge at the Mathematical Horizon' siding with Zazkis and Mamolo (2011), respectively.

The difference in perspectives between the three frequently cited descriptions of Horizon Content Knowledge is evident – besides the use of words – in the routines the researchers follow to exemplify the construct. For example, to illustrate how Horizon Content Knowledge might benefit teaching, Jakobsen et al. (2012) discuss an episode where primary school students were asked to divide a rectangle into four parts of equal area. One of the students, Maria, divided the rectangle in the way shown in Figure 3.3. The student explained that she knows that the parts do not look equal, but she claimed that she could make them equal by squeezing the lines closer together. The student's idea is correct and can be proven. The authors explain that when a line slides across a figure the area on one side can be thought of as a continuous function going from 0 to the whole area of the figure. Based on the intermediate value theorem there will be a line that cuts the shape exactly in half. Repeating this process for the two new shapes results in four shapes having equal areas. They continue:

> Experiences with the concept of continuity and different ways of thinking and talking about continuity would provide a teacher with resources for hearing mathematical ideas in Maria's talk — ideas related to major structures and developments in the discipline. […] Understanding the formalisms related to continuity can add precision to a teacher's thinking. Having language to talk about it casually yet with integrity can position a teacher to draw students' nascent attention to important mathematical ideas (Jakobsen et al., 2012, p. 4638)

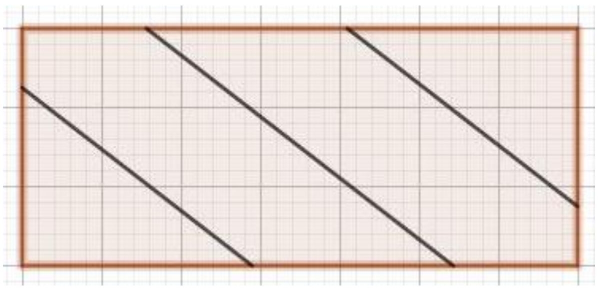


Figure 3.3 Adaptation from Jakobsen et al. (2012)

The excerpt depicts the authors' interpretation of how Horizon Content Knowledge could help a teacher hear the student's idea and act accordingly. The concept of continuity is treated as an important mathematical idea appearing in many seemingly unrelated situations and not as a characteristic of formally defined functions. This might require a deeper engagement with the notion of continuity, possibly also at a meta-level, which is consistent with the use of the words "orientation" and "familiarity". The reference to "formalism" in their description suggests that the authors see formalism as a desirable rather than an essential element during this episode and probably in general for Horizon Content Knowledge. Finally, being able to address the idea “casually” but with “integrity” seems to be evidence of Horizon Content Knowledge for the researchers.

On the other hand, Zazkis and Mamolo (2011) seem to focus on properties of specific concepts rather than routines and metarules about mathematics. For example, the main episode discussed in the paper is around an activity where primary school students had to identify the number of triangles formed by the diagonals in a regular pentagon, in which the students’ answers varied. The authors then claim:

> The teacher, though she had not yet determined the number of triangles herself, immediately knew that both answers were incorrect. She recognized rotational symmetry of order 5 in the figure and, as such, she knew that the number of triangles should be a multiple of 5. (Zazkis & Mamolo, 2011, p. 10)

They describe how knowing about a property of a specific concept, rotational symmetry, could help the teacher determine if the answers were correct. However, it would be possible for the teacher to determine the correctness of the answers relying on her awareness of problem-solving techniques. According to Zazkis and Mamolo (2011) this knowledge came from a university course. They continue:

> With this understanding in mind, she helped students identify different kinds of triangles and where, with each triangle-shape found, there were 5 of the same kind. She led students to catalogue different shapes and account for them systematically. (Zazkis & Mamolo, 2011, p. 10)

Zazkis and Mamolo (2011) do not go into the details of how the teacher aided the students to find the different types of triangles. Since they do not discuss what knowing about rotational symmetry can add to teaching practice, in comparison to knowing the strategy to solve the problem, the application in the classroom seems coincidental. The focus of the work is to illustrate how Horizon Content Knowledge is conceptualised as applied Advanced Mathematical Knowledge. Thus, more attention is given to exemplifying the role of the advanced content in teacher’s knowledge and not necessarily to how this advanced content could be communicated with students.

The above descriptions of Horizon Content Knowledge rely on the researchers' perceptions of the horizon. To avoid the vagueness created through the use of metaphors, it is appropriate to consider what the Mathematical Horizon is and what it stands for outside of the metaphor. Naik (2018), in line with Ball and Bass' (2009) and Jakobsen et al.(2012) ideas, defines Mathematical Horizon as:

> the projection of mathematical meanings, topics and structures in the curriculum in the discipline of mathematics to be able to learn the school mathematics **meaningfully** [emphasis added] (Naik, 2018, p. 60)

and then proceeds to define:

> a horizon encounter (or an encounter with the horizon) to be a situation in which a teacher enacts or engages with curriculum and students' thinking reaching to its meaningful potential. Therefore, the conception of HCK in this study involves the mathematical knowledge that teachers draw upon when navigating encounters with the mathematical horizon (Naik, 2018, p. 62).

Naik's (2018) definition does not rely on a metaphor about the horizon. By situating the topics in relation to the curriculum and the way it is enacted in a mathematics classroom, the researcher makes it clear that the focus is on the current moment of instruction. Naik's (2018) view of Horizon Content Knowledge is situated in the teaching practice. However, what is meant by projecting mathematical meaning and meaningful teaching and learning is not explicitly defined.

The narratives described so far conceptualise Horizon Content Knowledge as a domain of the Mathematical Knowledge for Teaching model. Fernández and her colleagues (2011) proposed a reconceptualisation of Horizon Content Knowledge "as a mathematical knowledge that actually shapes the Mathematical Knowledge for Teaching from a continuous mathematical education point of view" (Fernández et al., 2011, p. 2645). From this perspective Horizon Content Knowledge is not part of the Subject Matter Knowledge domains but rather a mixture of mathematical and pedagogical knowledge regarding the connections between the previous, current and following mathematical levels. Nonetheless, there is a common characteristic of Horizon Content Knowledge among most of the conceptualisations cited so far. Horizon Content Knowledge is associated with connections across mathematics that can go beyond specific mathematical content, hinting that its nature is different than the other subdomains of Mathematical Knowledge for Teaching model. I argue that a discursive approach could address the gap between knowledge and practice identified in the literature under the umbrella of sociocultural perspectives, specifically, discourse and teaching practices. I suggest that moving the attention from knowledge to communicational patterns between the teachers and students could provide new evidence regarding what the 'mathematical horizon' is and how this professional discourse is utilised in teaching.

Although the Mathematical Knowledge for Teaching model has evolved throughout the years to encompass sociocultural perspectives about teaching mathematics (e.g., Hoover et al., 2016), the above descriptions and definitions were developed within a cognitive discourse about teaching and learning mathematics with emphasis on measurement and evaluation of teachers' essential mathematical and pedagogical knowledge. In the following section, I focus on the commognitive adaptation of Horizon Content Knowledge – Discourse at the Mathematical Horizon. I argue that Cooper's (2016) work provides a solid ground to address the limitations mentioned in the literature review. I discuss the conceptualisation of Discourse at the Mathematical Horizon (Cooper, 2016) and propose a refinement which would allow me to operationalise the construct in the context of a secondary school mathematics classroom.

3.3. Refining Discourse at the Mathematical Horizon

In Section 3.1.3, I summarise the adaptation of Mathematical Knowledge for Teaching model into Mathematical Discourse for Teaching (Cooper, 2016) using a commognitive perspective. I also discuss how more recent developments in the theory of Commognition provide the ground to operationalise the model in a variety of situations such as during and in preparation of lessons. Here, I focus on Discourse at the Mathematical Horizon, I examine its initial conceptualisation in comparison to Horizon Content Knowledge and propose a refined working definition to be operationalised in subsequent chapters of this work.

Discourse at the Mathematical Horizon is briefly described as the "patterns of mathematical communication that are appropriate in a higher grade level" (Cooper & Karsenty, 2018, p. 242). The description seems to focus on the mathematical discourse of the teacher to gear students towards future learning (i.e., higher grade level). However, a more detailed description, included in Cooper's (2016) thesis, provides a more accurate portrayal of the concept:

> Discourse at the Mathematical Horizon [DMH] has to do with looking ahead at what students will be learning in the near future. Yet what students are learning today was once on the horizon. Hence looking ahead and looking back are different aspects of DMH. Though DMH is considered an aspect of Mathematical Discourse, both looking ahead and looking back have pedagogical implications. Looking back may advise teachers how to base new ideas on their students' prior knowledge, whereas looking ahead may advise them how to pave the way for what is to come, that is, how to prepare what will later serve as prior knowledge. (p. 195)

The difference between the brief and the elaborated descriptions of Discourse at the Mathematical Horizon is indicative of the complexity of the concept. The latter quote clarifies that "looking ahead" as well as "looking back" are both parts of Discourse at the Mathematical Horizon. The claim is in line with Ball and Bass'(2009) ideas about Horizon Content Knowledge. However, describing the horizon in relation to "higher grade level" creates a paradox. "Looking back" might be about communications regarding mathematical objects that students have encountered in the past following patterns of communication that are appropriate for the grade level that they are at the moment of instruction. In this case, the discourse of the teacher, and the students, is indeed higher in the sense that previous discourses are now subsumed in the current discourse. On the flip side, "looking ahead" to what the students might encounter in the future includes communications about (mathematical) objects which follow patterns of communication that resemble those in a higher grade level, without necessarily engaging in more advanced mathematical talk. In such cases, the mathematical communication would not be appropriate to be used in a higher grade level.

I argue that a refined definition of Discourse at the Mathematical Horizon could help in balancing the variations between the descriptions. Moreover, the latter quote depicts Discourse at the Mathematical Horizon in relation to "learning in the near future". The idea was initially developed in the context of primary school mathematics teaching and teachers' discourses which could explain the focus and influence of compulsory primary and secondary education in the conceptualisation. However, I propose that an adaptation of Discourse at the

Mathematical Horizon to account for secondary mathematics teachers' discourses should extend beyond learning in compulsory mathematics education and encompass ideas a student might come across during tertiary education or in every day and professional life.

Previous work on the conceptualisation of Horizon Content Knowledge hints a uniqueness in communicating primitive mathematical ideas. For example, Horizon Content Knowledge is thought of as the kind of knowledge that aids the teacher to hear mathematical significance in their students' words (Ball & Bass, 2009). Especially in the vignettes used by researchers to showcase instances of HCK narratives such as:

> "[h]aving language to talk about [continuity] casually yet with integrity can position a teacher to draw students' nascent attention to important mathematical ideas and to help students find language to express their emerging thoughts" (Jakobsen et al., 2012, p. 4638)

My interpretation of these works is that the communication between the teacher and her students depends on established patterns of communication within a specific classroom. It depends on the narratives and routines the students are familiar with, yet, having in mind the changes of the metarules in the future and preparing the students for these changes when the time comes.

I, now, propose a working definition of Discourse at the Mathematical Horizon that might address the lack of clarity mentioned above. The first step I took to resolve the issues was to shape my standpoint.

- Drawing upon Cooper's (2016) work, I adopt an elementary to advanced perspective, similar to Ball and Bass (2009) and Jakobsen et al. (2012), keeping the communication between the teacher and her students at the centre of attention.
- I situate Discourse at the Mathematical Horizon in relation to the mathematical discourse of the students as predicated by the curriculum and teaching instructions.

Examining Horizon Content Knowledge through a commognitive lens led to the operationalisation of Discourse at the Mathematical Horizon as patterns of communication that are unique to the specific moment of instruction. Thus, the rationale behind the following working definition of Discourse at the Mathematical Horizon is that the mathematics classroom discourse is bounded by what the students are expected to learn based on their age and curriculum. I refer to mathematical communications that are in line with curricular instructions provided for a topic and a specified age group of students as *mathematics of the moment.* The term is adopted from Ball and Bass' (2009) description:

> It engages those aspects of the mathematics that, while perhaps not contained in the curriculum, are nonetheless useful to pupils' present learning, that illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the *mathematics of the moment* (para. 17, my emphasis).

Therefore, I provisionally define Discourse at the Mathematical Horizon to be *patterns of mathematical communication that are <u>unique</u> to the moment of instruction and incommensurable to the mathematical discourse of the classroom as <u>predicated</u> by the curriculum.*

I claim that Discourse at the Mathematical Horizon is incommensurable to the expected discourse of the classroom, and not incompatible. Specifically, Discourse at the Mathematical Horizon could contribute to teaching in ways that might seem to diverge from what the students are expected to learn. However, the alternative ways sustain mathematical integrity and are appropriate considering the students' current discourses[14].

The definition encompasses Wasserman's (2018) local and nonlocal mathematical neighbourhoods. However, the neighbourhoods are signifiers of the relation between mathematical objects, whereas, what is earlier defined as mathematics of the moment signifies a relation between discourses. Thus, any communication beyond the mathematics of the moment is an opportunity to engage in Discourse at the Mathematical Horizon. Communications beyond the mathematics of the moment could take place in the classroom, between a teacher and her students, between colleagues in relation to a specific topic being taught, in preparation of a lesson, or during professional development activities between teachers and instructors. I suggest that the working definitions proposed could be operationalised in a variety of situations to account for Discourse at the Mathematical Horizon.

[14] Mathematical and otherwise.

I theorise that Discourse at the Mathematical Horizon includes both mathematical and pedagogical elements. A rationale for the conjecture could be that the discourses of the model are overlapping. In order to elaborate on the claim, I propose taking a closer look at the mathematical and pedagogical elements of Discourse at the Mathematical Horizon as they arise from the literature and the working definitions.

Naik (2018) suggest that Horizon Content Knowledge is knowledge situated in the teaching practice. Moreover, Fernández et al. (2011) propose that Horizon Content Knowledge shapes the Mathematical Knowledge for Teaching model because "it must be present in every in-action category in order to attend transition" (p. 2645). The views of Fernández et al. (2011) and Naik (2018) are in line with teachers' discourse about upcoming changes in metarules of their students' mathematical discourse. The presence of mathematical elements in Discourse at the Mathematical Horizon is self-evident. In addition, the teachers need to 'see' pedagogical relevance between what they are teaching and the mathematical ideas which are part of their mathematical discourse beyond the mathematics of the moment. Specifically, teaching as a discursive practice involves communication between the teacher and her students. Each interlocutor takes turns to speak, and each utterance is an attempt to respond to one another. Therefore, for a conversation beyond the mathematics of the moment to take place, the teacher needs to deem a mathematical idea or practice relevant in the situation in which the discussion is taking place, including the mathematical discourse of her students. The teacher does not necessarily have to be aware that the communication is beyond the mathematics of the moment. As an insider to the conversation, she might not realise the significance of her actions in real time. This argument extends to the planning of lessons and discussions about teaching. The interlocutors must deem the idea or practice relevant to teaching and the conversation will be situated in relation to students' learning. The conjecture should be explored through empirical evidence.

I operationalise the notion of Discourse at the Mathematical Horizon by first looking for opportunities to go beyond the mathematics of the moment in the mathematics classroom, to explore whether, there is sufficient evidence to support a need for Discourse at the Mathematical Horizon as essential discourse for teachers. Thus, I investigate the patterns of communication between teachers and students during typical mathematics lessons. For the communication beyond the mathematics of the moment to be effective, the teacher and the students need to have a common language that makes sense at both ends. Achieving effective communication in this case is a responsibility of the teacher as an expert participant to the discourse, which requires the teacher to be familiar with ways of making the mathematical content accessible to her students. Therefore, to operationalise Discourse at the Mathematical Horizon in the secondary school mathematics classroom, I propose the use of the notion of intersubjectivity[15]. Intersubjectivity has several definitions (Gillespie & Cornish, 2010) which are often based on the idea of negotiating a shared situation definition (Wertsch, 1984) or a shared understanding (Cobb & Bauersfeld, 1995).

Within the theory of commognition, intersubjectivity is defined as "an action that makes sense from the perspective of two discourses – the learner's and the expert's–which may be incommensurable"(Cooper & Lavie, 2021, pp. 8–9). The idea was used in Cooper's (2016) doctoral work in relation to Mathematical Discourse for Teaching model and its operationalisation was elaborated in Cooper and Lavie (2021) as "an action that is drawn from the learner's precedent space, yet can be seen as appropriate in the new discourse" (p. 8) through the use of discursive elements (words, visual mediators, narratives and routines) that carry meanings from one discourse to the other.

A goal of my work is to further refine the provisional definition of Discourse at the Mathematical Horizon through empirical evidence and propose ways of aiding teachers to develop their discourse at the Mathematical Horizon, through professional development activities. In the next chapter, I describe the research design and the methodology of the work.

[15] Intersubjectivity is linked with Zone of Proximal Development (Vygotsky, 1978) used to signify the distance between independent learning and learning in collaboration with adults or experienced peers. The focus of this work is teaching. The work does not explore the implications of Discourse at the Mathematical Horizon in students' learning. Discussions about emerging implications and further research are included in Chapter 8.

4 Methodology

This chapter discusses the research design and methodology of the study. I begin by describing the research design with a reminder of the research questions. Then, I offer an overview of the context of the study. Next, I discuss the demographics of the study, followed by a discussion of the methods used, before commenting on my role as a researcher. Then, I discuss the data analysis, I outline my actions to ensure the trustworthiness of the study, I reflect on ethical considerations and explain how I addressed methodological issues. Finally, I reflect on my experience of doing research during the COVID-19 pandemic and I outline how the experience affected the design of the study and enriched my experience as a researcher.

4.1. Research Design and Research Questions

The purpose of this study is to achieve a more nuanced understanding of what was defined earlier as communication beyond the mathematics of the moment and teachers discourse at the Mathematical Horizon. In particular, I aim to identify opportunities for teachers to engage in conversation that goes beyond the mathematics of the moment and explore the characteristics that are unique to Discourse at the Mathematical Horizon in order to refine the original working definition. Finally, I aim to propose a design and pilot materials to be used in professional development settings to enrich teacher's discourse at the Mathematical Horizon. Therefore, with this study I address the following research questions and sub-questions (also presented in Chapter 1):

RQ 1 What are the characteristics of discussions *beyond the mathematics of the moment*?

1a. What taken and potential opportunities for discussion beyond the mathematics of the moment can be identified in everyday teaching practice?

1b. How could teachers and students communicate effectively when opportunities for discussion beyond the mathematics of the moment are taken?

1c. What experiences shape individual discourses at the Mathematical Horizon?

RQ 2 How can empirical evidence be used to create practice-based resources for developing teachers' [d]Discourse[s] at the Mathematical Horizon?

The first research question is addressed through three sub-questions that would allow linking the mathematics classroom Discourse and Discourse at the Mathematical Horizon. The first sub-question (1a) aims to identify characteristics of the mathematics classroom Discourse beyond the mathematics of the moment. Whereas the third sub-question (1c) aims to identify characteristics of Discourse at the Mathematical Horizon. Studying the effectiveness of the communication beyond the mathematics of the moment between teachers and students in the second sub-question (1b) would provide evidence of how the two discourses interrelate. The second research question explores ways of using empirical findings in future professional development and research activities. I am using the abbreviation [d]Discourse[s] to signal the difference between Discourse and individual discourses. I aim to propose a design for creating resources that address issues which are considered related to Discourse at the Mathematical Horizon for a community of teachers (here, secondary school teachers in England).

My aim and the nature of my research questions suggest a qualitative research methodology in order to explore how the participants "interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences"(Merriam & Tisdell, 2016, p. 6) An interpretive approach (Stake, 2010) is used to construe the observed situations to account for certain elements of teaching as a discursive activity. With the theoretical framework in mind, this study aims to answer the above research questions through my own informed interpretation, as a researcher, of the data using thematic and discourse analysis – specifically, commognitive analysis (see section 4.6). Moreover, the research design includes interviewing the participants, seeking their own reflections and interpretations of their actions in the classrooms, and their experiences as teachers of mathematics.

Due to the nature of this study, any results presented as answers to the research questions are bound to the context in which the study took place and my identity as a researcher. Therefore, Section 4.2 addresses the key characteristics of teaching and learning mathematics in England. In Section 4.5, I also describe my position as a researcher within that context.

4.2. Context of the study

The following section provides an overview of the educational system in England (Section 4.2.1), the structure of the National Curriculum in Mathematics (Section 4.2.2) and the available routes for becoming a Mathematics Teacher and some key workforce demographics (Section 4.2.3).

4.2.1. *The English Educational System*

The compulsory education in England spans from 5 to 16 years of age and consists of two educational levels: Primary (ages 5 to 11), and Secondary (ages 11 to 16), followed by optional enrolment in Further Education (ages 16 to 18). Primary and secondary education in England is typically based on the National Curriculum which is separated into blocks called Key Stages (KS) with mathematics being one of the compulsory subjects. Primary education includes KS 1 and 2 and students are expected to sit Standard Attainment Tests (SATs) at the end of each Key Stage. Secondary education includes KS 3 and 4 and students are expected to sit exams to obtain a General Certificate of Secondary Education (GCSE) in at least 5 subjects including English, Mathematics and Science. Typically, students sit their GCSE exams at the age of 16 but it is possible for students to sit their exams earlier or re-sit them at a later stage to improve their scores.

Once students have completed their secondary education, they can stay in full time education until the age of 18. Post-secondary education is called Further Education and includes 3 routes that lead to technical or applied qualifications for students: applied general qualification to continue in Higher Education (A-Levels), level 2 technical certificates and level 3 technical levels to specialise in a specific technical profession. Table 4.1 summarises the information about educational levels and examinations. Alternatively, individuals who do not wish to continue in full-time education can start an apprenticeship or volunteer while in part-time education/training until the age of 18.

Thus far, I have outlined the general characteristics of the English educational system. However, students schooling experience might vary depending on the type of school in which they are enrolled. Specifically, there are six main types of schooling: attending state schools, faith schools, academies, free schools, independent schools or being home-schooled (https://www.gov.uk/types-of-school)[16]. State schools offer free education and are under the

[16] These types of schooling refer mainly to primary and secondary education. Further Education includes more options for example course run by universities, colleges and sixth forms. Detailing these differences is beyond the scope of the study.

control of the local authorities. State schools have to follow the national curriculum. Faith schools are state schools who have their own admissions criteria and follow the national curriculum, but they can choose what to teach in religious studies. Academies are independent schools funded by the government but run by academic trusts. Free schools are non-profit organisations that are funded by the government but are not run by local authorities. All state schools are inspected by the Office for Standards in Education (OFSTED). Academies and Free schools have the option of not following the national curriculum. Independent schools are private, fee-paying schools, and do not have to follow the national curriculum or be submitted to inspection by OFSTED. Finally, parents and students have the option of home-schooling, this type of schooling is beyond the scope of the study.

The overview of the types of schooling indicates the complexity of studying the actions of teachers beyond the curriculum. Some schools have much more control over what and how they teach compared to others. Despite that, many schools choose to follow the suggestions of the national curriculum and the exam boards for the core subjects, i.e., English, Mathematics and Science. In the following section, I discuss the national curriculum in Mathematics and the guidelines of the different exam boards.

Table 4.1 Stages of full-time education in England and examinations

Education Level	**Key Stage (KS)**	**Age**	**Year of Study (Y)**	**Exams**
Primary	KS 1	5 - 7	1	
			2	SAT1
	KS 2	8 - 11	3	
			4	
			5	
			6	SAT2
Secondary	KS 3	11 - 14	7	
			8	
			9	
	KS 4	15 - 16	10	
			11	GCSE
Further Education	-	16 - 18	12	AS-Levels (optional)
			13	A-Levels / Core Maths / BTEC etc

4.2.2. *The National Curriculum in Mathematics*

The National Curriculum of England is the policy framework intending to offer students “an introduction to the essential knowledge that they need to be educated citizens” (Department for Education, 2014b, p. 6) The three aims of the National Curriculum in Mathematics are for students to:

- become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately.
- **reason mathematically** by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language.
- be able to **solve problems** by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication, including breaking down problems into a series of simpler steps and persevering in seeking solutions. (Department for Education, 2013b, 2013a, 2014a).

The students are also expected to “apply their mathematical knowledge wherever relevant in other subjects”. The specified applications included in KS 1 and 2 with basic applications in science (Department for Education, 2013b), in KS 3 the applications are extended to geography and computing (Department for Education, 2013a) and finally in KS 4 extended to finance and economics (Department for Education, 2014a).

Table 3.1 list the topic areas covered during the four Key Stages. An important structural difference between the curricular materials of the Key Stages is that the programmes of study for KS 1 and 2 are set out by Year of study (Y) and include more detailed instructions as to what should be taught, and when, compared to the programmes of study for KS 3 and 4 that are less detailed and are set out by Key Stage. Both primary and secondary schools have the flexibility to teach subjects earlier or later within each Key Stage.

Table 4.2 Topic areas of National Curriculum by Key Stage

KS 1 and 2	**KS 3 and 4**
Number	Number
Measurement	Algebra
Geometry	Ratio, proportion and rates of change
Statistics	Geometry and measures
Ratio and Proportion (only Year 6)	Probability
Algebra (only Year 6)	Statistics

Each school is responsible for creating and publishing online their own 'school curriculum' which might or might not follow the guidelines of the National Curriculum depending on the type of schooling offered. Schools are in charge of creating their own resources or using and combining materials available by educational websites, publishers, university projects etc. Moreover, schools choose the exam board(s) which offers GCSE exams for their students. Currently, the three most popular exam boards in England are Pearson Edexcel (https://qualifications.pearson.com/en/home.html), AQA (https://www.aqa.org.uk/) and OCR (https://www.ocr.org.uk/). The choice of exam board also influences the development of the school curriculum. Each exam board offers guidelines for schools to help best prepare the students for the exams and additional resources for teachers and students. Finally, since the National Curriculum only covers the compulsory education between the ages of 5 and 16, exam boards influence the curriculum in Further Education.

4.2.3. *Mathematics Teachers*

According to national statistics available at the start of the data collection (November 2019), 75.8% of the school teachers were female, 78.5% of the workforce were white British (https://www.ethnicity-facts-figures.service.gov.uk/workforce-and-business/workforce-diversity/school-teacher-workforce/latest#by-ethnicity-and-gender). Moreover, only 15% of the teachers identified themselves as being a minority and fewer than 1% of the workforce held a non-UK qualification (https://explore-education-statistics.service.gov.uk/find-statistics/school-workforce-in-england). The specific demographics about mathematics teachers are not provided in the national statistics.

Becoming a qualified teacher In England is typically a two-step process. First, the individual needs to complete an undergraduate degree in Education for Primary Education, or in a related discipline to the subject they wish to teach for Secondary and Further Education. Specifically, if someone wishes to become a mathematics teacher, they are expected to have a degree in Mathematics or a degree which is at least 50% mathematics-focused (e.g., science, economics, engineering etc). Second, the individual is required to undergo Initial Teacher Training (ITT). At the time of this writing, there are three main routes for ITT: through a University-led course to obtain a Postgraduate Certificate in Education (PGCE), through School-Cantered Initial Teacher Training (SCITT) or through School Direct Initial Teacher Training (School Direct ITT) (https://www.gov.uk/government/collections/initial-teacher-training). When teachers complete the requirements to become qualified teachers, they obtain a Qualified Teacher Status (QTS). Having a QTS is essential for being employed by state schools, however, private schools and academies might choose to employ individuals regardless of their qualification status if they fulfil other job criteria.

The main focus of the programmes is for prospective teachers to develop pedagogical skills and practice subject-matter skills developed in previous stages of their education. All three routes lead to acquiring a QTS. However, SCITT and School Direct ITT routes do not necessarily award a PGCE certificate. The main difference between the three routes is the amount of teaching and seminars offered during the training year. The ITT options are funded from the Department for Education (DfE) and UK-nationals are eligible for bursaries during the training year. There are also fee-paying options for ITT training. Non-UK citizens can apply for funding to cover the fees of the training however the fee-paying options are less. (https://www.gov.uk/government/publications/train-to-teach-in-england-non-uk-applicants/train-to-teach-in-england-if-youre-a-non-uk-citizen). It is also possible for citizens of the European Economic Area, United States, Canada and Australia to obtain a QTS if they are already qualified in their home country (https://www.gov.uk/government/publications/apply-for-qualified-teacher-status-qts-if-you-teach-outside-the-uk/routes-to-qualified-teacher-status-qts-for-teachers-and-those-with-teaching-experience-outside-the-uk#apply-for-qualified-teacher-status-qts).

England faces an issue of teacher shortage, especially in STEM (Science, Technology, Engineering and Mathematics) Education. Thus, partly due to issues in recruitment and retention (Allen & Sims, 2018; Royal Society, 2018), mathematics teachers do not always have a degree in mathematics or a related subject. Non-specialist or out-of-field prospective teachers are required to have an A-level in mathematics and to attend a Subject Knowledge Enhancement (SKE) course prior to their ITT. While the length of ITT is one year, the length of SKE courses varies substantially among the providers. Mathematics enhancement courses are found to be a good strategy to address teachers' shortage (Warburton, 2014). However, research indicates a need for more research on the learning opportunities provided during the various programmes (Clarke & Murray, 2014).

Mathematics teachers begin their careers as Newly Qualified Teachers (NQTs) and can progress to senior teachers and Heads of Departments and Principals through years of experience and professional development. Teachers might also choose to change careers. Some teaching related options include becoming private tutors, working in student support or becoming teacher educators. Specifically, teacher educators who work in ITT are typically expected to have a QTS and teaching experience in schools.

Throughout their career mathematics teachers are expected to take part in professional development activities. Similar to the initial teacher training routes, there is a variety of Continuous Professional Development (CPD) activities offered to mathematics teachers by Universities, schools, professional development centres and maths hubs - programmes coordinated by the National Centre for Excellence in the Teaching of Mathematics (NCETM) aiming to improve mathematics education in England.

4.3. Participants

The participants of this study are 12 mathematics teachers and teacher educators in different stages of their careers and from a range of mathematical and social backgrounds. Also, during the lesson observations, there are in total 60 students participating, from 3 different classrooms. At the time of the data collection, 8 of the participants were working full-time as mathematics teachers, 3 were working for PGCE courses at universities across the country and 1 was a teacher in Further Education but also involved in the delivery of a Teacher Subject

Specialism Training (TSST)[17] programme. The methods used in the study include interviews, lesson observations, and focus group discussions with the teachers. The participants could choose their level of involvement in the study by opting in to participate in different stages of the study. The teachers' choices depended on their availability but also on students', and their parents', willingness to take part in lesson observations.

To recruit participants, I first approached teachers and teacher educators who worked at institutions in areas to which I had access via public transport, using an opportunistic sampling method (Tracy, 2012, p. 134). However, I aimed to capture a range of experiences and cultures. After receiving ethics approval, I sent out emails with information and a poster about my study to selected schools and asked them to share my email with their maths departments. I also contacted teacher educators by emailing them directly. In this way, I managed to secure a first group of participants, whom I met for an interview and to discuss the possibility of conducting lesson observations. Meeting the teachers in their work environment gave me the opportunity to meet and talk with other teachers about my study and to recruit more participants. I secured more participants, with the sampling technique of snowballing (Tracy, 2012, p. 136). With the snowballing technique, people already participating share information about the study with others who might be interested. Thus, I asked the teachers and teacher educators to share information about my project with colleagues who might be interested in participating.

In the following sub-sections, I provide information about the participants and their involvement during the different stages of the study.

4.3.1. *Participating Teachers*

From the 9 teachers participating in this study, 1 identified as female and 8 as male. At the time of data collection, the teachers were working at 5 schools[18] in and around Norfolk (see Table 4.3). Table 4.4 includes the pseudonyms and a summary of the profile of participating teachers.

[17] TSSTs are professional development programs focusing on improving "subject matter knowledge for non-specialists and returning-teachers" (https://www.gov.uk/guidance/teacher-subject-specialism-training-courses).

[18] The names of the teachers are not directly associated with the schools to protect confidentiality.

Table 4.3 Types of schools in the study

School	Type
School 1	Academy and Sixth form
School 2	Academy and Sixth form
School 3	Sixth form
School 4	Independent School
School 5	Faith School

Table 4.4 Profiles of participating teachers

	Pseudonym	Gender	Profile
1	Noah	Male	Noah acquired an undergraduate degree in Mathematics and Physics. At the time of the interview, he had 23 years of teaching experience and was the Faculty Leader for Mathematics in his school.
2	Eric	Male	Eric has an undergraduate degree in Economics and a Masters degree in Education. At the time of data collection, he had 7 years of teaching experience and was the Subject Leader for Mathematics in his school.
3	Marcus	Male	Marcus has an undergraduate degree in Mathematics and a Master's degree in Education. At the time of the interview, he had 6 years of teaching experience and worked as a mathematics teacher in a 6^{th} form. He was also a TSST co-ordinator and maths outreach lead.
4	Liz	Female	Liz has a degree in Mathematics. At the time of data collection, she was an NQT and had only one year of teaching experience. She has also worked in the industry as an analyst for 20 years.
5	Nick	Male	Nick has a degree in Environmental Sciences from a non-UK institution and a PhD in Environmental Science acquired in the UK. He has 21 years of teaching experience in secondary and tertiary education. At the time of data collection, he was an Advanced Skills Mathematics teacher.
6	Scott	Male	Scott has an undergraduate degree in Sports Sciences and has 17 years of teaching experience teaching Physical Education, Humanities and Mathematics. He has also worked as a fitness instructor for 7 years.
7	David	Male	David has a degree in Mathematics. At the time of the interview, he was the Head of Mathematics at his school and had 23 years of experience teaching Mathematics. Before training as a teacher, he had also worked in banking for 1 year.
8	Alex	Male	Alex acquired a mathematics degree from a European institution and a Master's in Mathematics Education from an institution in the UK. At the time of data collection, he had 2 years of experience as a mathematics teacher.
9	Sam	Male	Sam works as a mathematics teacher. He only participated in the focus group and no further details were gathered for this participant.

In Table 4.5 the reader can find information about the choices of the teachers to participate in one or more methods of data collection. 8 of the teachers were interviewed, 3 of them were also observed during lessons and 4 teachers participated in the focus group discussion. It is worth acknowledging that 6 of the teachers originally agreed to participate in lesson observations however for 3 of them (Noah, Eric and David) that was not possible due to limited parental consent (< 2/3 of students) or COVID-19 school closure (see also Section 4.10.1). After informing the students and their parents and collecting the parental consent forms, the observations went ahead with 3 of these teachers (1 female and 2 male).

Table 4.5 Levels of participation

	Pseudonym	**Interview**	**Observation**	**Focus group**
1	Noah	✓		
2	Eric	✓		✓
3	Marcus	✓		
4	Liz	✓	✓	
5	Nick	✓	✓	✓
6	Scott	✓		
7	David	✓		
8	Alex	✓	✓	✓
9	Sam			✓

At the time of the interviews the participants were teaching mathematics in KS 3, KS 4 and/or in Sixth form (from Year 7 to A-level). Recruiting participants who teach in different levels provided more opportunities for discussions across mathematical topics, applications of mathematics and their teaching in different levels.

Despite using a convenience sample, the recruitment of the teachers followed criteria to ensure that the teachers represented different mathematical and cultural backgrounds. For example, having a degree from a discipline other than pure mathematics and/or representing minorities. Their level of experience ranged from NQT to AST (Advanced Skills Teacher) and three of them were Heads of department during the time of data collection. Finally, one of the teachers was also involved in developing and delivering TSST programmes for in-service non-specialists and had experience as a teacher educator. In the following section, I summarise information about the participants who at the time worked as teacher educators full-time.

4.3.2. *Participating Teacher Educators*

In addition to the teachers and students, 3 teacher educators took part in the study, 1 identified as female and 2 as male. All three teacher educators had prior experience as secondary school teachers. At the time of the empirical study, the teacher educators worked in three universities across England, leading PGCE courses for prospective secondary mathematics teachers. All of them had a degree in mathematics. Two of them also had doctoral degrees in Mathematics and Mathematics Education respectively. Table 4.6 summarises this information. The teacher educators agreed to be interviewed about their experiences teaching both secondary school students and prospective teachers[19].

Table 4.6 Profiles of participating teacher educators

	Pseudonym	**Gender**	**Profile**
	Violet	Female	Violet has a degree in Mathematics and was PGCE secondary lead. She has 10 years of experience teaching in secondary schools across the country and 14 years working as a teacher educator.
	Damian	Male	Damian has a degree in mathematics and was working as PGCE director. He has 10 years of experience as a teacher in secondary school. Also, he has a PhD in Mathematics.
	Thomas	Male	Thomas has a degree in mathematics and worked as Senior Lecturer, PGCE secondary course leader at the time. He has 11 years of teaching experience in secondary school. Also he has a PhD in Mathematics Education

4.3.3. *Participating Students*

The focus of this study is teachers' practices in relation to their Discourse at the Mathematical Horizon. Since discourse is "made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (Sfard, 2008; p. 297), the contributions of the students are key in observing elements of teachers' practices during discussions beyond the mathematics of the moment. Interacting with the students is only one of the stimuli of

[19] Reminder: one of the participants who was classified as teacher had also experience as a teacher educator.

triggering teachers' actions - others can be for example a task, a reading or a video, students' voluntary contributions are important to the study. However, not all the students present participated in the study.

As mentioned earlier, I observed the lessons of 3 of the teachers. In total, I collected consent forms from 60 students and their parents. Table 4.7 includes information about the students of these three classes.

Table 4.7 Information about participating students

Pseudonym of Teacher	Number of participating students	Year
Alex	18	9
Nick	22	7
Liz	20	7

All three classes followed the Key Stage 3 (KS 3) curriculum at the time and the schools followed schemes of work provided by Pearson Edexcel. On average each classroom had about 25 students present in each lesson. With Alex's classroom presenting the most variations, with students often being absent or being removed from the classroom periodically[20] as a form of punishment for inappropriate behaviour. In Nick's classroom 22 students consented to the lesson being observed, however, 7 of those students indicated that they did not want to be audio-recorded. Similarly, in Alex's classroom 2 of the 18 students preferred to not being audio recorded. In these cases, the microphones were positioned away from the students and the contributions of those students are accounted for using only the notes taken during the lesson observations. In Section 4.8 one can find more details about protecting the information of non-participating students' identities during the lesson observation.

4.4. Processes and methods of data collection

The focus of the study is to explore and identify the characteristics of opportunities mathematics teachers have in order to engage in mathematical conversations beyond the curriculum and teaching instructions with their students. To acquire sufficient and reliable

[20] Standard practice used in Alex's school to discipline students.

experiential insights into the matter, I collected data from lesson observations and interviews with teachers and teacher educators. In addition, I aim to develop a design of professional development resources that could be used to aid the development of teachers' discourses at the Mathematical Horizon. In the context of the study, I piloted one of the resources created with a group of teachers in the form of focus group discussion. In this section, the reader can find information about my course of action, methods and tools used.

4.4.1. *Lesson Observations*

As mentioned in Section 4.3, I conducted lesson observations with 3 teachers: Alex, Nick and Liz. In Alex's and Nick's cases, I observed one lesson per week for 5 weeks. While in Liz's case, I observed 1 or 2 lessons per week for 4 weeks. Table 4.8 includes a list of the topics covered during the observations quoted from the lesson slides.

Table 4.8 List of lesson observations

Teacher	Students	Topics of lesson in chronological order	
Alex	Year 9	A1	n^{th} term of arithmetic sequences
		A2	n^{th} term of quadratic sequences
		A3	Times series graphs
		A4	Pie charts
		A5	Frequency tables
Liz	Year 7	L1	Vertically opposite angles
		L2	Angles [in circles]
		L3	Angles in quadrilaterals
		L4	Calculating angles
		L5	Angles Investigation
		L6	Geometry and algebra
Nick	Year 7	N1	Pie charts
		N2	Revision
		N3	Angles and parallel lines (1)
		N4	Rounding and Estimating
		N5	Angles and parallel lines (2)

The lesson observations were audio-recorded using three microphones: one worn by the teacher and two on either side of the classroom. In addition to the audio recorded lesson observation, I conducted one pilot lesson observation in each classroom, to familiarise myself with the environment, develop and practise a protocol for keeping notes and informing the students about my presence in their lessons. The pilot observations were not audio-recorded.

During the lesson observations I was keeping notes of what was happening in the classroom to help me track the conversations in the audio-recordings and capture visual elements of the lessons (e.g., slides and notes on the board, seating plan etc). The notes were taken on a laptop PC with touch screen using Microsoft® OneNote (https://www.microsoft.com/en-gb/microsoft-365). The advantages of digital note taking were that the notes were automatically stored in a secure encrypted folder and that I could use technology to improve the overall clarity of the notes by incorporating typing and drawing with a stylus. Figure 4.1 depicts the outline of the protocol I developed for taking notes during the observations.

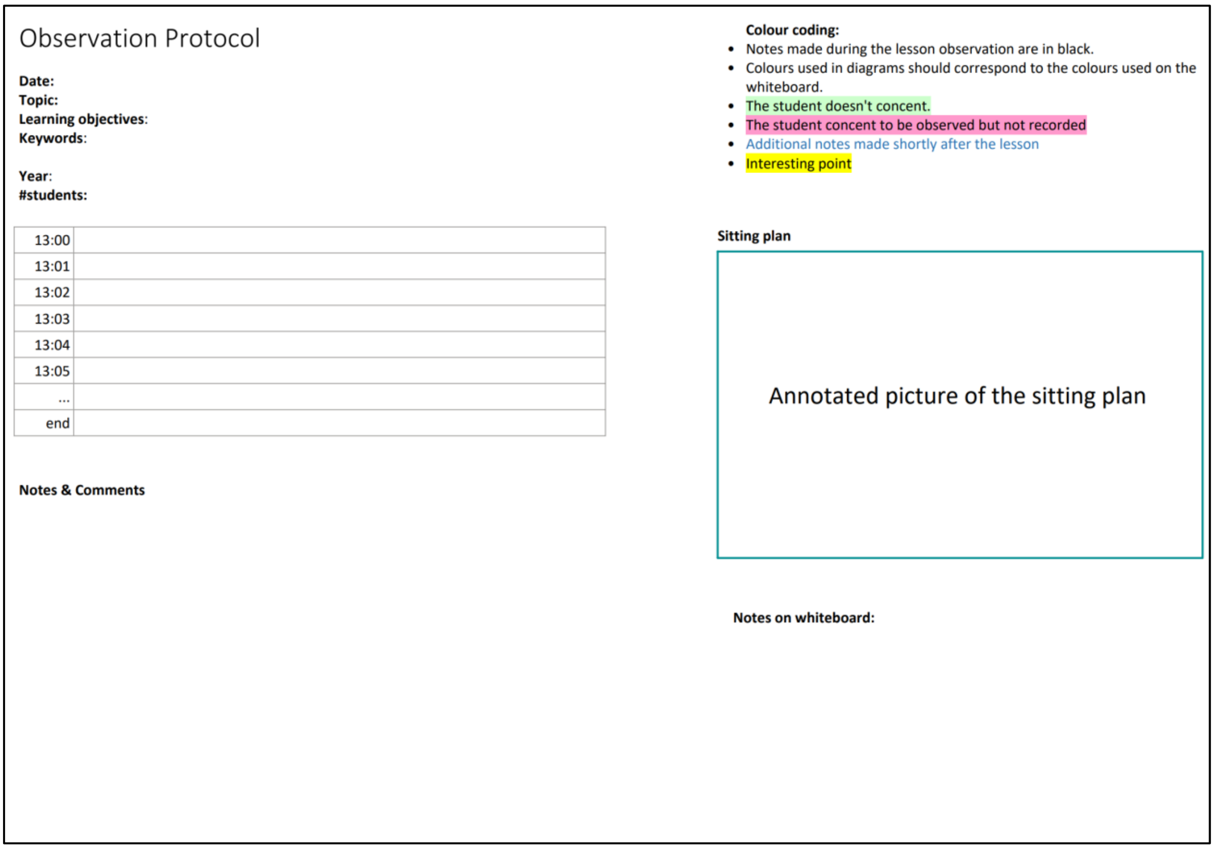


Figure 4.1 An outline of the observation protocol

The protocol includes contextual information about the lesson, an annotated picture of the seating plan of each classroom, a section to hand-write the notes on the board and a table to track the interactions during the lesson. Specifically, I asked each teacher to share with me the seating plan with the names of the students then I highlighted the names of the non-participating students. During the lesson, I used the seating plan to track the non-participating students and take a note of when they speak to remove their contributions in case it was captured in the audio-recordings. On the left-side of the page (see Figure 4.1) I created and used a table to time-stamp my notes during the lesson. In the table, I was writing the comments about specific moments during the lesson and noted the times when a non-participating student was speaking with the teacher in small groups or whole class discussions. Finally, the notes and the audio-recordings were revisited shortly after each lesson to make additional comments and produce factual accounts of each lesson.

Supporting data were collected from the schools' websites (e.g., school curriculum) and the Edexcel teaching resources and guidelines. In addition, Alex shared with me drafts of the slides for all the lessons he was planning to teach that year.

4.4.2. *Interviews*

The purpose of the interviews was to capture the teachers and the teacher educators' views and experiences regarding discussions beyond the mathematics of the moment. The interviews were semi-structured (Mann, 2016). Specifically, I prepared a plan that included a set of nine indicative questions, to be used as possible probes for discussion during the interviews (see Appendix V). However, the questions were personalised and adapted depending on the flow of the discussion.

The interviews lasted between 35 and 70 minutes, depending on the availability of the participant and the flow of the discussion, and consisted of four parts. At the start of each interview, I asked the teachers to tell me about their journey to becoming mathematics teachers (and when appropriate teacher educators). The purpose of the question was to initiate a conversation and collect information about the teaching experience of the participants and their mathematical background. The second part of the interviews consisted of three main questions and aimed to collect views and experiences about making connections across and beyond the curriculum, discussing ideas beyond the mathematics of the moment. The third part of the interview was different for teachers and teacher educators. It included a discussion of a mathtask for teachers or a discussion about their role as teacher educators (see also Sections 4.4.3 and 4.4.4). For teachers, the aim of this part of the interview was to introduce the mathtask resources and promote participation in focus group discussions. For teacher educators, the aim was to capture their views about teacher education in general and in relation to subject matter beyond the curriculum in particular. Finally, the fourth part of the interview was a hypothetical question requiring the teachers to reflect on what they would advise their students as they leave secondary/further/tertiary education. The aim of the question was to collect information regarding what the participants value as essential learning.

The face-to-face interviews were recorded using two digital audio recorders. One was used as the main audio-recorder and the second one used as a backup solution to mitigate the risk of losing data due to technical issues (e.g., low battery level, misfunctioning equipment). The online interviews were conducted through Microsoft® Skype for Business (https://www.microsoft.com/en-gb/microsoft-365). The first two online interviews were recorded using the built-in recording tool on Skype and the following ones using the audio and video editing software Camtasia® (https://www.techsmith.com/video-editor.html).

4.4.3. *MathTASK Design*

A part of the study is focused on creating vignette-based professional development resources aiming to foster opportunities for teachers to discuss mathematical ideas and practices beyond the mathematics of the moment and to develop their discourses at the Mathematical Horizon. Here, discourses are in plural with the intention of highlighting that Discourse at the Mathematical Horizon has different individualised aspects. My aim is to create resources based on data and findings from the lesson observations and interviews that could be used to facilitate conversations among teachers and between teachers and teacher educators. To do so, I based my design on the MathTASK principles (Biza et al., 2018, 2021). As a team member of the MathTASK programme, I took part in the development of mathtask and the organisation of professional development activities for teachers beyond the scope of my personal research project.

MathTASK resources, thereafter, called mathtasks, are presented to teachers as short narratives of a classroom situation starting from a mathematical problem followed by a hypothetical but realistic classroom dialogue and a series of questions. The design of mathtasks follows 5 principles:

- The mathematical content of the mathtask involves “a topic or an issue that is known for its subtlety or for causing difficulty to students” (Biza et al., 2018, p. 61). The mathematical topic can be chosen from the literature and/or teaching experience.

- The fictional dialogue reflects the topic or issue and provides an opportunity for the teachers to reflect and demonstrate ways in which they would act in the same situation.
- The fictional dialogue involves pedagogical approaches that "concerns mathematical, pedagogical and epistemological issues that are known for their subtlety or for being challenging to teachers" (Biza et al., 2018, p. 61)
- "Mathematical content and student/teacher responses provide a context in which teachers' knowledge, beliefs and intended practices (mathematical, pedagogical and epistemological) that are allowed to surface (MKT, HCK, KQ)"[21] (Biza et al., 2018, p. 61)
- The mathematical content and the dialogue are contextualised to the curriculum and the educational context.

The MathTASK principles make provisions to foster Horizon Content Knowledge (HCK in the above quotation) in the general principles. However, the principles do not specify how this can be achieved. Using findings from the present study might illuminate this aspect of the MathTASK principles.

Initially, the research plan included a more substantial use and development of mathtasks for bringing up Discourse at the Mathematical Horizon. However, due to the burst of the COVID-19 pandemic the plan had to be adapted (see also Section 4.10). After the changes the plan went ahead with trialling a pilot mathtask in a focus group discussion with four teachers.

The mathtask (Figure 4.2) was created during the first year of my doctoral studies and finalised in June of the following year for the purpose of the pilot. The idea for the mathematical problem came to me while studying the literature. In particular, while I was studying the National Curriculum and the available textbooks. The mathematical problem posed in this instance is an adaptation of an activity included in a textbook for Year 8 students, in the STP Mathematics 8 (Bostock et al., 2014). The problem was part of a series asking the students to draw shapes and calculate their area. This activity stands out because, in contrast with other similar activities directed towards Year 8 students, it gives no specific instructions

[21] MKT: Mathematical Knowledge for Teaching, HCK: Horizon Content Knowledge (Ball et al., 2008), and KQ: Knowledge Quartet (Rowland et al., 2005)

as to what the dimensions of the square paper should be leading potentially to a situation where students could use different units depending on availability of graphing/square paper.

I recreated a similar activity to be used in the pilot having in mind a fictional moment of contingency where two students realise that their diagrams are different yet they both reach a correct numerical answer in different units. The original idea for the mathtask was fine-tuned closer to the date of the focus group following the preliminary analysis of the lesson observations. Based on the preliminary analysis of the lesson observations, the tacit mathematical idea presented in the dialogue is in line with one of the themes identified as opportunities to go beyond the mathematics of the moment and the suitability depends on the teaching style. This situation might be plausible in a classroom where exploration is encouraged by the school, the teacher and the available resources (e.g., use of different types of graph paper or use of dynamic geometry software). In addition, the dialogue was adapted to be more realistic and make use of words typical for students in this age group. Prior to the focus group discussion, the mathtask was discussed and refined through discussions with other team members and members of the Research in Mathematics Education (RME) group at the University of East Anglia. Figure 4.2 depicts a printable version of the pilot task.

The Area of a Triangle

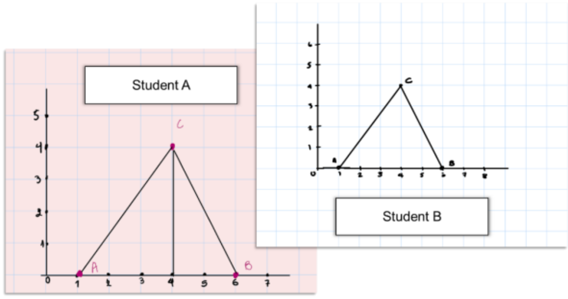
In a Year 8 class, the teacher gave the students the following mathematical problem:

Use squared paper to draw axes for x and y from 0 to 6 using 1 square to 1 unit. Find the area the triangle ABC with A(1,0), B(6,0) and C(4,4) *

Students A and B work on the problem when the following conversation takes place:

Student A: I found 10!

Student B: Me too! Wait … That can't be right! Your triangle is much bigger than mine… (Student A puts the two triangles side by side as in the figure below) How can that be possible?



Student A: I don't know … I'm sure about my answer … you see … the height is 4 and the base is 5 …

Student B: Yes, I did the same … so weird!

The teacher overhears the conversation and joins in.

Teacher: What is weird?

Student A: Emm … we both have found that the area is 10 … but our triangles are not the same. Is it because our drawings are not accurate?

Teacher: No! Both are fine! It's just because you have different squared paper. Your paper is in centimetres and yours is in inches.

Student A: Oh, I see … ok so mine is 10 square inches and yours …

Student B: Yeah, yeah … I get this, but still it is weird, isn't it? The area is the same, but it isn't, is it? Your triangle is larger than mine so you should have larger area. Oh … I am so confused …

Questions:

a. Solve this mathematical problem bearing in mind that this is a Y8 lesson.
b. Which parts of the conversation attract your attention?
c. If this conversation has taken place in your class, how would you respond to student B and to the whole class?
d. What would be the focus of your response on this topic?

Figure 4.2 Printable version of pilot mathtask

4.4.4. *Pilot Focus Group*

As part of the data collection, the mathtask in Figure 4.2 was trialled with a group of four teachers in an online focus group discussion. The focus group discussion took place in the form of a workshop where the teachers would discuss the different parts of the mathtask similar to how mathtasks are used in professional development activities (Biza et al., 2021). The aim of the focus group was to collect data about the potentials of the specific mathtask that would inform the design of more professional development resources. Specifically, the objectives of the focus group were to explore (1) whether the pilot mathtask can engage the participants in a debate about opportunities for discussions beyond the mathematics of the moment and (2) whether there are any observed discursive shifts in the mathematical and pedagogical narratives of the participants during the focus group.

The focus group took place on a platform called Blackboard Collaborate (https://www.blackboard.com/en-uk/teaching-learning/collaboration-web-conferencing/blackboard-collaborate) and lasted 1 hour and 30 minutes. The session was recorded using Camtasia®. Prior to the focus group discussion, I planned the structure of the focus group, familiarised myself with online data collection and the pieces of software that I was going to use. I also considered the possibility of asking a colleague to be present during the focus group to take notes. However, that was not possible due to time constrains as the focus group had to be prepared and conducted earlier than it was initially planned. During the focus group, my role was to be the facilitator of the discussion and take notes whenever possible.

The layout of the focus group included an introduction to the Collaborate platform, an icebreaker activity and then the discussion of the mathtask in parts. First the teachers were asked to solve the mathematical problem without having seen the dialogue and reflect on and share their thoughts about the problem. Next, the teachers were presented with the dialogue and the questions and were asked to spend time individually answering the questions. Then, we discussed the answers as a group. Finally, the teachers were asked to reflect on the mathtask in terms of content, structure and as a conversational piece.

4.5. Role of researcher

To better convey the results of the current study, I believe that the reader needs to be aware of my position and identity as a researcher, teacher, and mathematics learner. In parallel, I discuss my roles as a non-participant observer in the classroom and as an interviewer.

My degree in mathematics allowed me to have 'an informed view of the mathematical landscape' during the data collection and data analysis. During lesson observations, I was able to spot mathematical practices and ideas that are related to the topic of the day (mathematics of the moment) and identify potential episodes to be analysed. Similarly, during the interviews with the teachers and teacher educators, I was able to engage in rigorous mathematical conversation. For example, when the participants were talking about history of mathematics, advanced mathematical concepts or applications of mathematics that are not included in the secondary curriculum. Finally, during the data analysis, my advanced mathematical discourse helped me to pick up comments made by the teachers during lessons and interviews and to identify the potentials of mathematical discussions even when they were not fully addressed or noticed by the teachers.

My mathematical and pedagogical experience, both in Greece and in the UK, gives me an insider's view of what teaching and learning mathematics might look like in different contexts, whilst paying attention to the different discursive rules that apply in those different contexts. Through my work as a tutor, I have experienced the difficulties of teaching, for example, in planning lessons or attending to the needs of students, and the importance of the context in making decisions about what and how to teach. Throughout my PhD I was working as a sessional mathematics tutor for the Learning Enhancement Team which is part of the Student Services at the University of East Anglia (UEA). This experience gave me an extra advantage. I have had many chances to notice where university students struggle with mathematics during their studies, and I had the chance to engage with mathematical discourse in other disciplines. Being in regular communication with university students and lecturers from different disciplines provided insights into the different rules of mathematical discourse in other disciplines. Having these insights were also useful during the interviews with participants whose background in mathematics was different to mine, e.g., teachers that were educated in the UK or who had a degree in a discipline other than Mathematics. During data analysis, this additional information aided me in the identification of potential opportunities to go beyond the mathematics of the moment in a way that is relevant to what students might find useful in the future.

A commognitive researcher needs to be able to assume both an insider's and an outsider's perspective to the discourse under study. In the previous paragraphs, I have listed elements that make me an insider to the discourse. Now, according to Sfard (2008), the researcher "who, for the sake of sharper distinctions and insights, tries to assume an outsider's outlook at this discourse must proceed with her investigations ignoring the question of what kinds of objects mathematists are playing with" (p. 130). However, as a mathematics graduate, I have a specific mathematics discourse. Likewise, my studies in mathematics education and my teaching experience in Greece and in the UK have contributed in my mathematical discourse for teaching. To overcome the challenge of keeping the outsider's perspective, I am putting myself "in the position of a perfect beginner" (Sfard, 2008, p. 130), or in other words, to take in objects that I am familiar with through the experiences of others. For instance, during the interview with the teachers and teacher educators, I had been avoiding the use of mathematical and mathematics education terminology except when the participant was using a term. I tried to keep concepts open for discussion while I was asking the participants elaborative questions to get more insights into what they mean.

My mathematical and pedagogical experience in both Greece and in the UK allows me to explore the potentials of a conversation by combining information from different contexts. That is particularly useful in the use of mathematical terminology in two ways. Firstly, there is a significant number of mathematical objects that are signified by words with Greek roots. Being a native speaker gives me additional information about these objects - e.g., the words polygon, geometry, or dodecahedron – which non-Greek speakers might ignore. Secondly, in cases where the terminology is different in Greek and English, I am in a position to compare the two if necessary and get linguistic information from both. For example, the words 'reciprocal of number a' and 'αντίστροφος του αριθμού a' [antístrofos tou arithmoú alpha] describe a number b with the property $a \times b = 1$. To me both words are very useful. The English one gives me the information that a is related to b as a pair, and the Greek word, that comes from the composition of the words 'counter' and 'turn', hints at the rule of turning the fraction upside down. Being aware of these linguistic differences in some cases proved very useful, giving me opportunities to think whether this additional information is useful to the students or how one could explain the meaning of terms in different ways.

Being educated in Greece, I am not accustomed to terminology and practices used in schools in England. For example, in contrast to the UK context, formal proving is a very common practice in secondary schools in Greece. However, topics such as probabilities and statistics are only introduced to Greek students at a later stage of their studies. Therefore, my teaching practices and mathematical discourse differ from those of my participants not only in word use but also in the routines that constitute a narrative as mathematically acceptable (endorsed). As a classroom observer, I assumed a 'perfect beginner' standpoint experiencing already 'known ideas' for me, from a different teaching perspective having genuine interest in what was happening around me. Having the mathematical but not necessarily the appropriate pedagogical discourse about topics in the UK curriculum allowed me to have meaningful conversations with the teachers and teacher educators. During the interviews, my position was clearly not to provide the participants with feedback about their actions in the classroom rather to ask them to reflect on their experiences. Having no insider's pedagogical discourse on the subjects discussed, I showed my genuine interest and asked for clarification when needed. I believe that my position as non-expert in teaching made them more relaxed and willing to talk about their experience. In addition, I could easily identify key mathematical ideas to focus on during the conversation and let the participants explain to me mathematical and pedagogical aspects of these ideas without me having many preconceptions of how these ideas might be introduced to students in England.

As an observer in the classroom, my aim was to be seen as an outsider both to the students and the teachers I observed. That is, to be seen as someone being there to learn what is happening during a lesson and not to participate in it. Nonetheless, my presence in the classroom inevitably interfered with the lesson on some occasions (see also Section 4.9).

My age, gender and nationality also contributed to my interactions with the participants. Being a young, female researcher and mathematician influenced the way I positioned myself during the interviews. I believe that I was seen by some participants as less authoritarian or judgemental due to that. Also, a few of the participants seemed to acknowledge the fact that I was a newcomer in the UK, for example by giving me detailed information about the context.

Finally, it's worth mentioning that my appreciation for mathematics and the work of teaching is also related to growing up in a family with a lot of STEM teachers. Especially my mother has always been a role model as a woman, mathematician, and teacher.

4.6. Processes and methods of data analysis

In this section, I describe the phases of the analysis and the steps I followed within each phase. The data were analysed using the software ATLAS.ti (https://atlasti.com/) and Microsoft® OneNote.

4.6.1. *Preliminary analysis of the lesson observations and interview data*

The first stage of the analysis includes a preliminary thematic analysis of the data collected during the lesson observations and the interviews, i.e., audio-recordings, observation notes, and teaching materials. The aim of the preliminary phase was to familiarise myself with the data and identify potential episodes of interest and emerging themes. The preliminary analysis informed the later stages in two ways. First, themes and relations identified were used as visual mediators to develop my strategy during the analysis. For example, using diagrams created in ATLAS.ti, I identified similarities and differences between what the teachers reported during the interviews and what was observed during the lessons. Second, reflecting on the processes I followed during the preliminary analysis, I identified gaps in the processes I followed and ways in which I could improve. For instance, during the preliminary analysis phase, I found that I tended to use secondary data (e.g., reports from the DfE, guidelines from exam boards and the schools' curricula) to support episodes that I already found interesting based on my existing understanding of the curriculum and pedagogical practices. Therefore, in the main phase of the analysis, I decided that it would be good practice to examine all the interactions in the classroom and the views of the teachers in a way that was more rigorous through systematic analysis and reduction of bias. I achieved this by consulting secondary data (e.g., documentation from the school, lesson slides etc) for every mathematical topic identified and not only for the ones that appeared to be related to ideas and practices beyond the curriculum.

4.6.2. *Analysis of the lesson observations*

The analysis of the lesson observations consists of five steps (see Table 4.9). The data that were analysed include observation notes, teaching materials and audio recordings of the lessons and the interviews with the three teachers, Alex, Liz and Nick, as well as secondary data (e.g., schools' curricula and additional resources available to the teachers). The audio-recordings from the lessons were analysed as audios and only part of the data was transcribed. The decision was made in order to preserve the exact discussions as much as possible, including any parallel discussions that were taking place among students as a reaction to what the teachers were saying. The software ATLAS.ti was specifically chosen for having user-friendly tools to analyse audios. An earlier version of the method of analysis was presented in CERME12 and is included in the proceedings for publication (Papadaki & Biza, 2022).

Table 4.9 Steps of the analysis for the lesson observations.

Analysis of the lesson observations	
Step 1	Creating a template of the structure of each lesson according to the teacher's actions (e.g., initiating of the lesson, talking with groups of students, addressing the class etc).
Step 2	Identifying the mathematical topics and practices mentioned and/or discussed during the lesson.
Step 3	Identifying episodes with opportunities for discussion beyond the mathematics of the moment. Identifying the topics that do not match the topic of the day or teaching instructions and recommendations (consulting secondary data).
Step 4	Categorising the opportunities to go beyond the mathematics of the moment in taken, potential of the discussion and potential of the task
Step 5	Discursive analysis of the opportunities. Examination of the word use, visual mediators, narratives, routines in relation to intersubjectivity.

The first step was to create a template for each lesson to study teachers' contributions by dividing each lesson into sections according to the actions of the teacher. Table 4.10 includes the codes used and brief descriptions of how they were used to divide the lessons in smaller sections. The codes were decided based on common themes in the structure of the lessons, e.g., groupwork, introduction of new topic etc. During this step, I also identified the parts of the post-lesson interviews that corresponded to the sections identified.

In the second step of the analysis, all sections and corresponding parts of the interviews were coded in relation to the mathematical objects (e.g., angles) and/or practices (e.g., measurement of angles) accounting for the discursive elements (word use, visual mediators, narratives, and routines) identified. Specifically, for each section identified in the first step, I highlighted the mathematical narratives produced by the teacher and/or the students, identified the word use, the routines and, from the notes and the teaching materials, the visual mediators used. Then, I proceeded with coding the section according to the mathematical topic or practices these discursive elements accounted for.

Table 4.10 Descriptions of the codes used for sectioning the lessons according to teaching actions.

Code	Description of the actions included
Structural elements of lesson	• Procedural talk to initiate or terminate a lesson • Taking the register • Collecting and distributing learning materials/homework
Addressing the class	• Procedural talk to prompt students to work in groups • Procedural talk to get the students' attention back from individual/group work or prompt students
Talk about topic	• The teacher introduces a new topic to the class • The teacher revises a topic for the class
Introduction of activity	• The teacher introduces an activity to the students
Observing students work	• The teacher walks around observing the students' work without interacting with them
Talking with (group of) student(s)	• Communication between the teacher and an individual student or a small group.
Not interacting with students	• The teacher works on the computer • The teacher talks with an external (e.g., another teacher) • The teacher talks to the observer
Prompting discussions about mistakes	• Prompting students to identify (possible) mistakes that might have been made • Asking the students to notice an error
Providing clues	• Providing hints to help the students with the activity
Commenting on a challenge	• The teacher comments on a challenge given to the students (e.g., level of difficulty, relevance to the topic of the day etc)
Working as a class	• Solving (part of) the problem together with the students as a class
Giving an answer to the class	• Giving an answer to an activity on the board without the contribution of students

The third step of the analysis concerns the identification of taken and potential opportunities to go beyond the mathematics of the moment. By comparing the identified mathematical topics and practices discussed during the lessons with the topic of the lesson, as identified by the teacher and secondary resources, I eliminated the sections which were directly related to what the students are expected to learn during the lesson or consecutive lessons on the topic. I regarded the remaining sections as episodes that included opportunities to engage in communication beyond the mathematics of the moment and proceeded to analyse them further. For example, during a lesson on time graphs any discussion about the use of time graphs was regarded as directly related to the lesson, however, more general comments about the use of time graphs in quantitative studies exceed the requirements of the curriculum and thus could present an opportunity to go beyond the mathematics of the moment. The second and third steps of the analysis were introduced to improve the rigor of the analysis as identified and discussed in Section 4.6.1.

In the previous step, the identified opportunities were considered only in relation to the mathematical topics and practices mentioned. In my attempt to consider the identified episodes in relation to the discursive interactions between the participants, I realised that my interpretation of the opportunities based on the focus of the study did not always align with the actions of the participants. Teachers' decision making is a highly complex process that involves various factors and considerations. This complexity should be acknowledged despite the specific focus of the study. Moreover, the interpretations are bound to my mathematical and pedagogical discourses. Thus, although my interpretations are grounded in the curricular context and my discussions with teachers, alternative interpretations of the episodes are possible.

For transparency purposes, I introduced a distinction between the actions of the participants and alternative courses of action I considered based on the focus of the study. In the fourth step of the analysis, I categorised the identified opportunities in three groups: *potential of the activity*, *potential of the discussion*, and *taken*. Specifically, opportunities were categorised as *potential* when the discursive actions of the teacher and the students did not lead towards a discussion beyond the mathematics of the moment. Potential opportunities were further divided into *potential of the discussion* (i.e., the mathematical object or practice was briefly mentioned by the teacher or the students during the lesson without further discussion) and *potential of the activity* (i.e., the mathematical topic or practice was identified during the analysis based only on the discursive elements of a task, the teaching materials or other peripheral stimuli). Finally, opportunities were categorised as *taken* when the discursive actions of the teacher – as the expert of the classroom – lead the classroom discussion beyond the mathematics of the moment (i.e., explicitly address the mathematical topic or practice identified during the third step of the analysis). Table 4.11 presents an overview of each category.

Table 4.11 Description of each opportunity category

Category	**Description**
Taken	The mathematical object or practice is made explicit and addressed by the teacher during the lesson
Potential of the discussion	The mathematical object or practice is made explicit in the lesson but is not addressed by the teacher
Potential of the activity	The mathematical object or practice is not made explicit during the lesson, it is only attributed by the analysis

Finally, in the fifth step, I revisited the interactions between a teacher and the students during each episode to identify deviations or alignments in communication from the intersubjectivity perspective (Cooper & Lavie, 2021). To do so, I compared the discursive elements of the teacher's and the students' talk in the episodes. My focus was on the way they communicated to establish a shared understanding of the topic under discussion. I based my analysis on the operationalisation of intersubjectivity as a discursive "action that is drawn from the learner's precedent space, yet can be seen as appropriate in the new discourse" (Cooper & Lavie, 2021, p. 8) through the use of words, visual mediators, narratives and routines that carry meanings from one discourse to the other (see also 3.3).

The episodes were further analysed alongside the interview data from all the participants to find themes which connect the mathematical characteristics among the identified opportunities to go beyond the mathematics of the moment compared to the requirements of the curriculum (see Section 4.6.5).

4.6.3. *Analysis of the interview data*

The interviews with teachers and teacher educators were analysed as one set of data. I made the choice to treat the two groups as one – broader – group of people involved in teaching. All 4 participants involved in teacher education started their careers as secondary school teachers and one of them still identified as a teacher at the time of the interviews. During the interviews the teacher educators reflected equally on their experiences as teachers and as teacher educators, thus, I see the teacher educators as teachers who have chosen to extend their career without necessarily changing career paths as would someone who had left teaching to join, for example, market research.

The analysis of the interview data took place in four steps (see Table 4.12). Firstly, I performed a thematic analysis of the interview data to identify the topics and practices that the participants mention as relevant to their teaching. I coded the topics according to the mathematical topic or practice by tracking the discursive elements of the participants' talk. During the second step, I identified which of the topics and practices are related to ideas beyond the curriculum by consulting again secondary data (e.g., curriculum and/or teaching guidelines).

Table 4.12 Steps of the analysis of the interviews

Analysis of interview data	
Step 1	Thematic analysis to identify the mathematical topics mentioned.
Step 2	Identifying which of the topics and practices could be related to ideas beyond the mathematics of the moment.
Step 3	Thematic analysis of the factors teachers mention as relevant to their teaching practices beyond the mathematics of the moment.
Step 4	Discursive analysis of the excerpts identified in steps 2 and 3.

The third step included the identification of the limiting and enabling factors which the participants mentioned when considering discussing ideas and practices beyond the mathematics of the moment. Using thematic analysis, I first identified the factors mentioned across the interviews. The factors are categorised into three main themes: *Teachers*, *Students* and *External*. Figure 4.3 illustrates the identified factors.

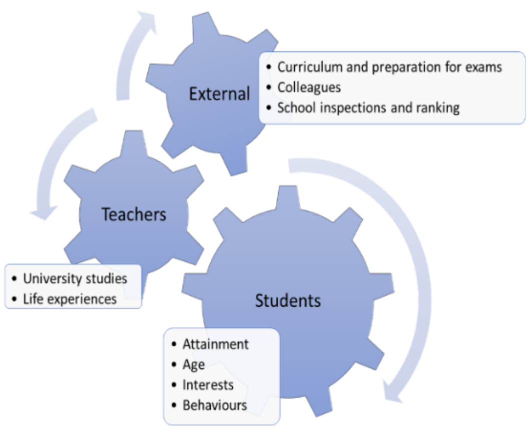


Figure 4.3 A visualisation of the factors and their interactions

Through the process, I noticed that although teachers often mention the same or similar factors they do not always agree as to whether these factors are limiting or enabling. Therefore, I proceeded to a fourth step and performed a discursive analysis of the excerpts identified in the second and third steps to explore possible links between teachers' mathematical and pedagogical discourses and the variations in the reported factors. The discursive analysis focused on identifying similarities and differences between the mathematical and pedagogical narratives of the participants during the interviews. Specifically, the mathematical narratives were analysed in relation to Disciplinary Discourses (e.g., Mathematics, Economics, etc) and colloquial. While the pedagogical narratives where analysed in relation to their alignment with the spectrum Pedagogical Discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022).

4.6.4. *Analysis of the pilot mathtask and focus group*

The final set of data that was analysed was the focus group discussion based on the pilot mathtask. The aim of the analysis was twofold: first to evaluate whether the implementation of the pilot mathtask could provide a ground for aspects of Discourse at the Mathematical Horizon to be discussed as outlined in the objectives mentioned in Section 4.4.4 and second to identify elements in the initial design that could be improved. The first step, towards this aim was the a-priori analysis of the mathtask, that took place before the events of the focus group. The second step of the analysis was to produce a self-reflective account of the pilot shortly after the pilot took place. Then, I left the data aside for 6 months. Distancing myself from the data and the analysis of the pilot helped me step back from the stress associated with the changes in the design due to the pandemic and look at the data more objectively later in light of emerging findings from the analysis of data from the interviews and the lesson observations. Thirdly, I analysed the contributions of the participants and their interactions based on findings emerging from the analysis of the lesson observation and the interviews[22] in relation to the objectives of the focus group. At the end of the analysis a second reflective account was produced about my thoughts of the events of the pilot in the long term.

4.6.5. *Synthesising*

[22] At that time, the analysis of the lesson observation and the interviews was completed and first drafts of the Chapters 5 and 6 were produced.

In this section, I briefly describe how the analysis of the different sets of data came together to address the research questions. The empirical results from the lesson observations and the interviews were consulted to address the first research question. This was achieved through the triangulation of the data sets. Specifically, I compared the codes from the interview data with the codes of the lesson observations to identify the themes that connected the mathematical topics and practices that were identified as opportunities to go beyond the mathematics of the moment. Table 4.13 includes the themes and sub-themes identified after triangulating observation and interview data. In the cases where the participants were observed and interviewed, I used data from the lesson observations to support or contradict statements made during the interviews regarding their practice. In Chapters 5 and 6, I present key findings through short episodes from the lesson observations and the interviews. The discussion of the episodes is based on the commognitive analysis of the excerpts (see also Appendix VI).

Table 4.13 Beyond the mathematics of the moment themes

Theme	**Description**
Across Curriculum	Utterances regarding topics and practices students come across multiple times during their compulsory mathematics education – not specific to one educational level, i.e. vertical curriculum (Shulman, 1986). <u>Sub-themes:</u> principles; reasoning; working mathematically; problem solving
What comes next	Utterances regarding mathematical ideas and practices students will come across in later stages of their education or in personal and professional life. <u>Sub-themes:</u> following curriculum blocks; further and tertiary education; professions (Interviews only); advanced computational techniques
Conventions	Utterances about the rules of mathematical communications. <u>Sub-themes:</u> notation and social contracts; definition; naming mathematical objects; history
Applications	Utterances about applications of mathematics in other contexts, including: references to other curriculum subjects i.e., lateral curriculum (Shulman, 1986), professions and everyday activities. <u>Sub-themes:</u> Cross-curricular; professions; everyday life; extracurricular activities

The episodes along with the findings of the analysis contributed towards the second research question. Specifically, the development of guidelines for designing mathtasks based on empirical data of the study. The episodes were used as inspiration to create practice-based imaginary dialogues, I call vignettes[23]. A vignette is a snapshot, a brief story "contributing little to experiential knowledge but bringing to life an issue central to the research or one that illustrates the complexity" (Stake, 2010, p. 171). The preliminary findings were used to refine the type of questions asked on the mathtasks to promote discussion. In Chapter 7, I present the findings from the focus group discussion on the pilot mathtask and the refined guidelines for the proposed design.

4.7. Trustworthiness of the study

The terms 'validity' – accuracy of results – and 'reliability' – consistency of the results – are associated with quantitative research. The use of the terms in qualitative research could imply a positivist perspective of a universal truth (Guba & Lincoln, 1994). In sociocultural research, the notion of trustworthiness has replaced the notions of validity and reliability. There are four aspects of establishing trustworthiness: credibility, dependability, transferability, and confirmability (Nowell et al., 2017; Shenton, 2004). Next, I will describe the main strategies I employed to achieve the trustworthiness of the current study.

- I developed "familiarity with the culture of participating organisations" (Shenton, 2004, p. 65). During my first year of study, I spent time learning about the English educational system, studied the curriculum and the guidelines of the various exam boards. Prior to the data collection, I met with the participants in their institutions whenever that was possible. Doings so, I also spent time observing the way they work within their schools and departments. In addition, I familiarised myself with the qualities of each school and I went through the information shared on the schools' websites.
- Triangulation of the data is important to achieve credibility and confirmability. In this study I used three primary different methods, i.e., lesson observations, interviews and focus groups, and primary and secondary data sources. The primary sources include

[23] From now on, I use the word 'vignette' to differentiate the fictional dialogues from the excerpts taken from empirical data which I refer to as 'episodes'.

notes and recordings from lesson observations, interviews and focus groups, and secondary sources include supportive materials, such as teaching materials shared by the teachers, the curriculum and guidelines, textbooks and information taken from the schools' websites. Moreover, the participants had the option to opt-in to review the interview and focus group discussions' transcripts, as a strategy of participant validation (Birt et al., 2016). Therefore, a sample of the data was validated based on the preferences of the participants. Participant validation was not employed for lesson observations due to time constrains, e.g., audio-recordings of lesson observations take significantly more time to transcribe than interviews, practical and ethical considerations, e.g., asking students to validate their contribution is a complicated process, it is not appropriate for the teacher to validate the contribution of the students.

- I use diagrams and provide reflexive comments about my thought processes and decisions at each stage of this work to establish an "audit trail" (Nowell et al., 2017; Shenton, 2004). I also recognise the limitation of this study (see also 4.9 and 8.3). These actions contribute to the transparency of the study by offering information to the reader that would help them make their own judgement about the procedures and methods to confirm or question the results.
- I acknowledge my position as a researcher and how I use commognition, as a lens for analysing data but also as a lens for critically reviewing and reporting on the literature around the 'horizon' (see also Section 3.2 and 4.5).
- In this piece of work, I aim to provide rich descriptions of the data so that the reader can form their own opinion about the data and my interactions (Stake, 2010). I provide rich descriptions of the data in chapters 5, 6, and 7, including lengthy quotations from the lesson observations, the interviews and the focus group discussions.
- I used negative case analysis (Shenton, 2004) to identify emerging findings and detect assumptions and refine the preliminary conceptualisation of Discourse at the Mathematical Horizon. Negative case analysis is a strategy for achieving trustworthiness through the analysis of data that do not fit the emerging findings. The process contributes to the revision and refinement of the results. The negative cases are reported as episodes under the emerging themes in the following chapters.

- I regularly discussed my project with my supervisory team and submitted my work for "peer scrutiny" or "peer examination – asking colleagues to comment on the findings as they emerge" (Merriam, 1998, p. 204). Discussing my findings with my supervisors and other members of the RME group at the University of East Anglia provided me with opportunities to reflect on my ongoing work with particular focus on the mathematics and their teaching. My discussions with fellow postgraduate researchers in the School of Education and Lifelong Learning broadened my horizons beyond Mathematics Education research.
- Finally, I presented the stages of the conceptualisation and operationalisation of Discourse at the Mathematical Horizon, and early findings in national and international academic conferences (Papadaki, 2021, 2019; Papadaki & Biza, 2020, 2022).

To conclude, in addition to my personal efforts to ensure the trustworthiness, the study was examined and approved by the Ethics Committee at the School of Education and Lifelong Learning which among others requires that the participants are fully informed about their rights to withdraw their participation at any stage and opt in to review transcripts of their interview or summaries of the lesson observations, when applicable. More details are given in the next section.

4.8. Ethical Considerations

Following the university regulations, the ethical aspects of the present study were addressed in the application to the School of Education and Lifelong Learning Research Ethics Committee (see Appendix I for the application and Appendix IV for the letter of approval). Due to unforeseen circumstances, part of the interviews had to be rescheduled and conducted online. The Committee was informed, and minor changes were applied to the original application with Chair's approval. In addition, I underwent a Disclosure and Barring Service (DBS) check to ensure that I complied with schools' safeguarding procedures.

In order to commence with the empirical work, I had to recruit participants and gather the necessary adult and parental consents (forms included in Appendix III). A first step towards recruiting participants was to circulate, via email to schools or in person, letters of invitation that included a poster of my study and the information sheets for teachers and parents (included in Appendices II and III) to schools around the area. As a researcher I have the "obligation to outline fully the nature of the data collection and the purpose for which the data will be used to the people or community being studied in a style and a language that they can understand" (Boeije, 2009, p. 45). The poster and the information sheets included the aims of the study, the responsibilities of the participants and the researcher, and the use of data in the thesis and other publications. Teacher educators were later contacted using opportunistic and snowballing methods. To reach out to teachers and teacher educators I used information publicly available on the websites of the organisations. When the snowballing method was used, I asked the participants to share my poster and details with their colleagues and ask them to contact me if they were interested. In addition, the participants (or their parents when appropriate) could contact my supervisor or Head of Department in case they had any concerns about the study.

After an initial meeting with the teachers and before I started the lesson observations, consent forms were given to the parents of the students. Despite this study being about teachers, the contributions of students are essential to trigger opportunities for discussion in the classroom. The students' input was voluntary and audio-recorded. To gain access to a classroom I had to have the consent of at least 2/3 of the students of that classroom and their parents. In any other case, the risk of capturing discussions with non-participating students was high. My initial intention during the planning of the data collection was for the participants to choose if they preferred to be video- or audio- recorded. However, in lesson observations video-recording proved to be a difficult task due to the amount of people involved. The teachers that initially agreed to be observed during the lessons suggested that choosing to only audio-record the sessions would give a higher chance of parents and students agreeing to participate the study. So, all the observations were eventually only audio-recorded. These recordings were accompanied by observation notes and, in some cases, the resources that the teachers agreed to share with me.

During the lesson observations and the interviews, I attempted to get as close to the teaching as possible without being intrusive (Stake, 2010). All the participants were made aware that their contribution was completely voluntary and that it would not affect their studies, their relations with the school/institution, or the University of East Anglia. That clause proved to be very important during the burst of the pandemic and the school closure, when teachers had a lot of additional responsibilities. During that time, teachers that had initially agreed to be interviewed or observed withdrew their consent due to time limitations or other issues, e.g., schools' online safeguarding policies (see also Section 4.10).

All the data were collected and stored electronically in a password protected folder on my computer, and on an encrypted hard drive locked in a cabinet. These files will be deleted 10 years after the end of the work. The consent forms collected from the participants are also stored in a locked cabinet. For the recordings of the interviews and the observations were used up to three discreet audio digital recorders. During the face-to-face interviews two recorders were put on the table close to the interviewer and the participant. For the lesson observations, one recorder was worn by the teacher and two more were put on either side of the classroom as backups. The data were transferred to the secure folders soon after the recordings took place and then deleted from all three devices. For the recordings of the online interviews, I used initially Microsoft® Skype and later a software called Camtasia® (see also Section 4.4.2). For my personal notes, I used Microsoft OneNote linked with my institutional Microsoft account. These notes were later exported to PDF format to be analysed. For the analysis of the data, I used the qualitative data analysis software ATLAS.ti, files produced by this software were also stored in a secure location on my computer. All the pieces of software that were used during the stages of data collection and data analysis are approved by the University of East Anglia as safe and compliant with the General Data Protection Regulations (GDPR).

To ensure confidentiality, the data were anonymised using aliases. The adult participants were assigned randomised names as pseudonyms. Students are referred to as "Student [Number]", the number corresponds to the order in which the students speak during a classroom episode. Since the individual characteristics of the students are not essential to the study, the numbering of the students resets after each episode as an extra step to ensure that the anonymity of the students is preserved. In addition, the schools or institutions of the participants are not mentioned in the study. To minimise the risk of capturing data from non-participating students, I used a protocol in my notes to make sure I kept track of their interactions with the teachers and then editing their contribution out again using the software Camtasia®. Accounts of what was happening during these moments are included in my notes without directly quoting the students’ words and analysed if determined to be essential to the study.

Prior to the analysis, parts of the raw data (notes, transcripts and audio-recordings) were only shared with my supervisors. The anonymised and analysed data were presented in conferences of the British Society for Research into Learning Mathematics (BSRLM) in 2019 and 2020, the 10th Young European Researchers in Mathematics Education (YERME) Summer School (YESS 10), the 44th conference of the International Group for the Psychology of Mathematics Education (PME 44), the 12th Congress of the European Society for Research in Mathematics Education (CERME12) and in the meeting with the Research in RME group of the university.

In addition to the measures I am required to follow by the guidelines for conducting educational research in the UK (British Educational Research Association, 2018). I felt the need to make sure that my participants felt reassured that there would not be any data breach especially when the data collection moved online. I dedicated time to make sure that the platforms I was using are indeed sufficiently encrypted and that the recordings are not saved on cloud storage indefinitely. For example, in one of my first online interviews I used Skype. At that time, Skype was one of the few pieces of software available that had clearance from the university as safe to use. Nonetheless, recording a meeting through Skype sends a copy of the recording to all the participants on the call and stores the video on the Microsoft cloud server for a limited period. I consider this to be a potential breach of confidentiality when interviewing more than one participant at a time, e.g., focus groups. Therefore, I decided that in future interviews I would use a separate software, Camtasia®, which did not have the aforementioned limitations and was also approved by UEA, to record the calls.

I feel responsible for any personal information shared during the recordings either voluntary or unintentionally during the interviews and the lesson observations. Therefore, I had to make the decision whether I should keep or cut and delete certain parts of the data that included personal information about the participants. My decision was to delete prior to the analysis any parts that are not directly related to the study. As a result, there are some gaps in the recordings used in the analysis.

Finally, I have a responsibility as a researcher to contribute to the improvement of teaching. Doing research on teachers and their practices raises ethical concerns as it might impose unrealistic standards and exacerbate the pressure to perform (Busher & Cremin, 2012) especially for those in the early stages of their career (Lambert & Gray, 2022). It is important for me to consider the potential impact of the study on the wellbeing of my participants and those within the teaching profession, to acknowledge the complexity of teaching and to prioritise ethical considerations throughout the research process. It is also my responsibility to remind the reader that any reports and findings are limited by the focus of the study.

4.9. Addressing methodological limitation of the study

In this section, I discuss the limitations of the study in terms of the methodology. Due to its nature, my study is small scale and context specific. Using a qualitative methodological approach required a smaller number of participants. Developing my methodological approach, taking into consideration the characteristics that make teaching mathematics in England unique was essential in ensuring the trustworthiness of the results. For example, the mathematical and colloquial discourse of the participants, the diverse mathematical background of the teachers, the structure of the curriculum and the plethora of alternative teaching and learning resources are characteristic of the English educational system. Although it is possible to find some of these characteristics in other contexts, the combination and interactions between the characteristics is unique to the context of England. Therefore, the empirical results cannot be generalised or transferred to account for a context other than the one in which the study took place. The method of analysis and theoretical findings, however, can be applicable in other contexts following adaptations and refinements to account for the contextual variations (see also Section 8.3).

One limitation of the data collection process was that a number of students and parents, who were invited to take part in the study, did not return their consent forms. Because I was seeking opt-in consent, I had to treat these students as non-participating. To ensure that data from non-participating students were not included in the analysis, I went over the recordings and my notes from the lesson observations and deleted the parts of the audio that included excerpts where non-participating students were communicating with their teachers or other students. This process left some gaps in the interactions. The missing recordings were substituted with factual accounts of the interactions to preserve to continuity of the dialogue (see also 4.8). The aim of the analysis is not to account for all the interactions between the teachers and the students, rather, my aim is to identify interactions that have the potential to address mathematical topics that are not included in the curricular materials. Therefore, the situation did not hinder the clarity and consistency of the findings.

The majority of my teacher participants are males. There are only two female participants which does not necessarily represent the ratio between female and male mathematics teachers and teacher educators. At the time of the data collection, I prioritised securing interviews and observations with participants with diverse ethnic, cultural and mathematical backgrounds, which might explain the numbers between the genders being disproportionate.

The study did not include post lesson interviews after every lesson with all the teachers. Due to the tight schedule of the participants, out of the 3 teachers who were being observed, Alex and Nick agreed to have one longer interview at the end of the period of observations where they were asked some additional reflective questions about the lessons that were observed. Liz agreed to an interview before the period of lesson observations commenced and post lesson interviews once a week. The time gap between the observations and the interviews could potentially have resulted in less detailed accounts. To account for the shortcomings, I tried to arrange the interviews as close as possible to the end of the period of the lesson observations. Also, I summarised the main events of the lessons to the teachers before asking them related questions.

In some parts of the interviews and/or the observations I noticed that the teachers were feeling slightly unconfortable sharing some of their thoughts with me, I was on purpose trying to encourage them to share their experiences. There were also some incidents where the teachers acknowledged my presence in the classroom. I made sure that I took notes of the occasions that I noticed such behaviours and acknowledged them in my descriptions of the incidents and the analysis presented in this work. Moreover, the curiosity of the students in what I was doing there or their fear of me recording the lesson was one of them. The Hawthorne effect (Cook, 1962) in research refers to the phenomenon of participants changing their behaviour when they are being observed by someone. To minimise these instances, I was sitting at the back of the classroom when possible and I asked the teachers to give me five minutes of their time. During these five minutes, I explained in simple terms to the students what this study is about and showed them a sample of what I was typing on my laptop during the lesson and how I could secure confidentiality. Then I gave the students time to ask me additional questions that I tried to answer as simply as possible. There were also a few occasions where teachers were making direct eye contact and, rarely, addressing me while teaching. In most of these cases, my interaction with the teachers was about technical issues or to reassure their students that my presence would not have any effect on their learning. However, there was one instance when a participant was clearly asking me to participate in the lesson, which I did.

Finally, limitations due to Covid-19 school closure include necessary adaptations in the research design and methods used, limitations in the available online tools, increased time limitations and dropouts and overall delays in the completion of the project. More details are discussed in Section 4.10.

4.10. Doing research during the COVID-19 pandemic

The breakout of COVID-19 happened in the middle of the empirical work of the present study. The effects of the new situation on the process of data collection started to show in February of 2020, when 2 of the teachers who had agreed to participate in the study were asked to self-isolate after their trip abroad during the half-term. On that occasion, our meeting was postponed for a couple of weeks. At that time, no one could really predict the radical changes that were about to happened in day-to-day life. It the following paragraphs, I reflect on my experience and actions, and give my account of the pandemic and how it affects the present study.

4.10.1. *A timeline of the circumstances*

In March of 2020 the university asked their staff and students to work from home and moved teaching online. Later that month, the UK government announced the first lockdown and school closure. Subsequently, the university reviewed regulations about data collection asking the staff and postgraduate taught and research students to proceed with online data collection only. Proceeding with online data collection, I was required to apply for a review of my initial ethics application to the ethics committee of the School of Education and Lifelong

Learning. Due to new regulations and school closure, some of the planned lesson observations could not go ahead. For several weeks and until safeguarding issues were resolved, many teachers had very limited interactions with their students mostly through written correspondence. Apart from practical reasons (safeguarding online, review of the ethics, etc), online lesson observations are not suitable for my study. Mathematical conversations of beyond the mathematics of the moment require a certain degree of confidence both in the subject and the means of communication. The transition to online teaching is evidently a difficult process for the teachers (Kim & Asbury, 2020). Some of my participants admitted to having difficulties adjusting to the new ways and managing conversations with their students was particularly hard especially in the beginning. Moreover, their attention and interest had been drawn to different problems such as ensuring that their students would not fall behind, preparing online materials and adjusting to the new normal. Therefore, the opportunities for open debates between the teacher and the students were in most of the cases very limited. Some of the participants opted out due to their sudden and extensive workload or personal reasons. Regardless of that, I managed to conduct a few more interviews online. However, the interviews that went ahead had to be delayed due to the necessary amendments and to accommodate the tight schedule of the participants.

Despite my best efforts to adapt and mitigate the impact of COVID-19 and the support of my supervisory team in the process, I was not able to proceed with the research plan as it was at the time. The proposed methodology of the study was initially a type of design study. The original research plan included elements of task design, where professional development tasks (mathtasks) informed by the lesson observations were meant to be designed and tested in multiple focus groups. The tasks were going to be trialled with small groups of teachers in June, July and September. Limited to the classroom data I had at the time, I decided to finalise only one task that I had originally created during my first year of study as a pilot to be trialled with a group of 4 teachers in July. After the pilot, I attempted to organise more focus group discussions with teachers to try out more mathtasks, but this attempt failed. Design research requires participants to have a higher level of commitment throughout the duration of the empirical study. Under the circumstances, the teachers could not have such commitment and the trials had to stop.

In September of 2020, the schools opened and remained open during the second lockdown. During that time, I tried to approach more participants. If I could have secured more participants, I would then have had to apply for special permission from the university to conduct face-to-face data collection. However, the attempt was not fruitful. It was very difficult to gain access to classrooms or to arrange face-to-face focus groups and interviews with teachers because of the risk of spreading the virus. There were some initial discussions about resuming the data collection later in the year. Yet, the announcement of the third lockdown that included moving teaching online once again changed the plans indefinitely. During that time, I proceeded with the data analysis of the interviews and the lesson observations. After careful considerations and discussions with my supervisory team, in October 2021, I took the decision that I would not make any further attempts to collect more data and I would focus solely on the analysis and the writing of my thesis. The analysis of the data I collected before and at the start of the pandemic provided substantial results. However, this decision had an impact on the design of the study. In particular, the task design elements of the study had to be limited to the conceptualisation of the design principles and the preliminary results available from the pilot mathtask focus group.

4.10.2. *Adaptations*

Since March 2020 and throughout the study, I had to make several adaptations to the research design and the plan to mitigate the impact of COVID-19. Regarding the design of the study, the first action that I had to take in order to continue the data collection during the lockdowns and social distancing measures was to change the mode of communication with the participants from face-to-face to online. That included informing the ethics committee of the department and making amendments to the application; exploring my choices of safe and approved, by the university, pieces of software; familiarising myself with the use of the online tools; and reading relevant literature. As mentioned in the previous section, moving the lesson observations online was not possible or desirable in the context of this study. Subsequently, I had to adjust the methods used. After some thought and with the support of my supervisors, I decided to limit the data collected through the observations and aim to have more interviews with teachers and teacher educators in order to gain insights into what was happening in the classroom through their accounts and reflections. Moreover, I decided to limit the task design component of the study. I had to make some changes in the research questions and the research plan. In the original research plan the main data sources for answering the research questions were the lesson observations and the delivery of the designed tasks in groups of teachers. Those components of the study had changed sufficiently due to COVID-19, so I had to evaluate the research questions to reflect the new research design and what was possible in the new circumstances.

To complete the study in a timely manner, adaptations in the plan were in order. After some failed attempts to conduct more online interviews and focus group discussions using mathtasks, I had to pause the data collection at the end of July of that year. The aim was to resume the data collection sometime in the following academic year if the situation had improved. Despite my best efforts to get in contact with teachers and schools, I had difficulties doing more lesson observations especially when teaching was moved again online. After careful consideration, a decision was made to end the data collection and focus solely on analysing and reporting on data I had at the time. This decision was not easy and took several hours of discussion with the supervisory team and personal thinking. However, the amount and quality of the data were determined to be sufficient for the adapted design of the study. In parallel, I was transcribing and analysing the data that I had already collected and was writing up parts of the thesis.

4.10.3. *Reflections on the impact of the pandemic on my progress*

Maintaining focus and interest in the topic was at times particularly hard. Adjusting to the new style of living and working from home, worrying about my family and loved ones or craving face-to-face, social interactions were a few of the reasons I felt less motivated. Most importantly, though, I was experiencing a world changing before my eyes more quickly than ever. I was always interested in issues related to politics and human rights and indisputably I had even more reasons to focus on them at that time. Through my contact with teachers and schools, I witnessed how different online teaching was from school to school and how the pandemic amplified inequalities in education. Through discussions with my participants, I could see how teachers in schools that were considered ‘posh’ adapted easier and more quickly to online teaching. All their students had the appropriate equipment and felt much more confident. While other teachers were worried about their students’ progress and wellbeing at home. In the short term it looked like I was losing my focus on the study. Changing the plans about the design elements of the study, I felt very stressed and discouraged at the time. I was looking for ways that my study could produce more than new knowledge for the research community. Reciprocity is always important for me, and it was especially important at that time. Abandoning the plan of a design research was not easy, I did not feel ready to do data analysis and I felt I was forced to do it in the worst possible time. Not being able to go along with this plan affected my motivation. However, a time came, and my thinking about the study matured. I had opportunities to pause and think about how this piece of work fits in and is still relevant to the new era and I resumed my plans for a design study extending past my PhD project.

In the following chapters, I present and discuss empirical results of the study. In Chapter 5, I explore taken and potential opportunities to go beyond the mathematics of the moment in the mathematics classroom. In Chapter 6, I explore the participants' reported experiences which influence individual discourses at the Mathematical Horizon. In Chapter 7, I focus on the idea of creating and using resources to explore and develop Discourse at the Mathematical Horizon using data and empirical findings of the study. In chapter 8, I discuss the substantive, theoretical and methodological contribution of my study; the limitations of the study and discuss ideas for further research. Finally, I provide a reflective account of how conducting this study contributes to my journey as an early career researcher beyond the impact of COVID-19.

5 Exploring opportunities for discussions beyond the mathematics of the moment

This is the first of two chapters focused on exploring the characteristics of discussions beyond the mathematics of the moment (RQ 1) based on empirical data collected through lesson observations and semi-structured interviews. The chapter addresses specifically the sub-questions:

1a. What taken and potential opportunities for discussion beyond the mathematics of the moment can be identified in everyday teaching practice?

1b. How could teachers and students communicate effectively when opportunities for discussion beyond the mathematics of the moment are taken?

In Section 5.1, I elaborate on how I approach the analysis of data that are relevant to answering the above research questions. Then in Sections 5.2 to 5.5, I discuss results emerging from the analysis of the lesson observations and interviews with the participants. The sections correspond to emerging themes of the analysis: *Across the mathematics curriculum* (Section 5.2); *What comes next* (Section 5.3); *Conventions* (Section 5.4); and *Applications* (Section 5.5). Finally, section 5.6 summarises key findings on the discussed opportunities to go beyond the mathematics of the moment leading to further study of the topic in subsequent chapters.

5.1. Synopsis of the data analysis

For the purposes of the present study, identified opportunities to go beyond the mathematics of the moment include both taken and potential opportunities to address a topic or a practice that is beyond the mathematics of the moment. The opportunities are identified by comparing discursive elements (i.e., word use, visual mediators, narratives, and routines) with elements of the predicated by the curriculum mathematical discourse of the classroom (see also Section 4.6.2). During the analysis, the identified opportunities were categorised into taken, potential of the discussion and potential of the activity (see Table 5.1). The purpose of the categorisation is to distinguish between the actual discursive actions of the participants and alternative courses of action I considered based on the focus of the study. The analysis of all three categories acts as a stepping stone towards exploring what experiences of teachers influence their engagement in discussions beyond the mathematics of the moment in Chapter 6. Table 5.1, also included in Chapter 4, describes the meaning of each category in more detail.

Table 5.1 Description of each opportunity category

Category	**Description**
Taken	The mathematical object or practice is made explicit and addressed by the teacher during the lesson
Potential of the discussion	The mathematical object or practice is made explicit in the lesson but is not addressed by the teacher
Potential of the activity	The mathematical object or practice is not made explicit during the lesson, it is only attributed by the analysis

The identified opportunities to go beyond the mathematics of the moment are further categorised into four themes according to common mathematical characteristics compared to the requirements of the curriculum among them: *Across the mathematics curriculum*; *What comes next*; *Conventions*; and *Applications*. To illustrate the elements of each theme, in the following sections, I present episodes from the lesson observations with the three teachers.

Table 5.2 includes information about the topics covered during the observed lessons of the three teachers. As depicted in Table 5.2, two of Alex's lessons (A1, A2) are related to sequences that are part of Algebra while the other three (A3, A4, A5) are part of Statistics in the National Curriculum. All six of Liz's lessons (L1 to L6) are part of the Geometry and Measures area of the National Curriculum, with the final lesson (L6) of the series to be about connecting geometry with algebra. Finally, Nick's observed lessons are a combination of lessons of three different topic areas: Statistics (N1), Numbers (N2, N4), Geometry and Measures (N3, N5). The codes in front of each lesson will be used within the chapter to locate the episodes discussed in relation to the topics covered during the observed lessons.

Here, the chosen episodes showcase both examples and counterexamples of identified opportunities to go beyond the mathematics of the moment. The analysis of both examples and counterexamples, i.e., negative cases (Shenton, 2004), helps in identifying key characteristics in the engagement of the participants. The decision was made to avoid unintentionally presenting discussions beyond the mathematics of the moment as panacea for bridging the gaps across educational levels. From a methodological perspective, reporting negative cases is a strategy for achieving trustworthiness by putting forward and discussing narratives that challenge hidden assumptions (e.g., discussions are always effective, or beneficial).

Table 5.2 List of lessons

Teacher	Students	Topics of lesson in chronological order	
Alex	Year 9	A1	n^{th} term of arithmetic sequences
		A2	n^{th} term of quadratic sequences
		A3	Times series graphs
		A4	Pie charts
		A5	Frequency tables
Liz	Year 7	L1	Vertically opposite angles
		L2	Angles [in circles]
		L3	Angles in quadrilaterals
		L4	Calculating angles
		L5	Angles Investigation
		L6	Geometry and algebra
Nick	Year 7	N1	Pie charts
		N2	Revision
		N3	Angles and parallel lines (1)
		N4	Rounding and Estimating
		N5	Angles and parallel lines (2)

The episodes presented in each section include an elaboration of why they were found to contain opportunities for discussions beyond the mathematics of the moment and are considered in relation to the discursive analysis of the narratives and intersubjectivity. In particular, I look at the word use, visual mediators, narratives and routines used by the teacher and the students in relation to the expected mathematical discourse of each classroom (see also Section 4.6.2) according to the National Curriculum, and the guidelines of the schools and the exam boards. For the purposes of the study, effective communication is defined as an interaction between the teacher and the student that provides opportunities to learn. To explore the effectiveness of discussions beyond the mathematics of the moment, I analyse the utterances in relation to intersubjectivity, i.e., the use of words, visual mediators, narratives, and routines that make sense from both the perspective of the students and the teacher. The analysis of the identified episodes is bound by my own mathematical and pedagogical discourses. Alternative interpretations are possible.

In the following sections, I discuss the findings emerging from the analysis of the lesson observations and interviews with the participants.

5.2. Ideas which run across the mathematics curriculum

During the interviews teachers talked about connecting ideas that run across and evolve throughout the National Curriculum for mathematics. For example, Thomas mentioned justification and proof, Nick mentioned plotting graphs, and Liz talked about problem solving. They also mentioned ideas that run across a topic area of the curriculum, for instance, Eric mentioned using factorisation in different types of algebraic problems. During the lesson observations, connections with broader ideas related to the topic of the day were found to provide opportunities to go beyond the mathematics of the moment, however, not necessarily beyond the requirements of the curriculum for KS3 and KS4.

In this section, I discuss five of the episodes identified to provide opportunities to go beyond the mathematics of the moment to address ideas and practices that are relevant throughout secondary education.

5.2.1. *"Not drawn to scale": Routines of estimating, measuring, and calculating*

All six of Liz's lessons were on geometric topics therefore the analysis arose an opportunity to discuss variation in mathematical practices that was present across a cluster of lessons (L1, L2, L3 and L4 in Table 5.2). The activities Liz gives to her students during these four lessons include the instructions "find the angles", or "what is/are the…" (Figure 5.1). She also uses the word "find" when she verbally instructs her students to work on problems. Her students interpret this instruction in various ways, sometimes as an invitation to perform calculation routines, but also as an invitation to perform a measuring routine (potential of the discussion: dialogue from L1, Table 5.2):

Student: Miss, for number 6, what angle are we measuring? [Figure 5.1, Part A]

Liz: Are we measuring? [pause] Do you mean which one are we trying to.

Student: Yeah, which one of these.

Liz: Calculate?

Student: Yeah, which one are we calculating?

other times as an invitation to estimate (potential of the discussion: dialogue from L2, Table 5.2):

Student:	Can we do a rough estimate? [Figure 5.1, Part B]
Liz:	OK, [Student], I'm not looking for a rough estimate.
Student:	How can we then determine this?

or both (potential of the discussion: dialogue from L2, Table 5.2):

Liz:	And so how did you come up with those three numbers for the triangle? [Figure 5.1, Part B]
Student 1:	I guessed and used my protractor.
Liz:	Right. So, you're not guessing. That's not what we do.
Student 1:	It is.
Student 2:	Yes. It's a guess on what I think.
Liz:	Mmm.

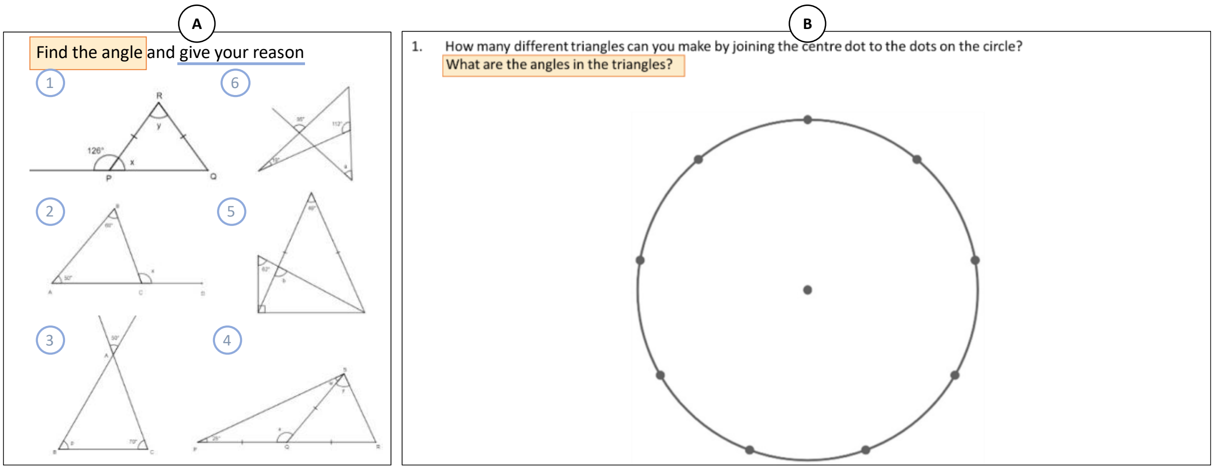


Figure 5.1 A recreation of the activities Liz's students interpreted as estimating or measuring tasks. Part A recreation of main activity in L1, Part B recreation of the Nine Point Circle activity in L2.

According to the national curriculum the students are expected to be able to calculate, measure and estimate angles (Department for Education, 2013a). In the guidelines followed by the school, under the umbrella 'Geometry and Measures' are included skills such as for students to "make sensible estimates of a range of measures in real-life situations" (AQA, 2015, p. 140), "measure and draw angles to the nearest degree" (AQA, 2015, p. 117) and "work out" or "calculate" (AQA, 2015, p. 119) various angles.

In the lessons I observed, the students were in the transition stage from measuring or estimating routines to calculating. Particularly, in one activity used in L4 (Table 5.2) all three routines were considered acceptable. During the activity Liz gave her students a diagram of a compound shape printed on isometric dot grid paper (Figure 5.2) without written instructions and asked them to "just write down everything that [they] can say about it". Liz had the following conversations with two students during group work (taken opportunity: dialogue from L4, Table 5.2):

Liz:	You could find the angles. [the activity is printed on a dotted paper and there are no labelled angles (Figure 5.2)]
Student 1:	Miss, I think you are asking me an impossible question.
Liz:	You could measure them. Do you have to measure them?
Student 2:	Well, I can find 4 already, boom, boom, boom, boom [points to four right angles].

During the whole class discussion of the same activity Liz tells the class:

Liz:	Um, [Student 1] and [Student 2] decided that they would find all the angles that they could find. Did you measure them or did you try and calculate them?
Student 2:	Um, we calculated some of them and measure some of them.

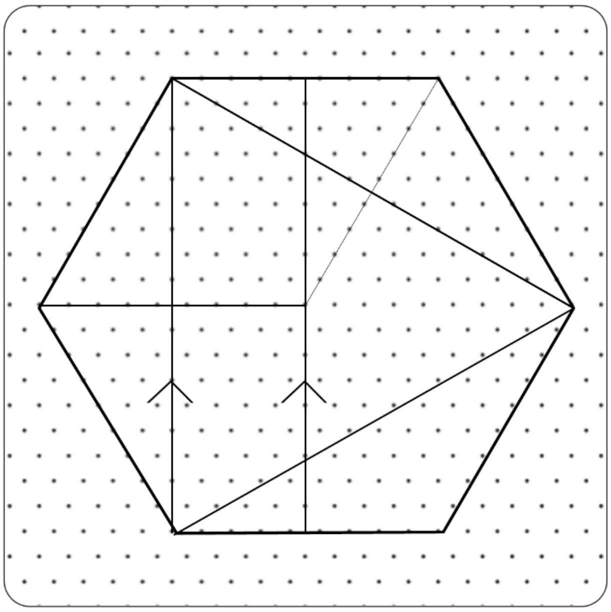


Figure 5.2 Recreation of the compound shape activity

In questions included in past GCSE papers, the instruction given are highly specific to decide what the appropriate routine is, either by the use of the words 'estimate', 'calculate' or 'work out', and 'measure', or additional instructions 'not drawn to scale' and 'not accurately drawn'.

In all the excerpts seen so far, the routines of measuring and calculating are rituals as Liz is the one that suggests to the students what they need to do to "find" the angles. The students are able to follow instructions when a problem clearly indicates which routine they should follow. However, when the instructions are not clear, the students will not necessarily use the routine which the teacher expects them to engage with during the task. The metarules between these three routines (i.e., estimating, measuring and calculating) are not necessarily revealed to the students at this age. For Liz, the word "find" can trigger different routines depending on the context of the question and whether the answer needs to be exact. Liz tries to convey the metarules that determine whether "find" is an invitation to measure or calculate. For example, pointing out the information that the diagrams are not drawn to scale (taken opportunity: dialogue from L1, Table 5.2):

Liz:	Oh! Ah! Really, really, shh, really interesting question from [Student]. Do you have to draw them to scale? Um, on any of these [Figure 5.1, Part A]?
Student:	Oooh....
Liz:	Shh. On these types of questions. It will usually say on a GCSE paper not drawn to scale. It usually appears somewhere. And it tells, it means don't try to measure the angle because that's not the way that they want you to deduce the answer. They want you to use your angle facts to be able to give the answer. No, I don't expect you to draw them to scale. A sketch is absolutely fine.

During the lessons, when the information is given as part of the instructions, the students perform calculation routines and they can also spot whether their calculations agree with the given diagram. For example the following observation was made by three different groups of students independently (taken opportunity – inaccuracy of diagram/potential of the discussion – explore the results of different routines: dialogue from L3, Table 5.2):

Student:	Miss, I think x and 71 degrees is the wrong way around. Cause I think where the x is (Figure 5.3).

Liz:	[Laughs] It would make, it would make more sense, wouldn't it?
Student:	Yeah.
Liz:	From the diagram point of view.

Liz chose to take this opportunity to discuss with the groups, and later with the whole class, the inaccuracy of the diagram when compared with the result of the calculation. In addition, I claim that the episode offers a potential opportunity to further explore different results one could get by performing the three routines (i.e., calculating, estimating, measuring).

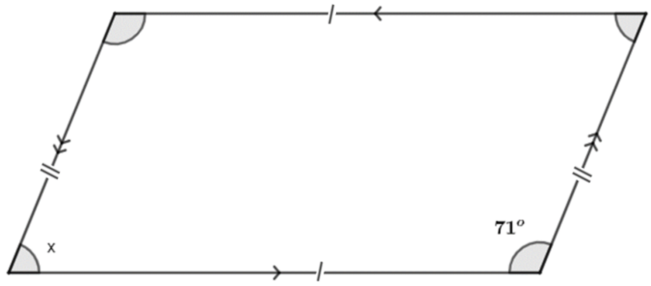


Figure 5.3 Recreation of the parallelogram which has labeled an obtuse angle as 71°

Another way Liz attempts to lead her students to consider doing calculations is by stressing the accuracy of the results (taken opportunity: dialogue from L3, Table 5.2):

Student:	Miss, the 1 degree angle must be tiny.
Liz:	Where is your protractor?
Student:	Here. Is that even, like, physically possible to draw as a human?
Liz:	Well, I think it depends on how thick your pencil is.
Student:	If I do that.
Liz:	Cause look, how thick your pencil is.
Student:	That's bigger than 1 degree is.

And (taken opportunity: dialogue from L2, Table 5.2):

Liz:	[laughing] If I'll do that. If you draw it and you measure it with a protractor, is your answer going to be guaranteed exactly the same as [Student 2]'s.
Student 1:	No.
Liz:	No.
Background:	Guaranteed!

Moreover, in activities where the students are asked to estimate angles Liz shows genuine excitement when a student find an exact result (taken opportunity: dialogue from L1, Table 5.2):

Background:	What? [with emphasis]
Liz:	She got it exactly right! Well Done! [claps]
Background:	[clapping]

Estimating, measuring and calculating are routines of quantifying an object, in this case an angle. Estimating relies in one's perception of size to deduce an acceptable answer. Measuring is performed by using available and appropriate tools. Calculating is performed by combining facts under well-defined rules. Although 'Geometry and Measure' is an umbrella topic in the National Curriculum, generally in geometry estimating or measuring routines are not considered appropriate. However, estimating and measuring are important in everyday life and professions such as architecture and engineering. The accuracy of the measurement depends on the accuracy of the tools and techniques and characterise a specific (physical) object (the angle drawn on the board or the notebook, or an angle on a flat surface in the house, etc) whereas a calculation does not depend on the physical tools and can be used to quantify both physical and mathematical objects.

Liz uses the word "find" to instruct the students to perform calculating or measuring routines. For her, the initiation of the appropriate routine depends on signifiers such as the sufficiency of notation and description of the problem or whether the shape is printed on a dotted paper. The students rely on the routines performed previously during the lesson, the presence of the phrase "not drawn to scale" or Liz's approval. The instruction to "find" an angle, on its own, is not sufficient to achieve intersubjectivity between Liz and her students. Liz highlights the differences to her students as an attempt to help them decide for themselves what is best to use. Words such as "guaranteed" and narratives such as "not drawn to scale" or "do you have to measure them?", if they are discussed, act as intersubjective mediators or rituals between Liz and her students, drawing from the students' precedent space, attending to the need for accuracy and offering opportunities for metalevel learning.

In a nutshell, the excerpts presented in the section illustrate opportunities to discuss ways of identifying and interpreting signifiers (e.g., words or visual mediators) to perform the expected routine. The opportunities were present across multiple lessons, sometimes multiple times within one lesson; however, they were not always taken. Negotiating the meanings of the words “find”, “work out”, “calculate”, “estimate” and “measure” with the students provides an opportunity to go beyond the mathematics of the moment and develop routines which would help them decide whether they should measure, estimate or calculate without relying on instructions from the teacher. Achieving effective communication can lead the students to perform the routines of estimating measuring and calculating as explorations rather than rituals.

5.2.2. *What lies behind “butchered mathematics”: The role of achieving intersubjectivity*

The following episode takes place in one of Alex’s lessons with his Year 9 students on the topic of finding the n^{th} term of a quadratic sequence (A2). Alex attempts to show the students an alternative method to calculate the formula of the n^{th} term of a quadratic sequence than the one proposed in the resources of the school. The significance of the episode is that it showcases a taken opportunity to go beyond the mathematics of the moment which did not materialise in the way Alex intended. The episode illustrates the aspirations of the teacher and the considerations he made in preparation of the lesson and in anticipation of students’ difficulties. At the same time, the episode emphasises the role of intersubjectivity relative to the overall complexity of teaching.

Typically, the students in Alex’s school are introduced to the n^{th} term of a quadratic sequence formula by using a procedure that is expected to be memorised. To show the steps of the procedure, I will use the example, which was also used by Alex during the lesson:

$$What\ is\ the\ n^{th}\ term\ of:$$
$$4 \quad 13 \quad 26 \quad 43 \quad 64$$

According to the typical method, as illustrated in the documents Alex shared with me, the solution follows the steps:

- <u>Step 1</u>: We find the first differences between the 5 terms, if the differences do not result to the same number, the sequence is a quadratic sequence and we calculate the second differences (the differences of the first differences).

$$4 \quad 13 \quad 26 \quad 43 \quad 64$$

$$9 \quad 13 \quad 17 \quad 21$$

$$4 \quad 4 \quad 4$$

- Step 2: Divide the number on the second difference by 2. The result is the coefficient of the n^2. Here, $4 \div 2 = 2$. So, the sequence has $2n^2$.
- Step 3: We create a table with the first few terms of the original sequence in the first row, the terms of the an^2in the second, and the difference between each row in the third (see Table 5.3).

Table 5.3 Difference T-2n²

$\boldsymbol{T}$	4	13	26	43	64
$\boldsymbol{2n^2}$	2	8	18	32	50
$\boldsymbol{T - 2n^2}$	2	5	8	11	14

- Step 4: The sequence $T - an^2$ is an arithmetic sequence for which the nth term formula can be calculated using a known procedure. Here, the arithmetic sequence is $3n - 1$.
- Step 5: The answer is the sum of the sequences found in steps 2 and 4. Here, $2n^2 + 3n - 1$.

Alex decided to use a different approach than the one usually taught in his school to introduce the topic. In previous lessons, Alex and the students had worked on finding the first terms and the formula of the nth term of arithmetic sequences and finding the first terms of a quadratic sequence given the formula (Figure 5.4, part A). Using an example from arithmetic sequences, Alex introduced the topic of the day and linked it with the topic of the previous lesson on quadratic sequences. On the slides (Figure 5.4, parts B, C and D), Alex used arrows to indicate the relationship between the processes and colours to highlight the coefficients with consistency. Alex tells the students:

Alex: Very good, awesome guys. And this is why we have as before a squared term, n squared, and this is what we are going to try and find. What is this "a"? What is this "b"? and what is the "c"? [Figure 5.4, part D] OK, this will be our aim.

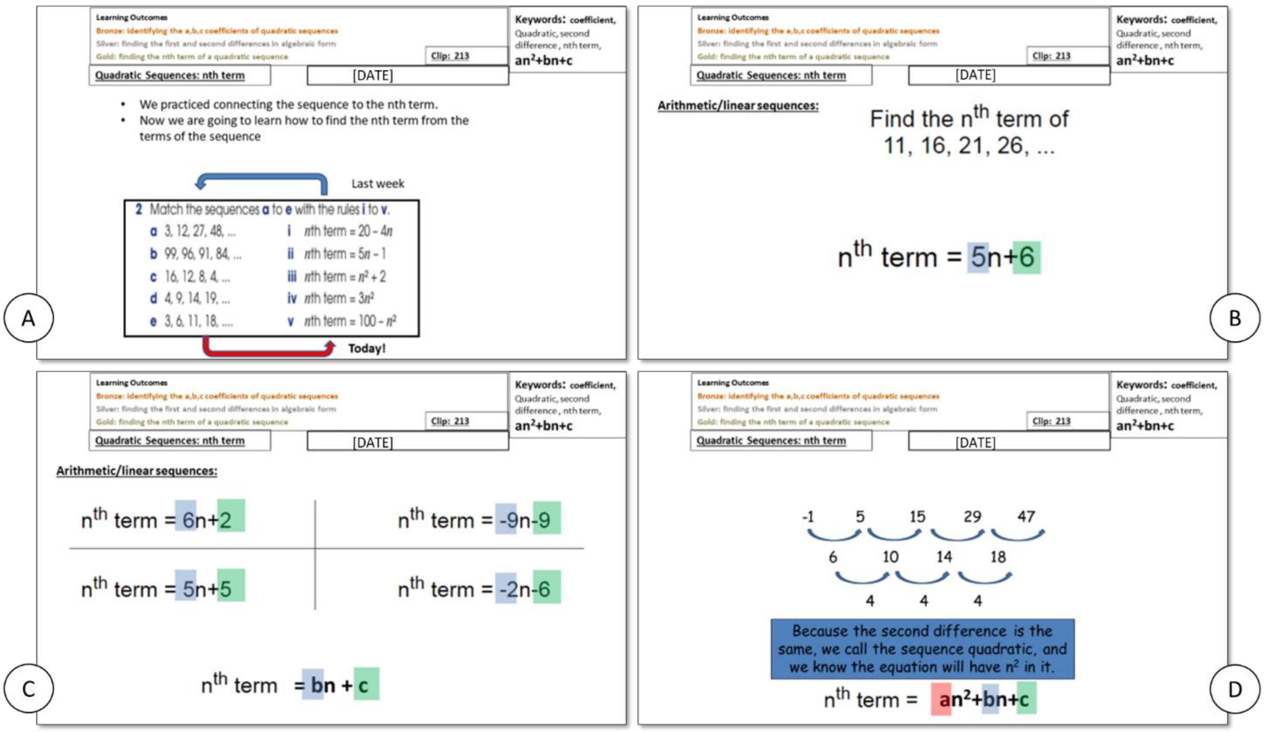


Figure 5.4 Alex's introduction to n^{th} term of a quadratic sequence slides

Then, Alex presents to the students the main activity of the lesson (Figure 5.5). The activity is introduced as a practice of the learning outcomes of the previous lesson, and it is divided into bronze, silver, and gold (Figure 5.5) for the students to choose. There is also a last question that is the general formula an^2+bn+c, that he invites all his students to try.

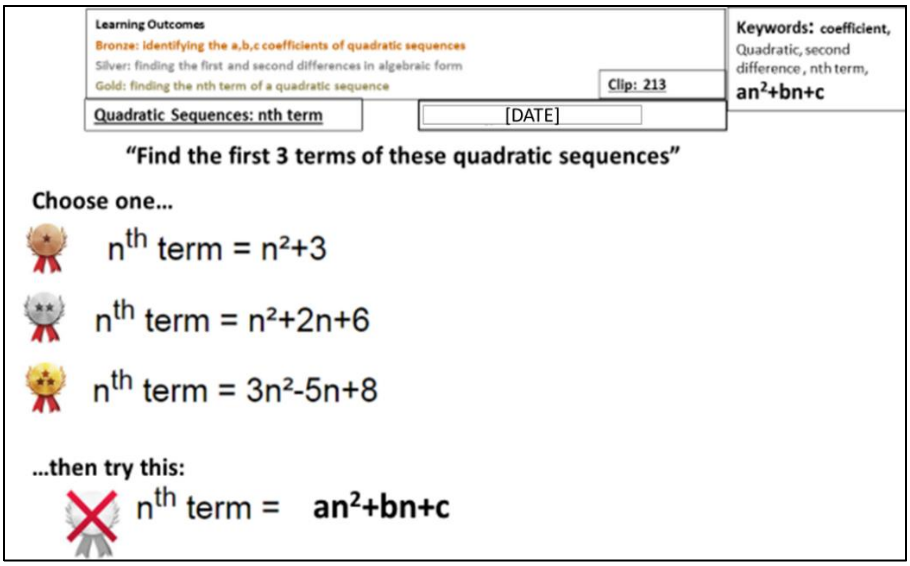


Figure 5.5 Activities on quadratic sequences slide

The students work individually for some time and then Alex asks them to help him write the solutions for bronze, silver, and gold on the board. Alex, then, addresses the students again to discuss the final activity:

Alex: Now, guys, that you figured this out. We are going to do the most annoying one of them all. And this is why I want your attention, OK? Because I taught last year, well, actually, two years ago, year 11s that they were preparing for the higher set and they got really confused with this, we had to do a couple of lessons. So, if you don't get it the first time, that's still OK. OK? So it's gonna be a bit difficult.

[the discussion was interrupted]

Alex: So first of all, guys, who can tell me, because many of you actually got this right, who can tell me what would my first term be here? Hands up, guys, what is my first term? Someone would like to say that? [Student 1]?

Student 1: “n”.

Alex: “n” equals?

Student 1: One.

Alex: One, right? This is my first term, so I'm going to ask [Student 2]. [Student 2] if “n” is one what is “a” “n” squared?

Student 2: “a” times 1.

Alex continues asking students to help him find the three first terms of the general formula of the sequence, then the first and the second differences. Throughout this time, he writes on the board (Figure 5.6).

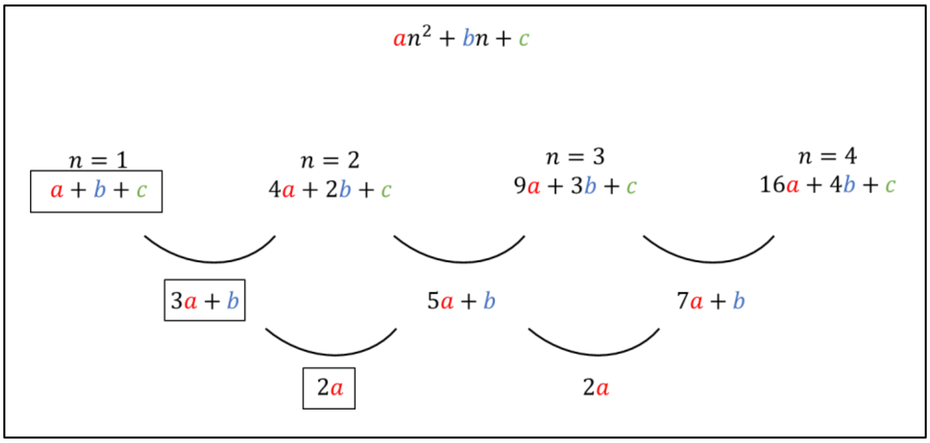


Figure 5.6 Recreation of what Alex wrote on the whiteboard from the observation notes

However, the discussion is interrupted three times for a couple of minutes each time due to the behaviour of some students. Then he asks the students:

Alex: Guys, do we see any patterns here?

Background: Yeah.

Alex: Who can see a pattern? Yes, [Student 1]?

Student 1: It goes up in 2.

Alex: It goes up in two what? Two?

Student 1: “a”.

Background: “a”.

Alex: Two a, very good, [Student 1]. So, do you remember the first differences?

Background: Yeah.

Alex: They looked a bit messy and disgusting. However, guys, the second differences are all nice and calm.

[The discussion was interrupted by a student asking about detention]

Alex: So, guys, this is all we had to do. All this mess was done for this reason. And we are going to need, [Student 2], these three terms to compare them with our actual sequence. So, all this discussion we have for the last ten minutes was to understand how can I construct this thing? OK? And don't worry, I have this printed for you so we can just glue it in and it's in colour as well.

[…]

Alex: So, guys, this is what we need to do. We've made this, [Student 3], so every sequence we can take we can complete one by one the second difference 2a and six each. [Student 3] that's a warning.

Student 3: [Student 4] is distracting me, sir.

Alex: [Alex does not respond to Student’s 3 comment and continues] And then we have the first term with our first term here. So that's what we will be aiming at. This 9 will equal to this. And this one here would be equal to my first term, [Student 5]?

Student 5: How do we know this equal to this?

Alex: Very good, because, [Student 3], here, this is the general form of any quadratic sequence. It would look like this “a” “n” squared plus “b” “n” plus "c". That is the general formation of it. And what we're trying to find for this specific sequence, what is my "a" my "b" and my "c".

Alex tells the students that this is a topic which others had found challenging in the past. As an attempt to help his class, he decided to follow a different method to teach this lesson than the one proposed by the school. In his interview he calls the method used in the school a "butchered method":

Alex: So basically, I have this concept, […], of butchered mathematics. The mathematics that you learn so you can get your job done. This mathematics usually is connected with quick and simple tricks that usually connect with a mnemonic rule, like BODMAS, that is one way of butchered mathematics, at least from my perspective. […]. Anyway, so rules that can be easily identified and shared. They are set, a set of steps that can happen and will give the instructions, an algorithm. I do not disagree with this as a general rule, what do, I do disagree is that sometimes these algorithms, and I think this is the basis of what we were discussing before, sometimes these algorithms impede the understanding and impede the conceptual understanding. And therefore, when it is time for me to do my assessment, when it's time for me to follow the rule, I've been told I will forget it. Because there is so many rules, so many stuff I have to remember, I will forget it. So, this is my perspective. If I can find a different again algorithm, obviously, a different way to solve a problem that also highlights or shows some concepts, I would ideally use that one as a teacher.

The method he follows is based on showing the students some of the basic ideas that are behind what he calls the "butchered method". Alex told the students that he is planning to revisit the topic in a following lesson to discuss "a shorter way". Although this method is not beyond the curriculum, and it is used in other schools, the method is beyond what the students might come across in the following years if they are taught by a different teacher in the same school.

Alex attempts to introduce to the students a method for finding the formula for the n^{th} term of a quadratic sequence which does not rely on memorisation of seemingly arbitrary steps.

Alex's method includes all the essential steps required to explain why the method works (e.g., why the coefficient of $x^2$is equal to the second difference divided by two?) if the explanation of the algorithmic method is followed by meta-comments to the students. Therefore, the episode is coded as an opportunity taken. However, the opportunity did not materialise in the way the teacher might have intended. For example, Alex tells the students that they can "compare" the diagram with any other diagram of first and second differences to find the coefficients. Alex initially does not disclose to the students why they can do so, and the students are not yet ready to come to this reasoning on their own accord. At this stage, students have only come across some examples with formulae of the n^th^ term of a quadratic sequence and calculated the first couple of terms (students' precedent space). Students' discourse about quadratic sequences does not necessarily include the realisation that any object referred to as a "quadratic sequence" is associated with the visual mediator $an^2 + bn + c$. The following discussion between Alex and one of the students illustrate this observation:

Student:	I don't understand at all! [Alex is moving towards her]
Alex:	So, you don't understand at all?
Student:	I understand, um, the first little like tree.
Alex:	Yes! That's a nice word to say it.
Student:	I don't understand, like, how we get the other one and try to
Alex:	OK, so do you understand how to get this tree? So, what is the difference from 4 to 13. 13 take away 4 is?
Student:	One?
Alex:	One. [Alex does not acknowledge the error in the calculation] What is the difference from 13 to 26?
Student:	13.
Alex:	There you go. 26 to 43?
Student:	17.
Alex:	Very good. 43 to 64?
Student:	21.
Alex:	There you go.
Student:	But how do I get these numbers.

Alex: Very good. What's the difference from 1 to 13?

Student: Yes, but how do we get to start with.

Alex: Oh, these numbers? You were given. [Student], you are given these numbers.

Student: Ok.

[Alex interrupts the conversation to ask other students to behave]

Student: In an exam, will we be given both?

Alex: No in an exam, you have to construct this tree yourself. This seems very, very strange now, but look, you can have it here. Look, at it now, I'll speak to your classmates, and I'll come back. I know it is confusing.

In the above excerpt, Alex and the student negotiate the role of the diagram in Figure 5.6. The student refers to the diagram as a "tree". Alex admires her choice, and he also adopts the name "tree" towards the end of the discussion. The use of the word "tree" from both Alex and the student shows evidence of intersubjectivity. In this instance, there is not sufficient evidence to analyse the use of the word and the visual mediator (Figure 5.6) further. Next, Alex ends the conversation with a promise to come back to her once he resolves a matter that arose with a couple of other students.

Throughout the lesson the discussions were frequently interrupted or cut short while Alex was trying to make students behave appropriately and participate in the lesson. See for example the dialogues presented here, Alex is calling out students' names to try and get their attention without asking them to contribute to the discussion. In the dialogue above, Alex appears to have his attention split between his discussion with the student and the behaviour of other students in the background (e.g., Alex does not acknowledge students' numerical mistakes, such as $13 - 4 = 1$). In my notes I wrote the following reflections about the dialogue:

12:51 A student asked a good question, but the teacher had to stop to change the seats of two students that were talking with each other

12:52 -

12:53 The student asked the question again after the interruption

12:54 -

12:55 -

12:56 The student says to her friend that she is still a bit confused.

For the next ten minutes, Alex lets the students work in small groups on finding the formula of various quadratic sequences using the diagram in Figure 5.6. During that time, he walks around the tables either helping students on the task or persuading others to behave and engage with the task. At the end of the lesson, Alex tries one more time to bring the attention of the students to how the method works:

Alex: These five minutes guys, this is what I want you to think. Can you pay attention please? This part here [shows Figure 5.7] is like the blueprint, [Student 1]. Every sequence that is a quadratic sequence will follow these differences. So, to actually, this way, shh shh, to actually guys figure it out what is my a what is my b and what is my c, so I can find the nth term, it's like I copy this and I pasted it on top of this. Yeah? So, [Student 2], let me use a lot of colours, this green will fall on top, which part guys? this green the 2a will always equal my second difference. Now, [Student 3], can you tell me which number on that side would correspond to three 'a' plus 'b' (Figure 5.7)?

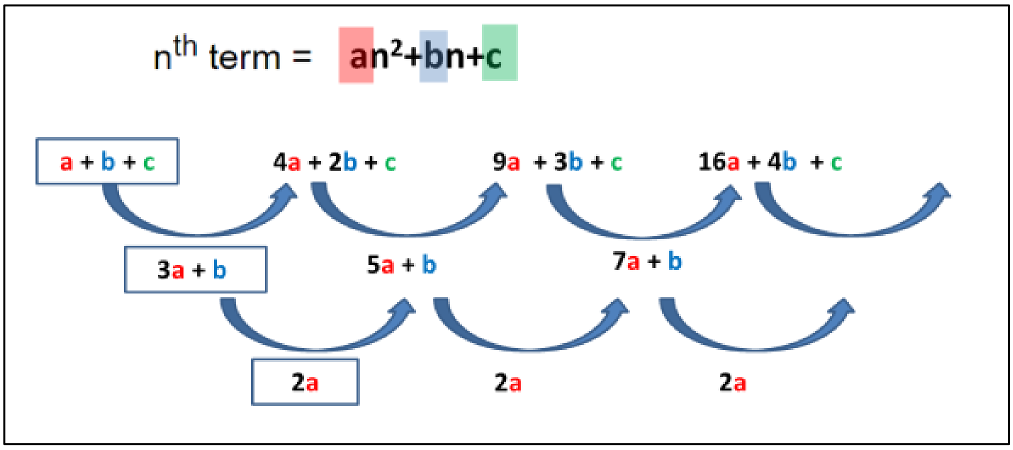


Figure 5.7 The "blueprint" of the quadratic sequences as shared by Alex

Alex calls the visual mediator he used (Figure 5.7) the "blueprint" of quadratic sequences. He had also printed the "blueprint" and asked the students to glue it on their notebooks (Figure 5.7). Focusing on the visual mediator, Alex's preparation for the lesson and attention to detail is evident. Comparing Figure 5.6 and Figure 5.7, it is noticeable that Alex chooses the same colours to represent the terms 'a' (red), 'b' (blue) and 'c' (green) in his hand-drawn diagram and the one he prepared and printed. Naming the diagram "blueprint" is considered also an indication that Alex tries to explain to the students the role of the diagram using familiar words. Both the visual mediator and the word use show Alex's attempts to achieve intersubjectivity. Despite that, based on Alex's accounts during the interview the overall outcome of the lesson did not materialise the way it was envisioned.

Throughout my visits to Alex's classroom, I witnessed Alex dealing with very challenging classroom management situations (e.g., extensive or confrontational talk among students, refusal to engage in the activities of the lesson). The episode discussed here illustrates Alex's efforts to engage students in a discussion beyond the mathematics of the moment while his attempt is interrupted multiple times to address pressing issues. Thus, discussing the reasoning behind the "butchered method" for finding the formula of the nth term of a quadratic sequence appears to be fragmented throughout the lesson. Intersubjectivity was not achieved in this episode despite Alex's preparation and considerations to this aim (e.g., adopting students' use of word, consistent use of colours and visual mediators, choosing words with familiar meaning). Alex's actions were not fully developed during the limited time left for mathematising between the interruptions. In a nutshell, in the episode Alex chooses to use a different approach from the one used in his school, informed by previous experience, to teach his students the formula of finding the n^{th} term of a quadratic sequence. Compared to the one preferred by the school, Alex's approach allows the comparison of the general quadratic formula and each individual case. Also, compared to other approaches to teaching the topic, it is easily generalisable for cubic or even higher sequences (Foster, 2004) that the students might come across at higher educational levels (e.g., studying mathematics at university). The episode showcases that taking an opportunity to go beyond the mathematics of the moment does not necessarily leads to students' engagement with the topic or opportunities to learn. Moreover, the episode illustrates how well-intended by the teacher intersubjective discursive moves can be obscured (or supported) by non-mathematical classroom interactions, including students' behaviour. Thus, the analytical tool used to account for intersubjectivity in the present work should also take into consideration the non-mathematical discursive elements that impact the effectiveness of the communication during particularly complex teaching situations.

5.2.3. *"Are we doing the correct question?": Managing equivalent approaches to the same problem*

The following episode is a potential opportunity of a discussion which takes place in Alex's classroom during the discussion of a starter activity (A4). The starter includes 8 revision questions. Alex chooses a random seat number from a ballot and asks the student on that seat to point out a question that either they had struggled with, or they think someone else might struggle with. The student pointed out a question asking to "work out 34÷0.2".

Alex:	Shhh, year 9s your attention, please, now, on the board question four. This was a similar question we did on our homework lesson. Let's go [Student 1].
Student 1:	Because five nought point twos make one [5×0.2=1] you'd do 5 times 34 [5×34].
Alex:	Excuse me, are we doing the correct question? Question four?
Student 1:	Yes, we are.
Alex:	Could you repeat again what was your process?
Student 1:	So, you've got nought point two.
Alex:	Nought point two. [writes on the whiteboard what the student is saying]
Student 1:	Five nought point two equals one [5×0.2=1].
Alex:	Ok.
Student 1:	Correct? Then you can just do 5 times 34 [5×34].
Alex:	You can just do five times thirty-four.
Student 1:	Yes.
Background:	I don't think that is working.
Student 1:	It does work!
Background:	[inaudible, students talk overlapping]
Alex:	Excuse me, shh shh shh, five times 34 equals a hundred and?
Student 1:	70.
Alex:	Very good. shh, [Student 2]?
Student 2:	I found it in a different way.
Alex:	Go for it!
Student 2:	So, you divide it by two.
Alex:	you divide thirty-four by two [34÷2].
Student 2:	And then, and then.

Background: Times it by ten.

Student 2: Divided, times it by 10[34÷2×10].

Background: Why?

Student 3: Cause we do.

Background: Why times it by ten?

Student 3: Because we divide by two. Then it's zero point two, you see?

Background: Pardon?

Alex: Go, go ahead. [Student 3], would you say that again please?

Student 3: he was dividing it by two and because the actual thing is zero point two then you times it by ten.

Alex: Very good. Year 9s, on our homework this Friday, we discussed [student 3]. That when I am dividing with decimals, I can remember that division also is a fraction and I can see some of you have followed this method that we revised again, [student 4], this Friday. So, [Student 2], if it's a fraction, I can find an equivalent fraction by times-ing numerator and denominator by ten. Exactly what you told me to do here and di-, divide three hundred and forty by two to get one hundred seventy.

The episode depicts an opportunity, coded as potential of the discussion, for Alex and his students to discuss different approaches to solve the same problem. Based on my analysis of the episode, the two methods proposed by the students can be linked by discussing the underlying idea of equivalent fractions: $\frac{a}{b} = \frac{a\times c}{b\times c}$. In particular:

$$34 \div 0.2 = 34 \div \boxed{\frac{1}{5}} = 34 \div \boxed{\frac{2}{10}}$$

and $\frac{2}{10} = \frac{1\times 2}{5\times 2}$.

It is not known what Alex had in mind when he commented that “division is also a fraction” from a pedagogical perspective. Alex states his realisation of division as a fraction, which in this case would trigger the routine of multiplying by the reciprocal. However, the comment is not elaborated upon. Such discussions are coded beyond the mathematics of the moment because talking about the similarities and differences of the approaches relies on meta-discourse about equivalent fractions and proper fractions. Both methods proposed by the students will give the same correct result. However, Alex does not seem to recognise the method proposed by Student 1 and focuses on the method proposed by the other students. In both methods, the students describe the steps they followed to reach the result but there is not discussion about how the methods work or reference to equivalent fractions. Thus, it is not appropriate to speculate about the students' realisations of the division during the task. In particular when one of the students asks "why" the response he gets from his classmate, "cause we do", suggests that the students perform the routine as a ritual. Therefore, the final comment from Alex seems to not follow what the students were talking about when solving the problem.

In a nutshell, making connections between different methods the students use could provide opportunities for discussions beyond the mathematics of the moment. The opportunity presented in this episode is considered as potential of the discussion. Although Alex and the students agree on the results produced, they do not proceed to compare the two methods proposed by the students. Intersubjectivity is not achieved in terms of elaborating upon why the two methods are equivalent by drawing on the two students' precedent space. The use of equivalent fractions is one suggestion to assist the process of saming the two methods which arises from the analysis of the task situation. Thus, I do not have sufficient evidence from the lesson to consider whether intersubjectivity would have been achieved if this or any other suggestion was materialised.

5.2.4. *'Framed' principles: The potentials of communicating about what is always there*

The following episode illustrates an opportunity for discussion to go beyond the mathematics of the moment provided by non-verbal stimuli present in the mathematics classroom.

Before every lesson (N1 – N5), Nick prepares his whiteboard adding essential information (date, title, and keywords) including a drawing of two frames. In each lesson, he draws a picture of two frames and inside the frames he writes the principles which connect addition to multiplication and multiplication to powers of numbers (Figure 5.8).

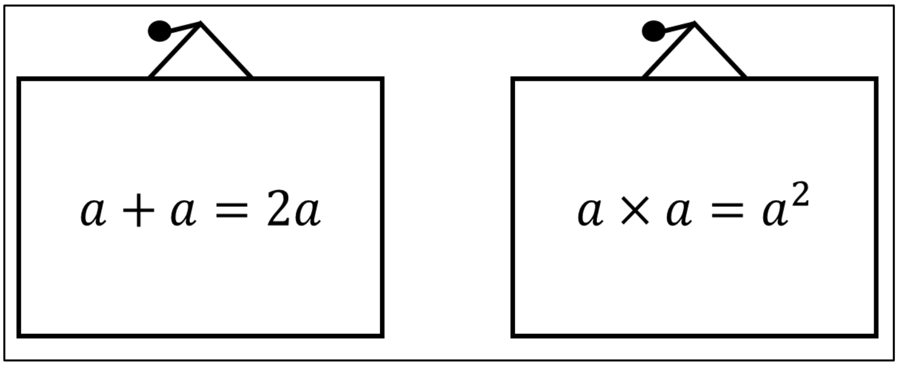


Figure 5.8 A recreation of the 'framed' principles on the whiteboard from the observation notes

Nick seems to consider this information relevant for any of his lessons. However, the ‘framed’ principles were not discussed with the students explicitly during the lesson observations. There is only one occasion where Nick points towards the drawing on the wall prompting a student to give the correct answer. The activity of preparing the board with essential information in every lesson is one of Nick’s teaching practices. Thus, the opportunity identified is categorised as potential of the activity.

The connection between addition and multiplication and multiplication and powers is present throughout the National Curriculum. The first time students come across these properties is in KS2, by the end of Year 6 students are expected to be able to interpret algebraic expressions such as the ones ‘framed’ on the board (Department for Education, 2013b). Discussing the relevance and the use of the properties in different situations, could provide an opportunity to discuss the universality of the properties. For example, using the properties in numbers (e.g., $3 \times 3 = 3^2$), in algebra (e.g., $x \times x = x^2$) in geometry (e.g., the formula for the area of the square is a special case of the formula for any quadrilateral where height is equal to the base) or in probability (e.g., the probability of getting heads twice is in two tosses is $0.5^2 = 0.5 \times 0.5$) and so on. The drawing could act as a visual mediator representing the properties as tangible objects, i.e., ‘frames’, helping the students identifying and saming the underpinning ideas in different situations.

It is not known what motivated Nick to include the ‘frames’ in every lesson. In the interview, he did not talk about this action, which might be an indication that his motives for including the ‘framed’ properties in every lesson are not related to making connections. Nick was not asked explicitly to elaborate on the ‘framed principles’ which should be taken into consideration. As a result, there is not enough evidence to assume that he had a motive that is related to his discourse at the Mathematical Horizon. His actions might stem from seeing the students struggle recalling the properties. Nick's gesture of 'framing' the properties could indicate that he regards them as relevant in each of his lessons.

In a nutshell, the gesture of 'framing' is a different way of communicating to students the significance of a given property or idea and act as a visual mediator that the students will come across in different situations aiding in saming and generalising. Similar actions such as framing posters or students artistic work on the wall could provide opportunities to discuss the larger significance of a topic of the day. For example, Liz was also observed to direct her students to look at displayed maths-related crafts on the wall twice and commented on them in relation to the topic of the day. The 'framing' action is coded as potential of the activity since the visual mediators used in this non-verbal communication could provide an opportunity for connecting ideas which act as 'building blocks' across activities. Identifying the 'building blocks' is an indication of acknowledging their importance in mathematics. Communicating such connections with the students is a potential opportunity to go beyond the mathematics of the moment.

5.2.5. *"And you might even want to twist your book around": Routinising gestures as a practical technique for problem solving*

Gestures, seen as visual mediators performed through body movements, were found to be one of the elements that could aid discussion beyond the mathematics of the moment. The following features a selection of episodes where the teachers and the students use gestures such as covering part of a shape or an equation, or turning the page, allowing them to concentrate on parts of the problem at a time.

A first example of these gestures was observed when Liz was discussing with a group of students about the shape in Figure 5.9 during the lesson L1. The shape was part of an activity where the students were asked to calculate the angles on the given shapes by using appropriate geometric reasoning.

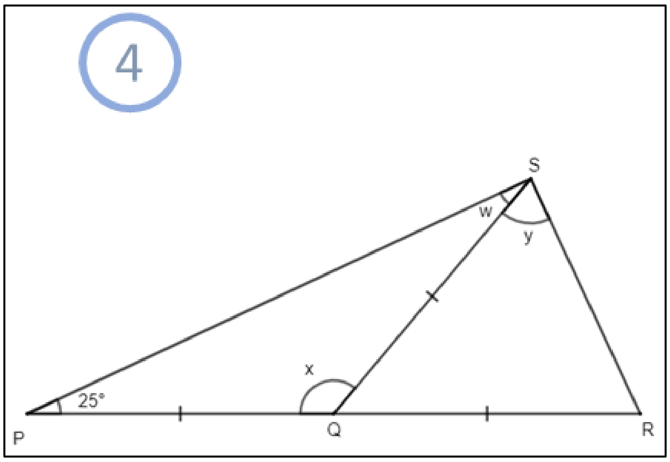


Figure 5.9 Recreation of the 4th shape given for the activity in Liz's lesson

Student 1: We didn't get that bit [points to notebook].

Student 2: Number 4 [shape in Figure 5.9].

Liz: Right. What can you work out? Where's, where's the starting point?

Student 1: I don't know, I did those three lines are the same size.

Student 2: There is something to do with the lines.

Student 3: But is, that angle is there as well.

Liz: What about if I cover up that bit [covers one of the two isosceles triangles].

Student 3: You mean, X, um, ehm...

Student 2: Those two are the same.

Student 1: It's an isosceles triangle.

Liz: That does mean. [overlap]

Student 1: Isosceles.

Liz: Yeah, yes. So those dashes tell you that these are equal lengths. If I cover that bit up. [knock as Liz puts her hand on the figure]

Student 1: That angle is also 25 degrees, the top there.

Student 2: Oh [overlaping].

Student 3: Oh.

Student 2: Thank you!

Liz: That's ok! it does help actually sometimes if you cover bits up and try and work out what you can see.

Student 2: Yeah.

Liz has similar conversations with other groups of students, and, in some cases, she also suggests to them to “twist it [shape/notebook] around” in order to view the shape from a different angle. She also makes the following suggestion during the whole class discussion:

Liz: Yes. Good. Um, question 4, this, this one caused a little bit of confusion. Um, and I suggested to a couple of people that it's nice to, you might want to cover some bits up and you might even want to twist your book around so that you can sort of see what you've got.

Liz’s students can routinely use gestures such as covering a page, highlighting part of the shape with colours or turning the paper, to zoom in and out of particular areas of the problem. The students are familiar with the idea that shapes do not change if we rotate or flip them, and they have learnt to recognise congruent triangles and identifying them as ‘the same’. There are other instances where the students use the gesture of covering part of the shape as a routine to identify components of a compound shape without any prompt from Liz:

Student: Cause there's a big one [triangle] this where [points on the piece of paper].

Liz: Oh, yes.

Student: And then when you turn it [paper] around, there's another big one [triangle] this where [points on the piece of paper].

However, the students cannot yet use the gestures to reason in unfamiliar situations. The following excerpt is representative of discussions between Liz and her students when students were asked to draw an isosceles triangle LMN where $LM = MN$.

Student 1: When, when you draw the triangle LMN, um, how, where would the angles go? So, L was, would be the top one?

Liz: Oh, no, it doesn't matter, because if you do it that way [paper moving sound] I could just twist that around and then L would be the top one over, just twist it around and L wouldn't be at the top anymore. So which way round you put them? But.

Student 2: [student interrupting the conversation].

Liz: Sorry, sorry, [Student 2], um, what is important is that you can label up 'L' M' 'N' but if it's told you that it is isosceles and LM is equal to MN, then you've put those marks in the right place, that's so important.

Figure 5.10 illustrates my interpretation of what Liz tried to communicate to the students, that the orientation of the shape does not affect the equal sides. The students do not seem to recognise that they can turn their piece of paper to reason why labelling does not depend on the orientation of the shape. Liz's gesture of turning the paper indicates that she is saming the action of moving the shape on a piece of paper – having the desk as the plane – and drawing a congruent shape on the piece of paper – where the paper represents the plane. However, these actions are not the same for the students as the metarule required to view congruent shapes as equivalent is not yet part of the students' discourse.

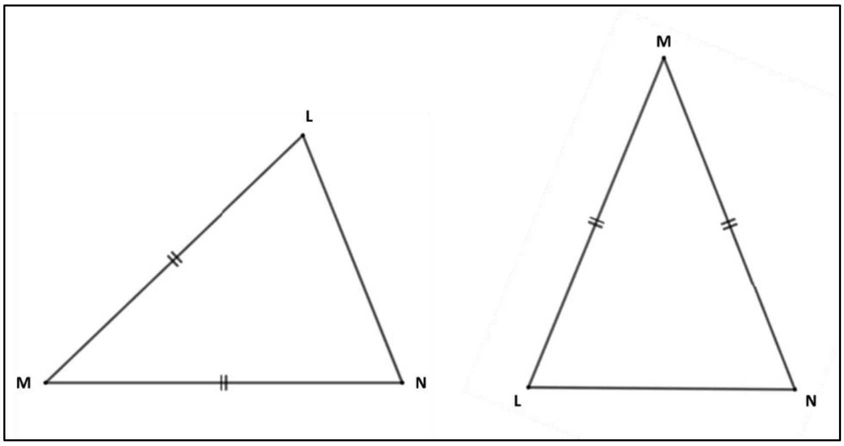


Figure 5.10 My creation of two equivalent constructions of triangle LMN with LM=MN

Another gesture of covering part of the work was observed in Alex's classroom when he and the students were rearranging an equation to make "x" the subject. In this case, Alex and the students talk about and use the gestures of "highlighting", to identify the variable, and "slapping the board", to isolate the variable. Alex points out to his students that his gesture of "slapping the board" is made in a "strategic way", however, he does not describe his strategy to the students in detail. This is a revision activity so the students and Alex might have discussed the strategy already in a different lesson.

The students and the teachers were observed to use gestures incorporated in routines during the lessons. Specifically, Liz and her students use gestures to argue in complex geometric problems and Alex to solve equations. According to Sfard (2009),"gestures are crucial for the effectiveness of mathematical communication” (p. 197). The gestures here could be used as tools to progressively allow students to manage working with more complex shapes or algebraic expressions just by recalling and not actually doing them. For the teachers acknowledging and addressing students’ difficulties is part of their discourse about Content and Students. Using techniques such as the gestures described above could be seen as solely pedagogical tools to help students overcome their difficulties in a particular set of problems. However, the use of gestures followed by “meta-comments” shows their significance beyond the particular problem. In this case, gestures are considered intersubjective. This use of gestures could provide an opportunity to discuss more general mathematical practices used in problem solving. For example, Liz commenting on the usefulness of the gesture in solving geometric problems or Alex reminding to students that way he “slaps the board” is not random but “strategic”.

In a nutshell, the section presents a collection of taken opportunities to go beyond the mathematics of the moment by discussing and developing the use of gestures as practical techniques which can be used in problem solving. Developing problem solving skills and techniques is a requirement included in the English curriculum. Problem solving spans across topics and is indicated by the participants of this study as an area of the curriculum that is important for their students as a life skill. The gestures are particularly useful in aiding students to “select appropriate concepts, methods and techniques to apply to unfamiliar and non-routine problems” (Department for Education, 2013a, p. 5). Therefore, gesturing supported by meta-comments could be considered as a visual mediator to achieve intersubjectivity between teachers and students to discuss mathematical practices such as simplifying a problem into steps, and ideas, such as that the orientation of a shape does not affect the outcome of the problem.

Overall, the theme *across the mathematics curriculum* includes opportunities to discuss mathematical ideas and practices that run across the mathematics curriculum for KS1, KS2, KS3 and KS4, e.g., discussing and developing explorative routines that could be used in problem solving, discuss connections between topics and methods included in the curriculum or discussing key ideas behind procedures that are usually taught as algorithmic processes. Another important finding which is prominent through the collection of the episodes is the importance of effective communication, i.e., providing opportunities to learn. Specifically, taken opportunities to go beyond the mathematics of the moment can be effective when intersubjectivity is achieved between the interlocutors. Here, teachers and students are observed using (or trying to use) intersubjective words, visual mediators and gestures to reach to an agreement leading to changes in the students predicated by the curriculum mathematical discourse for their age and perceived 'ability'.

5.3. What comes next

The analysis of the interviews highlighted discussions about whether it is possible to talk with students about mathematical ideas and practices that are considered advanced for their age. During the interviews, the participants who indicated that it is possible to bridge connections with advanced content gave examples such as discussing more advanced results of the content they are teaching (e.g., extending the discussion of group theory to Galois theory), fundamental ideas of a branch of mathematics (e.g., what is the area of interest of number theory) and proofs of propositions included in the curriculum (e.g., deriving the formula for Poison distribution). The theme *what comes next* also includes mathematical ideas and practices which students might come across outside the classroom. For example, Alex, David and Liz mention during their interviews that they share resources (e.g., mathematics communication videos or articles etc) with their students to spark their curiosity about mathematics. The following episodes depict instances from the lesson observations that the discussion could provide opportunities for a teacher and the students to communicate about mathematical ideas and practices which the students might come across in later stages of their education and life. Next, I present four episodes from lesson observations where there are opportunities, not always taken, for teachers and students to discuss ideas and practices that the students might come across later in their life and education. Section 5.3.4 presents an example of a negative case (Shenton, 2004) which highlights that such discussions are not always to the benefit of students' learning.

5.3.1. *"Can you work out the angles and shapes with more sides?": Blending spontaneous actions with planned activities*

The following episode depicts an attempt made by Liz to connect a proof of the sum of angles in quadrilaterals with a general proof of the sum of angles in any polygon is $(n-2) \times 180^{o}$ where n is the number of sides in a given polygon. The episode took place during a lesson on the characteristics and the sum of angles in quadrilaterals (L3). The triggering event for this episode was a discussion about the justification of sum of angles in quadrilaterals (see also Section 5.4.5). Liz had drawn on the board two different justifications as to why the sum of angles in quadrilaterals equals 360° (Figure 5.11). At the end of the discussion Liz gives verbally the students the following activity:

Liz: […] Um, I'd like you to do this actually. Can you work out the angles and shapes with more sides? So, using this, using either of these [points at the diagrams on the board – see Figure 5.11], can you work out the angles, in shapes that might have five, six, seven, eight sides? And if you've got enough, can you see if you come up with a rule?

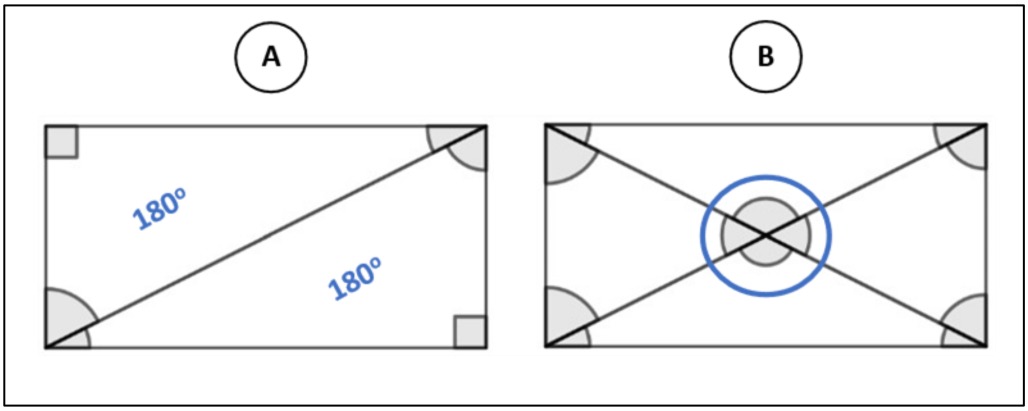


Figure 5.11 Recreation of Liz's drawing on the board from observation notes depicting two ways of using the sum of angles in a triangle to prove that the sum of angles in a quadrilateral is 360°.

According to her reflection during the post-lesson interview this part of the lesson was not planned. From the interview, Liz seems to have expected that her students will be able to work independently to explore the general proof for any polygon.

Liz It was very disjointed, um, and it wasn't particularly well planned, um, because I had these two topics I wanted to explore, that are usually done over two lessons. But I didn't think it would take them that long to pick it up. So, I ended up just shoehorning it into one lesson, um, and then I went a bit off track with the… getting them… I thought they would be ready to explore the concept of the number of triangles in a shape. But a lot of them were not very happy with that idea. So, we ended up kind of stepping back a bit and just going on and doing some questions at the end.

The students are expected to learn the justifications of the propositions regarding the sum of angles in triangles, quadrilaterals and in general polygons in different stages throughout KS3. Specifically, the students are expected to learn how to “use the sum of the interior angles of a triangle to deduce the sum of the interior angles of any polygon” (AQA, 2015, p. 119) by the time of their GCSE exams. This particular group of students has already come across the idea that the sum of angles in triangles is 180^o as an angle fact and they are expected to learn it’s proof in Year 8 after discussing properties of parallel lines and transversals.

During the lesson where the episode took place the students see for the first time two equivalent justifications for the sum of angles in quadrilaterals (Figure 5.11) and according to the program of study followed by the school students are expected to learn the general rule for any polygon in year 9. Liz attributes the difficulty of the students to produce an algebraic narrative (“a rule”) for the sum of angles in polygons because they were not ready to explore the number of triangles in a shape. However, the analysis of the audio-recordings shows that the students felt confident with ‘splitting’ polygons into triangles appropriately, but experience difficulties generating an abstract rule which would describe their results. The following excerpt is indicative of the conversations between Liz and students while they were working on generalising their results to produce the rule:

Student 1: I think I found a really, right, cheat-y way to do it.

Liz: OK.

Student 1: So, hmm, it's quite hard to explain, but on the first one there is one, eight, 'o' [180], and then so it goes eight six four two, 'o'[8,6,4,2,0, - looking at the tens of each sum]. Eight, six, four, two, 'o' [8,6,4,2,0]. So, and on here it was one three five seven nine zero [1,2,5,7,9,0 – looking at the hundreds of the sums]. So, it's going up two every time.

Liz: OK.

Student 1: But it's not really a rule but there's a, that's a very simple way to work out. But I think I did.

Liz: But the difference is that what I wanted to be able to say to you is I've got a shape with 150 sides.

Student 1: Yeah.

Liz: And then you tell me how many angles it's got.

Student 1: Ok.

Liz: Really quickly.

Student 1 notices a pattern in the digits of the sums which is the result of adding 180 for each additional side. But fails to notice the difference of 180 between consecutive sums.

More discussions with groups and individual students show that the students are familiar with the routine of "dividing" the shape into triangles (see also auxiliary lines in Section 5.3.3). However, the students did not manage to produce an algebraic narrative about the angles on their own as Liz might have anticipated. Finally, Liz prompts the students to create a table (Figure 5.12):

Liz: […] So I think what would be useful is if you write yourselves up a table that says this is the number of sides [writing on the board]. So, in my triangle, I've got three sides, and then these are the angles, what are the angles in a triangle?

Student 1: 180.

Liz: Hundred and eighty, four-sided shape, three, sixty and in a five-sided shape?

Student 1: 540

Liz: 540.

Student 2: Miss?

Liz: In a six-sided shape?

Student 2: I found the pattern.

Liz: What's the pattern?

Student 2: Adding 180 to it each time.

Liz: Yeah, adding 180 each time. There's one other thing that's really useful to know about this rule is how many triangles split it into. So, the number of triangles. So, the number of triangles in a triangle?

Student 3: Three.

Student 1: One.

Liz: Just one, is it? You can't split it. How many triangles did I make, turn my quadrilateral into?

Student 1: Two.

Liz: Two. My pentagon?

Student 1: Three.

Liz: Three. My hexagon?

Student 1: Four.

Liz: See if you can find a rule using that information.

Sides	Angles	Number of triangles
3	180	
4	360	2
5	540	3
6	720	4

Figure 5.12 Recreation of the table Liz draw on the board

Liz lets her students continue working on the problem a little more. By that point most of the students have identified or heard what the pattern is, thus found a recursive rule. However, Liz is expecting the students to make one more connection between the number of triangles and the number of sides to produce an algebraic narrative that describes it. Finally, one of the students described the pattern between the sides and the number of triangles to Liz and she then shares it with the rest of the class and moves on to the next part of the lesson:

Liz: Shhh. So, um, [student]'s rule said, I'm gonna take the number of sides of my shape. I'm gonna subtract two, because that tells me then how many triangles I've got and then I'm gonna multiply it by 180. And I'll show you, we normally write it [writes on the board] n minus two, right, multiplied by 180. So, any shape. So, if I've got a 10-sided shape, um, I will then know the angles straightaway are gonna be 8 times 180.

Despite Liz initial disappointment after the lesson, the students are able to describe the steps of the rule in a following lesson (L4) which indicates that the episode offered an opportunity for explorative learning. However, the students again fail to produce an algebraic narrative, but the majority of them could describe that they have to multiply the number of triangles with 180^{o} and could recall that the number of triangles is two less than the sides.

Based on students' precedent events, they are familiar with using variables to describe sides and vertices algebraically. But, in this case, the variable does not describe an unknown length or angle. Moreover, the students are not yet able to perform the routines required to describe complicated geometric results algebraically. Liz introduces the idea to her class for the first time in L6, i.e., two weeks after the current episode takes place. The use of the table as a visual mediator was how Liz could achieve intersubjectivity. In her interview Liz said that she will try the same activity again in the future with the same students possibly dedicating a whole lesson and giving the students printed shapes “to cut it up into triangles” before attempting to derive the rule. Her intention indicates her belief that the students did not achieve the goal she set for the activity and that she could try to achieve the goal at a later stage by using a bigger variety of tools and visual mediators. Despite Liz’s convictions, the overall episode was coded as a taken opportunity for the students to discuss justification and proof beyond the mathematics of the moment as the students are able to recall and use the pattern in subsequent lessons.

Liz and the students discuss only the justification of the rule if the polygons are divided into triangles from one of the vertices of the polygon. A justification using a division similar to the one used in the second diagram (Figure 5.11, Part B) would not lead to the same observation about the number of triangles (see for example Figure 5.13). During the lesson, at least one student was observed to use the second diagram (Figure 5.11, Part B) following the teacher’s suggestion to use “either of these [diagrams on Figure 5.11]” during the introduction of the activity. Therefore, discussing the alternative diagram was coded as a potential of the discussion which was not followed up by Liz.

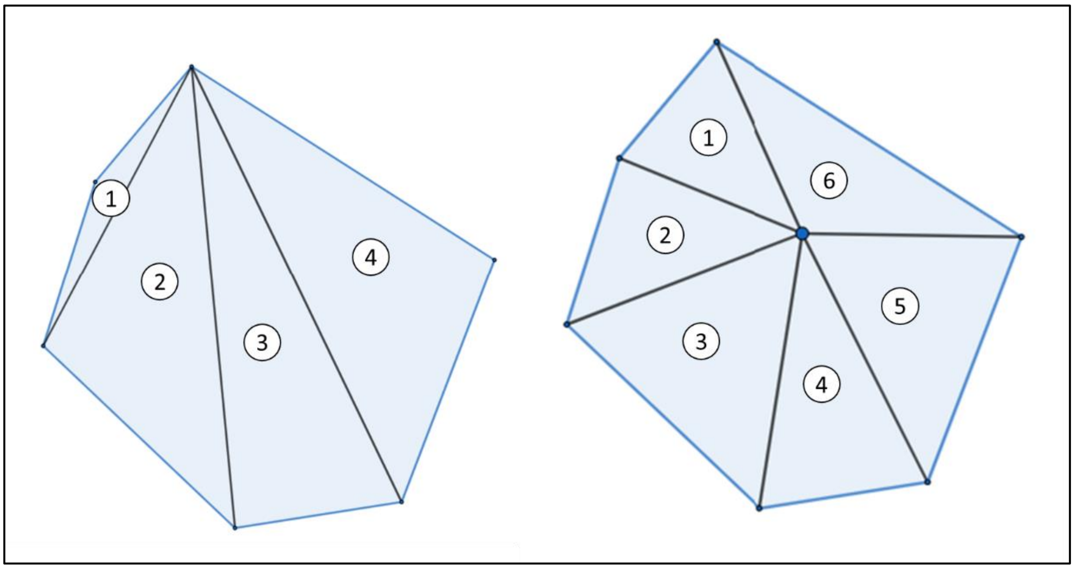


Figure 5.13 Two ways of dividing a hexagon into triangles

In a nutshell, the episode was chosen for two reasons. First, the analysis of the data indicates that opportunities to go beyond the mathematics of the moment can be part of the planned lesson or spontaneous actions of the teachers. The episode also depicts in detail the actions of the teacher when she decided to diverge of the original plan there and then and her reflections afterwards. Secondly, in parallel to the main episode there is a second issue that is left unexplored, i.e., what would happen if the students do not divide the polygons into triangles using one of the vertices? Therefore, the episode illustrates the complexity of taking opportunities on the spot which might lead to actions that are not as well thought or implemented as a planned lesson might be.

5.3.2. *The "nine-point circle": An early introduction to advanced topics*

The following episode takes place in Liz's classroom during the lesson L2 where Liz used an activity to introduce her Year 7 students to a theorem expected to be taught to Year 9 students. Figure 5.14 depicts the main activity of the lesson given to students in four parts.

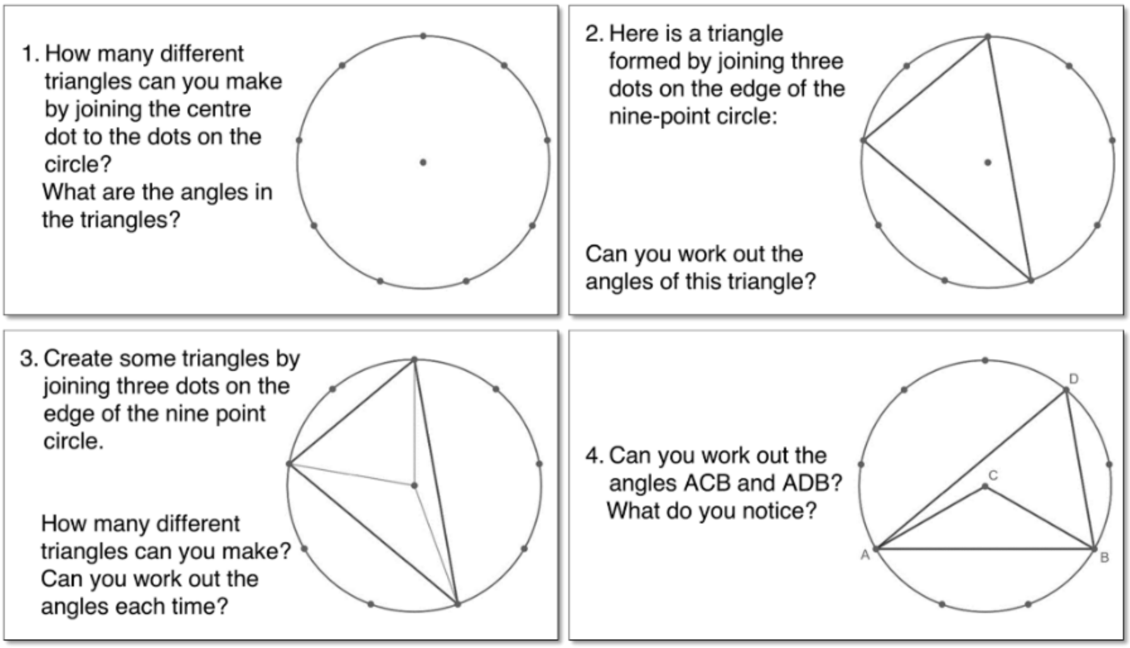


Figure 5.14 Recreation of the 'nine-point circle' activity slides from the observation notes

The activity is chosen by Liz specifically for this group of students, as indicated in her post-lesson interview:

Liz: Em, so there is a set of lessons that are pre planned available to me for this particular group. I won't do all of them because some of them will just be too easy for them. And I don't do them necessarily in that order that is given to us. […] So that nine-point circle one that uses that. It went into subtended angles. That isn't something I would do with a mixed ability. That was something I found and thought they might enjoy. So, you have something that they are available for you, but you change it depending on th-the class that you are.

The parts of the activity include a common visual mediator, a circle with nine equally spaced points on its circumference. Liz referred to this shape throughout the lesson as "the nine-point circle". First, the students were asked to identify all the different triangles with one vertex on the centre of the circle and the others on two of the nine points and then calculate the angles for each one. Then, Liz asked the students to find a way to calculate the angles in a triangle with all the vertices on the circle. Next, the students were asked to construct more triangles by joining three points on the circle and then to calculate their angles. Finally, Liz asked the students to calculate the angles ACB and ADB and tell her what they notice.

Liz uses the 'nine-point circle' to introduce her students to a theorem about angles in circles also known as the Star-Trek lemma:

> The angle subtended by an arc at the centre is twice the angle subtended at the circumference.

The Star-Trek lemma is part of the 'circle theorems' included in the highest band of the GCSE requirements. According to the English National Curriculum, the Star Trek Lemma, called as such due to the resemblance of diagrams with the Enterprise logo in the popular series, is one of the theorems that only highly attaining students are expected to learn to prove and apply during KS4. According to the school's program of study, Liz's students might come across circle theorems in Year 11 (age 16), but not at Year 7. Liz presents the last part of the main activity (part 4, Figure 5.14) as a challenge to the students:

Liz: So, I'm going to give you one last challenge, which is to take a look at this. [Liz puts up the slide with part 4 of the activity]. And you're going to need to start calculating the angles and see if you notice anything. So, I'd like you, actually we can start from, we've got a starting point, haven't we? We know this angle here.

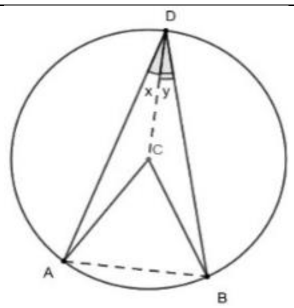
For this lesson, the opening slide includes only the starter. The title of the lesson is not included in contrast to Liz's usual practice to include the title, keywords and a learning outcome in her opening slide. When one of her students asked her what the title for the lesson is, Liz prompted the student to write down "angles" in their notebook, adding that she combined more than one lessons.

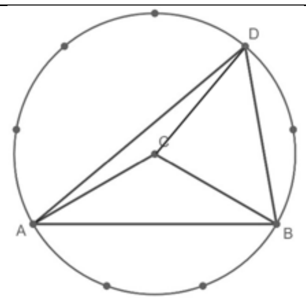
The analysis shows that with the help of the nine-point circle and building upon the previous parts of the activity, Liz and the students have the potential to go through the main ideas of the proof of the Star-Trek lemma despite not having yet engaged with algebraic routines and relevant terminology (e.g., arc, subtended, etc). In the first three parts of the activity Liz and the students go over the basic tools required to calculate the angles in the final one but also understanding the basic steps for proving the lemma. In the first part, Liz and the students discuss the key facts they need to calculate central angles:

Liz:	Right, the next. So, I've got four different triangles. Um... And then there's two key facts that we need to think about if we are trying to work out the angles. Um, [student], can you share what you said earlier?
Student:	Um, the angles around a point measure 360.
Liz:	Angles around a point measure three hundred and sixty. Which is a bit like the clockwork we did, wasn't it? Remember when we were looking at what did the hands on a clock make? What angles do they make?
Student:	Oh, yeah, oh yeah.
Liz:	So that's a kind of a clue. But the other thing that might help us here, what kind of triangles have I drawn?
Student:	Isosceles.
Liz:	thanks, [Student], yeah, they're isosceles triangles because, [Student]?
Student:	Two sides are the same one.
Liz:	Two sides, when I'm going from the centre point out to the edge of my circle, that, that line there is exactly the same distance is that line there. So, they are isosceles. So, they are your two clues. See if you can work out what the angles are on all four triangles, please.

In the second part, the class explores the use of auxiliary lines to calculate angles in inscribed triangles. In the third part, the students get the opportunity to explore the process in different triangles formed by joining three points of that circle and providing them an opportunity to accept that the routine is applicable in other cases, thus that it could be used in generalising the observation (see also Section 5.3.3).

Table 5.4 A side-by-side comparison of the discursive elements of the 'nine-point circle' activity and the Star-Trek Lemma

<table>
<tr><th></th><th>Star-Trek Lemma</th><th>The nine-point Circle</th></tr>
<tr><td rowspan="2">Keywords</td><td>lemma
centre
radius
circumference
arc
subtended
twice</td><td>“Challenge”
“the nine-point circle”
“centre dot”
“point”/ “dot”
“double”</td></tr>
<tr><td colspan="2">circle
angle
triangle</td></tr>
<tr><td>Visual mediators</td><td>

ADB=x+y (example)</td><td>

ACB=120 ^{o} (example)</td></tr>
<tr><td>Narratives</td><td>Draw AB and the radius CD.
On the diagram angle ADC=x and CDB=y.
Therefore, ADB=x+y
Angle CAD=x because the triangle ACD is isosceles, CA=CD radius of the circle.
And DCA=180o-2x (i)
Also, angle DBC=y because the triangle DCB is isosceles, CB=CD radius of the circle.
And BCD=180o-2y (ii)
ACB+DCA+DCB=360^{o} because angles around a point sum up to 360^{o}.
From (i) and (ii):
ACB+180o-2x+180o -2y=360^{o}
ACB=2x+2y
ACB=2(x+y)
ACB=2ADB</td><td>[Splitting the shape into the triangles ACD, ACB and CDB.]
Liz: Two sides when I'm going from the centre point out to the edge of my circle, that line there is exactly the same distance as that line there.
DCA=160^{o} [part 1/angles around a point]
ADC=10^{o} because the triangle is isosceles.
ACB=120^{o}
BCD=80^{o} [part 1]
CDB=50^{o} because the triangle is isosceles.
ADB=60^{o}
Student: Mmm, Double!</td></tr>
<tr><td rowspan="2">Routines</td><td>Naming unknown angles
Rearrange algebraic equivalence</td><td>Naming unknown angles with the purpose of calculating them
Calculating angles</td></tr>
<tr><td colspan="2">Using angle facts
Dividing the shape in known parts</td></tr>
</table>

Table 5.4 illustrates a reconstruction of the narratives observed during the lesson (right column) alongside a canonical proof of the lemma seen in later Years (left column). The merged columns include the common keywords and routines between the classroom activity and the proof of the lemma. To produce the reconstructed narrative, parts of the recording and the notes are collated in a way that corresponds with the steps of the proof. Throughout the lesson, Liz and the students communicate using the words ‘angle’, ‘triangle’, ‘circle’ and ‘point’ or ‘dot’. The new narratives about angles can be negotiated through routines and other narratives endorsed in previous lessons or earlier parts of the activity. In the classroom discourse students engage with simple algebraic routines based on the visual mediator of the nine-point Circle in comparison to the proof of the lemma seen in later in stages of KS3 and KS4 education.

The students are not yet familiar with advanced algebraic routines, such as rearranging algebraic equivalences with more than one variable. However, with the mediation of ‘the nine-point circle’ Liz and the students communicate about the Star-Trek lemma and its proof, using an example where the angles can be calculated. Liz presents the activity as a “challenge” to her students, rather than as a fact or a theorem that they need to memorise. Liz’s students are familiar with routines such as using angle facts to substantiate their actions or naming unknown angles towards calculating them. Liz concludes:

Liz: […] this angle here [centre] will always be double the size of that one there [circumference].

Phrases like “this angle here” and “that angle there” act as placeholders for missing words that the students might come across in the future. Liz is aware of the words but does not name them to her students. Yet, the constructed narrative make sense both from the perspective of Liz, as an application of the Star-Trek lemma, and to the students, as an observation that can be confirmed following the steps of the activity. At the end, Liz makes a comment to her students about the activity being an advanced (grade 9) GCSE theorem.

Liz: Um, so this is probably a grade nine GCSE thing called

Student: Grade nine? Grade nine?

Liz: Called circle theorems.

Student: So, if you got that right, so if you get a question like THAT [exclamation]

Liz: You know, you don't, you don't get one question at GCSE, [student]. It isn't a nine or a fail. Um, so this, there is a whole scheme of what that looks looks at angles and circles.

Her comment shows awareness that this is an application of the Star-Trek lemma and how it relates to students' learning in the future. The reaction of the student might be an indication that he found the activity easy for an advanced GCSE topic. Despite that, the data do not provide evidence as to how other students feel about this comment.

The activity and Liz's actions attempt to gear her students towards constructing a narrative about the two angles by the end of the lesson. Actions beyond the mathematics of the moment are observable here in the way that she engages students with mathematical content that they will see in the future in ways that are accessible to them.

Another potential opportunity emerging from the activity is to use the same visual mediator, the 'nine-point circle' to introduce similar observations about other circle theorems and/or exploring the angles of cyclic polygons. As mentioned earlier, only highly attaining students are expected to learn about circle theorems, including cyclic quadrilaterals, in KS4.

In a nutshell, the episode discussed here depicts an instance where young students are introduced to an advanced geometric theorem and its proof, typically only taught to students with higher predicted grades. The students have limited vocabulary and precedent space compared to what is expected before introducing the Star-Trek lemma at Year 11. For example, the students have not yet come across, angles in circles, or the word 'subtended' and they are not able to use it in their narratives. However, using the appropriate visual mediators the teacher managed to set up an opportunity for the students to explore the key ideas of the theorem.

5.3.3. *"He said, can I split it up into other triangles?": Using auxiliary lines in problem solving*

This section concerns episodes where Liz and her students use auxiliary lines as a routine in problem-solving. During the lesson observations, Liz's students are asked to explore unfamiliar to them situations where the use of auxiliary lines are necessary. I have identified instances where using auxiliary lines was required as part of the solution of a problem in three of Liz's observed lessons. Here, I discuss one of the instances as an example.

The episode took place during the nine-point-circle activity (L2) presented in Section 5.3.2 (Figure 5.14). The completion of the activity requires drawing auxiliary lines (i.e., additional lines) in order to divide the inscribed triangle in part 2 appropriately and calculate its angles. The process of drawing auxiliary lines was used as an approach to tackle an open problem.

Liz: Yeah. Right. [slide of part 2 on the board] So, I've got a slightly trickier question for you now. So here is a triangle formed by joining three dots on the edge of a nine-point circle, nine-point circle [points towards the diagram]. However, this time it doesn't go through the centre. Can you work out the angles of this triangle?

Initially, Liz gives the students time to explore possible ways of tackling the problem. During that time the students try to measure or estimate the angles of the given triangle. Approximately 5 minutes later, she addresses the class:

Liz: […] it's taken him [referring to Student 1] quite a long time to ask me a really, really good question. But, I think I should share it with everybody. He said, can I split it up into other triangles?

Student 2: Oooh.

Auxiliary lines are not explicitly mentioned in the National curriculum. However, problem-solving techniques that require the use of auxiliary lines are taught to students throughout the primary and secondary education. Specifically, the process of drawing additional lines is not new to Year 7 students. As part of their primary and secondary school education, the students are expected to use auxiliary lines for the purposes of calculating the area of composite shapes or identifying lines of symmetry. Also "highly attaining" (Department for Education, 2014a, p. 3) students are expected to learn to use auxiliary lines when proving certain theorems, in known situations, by the end of KS4, but not at Year 7. The use of auxiliary lines in unknown contexts, such as the context of this activity, is not expected by secondary school students. However, students might come across problems that require their use during their Further Education courses and at university level if they choose to pursue a degree in a STEM related subject. Drawing auxiliary lines to modify the triangle in an appropriate way is a routine. For Student 1, the initiation of this routine is prompted when other routines failed.

However, he requested Liz's approval before moving on. For other students the initiation was prompted by the teacher:

Liz:	Yeah, I'm not going to tell you the answer. I'm going to show you the clue. So, this is the clue. [draws three line segments from the centre to the edges of the triangle, Figure 5.15]
Student:	Oh.
Liz:	[…] if you can split it into other triangles and using the triangles where we can where we can start. We know some facts, don't we? An isosceles triangle is nice. I'm not trying to find three unknown angles now, am I? I know that two are the same. […] plus we did all that work beforehand.

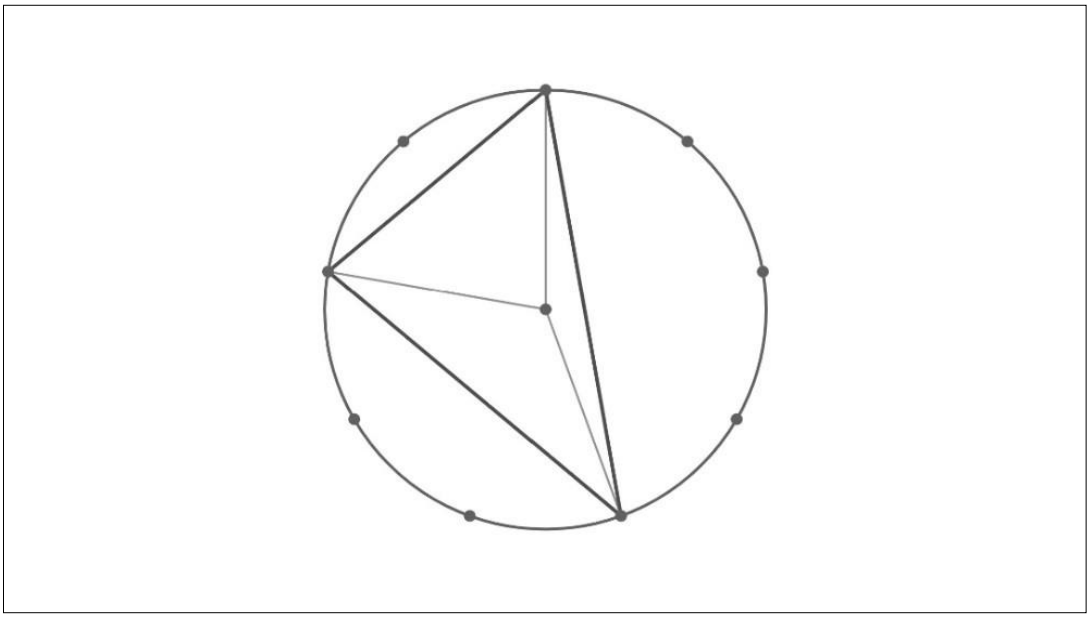


Figure 5.15 Recreation of the shape with the line segments drawn by Liz on the board. From observation notes

Here the routine is presented to the students as a "clue". The use of the word could signify a step to a direction that was previously hidden from the students who are now invited to follow this routine to complete the task. In all the episodes where auxiliary lines are used, Liz and the students refer to the routine using the words "split", "chop up" or "divide". Here, the use of the phrase "split up into other triangles" and the absence of words like 'draw', 'line segment', that are usually present in formal geometric proofs, indicates that the discussion is mediated by the specific drawings of the activity (Figure 5.15). I interpret the use of these words as evidence of intersubjectivity. For the students, this activity is a new situation where they need to draw additional lines to solve a problem - similar to what they are doing when they are asked to calculate the area of a composite shape (precedent event). While for Liz, this is a problem-solving routine towards the modification of a shape in order to use known results.

This episode was chosen because it illustrates an aspect of use of auxiliary lines that is not present in the others. In the third part of the activity, where the students were required to construct their own triangles by joining three points on the nine-point circle, one of the students constructed a triangle in a way that the centre of the circle was outside the triangle (Figure 5.16). The student asked Liz for help:

Student:	Mine is out of the.
Liz:	Oh, now, this is interesting. Can you now. Now you haven't got it in the centre. Can you still, can you, is there anything you can do?
Student:	Yes, you can still draw the triangles and know what that angle is. Only [mumbling/inaudible] I don't know.
Liz:	Have you got a pencil? Why don't you try drawing some triangles in a pencil, then you rub them out.
Student:	I can't draw it inside the [triangle], cause you have to make it to the point. So, it'll be like... That. But how on earth is that.
Liz:	How is that going to help?
Student:	It's not.
Liz:	No.
Student:	I actually need to have a point, you can't draw it inside.
Liz:	What about if we... made it quadrilateral, would that help?
Student:	Oh, yeah!
Liz:	I don't know.
Student:	Then I'll, then you have to decrease 30 from 360.
Liz:	Hh-mhh.

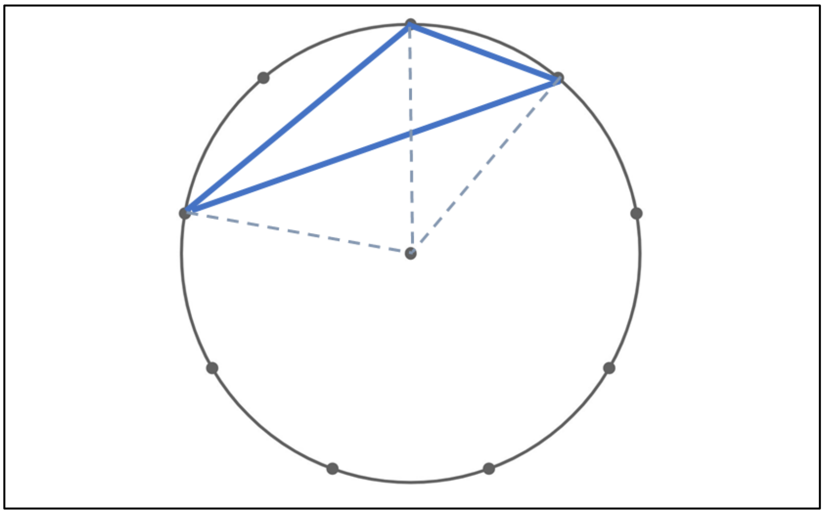


Figure 5.16 Recreation of the shape Liz and the student talk about from the observation notes. The dushed lines were added during the analysis to aid the visualisation of the discussion.

In contrast to the previous uses of auxiliary lines in this and other lessons, this is the only instance where the routine is not performed to “split” a shape. In this case, Liz prompts the student to use auxiliary lines to “make it quadrilateral”. In general, auxiliary lines can be used in geometric problems to transform the given problem to an equivalent that can be solved using established properties. Discussing this example with the student and potentially bringing it for discussion to the whole class could provide an opportunity for the students to expand the applicability of the routine.

In a nutshell, Liz presents the students with a broad spectrum of situations that promote the routine of auxiliary lines beyond the requirements of the curriculum for KS3 and KS4. Observing similar uses of auxiliary lines on many occasions during Liz’s lessons suggests that engagement with this routine was not coincidental and supports meta-level learning. Liz expects her students to engage in the routine as a technique which supports exploration in problem-solving. The instance presented here illustrates her teaching practice of providing gradual support to her students, of expanding the applicability of the routine and of establishing this process as conventional in geometric reasoning.

5.3.4. *“I can show you how to do that without calculator”: Displaying mental maths algorithms*

The following episode depicts a situation which although the discussion goes beyond the mathematics of the moment, the actions of the teacher are not aimed towards students’ learning. Thus, the interaction should be questioned as to whether it is an effective form of communication beyond the mathematics of the moment. The conversation started with Nick trying to convince his students to avoid mental calculations when working with decimals (N2):

Nick: What I mean is if you don't do it like this will be longer. Use column methods, I personally cannot physically with my age, with my experience of mathematics, I cannot do that. I can do harder than this, but not this one [points at a calculation with decimal numbers which appears on the board]. Brain is not designed like this. [pause, writes on the board $\sqrt[3]{12167}$] So, this, I can do something like this, I can work out the roots of this number, but I cannot do that one [calculation with decimals]. I can work out what number I multiply by itself three times to give me that number, without calculator, but I cannot do that easy, easy one [calculation with decimals]. I can do even bigger than this number. I can do, maybe, this number [writes on the board $\sqrt[3]{857375}$]. I can do this number, which is 857375, the cubic root of that. Maybe you think how you don't get this, but even the square numbers, which are much easier, it can be done.

Probably to impress the students, and possibly also me, the observer, Nick chooses a 6-digit number as an example proposing that he can calculate its cubic root mentally. Then, he proceeded explaining the method to me while the students were watching:

Nick: I can show you how to do that without calculator [addressing me specifically].

Evi: Is there a trick?

Nick: Not a trick, you need this. This one is 1 [$1^3 = 1$]. This? this one is this [$2^3 = 8$]. This one is that [$3^3 = 27$]. Yes? This one is 64 [$4^3 = 64$]? Yes?

Evi: Yes.

Nick: This one is one two five [$5^3 = 125$]. This one, to my knowledge, is one two six, two one six [$6^3 = 216$]. Yes? This one, to my knowledge, is three four three [$7^3 = 343$]. Yes? This, this one, to my best knowledge, is five one two [$8^3 = 512$]. And this one [9^3] is seven, seven, nine times nine times the size of the one [9×27]. I think it has to be one here. Yes? No, that would be

	nine times eighty-one that would be 9. Now, the three will be, it should be, yeah, seven two nine. Yep.
Evi:	Yes.
Nick:	Seven two nine. All you need is these numbers to remember, now you can do this one [shows $\sqrt[3]{857375}$]. Any number, cubic number, is two digits, or more than two digits. Yes? Now what is this number here [units]? 5. Which one is 5 here?
Background:	Five!
Nick:	That is 5. So, the first one is 5. Shhh. This one is 5. Now you cross those three numbers [3,7,5], you have eight five seven, where is your eight five seven? Your eight five seven? That would be 9, I think it should be nine, yeap it should be nine. Because physically you're between this and that and the highest of that is this one. So that should be 9. So, there should be nine if you multiply 95 times 95 times 95. Could you do that please with your calculator?

Like more popular algorithms such as the ones for addition and multiplication this method requires memorising, in this case cubic numbers. This method is not typically taught as part of the National Curriculum. However, it is an algorithm that can be found in extracurricular resources targeting the general public or math enthusiasts. The choice of number might not have been arbitrary as the result is a whole number and thus can be calculated following the proposed method. It is unclear from the data if this example convinced the students to try working out basic problems using pen and paper. Some of the students seem to pay attention on his explanation on the board. Yet, Nick's attempt to convince the students appear to be superficial, and there is no evidence of intersubjectivity during the episode.

In a nutshell, the episode is included here to highlight those discussions beyond the mathematics of the moment that are not necessary to the advantage of students. A controversial point of this excerpt is that Nick interacted mostly with me during the exchange although there were a couple of students observing him explaining his process. Unlike other episodes, this one does not seem to be connected to a specific topic or process the student might come across during their education or later in life. It is, however, an interesting example to showcase the sides of mathematics that do not make it into the curriculum, but an algorithm might one come across on the internet.

Overall, the theme *what comes next* includes opportunities to discuss ideas and practices which students might come across in later stages of their mathematics education or through activities of public engagement with mathematics (e.g., open day events, videos, etc). The episodes discussed under the theme also illustrate how taken opportunities to go beyond the mathematics of the moment can be a spontaneous act or planned actions of the teacher. Finally, it is important to acknowledge that discussions beyond the mathematics of the moment are not always seen as opportunities to engage the students. They could also be attempts to impress them or other observers. In such cases, the achievement of intersubjectivity is not considered as the teacher and the students are not mutually engaged in the discussion.

5.4. Mathematical conventions

As part of the KS3 and KS4 education, the students are expected to be able to identify mathematical objects (e.g., types of quadrilaterals, quadratic expressions) and use standard notations (Department for Education, 2013a, 2014a). However, the students are not expected to learn formal definitions of mathematical objects. The section includes findings from the analysis of examples around the theme of mathematical conventions. The theme emerges both in the interviews with the teachers and the lesson observations. For example, Thomas, when asked to explain what comes to his mind by the phrase "connections across educational levels", says:

Thomas: One might be, um, the <u>significance of notating</u>. So that will be, um, for example, the idea that when something is, um, I become aware of something or something is particularly significant, there is real power in naming it and giving it a name. And it allows me to use it and work with it and go forwards. After naming it, and that could be naming, but it also can be notating. So, I might give a notation for things. […] I think maybe in relation, could be in relation to notation, but something, um, <u>process is becoming objects</u>. So that's I, I count. And then, and then I will give, I would notate that, I would say that's three. So, the process of counting becomes

> an object, which is the number three, but initially that starts with a process. […] But then it's an object that I can work with and deal with and so on. […] Um, so I don't know, that there are some, some things that come to mind. So, they're not necessarily items of the curriculum. But it seems to me about things that are fundamentally mathematical and so they do run it all the levels.

The examples discussed in the following sections investigate the opportunities teachers have to go beyond the mathematics of the moment and give meaning to mathematical conventions such as naming, notating, and constructing working definitions of mathematical objects. Taking such opportunities poses the introduction of a new mathematical convention as a community-building activity rather than an arbitrary rule the students need to follow.

5.4.1. *"I was just wondering, why is it they always use the bad letters?": Labelling practices in mathematics*

The two episodes described here, depict potential opportunities for Liz and the students to discuss labelling, specifically, the use of letters in describing geometrical objects. The first episode provides a potential opportunity to address the need of using such labelling. While the second one, provides a potential opportunity to discuss the use of specific letters.

The first episode takes place between Liz and a group during L2. Figure 5.17 depicts the starter activity the students were working on when the discussion took place.

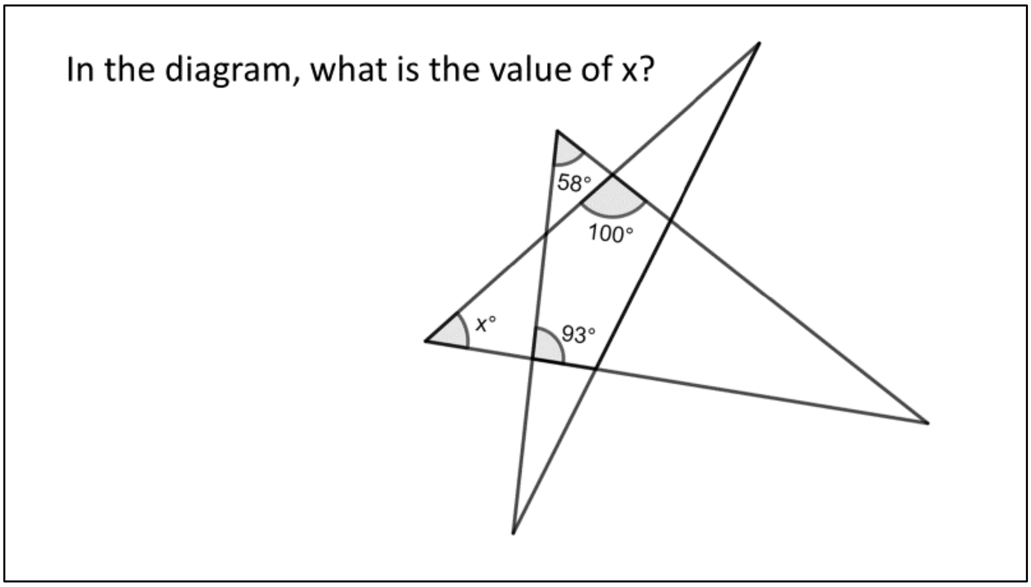


Figure 5.17 Recreation of the starter from observation notes

Student: [x is] 51. but I don't know how to explain it. So, what I found, how I found it is 93 add 58 is [overlap student talk inaudible, the student is thinking] 151.

Liz: Mm hmm.

Student: So, this I think will be 19, Sorry 29.

Liz: Mm hmm.

Student: Because it's 180 degrees.

Liz: Mm hmm.

Student: And then 100 add 29. A hundred and twenty-nine. And it's gonna be 51. Because that's what it needs to add to 180.

Liz: OK.

Student: I just don't know how to explain it.

Liz: Um, you can say red triangle, or which one did you do first?

Student: The green one.

Liz: The green triangle angles must equal 180. Red Triangle angles must equal 180.

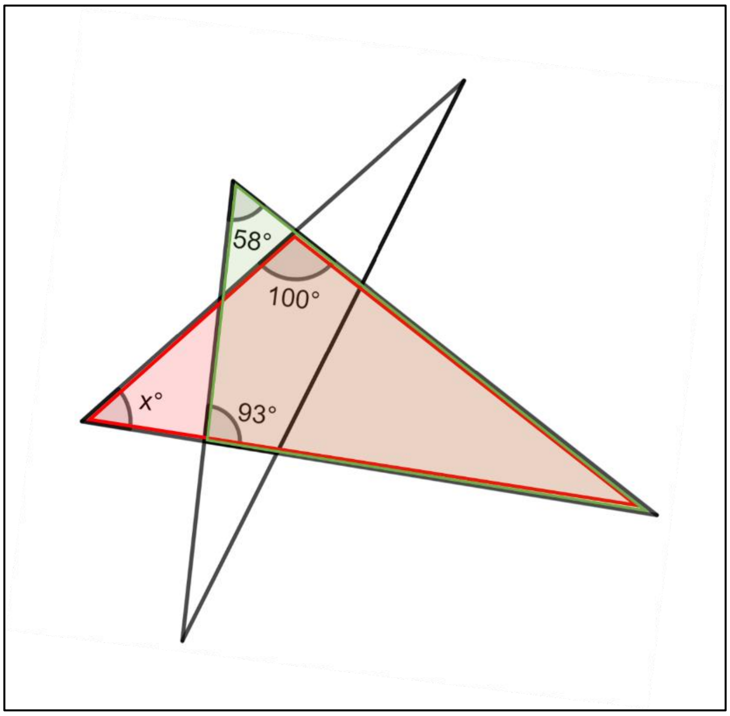


Figure 5.18 Recreation of the shape on students work from the observation notes

The activity is given to the students without any additional labelling apart from angle x. According to the curriculum, the students are expected to be able to "use the standard conventions for labelling the sides and angles of triangle ABC" (DfE, 2013, p. 8) by the end of

KS3. However, throughout my presence in this classroom, Liz and the students worked with labelled shapes and shapes without full labelling. Specifically, using a hybrid of labels and colours, when letters are given as part of the given question, or only using colours, when they are not. Liz never objected to the use of colours or descriptions of the mathematical objects (e.g., “the triangle there”). On the contrary, she regularly complemented the students on their use of colours as visual mediators and engaged in this routine of labelling using colours herself when discussing the activities on the interactive whiteboard.

This Year 7 class is still in a transitioning phase from KS2 to KS3 which might explain Liz’s choices. In the dialogue the student is using the word “this” and gesturing to describe her work to Liz, despite that she has already used colours (red and green) to highlight parts of the diagram in her notebook (Figure 5.18). The students recognises that she needs a better way to explain her work on paper. Liz’s proposal to use the colours as labels for the triangles gives the student a way to communicate her results without using the standard notation.

During the analysis this opportunity was coded as ‘potential of the discussion’ because I see it as an opportunity for Liz to then propose to this student, and possibly the rest of the class, that using standard notation even when this is not asked explicitly by the activity, is a standard mathematical practice in communicating geometric results.

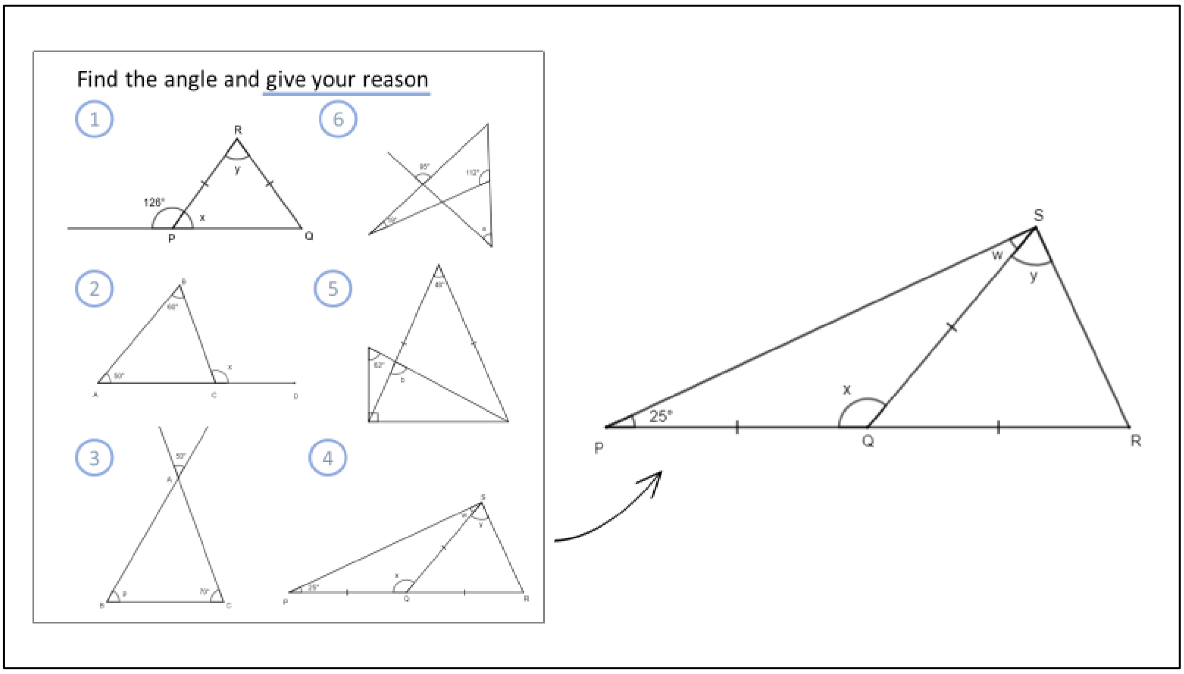


Figure 5.19 Recreation of the activity given to students from the observation notes with enlarged depiction of shape discussed

The second episode I wish to discuss here, shows students' reaction to certain choices of letters that seem contradictory to them. The conversation took place when students were asked to work in the activity in Figure 5.19 as part of a lesson on vertically opposite angles (L1). Looking at the Figure 5.19 one can notice that some of the diagrams do not include any labelling for the sides or angles while others do. The discussion concerns specifically the shape numbered 4[24].

Student 1:	I was just wondering why is it they always use the bad letters?
Liz:	Why do they?
Student 1:	Like 'x', 'y' and 'w'. Why would they use those letters?
Student 2:	Why can't they use 'M'? 'M' is better.
Liz:	Oh, they sometimes do. Oh, and sometimes they use Greek letters.
Student 1:	It's like 'S', 'Q' and 'P'.
Liz:	[Laughs].
Student 1:	No one likes those letters!
Student 2:	Yeah, we only like letters like 'M'.
Liz:	[Laughs] Being that discreet about it, you can write a letter to the GCSE, um, the people that write the GCSEs and say you don't like the letters 'S', 'Q', 'R', they may change it.

The activity includes 7 diagrams for the students to calculate the missing angles. Some of the diagrams are fully labelled while others are not. Here the group of students is concerned with the use of specific letters such as 'P', 'Q' or 'R'. The right and wrong way of labelling geometrical objects, however, depends less on the choice of letters and more on their use in accordance with conventions that are not yet fully described to the students. Seeing mathematics as language, one could say that conventional notation and typography is merely equivalent of the syntax and grammar in language learning. Seeing mathematics as an act of communication, though, the use of a symbolic language that might have already been linked with other uses such as the composition of words could make someone question its use to convey mathematical ideas.

[24] Liz numbered the shapes shown on the interactive whiteboard upon requests from her students.

The students of this episode do not disclose why they think specific letters as “bad”. In contrast with learning to use the alphabet for the first time here students have already some experiences with those letters and how they are used to form words (in English for example) the idea that letters ‘X’, ‘Y’, ‘W’, ‘S’, ‘Q’, ‘P’ are “bad” letters, might be linked with their experience with those letters in English language learning. An alternative possibility is that some of these letters might get confused with other mathematical symbols (e.g., ‘q’ with 9 or ‘Z’ with 2 in handwriting).

Liz’s remark about the use of Greek letters is twofold. On one hand, the remark could be used to signal the students to a different way of using letters in mathematics. On the other hand, it can also be seen as a remark that the students will come across more seemingly contradictory notation in the future. The students use of the pronoun “they” indicates that the students recognise that this is a choice made by a group of people. Liz attributes the choice of these letters to the “people that write GCSE”. This remark could potentially soothe students’ curiosity of the somewhat arbitrary choice when is used by others. However, when engaging in the routine of labelling geometric shapes, the choice of numbers depends also on the context and the parameters such as the number of objects that need to be labelled which is not discussed here.

The analysis unsurfaced more episodes similar to the ones provided to the reader in this section. I am sharing these two episodes as representatives of the category about labelling conventions. Other examples, include the use of three ordered letters (e.g., angle ABC) to refer to an angle versus a lower-case letter (e.g., angle b), referring to mathematical objects (e.g., triangles and angles) by describing their position and orientation within a given diagram and referring to angles by value. Discussion of these or similar topics could give students a sense of meaning on mathematical notation and why we need to name points, variables or special numbers. The episodes are all coded as potential of the discussions since Liz did not make any attempt to challenge the students’ current routines.

In a nutshell, the two episodes presented here indicate that discussions between the teacher and the students can unearth opportunities to introduce and negotiate appropriate notation. For example, when a shape is not labelled the students choose to use colours to help them solve the problem and communicate their findings. Moving from using colours to distinguish between shapes by labelling the vertices and referring to the shapes using the appropriate series of letters, requires a discursive shift from seeing the labels as arbitrary choices of others, often experts, to mathematical conventions which they use as mathematists (i.e., participants in mathematics discourse).

5.4.2. *"It's because they meet at a vertex ": The naming practices in mathematics*

The following episode takes place during the introduction of a lesson about vertically opposite angles and their properties (L1) and illustrates an opportunity to go beyond the mathematics of the moment by discussing the choices made in naming mathematical objects. Figure 5.20 depicts a set of vertically opposite angles similar to what Liz has drawn on the board. Liz asks her students if they can explain to her how they know that vertically opposite angles are equal.

Liz:	[Student 1] tells me that vertically opposite angles are equal. How do you know that they are equal?
Student 1:	Because they are.
Liz:	Because they are equal. Have we got any other reasons why are they equal? [Student 2]?
Student 2:	Because they can cone on the same point.
Liz:	Oh yeah. So, this is, this is quite nice, isn't it? This bit here, what do we call that in maths? Do you know what we call that?
Student 3:	A vertices.
Liz:	Oh, a vertex [stresses X] yeah. Which is why it's called vertically opposite. It's because they meet at a vertex.

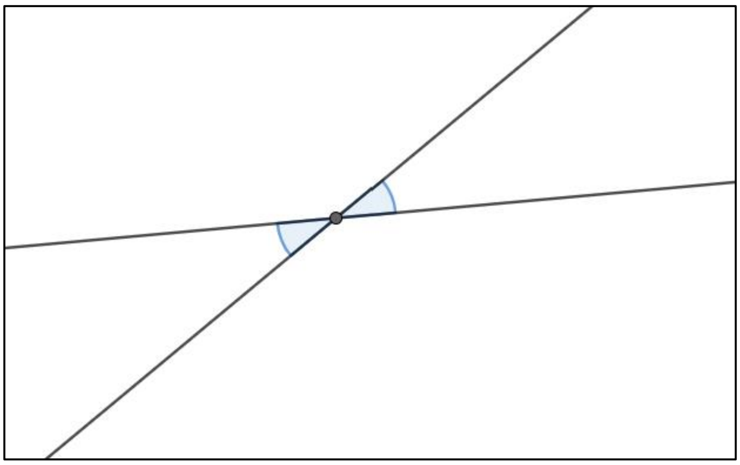


Figure 5.20 Recreation of Liz's drawing of vertically opposite angles on the whiteboard

According to the national curriculum students are expected to be able to draw and recognise vertically opposite angles in diagrams and find the missing angle as part of their KS2 education. In KS3, the students revisit this topic as part of their GCSE assessment in KS4. However, the students are not expected to learn a definition about vertically opposite angles. Specifically, the only direct reference to vertically opposite angles in the national curriculum reads: "recognise angles where they meet at a point, are on a straight line, or are vertically opposite, and find missing angles" (Department for Education, 2013b, p. 44). Therefore, it appears as that the students are expected to recognise vertically opposite angles based on their visual representation, in particular their relative position between the lines.

In the episode, Liz reacts to Student's 2 use of the term 'cone' pointing out to that the point of intersection of the two lines is called a vertex and subsequently that vertically opposite angles are called as such. The meaning of the word 'vertically' in the definition of this pair of lines can be found in resources for teachers, e.g., "pair of equal angles between two intersecting straight lines" (NCETM, 2014, p. 90). The students are not expected to be taught a formal definition at this age. Therefore, it is expected for students to recognise a mathematical object by its visual mediators. In this case, the object-level rule for Student 2 to identify vertically opposite angles is the resemblance of the diagram to a cone, or, for other students, an 'X'[25]. However, Liz draws attention to the point of intersection of the two lines, the vertex, shifting the attention from the overall look of the diagram to a specific point that is key in the definition. For Liz, two angles are vertically opposite when two lines intersect, following the formal definition. Liz and Student 2 identify vertically opposite angles based on different rules about the object. The use of the different object-level rules indicates that Liz and the students narrative stem from discourses with different metarules about identifying regularities of the object. For students, the metarule is the overall shape of the objects while for Liz is through the formal definition. Students are not aware of formal definitions yet. However, the communication between them is effective, making the situation intersubjective. Using a diagram of vertically opposite angles on the board as a visual mediator, Liz guides the students to move from looking at the overall shape (precedent event) to looking at the area around the vertex. Liz's action suggest that she took the opportunity to

[25] Not seen here.

discuss with students' ideas that are not part of their education at this stage as they are not required to understand or use formal definitions. Thus, the opportunity identified in this episode is classified as taken.

In addition, it is possible that the students recognise the word 'vertically' in the name with its colloquial meaning, making the name of the object appear arbitrary. According to the Cambridge dictionary:

Vertically: straight up or at an angle of 90° to a horizontal surface or line[26]

and stems from the word:

Vertical: standing or pointing straight up or at an angle of 90° to a horizontal surface or line[27]

However, in vertically opposite angles the word vertically stems from the word:

Vertex: The point at which two or more lines intersect. Plural: vertices. (NCETM, 2014)

Identifying an opportunity to share the meaning of the word as a response to students' description of the property is related to Liz's Discourse at the Mathematical Horizon. Liz's goal is not necessarily to make the students change the metarule which they use to identify the vertically opposite angles, as this is not part of their curriculum. However, identifying the correct stem of the word could add to the experience of the students with the object.

In a nutshell, the teacher in this episode took the opportunity to reason with her students the naming of a specific mathematical object. The episode goes beyond the mathematics of the moment because the students are not expected to learn a definition about the object. However, by discussing the origin of the name, the students could explore alternative metarules for recognising the mathematical object which metarules do not rely on the overall appearance of the object.

[26] https://dictionary.cambridge.org/dictionary/english/vertically
[27] https://dictionary.cambridge.org/dictionary/english/vertical

5.4.3. *"How are the insides more than the exterior": Negotiating the endorsement of narratives*

The previous episode showcases how Liz informed her students about the origins of the name of a known object and used the naming to support an argument about its formal definition. Next, I am presenting an episode where the students are trying to describe an unknown mathematical object based on its name and how this routine provided an opportunity to collectively substantiate a definition of a new object, exterior angles.

The episode takes place in Liz's classroom. The lesson is titled "Angles Investigation" (L5) and the keyword Liz uses for this lesson is "exterior angles". For this lesson Liz has rearranged the desks so as the students are seated in four small groups. She distributes printed hand-outs with the activity to the students and tells them that she expects them to work on the activity with "resilience", without asking her questions. She tells the students that she will not be answering any of their questions unless she things that she is the only person in the class that can give an answer. In the post-lesson interview Liz talks about her intentions for the lesson:

Liz: Well, the intention of the lesson was to try and get them to stop asking me so many questions, cause I think that they know much more than they do, particularly these two girls [Liz made a gesture showing me two empty chairs where two girls were sitting earlier] their first port of call is me always. I don't understand. I don't see. How I'm supposed to tackle this. So that really was the intention because I kind of knew that they can do that type of stuff. Em, so I did like the fact that I didn't allow them to ask me any questions. And then I did try and say, OK, you can ask questions and see if somebody else can answer it before necessarily me. And I liked the fact that the one question about the exterior angles was the one I was expecting to come up, and that was the one I was going to have to explain because I've never told them about exterior angles.

The activity consists of a series of propositions that are called 'clues' and the goal of the activity is for the students to identify a hidden message written in numbers. The clues are fragmented information about different triangles that the student should construct and match the numbers to the corresponding letters of the vertices of the triangles. One of these clues is:

“The triangle CED and D has an exterior angle 112º”

While the students are working in groups one of the students have the following conversation with Liz:

Student: How do you find that? The exterior angles.

Liz: Auh, this is the question I've been waiting for.

Student: And then, how are the insides more than the exterior? [Figure 5.21, A]

As a response Liz instructs the students to stop working and have a whole classroom discussion about exterior angles. On the board Liz draws two triangles both labelled CED as represented in Figure 5.21 without adding any angles just yet.

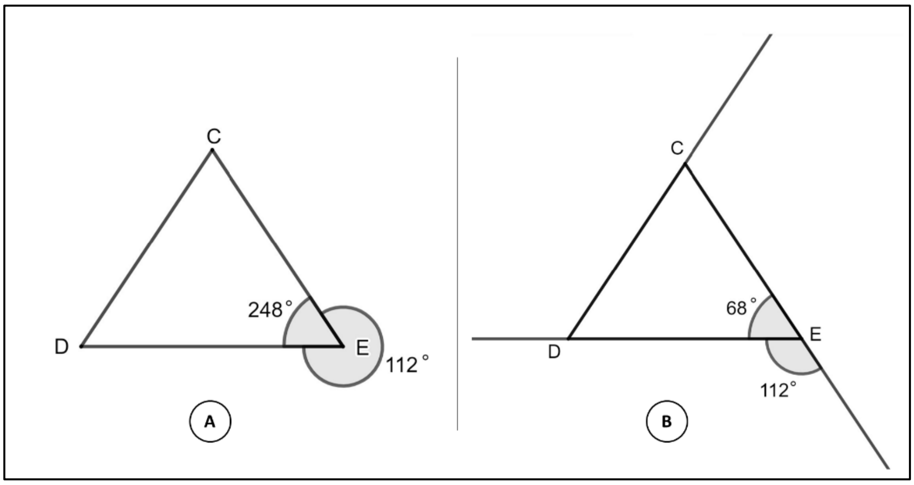


Figure 5.21 Recreation of the two figures Liz draws on the board from the observation notes

Liz: Folks, I'm just going to stop you for a minute. Just look this way. I've had one question and I wasn't sure if I was the only person in the room that would be able to answer it. So rather than you all asking the same question, I will answer this one as a group. Quite early on, it talks about Triangle CED and D has an exterior angle at one hundred and twelve. And I've seen quite a few people do this. [on Figure 5.21, A, Liz marks the angle out the point E which is located outside of the triangle as 112º] 112, which seems pretty reasonable, doesn't it? Which means, this angle here is what?

Student 1: 180 minus. [Student 1 refers to Figure 5.21, B]

Liz: <u>This is angles around a point</u> [Figure 5.21, A], not angles on a triangle.

Student 1:	I thought those angles were on a straight line [Student 1 refers to Figure 5.21, B].
Liz:	Well, <u>let's say, an exterior angle goes all the way around and then this bit here</u> [Liz makes a gesture drawing the attention of the students back to Figure 5.21, A]. So, if this is angles around a point, what is this angle here?
Background:	[students talk overlapping].
Liz:	I probably broken maths OR [raises her voice] I will define an exterior angle for you. Let me just go back a bit. Did, so, the people that have done loads of work, did you work this out?
Background:	No.
Liz:	Did you just accept that that was two hundred and forty-eight and auh, ok, that's fine? [Liz refers to Figure 5.21, A]
Background:	No
Student 1:	We did pretty much angles on a straight line.
Liz:	So why did you choose angles on a straight line?
Student 1:	Because it seems, it seemed more logical.
Liz:	It seemed most logical?
Student 1:	Yeah, because, well, if you have an angle around, somewhere on the outside of the triangle, then, well the inside of triangle if it's like that, has to be lower than 90. Well.
Liz:	OK, hold on, [Student 2] have you got something to add?
Student 2:	[student inaudible].
Liz:	Yeah. So, the exterior angle, um, is this one here. So, on any time when you've got any, um, you just you keep extending these around and this and this relates to the exterior angle here, which means that one is 68.
Background:	Ooooh.
Liz:	OK, so, that was something I wasn't sure if you'd have known about that. I don't think anybody necessarily, I like the idea that the boys down in this corner, um, just decided that that must be what it is. Without, um, really think, without really thinking and I just decided that must be it because it couldn't possibly be the other way. But that is that's why the exterior angle, it looks like that. Ok, of you go. Keep going.

The students are expected to learn about exterior angles in KS3. For this particular group of students, properties of triangles, including exterior angles are taught in Year 7. In addition, Liz included ‘exterior angle’ as a keyword on the opening slide of the lesson, making it clear that she intended it to be the topic of the day. However, she invites the students to explore what this new mathematical object could be before instead of giving them a definition right away. Thus, in terms of object level learning, the discussion does not go beyond the mathematics of the moment. However, the way Liz introduces the students to the new mathematical object provides an opportunity to discuss the metarules behind endorsing new narratives, here, definitions in accordance to previously established narratives, in a way that promotes more central participation in the mathematical community of the classroom.

As depicted in the conversation above, the definition of an exterior angle could seem contradictory to the students. In this episode, the narratives about exterior angles debated follow two different metarules. The first metarule is identifying exterior angles based on colloquial interpretations of the word ‘exterior’ [Figure 5.21, A], which contradicts the consistency of geometry and the other one identifying exterior angles by noticing the contradictory narrative and looking for alternatives [Figure 5.21, B], which contradicts the colloquial use of the word “exterior”. In contrast to the example presented in Section 5.4.2 about vertically opposite angles, here the naming of the mathematical object contradicts the definition of the word “exterior”, thus, Liz expects her students to endorse the correct narrative by rejecting the problematic one.

During the discussion, narratives such as “I probably broken maths”, “it seemed more logical” and “just decided that must be it because it couldn't possibly be the other way” signify Liz’s attempt to show that the new definition should not contradict previously established properties of triangles. The sentence “This <u>is</u> angles around a point” signifies that she sees the property as an object, while Student 1 says “I thought <u>those angles were</u> on a straight line” indicating that he is talking about two angles and not the angle fact. When Student 1 refers to the routine they performed in their group uses the verb “did” indicating an action. Liz uses the two diagrams (Figure 5.21) and tries to draw the attention of the students to the contradiction using known properties of angles. Thus, drawing from students’ precedent space. Intersubjectivity in this episode is achieved using the two diagrams (Figure 5.21) as visual mediators. For the purposes of this lesson, the goal of the discussion is for the students to learn how to construct and calculate exterior angles. This goal is addressed when Liz defines exterior angles for all the students in the classroom. The discussion highlights the importance of building upon the theory when defining a new mathematical object. However, the use of the word 'exterior' might still seem arbitrary to the students which could potentially influence the objectification of exterior angles.

In a nutshell, the episode exemplifies an opportunity to go beyond the mathematics of the moment presented when the students come across an unknown mathematical idea during an explorative activity. The significance of the episode lies behind the teacher challenging her role as the "ultimate substantiator" (Sfard, 2008, p. 234), i.e., the person responsible to assess the endorsement of a mathematical narrative. The students get the opportunity to explore and discuss possible definitions and to agree not to "break maths".

5.4.4. *"Shall we see what Google says?": Endorsing working definitions*

The episode discussed in this section regards again evidence of the teacher challenging her role as the ultimate substantiator and attempts to let the students collectively produce a definition which is beyond the requirements of the curriculum for KS3. In contrast to the previous example where the endorsement of the narrative is based on visual mediators produced by the students, here, Liz and her students look for the ultimate verification in Google. Thus, the search engine takes the role of the ultimate substantiator.

As an introduction to the lesson on the properties of quadrilaterals (L3), Liz asks the students to produce working definitions of parallel lines:

Liz: Um, before we do this, I'm just going to have a quick chat with you about parallel lines. Actually, I'm not gonna have a chat with you. You're gonna have a chat between yourselves. Well, I'd like to do is, um, write a definition of parallel, then, swap it with the person next to you and see if we can get the best definition as a class. So quick keep swapping them around, see if we can keep improving on your definition.

The routine of defining goes beyond the mathematics of the moment as they are not familiar with the metarules involved in producing a definition that will be accepted by the mathematical community at large (e.g., a mathematics teacher or a mathematician). The students are not expected to learn formal definitions during their secondary education.

However, they are expected to be able to recognise a pair of parallel lines, to be able to draw parallel lines and later will learn to use angle facts of parallel lines (Department for Education, 2013a, 2014a). Asking the students to produce a working definition about parallel lines, however, has pedagogical implications. Liz will be able to identify aspects of the students' precedent space (Lavie et al., 2019) about parallel lines and communicate about them as part of the lesson of the day. The students give a variety of answers:

Liz:	OK, folks, let's hear your definitions. Shh. Let's go. [Student 1], what did you put?
Student 1:	Um, parallel is two things that are opposite but will never meet.
Liz:	Opposite but will never meet. This word opposite popped a few times actually. [Student 2], what do you…?
Student 2:	Um, two lines that are adjacent to each other.
Liz:	Oh, adjacent, that's a nice word. What does adjacent then mean? [Student 3]?
Student 3:	Um, two things next to each other?
Liz:	Yes. So, they're next to each other. [Student 4]?
Student 4:	Well, paral-, parallel lines are two opposite lines that never meet. And they're normally found in, um, shapes that have an even amount of sides excluding the circle, which, yeah, it just bends reality.
Liz:	[Laughs] Um, But we, we, [Student 4] and I were talking about this, [Student 4] said to me, um, all shapes with even sides.
Student 4:	Most.
Liz:	Have parallel lines.
Background:	[inaudible].
Liz:	Yeah, I think we can probably break that quite quickly, but we'll come on and have a look at that, [Student 5]. You got anything to add to this?
Student 5:	Um, I've got parallel lines are lines that are [inaudible].
Liz:	Yes, so we are all similar. There's just one missing bit.
Background:	They are pairs.
Liz:	Oh, go on [Student 6].

Student 6:	Um, parallel lines are two lines that will never cross and that for the distance between them is always the same.
Liz:	Oooh, OK. Yes. This, this is their thing, isn't it? It's the distance between them as well. That they will.
Background:	[inaudible].
Liz:	Yeah.
Liz:	Um, I don't, should I, you know what I'm gonna get Google up, shall we see what Google says? Um, definition of parallel lines, lines on a plane, which we just mean as a flat surface. So, lines are always the same distance apart and that, yeah, they never meet.
Background:	Wow.

The definition found online reads:

> Parallel lines are lines in a plane that are always the same distance apart. Parallel lines never intersect.

At this stage, the students are familiar with the main words used to define parallel lines, e.g., “lines”, “distance” etc. However, the students do not know words that are peripheral for a working definition at this age, e.g., plane. Intersubjectivity in this case is achieved through the use of words that are familiar to the students in combination with the introduction of new vocabulary. The students are using words such as “opposite” or “adjacent” to describe the lines, Liz, referring to a definition found on google, calls them “line on a plane” that she then explains to the students. Moreover, the formal definition found online describes the property of parallel lines as being “the same distance apart”. The observation that the lines do not intersect is given as an additional property included in the definition on Google. Liz rephrased the property to match the word use of the students.

An additional potential of this discussion is that Liz and the students could take the opportunity to discuss how the two properties of parallel lines are equivalent and interchangeable. However, this does not seem to be the intention of Liz at this time as she expects from her students to give both properties as part of their working definitions.

In a nutshell, the significance of the episode is the use of Google as a source which informs discussions beyond the mathematics of the moment. Here, a tool, the search engine, is treated as the "ultimate substantiator" (Sfard, 2008, p. 234), confirming the suitability of students' narratives. However, the tool is not necessarily used critically. Google is mentioned by Liz and other teachers during their interviews (see also Section 6.5). During the lesson we observe Liz trying to explain words used online which the students might not understand the meaning of. However, we do not see the teacher challenging the information given online. For example, the definition provided by Google includes two equivalent mathematical definitions of parallel lines thus includes information that are redundant from an expert's perspective (e.g., a mathematician).

5.4.5. *"Why is a degree a degree?": Using history of mathematics in teaching.*

Apart from the choices of names for mathematical objects, other mathematical conventions might seem arbitrary to the students. The following episode depicts one such situation where Liz draws on history of Mathematics to give meaning to the choice of degrees as a unit of measuring angles.

During the lesson L3, Liz creates two diagrams on the board (Figure 5.22) and explains to the students why the first diagram justifies that the sum of angles in a quadrilateral is 360°. The discussion takes place right after Liz asks the students to justify why the second diagram also "tells" that the sum of angles in a quadrilateral equals 360°. Liz asks the students to work individually or in groups for a couple of minutes.

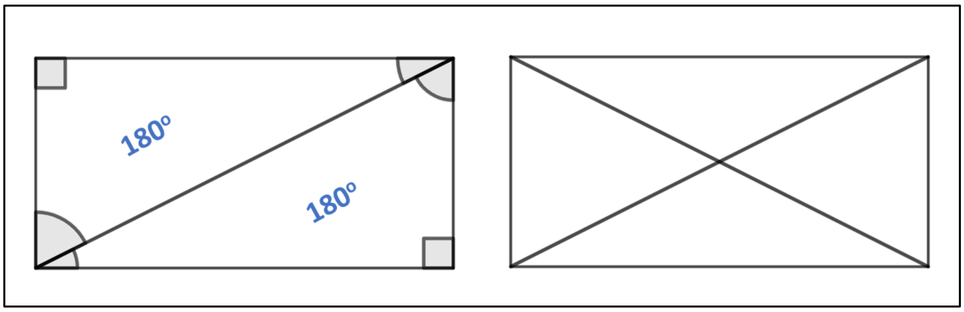


Figure 5.22 Recreation of Liz's drawing on the board from observation notes

While students work on the activity, one group of students have the following conversation with Liz:

Student 1:	On the whiteboard, when you said, why is it only 180 degrees in the triangle?
Liz:	Oh, there's 180 degrees in the triangle because.

Student 2: Of the equilateral triangle.

Liz: All the all the corners of the triangle, when you put them together, will make a straight line.

Student: But why?

Liz: We'll do that, we'll do that one day.

Student 1: Why is it, why is the straight line 180? [overlap].

Student 2: [inaudible]

Liz: Why is it three hundred and sixty? I think it's to do with Babylonian maths.

Student 1: Why? What is that?

Liz: So Babylonian, um, maths, many thousands of years ago used base 60. So, you know, we use base ten in our number system?

Student 1: Hmm.

Student 3: Yeah.

Liz: So it goes ten, one, ten, a hundred, a thousand. They used base 60.

Student 3: Will we do that?

Liz: Which is, and it's short of linked with why we've got 12 numbers around the clock.

Student 1: Sixty is a really bad number.

Liz: And they used 360 degrees. The 360 is, could have been any number.

Student 1: Exactly!

Student 3: Like sixty seconds in a.

Liz: Yes. Yes. Yeah. Yeah.

Student 1: But why, why is a degree a degree? [giggles] Why is it degree? Like the amount of it.

Liz: Why, why is it, the amount of.

Student 3: Because that's, that's just like the angle of the [overlapping]

Liz: Because they took, because they took a circ-, they took a full point and just divided it up into a number that they wanted to divide up into.

Student 1: But why?

Liz: It could have been four hundred. It could have been four hundred and fifty.

Student 1: Exactly!

Liz: But because of that, there's all these, all these facts that we can use. So, a straight line would be half of that.

Degrees is introduced to the students as a unit to measure angles in primary school. By Year 7, gradually the students are expected to be able to use tools to measure angles, estimate angles in degrees and calculate angles using learnt angle facts (Department for Education, 2013b). The students have not yet come across the number π, and thus are not familiar with radians. The question of the students is not particularly linked to this specific lesson, and it could have been triggered in different scenarios. A possible explanation as to why this unit seems particularly counterintuitive to the students is that in contrast to the other units used in mathematics (e.g., metres), significant angles, e.g., right angle and straight lines etc, are measured as multiples of 30 degrees (e.g., 90 and 180). Liz address this by pointing out that the origin of this unit is related with the sexagesimal number system. The reference to a clock might be an attempt to link the degrees with the unit of time that is based in similar principles.

As the discussion progresses, Liz's explanations to students' questions change. Progressively, she moves from narrating the basic idea behind the proof that a triangle has a sum of angles 180^o to history of mathematics and the Babylonians to give a reasoning for this seemingly arbitrary choice. At the end she closes the conversation talking about the arbitrary choice of the amount and the angle facts.

Liz and the students conclude that the choice was arbitrary. However, the metarules behind their agreement is different. Liz is familiar with different number systems as well as with different units of measuring angles (meta-discourse of measuring). On the other end, the students are only familiar with the decimal number system and only with degrees as a unit for measuring angles (students' precedent space). Liz and the students agree at the end of the discussion that the choice is a convention (intersubjective narrative). Although there is no indication about other units during this conversation, accepting that this choice can be changed might aid the students when they will come across a different unit for measuring angles, radians, in the future. Thus, the episode is coded as taken opportunity to go beyond the mathematics of the moment.

Another interesting point of this conversation is Liz's final remark:

Liz: But because of that, there's all these, all these facts that we can use. So, a straight line would be half of that [360°].

It is not obvious what her intentions of making this remark are. It might be to bring student's attention back to their task, or to highlight the usefulness of having a unit as a point of reference. The use of the word "fact" is common in the mathematical classroom discourse in England. The word refers to properties (e.g., for angles, lengths, numbers etc) that the students are expected to memorise and use without necessarily being proven. The word indicates that the property is 'a given'. However, the properties of the angles can be used despite of the arbitrary choice of unit, e.g., angles on a straight line add up to 180°, π, $1/2$ turn or $\tau/2$ ($\tau = 2\pi$), or 200^g. These different units are particularly useful to some things and less so to other. The choice of units depends on our communicational and practical needs. For example, there is a debate among mathematicians about the use of π and τ. Similarly using degrees or turn as a unit in everyday mathematics depends on our purposes, for instance 360 degree has a large number of divisors resulting to having angles that are measured in whole numbers (e.g., calculating 30°+60°+90°+180°=360° is easier to calculate in pen and paper than 1/12 turn+1/6turn+1/4turn=1turn, but, in verbal communications, we might use phrases such as 'a full turn' or 'turned 360' interchangeably). The use of different units affects only the way the "facts" are expressed but not the actual geometric properties.

In an interview that took place before the observation of the lesson Liz makes a remark about talking to a different group of students about Babylonian mathematics:

Liz: After a few times when, um, we were looking at the different bases of numbers, it was an investigation that I taught to a year eight group. And, um, we started talking about what our number system, the fact that it's base 10, and then we looked at what binary. Um, and then I started talking to them about Babylonian mathematics and I didn't, and this is what I was saying, I don't ever plan for this. So, I don't do enough research before I go in. But I remember during my degree that we looked at Babylonian mathematics and, uh, their number base system. And so, I end up talking to the pupils about it and I'm Googling at the same time. I'm trying to remember. Yeah. Okay. Yeah, that's right. I am telling the right thing.

It is not clear whether Liz's response was influence by her recent exploration of the different number systems as part of another lesson or whether that was a remark that she remembers from her undergraduate studies. The use of history of mathematics as a tool to convey meaning about mathematical conventions is talked about both by Liz and David. David, give's specific examples from the history of mathematics that he uses with his students, such as the history of negative numbers around the world, looking at and deciphering original manuscripts, the "cult of Pythagoras", the contribution of John Napier with logarithms "to speed up calculations" and finally the origins of trigonometry from India to Middle east, North Africa and "then it comes into Europe". Through the examples chosen by Liz and David, it is observed that both teachers recognise and try to present mathematics as a discipline that evolved through humans and culture.

In a nutshell, the significance of the episode presented here is the use of history of mathematics. Using history of mathematics in discussions beyond the mathematics of the moment appear to be a favourable teaching practice among the participants of the study (see also Chapter 6). Although the reference to Babylonian mathematics is only theorised by historians, as it is likely that more than one civilisations came to similar realisations independently, the information is often sited in online articles and blogs about history of mathematics[28]. Ultimately, the use of degrees is an agreement among humans that remains useful due to real life applications (e.g., astronomy) and due to the big number of divisors.

Overall, the theme *mathematical conventions* includes opportunities to discuss the reasons behind agreements and choices of the mathematical community that might otherwise seem out of place to students (e.g., notating and labelling, names and definitions of mathematical objects). The episodes identified under the umbrella of the theme depict identified opportunities to discuss the role of history and culture, and to portray mathematics as a communicational activity. When such opportunities are taken, the introduction to new mathematical conventions could take place as a community-building activity (e.g., creating shared working definition, or agreeing not to 'break the rules' of mathematics). Finally, it was also found that during those moments, the teacher rejects the temporarily role of the

[28] See for example, https://nrich.maths.org/6843

"ultimate substantiator" (Sfard, 2008, p. 234) to encourage students' explorative participation.

5.5. Applications of Mathematics in other disciplines and everyday life

Thus far, the themes discussed concern examples of internal consistency of mathematics (e.g., connections between mathematical ideas, maintaining consistency or following mathematical conventions). However, secondary school students are expected to learn about applications of mathematics in other disciplines. For example, the National curriculum for KS3 states:

> Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas. […] They should also apply their mathematical knowledge in science, geography, computing and other subjects.(Department for Education, 2013a)

The quotation suggests that making intra- and extra- mathematical connections are equally important from the perspective of curriculum developers. However, the programmes of study published in the National Curriculum do not include specific examples of applications that should be learnt as part of secondary education. Thus, the choice of examples is left primarily to the exam boards, the schools' mathematics departments and to individual teachers.

During the interviews the participants indicated examples from their personal and professional experiences that they see as having mathematical meaning. For example, Alex talked about dancing as sequence of steps, Damian about the use of graph theory in telecommunication networks, and Scott about performance analysis. Also, the participants gave examples indicating that they are aware of the applications of mathematics in the curricula of other subjects (e.g., science, geography) and in careers (e.g., economics, engineering, plumbing).

The final group of episodes I wish to discuss, constitute examples of identified opportunities for discussions about the applications of mathematics in other disciplines and in life. In comparison to previous themes, the majority of the episodes identified under the theme *Applications of Mathematics* include opportunities that were identified considering alternative courses of actions thus categorised either as potential of the discussion or potentials of an activity.

5.5.1. *"If you work in biology and science, you will see these things happening": Unsubstantiated claims*

The following discussion which takes place in Alex's classroom, showcases a potential of the discussion to touch upon applications of sequences in science which was not followed up by the teacher. The discussion occurs towards the end of the lesson A1 about the n^{th} term of arithmetic sequences, as part of the closing activity.

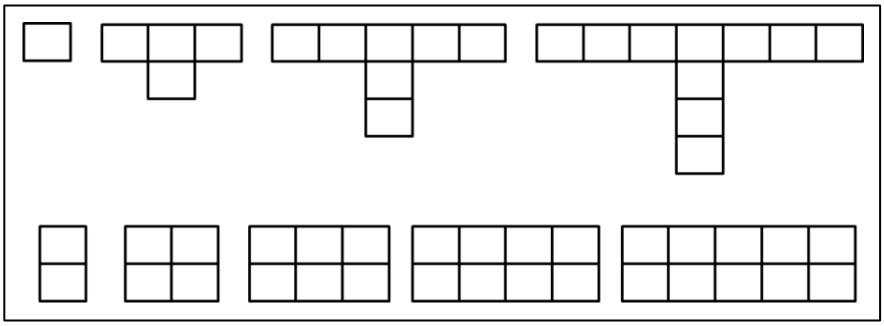


Figure 5.23 Snapshot from the slides Alex was showing to students during the dialogue

Alex: […] We have some shape's guys. Again, using the sequences, are used, [Student 1], to create different shapes or to explain how are these shapes created? If you work in biology and science, you will see these things happening, yeah? Or see it, how anything in the world grows. Yeah? They follow specific sequences. So, we're going to start with the basics, the ones we're doing today and then as we move throughout the week, see more difficult things. So, let's look at this one. Who can tell you what would the next shape looks like? [see Figure 5.23]

Student 2: I'd look bigger.

Alex: It looks bigger. What do you mean by that, [Student 2]?

Student 2: I would add another one on each side.

Alex: Very good a visa[29] for [Student 2]. Guys, that's what's gonna happen. Yeah? We're gonna add the next one. […]

The discussion continues with Alex and the students working out patterns in geometric objects shown on the board and confirming the next term of each sequence. Alex has plans

[29] Alex uses an award system with his students. Students get a "visa" every time they give a correct answer.

to give students some more examples to practice (Figure 5.24) but they run out of time at the end. The activities that Alex used come from the school's repository.

In the episode, Alex seems aware of applications of sequences "in biology and science". However, during the discussion between him and the students, Alex does not elaborate his claims. The visual mediators provided during the activity do not stimulate the discussion as they do not seem to provide an opportunity to discuss the patterns in the context of science.

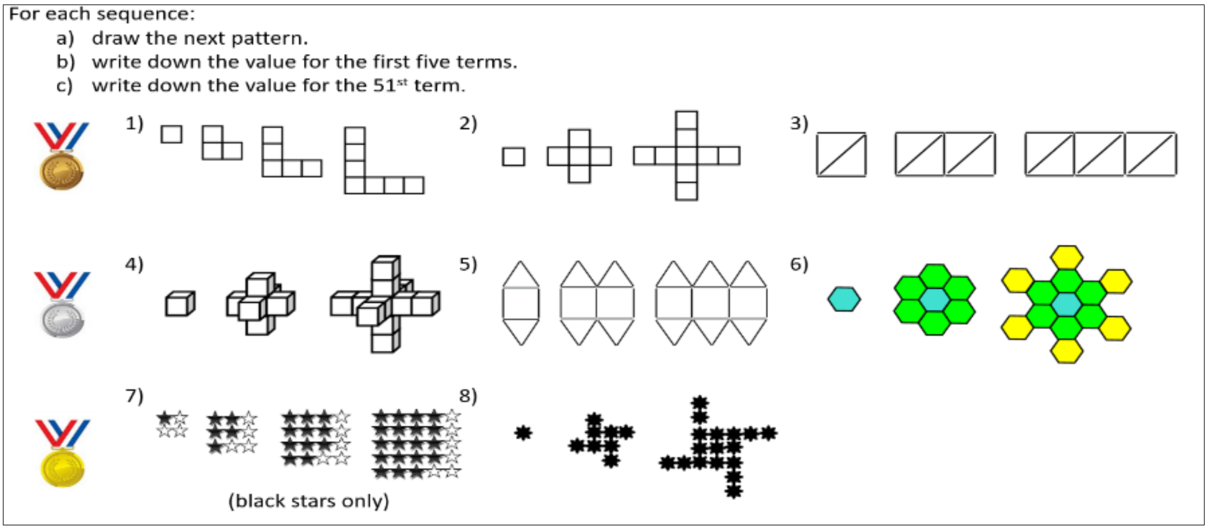


Figure 5.24 Snapshot from Alex's slides of the activity

In a nutshell, the episode was chosen because it illustrates instances where unsubstantiated statements avert opportunities to go beyond the mathematics of the moment from materialising. Here the unsubstantiated claim appears to be an attempt of the teacher to catch the attention of the students with no intention to discuss applications of mathematics.

5.5.2. *"Crumbled" axis: An opportunity to discuss manipulating graphs in statistics.*

The episode discussed in this section is an example of a potential opportunity to discuss everyday implications of mathematics as seen in school. The triggering event was a remark from Alex made during the whole class discussion on Time Graphs (A3):

Alex: Before I let you go on your worksheets, guys, I just want to remind you that we saw this random symbol last time [Alex draws on the whiteboard Figure 5.25]. As I was plotting my graph, on my scatter plot, if you remember, I had crumbled this information here [points at the zig-zag part on the y-axis]. We do this to signify that I don't really care about values from here downwards, I really want to, I really want to know everything that starts from, what is my lowest value? 20. What is my highest value? Oh, 45, oh, sorry 47. So, I know that I am gonna start from 20 all the way to 50. So, I don't really care about all that's going on down there because nothing is really going on down there. Yeah? Again, on the other side [points to x-axis], in this case I have years all the way from 1 to 10 so I don't have to crumble this bit here, [Student 1]. I just can go straight ahead. Usually once one thing is crumbled so is the other. But that depends on your work. Just I wanted to make you aware, guys, that this situation might appear, I don't think on your worksheets.

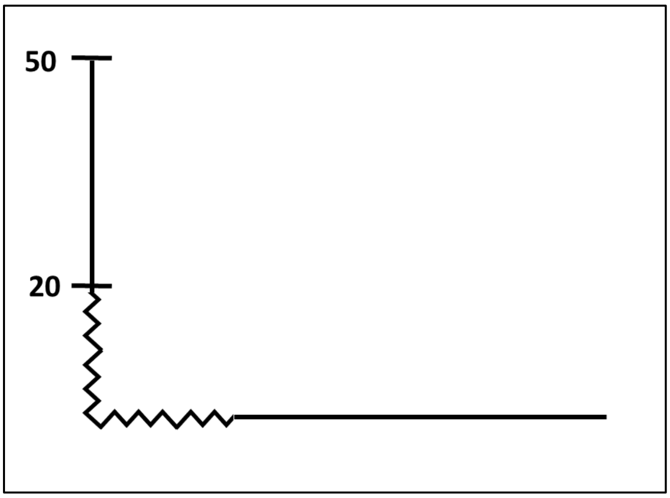


Figure 5.25 Crumbled axis. Recreation of Alex's drawing on the board.

The episode hints to an opportunity to discuss with the students how graphs such as the one in Figure 5.25 could be used to manipulate the publics interpretation of a graph and how one can overcome the misleading information[30] by focusing on the mathematics represented (potential of the discussion).

As part of the KS3 curriculum the students are expected to be able to interpret different types of graphs in statistics. The information that "crumbled" graphs might be used to influence one's interpretation will not be needed to the students for their exams and is not explicitly mentioned in the curriculum or the guidelines. Therefore, such discussions could cross the limits of the curriculum in a way that gives the students useful tools for their everyday life.

[30] Some real life examples of misleading graphs can be found here: https://www.statisticshowto.com/probability-and-statistics/descriptive-statistics/misleading-graphs/

The visual mediator brought for discussion by Alex is frequently used in media or marketing to influence people’s perceptions about the magnitude of the results of descriptive statistics.

In a nutshell, the discussion broaches a matter that goes beyond mathematics to social issues. Alex chooses to keep the conversation within the focus of the lesson and lets the students practice on the questions of the worksheet. Thus, episode does not offer further evidence to discuss the potential opportunity.

5.5.3. *“[She] is gonna show us some Origami”: Exploring geometric ideas in arts*

The following episode is an example of bringing for discussion the contribution of mathematics in art, specifically in origami. The episode takes place in Liz’s classroom during the lesson L6, and it showcases a spontaneous decision of the teacher after a conversation she has with a group of students during group work. When Liz passes by the group’s table, one of the students tells Liz that their classmate is going to teach the group how to make an origami turtle after the lesson. As a response, Liz asks the student if she would like to take the last 10 minutes of the lesson to show the whole class how to make this origami art and the student agrees.

Toward the end of the lesson, Liz addresses the class:

Liz: Um, right now though, [Student] is gonna show us some Origami.

Background: Wooh.

Liz: So, do you remember we made these [points at some fold and cut craft shapes decorating the wall]? Which day did we do this?

[Omitted discussion between Liz and the students negotiating the date]

Liz: So, we made all these really lovely shapes [points at the window, Figure 5.26] and, um, [Student] is gonna make, she's taken, she's got a book out and she's found out how to make a [pause] turtle, thing.

Liz gives her students pieces of A4 paper. Then, Liz and the students are gathered around the girl while the student explains the folds. The involvement of Liz in the following episode is limited, mainly keeping the students in order, repeating the instructions when needed and supporting individual students.

Liz’s decision is not necessarily unusual, especially with this group of students as indicated in her interview. She and her students used folding techniques to create symmetrical cuttings on pieces of paper that resemble snowflakes (see Figure 5.26). The walls of the classroom are decorated with this and more mathematical art she and her students have made throughout the year. The creations could potentially serve as visual mediators in communicating the use of mathematics in creative activities (see also Section 5.2.4). In addition, creating the art engages the student in non-typical mathematical routines, for example here following a series of folds and/or cuts to create a shape.

Figure 5.26 Students crafts decorated in Liz's classroom

The art of origami is studied mathematically, and the patterns created by the folds are used in many applications (e.g., in engineering). While for students this might be a purely creative activity for Liz it has mathematical importance as it provides opportunities to learn. Using the stages of creating the “turtle” as visual mediators, Liz and the students endorse narratives about the folds using mathematical terminology and justification. One opportunity for learning can be observed in the following discussion with one of the students:

Liz: Have you made a square [Student]?

Student:	You need to, um, I don't know how, how to measure this. Is that a square?
Liz:	So, if I do this [folds the paper so the short edge lays on top of the long one].
Student:	Yeah.
Liz:	And I open it up. What kind of shape have I got here?
Student:	What?
Liz:	A square.
Student:	How many lines have you got?
Liz:	No, I didn't. I just folded the triangle over. This edge [long side]. This, to make a square, I need this le-, edge to be the same as this one here.

Although Liz did not plan the origami activity as part of her lesson, her use of mathematical vocabulary (e.g., "square", "line") indicates that she recognises the activity as mathematical. In addition, the quick change of choice "le-[ngth], edge" might be an indication of intersubjectivity, aiming to communicate with the student using a word that has double meaning (i.e., edge of shape, edge of paper). Liz takes the opportunity to discuss with the student how to apply geometry to create a square from a rectangular piece of paper. It is unclear from this conversation whether the student accepted Liz's proposal or completed the task as a ritual and continued creating the origami.

In a nutshell, the episode exemplifies an opportunity to discuss an application of mathematics in disciplines that are often overlooked, for example, here, applications in arts and design. Liz's actions, although spontaneous, maintain focus on mathematics attempting to communicate to students the ideas behind the folds and symmetry. Moreover, the episode adds to research on the use of origami in mathematics education (e.g., Arıcı & Aslan-Tutak, 2015; Cipoletti & Wilson, 2004) by providing an example that presented an opportunity to use mathematical vocabulary to talk about tangible objects, as opposed to using the object to build mathematical vocabulary and spatial visualisation.

5.5.4. *"Are we pretending to have the coronavirus now?": Preparing students for the unknown*

The episode described below takes place at the end of the last of Liz's lessons I observed (L6). The discussion is about what they are going to do during the following lesson which is going to take place in the library. Liz introduces the activity to the students to avoid doing so while they are spread around on the computers in the library:

Liz:	Right, I promised you that we would be doing something really fun, next lesson. So, you know how, um, every time I come into a classroom there's a group of students to tell me that they've got the coronavirus.
Student 1:	Are we pretending to have the coronavirus now?
Liz:	We're gonna do a pro-, a little mini project next lesson on, um, containing the virus.

Liz broadcast a video on the interactive whiteboard with snapshots of a game and then she explains the students what they are going to work on during the following lesson. The game is called ‘Outbreak’[31] and pre-existed the pandemic.

It is not specified whether playing this game as part of a mathematics lesson is Liz’s choice, or a strategy of the school to prepare the students for the pandemic. The episode, however, takes place in mid-February 2020, before the first national lockdown. Looking at this episode retrospectively the game included aspects that were soon to be parts of students’ everyday life. The three actions of the game required the students to create an antidote, triangulate the position of infected people and propose a strategy for vaccinating people with different types of vaccines.

The activity could potentially bring for discussion social issues that might be intertwined in real life problem solving. One such opportunity is taken in the lesson I observed. During the activity the students are expected to find the best strategy of using resources to vaccinate people. Liz and the students discuss who should be prioritised in vaccinations.

Liz:	Um, because there's a cost to vaccinating the people. Um, so there's a cost of vaccinating the people, there will be two vaccines as well, I believe one will be really good work and the other one is not going to be so great. Um, but you need to decide who you're going to vaccinate. So who, who do, who do you vaccinate when we?
Background:	The young people.
Background:	Yeah.

[31] After the outbreak of the Covid19 pandemic the developers of the game released a statement that the game should not be used as source of information for Covid-19

Background:	Yeah.
Liz:	Because, one of my favourite ones is gonna be the, the block that says would you vaccinate your teacher? [laughs].
Background:	Woh!
Background:	No.
Background:	No.
Background:	No
Liz:	[Student 1], who would you vaccinate then?
Student 1:	Um, if there is like two people within, in Britain, I'll vaccinate because then they won't spread anymore.
Liz:	Auh, the people that you found. Who asked who, how do we tend to vaccinate first?
Background:	Youths.
Background:	No, elderly people.
Background:	Kids.
Liz:	Well, I know what [Student 2]'s mom does. [Student 2], who would you vaccinate?
Student 2:	I don't know.
background:	[laughing, mumbling].
Background:	Your mothers!
Liz:	Who are the people that treat the ill people? Doctors.
Background:	Will you vaccinate the doctors?
Background:	Oh yeah.
Liz:	You don't have to. It's your decision who you vaccinate.
Student 3:	[inaudible/overlapping] vaccinate the queen and stuff like that, and the more important people.
Liz:	Auh, and politicians and people like that. Well, anyway.

Understanding the context of the problem, which is probably very new to them, is essential for solving the problem. During this conversation there is limited use of words with mathematical meaning compared to other parts of the lesson. At this time Liz and the students discuss social issues and the role of mathematics in solving problems created in a social context:

Liz:	There's loads of maths in it. Who asked me about the maths? So, there's coordinates, there's angles and bearings.
Background:	Angles?
Liz:	Yeah, and when you make the antidotes, we will be looking at fractions and ratios and it's problem-solving.

The students seemed startled when Liz tells them what mathematical topics are involved in playing the game. Unfortunately, I was not present in the lesson when students play the game. The limited data collected as part of the pre-lesson discussion, however, indicate that with appropriate mediation, the activity could provide opportunities for Liz and the students to discuss how mathematics is used in fighting an outbreak.

In a nutshell, the episode exemplifies the existence of opportunities in the mathematics classroom to proactively engage in discussions about social matters. Using mathematics that the students have already encountered (e.g., ratios, coordinates etc), students can expand their precedent space (Lavie et al., 2019) in ways that enable non-routine problem-solving in social context. For example, here, budgeting the rollout of the vaccine effects and prioritising social groups.

Overall, the theme *applications* includes opportunities for discussing the applications of mathematics in other disciplines, careers or everyday activities. In contrast to the other three themes identified, here, the focus is not on a specific mathematical idea or practice and its use. Rather, the focus of potential discussions about how branches of mathematics are used to various activities (e.g., geometry in creating art, statistics in media or algebra in biology, science, and medicine). Despite limited evidence of teachers noticing and addressing opportunities to discuss applications of mathematics, the theme is an integral part of going beyond the mathematics of the moment as the students might come across such ideas as part of their curriculum in other subjects (e.g., biology or physics) or during their life beyond secondary education (e.g., at university, career and personal life).

5.6. Summary of findings

In this chapter, I explore taken and potential opportunities for teachers and students to communicate beyond the mathematics of the moment emerging in secondary school mathematics classrooms. My analysis of interview data and classroom observations focuses on identifying specific moments during lessons, i.e., episodes, where the activities and/or discussions between a teacher and the students could initiate conversations beyond the topic of the day.

The findings indicate that opportunities identified in the data fall into four themes:

- The first theme includes discussions about **ideas and practices that run across the mathematics curriculum**. This type of discussions aims at communicating central mathematical ideas and practices included in the curriculum through examples from what the students are learning at the time. Specifically, these discussions contain opportunities to make metalevel comments about ideas students are expected to come across several times by the end of their secondary school education (e.g. problem solving) which the students might not realize as the same, and to connect ideas of what the students might have already come across in the past and what they may see again in the future. For example, Liz and her students negotiate the metarules of the routines of calculating and measuring (see Section 5.2.1) and Alex tries to show his students a technique to construct a formula rather than memorising it (see Section 5.2.2).
- The second theme includes discussions about **ideas and practices that the students might come across in later stages of their education and/or through activities of public engagement with mathematics** (e.g., science festivals, open days, videos etc). During these discussions the communication is focused on connecting the topic of the day with topics or practices that might be part of the students' education in the future or encountered in their free time. For example, later in the next Key Stages (e.g., Section 5.3.2), in a Mathematics course at University (e.g., Section 5.3.3) or outside the classroom (e.g., Section 5.3.4).
- The third theme includes discussions about **mathematical conventions**. This type of discussions aims at communicating mathematical warrants through examples from what the students are learning at the time, for instance, aiming to give reasoning behind the definitions and naming of mathematical objects which might seem

arbitrary to students. In contrast to discussions about ideas and practices that run across the curriculum, discussions about mathematical conventions could extend to ideas and practices not included in the National Curriculum (e.g., defining). The analysis indicated examples where teachers use episodes from the history of Mathematics (e.g., Section 5.4.5), links between the naming of an object and root words (e.g., Section 5.4.2) and constructing working definitions (e.g., Sections 5.4.4 and 5.4.3).

- The final theme includes discussions about **applications of mathematics**. These discussions focus on demonstrating the use of mathematics in other school subjects (e.g., Section 5.5.1), in other disciplines and professions (e.g., Section 5.5.4) or in students' everyday activities (e.g., Section 5.5.3). In contrast to the previous themes which focus on understanding and appreciating mathematics as a discipline, this theme concerns opportunities for making extra-mathematical connections. Here, the focus is on discussing about mathematics with students and how they 'work' in the background of other activities e.g., in technology, science or in crafts, as opposed to discussing the mathematical activities as abstract stand-alone tasks. The episodes identified and discussed under this theme often depict potential opportunities identified to make connections with applications of mathematics beyond what is predicated by the curriculum. However, the findings highlight the importance of discussions beyond the mathematics of the moment in balancing inconsistences in instructions about real-world applications in curricular documentations (Smith & Morgan, 2016).

The opportunities are presented in both planned and spontaneous actions of the teachers. Thus, discussions beyond the mathematics of the moment could be both part of daily practice or in response to moments of contingency. Moreover, some episodes include multiple opportunities to discuss different aspects of the same idea which are not always noticed or addressed (e.g., Section 5.3.1).

The episodes were further categorised into taken opportunities, potential of the discussion, and potential of the activity. Studying both taken and not taken opportunities (i.e., potentials of the discussion and potentials of the activities) aids in identifying key characteristics associated with discussions beyond the mathematics of the moment. Specifically, the characteristics include: the role of intersubjectivity, and the use of gestures. In taken opportunities, the teachers attempt to help the students build routines, work and communicate mathematically beyond curriculum requirements. However, taking an opportunity to go beyond the mathematics of the moment does not necessarily benefit students' learning and experience with mathematics (e.g., Sections 5.2.2 and 5.3.4). Teachers and students often communicate using words, visual mediators, narratives and routines that bear different meanings from the perspectives of the teacher and the students. In the episodes where intersubjectivity is achieved, taken opportunities to go beyond the mathematics of the moment seem to result in effective communication. Moreover, gestures were used to communicate about mathematical objects and practices that are peripheral to the topic of the day and perhaps unfamiliar to the students (see Section 5.2.5). According to Sfard (2009), "words and gestures remain in a symbiotic relation and, in many situations, act as a "backup" one for the other" (p. 199). Therefore, gestures can also be used to achieve intersubjectivity.

The study of taken and potential opportunities to go beyond the mathematics of the moment indicates the engagement in explorative teaching practices. For example, in Section 5.2.2, Alex deviates from the preferred teaching method of the school for a particular topic, or in Sections 5.4.3 and 5.4.4, Liz challenges her role as the "ultimate substantiator" (Sfard, 2008, p. 234) in her classroom. The role of explorative pedagogies will be further discussed in following chapters.

Finally, potential opportunities can be found across all four themes, in some cases the opportunities are not mentioned by the teacher (e.g., Section 5.5.2) while in other cases they are acknowledged and dismissed in favour of attending to other aspects of teaching (e.g., Section 5.4.1). Feeding forward, I would like to explore what informs teachers decisions to go beyond the mathematics of the moment. Specifically, whether there are patterns in teachers' mathematical and pedagogical discourses that are associated with noticing and following up on opportunities to go beyond the mathematics of the moment.

6 Exploring teachers' discourses at the Mathematical Horizon

In this chapter, I continue my exploration of the characteristics of discussions beyond the mathematics of the moment (RQ 1). Here, I wish to further explore the experiences which the participants reported to have influenced their individual discourses at the Mathematical Horizon. In particular, this chapter addresses the sub-question:

> **1c.** What experiences shape individual discourses at the Mathematical Horizon?

In Section 6.1, I elaborate how I approach the analysis of the interview data in relation to the above research question. Section 6.2 presents examples of the views of participants regarding how their experience with mathematics and other disciplines in tertiary education informs their teaching. In Section 6.3, I discuss the analysis of excerpts from three mathematics graduates to illustrate the variations in their discourses beyond the mathematics of the moment. Section 6.4 highlights the influence of professional experiences in the development of individual discourses at the Mathematical Horizon. In Section 6.5, I discuss the role of reading books and sourcing information on the internet. Section 6.6 presents influences within the school environment such as interactions with colleagues and collaborations. In Section 6.7, I focus on narratives about the curriculum and in Section 6.8 on narratives about the students in relation to discourses at the Mathematical Horizon. Finally, Section 6.9 summarises the key findings of the chapter.

6.1. Synopsis of the data analysis

During the interviews the participants mentioned various factors that inform patterns of communication beyond the mathematics of the moment. Figure 6.1, also presented in Section 4.6.3, depicts the three factors (teachers, students and external) and their sub-themes. The initial steps of the analysis of the interview data indicated that the factors mentioned interact with each other to inform teachers narratives beyond the mathematics of the moment. However, the factors do not influence teachers' narratives in a unique way. For instance, participants who may agree on the same factors influencing their practices beyond the mathematics of the moment, but not agree as to whether these factors are limiting or enabling. Therefore, to move beyond the initial observations, I conducted a discursive analysis of the excerpts from the interview to identify similarities and differences between the participants mathematical and pedagogical discourses which could account for the inconsistences found in the preliminary analysis of the factors.

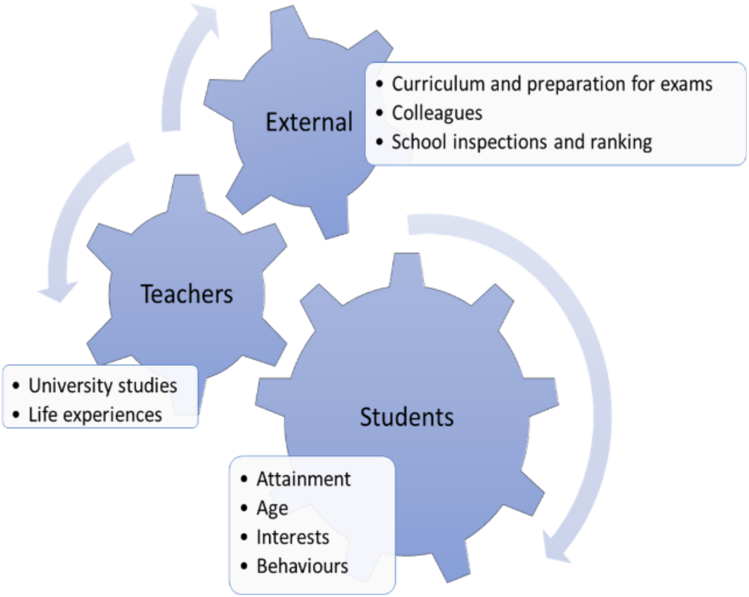


Figure 6.1 Factors reported to influence teachers' decisions of going beyond the mathematics of the moment

During the stage of the discursive analysis, the mathematical narratives were analysed in relation to Disciplinary Discourses (e.g., Mathematics, Economics, etc) and colloquial. While the pedagogical narratives where analysed in relation to their alignment with parts of the spectrum of Pedagogical Discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022). Specifically, narratives which present more traditional teacher centred values are positioned towards the Delivery end of the spectrum while narratives which support students explorative learning towards the Exploration end. The majority of the participants expressed both narratives that are in line with Exploration Pedagogical Discourses and narratives that are in line with Delivery Discourses.

The analysis suggests that the inconsistencies in the narratives across participants can be due to variations in the participants' mathematical and pedagogical narratives which can be attributed to their educational, personal, and professional experiences in relation to discussions beyond the mathematics of the moment. In the following sections, I present the findings of the discursive analysis and discuss the variations observed between the participants starting from influences of their university studies.

6.2. Influences of a teacher's university studies in teaching mathematics

The section presents the views of the participants regarding how various experiences in tertiary education inform teaching. The perceived influences of the university studies mentioned by the participants vary depending on the discipline of study and can be linked both to object-level learning and meta-level learning at university level. The emerging findings draw attention to the diversity in the mathematical backgrounds of secondary school mathematics teachers.

In particular, views about the influences of having studied pure mathematics at university level were found to be linked to object level learning and meta-level learning that was enabled due to the course of study. Specifically, an emerging view which was attributed to object level learning of mathematics at university level – i.e., the enrichment of teachers' discourse about a specific topic they teach with advanced vocabulary, notation, narratives and routines – is having a potential to identify topics or relationships between topics and locate how ideas evolve within and beyond the National Curriculum. Some of the teachers mention this skill directly, such as Liz:

Liz: [S]ometimes what I've already been talking about is when would you study this thing? And I think that's all I'm saying is that, you know, actually you probably wouldn't do that until you were at degree. Um, whereas if you wanted to explore this a bit more, you should do an A-level in it. Um, but that's all I'm doing is I'm picking where my learning happened and then trying to tell them [her students] where that's what, where they would learn the next thing.

Identifying where "learning happened" indicates an action of 'looking ahead' at what the students might come across in the near (A-level) or distant (university) future. This action relates to the theme *what comes next* discussed in Section 5.3. Thus, Liz's narrative is considered of interest to the study. In other cases, teachers give specific examples that indicate such potentials, such as seeing the relevance of advanced group theory discussed in Section 6.3. These narratives are associated with object-level learning at university level and indicates that the teachers notice how curriculum topics, "in either the same form, depending on the module, or in a diluted form" (David), are linked with topics seen in their university course.

Views associated with meta-level learning at university level, i.e., changes in the metarules of the teacher's mathematics discourse, include narratives about mathematics as a discipline, its structure and the engagement in the community of mathematics discourse. For example, Alex describes how through his studies he gained an awareness of working mathematically which helps him in managing expectations of others:

Alex: So, having confidence, because sometimes as a teacher, you are, well, at least the students see you as a person that knows all the answers and then you get stressed and you're like, oh, no, maybe I should know this, oh, what am I going to do? No, now, I don't remember. But I think through university and now through these years of teaching, I think it is very important for us, the teachers, to understand and project to students the following. We are not the answering machines. We cannot solve this any faster than you can. We just have a bit more experience. And this is what I tend to highlight so many times, whenever a, a situation is given for me to show the students that, that no one is expecting from you, as no one is expecting from your maths teacher, to know this by heart. No one is expecting from you to do this in seconds. Let's take a piece of paper. Let's read the question. Let's read it again. Let's try a small example. Let's draw on the side. Let's mess it up again. Oh, look, we missed this or we made a mistake. Let's fix it.

In this quotation, Alex describes problem solving as an activity that people engage with, rather than professing a skill. He attributes his understanding of working mathematically to his engagement with mathematics at university level which influenced his perception of his role as a teacher. He rejects 'traditional' pedagogical maxims, indicative of Delivery Pedagogical Discourse (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022), such as teachers as a source of knowledge or the focus on performance.

Similar views can be observed in the narratives of Damian, David, Liz, and Thomas. A common theme across the five teachers is that they associate their views about mathematics with their roles as teachers. The views appear to have developed through their engagement with teaching and not solely during or due to their studies.

The findings discussed so far was specifically about studying for a degree in Mathematics. However, 7 out of the 11 participants mention instances which illustrate potential influences of having studied other subjects before their initial teacher education course. For example, Damian describes how he initially viewed his university mathematics discourse as insufficient for teaching during his ITT in comparison to out-of-field prospective mathematics teachers who had gone through a SKE course involving “more pedagogical thinking”:

Damian: [W]hen I started teaching, I almost had the opposite of a chip on my shoulder about my mathematical kind of qualifications. In the, I trained, when I did my training year, I was with a couple of, a few people who hadn't done maths, undergraduate maths, they've done A-level, but, and stopped their maths studies, so they'd had to do a subject knowledge enhancement course before starting the teacher training. And part of that involved a bit more pedagogical thinking, thinking about people learning maths rather than just doing the maths, and when in, you know, they seemed to be miles ahead of me in the classroom. […] I was a very pure mathematician, so some something [laughs]. So that must’ve shaped the, what I can offer, I guess.

Both Damian and Thomas highlight pedagogical influences that teachers with degrees in other subject have. Specifically, sensibilities regarding learning mathematics and difficulties of the students. In relation to the mathematical content, Damian, Marcus and Thomas mention that teachers with degrees in other disciplines could talk with their students about examples of applications of mathematics. However, only Marcus gives a specific example related to teaching mechanics to substantiate the claims:

Marcus: We've got, this year it's a physics teacher come and teach maths, he wants to teach the maths. So, he teaches some of the Year 12 maths with us. And, I'd, I'd argue, I think that's, the bit he's finding trickiest, is just seeing something and then being able to go 'well, actually, I could ask this question in a slightly different way' that's been missing. And I think that's what he's struggling with. He's, he's able to be quite formal lang with it, and know exactly what he wants to ask because he's struggling with that. Um, now, he's a physicist, so when it comes to the mechanics side, he's been very good. I think he's looking forward to that, cause he'll probably have to bring so many more connections that I can't bring in because he knows all these different things which relate to mechanics. So that's where I could be probably not quite as good of mechanics because I can apply to the different exam-style questions. I know different things, but beyond that, I don't really.

The excerpt highlights an instance where the mathematical background of a teacher affects their expertise in specific topics of the curriculum and how this expertise could be used to enhance the students' experiences with the subject. The observation is also supported by the statements of participants who hold degrees in other subjects. For example, Nick mentions choosing examples from science and physics when teaching the topic of 'rearranging formulae' and Scott mentions specific examples from sports science when he talks about teaching statistics. Eric also gives an example where he finds his university studies relevant to his teaching of calculus, but he is more reluctant when asked whether he finds the mathematics he studied during his undergraduate in Economics relevant to his teaching:

Eric: I think the only time I've ever done that [using examples applied in economics] is when I'm teaching A-level and I'm doing, um, calculus. And I'm doing some maximising minimum- mising problems. And students ask me, and I will then say, for example, you can have a profit function and if you differentiate that to maximise the profit or whatever. So, there's a very few occasions that I do, and I think that's one of the only ones. But I think that one might be a weakness in my ha-, my part, because I haven't searched those connections enough. Um, and it was a long time ago. You know, like my, I studied economics 15 years ago.

Eric reflects on this as a weakness on his part potentially considering that there might be more links between his undergraduate studies and the mathematics he is teaching. In contrast to Nick and Scott who embraced their expertise in other subjects, Eric distances himself from his studies and his identity as a holder of economics major.

The examples given by the three teachers are related to specific topics thus to object level learning at university level which in this case includes applications of mathematics in the discipline of study. In contrast to mathematics majors, Eric, Nick and Scott do not explicitly express views that could be linked to meta-level learning due to their studies. However, their views about mathematics as a discipline, its structure and the forms of engagement include narratives which link back to their discipline of study and their experience teaching. Studying the discourses of the three teachers about mathematics and pedagogy during the interview two patterns can be observed. The first pattern regards their specificity in providing examples of applications of mathematics. In contrast, the second pattern regards the lack of specificity in providing examples of connections between abstract mathematical ideas or practices. When discussing abstract mathematical ideas and practices during the interviews, Eric, Nick and Scott put forward generic narratives which are often accompanied by pedagogical language. To illustrate the two patterns, I draw examples from Scott's interview. Table 6.1 presents part of the analysis of Scott's interview to illustrate the specificity of the examples about applications of statistics in sports.

Table 6.1 Excerpt from the interview with Scott

Utterances		*Descriptors*
Scott:	[…] I'm always sort of like if I'm talking about maths in context, my first go to is sports because there's so much maths in sports. And I was talking to one of my groups earlier this week about it.	
	So, they're doing averages and data, for example, and so we're looking at data, how it's organised and represented in tables and charts, <u>blah, blah, blah.</u>	*Identifying the mathematical topic*
	And then, one of the things I said to my year 9 set 2 group was you're doing a form of **performance analysis**. Here's some data and information that's relative to sport. OK, where would you see that in real life?	*Introducing the context*
	We talked about it. Tennis, first a percentage, **aces**, **serve- with forehand**, **serves with the backhand**. Football passes, **number of passes**. All these data that's been used and collected and analysed.	*Providing examples within the context*
	Well, you don't see it like how we do it on the board. But, <u>because of technology</u> and, there is <u>different</u>, you know, <u>means for it</u>, <u>but maths is there</u> all the time and that's just performance analysis and looking at data. What does the data mean? So, it goes on in front of you all the time,	*Handling calculations with technology*

that's what I was, um, convey to the students, about how it's prevalent in real life.	
And then if I wanted to, I could always talk about, for example, sport as in **minutes played**. Or let's consider sport when it's played. So, **football pitch**, **a formation**, **tactics**, **direction of movement**, which occurs in all the **invasion games**, angles, **angling your run**, force, power when you want to jump ahead of the ball. Basketball, pass the ball is so maths.	*Providing examples within the context*

<u>underlined</u>[32]: averting the discussion about specific mathematical ideas and practices
***bold**: vocabulary from Sports Science*

Scott provides various examples of applications of mathematics in sports without having to pause and recall, which is indicated also by the lack of filler words in the excerpt. He brings vocabulary from the Sports Science Discourse (e.g., "performance analysis") and confidently integrates talk about his teaching. Regarding the lack of specificity in providing examples of connections between abstract mathematical ideas and practices, he averts from being specific about the mathematics involved either by interrupting his description of the topic being taught using the idiom "blah, blah, blah" or by stating that the calculations he is teaching are handled by technology. Scott concludes:

Scott: I could spend, God knows how long it would take, talking about sport and I could talk about it in the maths context because I understand how the maths works, because after having taught and played and I can see that the maths is from a playing field perspective, to a player perspective to a, if you want to talk about force and power, if you want to talk about, you know, the physics of it and the mechanics of it. So, it would be easy for me because obviously I've done it, play that and seen it.

Scott utterance "I know how maths works" is interpreted as a comment that describes his view about mathematics as a discipline. He links his claim with his experience teaching (e.g., "after having taught") but also through his engagement with sports (e.g., "I've done it, play that and seen it"). However, Scott does not elaborate his claim. Similarly, when he is asked

[32] I use **bold** and <u>underlined</u> notation to highlight key elements of the dialogue. The notation a simplification of the coding used in the analysis. Thus, the use might differ in each case and is stated in text under the table.

about teaching the connections between Algebra and Geometry, his example lacks specificity and coherence:

Scott: That was, some do, I mean, some, some students do [understand the connection between Algebra and Geometry]. Could be, um, maybe as they get more <u>emotionally intelligent</u> as they get older, up through the years. […] And it depends on the curriculums. So, for example, Geometry and Algebra being **fused** together. You can leave that step for into a year 7, year 8 head, but the-, not really going to be introduced to it fully until they get to GCSE. That concept. If it's a <u>top set</u> group, they might get that already. You, they might already be doing that sin-, in key stage 3. You could introduce them to a concept of geometry and algebra. Or even just a simple term and how it works. That's <u>mastery</u>.

Scott's response here is less specific in comparison to the earlier one (Table 6.1). He does not talk about specific examples he might use in his lessons. He uses the word "fused" (bold in excerpt) to refer to connections between Geometry and Algebra. The use of the word "fuse" in conjunction with the lack of specific examples might indicate that connections are views as superposing of two curriculum topic areas. Moreover, his pedagogical discourse is very dominant in this quotation (underlined in excerpt). He refers to "mastery" (underlined in excerpt), referring to the Mastery Framework (NCETM, 2017), and "emotional inteligen[ce]" related to age and attainment to support his statement about when students are introduced to connections between Algebra and Geometry included in the curriculum.

I regard the evidence from Eric, Nick and Scott as examples of the diversity in discourses that is introduced as mathematics teachers enter their careers through different routes. The final observations, however, should not be regarded as a characteristic unique to non-mathematics graduates. Similar observations are made in Violet's case, who has a degree in mathematics. Specifically, Violet's narratives about connections between topics lacked specificity and were justified using pedagogical language associated with the Discourse of the National Curriculum (see also Section 6.7).

Overall, this section discusses the views of participants regarding how their experience with mathematics and other disciplines in university education informs their teaching. The emerging findings suggest that the influences of teachers' university education mentioned by the participants link to object-level and meta-level learning at university. Influences of object-level university learning, i.e., the enrichment of teachers' discourse about a specific topic they teach with advanced and/or applied vocabulary, notation, narratives and routines, are in line with Zazkis and Mamolo (2011) remarks about Knowledge at the Mathematical Horizon as an application of Advanced Mathematical Knowledge (Zazkis & Leikin, 2010). However, as discussed in Chapter 3, evidence from previous studies suggest that such applications are limited mostly to discussions with gifted students or during maths clubs (Yan et al., 2021). Views about the influences of meta-level learning about mathematics at university level in teaching, i.e., changes in the metarules of the teacher's mathematics discourse including narratives about mathematics as a discipline, its structure and the engagement in the community of mathematics discourse, echo the descriptions of Horizon Content Knowledge by Ball and Bass (2009) and Jakobsen et al. (2012) which adopt an holistic view of mathematics as a discipline. The findings draw attention to the diversity in the mathematical backgrounds of secondary school mathematics teachers. Specifically, how the university studies might inform – but not in a systematic way (see Section 6.3)– the ideas, practices, and applications beyond the mathematics of the moment which are noticed and discussed with the students. The observed diversity in the mathematical discourses of the participants is consistent with findings from research in university mathematics education. Specifically, the literature regarding the representations of mathematical content in different university courses (e.g., Hitier & González-Martín, 2022) and the role of advanced mathematical content in initial teacher education (e.g., Büdenbender-Kuklinski et al., 2022; Pinto & Cooper, 2022)

6.3. Bringing experiences with advanced mathematical content in teaching

The previous section discussed the views of the participants regarding the relevance of their degree to their teaching. The findings suggest that all the participants recognise some links between their practice and their undergraduate studies. The comparison between the discourses of mathematics and non-mathematics graduates indicates differences on what the teachers value as relevant to their teaching. However, differences can also be observed among mathematics graduates. Here, I present the analysis of excerpts from three mathematics graduates talking about related advanced mathematical topics (Group Theory and Galois Theory). The following examples showcase ways that mathematics graduates could bring their educational experiences with abstract mathematical content to their teaching practices. The examples are taken from the interviews of Alex, David and Damian and show how their decisions are connected to their learning of advanced mathematical topics but vary according to their views about mathematics and pedagogy.

David and Damian mention the introduction of Group Theory being part of the A-level curriculum and both recognise and reflect on elements of Group Theory and its applications from their university studies. They also indicate that they have discussed these ideas with their A-Level students in the past. Damian also talks about how he used to introduce elements of Group theory to Year 6 students on their visits to his secondary school when he worked as a teacher. Table 6.2 and Table 6.3 present parts of the analysis of the interviews with David and Damian respectively.

Table 6.2 Excerpt from the interview with David

Utterances		*Descriptors*
David:	I mean, even group theory, you know, that, it's not on the syllabus as we offer the students now, but up until a couple of years at A-level you would always finish further maths with an introduction to group theory	*Identification of topic*
	and it was really cool, um, yeah.	*Appreciation of topic*
	And then at uni there was lots of **numbers**, **groups** and **codes** and things, so, yeah.	*Link to own university learning*
	[…]	
	So, going back to group theory, oh this is great, um, we are only you gonna dip our toe in it and you would really like, it's very doable and fun,	*Appreciation of topic*
	but, yeah, you will look at this much further in, you know.	*Link to university content*
	One of the things I remeber from group theory was, um, some guy did a lecture called **the longest proof** in Maths and it's 10000 pages, and it was within group theory.	*Link to own university learning*
	So, I sort of reference that, no, no more details. um, but talk about what might lie ahead.	*Link with teaching practice*
bold: Keywords linking to advanced mathematical discourse		

David's and Damian's discourses about advanced mathematics seem to be a factor in recognising relevance of the more advanced ideas to what they teach. In both cases, they start their account by identifying the curriculum topic followed by a reflection to their own university learning (i.e., link to own university learning) and projecting it to the future learning of their students (i.e., link to students' university learning). However, the integration of their advanced mathematical discourses in their teaching practices differs. The differences can be linked to their interests and their participation in the community of advanced mathematical discourse.

Table 6.3 Excerpt from the interview with Damian

Utterances	*Descriptors*
Damian: I could in in A-Level there's hints of group theory.	*Identification of topic*
So you could kind of you could kind of talk a little bit about **Galois Theory** and how it is connected with, um, kinds of **symmetries** and **solutions of equations**, um, and and sort of, yeah.	*Link to own university learning*
So, I've, so, hopefully you're giving them these hints as to how how things are interconnected at a later stage.	*Link to university content*
[…]	
I mean, another thing that I've done, which is which was quite interesting, um, was, you know, do kind of fun lessons for year 6 is, when they come up to year 7, uh, to kind of see. So, changing from primary school to secondary school to see what secondary school maths is like,	*Setting the context*
um, is you do a session on, um, on group theory, but just kind of doing symmetries, um, **symmetries** of, **of a square**. Um, and so you introduce them to basic group theory and tell them, yeah.	*Identification of topic*
I mean, I remember having these groups of 10, 10 year olds, I suppose, you know, 10 year old sort of chant- chanting **isomorphism** [laughs] which is [laughing, inaudible].	*Link to university discourse*
Um, so, you know, you can show me that two two two different sets of of objects, you know, symmetries of the square, and another group, the other group, oh, oh **isomorphic**.	*Link to university discourse*

bold: Keywords linking to advanced mathematical discourse

Specifically, David consistently mentions throughout his interview the use of historical elements as one of his teaching practices and he is very passionate about it. Here, he mentions sharing with his student the fact that one of the theorems in Group Theory is known to have the longest proof in mathematics. His enthusiasm is also evidenced in the excerpt (i.e., appreciation of topic). Damian extends the discussion to Galois Theory which is the branch of mathematics that connects field theory with Group Theory and thus a meta-discourse of

Group Theory. He also mentions talking about Group Theory to students as young as 10 years old. Damian laughs as he describes how the students were “chanting isomorphism” perhaps acknowledging that the use of the word “isomorphism” was ritualistic but at the same time possibly enjoys recounting his experience.

The differences in the word use are potentially relevant to their engagement in the community of advanced mathematics discourse. Specifically, Damian, who used to be a research mathematician and thus a central participant in the discourse, uses technical vocabulary associated with Group Theory and Abstract Algebra (e.g., symmetries and isomorphism). In comparison, David identifies himself as mathematics enthusiast during the interview, thus, his engagement with advanced mathematics discourse is interpreted as peripheral. David’s word use is less technical. For example, he mentions “numbers, groups and codes” which could be interpreted as a reference to the application of finite groups to Cryptography, however, he does not specify the link between the three words.

In addition to David’s and Damian’s accounts, Alex mentions Galois Theory during the interview (Table 6.4). However, Alex’s account differs from the other two as he resonates with different aspects of his experience of learning about advanced mathematics. Specifically, he projects his views about mathematics to a historical account about Galois and the legacy of his work. However, he does not mention why he finds the particular example relevant to his teaching practice.

Another reason why the excerpts from Alex’s interview was chosen to be discussed in the section is that his engagement with Galois theory was not part of a university course. At the time of the interview, Alex was studying Galois Theory in his free time through an online course available on YouTube[33]. The excerpt is chosen here to highlight that the teachers’ engagement with advanced mathematical discourse does not necessarily take place in a university setting. Alex expresses his view about Mathematics by making meta-comments about the historical account in relation to the development of mathematics throughout the years.

[33] The video lectures are available here: https://www.youtube.com/watch?v=Buv4Y74_z7I&list=PLTmQXJCN3JDOWYTDtl2qEXWqcJ48l6ec_&ab_channel=ProfessorMacauley

Table 6.4 Excerpt from the interview with Alex

	Utterances	*Descriptors*
Alex:	Oh, it's nice, because we were [him and I, off camera] talking about **Galois** and what he said? The **Galois Theory**.	*Link to independent learning*
	He said on that time of his death. He said, I hope someone finds this interesting and spends time with it, or whatever, from this nonsense.	
	And I think this is it.	
	Like what mathematicians have been doing for millennia maybe had some basics on maths, maybe had some basics on human structure and human life, maybe organising book-counting how much food they had, maybe measuring lengths and so on.	*Comment about the development of mathematical ideas*
	And now it has transcended this	*Alienation of mathematics*
	and it might just stay there, people that, maybe there is maths now that only five people in the world can understand now,	
	but maybe in the future, this type of maths that they're working with will be foundations for something else. It might not be. It's just weird.	*Comment about the future of mathematics*

bold: Keywords linking to advanced mathematical discourse

Overall, in this section I discuss the analysis of excerpts from three mathematics graduates talking about their experiences with Abstract Algebra, specifically Group Theory and Galois Theory, in relation to their teaching. The emerging findings highlight that what mathematics graduates find relevant to their teaching varies across participants. In all three cases, the participants show an interest to the topic. Alex and Damian focus on historical accounts related to the topic, while Damian brings forward key mathematical ideas. However, Alex would not discuss his ideas with his students directly, but his view informs his actions when he is teaching, David would discuss aspects of history of Mathematics with A-level students and Damian, would try to teach elements of Group Theory to very young students. Moreover, their views are not exclusively linked to their undergraduate studies but can also stem from independent engagement with advanced topics. In Damian's case that is being involved in research, while, for Alex it is due to individual studies. Thus, the findings suggest that undergraduate studies could inform but do not shape teachers discourse at the Mathematical Horizon in a systematic way. In light of this observation, the conceptualisation of Horizon Content Knowledge proposed by Zazkis and Mamolo (2011), as an application of Advanced

Mathematical Knowledge applied in teaching, echoes only part of the experiences with advanced mathematics discourse mentioned by the participants of the study.

6.4. Life experiences: "it's a kind of hybrid of awarenesses"

This section draws attention to the role of personal and professional experiences of the participants which they view as relevant to their teaching beyond the mathematics of the moment. The emerging findings suggest a shift in the discourse of the participants where certain experiences are attributed pedagogical meaning.

In addition to influences of undergraduate studies discussed in previous sections, all the participants mention experiences from their personal or professional life – prior to becoming teachers – which they inform their discussions with the students beyond the mathematics of the moment. For instance, Liz mentions that she talks to her students about applications of mathematics learnt through discussions with her husband who works in statistical programming. Eric, when he is teaching hypothesis testing, uses an example from his personal life to highlight why it is important that the statistics are accurate. Finally, Alex and Nick who were born and educated in other countries compare their experiences with educational systems in their home countries to the English and comment on the different mathematics curricula. However, the participants do not necessarily link every experience with their teaching. Violet and David, for example, mention briefly working in industry before becoming teachers but they do not mention this experience as relevant to their teaching practices later on. In contrast, Liz mentions that she uses examples from the mobile phone industry where she used to work before she trained to teach. The latter example suggests a shift in the teachers' discourse which attributes certain experiences pedagogical meaning thus integrating them in their teaching practices.

To illustrate the integration of professional and personal experiences in teaching practices, I use Damian's reflective account about how his experience as a researcher in Pure Mathematics informed his teaching. Damian compares doing research with teaching multiple times during the interview and gives a very detailed account as to how his prior experiences with mathematics in academia link to mathematics taught in school. Table 6.5 includes part of the analysis of excerpts where Damian talks about his experiences. His account can be used to trace whether his views stem from being a researcher, a teacher or both. His reflection was triggered while sharing how he felt that "having a really high level of mathematical thinking" was a disadvantage when he started teaching.

Table 6.5 Excerpt from the interview with Damian

Utterances	*Descriptors*
Damian: I think it [experience in research] does give you an **overview of what, what mathematics actually is** and why, why I think it's a valuable thing for humans to do, um, in terms of **analysing the world** around us and hopefully, uh, **coming up with solutions for some of the problems** we see, um, and being a very powerful way of abstracting, **abstracting patterns**.	*View about mathematics*
Um, and so, there's, there's sort of, the **process of mathematical thinking**, I think is, like I said earlier, is of, of really high importance to humanity [laughs].	*View about doing mathematics*
So, so, I, I don't, I'm not sure whether, whether that's what I get from having studied the PhD. Um, I guess I'm very aware of what, of what that process looks like, um, and feels like, um, which maybe people who had only been exposed to school maths wouldn't have or even people who had just done even just an undergraduate degree.	*Reflecting on PhD experience (affordances)*
[…]	
I sort of, […], come to think actually there are good advantages of having a deep level of maths. But, I also think that some of the things, some, some of the kind of the personality traits that make a successful mathematician, **kind of being able to focus really, really strongly on one thing for quite a long time**, I think is a, something that mathematicians often can do, um, to the exclusion of focussing on other things is not useful in the classroom.	*Reflecting on PhD experience*
You need to be able to focus on, on lots of different things at the same, at the same time.	*Reflection on teaching experience*
You need to be able to work one on one with the child trying to <u>diagnose</u> that particular <u>misconception</u>, at the same time <u>scanning the classroom</u> to make sure everyone is staying on task, and um,	*Reflection on teaching experience*
so you kind of need a <u>split focus</u>, which I, you know, I had to work to to develop that a bit more, having, having spent, um, you know, years as a mathematician, just focussing on one thing, is hard [laughs] as that goes for ages	*Differences between teaching and research*
[…]	
Um, and I was just, um, that made a connection between doing maths, you know, that is the same	*Similarities between teaching and research*

activity, you can, you can, well, it can be, school mathematics can be the same activity as as as research.	
In the sense, in as much as doing art at school can be the same activity as doing art as a practising artist, which which, I guess, is sort of tries to tries to be. I don't think maths in schools does try to be like doing maths in a research capacity.	*Similarities in the engagement with mathematics*

<u>underlined</u>: trades of a teacher
bold: trades of a mathematician

Damian juxtaposes the advantages and disadvantages of the experiences as a researcher in relation to his career as a teacher in a reflective way. He accounts of his trades as a researcher (bold) and as a teacher (underlined) comparing the similarities and differences of the two activities. However, he states in two occasions during the interview that is difficult for him at times to "disentangle" what "awarenesses" (sic) he has from being a mathematician and what from being a mathematics teacher. Proposing the idea of a unique, "hybrid", awareness gained during his unique path. The following excerpt summarises the key points of the analysis of the excerpt where Damian exemplifies the role of his "hybrid" awareness.

Damian: But I guess, what things, I would say is, it's a kind of hybrid of awarenesses (sic) from both the mathematician and the teacher in the, I'd say, you know, if you say four eighth is the same as one half, then you're obviously lying to a child and the child can see that because one the symbol is it just completely different. They're not, they're not the same symbols [laughs], so they're not the same. Uh, so at some level, you, anyone who says that they are the same has developed an awareness a sort of, they, they can see that as an equivalence class, they see four eights, and they see it's, it's one half straight away. And so, they have this, they made, they probably haven't ever heard, well, you know, most, lots of people can come to fractions and have never heard of equivalence classes, but they think of them as being the same. um, so I would sort. But. Not, you know, not everyone, certain kids will never see them as the same and they'll always see the four eights as representing four out of eight things, um, and they won't be able to shift that at all. And so, I kind of talk about that as a potential stumbling block for kids. Um, yeah, I guess, yes, I guess I guess it's, again, it's something, I think

> my awareness of it informs everything I sa-, what, what I say, um, a little bit at least.

The "awarenesses" described in this excerpt do not fit into the domains of Common Content Discourse or Specialised Content Discourse as they are not necessary for educated adults neither typical of a mathematics teacher. They fit however, the preliminary definition of Discourse at the Mathematical Horizon as *patterns of mathematical communication that are unique to the moment of instruction and incommensurable to the mathematical discourse of the classroom as predicated by the curriculum*. Specifically, Damian's advanced mathematical discourse is very prominent in the excerpt, he gives a specific example that illustrates the different realisations of fractions using corresponding vocabulary, that is, "they are not the same symbol" (seen as distinct objects), "they are the same" (saming the objects), "fractions as equivalence classes" (encapsulating and reifying). At the same time, he links each representation with different levels of learning (i.e., in school, during university education, doing research) and shows pedagogical sensitivities regarding being truthful to what he says and what he expects from students. The "hybrid" awareness described by Damian is a shift from advanced mathematical discourse as a researcher to advanced mathematical discourse for teaching that combines both mathematical and pedagogical narratives under the same scope.

The choice of discussing in detail Damian's very reflective account was made intentionally to highlight the integration of professional and personal experiences in teaching practices as a "hybrid" awareness developed retrospectively through the engagement with teaching. When juxtaposed and discussed in general terms, e.g. in Table 6.5, Damian's experiences of doing research and teaching can be seen as two distinct activities which can be compared. However, in specific situations, Damian realises equivalent fractions as the same and distinct objects simultaneously by attributing pedagogical meaning to his standpoint. This observation indicates that Discourse at the Mathematical Horizon is an amalgamation of teachers' mathematical and pedagogical discourses.

Overall, the section highlights the influence of professional experiences in discussions beyond the mathematics of the moment and the development of individual discourses at the Mathematical Horizon. In this light, the variations in the narratives of the participants regarding the influences of university studies and personal or professional experiences to teaching beyond the mathematics of the moment can be explained as shifts of advanced or applied mathematical patterns of communication enriched with pedagogical meaning through the engagement with teaching. Finally, the findings suggest that a teacher's discourse at the Mathematical Horizon is evident when focusing on specific situations.

6.5. Sourcing information

In this section, I discuss the participants' narratives regarding information and resources from the internet and from books in relation to their teaching practices beyond the mathematics of the moment. The emerging findings suggest that the participants do not feel the need to rely on memorising details. Finally, the findings highlight the need to critically evaluate the sourced information in relation to their reliability and accuracy.

A source of information for the teachers is books. The participants mention recalling information beyond the mathematics of the moment from university textbooks, professional journals but also science writing books. For example, Damian shares some authors that he believes can offer an overview of advanced mathematics for teacher's that do not have a degree in mathematics but also mathematics education literature that he found useful in this transition from a researcher to a teacher of mathematics:

Damian: If you've read books by Marcus du Sautoy or um, Ian Stewart and sort of popular maths books, there's lots of, you know, you can talk about chaos theory and how that, you know, comes from the complex numbers, um, without having a higher degree in maths. […] I think, I think that was one of the things that brought me into teaching, the idea of that kind of, the mathematical thinking was, was what was of the most value and what I really, and, and I've gotten up from, from being a maths researcher, and I guess, I, I guess John Mason's book thinking mathematically sort of made that connect, cause that's the book that I read while I was training to teach.

Damian states that books could serve as alternatives for learning about advanced mathematics for teaching. He differentiates between "popular maths books", science writing books, which he thinks might be beneficial for teachers who have not studied mathematics at university and books he found useful as a researcher changing careers. Damian's reflection to his own learning through mathematics education books links to his description of "hybrid" awareness in Section 6.4 indicating that engagement with mathematics education literature acted as a bridge between his experience as a researcher and as a teacher.

Science writing books were also mentioned by participants with degrees in mathematics (e.g., Alex, David and Liz), thus, extending Damian's claim about potentially benefiting non-mathematics majors. Specifically, David describes how he reads science writing books and evaluate whether they can offer ideas to share with his students during the general lessons or add the book as a resource for the maths club at his school. Moreover, David talks about sharing videos and other information found on the internet with his students.

Table 6.6 Excerpt from the interview with David

Utterances		*Descriptors*
David:	I **read** stuff and I **reread** stuff.	
	So, yeah, the things like, things like, Makers of Mathematics is quite **dry** by Hollingdale[34], um, but that was first on the list of recommended reading in Cambridge I think, for maths. And that was pretty **comprehensive**. Then, there's more, um, **accessible** things.	*Evaluation of book*
	So, we tend, if I'm interested, I might tend to put it down in the library.	*Sharing*
	So, you've got things like, um, the Math Book by Clifford Pickover[35] is very **visual** so it's great for the students. There's loads of **interesting** stuff in there.	*Evaluation of book*
	So, it's just like cherry picking.	
	You know, I'll often, I'll often <u>watch</u> staff on YouTube and then get very <u>excited</u> about those, and, you know, might put, well,	
	if I'm doing it <u>at the minute</u> and I've seen a few things on there, I go oh, allright, I'm gonna put that into maths society, and we can <u>play it loads</u>.	*Sharing*
	Um, so, yeah, if it's maths teaching, I don't tend to look for history stuff in maths teaching. So, I don't go where's the history first. I tend to read about it first and then want to put it in.	*Sharing*
	So, there's loads of stuff I don't know any history about. But yeah, **I like reading maths books in history of maths**	

<u>Underlined</u>: Keywords indicating the engagement with online resources
bold: keywords indicating the engagement with books

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[34] https://store.doverpublications.com/0486450074.html
[35] https://www.unionsquareandco.com/9781402788291/

Table 6.6 presents the analysis of an excerpt where David explains how he is “cherry picking” resources he would share with his students and the maths society. The analysis of the excerpt indicates differences in the engagement of David with books (bold) and videos (underlined). Specifically, he describes reading books multiple times before deciding what to share with his students which indicates a cautious engagement with the books. In contrast, he describes choosing videos on the basis of getting “very excited” which indicate an emotional reaction. Moreover, David evaluates the content of the books he mentions, not necessarily in terms of the accuracy of the content, but in terms of the writing style (e.g., dry, comprehensive etc). In contrast, he does not comment regarding the videos he chooses. The ways he shares the information with students is different between books and video content. On one hand, David adds the books he finds interesting in the library to be accessed when needed. Later in the interview he mentions going to the library to access and use a specific book in a discussion beyond the mathematics of the moment with a student. On the other hand, sharing a video with the maths society depends on whether the content is related to the topic they are working on at the time. Finally, David mentions that he integrates references to the history of mathematics as he learns about it, and he is not systematically looking for elements of history in every lesson.

Alex and Liz also mention YouTube videos as resources shared with students. For example, Alex mentions regularly using videos to attract the attention of his students:

Alex: I don't know which videos you watched [during the observations], there is a band that I was amazed by called OK Go and many of their videos are very, very creative. […] that's why I like Ok Go so much, because they've actually done videos explaining the maths behind their concepts. And what they do is pushing artistic and creativity limits I never thought possible and then not do it, not only do it in a way that OK, look, read up this paper, what it did these guys do. It's actually enjoyable and it's actually amazing to watch. Therefore, if I find, basically, I just use these as an example. If I find something along the structure of this, that is something that would be my first, always on the back of my head. What kind of mathematics can I connect this? What kind? Can I show it to people?

In the excerpt, Alex contrasts video content with reading a paper, putting forward the idea that videos are more "enjoyable" to justify the use of videos in his lessons. Alex appears to evaluate the content before sharing it with students. He describes the process of choosing the content he is sharing by considering, first whether he found it appealing ("always on the back of my head"), then whether he can connect it to mathematics and finally whether if it is appropriate for the students. Similar to David's account, Alex's excitement about the video appears to play a role in his choices. The observation indicates that what the teachers choose to share with their students are influenced by their interests.

In England, it is common for teachers to have a computer and access to the internet in their classrooms. The use of videos discussed so far, requires teachers to search for the content in advance of the lesson. However, the access to online resources in addition to books could open more paths for teachers to reach for information beyond the resources they are given and expected to utilise in teaching. The use of the internet was found to play a role in accessing information during the lesson about ideas and practices beyond the mathematics of the moment. Alex, David, Liz and Scott mention using search engines, such as Google, to access information in order to answer student's questions that cannot be answered using just the syllabus. A similar observation was made in Section 5.4.4 which discusses an episode where Liz uses Google to search and present her students with a definition from Google. For example:

Scott: […] sometimes I've gone on Google and, um, you know, gone on Google and just research and close look or look there. Wh-, who uses algebra?

Also, Alex Liz and David mention searching for information about history of mathematics, when an opportunity to go beyond the mathematics of the moment appear during their lessons. For example, in Table 6.7, I present a short excerpt from the analysis of David's interview. David's utterance "let's google it" (bold) indicates is action of searching for the 'missing' information as well as inviting a student he was discussing with at the time to join him in his search.

Table 6.7 Excerpt from the interview with David

	Utterances	*Descriptors*
David:	And so spontaneously I went, [snaps his fingers].	*Spontaneous opportunity*
	Right, well, there was this mathematician, I forget his name,	*Recalling experience with topic*
	let's google it. I think it was, um, Yacob Bernouli or Jacob Bernoulli. But, I might be wrong.	*Using Google to refresh memory*

bold: mentions of Google

In addition, Liz mentions using the internet to find illustrations to show to students (e.g., more complex Venn diagrams) or to access information about a topic. Table 6.8 presents part of the analysis of the excerpts where Liz mentions using Google during her lessons.

Table 6.8 Excerpt from the interview with Liz

	Utterances	*Descriptors*
Liz:	Um, and then I started talking to them about Babylonian mathematics and I didn't, and this is what I was saying, I don't ever plan for this. So, I don't do enough research before I go in.	*Spontaneous opportunity*
	But I remember during my degree that we looked at Babylonian mathematics and, uh, their number base system.	*Recalling experience with topic*
	And so, I end up talking to the pupils about it and I'm **Googling** at the same time. I'm trying to remember. Yeah. Okay. Yeah, that's right. I am telling the right thing.	*Using Google to refresh memory*
[…]		
	And sometimes then, so in that instance, I literally just sat there	*Spontaneous opportunity*
	and went, I, I know that this [Venn diagramms of more than two sets] exists	*Recalling experience with topic*
	and I can pull it up and showed them some pictures.	*Using Google to access resources*
	Um, I mean maybe I did myself a disservice. I suppose. I, you know, if they asked me a question and I have got **Google** here,	*Using Google to access information*
	but we start talking about, you know, huge numbers and where they come from and why we named them, then we can explore it	*Recalling experience with topic*
	and sometimes we just explore it and I'll say to them, I don't know, so let's just read it together and see what the answers are.	*Using Google to access information*

bold: mentions of Google

The analysis of the accounts about the use of Google in teaching indicates a common pattern among the participants which occurs during unplanned events, e.g., an unexpected question. Specifically, the action of "Googling" is triggered by recalling an experience with the topic which the participants then try to refresh by using the search engine. The pattern indicates that the participants, who had at some point during their education or personal life come across an information, can refresh the memory in real time. In that sense, it might not be necessary for the participants to memorise details about ideas beyond the mathematics of the moment. Borrowing the phrase from the literature on Horizon Content Knowledge, the teachers have "an awareness of the large mathematical landscape" (Ball & Bass, 2009, para. 17) and use the internet to access details and support their arguments in the classroom. Alex describes this awareness as a "menu" to navigate online resources:

Alex: […] For me, it was the, the elements of Euclid and stuff. There are things that I use, for example, because I know some basis of the elements of [Euclid], oh I forgot who was it. So, this is what I have. I have maybe some. Maybe if I Google it, I will find what is there. Maybe I, I have like a menu and say, OK, if I search this, I can find it.

Alex, David, Liz and Scott seem to be very familiar with using Google during their teaching. The use of the search engine requires technical skills, such as identifying the topic and using the suitable keywords. Above all, it requires critical evaluation of the search results. In the excerpts discussed, here, the participants utterances do not indicate critical considerations of the results. Moreover, in Section 5.4.4 Liz uses Google as the "ultimate substantiator" (Sfard, 2008, p. 234) of the definition for parallel lines which appears at the top of the search which is not necessarily a definition that would be accepted by mathematicians. The findings highlight the importance of digital literacy of mathematics teachers in general and, in particular, in relation to their engagement with discussions beyond the mathematics of the moment.

Overall, sourcing information and resources from the internet and from books seems to have a role in the discussions of ideas and practices beyond the mathematics of the moment. This section illustrates that teachers do not feel the need to rely on distant memories when it comes to ideas beyond the mathematics of the moment. A familiarity with the potentials of the mathematics involved in the discussion and access to the resources could be a starting point for the teachers to develop their arguments. The teachers' mathematical interests influence what they are looking for when they search for details or inspirations on the internet or in books. In return, the available information could also influence teachers' discourses. For instance, one can find on the internet different and sometimes conflicting information. Therefore, it is important for the teachers to recognise the validity and relevance of the information to their teaching and be able to communicate information intended for the general public to their students in real time while maintaining mathematical integrity.

6.6. Sharing practices, collaborations and the value of institutional support

In the previous sections, I discussed the influences of university studies, life experiences and sourcing information from books and the internet in teachers' discourses at the Mathematical Horizon. Here, I discuss the participants' narratives about enriching their teaching practices beyond the mathematics of the moment through interactions with colleagues. Specifically, the emerging findings suggest that discussions between colleagues, lesson observations and teaching collaborations across departments could influence what the participants share with their students beyond the specifications of the curriculum, when to do so, and how. Thus, influencing their discourses at the Mathematical Horizon. However, the opportunities for collaborations can be affected by the lack of support from the school, the views of others and the participants' perceptions about their role as 'experts' in the mathematics classroom.

Positive influences mentioned by the participants include enriching teaching practices through discussions in informal settings, through lesson observations or collaborations between departments. In particular, David, Eric and Noah mention having regular discussions with other mathematics teachers in their departments. All three of them indicate that these discussions contribute to their or/and their colleagues learning. For example, Eric mentions a discussion he had with a colleague about teaching an A-level topic and his plans to bring the example to his Year 11 (GCSE) students to prepare them for what they might come across at a later stage. Specifically, Eric talks about the interaction with his colleague:

Eric: But at A-level, I'm still learning how to teach them. You know, I had a conversation the other day. How would you do this? I, I don't know if I can write it down [writes in his notebook: $(8+3x)^2$] but I had something, how, how are you to explain to take a factor of eight out of eight plus three X

> squared with someone. And we're debating whether you can ask the students what would be, what would this be, or do you just tell them that you can take a factor of that eight out, and then square it, so it's 16. And then divide.

Eric was debating with his colleague whether they should let their students explore how the expression can be factorised using what they have learnt from simpler cases or if they should tell them a process. But he does not state what was his position in this argument. When Eric says “so it’s 16” he probably means 64 but he performed a multiplication instead. As he is trying to recall and tell me the process, he multiplies by two instead of squaring the number 8. His mistake, although trivial, signifies ritualistic rules of procedures which might results in errors. Eric does not realise his mistake and continues to explain the process. He concludes that he would use a similar example with his top set year 11, preparing for GCSE exams, but he does not specify if he is planning to show them the method or let them explore how they could factorise it. In this example the discussion with his colleague helped him equip his ‘tool kit’ - precedent space (Lavie et al., 2019) regarding teaching- with an example that goes beyond the requirements of the curriculum in year 11 but it cannot be said whether it provided additional meaning to the process of factorising this type of examples for either of the collocutors.

Observing other teachers teach was mentioned by Marcus, who works both as a teacher and as an educator, as a mean for his own learning. Table 6.9 presents a summary of the analysis of the excerpt where Marcus substantiates his statement through a specific example.

For Marcus, interacting with non-specialist mathematics teachers as part of his role as an educator provides opportunities to expand both his mathematical and pedagogical discourses through the same activity. The example of simplifying complex fractions in the excerpt is usually taught in a ritualistic way, thus understanding why the method works is not part of what teachers need to explain to the students. Marcus reflects on his learning both in terms of content and teaching practices. Specifically, he explains through an example why division by a fraction is performed as a multiplication with the reciprocal of the fraction using mathematical vocabulary (e.g., equivalent fraction) and visual mediators (e.g., the symbol $\div$ and the representation as a complex fraction $\frac{a}{b}$ where a and b are fractions). Then he connects what he learnt through the lesson observation with teaching and indicating a shift in is practice as a result.

Table 6.9 Excerpt from the interview with Marcus

Utterances	*Written notes*	*Descriptors*
Marcus: Yeah, I think so. Um, particularly, when I've, I guess my thing is when I do run these events [for non-specialist teachers], I see some of the other teachers teaching.		
And I see things that they do that I would like and I hadn't thought of before.		
Um, I guess one example might be, I know this is somewhat done long time, and when you divide by a fraction, say, I don't know two thirds divided by five sixths.	Writes on a piece of paper: $\frac{2}{3}\div\frac{5}{6}$	
And now, I would always like, probably, why do you times by the reciprocal, why do you do that?		*Questioning mathematical practices*
And I've never really thought it through really.		
And just one of the teachers just wrote it like that [as a complex fraction].	Continues: $=\frac{\frac{2}{3}}{\frac{5}{6}}$	*Realisation of division as fraction*
So that's what it means when we divide. That actually, I could find equivalent fraction that times in the numerator and the denominator by six over five. And then my denominator there, well, that's going to be one, so that is to times by the reciprocal.	Continues: $=\frac{\frac{2}{3}\times\frac{6}{5}}{\frac{5}{6}\times\frac{6}{5}}=\frac{\frac{2}{3}\times\frac{6}{5}}{1}$ $=\frac{2}{3}\times\frac{6}{5}$	*Using discourse about equivalent fractions*
That's something I've never seen before		*Identifying shift in mathematics discourse*
[…]		
So, I think, it just helps me **change the practice**, just think of different things.		*Shift in teaching practice*
[…]		
Seeing other teachers and such. They- they've **pedagogical knowledge as well**, […].		*Link to pedagogical influences*
[…]		
Um, they've, what they bring out is arguably better ideas as to how, I		*Link to pedagogical influences*

	mean, **conceptually understand** it. I think they've got new ways of which I haven't thought of doing, I haven't seen before, I haven't the experience of, which helps with that.	
bold: indicators of pedagogical discourse		

The previous examples showcase instances where discussions with colleagues could influence decisions to go beyond the mathematics of the moment in a positive way. On the contrary, Alex, Damian, Marcus, Noah and Scott mention that a restricting factor for going beyond the curriculum during their lesson could be what is expected from them by the school. Marcus attributes the restriction to individual "headteachers" while Damian to external "pressure on schools […] to show progress in every lesson". Noah believes that in schools that only teach up to GCSE the students are not taught in ways that would link to their A-Level mathematics because this is out of the focus of the school. Scott mentions that when he was teaching in a middle school[36], he could take the students out in the field and explore patterns in nature which is something that feels he cannot do working in a secondary school (age 11 and beyond).

Scott: Then you go, you know let's go and do something different, I think it's right to do, I'd do it just to see. And then if you get a reaction [different voice] "oh why the pupils aren't blah, blah, blah", em, you know, alright, I guess, I guess we can't do that sort of thing. If I was in middle school that wouldn't be questioned.

Moreover, Alex's narratives that are found relevant to his discourse at the Mathematical Horizon are preceded or followed by narratives about what he is "allowed", "permitted" or "advised" to do. Alex and Scott's narratives indicate that seeking approval from more experienced colleagues or a school authority in relation to explorative activities. Therefore, negative reactions influence their choices to go beyond the mathematics of the moment.

Regarding collaborations with other departments teachers express various views that are linked with their mathematical and pedagogical discourses and their perception of expertise in the subject. For example, Nick says that as a school they "have extra curriculum, […] which

[36] A middle school is a type of school that offers upper primary to lower secondary education (students' age range between 8 and 14 years old)

is linking mathematics to geography, linking mathematics to history or linking mathematics to science”. Nick says that the mathematics department in his school has worked with the science department to prepare resources for rearranging equations in science. Also, the mathematics department in his school integrated in the homework an activity they call “root words” where students are given a word and they are looking for mathematical words stemming from the same root word. In this case collaboration between departments seems to be promoted externally by the school. Nick appears to be very confident with collaborating and sharing expertise with his colleagues, especially in science, “maths and physics is one topic”. Moreover, having studied for a PhD in Environmental Sciences might be contributing to his confidence taking part in these collaborations. His narratives about mathematics as a set of rules to be applied in problems is in line with his engagement with these extra-curricular activities. Eric, on the other hand, who professes narratives about mathematics being an “abstract subject” appears to feel less confident in making connection with other school subjects:

Eric: And sometimes mathematics is an abstract subject. So, there isn't, I think sometimes there's just simply isn't many links. […] Hmm, I have looked at linking it, linking to computer science, I'm doing sort of half maths, half computer science. So, I was having a lesson a week, every two weeks, where it's partly maths and partly computer science. And I've sat down with the computer science, head of computer science here before and talk to them about it. But ultimately, you need really good teachers to link it like that because you need to have an e-, be subject, need to be an expert in mathematics and an expert in computer science, which is really difficult. So, yeah, I do struggle to make it to other, other subjects.

Eric feels that he “needs to be an expert” in both subjects to make connections between them. In this case the collaboration between the two departments was based on Eric’s choice to ask for help and is not necessarily a regular practice within his school. In comparison to Nick’s example, where the school support and instigate opportunities to develop interdisciplinary activities, Eric does not see this collaboration as a mutual share of expertise. Moreover, Nick’s school support collaborations between the departments to develop extra-curricular activities, i.e., activities to be used with students outside the normal mathematics (or other) lessons. On the other hand, Eric is trying to incorporate the connections between mathematics and computer science in his mathematics lessons, which might be a factor which influence his need for expertise, in comparison to Nick.

Overall, the section highlights the stimuli of discussions between colleagues, lesson observations and collaborations between departments for the enrichment of narratives beyond the mathematics of the moment. In the examples presented, the participants learn as part of a community of mathematics teachers or from interactions with communities of teachers from other subjects (e.g., science, computer science). Thus, the interaction with colleagues influences individual discourses at the Mathematical Horizon. An important element of the findings is that learning at the Mathematical Horizon can occur in formal (e.g., higher education) or informal (life experiences, discussions with other teachers etc) situations and through participation in different communities of practice (Lave & Wenger, 1991). For instance, teachers can learn in a community within the mathematics department of their school with a common aim to improve their teaching practices, a cross-curricular community with a shared goal to identify links between their disciplines for teaching purposes, or a community consisting of experts (e.g., mathematicians) and teachers (Pinto & Cooper, 2022). Differences in participants' narratives about teaching practices beyond the mathematics of the moment can be linked in the support of their colleagues and/or the school and their perceptions about their role as 'experts' in the mathematics classroom.

6.7. Interpretations of the curriculum and curricular guidelines

In Section 5.2, I argue that an emerging theme of discussions beyond the mathematics of the moment includes opportunities to communicate about ideas and practices that run across the curriculum. Those opportunities include metalevel comments about ideas students are expected to come across several times by the end of their secondary school education in relation to the topic of the day, and to connect ideas of what the students might have already come across in the past and what they may see again in the future. In addition, Section 5.5 discusses opportunities to connect the topic of the day with mathematics used in other subjects. In this section, I discuss the narratives of the participants during the interview regarding connections between different topics of the curriculum, underlying ideas and connections of the mathematics curriculum with curriculums of other subjects. The emerging findings suggest that the views of the participants about the role of the curriculum and school guidelines and their understanding of the content link to their narratives beyond the mathematics of the moment.

Mentions about the curriculum in the interviews fall into two interrelated narratives: (a) underlying ideas and (b) connections between curriculum topics. Underlying ideas run across the different topic areas, in the National Curriculum the ideas explicitly mentioned include development of fluency, mathematical reasoning and problem solving (Department for Education, 2013a, 2014a). However, participants also reference underlying ideas different from the one stated in the official documents. The references to underlying ideas that are not explicitly stated in the aims of the curriculum are accounted for as teachers' personal interpretations of the curriculum and other guidelines which could lead to discussions beyond the mathematics of the moment. For example, Thomas refers to qualities he interprets as key elements of working mathematically. He references the idea of reasoning, which is explicitly stated in the curriculum, along with proving, notating and naming, and processes becoming objects as well as abstracting conjecturing and generalising. He links his argument to seminal Mathematics Education literature and not necessarily with official documents:

Thomas: […] Then John Mason's talked a lot about specialising generalising and so on. And I think that's true, that dynamic between looking at special cases, getting conjectures, being, generalising, but then often you're testing or generalising by going back to special cases as well. And that sort of dynamic, uh, I think is quite important. Yeah, no, I think yeah, I just think those things run through. No matter what's you're doing.

Other participants mentioned taking into consideration ideas and practices the students have come across with what comes next when preparing resources and planning their lessons, without necessarily explicitly talking with their students about possible connections or extensions of the topic. In those cases, narratives about underlying ideas and connections between topics are blended together. For instance, Table 6.10 illustrates part of the analysis of Marcus's interview.

Table 6.10 Excerpt from the interview with Marcus

Utterances	*Descriptors*
Marcus: Um, well, as an A-level teacher, generally, I mean, I think what I like to do is show that all of these things we do at some point connect to each other, say, with the <u>differentiation</u>, how we can link that with the <u>quadratics</u> we used and solved those.	
Um, obviously, the imp-, the obvious links you've got between <u>differentiation and integration</u> make sure they know that they're the inverses of each other.	*Link between learnt topics*
When we solve <u>quadratic inequality</u>,	*Link between learnt topics*
linking that to **sketching** a graph, which they've done before to help **visualise** it. And often bringing them sketches to graphs would be actually saying that probably the most important connection or try might be if they can visualise it, they're often much better at solving it.	*Underlying idea*
Um, again, going into further maths, a particular one I like is the connection between those that got <u>exponentials, cos, sin, McLarens</u> and all those other things you've done with differentiation.	*Link with topics to be learnt in the future*
I guess pull what they do, really, I think if you can show that they're not all standalone things and each lesson, try and bring previous stuff we've done and show how it relates to it. I think that's a very important thing.	

<u>underlined</u>: curriculum topics
bold: underlying ideas

In this example, Marcus identifies the links between differentiation and other topics to be taught in the future or learnt in the past. Marcus calls "obvious" the link between differentiation and integration and tries to highlight the importance of visualising a problem using graphs, which is an underlying idea related to working mathematically and problem solving. The connection between differentiation and integration is part of what the students are expected to learn as part of their A-level education. However, the connection with quadratic inequalities and the underlying idea of visualising the problem as a mean for finding the critical points of a function is not necessarily explicit. The content of A-level mathematics is not covered by the National Curriculum, however, the topics to be taught are specified by the different exam boards. The students can learn to solve maximising and minimising problems as a routine without looking the problem as an application of quadratic inequalities as learnt in KS4. Finally, Marcus mentions applications of differentiation in Further

Mathematics, e.g., McLaren Series. His example indicates that he takes into consideration both underlying ideas and making connections between topics in preparation of teaching.

Covering the content of the curriculum is one of the main responsibilities of the teachers and sometimes it can be a challenging one. Teachers who teach KS3, KS4 and in 6th form (Eric, Noah and Liz) mention how their teaching in different levels help them in choosing appropriate examples and methods. However, realising connections across topics was not necessarily found to be linked with teachers' experiences in teaching at different levels. For example:

Eric: I think I have to be conscious. Solving equations, for example, is a typical example of, it's really easy to get students to solve a difficult equation, equation, with X is, when there's a variable on one side and it can look really complicated. And I could teach a student to solve that. But ultimately, I know that's going to only get a student so far. So really, I wouldn't teach them with and say a function machine sort of way, I would rather get the students to solve it in a terms of you've got to divide by four to get, and like a methodical way of unpicking the question almost. So, then it gets in the skills to, to be able to solve more complex problems when there's a variable in two sides, for example. So, I think, the method you, I choose is because I teach from, I think, I teach from A-level all the way down to year seven.

Eric recognises the spiral nature of the curriculum and acknowledges that the students will come across more difficult equations in the future and plans appropriately. His plans include teaching the students methods that can be generalised rather than processes that work only in specific cases. The mention could be linked to the underlying idea of generalising mentioned by Thomas. Regarding connections across topics, Eric says:

Eric: But I'll try to say, if I'm working out, there's a lesson on finding the mean, I won't just have, I will try not to have a list of numbers and divide it by how many are. I will try to throw in algebra in every lesson that I do. So, one of them will be find the mean of three X plus seven X, three X or a seven X and five X. So, it's almost like a constant drip thing.

In his response, however, Eric does not acknowledge when or why finding a mean of algebraic expressions might be desirable. Noah, on the other hand, briefly explains why we might want to use algebra within probability:

Noah: We'll look at using tree diagrams using algebra if we don't know a particular outcome. But we know the final results. Um, Venn diagrams, we use a lot of Venn diagrams in probability and again, that, that leads very nicely into algebra. Yep. And we'll go beyond that right up to the, the very top end with our, our GCSE students and constantly stretch them with algebra within the probability and working backwards where we'll give them the solution and may have to undo the problem, work through it backwards to try and find one of the initial conditions.

The excerpts from Eric and Noah, here, might indicate that Eric in his example is superposing the topics rather than connecting key ideas between them, such as elements of problem-solving that are mentioned in Noah's response.

The analysis of the instances where the teachers make mentions of connecting what the students have learnt and what they will learn in the future shows that the differences between the teachers' responses can be attributed to their views about where these connections stem from. For example, in the excerpt from Noah's interview the connection between Algebra and Probability is through problem solving which is an underlying idea that runs across the mathematics curriculum.

Instances of teachers superposing connections between topics of the curriculum were found in the interviews of Eric, Nick, Violet and Scott. On all four occasions, the participants' mathematical discourse included unsubstantiated narratives about the connections. For example:

Nick: So practically so many areas of mathematics which we practise here, and we do here will have a clear connection. And that's why I said we are going to teach this topic because by teaching this topic here, we are going to be able to do the next one, which is completely different. Even when you look at handling data. Handling data is another topic, which physically it could be not related to anything. But deep down, deep down, you find maybe algebra

> is not very hard to connect with handling data. But then you have surely the the diagrams, the space and shape, and the real number is part of.

Nick begins by saying that the topic areas covered in his school “will have a clear connection”, the tense used could indicate that the connections are not yet clear. For him, the topic areas can be “completely different” and “not related to anything” but at the same time connected with other topics. His statement is contradictory. The contradiction can be attributed to viewing the mathematics curriculum as sets of rules needed for “facing harder and harder questions or harder and harder problem[s]”. Nick locates the connection to be “deep down” but he does not give a reason as to how the topics he mentions are connected perhaps because he does not endorse the connection himself.

Similar findings were present in Eric’s, Nick’s, Violet’s and Scott’s interviews. The findings suggest that these participants might not fully endorse the connections between the different topic areas and thus interpret the curriculum as a set of rules on what they are expected to do in such situations rather than a starting point to unpick underlying ideas.

Contrasting to the previous observation regarding Nick’s discourse about connections between topic areas of the mathematics curriculum. Nick provides specific examples of applications of mathematics in physics and claims that he incorporates those examples in his teaching of mathematics (e.g., when teaching rearranging equations) despite not being explicitly included in the National Curriculum. However, he was not observed making connections with content from the Physics curriculum during the lesson observations. The episodes discussed in Section 5.5, illustrate opportunities where the teachers could make cross-curricular connections. However, the analysis of the lesson observation indicates that this theme contains mainly potential opportunities (i.e., alternative courses of action) in comparison to the other three themes: across the mathematics curriculum (Section 5.2), what comes next (Section 5.3),and conventions (Section 5.4). In addition to influences of universities studies and collaborations between colleagues that were discussed in Sections 6.2 and 6.6, another possible influence could be interpreting the curriculum as following a specific working scheme. Specifically, Violet, who views teaching as to “follow schemes of work in school”, acknowledges that mathematics is used in other subjects. However, she distances herself from what is included in the National Curriculum of other subjects. Table

6.11 presents the analysis of an excerpt from Violet's interview where she talks about connections with art and science.

Table 6.11 Excerpt from the interview with Violet

	Utterances	*Descriptors*
Violet:	Just through my own teaching career, working with children. I'm a very creative person. I've got some experience of art when I was younger, and my maths classrooms used to be full of children's work.	*Link with personal experiences*
	You know, I like to have lots of display all around in the classroom because I felt that that really **motivated** children to see their own work on the walls.	*Link with teaching practice*
	So, I used a combination of children's work on the walls, but then also things like posters, you know, that we brought in from maths catalogues. So, it was a combination, really. Um, yeah.	*Link with teaching resources*
	I think it's really important for children to see **creativity** in maths. Maths isn't just about a textbook and just numbers or equations.	*Link with teaching resources*
	Maths is everywhere.	*Generic claim*
	It's out there.	*Generic claim*
	You know, maths is in arts, but it's also in sciences and all subjects across the mathematics curriculum.	*Link with other subjects*
	So, I think that's really important to teach children that.	
	Because, you know, we live in a society, I think certainly here in Britain where, you know, maths, you know, regarded as **very difficult** and **inaccessible** for many.	*Link with deficit perceptions about mathematics*
	So, it's important to show that maths is **accessible** actually, um, for all really.	*Generic claim*
	Even, you know, artists use some level of mathematics, when we talk about symmetry of rotation, for example.	*Link with other subjects*
bold: indicators of pedagogical narratives		

In Violet's office, where the interview took place, she has also displayed on the wall drawings of children that have geometric elements (e.g., display symmetries) and poems about mathematics. In her interview, Violet's pedagogical narratives are very dominant. Throughout her interview she talks about social justice, empowering students and seeing relevance of the topics outside the mathematics classroom, which are in line with Exploration Pedagogical

Discourse (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022). In Table 6.11, for instance, she talks about the notions of motivating the students, promoting creativity and accessibility which are not specific to teaching mathematics. Violet does not specify what she considers creative. She gives a generic example of why she believes that her practice of displaying students work on the wall alongside posters from catalogues motivates her students. The lack of consistency observed between her arguments about teaching mathematics and her exemplification of these arguments using specific examples indicate that her discourse about teaching in general and her discourse about teaching mathematics are not aligned.

Her narratives about mathematics include mentions of applications in other disciplines and school subjects. Here, she references the mathematics curriculum and not mathematical ideas included in other subjects. She says, “maths is in arts, but it's also in sciences and all subjects across the mathematics curriculum”. Later in the interview, Violet was asked if she thinks whether it would be useful for the students to see cross-curricular examples. In reaction, she prompts me to ask teachers of other subjects as “they'll know more about their own national curriculum and what maths is covered in those”. Her examples lack specificity which might indicate that her mathematical discourse is limited to the curriculum content she teaches. She fails to provide specific examples of how mathematics is used in science and history curriculum, that is in line with a view about the role of the teacher being to “follow schemes of work”, which contradicts her earlier claims indicating that she does not fully endorse exploration pedagogical narratives about making connections with other subjects. In Violet’s response what is covered in the national curriculum draws a clear line between an expertise in mathematics and an expertise in another subject.

An obstacle the majority of the participants mention which keeps them from engaging in discussions beyond the mathematics of the moment was the pressure of preparing students for the exams. Six of the participants mention that discussing explicitly with their students beyond “what is in the syllabus” is limited when students are in Year 11 preparing for their GCSE exams either because there is not enough time or because their students are “overloaded”. The obstacle can be attributed to viewing the curriculum as following schemes of work. Therefore, ideas and practices beyond what is included in their schemes of work are not a priority for the participants. The observation links back to influences within the school environment. Specifically, the views about the authority of the school and the focus on evaluation.

Overall, this section discusses the narratives of the participants regarding connections beyond the mathematics of the moment. In particular, I focus on narratives about connecting different topics of the curriculum and identifying underlying ideas. The emerging findings suggest that the differences between the participants' responses can be attributed to whether they view the connections stemming from underlying mathematical ideas or from teaching instructions. Specifically, the findings suggest that participants might not fully endorse the connections between the different topics, i.e., their mathematical discourse included unsubstantiated narratives about the connections, and thus interpret the curriculum as a set of rules on what they are expected to do in such situations rather than a starting point to unpick underlying ideas. Finally, lack of engagement in discussions beyond the mathematics of the moment, and especially cross-curricular connections, was found to be linked with participants narratives about lack of expertise, time or authority which can be attributed to interpretations of the curriculum as "schemes of work".

6.8. Narratives about students' abilities and engagement

This section discusses the participants narratives about their students in relation to their discourses at the Mathematical Horizon. In particular, the participants' perceptions of their students were found to be one of the factors that they take into account when initiating discussions beyond the mathematics of the moment. The analysis of the data indicated that participants' decisions to deviate from the lesson of the day is linked to narratives about the age of their students, students' behaviours and interests, and their perceived attainment. Thus, participants' discourses at the Mathematical Horizon can be subjectified when seen as relevant to their teaching of the topic of the day.

The participants' reasons as to why they might choose to go beyond the moment indicate their intentions to inspire or help their students go forward with their studies. In the interviews the responses were split. David, Damian, Liz, Violet, and Alex focus their reasoning more on inspiring or stretching their students whereas Eric, Marcus, Nick, Noah, and Thomas focus on helping their students succeed in their exams and university studies. Regardless of their reasons, taking into account students' learning and experience with mathematics is important for the participants in making the decision to share more than what they are instructed by the curriculum and school guidelines.

Liz and Alex mention that students' interests and behaviour during the lesson plays a role in their decisions to deviate from the planned lesson to have a discussion beyond the mathematics of the moment:

> Liz: I know that I can spend 5, 10 minutes talking about that with some of the students because the others are all very well behaved and getting on with the problems that were on the board. […] Um, I wouldn't necessarily then represent that back to the rest of the class because it will depend on whether that's, that connection would have meaning I think for the rest of the class.

Alex, who claims that attending to his students' poor behaviour was a priority at the time of the interview and the observations, he consistently mentioned students' behaviour as a limitation for what he is sharing with them. For example, in the following excerpt Alex describes how through his exposure to a different educational system and through his undergraduate studies he has seen proofs that are not part of the English curriculum, and he would like to share with his students, but he refrains from doing so because of his need to prioritise managing students' behaviour. Table 6.12 illustrates the analysis of an excerpt where Alex explains why he chooses to hide connections he finds relevant to the topics he teaches from his students.

Alex seems to recognise where ideas beyond the curriculum fit to his practice, but is "advised" to not discuss them with his groups of students (see also Section 6.6). He attributes this advice to the behaviour of his students. The challenges Alex faces in classroom management are also evident during the lesson observation. In Section 5.2.2, there is an example from the observed lessons where Alex's attempts to have discussions beyond the mathematics of the moment with his students are interrupted due to poor behaviour of some students in his class. Alex's conflicting feelings are an indication that he would like to share more ideas and practices beyond the mathematics of the moment in his teaching. However, at this time, he chooses to improve his teaching in relation to classroom management.

Table 6.12 Excerpt from the interview with Alex

Utterances		*Descriptors*
Alex:	I would like to make things, connections with the things you are not supposed to learn, […].	
	It's not that, oh, this person is smarter, that person is dumber.	*Rejects narratives about ability*
	I just think this goes back to what you said. What types of connections should be made.	
	Right now, as a teacher living in 2020 in this school,	*Reflects on current events*
	I am advised to make some connections and I am advised to not make other connections. Not that. Oh, Alex don't do this.	*Influences from colleagues*
	To the fact that I understand I have students in front of me, that probably don't want to be in here right now, probably want to be playing video games or messing around with one another, or whatever.	*Reflects on students' behaviours*
	Right now, I have a limited time.	*Reflects on current events*
	I should probably spend this time telling them what they are expected to know in this time and place of the world they live in.	*Implicit reference to curriculum*
	I'm happy to know, inside of me, but also sad at the same time, that what you're doing has connections, you cannot see it right now, […].	*Conflicting feelings*
	I can try and make the connections, but obviously there's the variable of they will not listen or not only they will not listen, but they will deconstruct their learning or their classmates'.	*Reflects on students' behaviours*
	And that's something I have [sighs] faced many times. And I'm trying to improve.	*Reflects on teaching practice*

Alex, David and Liz share narratives about identities they want their students to associate them with. Liz claims she is using her identity as a woman to empower her female students who do not feel confident about mathematics. David states that he enjoys being spontaneous in his lessons and talk to his students about the history of mathematics. Through the examples he shares with his students he would like them to say that "he is really passionate […] he's enthusiastic". Alex is trying to "create an identity" as a teacher to aid him manage his students' behaviour:

Alex:	I just try things, and the last couple of months, things that I've tried worked fantastically, other things, that I wished would work, didn't. So, I am adapting and trying things. What I have found is to create an identity, to, to have OK, now we're finished, if we tidy up quickly, Mr. [Surname] will show us a random video.

Alex states that he collects videos related to mathematics he finds interesting which he shows to students at the end of his lessons, if there is some time left (see also Section 6.5). Generally, however, he claims that he sees himself as "the tour guide". Table 6.13 includes the analysis of the excerpt where Alex presents a series of allegories to describe the roles of researchers and his role as a teacher.

Table 6.13 Excerpt from the interview with Alex

Utterances		*Descriptors*
Alex:	This is why maybe I landed to teaching at this part of my life. I don't think I'm the kind of person [who] is gonna be like, we are going to invent. We are going to venture something new.	
	I may be the tour guide and guide people in the cave {mathematics}. Okay, guys, this is it. Yeah? When your hearts hatch. No, don't go there. You cannot swim {understand} yet. No, no, no, no.	*Allegory of himself as a teacher*
	Em, Mr. Jeffrie's [fictional character {researcher}] down there right now digging really deep {contribution to knowledge}. He has a mask {appropriate equipment}. He has learnt scuba diving {advanced mathematics}. He's been almost king 10 times {awards and scholarship}. He knows how to kill dinosaurs {research skills}. He can go.	*Allegories for mathematicians*
	We? No, we're gonna stay here and walk {explore} around where there are benches {teaching resources}.	*Allegory of teaching and learning*
	This is what I think. And this is what I saw through university.	
	I didn't have the kind of passion other people did. Therefore, I thought I would not be valuable in a position like that. But I like the stories I like the concepts, so maybe that's what keeps me going.	
{ }: my interpretation of the allegory		

Through his emotional description it is evident that he sees himself being among his students, possibly protecting them in the "cave" while showing them the "stories" he likes. It is interesting how Alex describes the work of research mathematicians starting from scuba diving and having the required equipment to having accomplished mythical quests which possibly indicate his admiration for people that do research but also the complexity of the activity. It is not known whether Alex is aware of the work of Ball and Bass (2009) or whether the use of the analogy was incidental. However, his view of himself as a tour-guide, who has the role to keep his group of students ‘safe’ while roaming a deep cave, is the opposite of an “appreciative tourist” (Ball & Bass, 2009, para. 17). The two identities Alex presents about himself, as someone who is enthusiastic about mathematical ideas around him and as a tour-guide, are contrasting, possibly indicating a clash between his aspirations and demands stemming from his other roles as a teacher.

Another aspect the participants discuss is students’ age. For example, Liz distinguishes between teaching at GCSE level where she talks more about elements from the history of mathematics as she doesn’t “tend to pull much of that content down”. Whereas with A-Level students she talks about studying mathematics at a degree level, specifically to empower her female students, characteristically saying “yes, of course you can as a woman you definitely can do a maths degree”. David also mentions that his younger students (e.g., Year 7) “may not ask so many questions”:

David: So, in Year 7, a lot of them have dutifully written down minus an a minus makes a plus, plus and a minus makes a minus, plus and plus. And they've written it all down and it's great. But I've written next to them, to their notes, why is this? Can you tell me why? Because, in my head I think I've taught them it. And they may know it, but they haven't written it down. And I want them to think why is that? And for me, it's all about inner reflections in number-lines and things like this that make sense to me.

In this example, David claims that he has shared with his students the reasoning behind the properties of multiplication in real numbers, and he expect that his students will use this reasoning. He seems to feel responsible to ask questions about what students’ do not ask on their own. Whereas he claims that Year 11 students question more what they learn thus providing more opportunities to go beyond the curriculum.

Finally, discourses around attainment are dominant when teacher’s talk about their decisions to go beyond the mathematics of the moment. Narratives about attainment are expected to be found in the teacher’s responses, as for the majority of the secondary schools in England students are divided according to their grades. However, differences are observed in how the attainment of the students is perceived as a limitation. These differences are linked with observations regarding the characterisation of the teacher’s pedagogical discourse. For example, Liz, whose pedagogical discourse was positioned towards the Explorative side of the spectrum, still considers students attainment as a factor. She tries to “stretch” more her “top set” students than she would a “mixed ability group”. She mentions:

Liz: So, I do think that, you know, I teach at lower set year nine group, they have no interest in... There's, they have no curiosity about maths, I suppose. And maybe I should address that actually and think, well, I should be, I'm, maybe I should make them more curious. […] I think they would just say, ‘I'm not interested, it means, it's meaningless to me’. Um, but maybe I'm, maybe I'm, maybe I'm wrong. Maybe I should try and do something more with them.

Liz expresses her view that the attainment of the students is related to the students’ interest and curiosity but at the same time she challenges her view saying that she should probably do something about it and engage her lower attainment students.

Thomas, who proclaims to be “a believer […] that any idea can be accessible to anyone of any age in some meaningful way”, did not present any narratives about attainment in his interview when talking about his teaching. He says in his interview that the way to communicate with students beyond the mathematics of the moment is through “meta-comments”:

Thomas: Yeah. Because you can, um, you can give, sometimes while you're working on something you can make meta comments, about what it is that's happening now. So it could be that, um, it could be a meta comment about the fact that that sometimes it is really helpful to, to name things that can be said when something gets named. It's really helpful at times to give a notation to something probable. So there can be these, that comment that get said at the time that something happens, and then that comment can be said again later and later or later so that students become aware and begin to think about the possibility of of labelling, naming, um, and, um , and so on themselves.

Thomas, who's pedagogical narratives were categorised as Explorative (towards the end of the spectrum), maintained consistency and balance between his pedagogical arguments and his mathematical examples following those arguments. Thomas recognises, notating as a key idea in mathematics and a central step towards abstraction. Here, he explains his claims about making "meta-comments" using the example of notation. For him, the teaching practice of making "meta-comments" is longitudinal, repeated whenever there is an opportunity for the students to explore the use and relevance of the mathematical idea or practice. His professed action of meta-commenting could prepare the students for meta-level learning in the future. Meta-commenting was observed during opportunities to discuss ideas and practices that run across the curriculum, in teaching practices of Alex (e.g., 5.2.2) and Liz (e.g., 5.2.5). Moreover, he explains the processes he follows when he is looking for ways to communicate with his students:

Thomas: So, I was often trying to teach maybe parts of the curriculum that was considered too advanced for children of that younger age. And so, I was trying to find out a way in which I could make this accessible to them and seem as if it's relatively, um, straightforward. So, I was often looking down in terms of what is it within their experiences? Not always only mathematics experiences, but whether their experiences within them, their life and what they do in their life that I could find I could relate to this particular, um, topic. And so, so I often did, looking down, I think maybe looking down more than looking up. But however, I, I know that there are times that I do I am aware of things that are to follow, and sometimes it's quite interesting about whether you go you go to there what is to follow and then come back to what they say is maybe expected at the moment because they know that that places a context of something a bit more advanced. They don't have to understand the advanced, but they may be just an awareness that there is this stuff here. But then saying that an aspect of that is this that we work on at the moment. And so there are times I also am looking up and thinking about this is just the beginning of, of something else.

Thomas indicates that the process is not linear and it involves looking for experiences the students might have that he could use to communicate the ideas in an accessible way. Trying to identify experiences in his students' precedent space (Lavie et al., 2019) and projecting to what the students will learn in the future is an essential idea in the literature about the Mathematical Horizon (Ball et al., 2008; Ball & Bass, 2009; Cooper, 2016; Jakobsen et al., 2012). Moreover, his claim about finding ways to communicate advanced ideas with young students is in line with findings presented in Chapter 5 regarding the role of intersubjectivity in achieving effective communication beyond the mathematics of the moment.

On the opposite side, teachers whose pedagogical narratives were located mostly towards the Delivery Pedagogical Discourse, appear to perceive student's attainment as a barrier that cannot be removed. Eric, for example, mentions that he focuses on linking topics within the GCSE curriculum "especially the lower set because their retention is quite poor". When asked about making connections between the GCSE and the A-level curriculum he replied that "the split between Foundation Higher [GCSE] almost acts as a barrier to making that connection sometimes". In this case, the use of the word "retention" is significant because it indicates a focus in memorisation of rules as opposed to explorative learning. Similar observations are made in the interview with Nick who indicates that he would discuss connections between GCSE and A-level mathematics with his "bright" students, the one's that "are predicted G9 or G8". G8 and G9 refer to the grading system his school uses to predict GCSE scores for students, these scores are associated with "highly attaining pupils" (Department for Education, 2014a, p. 3) who should be taught additional topics according to the National Curriculum. Nick uses these scores to characterise his students' numeracy and literacy throughout his interview.

Overall, in this section I discuss the participants narratives about their students in relation to their discourses at the Mathematical Horizon. The emerging findings suggest that participants' decisions to deviate from the lesson of the day is linked to narratives about the age of their students, their behaviours and interests, and their perceived attainment. Subjectification of narratives beyond the mathematics of the moment are observed in the discourse of the participants who were interviewed. In other words, the perceptions of the participants about the students were found to be connected with their discourse at the Mathematical Horizon as a factor that impact what, when and how they share content beyond the curriculum with their students. The findings support the idea presented in section 6.4, which claims that Discourse at the Mathematical Horizon is an amalgamation of mathematical and pedagogical discourses.

6.9. Summary of findings

In this chapter, I explore the experiences which shape individual discourses at the Mathematical Horizon. The analysis of the interview data indicates diversity in the participants' narratives about teaching practices beyond the mathematics of the moment and thus, in their discourses at the Mathematical Horizon. The variations can be linked to narratives about experiences during their tertiary education, personal and professional life as well as their narratives about teaching, the curriculum and the students. The findings suggest that Discourse at the Mathematical Horizon is an amalgamation of Advanced Mathematical and Pedagogical Discourses. In the following paragraphs, I discuss in more detail the findings that support the latter claim.

The results of the analysis indicate that the participants' discourses at the Mathematical Horizon are influenced by their engagement with advanced – including applied – mathematics in both formal (e.g., university and professional development activities) and informal settings (e.g., independent learning, personal experiences, reading, searching online and/or discussions with colleagues). Sections 6.2, 6.3, 6.4, 6.5, and 6.6 illustrate the various ways in which teachers can develop their discourse at the Mathematical Horizon through different activities at various points of their teaching career. Moreover, the findings in Section 6.5 support the idea that the teachers do not necessarily rely on distant memories (e.g., from a university course). Rather, a familiarity with the potentials of the mathematics involved in the discussion and access to the resources could be a starting point to develop arguments beyond the mathematics of the moment.

The findings discussed here aid in mapping the fluid boundaries between University Discourse, Advanced Mathematical Discourse, Pedagogical Discourse and Discourse at the Mathematical Horizon which will be discussed in the following paragraphs.

Teaching and learning of advanced mathematics is often considered in the context of university education (e.g., Even, 2011; Pinto & Cooper, 2022; Wasserman et al., 2019; Zazkis & Leikin, 2010). Therefore, the descriptions of advanced mathematics are often focused on University Discourse, i.e., ideas and practices taught in institutions (e.g., university, college etc). For instance, Advanced Mathematical Knowledge (Zazkis & Leikin, 2010) is defined as "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (p. 264). However, the findings indicate that university studies, although important, are not sufficient to describe the advanced mathematical narratives of the participants which were found to influence discourses at the Mathematical Horizon. Thus, I propose seeing university studies as a threshold between elementary and advanced mathematics and expand the context in which advanced mathematical learning is considered. Under this assumption, applications of mathematics can be considered as elementary (part of primary and secondary mathematics education) or advanced (tertiary education and beyond). I view Advanced Mathematical Discourse as opposed to Advanced Mathematical Knowledge (Zazkis & Leikin, 2010), in the context of the study based on emerging findings as the Discourse which includes word use, visual mediators, narratives and routines which teachers might encounter through the engagement with advanced – including applied – mathematics both in formal and informal settings.

Whilst the Advanced Mathematical discourses of the participants were found to influence their narratives at the Mathematical Horizon, the observation is not enough to describe the actions of the teachers beyond the mathematics of the moment. The findings in Sections 6.6, 6.7 and 6.8 suggest that discourses at the Mathematical Horizon are influenced by pedagogical narratives, specifically, narratives about teaching and their role as teachers, the curriculum and their students. Thus, narratives beyond the mathematics of the moment can be subjectified, i.e., "the discursive focus shifts from actions and their objects to the performers of the actions" (Sfard, 2008, p. 113). Considering this evidence, the variations in the narratives of the participants regarding the influences of university studies and personal and professional experiences to teaching beyond the mathematics of the moment can be explained through discursive shifts that enrich advanced mathematical patterns of communication with pedagogical meaning through the engagement with teaching. The shift requires the introduction and use of intersubjective discursive elements that are not typical in advanced mathematical discourse but are relevant to the experiences of the teacher and the students. This claim is further elaborated upon in Section 8.1.2 where I propose a refined definition of Discourse at the Mathematical Horizon in light of the empirical evidence.

The pedagogical narratives of the participants were categorised and discussed according to Nachlieli and Heyd-Metzuyanim (2022) definitions of Delivery (traditional teacher-centred values) and Exploration (explorative learning values) Pedagogical Discourses. The findings suggest that participants whose pedagogical narratives in the majority align towards the Exploration end of the spectrum challenge views related to students' attainment, curriculum limitations and risk-averse pedagogies and claim intentions of including discussions beyond the mathematics of the moment in their teaching. On the other hand, participants whose pedagogical narratives in the majority align towards the Delivery end of the spectrum depict students' attainment and meeting the requirements of the curriculum as barriers for discussions beyond the mathematics of the moment. The observation is in line with Delivery Pedagogical Discourses which locate explorative discussions beyond the scope of an ordinary lesson in contrast to Exploration Pedagogical Discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022).This observation is also collaborated by the results of Chapter 5. In particular, the majority of episodes that demonstrated taken opportunities in Chapter 5 comes from Liz's lessons whose pedagogical discourse can be generally characterised as Explorative. Episodes from Alex's and Nick's lessons were fewer and coded as potentials of the discussion. During the interview, Nick's utterances evidenced narratives that are in line with Delivery Pedagogical discourse while Alex expressed conflicting narratives when talking about his aspirations as a mathematics teacher and dealing with classroom management. Alex's utterances evidenced both Delivery and Explorative narratives equally.

Finally, a signalling difference between University Discourse and Discourse at the Mathematical Horizon is the observation that teacher's learning at the Mathematical Horizon is not necessarily institutionalised. Specifically, changes in Discourse at the Mathematical Horizon can happen throughout their education, but also by interacting with colleagues, life experiences and engagement with books and other resources if the teacher sees a relevance of the advanced content in their teaching practice. Thus, Discourse at the Mathematical Horizon is evident when focusing on specific situations and interactions of the teacher with others (e.g., colleagues, educators etc) and/or resources. The final observation suggests that engagement in communities of practice (Lave & Wenger, 1991) could provide structured and longitudinal opportunities for teachers to develop their discourses at the Mathematical Horizon (see also Section 8.4).

In the next chapter, I explore and propose a design of professional development activities that could bring up opportunities for teachers to develop their discourse at the Mathematical Horizon.

7 Tasks designed to bring up Discourse at the Mathematical Horizon

The final empirical chapter of this work focuses on the idea of creating and using professional development activities – here mathtasks – to explore and develop Discourse at the Mathematical Horizon. In particular, this chapter addresses the question:

RQ 2 How can empirical evidence be used to create practice-based resources for developing teachers' [d]Discourse[s] at the Mathematical Horizon?

In the following sections, I elaborate how I approach the task design and the analysis and the evaluation of a pilot mathtask (Section 7.1). I present an a-priori analysis of a pilot mathtask (Section 7.2). Then, present a reflective account of the implementation of the mathtask in an online focus group (Section 7.3) and discuss the evaluation of the mathtask based on the objectives of the focus group and findings of the previous chapters (Section 7.4). Next, I propose a design to create professional development and research resources using empirical findings and data from the study informed by the evaluation of the pilot in the focus group. Finally, I summarise the key findings of the chapter (Section 7.6).

7.1. Synopsis of the data analysis and task design

Mathtasks are vignette-based resources designed based on the MathTASK principles (Biza et al., 2018, 2021). The MathTASK design consist of three elements: a mathematical problem, a fictional situation (vignette) and a set of questions. The mathematical and pedagogical content of resources offer contextualised opportunities for discussion of "a topic or an issue that is known for its subtlety or for causing difficulty to students" (Biza et al., 2018).

The first part of the chapter (Sections 7.2, 7.3 and 7.4) concerns the design, implementation, and evaluation of a pilot mathtask. To evaluate the design of the mathtask I conducted a focus group intervention with 4 teachers. The four teachers who participated in the focus group were Eric, who is head of department in his school, Nick, who is an advanced skills teacher, Alex, and Sam, who both have less than 5 years of teaching experience. Figure 7.1 depicts a printable version of the mathtask used during the focus group. Details about the design of the pilot mathtask and the focus group can be found in Sections 4.4.3 and 4.4.4. The mathtask was created during the first year of my doctoral studies and finalised in June of the following year. At the time, I conducted an a-priori analysis of the mathtask and set the objectives of the focus group intervention. Later, the a-priori analysis was revisited and rephrased in light of the findings presented in Chapters 5 and 6 to maintain consistency across the thesis. The implementation of the mathtask was analysed based on empirical findings in Chapters 5 and 6 in relation to the objectives. The second part of the chapter (Section 7.5) concerns the design of vignette-based resources to bring up discourses at the Mathematical Horizon. I draw on the MathTASK design principles and discuss my proposal based on the findings in Chapters 5 and 6 and Sections 7.3 and 7.4 and utilising the collected data from the lesson observations.

math TASK
@mathtask

The Area of a Triangle

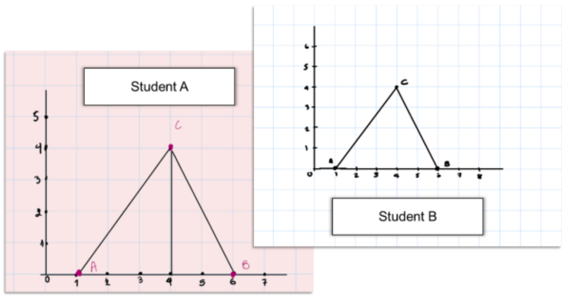
In a Year 8 class, the teacher gave the students the following mathematical problem:

> *Use squared paper to draw axes for x and y from 0 to 6 using 1 square to 1 unit. Find the area the triangle ABC with A(1,0), B(6,0) and C(4,4)* *

Students A and B work on the problem when the following conversation takes place:

Student A: I found 10!

Student B: Me too! Wait … That can't be right! Your triangle is much bigger than mine… (Student A puts the two triangles side by side as in the figure below) How can that be possible?



Student A: I don't know … I'm sure about my answer … you see … the height is 4 and the base is 5 …

Student B: Yes, I did the same … so weird!

The teacher overhears the conversation and joins in.

Teacher: What is weird?

Student A: Emm … we both have found that the area is 10 … but our triangles are not the same. Is it because our drawings are not accurate?

Teacher: No! Both are fine! It's just because you have different squared paper. Your paper is in centimetres and yours is in inches.

Student A: Oh, I see … ok so mine is 10 square inches and yours …

Student B: Yeah, yeah … I get this, but still it is weird, isn't it? The area is the same, but it isn't, is it? Your triangle is larger than mine so you should have larger area. Oh … I am so confused …

Questions:

a. Solve this mathematical problem bearing in mind that this is a Y8 lesson.
b. Which parts of the conversation attract your attention?
c. If this conversation has taken place in your class, how would you respond to student B and to the whole class?
d. What would be the focus of your response on this topic?

Figure 7.1 Printable version of pilot mathtask also depicted in Section 4.4.3

7.2. A-priori analysis of the pilot task

In this section, I revisit the main points of the conceptualisation and the finalisation of the task before the pilot along with an a-priori analysis of the mathtask.

The idea for the mathtask in Figure 7.1 came to me while studying the National Curriculum and the available textbooks. As a thought experiment, I was trying to create fictional moments of contingency stemming from mathematical problems that would be considered typical for a secondary school classroom in England. The mathematical problem posed in this instance is an adaptation of a series of activities included in the STP Mathematics 8 (Bostock et al., 2014) textbook for Year 8 students. The activities instructed the students to draw shapes based on the coordinates of their vertices and calculate their area. The activities stood out for me because they do not give specific instructions as to what the dimensions of the square paper should be in contrast to similar activities in the book. In Figure 7.2, I exemplify my claim using two activities from the book. The first activity (Figure 7.2, Part A), leaves the choice of the unit open for the students and the teachers, although, a complete answer to the problem requires a reference to the unit (e.g., 3 cm^2, 3 in^2 or 3 square units). However, the second activity (Figure 7.2, Part B) gives specific instructions that the unit should be 1 cm despite that the solution to the problem depends on the accuracy of the drawing and not on the specific unit. Leaving the choice open could lead to a situation where students could use different units depending on availability of resources.

A

6 Use squared paper to draw axes for x and y from 0 to 8 using 1 square for 1 unit.
Find the area of triangle ABC as a number of square units if the coordinates of the three corners are $A(2,4)$, $B(6,4)$ and $C(2,1)$.

(Bostock et al., 2014; p. 230)

B

In questions 18 and 19 use 1 cm to 1 unit.

18 Draw axes, for x from -5 to 5 and for y from 0 to 5. Draw triangle ABC by plotting $A(1,2)$, $B(3,2)$ and $C(3,5)$. Draw the image triangle $A'B'C'$ when triangle ABC is reflected in the y-axis.

19 Draw axes, for x from 0 to 5 and for y from -2 to 2. Draw triangle PQR where P is $(1,-1)$, Q is $(5,-1)$ and R is $(4,0)$. Draw the image triangle $P'Q'R'$ when triangle PQR is reflected in the x-axis.

(Bostock et al., 2014; p. 211)

Figure 7.2 Recreation of two activities in Bostock et al. (2014).

In Year 4, students come across the idea of coordinates and learn how to plot points in 2 dimensions (Department for Education, 2013b). In KS3 students are expected to work with coordinates in geometric problems, calculating lengths and the perimeter or the area of the shape created, and creating graphs of functions (Department for Education, 2013a). Therefore, the activity could take place during a lesson on calculating the perimeter and area of shapes on a coordinate system in Year 8. The method of solving the mathematical problem might depend on the program of study followed by the school and the allocated set for a specific group of students. The activities included in the STP Mathematics textbooks are targeted towards "a high level of achievement at Key Stage 3" (Bostock et al., 2014, p. vii), however, similar activities can be found in various curricular resources and thus the particular activity discussed here seems to be appropriate for students which might be considered as 'low achievers' as well. A student could solve the mathematical problem either by applying the formula for the area of the triangle $a = \frac{1}{2}h \times b$ where b is one of the sides and h the corresponding height. Alternatively, a student could calculate the area by counting the squares.

The fictional moment of contingency stems from two students realising that their diagrams are different yet they both reach a correct numerical answer without realising the reference to different units. The students are familiar with different number systems from KS2, they are expected to recognise different units, be able to measure using imperial and metric systems and recognise that the results would be different depending on the unit used. In the fictional situation, each student followed the same instructions in different systems resulting in the same numerical answer that might be contradictory based on their previous experiences. From an advanced perspective, if given a specific (tangible) object, e.g., a line segment on a piece of paper, measuring its length in inches or centimetres will result in different numeric values. However, the idea of choosing an arbitrary unit changes the representation of an abstract object, but not the numeric value of its length or the area. Therefore, choosing a unit to be represented by one inch or one centimetre to construct a line segment of length 2 units will result in different representations of the object. Here, choosing different units will result in different representations of a triangle with an area of 10 square units. Students of that age are transitioning from concrete (student's current discourse) to abstract mathematical objects and might not yet be aware of this tacit idea (meta-discourse). Within each system of measurement, the results are consistent. The contingency here is not realised until the two students look at each other's answers.

Exploring the contingency, is viewed as appropriate for studying Discourse at the Mathematical Horizon because discussing such issues is not on the focus of a lesson about area or coordinates, but it offers an opportunity to discuss the use of different systems of measurement, abstract units and the importance of referring to them when measuring or calculating the size of an object.

Following the thought experiment, I created a first draft of the mathtask by recreating the mathematical problem and a fictional dialogue aiming to highlight the contingency. I shared the first draft of the mathtask with my supervisors and RME group members for further discussion and received suggestions for improvement. Closer to the focus group, the mathtask was chosen and finalised among three mathtasks created for the purposes of the study to be the first one in a series of focus group interventions. Initially, my plan was to have more than one focus groups using the other two mathtasks (Appendix VII), but this plan did not materialise. The three mathtasks had similar objectives and focus. The mathtask in Figure 7.1 was chosen as the first one in the series for practical reasons, the mathtask in Figure 7.1 was the simplest, thus easier to finalise on time for the first focus group. The other two mathtasks were inspired by the work of Tim Rowland (e.g., Rowland, 2013) and developed in consultation with him and other members of the RME group.

At the time when the focus group was about to take place, I had completed a preliminary analysis of the lesson observations and the interviews that had taken place thus far. To maintain the consistency of the word-use and the narratives across the chapters, I now revisit my design intentions in light of the findings presented in Chapters 5 and 6. The preliminary findings suggested that the contingency represented in the dialogue was in line with a theme of opportunities to go beyond the mathematics of the moment by discussing *mathematical conventions*. The theme included instances of opportunities to discuss the meaning of mathematical conventions such as naming, notating, and constructing working definitions of mathematical objects (see Section 5.4). The fictional dialogue of the mathtask presents an opportunity to discuss the meaning of different units of measurement and present the idea of arbitrary units as a process of abstracting key elements of an adopted scientific convention rather than an arbitrary rule the students need to follow. Moreover, the preliminary data analysis suggested that discussions beyond the mathematics of the moment take place during explorative learning opportunities. Thus, the fictional dialogue might be more plausible in a classroom where exploration is encouraged by the school and/or the teacher and depends on the available resources (e.g., use of different types of graph paper or use of dynamic geometry software). Upon further analysis which took place after the events of the focus group, individual discourses at the Mathematical Horizon were found to be influenced by institutional factors (e.g., the priorities and support of the school), use of resources and the views of the teachers about their role and their students which might explain the initial claim that discussions beyond the mathematics of the moment were linked to explorative learning opportunities (see Sections 6.5 to 6.8). Finally, the dialogue was adapted to be more realistic and make use of words typical for students at this age group.

The mathtask in Figure 7.1 depicts the final version of the mathtask which was presented to the teachers in the focus group. It includes limited information about the students, amending any information about their gender, background or presumed abilities to remove unintended bias (Poulou, 2001) and allow space for teachers to debate. The goal of the dialogue in Figure 7.1 is to create discussion around the mathematical and pedagogical issues of the transition from known to arbitrary units, what stays the same and what could change. Specifically, the task can be used to discuss mathematical aspects of the transition to arbitrary units, e.g., magnitudes of units of measurement, arbitrary units, and abstraction. The dialogue can also serve as a starting point for discussions about teaching and learning, e.g., what the students are expected to learn, what they have learnt in the past or how to support them in learning about arbitrary units. The choice of questions follows the MathTASK design principles (see also Section 4.4.3) starting with a request for the teachers to solve the mathematical problem before commenting on the specific task situation presented in the fictional dialogue.

My overall aim in this chapter is to propose a set of design principles for creating professional development activities which could bring up teachers' discourses at the Mathematical Horizon. The basis for the design is inspired by mathtask activities. The MathTASK principles make provisions to foster Horizon Content Knowledge to surface (Biza et al., 2018, 2021). However, the principles do not specify how this can be achieved. The pilot mathtask was designed based on the MathTASK principles on a topic that goes beyond the mathematics of the moment. My aim was to use the mathtask in a focus group discussion and explore the potential of the design in order to improve and refine future designs. Specifically, the first objective of the pilot focus group was to explore whether the mathtask can engage the participants in a debate about opportunities for discussions beyond the mathematics of the moment. The second objective of the pilot focus group was to explore whether there are any discursive shifts observed in the mathematical and pedagogical narratives of the participants during the discussions. In the following section, I present a reflective account of the events of the focus group in relation to the objectives of the pilot and discuss how the observations can feed forward to inform the overall aim of the design in conjunction with findings from Chapters 5 and 6.

7.3. The implementation of the pilot mathtask during the focus group

In the following section, I offer a reflective account of the events during the implementation of the pilot mathtask during the focus group. In Section 7.3.1, I reflect upon whether the mathtask can engage the participants in debates about opportunities to go beyond the mathematics of the moment. Then, in Section 7.3.2, I reflect upon what discursive shifts can be observed in the mathematical and pedagogical narratives of the participants in relation to their discourses at the Mathematical Horizon.

7.3.1. *Engaging in a debate about opportunities beyond the mathematics of the moment?*

The section presents a reflective analysis of the debates of the participants during the focus group. I focus on the mathematical and pedagogical narratives of the participants and explore whether these narratives suggest that the conversation could foster individual discourses at the Mathematical Horizon to surface. The emerging findings suggest that the participants' narratives are in line with the findings in Chapters 5 and 6.

The participants were presented to the mathtask in two parts. First, they were presented with the mathematical problem alongside a question to solve it bearing in mind that it is designed for Year 8 students (Figure 7.1, question a), the results to the mathematical problems were discussed before the introduction of the dialogue. Next, they were presented to the dialogue and asked to spend some individual time to read the dialogue and reflect on it based on questions (b) to (d) in Figure 7.1. Their reflections were, then, discussed as a group commenting on each other's thoughts.

During the focus group the participants shared mathematical and pedagogical narratives that are in line with the descriptions of the sub-discourses of the Mathematical Discourse for Teaching (Cooper, 2016). While the participants were engaged with the first part of the mathtask, i.e., the mathematical problem, their narratives aligned with Common and Specialised Content Discourse (Mathematical Discourses) and Pedagogical Content Discourses. Narratives related to discussions beyond the mathematics of the moment and, thus, narratives that are in line with Discourse at the Mathematical Horizon were only observed after the introduction of the dialogue.

Towards the beginning of the focus group, the participants took some time to individually solve the mathematical problem and then the answer was discussed:

Nick:	It should be 12 I think, 14 or 12.
Alex:	Oops. I, I think I found 10
Eric:	I got 10. Yeah.
Me:	10?
Alex:	You scared me there, Sir [laughs].
Nick:	So, what's?
Sam:	Yeah, I got, well, 10 unit squared, if you want to be...
Nick:	Ok. Yes.

Solving the mathematical problem brought up discussions about the correctness of the numerical answer. In the above discussion, only Sam refers to the units, commenting “if you want to be…” possibly ‘precise’ but he is interrupted. However, the lack of reference to units could be because the focus was on comparing their calculations. The narratives indicate that at the time the teachers participated in a way “which educated adults participate in mathematical discourse.” (Cooper, 2016, p. 22). Despite being asked to solve the problem having in mind that it was intended for Year 8 students they do not appear to consider the content in relation to teaching at this stage. The participants were, then, asked if they would like to comment on the task and how they would use it in the classroom. The prompt led to teachers sharing narratives of the problem in relation to their teaching.

Nick was the first to speak and he focuses on different ways of using the problem referring to the ‘ability’ of the students:

Nick: Well, I think that, practically will be two, three ways of doing that, if you have, uh, low ability students which you have in a classroom, practically, better, plotting the points, joining the point[s] and maybe counting the squares? […] Before introducing the formula. Maybe then with, thinking of the formula and then applying the base and vertical height, and if you have a much advanced class, maybe they can do it without even any diagrams from the point you [connection dropped], if they can. Act like, like an extension without using the radical method.

Nick's narratives show elements of Specialised Content Discourse when he reflects on different ways of solving the problem. Nick matches the different methods of solving the problem with different 'abilities' of students, i.e., plotting points and counting squares – low ability, plotting points and using formula – medium 'ability', algebraically. The matching to 'abilities' suggests a teaching practice in English schools where students are segregated and taught in 'ability groups'. Thus, Nick's narratives show also elements of Discourse of Content and Teaching which includes narratives about teaching routines.

Supporting Nick's answer, Sam adds his comments briefly mentioning the choice of grids:

Sam: […] There's obviously a lot of places that they can go wrong or right. Um, you know, obviously you've got how they might draw the grids, the spacings, um, how they might plot the points could all lead to an incorrect answer, but like a correct method. So, there's a kind of, yeah. So, if you just, I mean, it could be quite nice to to give a class that you've maybe not taught before something like this because you can see how they might approach it. And obviously, if they've come across the formula before, they might kind of start to draw it, but then skip that stage, so it could be a quite interesting task to see them approach. I think the issue is that you would get in maybe doing things at different speeds if you give them more practical tasks, and that obviously could create different issues.

Sam mentions the choice of spacing on the grid, which is connected to the idea of units being a potential space for error, specifically "leading to an incorrect answer" but with "a correct method". This view is contradictory to his earlier mention of the answer being in 'units squared' if the problem is solved algebraically and not by counting squares. Sam suggests using the task as a diagnostic tool in a classroom. He characterises the problem as "practical task" and makes a comment regarding issues that could be triggered by the use of the activity depending on the "different speeds". Sam's comment might indicate a tendency to avert from "practical task[s]" to avoid alleged issues. However, Sam does not specify what constitutes the problem 'practical'. Sam's narratives are in line with the description of Discourse of Content and Curriculum as considerations of "which curricular materials do they value, which do they use, what is the basis for their decision, etc" (Cooper, 2016, p. 24).

Finally, Eric comments on changes he could make on the problem to adjust it for his students:

Eric: I think if I did it with one of my classes, depending on the, the ability, I would maybe want some. Um, maybe some scaffolding [connection break] you know, I might actually have have some of the, you know, the axes drawn out, for example, with the numbers on it, cause some might have trouble to, actually, draw the axis [connection break]. So, I might, for some of the weaker students I might potentially have the axis drawn out, and I would think about my questioning for the height. You know, how can you work out the height if you, you gonna do that, that they're going to get away from counting squares as well, I think. Um, so to mention ideas you could maybe, can you change a coordinate to maybe change the area of the triangle somehow? Or, I mean, I just. Can you change, coordinate C so the area is now 12, for example?

Similar to Nick and Sam, Eric also considers the problem in relation to perceived abilities. Eric's reflection on using the problem with his class focuses on adaptation and scaffolding. In particular, Eric suggests questions that he could ask students of different 'abilities' to prompt them and help them solve the task. Eric's narratives, here, align with Discourse of Content and Teaching as he suggests teaching practices he would have used, but also, Discourse of Content and Curriculum as he is considering the ways in which he could use the specific resource.

Eric, Nick and Sam comment the perceived abilities of students which might indicate that their narratives process elements of Discourse of Content and Students. However, there are variations in their narratives about their students. For Nick, the ‘abilities' of the students determine his expectations from them when engaging with the problem. For Sam, the problem is seen as a tool to determine the 'abilities’ of the students. While for Eric, the problem can be changed to build upon their ‘abilities’. This observation is significant and will be revisited later on in relation to Discourse at the Mathematical Horizon and observed shifts in the discourses of the participants in Section 7.3.2.

The responses of the teachers focused on using the problem as an activity on calculating the area of a triangle. Consistency within measuring systems was not mentioned prior to the introduction of the dialogue. Thus, the responses of the participants support the initial hypothesis that discussions about measuring systems in relation to the specific mathematical problem may be considered as going beyond the mathematics of the moment.

In question (b), “which parts of the conversation attract your attention?”, the teachers were asked to reflect on the dialogue and then discuss what part of the fictional conversation captured their attention. Eric and Alex focused on the use of the word ‘same’ and the lack of units. Specifically, Alex focuses on the part of the dialogue where the students compare their triangle and moves his attention into possible reasons why the triangles are different. Eric focused on the language and the role of the teacher as gatekeepers of mathematical meaning and terminology:

Eric: I, what struck me was two bits really. The first was, the first thing that Student A said when he said I found 10, as in, it didn't, 10 what? And it just shows how important that is, well apart from that, the triangles that we use and what we explain, and we explain the, the vocabulary we're using. So, it's not acceptable thing to say I found 10, you know, I found 10 centimetres squared or 10 inches squared. So, that first point really, to me, reiterated how important it is to be really specific with language when you are teaching, especially when modelling something.

Although Eric’s answer to the first part of the mathtask was also “I got 10”, Eric comments on the lack of units, and then focuses on the meaning of the word “same” moving his attention to his own practices. In relation to mathematical practices Eric describes the task as a modelling routine which indicates the realisation of the triangle as an abstract object which is signified by a drawing. It is unlikely however, that a Year 8 student will realise the task as a modelling routine thus from this perspective, Eric's narrative is advanced in comparison to students' learning. His attention to language use focuses more on teaching and the responsibility of the teacher to carefully set the rules of the mathematical discourse of the classroom. Eric's focus on language-use is in line with his earlier claims of paying attention to how he would phrase questions to help his students. His narratives echo findings discussed in Section 6.4 which indicates that Discourse at the Mathematical Horizon is an amalgamation of advanced mathematical and pedagogical discourse as the attention to mathematical vocabulary become relevant in relation to teaching.

On the other hand, Sam and Nick focus on the environment of the fictional scene and the confidence of the students.

Sam: Yeah, I mean, the things that attract my attention reading through, I thought the, um student A was very confident in their method. You know, they were like, well, I know the answer is 10. Uh, I also thought they were both very engaged in the task. And, and from the kind of the text, you know, they seem to be very happy to share ideas and talk to each other, um, but in terms of the confidence, it seemed that student B would be more doubtful of himself. Uh, so yeah, those are the parts that attracted my attention and obviously the, the kind of talking about the units and the fact that something is the same size or not the same size. […]

[…]

Nick: Yeah, I just want to touch a, one point, which, uh, if you like, uh, the technicality of the question in terms of the units, mentioned by, uh, everyone. Uh, I go for, that in this classroom, the environment is so encouraging for a student to discuss their answers, to share their ideas and to raise questions. And that what we always call it a good practise in classroom, which is behaviour for learning. So, students are not like, have my answer that's it, finished right or wrong, tick or cross, bye. This is a sort of dimension of, uh, talking about that and the dialogue of extending that to the teacher where he intervenes as well.

Both Sam and Nick briefly mention the different units in their answers. However, they appear to be focused to pedagogy and not to the mathematical content. They both express narratives about the students (e.g., students' confidence, engagement, participation) which could be linked with Discourse at the Mathematical Horizon, specifically, the subjectification of narratives beyond the mathematics of the moment. However, neither of the two participants appear attentive to discussions beyond the mathematics of the moment at this point. In their answers their mathematical and pedagogical narratives in relation to the dialogue are presented as two distinct points that attract their attention.

At this point, the teachers started discussing the issues presented in the fictional situation and started noticing the contingency, supporting that the choice of the fictional situation is an appropriate conversation piece. However, the participants are not yet engaged in a debate about discussions beyond the mathematics of the moment. In particular, the responses of the participants seem to have different focus.

The teachers are then asked to elaborate how they would have responded to student B and the whole class (Figure 7.1, question c). At this stage the participants answered the question by encompassing or rejecting each other's ideas on how to approach the contingency which goes beyond the mathematics of the moment.

Alex, who was the first to speak, said that he would give an "over-exaggerated analogy" as an example to the students. He draws on what Eric and Sam had mentioned earlier to build his argument:

Alex: […] Well, first of all, I would say well done to, that you found this issue and good job for discussing and and then say, OK, we have 10 and 10 and these these students chose different centimetres and inches or happened by accident. And then I would say, OK, let's say we have like a small bacteria doing the same question, and they would draw very little nano-squares of 10s. And then we have giants doing the same thing, working with 10 kilometres and drawing whatever. Then I would say yes, both have the area of 10. But look at the difference well done for finding the number. But as [Eric] said, it is very important for us to identify and make clear the situation. And, uh, I think that's what I would use. I can use this completely crazy analogy to also try and say, OK, guys, we're not going to have a discussion from the practical point of view. OK, stop what you're doing, try and pay attention here. Because, as Mr [Sam's Surname] said, this is this is a situation where you ideally want to have this class listening to you, but sometimes it doesn't happen. So, this, I think, covers both by talking about giants and microbes, in a situation where we're supposed to be doing maths, maybe it will get their attention for a couple of seconds. […]

Alex begins the narrative of what he would do by praising the students. He refers to Sam's earlier claim about the engagement of the student and reflects on his own experience with his students (i.e., "but sometimes it doesn't happen"). Alex's narratives are in line with findings from the lesson observation in his classroom (see, for instance, Section 5.2.2) and his interview where he talks about the challenges he faces due to students poor behaviour and low engagement (see Section 6.8). His discourse at the Mathematical Horizon in the excerpt is visible in the narratives about the "microbes" and the "giants". To produce such example, Alex must have an understanding of triangles as abstract mathematical objects and a realisation of the idea that within a system of measurement the results are consistent. The choice of example fulfils for Alex both mathematical and pedagogical purposes. Alex's proposed teaching practice is consistent with his narratives in the interview about attempts to build an identity as a teacher which would help him manage his students' behaviours (see Section 6.8). Here, Alex subjectifies his narrative by projecting his reflection to (imaginary) misbehaving students.

Nick challenges Alex's approach and chooses to respond using available mediums e.g., graph papers, the interactive whiteboard:

Nick: […] So practically instead of going for a crazy idea like [Alex], which is very difficult to prove in a classroom when you have like last five minutes, I would like quickly to find from the graph paper, which are available in the system [school's repository], […], one based on millimetre, one based on centimetre and count and draw a triangle, which will represent exactly the base of 6. Sorry the base [5, connection break]. This is what I went on to start with, try to, for. And then choose another next to it, different scale graph paper and also count, count for make the triangle so student B will be more convinced of rather than just say it verbally, he used different scale, use, different scale, he use[s] different, uh, sort of graph just show that on the screen, on the interactive whiteboard. And that could clear maybe the problem of most of the students and maybe student B will still be conf-, was still not [be] convinced. But here we are. We always have losers and winners in the game of education.

Nick touches upon practical issues e.g., time constrains and availability of resources to support his proposed teaching practice. Nick's response is focused on replicating the situation the students encountered with graph paper instead of square paper to highlight the different units. However, he does not address the tacit idea that the measurements will be consistent within each system of measurements. Nick's discourse at the Mathematical Horizon is not visible in the excerpt. Nick's conviction that there are "losers and winners in the game of education" is consistent with views he shared during the interview regarding students' 'abilities'. Also, his conviction is in line with his discourse about Content and Students discussed earlier in the section as viewing the student's perceived abilities determining what the students can and cannot do. The conviction might be a limiting factor as to what he sees relevant for discussing with the students as discussed in section 6.8.

Next, Sam draws on what was said so far, summarising the main points of the discussion as he noticed them. Then, he suggests a different approach considering the experiences of the students in primary school:

Sam: […] And so, I think we also mentioned the fact that sometimes we can, you know, forget that quite often the students are led certainly in primary school from a practical perspective, that kind of concrete, uh, whereas this is looking more like a representation. And it's that confusion between the practical element and what you actually have in front of you to just, you know, like formula. So perhaps that these both, these students have a good theoretical knowledge of base times height and half it. And so, yeah, how I would respond to student B is I would, um, what I said in my notes, I said, you know, what is it that you are confused about? You know, and obviously, depending on what they might say to that, you can obviously lead them, uh, to go back to address the un-, the unfinished question that student A had where he said, uh, I can see that I have 10 square inches and yours is what? [...]

Sam responds acknowledging the transition of the students from concrete to representational. His discourse at the Mathematical Horizon is hinted in the excerpt as he recognises that the drawn diagrams are representations of an abstract triangle while the students see them as tangible objects. However, Sam proceeds in proposing to dissolve the misunderstanding by highlighting that their answers are "unfinished" without a reference to specific units. Although, his use of the words 'concrete, and 'representational' might also have pedagogical underpinning. His observation is in line with the Concrete Representational Abstract (CRA) model (Yew Hoong et al., 2015) which proposes an approach for teaching mathematics in three phases as indicated in the name of the model and which is often featured during professional development activities across the country.

Thus far, Eric and Sam have shared conflicting narratives about the idea that if a given tangible object is measured in inches squared or centimetres squared will result in different numeric values. However, the idea of choosing an arbitrary unit changes the representation of an abstract object, but not the object itself. To prompt Sam and the rest of the participants to actively reflect on the idea that the numerical results are consistent in abstract units, I intervened and attempted to link the events in the dialogue with students' prior experience:

Me: Mm-hmm. Um, following that. It's, following what you said about the practicalities of using different units. Uh, I've seen many activities, especially in, in, uh, in textbooks for primary school that say picking a unit and then try to measure something, let's say measure the table. And there are a lot of examples where the students have to measure something with different units in imperial or in, uuh, in the metric, in the metric system, and they find different answers. In this case, though, the students find the same numerical answer. So, would that be something that the student, that could confuse the students? Because they are used to seeing examples where, uh, be, choosing different units result in different answers and not in the same answer? That's a question to all of you.

Sam: Yeah, yeah. Yeah, I mean, like [Nick], you, you can, you can go down that approach of, uh, kind of challenging their idea that it's the same because

> that's, you know, again, it depends on your focus of the lesson, because those two diagrams [in the picture of Figure 7.1] do not have the same area, do they? But they have the same [arbitrary] unit as you said. So obviously, another approach is you could get in to actually use a ruler, you know, actually measure it in the same means and say, OK, when you actually, if we agree that we do the base times the height end we halve it. Can you actually measure it in centimetres on your diagram? You measure it in centimetres on your diagram. And obviously, they'll find that they'll have different centimetre measurements. And obviously they will then be convinced that they are different. Uh, they will have a different area. Um, again, the issue, perhaps, if you go down that route, again, it depends on what you're trying to achieve in that time. But you, the student, might be more confused by the fact that they've got a different area. Which can be counted, but do you know what I mean, so you have two different issues. One is challenging the confusion within that context or two is what what is it you are trying to achieve and make them understand? You know, again, it could be something that perhaps you might return to that student a little bit later on, um, and try and clarify that rather than consume the whole class.

The intention of my comment was to draw attention to the aspect of Discourse at the Mathematical Horizon which is 'looking back' to students' experiences (see Section 3.3). Here, Sam's discourse at the Mathematical Horizon is more visible in comparison to his original answer. He points to the diagrams of the mathtask (Figure 7.1) and point to the contradiction of the diagrams not having the same area while if measured in arbitrary units they have the same area. Sam's response brought out the idea of directing the actions considering the focus of the lesson. He appears to recognise the opportunity to discuss the idea of working in different measuring systems. However, he appears reluctant to discuss the idea with students to avoid further confusion. Sam's narratives align with Delivery Pedagogical Discourses which view explorative discussions beyond the scope of an ordinary lesson (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022) and were found to avert from discussions beyond the mathematics of the moment (Section 6.9).

The conversation continues leading to the discussion of question (d) where the participants continued debating about ideas beyond the mathematics of the moment drawing upon each other's comments and challenging their views. To conclude, the reflective analysis presented in the section indicates that the mathtask could bring up for debate opportunities to go beyond the mathematics of the moment. In particular, the mathematical and pedagogical narratives of the participants indicated that the discussion moved gradually from broader discussions about mathematics and pedagogy during the first part of the task to narratives that were found to influence individual discourses at the Mathematical Horizon in Chapter 6, such as narratives about students' 'abilities', engagement, and the role of the teacher. The responses in question (d) will be further discussed in the following section to encompass the analysis of the second objective of the focus group regarding discursive shifts.

7.3.2. *Emerging discursive shifts in the discourses of the participants*

In this section I focus on the discursive shifts that were observed in the discourses of the participants during the events of the focus group. Through the reflective analysis of the recording three shifts were observed: attributing pedagogical meaning to mathematical narratives, endorsing teaching practices of others, and challenging their own risk-averse pedagogies. The first shift can be attributed to the engagement with the resource and the other two to the engagement of the participants in a debate regarding discussions beyond the mathematics of the moment.

One observed discursive shift was attributing pedagogical meaning to advanced mathematical narratives and occurred during individual engagement with the mathtask. Specifically, the teachers were asked to spend some time to read the dialogue and reflect based on questions (b), (c) and (d) in Figure 7.1. Then, the participants were asked to share their reflections to each question starting from question (b), "which parts of the conversation attract your attention?". Eric was the first to speak and draw attention to the use of the word "same" (see also Section 7.3.1), he also added:

Eric: […] So, that first point really, to me, reiterated how important it is to be really specific with language when you are teaching, especially when modelling something. And then I read, and then even the next thing down when it says, these are the same. I, um, I never even really thought about

> that as a teacher, really like, how, ok it is the same area, but it is not the same area. I really care for the language that we use, that the units of the area are the same, but actually the area isn't the same. And for me, it really, like, knock me on the head. I've got to be so careful with the language that we use, are using. Think about every single thing that you say because it can be misinterpreted or lead to scenes like this where students are even more confused. […]

Eric states that the fictional conversation triggered a new thought regarding his teaching. His comment is in line with findings in Section 6.4 which suggest that development of discourses at the Mathematical Horizon can be explained as shifts of advanced – including applied – mathematical narratives that are attributed pedagogical meaning through the engagement with teaching. Eric's attention to language is evident before the introduction of the dialogue which suggest that language use is part of his wider pedagogical narratives. His use of the word "reiterated" signifies that Eric shifts his advanced mathematical narratives about modelling with his teaching practices, specifically with the attention to language when talking to students about mathematics. His utterance about the area being "the same, but actually the area isn't the same" echo findings discussed in Chapter 5 about the role of intersubjectivity in discussions beyond the mathematics of the moment (Section 5.6). The shift in his narratives brings to light key ideas that Eric needs to address to achieve intersubjectivity with his students.

Eric's response is also consistent with earlier observations suggesting that a teacher's discourse at the Mathematical Horizon is evident when focusing on specific situations (Sections 6.4 and 6.9). The discursive shift which was observed in Eric's narrative is attributed to the engagement with the mathtask. As he was the first to speak, it is unlikely that his narrative was influenced by other participants. Eric was the only of the participants who professed narratives that suggest a shift in this respect. However, more discursive shifts were observed in the narratives of the participants during the debate about opportunities to go beyond the mathematics of the moment.

Two discursive shifts were observed and attributed to the engagement of the participants in a debate about ideas to go beyond the mathematics of the moment. One relates to the type of engagement of the teachers with each other by endorsing teaching practices of others and the other to challenging their previously stated risk-averse pedagogies.

To comment on the observed shifts that were attributed to the debate of the participants I need to summarise the events discussed in the previous section. In Section 7.3.1, I discussed the engagement in discussions related to the mathtask. The findings suggest that the participants draw upon and challenge what have been said before to share their reflections in questions (b) and (c). Alex, Eric and Sam share narratives which suggest that they acknowledge the relation between the different units and the idea of getting the same numerical answer despite the choice. However, Alex was the only one of the teachers that attempted to address the issue using an 'over-exaggerated analogy'. Sam also touched upon the idea of 'representation' however he was reluctant to share the idea in the class. Teachers appear to maintain their narratives, thus, there were no discursive shifts observed during the events discussed in Section 7.3.1.

The shift to endorsing narratives of others was observed towards the end of the discussion of question (c), when the teachers were asked whether they would take the discussion to the class, and in the discussion of question (d) regarding the focus of the answer. At this stage the participants' routines shifted from drawing upon what was said before to endorsing the narratives of other participants. For example, Nick, who initially challenged Alex's approach of using an 'over-exaggerated analogy', claims that he might tries Alex's approach:

Nick: [question b] So practically instead of going for a crazy idea like [Alex], which is very difficult to prove in a classroom when you have like last five minutes. […]

[question d] […] As I said before, the two different, uh, sizes of graph paper only if be given more time and do the crazy idea which was present presented by [Alex]. […]

Nick responding in question (b) by challenging Alex's "crazy" idea as unrealistic in a real classroom situation. In question (d) he revisits the "crazy" idea and appears to be more open to trying it out if he has more time to spend on the issue. The shift signifies that Nick considers changing his teaching practice if he has time to try new approaches which he might have avoided otherwise. Similarly, Sam appears to endorse Eric's narrative about attention to language used:

Sam: Um, yeah. One, one thing I would kind of just draw attention to is the question. The question itself, so the actual task itself said use square paper to draw axes for x y from 0 to 6, using one square to one unit and a bit like [Eric] said, it's, it's drawing back to that attention on the language that you use. And that is where, if you like, the two problems diverged. So again, it's not practical to theory is that this question was not specific enough. Um, it was specific enough to get an answer of ten units, but it's that, this is on the different units. Um, so again, that that that's the kind of focus of the, you might then say, you know, if you're just talking theoretically about having units and answer is that's the practically helpful. So yeah, I would just maybe draw attention to that. […]

Here again, Sam seems to recognise the opportunity for discussions at a metalevel. However, his focus moves away from his earlier claim that he would tell the students that their answers are in different units that was noticed in his earlier response. His latest narratives signify a shift from 'what' he would say to the students to 'how' he would say it, by drawing to the specificity of the question from a practical and a theoretical perspective. Eric's earlier claim about attention to language possibly triggered narratives that were not the result of individual engagement with the problem and the dialogue.

The final discursive shift that was observed during the focus group related to challenging pedagogical narratives that would refrain from discussions beyond the mathematics of the moment in the classroom. In Section 7.3.1, I discussed how Sam's final response introduce the idea of directing his actions considering the focus of the lesson and that he appears reluctant to discuss with students about the idea of abstract units in the context of solving the mathematical problem. I claimed that his narrative is in line with Delivery Pedagogical Discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022) and promised to discuss the observation further in light of the findings in this section.

Following Sam's comment, I asked the participants whether they would choose to discuss the issue with the whole class. Eric responded that his decisions would depend on knowing his students:

Eric: Well, yeah, I I would open up to the whole class, but it depends on the class, isn't it? You know, the class. I think that's why the more you teach a class throughout the year, the better. And it's always a difficult decision, right when to stop the class from the task they're doing. Is it more important than what they're doing? And is what I always struggle with that teaching to know when to stop the class? Um, and yeah, I would think of lots of things that I think I would. Yeah, my initial go would be to show both of them [students' diagrams] visually to the class and hold them up and open up to the class and actually say, […] Are they both 10? And then hopefully that would lead to that discussion. But. It leads me to think. Do you open up to the whole class? I think my gut is usually, I would do because if there is two people that obviously got quite a deep understanding of area of like, [connection break], of how, you know, it's quite difficult problem that they've managed to solve, but they lack in a fundamental understanding of... the difference between inches square and centimetres square. […]

Eric shares his "struggle" to decide when to open a discussion that he has with a couple of students to the whole class. Eric shares narratives that indicate he considers his class as a whole (e.g., "it depends on the class") but also individual students in a specific situation (e.g., "quite a deep understanding […] it's a quite difficult problem to that they've managed to solve"). His narratives indicate that his choice to engage in discussion beyond the mathematics of the moment is influenced by his narratives about his students, which is consistent with findings in Section 6.8.

All the teachers agree that their response to the students and their follow-up action will depend on their students and the focus of the lesson. The discussion of the previous question led very naturally to question (d), "where would be the focus of your response on this topic?". During this part of the mathtask, the teachers shared more views about how they would respond to the students and the whole class if they were presented with a similar situation. Finally, I asked the teachers if they would use this exact question with their students in the classroom. This question was not one included in the mathtask. However, it gave the opportunity to the teachers to elaborate on their earlier claims about the focus of the lesson and address other factors identified in Chapter 6, such as, external factors (here, OFSTED), experience and confidence. For example, Nick says:

Nick: I personally to eliminate problems like this. Instead of giving them graph paper. I'll give them square paper. Because with the graph paper you have a wide range of selection to choose your unit. You can use one centimetre, you can use it half centimetre, you can use it one millimetre. If you have a proper graph paper. So physically, as a teacher to eliminate such confusion from the beginning, I'll be clear, I'll give them square paper. So, a square paper, one unit is one unit, one block, one square. If I want to give them a graph paper because I don't have square paper, fine. I have to be also clear enough to say how many millimetres will be my unit on that graph paper. And in this way, from the beginning, guys, [Sam] touched it, it would be the start of the question. Well, the question itself or the instructions from the teacher, which could lead to such issue. Like, imagine if you have an **OFSTED** [inspector] on that class and you have something being raised like this, where you go? If you are not competent with immediately rectifying the problem, you could run with issues. With more students saying, like, I still don't get it, I still don't understand and things like this. So, it's like a starting point. Shall I give them clearance? But sometimes the future will be maybe trying to do that. Give him a different graph paper or give him different square paper to raise issues like this, and that will then lead to problem solving, will lead to dialogue in the classroom and will lead to all sorts of other activities. So, it might be different depending on the mentality of the teacher, what he want[s] by the end? An easy lesson, everything is done to the <u>rule</u> of what you do, squares and triangles will be the same size the formula applied, counting apply answer this second. Or create such an environment of... they can be confused and raise questions.

The dense excerpt above shows an instance of employing risk-averse pedagogies. Nick indicates his intention of paying attention to the resources that he will use (e.g., square paper) and his instructions to avoid raising issues that will require him to diverge from the topic of the day. He attempts to justify his choice by positioning an OFSTED inspector (bold) in the class while discussions such as the one in the dialogue could take place. He refers to the "competency" of the teacher to "rectify the problem". His example showcases the influences of institutional discourses in teaching beyond the mathematics of the moment which were discussed in Section 6.6. Specifically, OFSTED visits for evaluation are, for the majority of the schools in England, a very important event. The visits happen with short notice and schools prepare the whole year for the visit, in some cases even doing 'mock' visits where the teachers and the students prepare for the event. Therefore, a situation which the teacher might not be prepared to address during the lesson can be very stressful when the priority of the school, and consequently the teacher, is to showcase 'outstanding' performance. Based on findings discussed in Sections 6.6 and 6.9, lacking institutional support the teachers might engage in risk-averse teaching practices aligned with Delivery Pedagogical Discourses (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022) and feel less encouraged to take an explorative perspective and address discussions beyond the mathematics of the moment.

Nick mentions attempts to avoid discussions during moments of contingency and that he would try to control the environment where such discussions could take place. Nick considers using the mathematical problem in his class when the focus of the lesson is problem solving where he would encourage his students to ask questions and be confused. In contrast to a lesson on area where the teacher's "rule[s]" (underlined) guide the outcome of the activity. His narrative is in line with findings about interpretations of the curriculum (Section 6.7) in particular with narratives about the connection of ideas stemming from teaching instructions.

Nick's response ignited the discussion about risk-averse pedagogies and triggered a response from Eric.

Eric: Yeah, but I think [Nick] said what I was going to say. So yeah, it's an interesting point, isn't it about deliberately giving up different paper to promote discussion, isn't it? Because, I think, we quite often, [Eric and Alex are laughing], yeah, we quite often try to avoid these situations by giving out square paper, yes, but potentially, maybe we shouldn't. Maybe we should give out. Yeah, it's, it's a really interesting point [Nick] made.

Me: Mm-Hmm. Hmm.

Alex: I'm too afraid to follow that point, to be honest, I think it takes so much skill you have to be prepared for the, the chaos that will ensure if this happens. But it is an interesting perspective.

Eric stresses Nick's idea of creating an environment to encourage students to ask questions. He shifts the focus of the discussion from changing the mathematical problem to avoid confusion to embracing the possibility of having questions raised as a result of deliberately giving the students different types of square or graph paper. The teachers debated whether they could deliberately give different graph papers to the students to raise the issue and "promote discussion". The mentions of the teachers regarding whether they would invest in a discussion with the students are consistent with findings in chapter 6, supporting the idea that Discourse at the Mathematical Horizon is influenced by teachers discourses about their students (Section 6.8) and the environment they teach (Section 6.6). Eric reconsiders his own risk-averse position by challenging himself and the other teachers. Three of the four teachers, sided with the possibility of addressing the issue in a controlled environment if they wanted to focus and challenge students' "problem-solving skills". The narratives of the three participants appear to shift towards the end of the discussion and align with more explorative pedagogical narratives. However, Alex expressed his fears with this approach. Later, in the recoding, Alex mentions that he would try to have his "equipment sorted" in the future.

To conclude, the section presents a reflective analysis of the shifts observed in the narratives of the participants during the focus group. The reflective analysis indicates instances where the participants reconsider their original standpoint by attributing pedagogical meaning to mathematical narratives, endorsing teaching practices of others, and challenging their own risk-averse pedagogies. The observed shifts can be attributed to the engagement with the resource and the debate regarding discussions beyond the mathematics of the moment. In the following section, I evaluate the findings of Sections 7.3.1 and 7.3.2 in relation to the objectives of the mathtask and discuss the contribution of the pilot to the development of mathtasks to bring up discourses at the Mathematical Horizon.

7.4. Lessons learnt moving forward: Evaluation of the focus group intervention

The focus group discussion of the pilot mathtask provided feedback that can be used to improve the design of professional development resources to challenge narratives at the Mathematical Horizon and bring up teachers' discourses. During the focus group the teachers discussed the pilot mathtask starting from solving and reflecting on the mathematical problem. Then, the discussion revolved around the fictional dialogue. Following the questions as guidance the teachers' expressed their opinions on how the fictional situation could be handled and whether they would choose to address similar issues with their class.

With the mediation of the mathtask, the teachers and I recognised and discussed the affordances of raising issues such as the coherence of the answer in different number systems, and the potentials of choosing to share the issues with their students. The narratives of the participants were in line with mathematical and pedagogical narratives associated with Discourse at the Mathematical Horizon in chapters 5 and 6, including considerations about mathematical conventions, intersubjectivity, the students, the school environment, and the purposes of the lesson.

An interesting observation regarding the discussion that took place, was that the teachers initially noticed different elements of the dialogue. Specifically, Eric focused on the use of the word 'same', Alex highlighted the part of the dialogue where the students notice the inconsistency, Nick pointed out that the fictional situation appears to depict an encouraging environment for the students to explore, and finally, Sam noticed a difference between the confidence of the fictional characters. However, the narratives of the teachers progressively started converging around the issue of the units. The findings suggest that the mathtask aided in focusing the discussion on the idea of arbitrary units and kindled a debate about discussions beyond the mathematics of the moment.

The emerging findings suggest that the mathtask can be used as a tool to develop discourses at the Mathematical Horizon by bridging advanced mathematical and pedagogical narratives under a specific situation (Sections 6.4 and 6.9). Specifically, Eric noticed the ambiguity in the students' use of the word the "same" in the context of a modelling activity. He linked his observation with his teaching practice which would allow teachers and students to achieve intersubjectivity (Section 5.6). The observation is in line with Discourse at the Mathematical Horizon being an amalgamation of Advanced Mathematical and Pedagogical Discourses. Moreover, there is evidence that the teachers' discourses started to shift towards the end of the focus group, by endorsing ideas of others (e.g., Nick and Sam) or to challenge their own ideas about risk-averse pedagogical practices (Eric, Nick and Sam but not Alex). Specifically, the latter observation is in line with the influences of institutional discourses (Section 6.6), discourses about the curriculum (Section 6.7) and discourses about the students (Section 6.8) discussed in chapter 6.

After the analysis of the focus group data, the pilot was revisited and refined. Specifically, I decided to add a description at the beginning of the dialogue saying that the students were given different sizes of square paper on purpose. The decision was made in response to the references of the teachers to the focus of the lesson. This decision influences the dialogue in a way that does not present a contingent moment. However, it presents a planned action from the teacher to challenge the students. The participants can still negotiate whether they would act in a similar way or not in their classroom considering what they would do if they have given the students different types of square or graph paper.

One more consideration that will help improve the design of the pilot mathtask and future mathtask is the choice of questions. At the time of the pilot, I asked the teachers whether they would use the activity with their class after seeing the dialogue which was not one of the originally planned questions. However, this question brought up discussions about risk-aversion, fears and aspirations. For example, despite Alex proposing a very imaginative way of replying to the fictional students, he remains scared of having to face the same situation in a lesson with his classes.

Here, I would like to address some technical difficulties encountered during the online focus group. The pilot took place online, using Blackboard Collaborate, at a time where online sessions were something very new to both me and to the teachers. The platform then only offered the option of sharing slides and showing one person's video at the bottom of the screen. Therefore, as I had the slides on for us to see and discuss, we could not see each other's faces. Without being able to look at the faces of the participants, I felt unsure as to whether I should ask more questions or let the discussion flow. Not being able to see each other might have been an issue for the participants as well, limiting the interactions with each other. For example, I felt at times that the teachers' responses were directed only to me and not the whole group. However, the analysis of the recording does not indicate that my personal feeling is reflected in the data. To end on a positive note, I believed that the online tools we have at our disposal now are improving rapidly. Therefore, if similar discussions had to take place online in the future the aforementioned technical difficulties can be addressed.

To conclude, as discussed in Section 6.6, discourses at the Mathematical Horizon can be influenced and developed through collaboration with other teachers. I consider the emerging findings of the focus group as evidence that mathtasks could provide opportunities for the teachers to enrich their discourses at the Mathematical Horizon by endorsing collectively-built narratives while maintaining and showcasing unique characteristics of their individual discourses. Thus, I feel confident that the pilot offers evidence that MathTASK activities could be attuned to bring up Discourse at the Mathematical Horizon.

I now turn to proposing a set of design principles to create vignette-based professional development resources to bring up Discourse at the Mathematical Horizon. The design is based on the MathTASK principles, and it is informed by empirical findings discussed in Chapters 5 and 6 and the findings of the analysis of the focus group intervention.

7.5. Design of resources for research and professional development

In this section, I propose a vignette-based design for professional development resources oriented towards bringing up Discourse at the Mathematical Horizon. The use of the word 'bring up' has a double meaning here. First, the resources created could be used to develop or 'nurture' the discourse in professional development settings. Secondly, the same resources could be utilised as research tools for raising aspects of teachers Discourse at the Mathematical Horizon to be investigated through interviews or focus groups.

Vignette-based resources has been used in mathematics education for research, initial and continued education purposes (e.g., Beilstein et al., 2017; Friesen & Kuntze, 2021; Herbst et al., 2011; Skilling & Stylianides, 2020). The use of vignettes has been found useful in providing a specific scenario to collect information and prompt discussions for matters which might be complex, sensitive or controversial (Poulou, 2001). Based on the findings in Chapter 6, there is diversity in the discourses at the Mathematical Horizon which are often influenced by lack of institutional support and Delivery Pedagogical Discourses. I consider vignette-based activities appropriate to bring up discussions which challenge dominant narratives beyond the mathematics of the moment.

Mathtasks is a type of vignette-based activities which have the characteristic of starting first with solving a context-appropriate mathematical problem before moving to the discussion of a realistic fictional dialogue through reflective questions. As discussed in Section 7.2, the MathTASK principles (Biza et al., 2018, 2021) include provisions of fostering Horizon Content Knowledge. However, the principles do not specify how this can be operationalised. Drawing upon the MathTASK principles and empirical results presented here and in previous chapters, I propose a refinement of the design which could foster discourses at the Mathematical Horizon starting from systematically constructing dialogues to be used in vignettes, choosing appropriate mathematical problems to situate the vignette, and posing questions.

To exemplify the design and illustrate my arguments, I use one of the mathtasks created as an example[37]. The mathtask is inspired by one episode identified in Liz's lessons (see Section 5.4.5). Figure 7.3 depicts a printable version of the mathtask[38]. It is useful to have the resource presented in a printable way (e.g., to give out the resources after a workshop or collecting responses on a survey format). However, the mathtasks should not be solely associated with the printable. The activities are primarily designed to be discussed in sections: first the mathematical problem and its solution, and then the dialogue with the reflective questions.

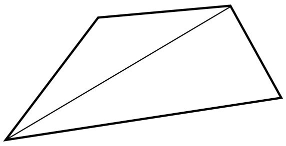
[37] The remaining of the tasks are reserved for future projects and publications.

[38] At the time of this writing, the mathtask has been through the initial cycle of the design. It has been discussed with my supervisors and the RME group members and refined based on their comments and suggestions.

Why is a degree a degree?

A Year 7 class is working on the following mathematical problem posed by the teacher:

Use the diagram to explain why the sum of angles in quadrilaterals is 360°.



The following conversation takes place between Student A and the teacher.

Student A: Why is it only 180 degrees in the triangle?
Teacher: Oh, there's 180 degrees in the triangle because all the corners of the triangle, when you put them together, will make a straight line. We'll do that, we'll do that one day.
Student A: But why? Why is it, why is the straight line 180?
Teacher: I think it's to do with Babylonian maths.
Student A: Why? What is that?
Teacher: So, Babylonian maths, many thousands of years ago used base 60. So, you know, we use base 10 in our number system? So, it goes one, ten, a hundred, a thousand.
Student A: Hmm…Yeah…
Teacher: They used base 60. Which is sort of linked with why we've got 12 numbers around the clock.
Student A: Will we do that? Sixty is a really bad number.
Teacher: And they used 360 degrees. The 360 is, could have been any number.
Student A: Exactly! Like sixty seconds in a clock.
Teacher: Yes. Yeah.
Student A: But why, why is a degree a degree? Like the amount of it.
Teacher: Because they took a full point and just divided it up into a number that they wanted to divide up into. It could have been four hundred. It could have been four hundred and fifty.
Student A: Exactly!

Questions:

a. How would you answer the mathematical problem posed by the teacher?

b. How would you respond to student A? Would you take this discussion to the whole class?

c. Do you think that explaining the origins of the mathematical ideas are important in mathematics teaching?

d. Would you use the same problem with your class? Why?

Figure 7.3 Printable version of the mathtask 'Why is a degree a degree?'

7.5.1. *The vignettes*

Vignettes are short open-ended stories based on research or real-life situations. The vignettes can be of different formats e.g., text, comics, animation or video each of which have both advantages and shortcomings (Chazan & Herbst, 2012; Herbst et al., 2011). MathTASK vignettes are presented in the form of a dialogue around a mathematical problem, usually written, accompanied if needed by images depicting students' work or writing on the board. Video-vignettes are used in MathTASK activities for the project Challenging Ableist Perspectives on the Teaching of Mathematics (CAPTeaM (Nardi et al., 2018)). When designing mathtasks for the purposes of my study, I consider the options of adapting the dialogues into audio or video format. However, I suggest using text vignettes because low video or image quality or background information (e.g., colours, noise, accents) might be a distraction or introduce bias (Chazan & Herbst, 2012; Herbst et al., 2011; Poulou, 2001). Moreover, text-based vignettes give the user more control over the fictional situation (Herbst et al., 2011). Indeed, the findings of the pilot focus group suggest that the written dialogue triggered the imagination of the teachers, see for example in Section 7.3.2 Nick positions an OFSTED inspector at the scene.

For this specific category of mathtasks I propose starting from creating the fictional dialogue before deciding on the mathematical problem. I am using episodes identified as opportunities to go beyond the mathematics of the moment (see Chapter 5). The dialogues are based on the data from the lesson observations. Therefore, the vignettes created are based on teaching practices and supported by research.

I propose that the criteria for selecting the dialogues to become vignettes arise from the process followed for the analysis of the lesson observations (Section 4.6.2). In summary, the steps of the analysis include the identification of teacher's actions and the mathematical topics and practices throughout the lesson, comparison of the different topics and practices to identify episodes of opportunities (taken, potentials of the discussion and the task) to go beyond the mathematics of the moment and finally, analysis of the episodes to identify the discursive patterns of the communication. The episodes identified, some of which are discussed in Chapter 5, can be seen as potential critical incidents, i.e., instances of decision making considering several conflicting motives, crucial to the teacher's school mathematics priorities, the development of classroom interactions and students' learning (Skott, 2001).

Scott (2001) suggests that reflecting on critical incidents “turn the classroom into a learning environment for teachers as well as for students” (p. 4). Likewise, reflecting on vignettes could turn a malleable fictional classroom into an opportunity for teachers to learn.

Table 7.1 depicts part of the fictional dialogue side-by-side with the original data from Section 5.4.5. The full side-by-side comparison between the original and the fictional dialogue can be found in Appendix VIII. To create the vignettes, I propose the following procedure:

1. Cleaning the chosen dialogue from filler words (e.g., uh, hmm etc) or any repeated phrases (e.g., “that's, that's just like the angle […]”) which are not essential to the progression of the discussion.
2. Accounting for and completing fragmented quotations (e.g., “like sixty seconds in a…” to “like sixty seconds in a clock”)
3. Highlighting the main points of the dialogue (e.g., keywords and central narratives).
4. Simplifying sentences and combining responses when they are similar (see Table 7.1).
5. Recreating any images depicting students’ or teacher’s work if necessary.

The procedure proposed here ensures that the dialogue is realistic as it summarises events from a real lesson and grounded on research on Discourse at the Mathematical Horizon. Next, I suggest a way for choosing a mathematical problem which would be appropriate as a starting point for the vignette.

Table 7.1 Side-by-side comparison of the observed and the fictional dialogue

<table>
<tr><th colspan="2">Observed dialogue</th><th colspan="2">Fictional dialogue</th></tr>
<tr><th>Teacher</th><th>Students</th><th>Teacher</th><th>Student A</th></tr>
<tr><td></td><td>St 1: On the whiteboard, when you said, why is it only 180 degrees in the triangle?</td><td></td><td>Why is it only 180 degrees in the triangle?</td></tr>
<tr><td>Liz: Oh, there's 180 degrees in the triangle because.</td><td></td><td rowspan="5">Oh, there's 180 degrees in the triangle because all the corners of the triangle, when you put them together, will make a straight line. We'll do that, we'll do that one day.</td><td rowspan="5"></td></tr>
<tr><td></td><td>St 2: Of the equilateral triangle.</td></tr>
<tr><td>Liz: All the all the corners of the triangle, when you put them together, will make a straight line.</td><td></td></tr>
<tr><td></td><td>St 1: But why?</td></tr>
<tr><td>Liz: We'll do that, we'll do that one day.</td><td></td></tr>
<tr><td></td><td>St 1: Why is it, why is the straight line 180? [overlap].</td><td rowspan="2"></td><td rowspan="2">But why? Why is it, why is the straight line 180?</td></tr>
<tr><td></td><td>St 2: [inaudible]</td></tr>
<tr><td>Liz: Why is it three hundred and sixty? I think it's to do with Babylonian maths.</td><td></td><td>I think it's to do with Babylonian maths.</td><td></td></tr>
<tr><td></td><td>St 1: Why? What is that?</td><td></td><td>Why? What is that?</td></tr>
<tr><td>Liz: So Babylonian, um, maths, many thousands of years ago used base 60. So, you know, we use base ten in our number system?</td><td></td><td>So, Babylonian maths, many thousands of years ago used base 60. So, you know, we use base 10 in our number system? So, it goes one, ten, a hundred, a thousand.</td><td></td></tr>
</table>

7.5.2. *The Mathematical Problem*

The findings in Chapter 6 suggest that Discourse at the Mathematical Horizon can be developed through discursive shifts that enrich advanced mathematical patterns of communication with pedagogical meaning through the engagement with teaching. Thus, professional development activities which aim to address Discourse at the Mathematical Horizon should include engagement both with mathematical and pedagogical content in specific situations. The mathtasks offer opportunities for the teachers to engage with contextualised mathematical and pedagogical content (Biza et al., 2018, 2021). The suggestion is also supported by the observed shift in Eric's discourse through his individual engagement with the resource in Section 7.3.2. Thus, the format of mathtasks is considered suitable for bringing up discourses at the Mathematical Horizon.

The MathTASK principles propose that the mathematical content should be contextualised to the curriculum and involve "a topic or an issue that is known for its subtlety or for causing difficulty to students" (Biza et al., 2018). In addition, I suggest that the mathematical problem should follow the subsequent criteria:

1. Relevance: The mathematical problem and the mathematical content of the dialogue should be related to one of the main mathematical themes identified in chapter 5[39]. Specifically, the mathematical content should include ideas and practices that run across the mathematics curriculum, or which students might come across in later stages than the one identified in the mathtask, mathematical conventions or applications of mathematics. Empirical data from my study can be used as inspiration for the mathematical problem.
2. Increased incidence: The MathTASK principles suggest that the mathematical content and the dialogue together provide the context in which the teachers are asked to reflect. The vignette of the mathematical problem and the dialogue should trigger discussions around the themes identified in Chapter 6. To provide opportunities for meta-level discussions, the idea beyond the mathematics of the moment in the vignette shouldn't be seen as relevant only to the specific problem. Therefore, the

[39] Or any other identified through further study.

mathematical content of the problem should not narrow the context in which the fictional discussion could take place to a single atypical situation. A practical reason for this suggestion is that during the pilot mathtask, the teachers initially proposed averting from using a task similar to the one presented in the resource. One way of avoiding that could be ensuring that the idea is not directly mentioned in the description of the task – if it is a concept – nor being the main methodology of solving the problem – if it is a practice).

A Year 7 class is working on the following mathematical problem posed by the teacher:

Use the diagram to explain why the sum of angles in quadrilaterals is 360°.

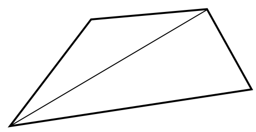


Figure 7.4 The mathematical problem of the fictional situation

To exemplify the criteria, I use the mathtask in Figure 7.3 and I zoom in to the mathematical problem (Figure 7.4). The mathematical problem depicted in Figure 7.4 is an adaptation from the data of the lesson observation where the dialogue presented in the vignette took place. In the data, Liz asked her students to draw a diagram explaining why the angles in a quadrilateral add up to 360°. Later on, she drew two diagrams on the board to help her students who were struggling to answer her open question. To simplify the problem, I posed a slightly different question using a similar diagram as an initiation of the activity. The problem is suitable for the student and can be part of a Year 7 lesson which suggest that the choice of the problem is consistent with the MathTASK principles.

The problem is related to Mathematical conventions (see Section 5.4). Using a property about the sum of angles in triangles which is not proven but rather given to students as a “fact” to argue about the sum of angles in more complex polygons, e.g., quadrilaterals, could make the student to wonder why the sum of angles is a specific number and where this number came from. I decided to include a diagram that had the auxiliary line drawn for the students. As discussed in Section 5.3.3, drawing auxiliary lines for purposes other than calculating areas is beyond the requirements of the curriculum which provides a different opportunity for discussion than the one presented in the dialogue. To avoid overloading the problem with more than one opportunity at a time, I decided to add the line as given. Thus, the mathematical problem fulfils the relevance criterion.

Finally, the focus of the dialogue is the choice of 1^{o} as a unit for measuring angles. The contingency stems from the students being required to apply and generalise a memorised, and not yet substantiated, property of the angles in triangles to justify their answers in a new situation. However, the contingency could manifest in other similar problems that requires the use of the fact in explorative ways. The mathematical problem can be viewed as a point of initiating discussion and not the cause of the contingency. Thus, the mathematical problem fulfils the increased incidence criterion.

7.5.3. *The questions*

The vignettes can be used to elicit responses through open-ended questioning (Poulou, 2001; Skilling & Stylianides, 2020). The questions might vary depending on the focus of the vignette and the mathematical problem, they can be predetermined, or they can be altered and adapted in real-time during the engagement with the resource. Having a predetermined list of questions has the benefit of comparing commonalities and differences in participants responses in research settings (Skilling & Stylianides, 2020). In professional development situations, unplanned interventions from the facilitator can be useful in the progression of the discussion, see for example my interventions in Sections 7.3.1 and 7.3.2. Therefore, I propose a structure for the questions which can be adapted depending on the use of the mathtask. The proposed structure is a combination of the general questions frequently found in MathTASK activities and questions that I have found to be promoting the discussion beyond the mathematics of the moment in the pilot focus group. Specifically:

- <u>Engagement with the mathematical problem:</u> Although the discussion could be triggered by different mathematical problems, asking the teachers to solve a specific problem provide a starting point for the teachers to reflect on the ideas involved for solving the problem and what they expect from the students. For instance, questions (a) in Figure 7.3 “How would you answer the mathematical problem posed by the teacher?” stems from the MathTASK principles (Biza et al., 2018, 2021).
- <u>Engagement with the dialogue:</u> A question which aims to provide a ground for the teachers to identify the main points of the dialogue, propose and discuss different approaches of tackling the situation. For instance, question (b) in Figure 7.3 “How would you respond to student A? Would you take this discussion to the whole class?” could trigger discussions about alternative explanations, e.g., focus on the factors of

60 and its multiples (particularly 180 and 360). Discussing different approaches with groups of teachers could expand the repertoire of plausible answers. Question (b) also stems from the MathTASK principles (Biza et al., 2018, 2021).

- Draw attention to teaching practices at the Mathematical Horizon: A question to trigger (internal) debate about affordances and shortcomings of teaching practices proposed by the dialogue and previous questions. For instance, question (c) in Figure 7.3 “Do you think that explaining the origins of the mathematical ideas are important in mathematics teaching?” provides an opportunity to discuss the role and the place of explaining how mathematical conventions are endorsed and reflect on the affordances and shortcomings of each one of the approaches discussed in question (b). For example, whether historical sources are accurate, or whether students might not see the relevance of the factors of 60 in this situation.
- Reconsideration of the situation: The final question should aim to draw the attention of the teachers to the potentials of the mathematical content involved in the vignette. For instance, question (d) Figure 7.3 “Would you use the same problem with your class? Why?”, aims to trigger a discussion about posing questions and problem which could promote discussions beyond the mathematics of the moment. In the pilot, a similar question brought up discussions about the aims and the focus of the lesson, confidence, and perceived abilities of the students. During a professional development workshop teachers can exchange their final thoughts and have a closing discussion about their experiences with the activity.

I argue that a predetermined structure for all the vignette-based resources aids in evaluating whether the resources is an appropriate tool for bringing up discourses at the Mathematical Horizon while maintaining the flexibility of adjusting them to account for the specific situation. For instance, following the structure would provide comparable data across the resources about teachers’ engagement and any observed shifts in teachers’ discourses.

7.6. Summary

The literature suggests that professional development activities aiming to develop an understanding of mathematics beyond the curriculum focus on bridging the gap between subject matter knowledge and practice, by bringing communities together (Larsen et al., 2018; Maass & Engeln, 2019; Nelson & Slavit, 2007; Pinto & Cooper, 2022), and/or by grounding the discussions on secondary school teaching practices (Wasserman et al., 2019). Additionally, findings in Chapter 6 suggest that Discourse at the Mathematical Horizon is an amalgamation of advanced mathematical and pedagogical discourses. Thus, activities aiming to engage teachers in discussions beyond the mathematics of the moment and trigger discourses at the Mathematical Horizon should provide opportunities for the teachers to engage with the mathematical content in the context of their teaching practice. The final empirical chapter of this work focuses on the idea of creating vignette-based resources to explore and develop Discourse at the Mathematical Horizon. The format is proposed based on the review of the literature and emerging findings in Chapters 5 and 6.

Mathtasks is a type of vignette-based activities which have the characteristic of starting first with solving a context-appropriate mathematical problem before moving to the discussion of a realistic fictional dialogue through reflective questions. Initially, I created a pilot resource drawing upon the literature and using the MathTASK principles based on the provision that mathtasks can foster Horizon Content Knowledge to surface (Biza et al., 2018, 2021). I later finalised the resource based on preliminary findings from the lesson observations and the interviews and used it in a focus group to explore the potential of the design.

After the events of the focus group intervention, I analysed the collected data based on the findings discussed in Chapters 5 and 6. The findings presented in Sections 7.3.1 and 7.3.2 suggest that the mathtask supported the engagement of the participants in debates about opportunities for discussions beyond the mathematics of the moment. Specifically, the mathtask triggered discussions about the influences of institutional discourses (Section 6.6), discourses about the curriculum (Section 6.7) and discourses about the students (Section 6.8) which are found to influence individual discourses at the Mathematical Horizon. Moreover, the discursive shifts observed during the focus group suggest that the mathtask can be used to encourage participants to bridge advanced mathematical and pedagogical narratives under a specific situation (Sections 6.4 and 6.9), draw attention to intersubjectivity (Section 5.6), endorse ideas of others (Section 6.6) and challenge their own ideas about risk-averse pedagogical practices (Section 6.9).

In section 7.5, I propose a design of vignette-based resources to bring up discourses at the Mathematical Horizon, drawing upon the MathTASK design principles, based on my research findings and utilising the collected data from the lesson observations. Specifically, a key characteristic of mathtask resources is that they combine engagement with contextualised mathematical and pedagogical content starting from a mathematical problem. This characteristic is considered appropriate for this type of resources as Discourse at the Mathematical Horizon can be developed through discursive shifts that enrich advanced mathematical patterns of communication with pedagogical meaning through the engagement with teaching (Sections 6.4 and 6.9). Empirical data from my study can be used as inspiration for the mathematical problem and the fictional dialogue (Sections 7.5.1 and 7.5.2). The adaptation of the mathematical problem and dialogues should fulfil the criterial of relevance and increased incidence to avoid narrowing down the applicability of the fictional situation (Section 7.5.2). Finally, I propose a structure for setting reflective questions that could prompt – yet not control – mathematically and pedagogically reach discussions about ideas and practices that were found to be related to discourse at the Mathematical Horizon (Section 7.5.3).

Proposing principles and creating the resources is the first step towards building a repository for professional development activities which bring up discourses at the Mathematical Horizon. Piloting more mathtask and addressing feedback is the next step of this attempt. Moreover, the way that these resources are implemented could lead to different interactions between the members of the group depending also on the aims of the facilitator. The resources can be used in a professional development setting or as research tools. In each case the aims are different, In the first case the facilitator might choose to intervene and highlight points to aid the discussion whereas in research the interviewer might choose to stay silent and let the participant reflect and narrate her views. Moreover, the proposed design is self-sufficient. It can be adapted into different contexts provided that the data reflect the educational system and generalised using a different format (e.g., videos, comics etc) or overarching design.

Finally, findings in Section 6.6 suggest that learning at the Mathematical Horizon can occur through participation in different communities of practice (Lave & Wenger, 1991). The findings of the focus group indicate that mathtasks could support the engagement of the participants with the activity and each other in the one-off event. Thus, it would be interesting to explore in the future whether longitudinal engagement with the mathtasks in group settings could encourage teachers to form a community within the group and gradually change the enterprise, the shared repertoire of the community (Wenger, 1998) and the discourse of the community with shared interest of introducing opportunities to go beyond the mathematics of the moment in their teaching practices.

8 Discussion and conclusion

In this chapter, I first synthesise the results presented in Chapters 5,6 and 7 and discuss the substantive, theoretical and methodological contribution of my study (Section 8.1). Then, I discuss implications for policy and practice (Section 8.2). I report the limitations of my study (Section 8.3) and discuss ideas for further research (Section 8.4). Finally, I reflect on conducting the study and how the experience contributed to my journey of becoming a researcher (Section 8.5).

8.1. Summary of key findings and contribution to the field

The present study sought to explore secondary mathematics teachers' communicational actions beyond the mathematics of the moment, i.e., discussions beyond what is proposed by the curriculum and teaching instructions. Building on the Theory of Commognition (Sfard, 2008), I use Discourse at the Mathematical Horizon as a theoretical tool to explore the engagement of teachers in the communications - verbal or otherwise - about mathematical ideas and practices that are not explicitly included in the curriculum guidelines for a specified year of study. Specifically, my research addresses the following research questions (also presented in Section 4.1):

RQ 1 What are the characteristics of discussions *beyond the mathematics of the moment*?

- **1a.** What taken and potential opportunities for discussion beyond the mathematics of the moment can be identified in everyday teaching practice?
- **1b.** How could teachers and students communicate effectively when opportunities for discussion beyond the mathematics of the moment are taken?
- **1c.** What experiences shape individual discourses at the Mathematical Horizon?

RQ 2 How can empirical evidence be used to create practice-based resources for developing teachers' [d]Discourse[s] at the Mathematical Horizon?[40]

[40] The abbreviation [d]Discourse[s] is used to signal the difference between Discourse and individual discourses.

The research questions were addressed in the empirical chapters (Chapters 5, 6 and 7). Specifically, I began exploring the characteristics of communications beyond the mathematics of the moment by identifying taken and potential opportunities for discussion (RQ 1a) and the elements that aid in effective communication between teachers and students (RQ 1b). Then, I studied the educational, personal and professional experiences that build a discourse around diverging from curriculum guidelines, i.e., Discourse at the Mathematical Horizon (RQ 1c). Finally, I proposed a design for professional development activities that could aid in enriching teachers' experiences beyond the mathematics of the moment and develop their individual discourses at the Mathematical Horizon (RQ 2).

In the following sections, I synthesise the key findings presented in the empirical chapters and focus on the substantive, theoretical and methodological contribution of the present work. In particular, I highlight the following aspects: a broader understanding of teaching practices that go beyond the mathematics of the moment; a refined definition of Discourse at the Mathematical Horizon; an expanded outlook on Advanced Mathematical Discourse; and a methodological approach to create professional development resources.

8.1.1. *Exploring teaching practices beyond the mathematics of the moment*

The study contributes to the Theory of Commognition by adding to the growing literature about teachers' discourses and teaching (Cooper, 2016; Cooper & Lavie, 2021; Gavilán-Izquierdo & Gallego-Sánchez, 2021; Heyd-Metzuyanim & Graven, 2019; Heyd-Metzuyanim & Shabtay, 2019; Lavie et al., 2019; Nachlieli & Elbaum-Cohen, 2021; Nachlieli & Tabach, 2019). In particular, the findings of my analysis offer insights regarding teachers' discourses at the Mathematical Horizon and their teaching practices that are not grounded in curricular instructions.

Discourse at the Mathematical Horizon was provisionally described in Section 3.3 as patterns of mathematical communication that are unique to the moment of instruction and incommensurable to the mathematical discourse of the classroom as predicated by the curriculum. To operationalise Discourse at the Mathematical Horizon using data from lesson observations and interviews, I adopted the term *beyond the mathematics of the moment* to describe mathematical communications that are not in line with curricular instructions provided for a topic and a specified age group of students. The description encompasses

Wasserman's (2018) idea of local and nonlocal mathematical neighbourhoods. However, the focus moves from specific mathematical objects and their temporal and developmental location within a curriculum to the discursive elements of the communication described in the following paragraphs. Therefore, any opportunity for discussion beyond the mathematics of the moment is seen as a situation to study teachers' discourses and teaching practices at the Mathematical Horizon.

Findings discussed in Chapter 5 aid in identifying opportunities for discussions beyond the mathematics of the moment emerging in secondary school mathematics classrooms (RQ 1a) and ways of achieving effective communications between the teacher and the students during those discussions (RQ 1b). The findings indicate that the opportunities identified in data from the lesson observations and the interviews with teachers and teacher educators fall into four themes:

- Discussions about **ideas and practices that run across the mathematics curriculum**. This type of discussions aims at communicating central mathematical ideas and practices included in the curriculum by making meta-comments on what the students are learning at the time. The findings under this theme reflect the claims that "the nature of horizon content knowledge is about understanding the motivation for given topics, having an intuitive grasp of core ideas involved, and being familiar with basic techniques developed to contend with the ideas." (Jakobsen et al., 2012, p. 4641). The theme bears comparison with the idea of vertical curriculum knowledge (Shulman, 1986). However, the focus here is on the communicational patterns between the teachers and the students. Thus, the familiarity of the teachers with the curricular materials was not explored.
- Discussions about **ideas and practices that the students might come across in later stages of their education** (e.g., in the next Key Stage, Further Education or a Mathematics course at university) **and through activities of public engagement with mathematics** (e.g., science festivals, open days, videos etc). Findings under this theme highlight the importance of considering the "large mathematical landscape" (Ball & Bass, 2009, para. 17) both in terms of students' formal education but also in relation to their everyday lives.

- Discussions about **mathematical conventions**. This type of discussions aims at communicating mathematical warrants through examples from what the students are learning at the time, for instance, aiming to give reasoning behind the definitions and naming of mathematical objects which might seem arbitrary to students. According to Nachlieli and Elbaum-Cohen (2021) introducing mathematical conventions is a change in vocabulary or visual mediators that calls for the substantiation of a historical choice and does not involve meta-level learning. However, I argue that negotiating with students a change in the vocabulary or visual mediators about seemingly arbitrary objects provides an opportunity for the students to shift their discourse about naming mathematical objects and notation. For example, Section 5.4.2 illustrates an opportunity for the teacher and her students to discuss the connection between vertically opposite angles with the word 'vertex' and to shift the metarules which the students use to identify the object.
- Discussions about **applications of mathematics**. These discussions focus on demonstrating the use of mathematics in other school subjects (e.g., lateral curriculum (Shulman, 1986), or connections with preceding and future topics in other subjects) , in other disciplines and professions or in students' everyday activities. Prior research on Horizon Content Knowledge is primarily described as an awareness needed for making intra-mathematical connections (Ball & Bass, 2009; Jakobsen et al., 2012). However, this theme concerns opportunities to make interdisciplinary connections complementary to prior research and the other three themes of this study which focus on meaning making and appreciating mathematics as a discipline. The findings around discussions about applications of mathematics highlight how discourses at the Mathematical Horizon could balance the variations in the operationalisation of real-world applications in curricular documentations (Smith & Morgan, 2016).

The classroom episodes discussed under each theme indicate taken and potential opportunities for the teachers to share, with their students, aspects of mathematics that are otherwise in the background of teaching, such as philosophical, practical, ethical or moral issues related to the study of mathematics. The opportunities identified compare with the vignettes used to exemplify Horizon Content Knowledge in research by Ball and Bass (2009) and Jakobsen et al. (2012), and what is defined as 'horizon encounters' by Naik (2018). My study proposes looking at the discursive characteristics of the identified opportunities to go beyond the mathematics of the moment to refine our understanding of the concept.

Among the identified opportunities to go beyond the mathematics of the moment are also instances where the discussions beyond the mathematics of the moment were not aimed for the benefit of the students (e.g., Section 5.3.4) or instances where the teacher and the students did not manage to communicate effectively (e.g., Section 5.2.2). Moreover, the opportunities that I identified as an observer were not always remarked on or addressed fully by the teachers (e.g., Sections 5.2.3 and 5.3.1). Rowland and Zazkis (2013) propose that the teacher's actions in contingent situations is informed by their mathematical knowledge, awareness of the potential of the opportunity and engagement with mathematical enquiry. From a methodological perspective, considering both taken and potential opportunities benefits the study in two ways. Firstly, distinguishing between the observed actions of the participants and alternative courses of action serves as a reminder of the complexity of teaching which might otherwise be obscured by the focus of the study. Secondly, exploring alternative courses of action addresses the limitation of looking at snapshots of specific lessons. Specifically, we can consider how a discussion might take place in different scenarios grounded on the available evidence (i.e., by examining brief classroom interactions and teaching materials). Studying both taken and potential opportunities provides a wider understanding of the situations where Discourse at the Mathematical Horizon might be relevant to teaching.

The findings suggest that an essential component of effective communications beyond the mathematics of the moment is the teachers' attempts to help the students build routines that would allow them to work and communicate mathematically beyond the curriculum requirements. During the episodes, teachers and students often communicate using words, visual mediators, narratives and routines that bear different meanings from the perspectives of the teacher and the students. In the episodes where intersubjectivity is achieved, namely, the communicational actions make sense from both teacher and student perspectives, taken opportunities to go beyond the mathematics of the moment appear to achieve effective communication. Moreover, my analysis indicates that gesturing was present in communications beyond the mathematics of the moment. Gestures, either physical (e.g., turning, covering the page), supported by verbal utterances (e.g., “splitting” a shape) or non-verbal communication (e.g., drawing on the board), were used to communicate about mathematical objects and practices that were beyond the topic of the day. According to Sfard (2009) “gestures are crucial to the effectiveness of mathematical communication” (p. 197). Moreover, attributing mathematical meaning to gestures was found to encourage experimentation (Yoon et al., 2011). Therefore, gesturing is considered another means of achieving intersubjectivity beyond the mathematics of the moment.

The notion of intersubjectivity is used in commognitive research to propose methods to support explorative meta-level learning (Cooper & Lavie, 2021). The findings of the study suggest that the intersubjective communications could support meta-level learning at the time of the event or act as precedent events for future learning. Thus, building on Cooper and Lavie’s (2021) work, I theorise that achieving intersubjectivity could support explorative opportunities for students to enter a new discourse not only concurrently but also in the long term through discussions beyond the mathematics of the moment.

8.1.2. *Refining the definition of Discourse at the Mathematical Horizon*

In the literature, the dominant narratives about Horizon Content Knowledge (Ball & Bass, 2009; Jakobsen et al., 2012; Zazkis & Mamolo, 2016) and Discourse at the Mathematical Horizon (Cooper, 2016; Mosvold, 2015) frame the constructs in relation to teachers’ understanding of mathematics for teaching. For example, descriptions include mentions of “awareness […] of the large mathematical landscape” (Ball & Bass, 2009, para. 17), “orientation to and familiarity with the discipline” (Jakobsen et al., 2012, p. 4642) or “patterns of mathematical communication that are appropriate in a higher grade level” (Cooper & Karsenty, 2018, p. 242). The findings of the study, however, provide evidence which contradicts a view of Discourse at the Mathematical Horizon as a primarily mathematical discourse. In the following paragraphs, I revisit the key findings which support the latter claim. I focus on the shift from ‘understanding’ mathematics to ‘discursive activity’ and propose a refined definition of Discourse at the Mathematical Horizon.

In Chapter 6, I explore the experiences which shape individual discourses at the Mathematical Horizon (RQ 1c). The analysis of the interview data indicates diversity in the participants’ narratives about teaching practices beyond the mathematics of the moment and thus, in their discourses at the Mathematical Horizon. The variations can be linked to educational, professional and personal experiences of the participants as well as their narratives about teaching, the curriculum and the students.

The analysis indicates that participants' discourses at the Mathematical Horizon are influenced by their engagement with advanced – including applied – mathematics in both formal (e.g., university and professional development activities) and informal settings (e.g., independent learning, reading, searching online and discussions with colleagues). Teachers' narratives about discussions beyond the mathematics of the moment include elements of history, development of the discipline, applications of mathematics, and perceptions about mathematics and the curriculum. Moreover, the narratives are contextualised (i.e., influenced by experiences and interests), subjectified (i.e., influenced by narratives about students and teaching,) and opportunistic (i.e., do not have a specific "place" within a lesson), sometimes being part of the teacher's toolkit to engage or help students and sometimes coming up during moments of contingency. Finally, narratives that align with Exploration Pedagogical narratives (Heyd-Metzuyanim & Shabtay, 2019; Nachlieli & Heyd-Metzuyanim, 2022) support the inclusion of discussions beyond the mathematics of the moment in the mathematics classroom. Thus, individual discourses at the Mathematical Horizon are contingent on broader pedagogical narratives.

The findings presented in Chapters 6 indicate that Discourse at the Mathematical Horizon can be viewed as the discourse that includes advanced mathematical patterns of communication enriched with pedagogical meaning through the engagement with teaching. Considering this evidence, variations in the narratives of the participants can be explained as an amalgamation of advanced mathematical and pedagogical discourses.

As mentioned earlier, I provisionally proposed a refinement of Discourse at the Mathematical Horizon as patterns of mathematical communication that are unique to the moment of instruction and incommensurable to the mathematical discourse of the classroom as predicated by the curriculum. Moreover, I theorised that Discourse at the Mathematical Horizon included both mathematical and pedagogical elements based on remarks about partly overlapping discourses for teaching in the definition of Mathematical Discourse for Teaching (Cooper, 2016; Mosvold, 2015).

Although any pattern of advanced mathematical communication[41] is incommensurable with the discourse of the classroom, the findings presented in Chapter 5 indicate that such communication might be effective when intersubjectivity is considered. The shift to Discourse at the Mathematical Horizon requires the introduction and use of intersubjective discursive elements that are not typical in advanced mathematical discourse but are relevant to the experiences of the teacher and the students. Thus, Discourse at the Mathematical Horizon is a meta-discourse of advanced mathematics where the patterns of communication are attributed pedagogical meaning. Specifically, Discourse at the Mathematical Horizon is the Discourse about the pedagogical implications of mathematical objects that are typically considered under Advanced Mathematical Discourse (see also Figure 8.1). Therefore, I propose refining the definition of Discourse at the Mathematical Horizon to reflect the findings presented in Chapters 5 and 6:

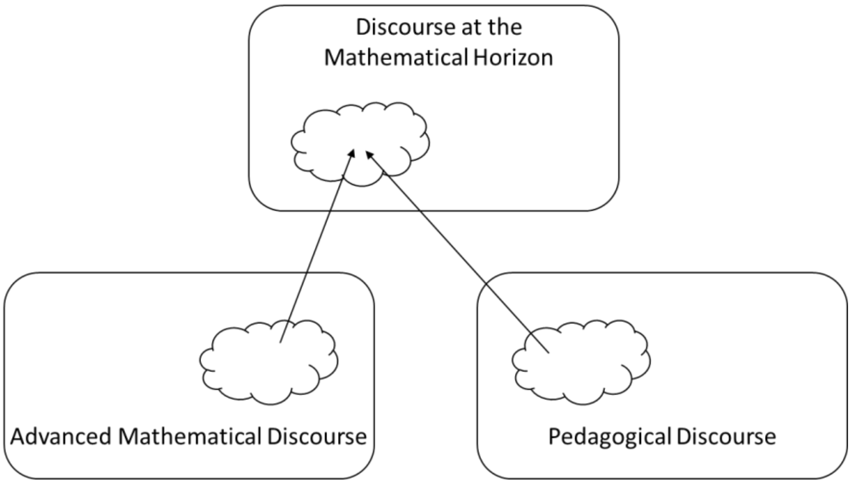


Figure 8.1 A visualisation of discursive objects at the Mathematical Horizon

> **Discourse at the Mathematical Horizon** is a meta-discourse of advanced mathematics where the patterns of communication are attributed pedagogical meaning and refers to teaching practices aiming to substantiate intersubjective narratives beyond the mathematics of the moment.

The refined definition of Discourse at the Mathematical Horizon addresses the gaps of the original definition (Cooper, 2016; Cooper & Karsenty, 2018) and the provisional refinement in three ways. First, the revised definition provides information about the type of

[41]Reminder: In this study advanced mathematical discourse is associated only with university mathematics.

communication. Specifically, in the provisional definition, I used the word ‘unique’ as a placeholder to signify that teaching practices at the horizon differ from common teaching practices in a mathematics classroom (e.g., attending to students’ errors). The findings suggest that achieving intersubjectivity is central to communicating effectively about ideas beyond the mathematics of the moment and aid the introduction of students in a new discourse (Cooper & Lavie, 2021). Second, in the revised definition narratives about the grade level of the students (e.g., curriculum instructions for a Year of study, predicted ‘ability’ group) are operationalised through the idea of mathematics of the moment. Finally, the definition highlights the importance of attending to the mathematical significance of a teacher’s contribution to discussions beyond the mathematics of the moment (mathematical narratives) and teaching practices (pedagogical narratives). Thus, the refined definition is in line with Fernández et al. (2011) reconceptualisation of Horizon Content Knowledge as shaping ‘in-action knowledge’ and Naik’s (2018) suggestion of introducing professional practice knowledge as a dimension of Horizon Content Knowledge. However, my focus is the discursive activity of the teacher and not ‘understanding’ as a separate entity contributing to teaching actions.

8.1.3. *Expanding our outlook on advanced mathematics for teaching*

In Section 8.1.2, I described Discourse at the Mathematical Horizon the discourse that includes advanced mathematical patterns of communication enrichedwith pedagogical meaning through the engagement with teaching. According to this description Advanced Mathematical Discourse is an essential pre-requirement of Discourse at the Mathematical Horizon.

Historically, advanced mathematics is considered in the context of higher education (e.g., Klein, 1908/2016; Tall, 1991). In particular, advanced mathematics for teaching is studied in relation to designing courses for initial teacher education or continued professional development (e.g., Even, 2011; Pinto & Cooper, 2022; Wasserman et al., 2019; Zazkis & Leikin, 2010). Therefore, the descriptions of advanced mathematics are often focused on University Mathematical Discourse, i.e., ideas and practices taught in higher education. For instance, Advanced Mathematical Knowledge (Zazkis & Leikin, 2010) is defined as “knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college” (p. 264). However, my findings indicate that university studies, although important, are not sufficient to describe the advanced mathematical narratives of the participants which were found to influence discourses at the Mathematical Horizon. Moreover, Pinto and Cooper (2022) evidence how bringing together communities of mathematicians and teachers provide opportunities to develop Mathematical Discourse for Teaching in ways traditional university courses do not currently offer.

In Section 6.9, I proposed seeing university studies as a threshold between elementary and advanced mathematical learning and expand the context in which advanced and applied mathematics is considered. Thus, I defined Advanced Mathematics Discourse as the Discourse which includes word use, visual mediators, narratives and routines encountered through the engagement with advanced – including applied – mathematics both in formal and informal settings.

Specifically, I suggest that:

> **Advanced Mathematical Discourse** includes mathematical narratives, word use, visual mediators and routines of expert mathematists that teachers endorse during their university studies (e.g., studying mathematics, science or sports science), or through professional experiences (e.g., working in research or in industry) and personal interests (e.g., reading non-fiction books, searching on the internet).

Under the proposed definition, University Mathematics Discourse is viewed as part of Advanced Mathematical Discourse which contains institutionalised mathematical narratives that vary depending on the programme (e.g., mathematics degree, Subject Knowledge Enhancement Course) and the discipline (mathematics, science etc). However, Discourse at the Mathematical Horizon is not necessarily institutionalised. Specifically, changes in a teacher's discourse at the Mathematical Horizon can occur throughout her education and career e.g., by interacting with colleagues, through life experiences and engagement with books and other resources. Moreover, the findings support the idea that teachers do not necessarily rely on distant memories (e.g., from a university course). A familiarity with the potentials of the mathematics involved in the discussion and access to resources could be a starting point to develop arguments beyond the mathematics of the moment.

The distinction between University Mathematics Discourse and Advanced Mathematical Discourse is important for this and future studies on Discourse at the Mathematical Horizon.

The distinction highlights the progression in the agency of a teacher over the mathematical narratives she shares with her students. Specifically, she matures from a peripheral participant in a University Mathematics Discourse, as an undergraduate student or trainee, to engaging with advanced mathematics out of interest, curiosity, or necessity to transforming advanced mathematical narratives into narratives accessible to the students. I argue that the distinction between the theoretical constructs will aid in addressing gaps between teacher education and teaching practices beyond the mathematics of the moment. For example, initial mathematics teacher education could alert prospective teachers to opportunities for lifelong learning and focus on reflective methods for identifying connections between advanced mathematical content and teaching, and techniques for using appropriate words and visual mediators and building narratives and routines to achieve intersubjectivity.

8.1.4. *Transforming empirical evidence into practice-based resources for teachers*

In chapter 7, I propose a design of vignette-based resources to bring up discourses at the Mathematical Horizon, drawing upon the MathTASK design principles (Biza et al., 2018, 2021), based on my research findings and utilising the data collected from the lesson observations (RQ 2).

A review of the literature suggests that professional development activities aiming to develop the understanding of mathematics beyond the curriculum focus on bridging the gap between subject matter knowledge and practice, by bringing different communities together either during the design phase (Larsen et al., 2018) or during the implementation (Maass & Engeln, 2019; Nelson & Slavit, 2007; Pinto & Cooper, 2022), or by grounding the discussions in secondary school teaching practices (Wasserman et al., 2019). However, the choice of activities in each case is different, sometimes focusing on a mathematical problem while in other cases on classroom situations. The empirical findings of the study reinforce the above suggestions. In particular, Discourse at the Mathematical Horizon is described as an amalgamation of Advanced Mathematical and Pedagogical discourses. The results discussed thus far suggest that the diversity in the discourses at the Mathematical Horizon can be influenced by (lack of) institutional support and Pedagogical Discourses. Moreover, changes in a teacher's discourse at the Mathematical Horizon can happen throughout her education and career and can be supported by interactions with colleagues, life experiences and engagement with books and (online) resources.

Following the idea of bridging the gap between education and practice, I side with the view that activities should be grounded in practice and propose that vignette-based resources are appropriate tools to bring up discourses at the Mathematical Horizon due to their usefulness in providing a specific scenario to collect information and prompt discussions for matters which might be complex, sensitive or controversial (Poulou, 2001). Additionally, MathTASK resources (mathtasks) have the distinctive characteristic of starting first with solving a context-appropriate mathematical problem before moving to the discussion of the vignette through reflective questions.

As a starting point for the proposed design, four teachers were invited to participate in an online focus group during which they discussed and reflected on their teaching using a pilot mathtask designed specifically for the purposes of the present study. The activity consisted of a mathematical task intended for Year 8 students, a fictional dialogue depicting a moment of contingency inspired by the mathematical task and a series of questions. The findings of the focus group suggest that the mathtask supported the engagement of the participants in debates about opportunities for discussions beyond the mathematics of the moment. The discursive shifts observed suggest that the mathtask can be used to encourage participants to bridge advanced mathematical and pedagogical discourses under a specific situation, draw attention to intersubjectivity, endorse ideas of others and challenge risk-averse pedagogical practices.

The findings of the focus group furthered the refinements of the design. Specifically, I propose a method for adapting the episodes identified during the classroom observations into realistic but fictional situations and a structure for setting reflective questions that could prompt mathematically and pedagogically reach discussions about ideas and practices that were found to be related to discourse at the Mathematical Horizon in Chapters 5, 6 and 7.

The proposal aims towards building a repository of resources to bring up discourses at the Mathematical Horizon. The designed resources can be used as methodological tools for research and professional development activities to stimulate discussions. The design can be adapted into different contexts provided that the data reflect the educational system and generalised using a different format or overarching design. Moreover, I claim that longitudinal engagement with the mathtask and others in the group (e.g., teachers, facilitators) could foster an environment that brings together different communities (e.g., researchers, teacher educators and teachers) and encourages the participants to form a community of practice (Lave & Wenger, 1991) with shared interest of introducing opportunities to go beyond the mathematics of the moment in their teaching practices. Thus, bridging the gap between education and practice by combining different approaches proposed in the literature (Larsen et al., 2018; Maass & Engeln, 2019; Nelson & Slavit, 2007; Pinto & Cooper, 2022; Wasserman et al., 2019). The proposed design and the implementation of the resources will be tested and improved in the future (see Section 8.4).

8.2. Implications for policy and practice

Alongside the theoretical contribution of the study, conceptual, methodological and empirical findings offer suggestions at policy and practice level. First, the study supports the idea of viewing and interpreting a curriculum as an act of communication, a contract, between policy makers and education professionals (Braslavsky, 2002; Stenhouse, 1975). As such, curricula shape and can be shaped by the actions of the contributors. Policy makers can influence what and how topics are taught by teachers in schools but also policy makers could – and maybe should – consider how teachers use the provided materials to build their teaching. In other words, policy makers could use input from teachers to create flexible curricula that could offer the teacher support to go beyond teaching a list of topics.

Secondly, my findings indicate that teachers learn – and incorporate to their teaching practices beyond the mathematics of the moment – experiences gained through their education but also through their personal and professional life. Moreover, the empirical evidence illustrates teachers taking initiatives to prepare their students for future learning, the "world of work" (Maass & Engeln, 2019) and living in future societies. These remarks highlight the need to invest in developing sustainable ways for teachers to enrich their discourses at the Horizon throughout their careers. The MathTASK design proposed in this work can be utilised to develop resources for both initial and continued teacher education. Specifically, initial teacher education should aim to bridge the gap between subject matter and practice through practice-based activities and alert prospective teachers to look for connections across the topics being taught in school, advanced mathematics and their applications. Regarding continued teacher education, the findings of the study imply the design and delivery of professional development activities to develop strategies that could aid teachers in taking the most out of past and future experiences which might be related to teaching beyond the mathematics of the moment. For instance, professional development activities could aim to support teachers in evaluating online information and resources, guided reflection, noticing (Mason, 2002) and building intersubjective narratives which could be shared with students.

Finally, the findings highlight the diversity in the individual discourses at the Mathematical Horizon. At the time of this writing, teacher education in England does not include provisions for enriching teachers' discourses at the Mathematical Horizon in a systematic way. I highlight the need to support initiatives for prospective and/or in-service teachers to discuss, reflect on and develop their own practices within their communities (e.g., colleagues) or across communities (e.g., prospective and in-service teachers, colleagues from different disciplines or teacher educators). The professional development resources proposed in this work can serve as medium to share individual practices beyond the mathematics of the moment with others and create a shared awareness of Discourse at the Mathematical Horizon.

8.3. Limitations

This section reflects on the limitations of my study and how they can be addressed to inform future research. The methodological limitations of the study are discussed in detail in section 4.9. Here, I wish to briefly revisit the adaptability of the study in different contexts in light of the revised definition of Discourse at the Mathematical Horizon. Specifically, the findings of the study are bound to the context in which schools in England operate and the National Curriculum. Thus, adapting Discourse at the Mathematical Horizon relied on defining discussions beyond the mathematics of the moment in relation to the local curricular guidelines. In this context, discussions beyond the mathematics of the moment are identified based on (1) the National Curriculum, (2) the guidelines provided by the exam boards, and (3) the resources suggested by the school in which the teacher was teaching at the time. Studies in other contexts could reveal elements of Discourse at the Mathematical Horizon which my study did not capture.

A second limitation is that identifying opportunities to go beyond the mathematics of the moment was not always an easy task. This was because the teachers did not always denote explicitly what the schools' expectations were. Talking to the teacher straight after the lesson could have helped identifying if a discussion was in line with the resources provided by the school. But, this was not always possible, due to the busy schedules of the teachers. In such cases, the decision was made by comparing the activities included in the episode with resources available to schools (e.g., books, teachers' blogs and well-known educational websites). Related to that, a third limitation is the timing of the interviews which varied across the participants depending on their availability. Addressing these limitations in future studies could help refine the process of identifying opportunities to go beyond the mathematics of the moment.

A fourth limitation is that all the lesson observations took place in Key Stage 3. In fact, even the teachers who initially agree to letting me in their classroom but later the observations did not take place either because of the limited interest of the students and their parents or due to COVID-19 school closure, suggested that I could observe their lessons in Key Stage 3. One possible explanation might be that the teachers felt more confident in letting me observe lessons where they were not pressured to prepare students for the GCSE exams. However, this is just a conjecture. Collecting more data from Key Stage 4 and even Further Education will help identify more opportunities to go beyond the mathematics of the moment and possibly examine whether the patterns of communication change across the different curriculum blocks.

A final limitation is related to the analysis of the data set of this study. Specifically, during the initial stages of the analysis I identified narratives and teaching practices that relate to agency and authority over the substantiation of the mathematical narratives in the classroom. For example, Liz frequently challenged her authority as the "ultimate substantiator" (Sfard, 2008, p. 234) in her classroom, and Alex mentioned multiple times that he is "advised" to avoid certain teaching practices with his students. Those and other examples, are briefly mentioned in this work as part of the broader findings regarding Discourse at the Mathematical Horizon. However, the data are not analysed or discussed to their full potential in this regard due to time limitations. Investigating the role of agency in narratives beyond the mathematics of the moment would be a promising direction to be addressed in the future.

8.4. Future research

In the future, collecting and analysing data from different curriculum blocks (e.g., Key Stage 4 and A-level) will help validate and refine the findings of this study. Moreover, further examination of existing data is planned in order to explore aspects which arose during the data analysis stage but were beyond the focus of the current study. These include (1) exploring the role of agency and authority in narratives beyond the mathematics of the moment, (2) exploring in more detail the role of non-fiction books and the internet as sources of information and channels of communication about ideas beyond the mathematics of the moment and (3) studying the role of intersubjectivity during discussions beyond the mathematics of the moment in relation to Zone of Proximal Development (Vygotsky, 1978; Wertsch, 1984) for students.

Finally, future research could focus on testing and improving the proposed design of professional development resources. The resources can then be used in professional development settings, such as the MathTASK programme (Biza et al., 2007, 2018, 2021), or as research tools to explore whether they could trigger longitudinal change in the narratives of the participants of future research. Specifically, the findings discussed in Chapter 6 suggest that Discourse at the Mathematical Horizon is evident when focusing on specific situations and interactions of the teacher with others and/or resources. Findings in Chapter 7 expand this argument by indicating that the mathtask successfully engaged the participants in debates beyond the mathematics of the moment, exchange of ideas and, among others, proposing ways of changing their risk-averse pedagogical practices. Thus, I conjectured that vignette-based resources could be utilised to provide structured and longitudinal opportunities for teachers to develop their discourses at the Mathematical Horizon in the form of a community of practice (Lave & Wenger, 1991) with a shared interest in introducing opportunities to go beyond the mathematics of the moment in their teaching practices. I intend to investigate the latter claim in a future project.

8.5. Reflections on my journey

The experiences I gained during my doctoral studies are of great value to me as an early career researcher. When I joined the School of Education and Lifelong Learning as a PhD student, I felt a misfit because of my background in pure mathematics and my exposure to a different educational system. In my first year, studying different theories in mathematics education and education enriched my understanding of the field and helped me express my aspirations using established educational narratives. Being welcomed in classrooms and hearing the stories and the thoughts of teachers and teacher educators has been a privilege. Being exposed to so many different narratives about mathematics and teaching expanded my own mathematical horizons in ways I had never imagined. My participants guided me along paths that I had never considered as a pure-mathematist or as a teacher. So, I felt great responsibility to accurately represent their ideas and views, protect their identities and draw trustworthy conclusions. During my PhD, I had multiple opportunities to present my work to others with similar interests in the department and at national and international conferences. I am very grateful to those who engaged with my work, questioned, and challenged me, giving me the opportunity to pause, think and reflect. On a greater scale, I had the opportunity of developing as a researcher at a time when academic culture and the world was rapidly changing. I thoroughly enjoyed partaking in meaningful conversations about equity, diversity, and inclusion. Finally, conducting research in the midst of a pandemic taught me how to deal with uncertainty, adapt my objectives and maintain a work-life balance. This journey made me grow as a person as much as a researcher and helped me develop a sense of belonging. My experiences will continue to shape my identity and practice as a mathematics education researcher.

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Declaration of published work

Parts of the research work for this thesis have been presented in national and international conferences and published in conference proceedings. The publications in the conference proceedings are reports of the ongoing analysis.

Conference Papers:

Papadaki, E. & Biza, I. (2022, March). Teaching beyond the mathematics of the moment, *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12).* https://hal.archives-ouvertes.fr/hal-03745470

Papadaki, E. (2021, July). Routines at the horizon: when and how a teacher can go beyond the mathematics of the moment. In Inprasitha, M., Changsri, N., Boonsena, N. (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education (Vol.1)* (p. 169). Khon Kaen, Thailand: PME. https://pme44.kku.ac.th/home/uploads/volumn/pme44_vol1.pdf

Papadaki, E. & Biza, I. (2020, March). Conceptualising the Discourse at the Mathematical Horizon: Looking at one teacher’s actions “beyond the mathematics of the moment”. *Proceedings of the British Society for Research into Learning Mathematics,* 40 (1). https://bsrlm.org.uk/wp-content/uploads/2020/05/BSRLM-CP-40-1-12.pdf

Papadaki, E. (2019, March). Mapping out different discourses of mathematical horizon. *Proceedings of the British Society for Research into Learning Mathematics,* 39 (1). https://bsrlm.org.uk/wp-content/uploads/2019/07/BSRLM-CP-39-1-07.pdf

Appendices

Appendix I: Application to the EDU Research Ethics Committee

UNIVERSITY OF EAST ANGLIA

SCHOOL OF EDUCATION AND LIFELONG LEARNING RESEARCH ETHICS COMMITTEE

APPLICATION FOR ETHICAL APPROVAL OF A RESEARCH PROJECT

This form is for all staff and students across the UEA who are planning educational research. Applicants are advised to consult the school and university guidelines before preparing their application by visiting https://www.uea.ac.uk/research/our-research-integrity and exploring guidance on specific types of projects https://portal.uea.ac.uk/rin/research-integrity/research-ethics/research-ethics-policy. The Research Ethics page of the EDU website provides links to the University Research Ethics Committee, the UEA ethics policy guidelines, ethics guidelines from BERA and the ESRC, and guidance notes and templates to support your application process: https://www.uea.ac.uk/education/research/research-ethics.

Applications must be approved by the Research Ethics Committee before beginning data generation or approaching potential research participants.

- Staff and PGR (PhD, EdD, and EdPsyD) should submit their forms to the EDU REC Administrator (edu.support@uea.ac.uk) and Dr Kate Russell (Kate.russell@uea.ac.uk) at least two weeks prior to each meeting.
- **Undergraduate students and other students must follow the procedures determined by their course of study.**

APPLICANT DETAILS	
Name:	Paraskevi Papadaki
School:	EDU

Current Status:	**PGR Student**
UEA Email address:	p.papadaki@uea.ac.uk
If PGR, MRes, or EdD/EdPsyD student, name of primary supervisor and programme of study: PhD in Education (Mathematics) Primary Supervisor: Dr Irene Biza	
If UG student or MA Taught student, name of Course and Module: N/A	

The following paperwork must be submitted to EDU REC **BEFORE** the application can be approved. Applications with missing/incomplete sections will be returned to the applicant for submission at the next EDU REC meeting. Please combine the forms into **ONE** PDF	
Required paperwork	**✓Applicant Tick to confirm**
Application Form (fully completed)	✓
Participant Information sheet and Consent Form (EDU template appropriate for nature of participants i.e. adult/parent/carer etc.)	✓
Other supporting documents (for e.g. questionnaires, interview/focus group questions, stimulus materials, observation checklists, letters of invitation, recruitment posters etc)	✓

2. PROPOSED RESEARCH PROJECT DETAILS:	
Title:	Implementing Tasks to explore teachers' discourse at the mathematical horizon

Start/End Dates:	1/10/2018 – 30/09/2022

3. FUNDER DETAILS (IF APPLICABLE):	
Funder:	UEA SSF Studentship
	Has funding been applied for? Application Date:
	Has funding been awarded? YES
	Project code if known:
Will ethical approval also be sought for this project from another source? NO	
	If "yes" what is this source?

4. APPLICATION FORM FOR RESEARCH INVOLVING HUMAN PARTICIPANTS:
Please use the guidance notes to support your application as this can clarify what the committee needs to see about your project and can avoid any unnecessary requests for further information at a later date.

4.1 Briefly outline, using lay language, your research focus and questions or aims (no more than 300 words).

The aim of my study is to explore in-service teachers' mathematical and pedagogical discourses while engaging with mathematical content that is important for the coherence of the curriculum across educational levels. I will call the engagement with content an engagement in discourse at the mathematical horizon. In relation to the mathematical content, this study is not focusing on advanced mathematics but rather on general mathematical ideas, principles and applications of mathematics that connect and structure the mathematical theory across the school years. My focus is to investigate teachers'

discourse at the mathematical horizon during moments of contingency in the classroom and during discussions with their colleagues while engaging with professional development activities created under the MathTASK programme.

To this purpose I will observe teaching sessions and then run professional development workshops with teachers in order to address the following research questions and sub-questions:

- What are the teachers' opportunities to engage in discourse at the mathematical horizon in the classroom?
  - What opportunities are given by the structure of the curriculum and the teaching resources?
  - What opportunities are presented during teaching?
- How do teachers engage in discourse at the mathematical horizon during MathTASK professional development activities?
  - What are the characteristics of teachers' discourse at the mathematical horizon?
  - Are there any discursive shifts observable during the MathTASK activities in the workshops?
  - What can we learn from teachers' experiences at the workshops?

4.2 Briefly outline your proposed research methods, including who will be your research participants and where you will be working (no more than 300 words).

- **Please provide details of any relevant demographic detail of participants (age, gender, race, ethnicity etc)**

My research will be structured as a design research. More specifically, I will design MathTASK activities (Tasks) based on classroom observations, brief interviews with the teachers and detailed study of the curriculum and the available materials. The Tasks will then be embedded in a series of professional development workshops to further investigate teachers' discourses.

The main group of participants will be 5 to 10 in-service secondary school teachers (Key Stage 3 and Key Stage 4) in Norwich and/or the surrounding area with different mathematical backgrounds and levels of experience. Recruiting participants that teach in different key stages might provide more opportunities for discussions about connections across mathematical topics and their teaching in different levels. The reasons that I am choosing Norfolk as the focus of my study is partly because of convenience but also because East of England is one of the regions with high percentage of out-of-subject mathematics teachers in KS3 and KS4 according to a recent Nuffield report. In order to recruit participants, I will be sending emails to stakeholders (head of schools, academies and/or trusts) informing them about my research project and asking them to contact me if they are interested and if possible to circulate this email to other people that might be interested (snowballing). If a school, academy or trust is interested in my project then I will send an email to the teachers directly if I am granted access to their email addresses or through the school (for more details please see the Appendix A).

The teachers will have the choice to participate in every phase or in parts of the project. Therefore, the number of participants might vary between the different phases. In order to meet the objectives of this study, there is a minimum of 10 classroom observations needed and a substantial overlap between the groups of teachers participating in each phase.

Capturing interactions between the teacher and the student is important to by research due to the nature of the key concept I am investigating. Therefore, students between the ages of 11 to 16 will also be participating in the first phase of the research. I estimate that the number of student-participants will be around 100, but the exact number is dependent on the number of students in the different classes and the number of teacher participants.

In detail, to explore teachers' opportunities to engage in discourse at the mathematical horizon, I will use the following research methods:

Classroom observations (2-3 months): The data collected in this phase will be audio- or video-recorded lessons, teaching and learning resources and my notes. The choice between audio- or video-recording the lessons will be made based on the preferences of the teachers and the parents (see section 4.10 and the consent forms for more details). Each teacher, and subsequently the students, will be asked to participate in 2 to 3 classroom observations. The aim of the observations is to gather information about teachers' discourses at the mathematical horizon. These data will be used to investigate the first research question and inform the design of the activities. The incidents that will be recorded during the observations will not be included directly in the MathTASK activities. Rather, they will be used to inform fictional situations grounded on plausible teaching incidents.

Short follow-up interviews with the teachers (approx.10minutes after each classroom observation): The interviews will take place a convenient time not long after each of the classroom observations. The interviews will be audio-recorded and their aim is to collect the teacher's view and reflections about the observed lesson (see Appendix C for an indicative sample of the questions). The data collected will be used to add-on or support the data from the observations.

Recorded workshops (5 to 6 workshops; 1 per month for 5 to 6 months): The workshops will be audio- or video-recorded (this will depend on the preferences of the participants). The data collected here will be audio- or video-recordings of the discussions, snapshots of the produced work and the whiteboard, online communication (such as Padlet), copies of teacher's diaries entries, and (online) questionnaire data (e.g. Qualtrics) and field notes. These data will be used to investigate the second research question and in particular the first sub-question.

Semi-structured interviews (in a period of 1 month): The interviews will be audio-recorded and the data collected will be used as feedback for the workshops and to further investigate teachers' discourse at the mathematical horizon. These data will be used to investigate the second research question and in particular the second sub-question (see Appendix C for an indicative sample of the questions).

If data are to be collected from online sources, this will be done by using platforms that align with the UEA data protection policy.

4.3 Briefly explain how you plan to gain access to prospective research participants. (no more than 300 words).

- **Who might be your gatekeeper for accessing participants?**
- **If children/young people (or other vulnerable people, such as people with mental illness) are to be involved, give details of how gatekeeper permission will be obtained. Please provide any relevant documentation (letters of invite, emails etc) that might be relevant**
- **Is there any sense in which participants might be 'obliged' to participate – as in the case of pupils, friends, fellow students, colleagues, prisoners or patients – or are volunteers being recruited?**

I will get in touch with gatekeepers (e.g. school heads, head of mathematics departments, heads of academy trusts, professional development providers, etc.) around Norwich to inform them about my research plan. I have prepared an initial letter of invitation (see the attached example). For the school heads, there will be a special section explaining my intentions and the purpose of classroom observations, especially in relation to the focus of my observations: the teacher of the class and their interactions with the students. If the gatekeepers agree my study to be conducted in their institution, I will circulate invitations for participation to teachers, for example through the head of school. The teachers are not in any way obligated to take part in my study.

For my visits at the school(s), I will work closely with the person that is responsible for the safeguarding.

Regarding the classroom observation, my presence will not affect any student's marks or classroom experience. All the students, and their parents, will be informed about my visit and the aims of the observations beforehand and will have the chance to ask me any question and will be asked to give their consent (see Appendix B). None of the students will be obligated to take part in my study. If they or their parents do not agree by signing the consent for their contribution in the classroom <u>will not be included in my data</u> (see section 4.10 for more details on the procedures in place).

4.4 Please state who will have access to the data and what measures will be adopted to maintain the confidentiality or anonymity of the research subject and to comply with data protection requirements e.g. how will the data be anonymised? (No more than 300 words.)

Only I and my primary supervisor will have access to the raw data. Information will be stored securely and participant's identity will not be disclosed, except as required by law. The data will be stored in a password protected OneDrive folder and also in an encrypted hard drive for backup. Study findings may be published, but participants will not be identified in these publications. There is a small risk that the teachers who will be participating in the workshops might be identifiable within their group. this risk is clearly stated in their PCFs. The data will be stored for a period of 10 years and then destroyed.

Anonymised extracts of the data might be discussed during the monthly RME and MathTASK group meetings for triangulation.

4.5 Will you require access to data on participants held by a third party? In cases where participants will be identified from information held by another party (for example, a doctor or school) describe the arrangements you intend to make to gain access to this information (no more than 300 words).

Materials will be sought from the schools online portal or inventory which includes teaching and learning resources and detailed guidelines of the school curriculum (no students grades, no pieces of students' responses to assessment). These data will be used to inform the design and the choice of the professional development activities. None of these data will be directly used in the workshops. If these materials include information such as the name of the school/trust or the name of a teacher, these information will be edited out of the copy and will not be included in the analysis.

4.6 Please give details of how consent is to be obtained (no more than 300 words).

Identify here the method by which consent will be obtained for each participant group e.g. through information sheets and consent forms, oral or other approach. Copies of all forms should be submitted alongside the application form (do not include the text of these documents in this space).

- **How and when will participants receive this material and how will you collect forms back in?**

Copies of the information sheets and the consent forms will be given to potential participants before the beginning of the fieldwork. I will make sure that the participants will have enough time to consider their involvement and sign the consent forms. The participants will be informed that data collected will be treated in the strictest confidence and will only be reported in an anonymised form. All potential participants will be asked to give their explicit, written consent to participating in the research, and, where consent is provided, separate copies of this will be retained by both researcher and participant. Participants will be made aware that they may freely withdraw from the project at any time without risk or prejudice.

Following that, the students and their parents will be asked to sign opt-in forms (the forms for the students and the parents are included in Appendix B-3).The information sheets and consent forms will be handed to the parents and the students through the school, again they will be informed about the process and anonymity of the data. Moreover, they will be made aware that they may freely withdraw their consent at any time. The parents and the students will be asked to return the forms to the teacher at a given date. The consent forms will be checked that include both the approval of the students (Appendix B-4) and their parents (Appendix B-3).

4.7 If any payment or incentive will be made to any participant, please explain what it is and provide the justification (no more than 300 words).

No payment will be made, but I may provide coffee and biscuits for the workshops.

4.8 What is the anticipated use of the data, forms of publication and dissemination of findings etc.? (No more than 300 words.)

The data collected will be used in my dissertation. Apart from that the findings will be presented in conferences, journal articles and/or in a book. Moreover, the data might be used for further development of resources (refining the Tasks, informing the design of future workshops). The identity of the participants will not be identifiable in the publications.

4.9 Findings of this research/project would usually be made available to participants. Please provide details of the form and timescale for feedback. What commitments will be made to participants regarding feedback? How will these obligations be verified? If findings are not to be provided to participants, explain why. (No more than 300 words.)

Findings of the study will be made available to participants in the form of an anonymised summary sent via emails at the end of the study. The participants will also have the opportunity to review the transcripts of the interviews if they wish. The participants will indicate to me whether they would like to receive the summary and/or review the interview transcript by ticking the appropriate box on the consent form along with the provision of their email address. Finally my thesis will be available online after its successful completion.

4.10 Please add here any other ethical considerations the ethics committee may need to be made aware of (no more than 300 words).

- **Are there any issues here for who can or cannot participate in the project?**
- **If you are conducting research in a space where individuals may also choose not to participate, how will you ensure they will not be included in any data collection or adversely affected by non-participation? An example of this might be in a classroom where observation and video recording of a new teaching strategy is being assessed. If consent for all students to be videoed is not received, how will you ensure that a) those children will not be videoed and/or b) that if they are removed from that space, that they are not negatively affected by that?**

My study involves classroom observations (Key Stages 3 and 4). The focus of the observation is the teacher's mathematical and pedagogical discourse beyond the curriculum, such kind of discourse can be observed during moments of contingency in the classroom. As a result, capturing the interaction between the teacher and the students is important. Therefore, it is most appropriate to an opt-in form for the parents. I will assure that the students will be guaranteed of personal anonymity and all the data obtained will be kept in a safe place, and will be destroyed after 10 years. If any lesson is video recorded the camera will be phasing away from the students, if possible. No snapshots of students faces will be included in future publications. On the day of the observation, I will locate myself in a position least visible to students so that the learning and teaching will be the least affected by my presence.

Regarding the method of recording the data, the teacher will initially have to choose between video- or audio-recording of the classroom. Then the parents and the students will be handed the appropriate form. If the students or their parents do not agree by signing the consent form, I will do my best not to include their contribution in the data analysis. To minimize capturing non-consenting students, I will make sure that I will position any recording device away from them. If any changes in the layout of the classroom are required, this will be done in collaboration with the teacher and only if there is no alternatives. I will keep track of their position in the classroom and keep notes of the times they contribute to the conversation. After the end of the lesson will edit out any of their questions or responses if they are accidentally recorded. This will ensure that these students will not be included in the data analysis, however I may analyse their influence in the conversation (e.g. the teacher's response to them). No one else will have access to the unedited files. In case there is a large number of non-participating students (more than 1/3 of the class) I will consider changing the type of recording or if necessary not observing this particular class.

I will be involved in the preparation and facilitation of the workshops. Consequently, my actions will also be part of the data. I will do my best for my research to fulfil the requirements of a high quality close-to-practice research. I will do my best to be as objective as possible and keep my personal expectations aside from the aims of the workshops. Another member of the MathTASK team might also be present during the workshops as an observer or as a facilitator if I have to be less involved. I will do my best to maintain the high quality of the workshops as an opportunity of professional development for the participants.

Finally, the teachers will also be able to attend the workshops even if they do not want to participate in the research. If some teachers do not want their responses to be included in the study, their response will be still discussed in the group. If they do not want their engagement with the Task and/or their participation in the group discussion to be recorded, they can still engage with the Task and discuss their response with a group and this discussion will not be captured by an audio or/and a video recorder.

4.11 What risks or costs to the participants are entailed in involvement in the research/project and how will you manage that risk?

- **Are there any potential physical, psychological or disclosure dangers that can be anticipated? What is the possible harm to the participant or society from their participation or from the project as a whole?**
- **What procedures have been established for the care and protection of participants (e.g. insurance, medical cover, counselling or other support) and the control of any information gained from them or about them?**

The only potential cost will be for the teachers to give me some of their free time for a period of time. There is no physical or psychological risk anticipated for the participants.

There is still a risk the participation of a teacher or the students that will be present in the room but do not wish to take part in my study to be recorded by a nearby audio/video-recorder. In this case what they will say will not been transcribed and reported directly but its influence to the conversation of those who have agreed to participate in the study will be recorded and analysed.

4.12 What is the possible benefit to the participant or society from their participation or from the project as a whole?

My study is a design research. Hence, it involves designing an intervention though which I will collect data to develop a theory. The Tasks will be designed based on the general principles and structure of the MathTASK programme. The intervention will be designed as a series of professional development workshops for the teachers. The workshops will attempt to address issues related to implementing a well-connected and coherent curriculum, as proposed by the new Ofsted guidelines. Participation in the workshops will give to teachers the opportunity to reflect on their practice, discuss issues related to the teaching and learning of mathematics. Moreover, the workshops will be specifically design aid the teachers to enhance their mathematical and pedagogical discourses based on their particular needs and the Ofsted requirements.

4.13 Comment on any cultural, social or gender-based characteristics of the participants which have affected the design of the project or which may affect its conduct. This may be particularly relevant if conducting research overseas or with a particular cultural group

- **You should also comment on any cultural, social or gender-based characteristics of you as the researcher that may also affect the design of the project or which may affect its conduct**

There is no evidence in the literature that my area of study is associated with any kind of socio-cultural or gender issues. Therefore, I expect no cultural, social or gender-based characteristics of the participants will affect the research design of the study. However, if any issues come up during the data collection, I will include them in the analysis. Moreover, I have to acknowledge the fact that I was lived and studied in Greece since I was born and I had only recently moved to the UK. I am familiar with the UK, regardless my position regarding mathematics and teaching of mathematics might be distingue from people that live in the UK for longer that I am.

4.14 Does your research have environmental implications? Please refer to the University's Research Ethics Guidance Note: Research with a Potential Impact on the Environment for further details. Identify any significant environmental impacts arising from your research/project and the measures you will take to minimise risk of impact.

No

4.15 Will your research involve investigation of or engagement with terrorist or violent extremist groups? Please provide a full explanation if the answer is 'yes'.

No

4.16 Please state any precautions being taken to protect your health and safety? This relates to all projects and *not just* those undertaken overseas.

- **What health and safety or other relevant protocols need to be followed e.g. a DBS for work in schools? Have you completed this?**
- **If you are travelling to conduct your research, have you taken out travel and health insurance for the full period of the research? If not, why not.**
- **If you are travelling overseas, have you read and acted upon FCO travel advice (https://www.gov.uk/foreign-travel-advice)? If not, why not. If acted upon, how?**
- **Provide details including the date that you have accessed information from FCO or other relevant organization**
- **If you are undertaking field work overseas you are required to submit a Risk Assessment Form with your application. This is even if you are a researcher 'going home' to collect data (check EDU REC website).**

I will complete a DBS if necessary once I get in contact with the schools. Each school might have specific requirements based on their safeguarding policy that must be taken into account.

4.17 Please state any precautions being taken to protect the health and safety of other researchers and others associated with the project (as distinct from the participants or the applicant).

No health and safety issues.

4.18 The UEA's staff and students will seek to comply with travel and research guidance provided by the British Government and the Governments (and Embassies) of host countries. This pertains to research permission, in-country ethical clearance, visas, health and safety information, and other travel advisory notices where applicable. If this research project is being undertaken outside the UK, has formal permission/a research permit been sought to conduct this research? Please describe the action you have taken and if a formal permit has not been sought please explain why this is not necessary/appropriate (for very short studies it is not always appropriate to apply for formal clearance, for example).

My research project is undertaken in the UK.

4.19 Are there any procedures in place for external monitoring of the research, for instance by a funding agency?

No

5. DECLARATION:

Please complete the following boxes with YES, NO, or NOT APPLICABLE:

I have read (and discussed with my supervisor if student) the University's Research Ethics Policy, Principle and Procedures, and consulted the British Educational Research Association's Revised Ethical Guidelines for Educational Research and other available documentation on the EDU Research Ethics webpage and, when appropriate, the BACP Guidelines for Research Ethics.	Yes
I am aware of the relevant sections of the GDPR (2018): https://ico.org.uk/for-organisations/guide-to-the-general-data-protection-regulation-gdpr/ and Freedom of Information Act (2005).	Yes
Data gathering activities involving schools and other organizations will be carried out only with the agreement of the head of school/organization, or an authorised representative, and after adequate notice has been given.	Yes
The purpose and procedures of the research, and the potential benefits and costs of participating (e.g. the amount of their time involved), will be fully explained to prospective research participants at the outset.	Yes
My full identity will be revealed to potential participants.	Yes
Prospective participants will be informed that data collected will be treated in the strictest confidence and will only be reported in anonymised form unless identified explicitly and agreed upon	Yes
All potential participants will be asked to give their explicit, written consent to participating in the research, and, where consent is given, separate copies of this will be retained by both researcher and participant.	Yes
In addition to the consent of the individuals concerned, the signed consent of a parent/carer will be required to sanction the participation of minors (i.e. persons under 16 years of age).	Yes

Undue pressure will not be placed on individuals or institutions to participate in research activities.	Yes
The treatment of potential research participants will in no way be prejudiced if they choose not to participate in the project.	Yes
I will provide participants with my UEA contact details (*not* my personal contact details) and those of my supervisor (if applicable), in order that they are able to make contact in relation to any aspect of the research, should they wish to do so. I will notify participants that complaints can be made to the Head of School.	Yes
Participants will be made aware that they may freely withdraw from the project at any time without risk or prejudice.	Yes
Research will be carried out with regard for mutually convenient times and negotiated in a way that seeks to minimise disruption to schedules and burdens on participants	Yes
At all times during the conduct of the research I will behave in an appropriate, professional manner and take steps to ensure that neither myself nor research participants are placed at risk.	Yes
The dignity and interests of research participants will be respected at all times, and steps will be taken to ensure that no harm will result from participating in the research	Yes
The views of all participants in the research will be respected.	Yes
Special efforts will be made to be sensitive to differences relating to age, culture, disability, race, sex, religion and sexual orientation, amongst research participants, when planning, conducting and reporting on the research.	Yes
Data generated by the research (e.g. transcripts of research interviews) will be kept in a safe and secure location and will be used purely for the purposes of the research project (including dissemination of findings). No-one other than research colleagues, professional transcribers and supervisors will have access to any identifiable raw data collected, unless written permission has been explicitly given by the identified research participant.	Yes
Research participants will have the right of access to any data pertaining to them.	Yes

All necessary steps will be taken to protect the privacy and ensure the anonymity and non-traceability of participants – e.g. by the use of pseudonyms, for both individual and institutional participants, in any written reports of the research and other forms of dissemination.	Yes

I am satisfied that all ethical issues have been identified and that satisfactory procedures are in place to deal with those issues in this research project. I will abide by the procedures described in this form.

Name of Applicant:	**Paraskevi Papadaki**
Date:	**27/06/2019**

PGR/EdD/EdPsyD/MRes Supervisor declaration (for PGR/EdD/EdPsyD/MRes student research only)

I have discussed the ethics of the proposed research with the student and am satisfied that all ethical issues have been identified and that satisfactory procedures are in place to deal with those issues in this research project.

Name of PGR Supervisor:	**Dr Irene Biza**
Date:	**27/06/2019**

MA taught/Undergraduate Supervisor declaration (for MA Taught/Undergraduate student research only)

I confirm that I have read and discussed the ethics of the proposed research with the student and am satisfied that all ethical issues have been identified and that satisfactory procedures are in place to deal with those issues in this research project. I also confirm that all of the relevant documents are appropriate to conduct the proposed research.

Name of Supervisor:	
Date:	

Appendix II: Recruiting participants

1. Draft email to stakeholders

Dear [name],

My name is Paraskevi (Evi) Papadaki, I am a postgraduate researcher in the School of Education and Lifelong Learning at the University of East Anglia. I am currently looking to get in contact with [schools/trusts] that might be interested in my research regarding mathematics teachers' knowledge and professional development.

The following is a brief description of my research project. I would much appreciate it if you could find some time to read and consider it.

The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content "mathematical horizon". My research will involve classroom observations to look for such opportunities, interviews with teachers and observations of a series of professional development workshops with MathTASK activities. The activities will be created specifically for this research and will be tailored to the needs of the participants. Therefore, if your school/trust is interested in taking part, you might be asked to grant me access to your teaching and learning resources but not to students' personal data.

The workshops will attempt to address issues related to implementing a well-connected and coherent curriculum, as proposed by the new Ofsted guidelines. Participation in the workshops will provide opportunities to reflect on mathematical and pedagogical values and to discuss issues related to the teaching and learning of mathematics.

Please find attached a poster that illustrates the objected and the intentions of this study. If you think that your school/trust might be interested in participating, do not hesitate to contact me at any time for more information or questions.

Please feel free to forward this email to anyone that you think might be interested in the project. Thank you for considering my offer.

Kind regards,

Evi Papadaki

2. Draft email to the teachers

Dear sir/madam,

My name is Paraskevi (Evi) Papadaki, I am a postgraduate researcher in the School of Education and Lifelong Learning at the University of East Anglia. I am getting in contact with you in regards with my research project.

I have been in contact with [your head of school/ the representative of XXX trust], [name], and they informed me that you might be interested in participating in the study I am contacted as a basis for a degree of Doctor of Philosophy (PhD).

The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content “mathematical horizon”. My research will involve classroom observations to look for such opportunities, interviews with teachers and observations of a series of professional development workshops with MathTASK activities. The activities will be created specifically for this research.

The workshops will attempt to address issues related to implementing a well-connected and coherent curriculum, as proposed by the new Ofsted guidelines. Participation in the workshops will provide opportunities to reflect on mathematical and pedagogical values and to discuss issues related to the teaching and learning of mathematics.

Please find attached a poster that illustrates the objected and the intentions of this study. If you are interested do not hesitate to contact me at any time for more information or questions. Thank you for considering my offer.

Kind regards,

Evi Papadaki

3. Draft poster

Implementing Tasks to explore teachers’ discourse at the mathematical horizon

The Ofsted guidelines highlight the importance of a coherently planned curriculum.
What does that mean for the mathematics teachers?
How do they have to act to implement a coherent curriculum?

What is the study about?

The aim is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels.

What does the study involves?

For you:
✓Complete your day to day classroom activities
✓Participate in a small number of professional development workshops
✓ Participate in interviews

For me:
✓Observing some of your lessons
✓Design and facilitate the workshops based on your needs
✓ Interview you

Why to get involved?

You will have the opportunity to reflect on your practice
Collaborate and discuss with your colleagues
Share what you know
Learn from others
Explore mathematical and pedagogical values related to the matter

Your contributions will be valuable for me and your colleagues!

Contact details:
Evi Papadaki
PhD Researcher
Email: P.Papadaki@uea.ac.uk
Tel: +44 (0) 7706268987

@mathtask
https://www.uea.ac.uk/education/mathtask

University of East Anglia

Appendix III: Information sheets and consent forms

1. Classroom observations and post-lesson interviews

Paraskevi Papadaki

PhD Researcher
[Insert date]

Faculty of Social Science

School of Education

University of East Anglia

Norwich Research Park

Implementing Tasks to explore teachers' discourse at the mathematical horizon

PARTICIPANT INFORMATION STATEMENT

(1) What is this study about?

The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content "mathematical horizon". My research will begin with classroom observations and brief interviews with the teachers to look for such classroom opportunities. I will transform these opportunities into fictional classroom situations to be included in a series of professional development activities. The design of the activities will be based on the principles of the MathTASK programme. Then, in-service teachers will be invited to participate in a number of workshops. The workshops will be designed to provide the teachers a secure place to collaborate, exchange ideas and broaden their understanding of the mathematical and pedagogical value of such opportunities.

You have been invited to participate in the first phase of this study (classroom observations and brief interviews) because you are a mathematics teacher. If you are interested to participate in the second phase a separate document will be distributed to you at a later date. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the study. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

Participation in this research study is voluntary. By giving consent to take part in this study you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined below.
- ✓ Agree to the use of your personal information as described.

(2) Who is running the study?

The study is being carried out by Paraskevi Papadaki, who is conducting this study as the basis for the degree of Doctor of Philosophy (PhD) at The University of East Anglia. This study is supported by an UEA SSF studentship and will take place under the supervision of Dr Irene Biza at the University of East Anglia.

(3) What will the study involve for me?

Your participation in this study will involve you completing your day-to-day classroom activities as normal. With your permission, I will observe about 2 lessons that you deliver on a topic that we will agree together before the observations take place. You will be asked to share with me the teaching and learning materials that you will be using for these lessons (no personal information of the students, no pieces of assessment). The focus of the observation will be the interactions between you and your students. I will audio- or video-record the lessons depending on your preference (please indicate your preference in the consent form). In addition, I will be taking notes to help me reflect on the observations later.

You will also be asked to participate in a short post-lesson interview about your experience during the lesson. The interviews will be done at the time and place convenient to you and will be audio-recorded. You will be able to review the interview transcript if you indicate so in the consent form.

(4) How much of my time will the study take?
Each post-lesson interview will last in approximately 10 minutes so that will mean a total of 20-30 minutes of your time. I will be in around 2 classes but this will be part of your normal everyday activities and so no additional time is required for you.

(5) Do I have to be in the study? Can I withdraw from the study once I've started?
Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with the researchers or anyone else at the University of East Anglia, the school and/or the XXX trust.

If you decide to take part in the study and then change your mind later, you are free to withdraw at any time. You can do this by simply telling me either via email, phone or in person. If you take part in an observation session, you are free to stop participating at any stage. If you decide at a later time to withdraw from the study, excluding your data will only be possible if the observation was videoed or you were individually identified at the time of the observation.

With regards to the interviews, you are free to stop at any time. Unless you say that you want me to keep them, any recordings will be erased and the information you have provided will not be included in the study results. You may also refuse to answer any questions that you do not wish to answer during the interview. With regards to the observations, if you wish to exclude any part of the data collected, I will remove those data. If you decide at a later time to withdraw from the study, your information will be removed from my records and will not be included in any results, up to the point I have analysed and published the results.

(6) Are there any risks or costs associated with being in the study?
Aside from giving up your time, we do not expect that there will be any risks or costs associated with taking part in this study.

(7) Are there any benefits associated with being in the study?

I hope that reflecting on your classroom experiences will be a rewarding act to take. Paying attentions to small details that you might otherwise overlook might inform the future delivery of similar lessons. Moreover, your contribution will aid in identifying, and better understanding, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum. Your opinion will contribute to the design of a series of professional development courses tailored to the needs of the teachers and the creation of activities based grounded in research but based on realistic classroom situations.

(8) What will happen to information about me that is collected during the study?

Data from my notes and audio or audio-recordings will be analysed. Outcomes of the analysis will be reported in academic and professional publications. Your information will be stored securely and your identity/information will be kept strictly confidential, except as required by law. Study findings may be published, but you will not be identified in these publications if you decide to participate in this study. In this instance, data will be stored for a period of 10 years and then destroyed.

By providing your consent, you are agreeing to me collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 2018 General Data Protection Regulation Act and the University of East Anglia Research Data Management Policy (2015).

(9)What if I would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me directly via email at p.papadaki@uea.ac.uk. If you would like to know more at any stage during the study, please feel free to contact Dr Irene Biza at i.biza@uea.ac.uk.

(10) Will I be told the results of the study?

You have a right to receive feedback about the overall results at the end of the study. You can indicate to me that you wish to receive feedback by ticking the relevant box on the consent form. This feedback will be in the form of an anonymised up to three-page summary on the completion of the data analysis. Moreover, after the successful completion of my studies, my thesis will be available.

(11) What if I have a complaint or any concerns about the study?

The ethical aspects of this study have been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

If there is a problem please let me know at the following address:

Paraskevi Papadaki

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0)1603 591741

Email: p.papadaki@uea.ac.uk.

If you would like to speak to someone else you can contact my supervisor: Dr Irene Biza email: i.biza@uea.ac.uk Tel: +44 (0)1603 59 1741

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact please contact the Head of the School of Education and Lifelong Learning, Professor Richard Andrews, at Richard.Andrews@uea.ac.uk.

(12) OK, I want to take part – what do I do next?

You need to fill in one copy of the consent form and contact me via email or phone so that we can arrange a meeting for collection of the consent form and for further exchange of information. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARTICIPANT CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.
✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia, the school and/or the XXX trust, now or in the future.
✓ I understand that I can withdraw from the study at any time.
✓ I understand that I may stop participating in an observation at any time if I do not wish to continue. I also understand that it will not be possible to remove my data unless the observation is videoed or I am individually identified in some way.
✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.
✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my name or any identifiable information about me.

I consent to:

- **Audio-recording** YES ☐ NO ☐
- **Video-recording** YES ☐ NO ☐

- **Classroom Observations** YES ☐ NO ☐
- **Access to teaching and learning materials** YES ☐ NO ☐
- **Audio-recording of the interviews** YES ☐ NO ☐
- **Reviewing transcripts of the interviews** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

....

Signature

....

PRINT name

....

Date

PARTICIPANT CONSENT FORM (2nd Copy to Participant)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.
✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia, the school and/or the XXX trust, now or in the future.
✓ I understand that I can withdraw from the study at any time.
✓ I understand that I may stop participating in an observation at any time if I do not wish to continue. I also understand that it will not be possible to remove my data unless the observation is videoed or I am individually identified in some way.
✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.
✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my name or any identifiable information about me.

I consent to:

- **Audio-recording** YES ☐ NO ☐
- **Video-recording** YES ☐ NO ☐
- **Classroom Observations** YES ☐ NO ☐

- **Access to teaching and learning materials** YES ☐ NO ☐

- **Audio-recording of the interviews** YES ☐ NO ☐

- **Reviewing transcripts of the interviews** YES ☐ NO ☐

- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

....

Signature

....

PRINT name

....

Date

2. Interviews with teachers and teacher educators

Paraskevi (Evi) Papadaki

PhD Researcher
[DATE]

Faculty of Social Science

School of Education

University of East Anglia

Norwich Research Park

Norwich NR4 7TJ

United Kingdom

Implementing Tasks to explore teachers' discourse at the mathematical horizon

PARTICIPANT INFORMATION STATEMENT

(13) What is this study about?

The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content "mathematical horizon". My research will begin with brief interviews with teachers and teacher educators and classroom observations to look for such classroom opportunities. I will transform these opportunities into fictional classroom situations to be included in a series of professional development activities. The design of the activities will be based on the principles of the MathTASK programme. Then, in-service teachers will be invited to participate in a number of workshops. The workshops will be designed to provide the teachers a secure place to collaborate, exchange ideas and broaden their understanding of the mathematical and pedagogical value of such opportunities.

This Participant Information Statement regards only the interviews to which you have been invited to participate because you are a mathematics teacher or/and educator. If you are interested to participate in the following phases of the study, a separate document will be distributed to you at a later date. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the study. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

Participation in this research study is voluntary. By giving consent to take part in this study you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined below.
- ✓ Agree to the use of your personal information as described.

(14) Who is running the study?
The study is being carried out by Paraskevi (Evi) Papadaki, who is conducting this study as the basis for the degree of Doctor of Philosophy (PhD) at The University of East Anglia. This study is supported by an UEA SSF studentship and will take place under the supervision of Dr Irene Biza at the University of East Anglia.

(15) What will the study involve for me?
Your participation will involve having an interview with me. These will take place in a quite space of your choice at a time that is convenient to you and the interviews will be audio recorded. You will be asked questions relating to opportunities for making connections across educational levels within your classroom and your opinion about use of MathTASK activities as a tool for professional development. You will be able to review the transcript of your interviews, if you wish to ensure they are an accurate reflection of the discussion.

(16) How much of my time will the study take?
It is expected that the interview will take between 30-40 mins.

(17) Do I have to be in the study? Can I withdraw from the study once I've started?
Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with me or anyone else at the University of East Anglia, your school/institution or any school trust.

If you decide to take part in the study and then change your mind later, you are free to withdraw at any time. You can do this by simply telling me either via email, phone or in person. You are free to stop the interview at any time. Unless you say that you want me to keep them, any recordings will be erased and the information you have provided will not be included in the study results. You may also refuse to answer any questions that you do not wish to answer during the interview. If you decide at a later time to withdraw from the study, your information will be removed from my records and will not be included in any results, up to the point I have analysed and published the results.

(18) Are there any risks or costs associated with being in the study?

Aside from giving up your time, we do not expect that there will be any risks or costs associated with taking part in this study.

(19) Are there any benefits associated with being in the study?

I hope that reflecting on your teaching experiences will be a rewarding act to take. Moreover, your contribution will aid in identifying, and better understanding, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum. Your opinion will contribute to the design of a series of professional development courses tailored to the needs of the teachers and the creation of activities based grounded in research but based on realistic classroom situations.

(20) What will happen to information about me that is collected during the study?

Data from my notes and audio-recordings will be analysed. Outcomes of the analysis will be reported in academic and professional publications. Your information will be stored securely and your identity/information will be kept strictly confidential, except as required by law. Study findings may be published, but you will not be identified in these publications if you decide to participate in this study. In this instance, data will be stored for a period of 10 years and then destroyed.

By providing your consent, you are agreeing to me collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 2018 General Data Protection Regulation Act and the University of East Anglia Research Data Management Policy (2015).

(21) What if I would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me directly via email at p.papadaki@uea.ac.uk. If you would like to know more at any stage during the study, please feel free to contact Dr Irene Biza at i.biza@uea.ac.uk.

(22) Will I be told the results of the study?

You have a right to receive feedback about the overall results at the end of the study. You can indicate to me that you wish to receive feedback by ticking the relevant box on the consent form. This feedback will be in the form of an anonymised up to three-page summary on the completion of the data analysis. Moreover, after the successful completion of my studies, my thesis will be available.

(23) What if I have a complaint or any concerns about the study?

The ethical aspects of this study have been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

If there is a problem please let me know at the following address:

Paraskevi(Evi) Papadaki

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0)1603 591741

Email: p.papadaki@uea.ac.uk.

If you would like to speak to someone else you can contact my supervisor: Dr Irene Biza email: i.biza@uea.ac.uk Tel: +44 (0)1603 59 1741

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact please contact the Head of the School of Education and Lifelong Learning, Professor Nalini Boodhoo, at N.Boodhoo@uea.ac.uk.

(24) OK, I want to take part – what do I do next?

You need to fill in one copy of the consent form and contact me via email or phone so that we can arrange a meeting for collection of the consent form and for further exchange of information. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARTICIPANT CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.
✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researcher or anyone else at the University of East Anglia, my school/institution and/or any school trust, now or in the future.
✓ I understand that I can withdraw from the study at any time.
✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.
✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my name or any identifiable information about me.

I consent to:

- **Audio/Video-recording of the interviews** YES ☐ NO ☐
- **Reviewing transcripts of the interviews** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?**
  YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ☐ Email:

____ ____

____

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…………………………………………………………………………………

Signature

…………………………………………………………………………………

PRINT name

…………………………………………………………………………………

Date

PARTICIPANT CONSENT FORM (2nd Copy to Participant)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.
✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia, my school/institution and/or any school trust, now or in the future.
✓ I understand that I can withdraw from the study at any time.
✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.
✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my name or any identifiable information about me.

I consent to:

- **Audio/Video-recording of the interviews** YES ☐ NO ☐
- **Reviewing transcripts of the interviews** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?**
YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ☐ Email:

____ ____

____

____

…………………………………………………………………………………

Signature

…………………………………………………………………………………

PRINT name

…………………………………………………………………………………

Date

3. Online Workshops

Paraskevi Papadaki

PhD Researcher
[DATE]

Faculty of Social Science

School of Education

University of East Anglia

Norwich Research Park

Norwich NR4 7TJ

Implementing Tasks to explore teachers' discourse at the mathematical horizon

PARTICIPANT INFORMATION STATEMENT – In-service Teacher workshops

(1) What is this study about?

The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content "mathematical horizon".

My research begun with classroom observations and brief interviews with teachers to look for such classroom opportunities. I transformed these opportunities into fictional classroom situations to be included in a series of professional development activities. The design of the activities is based on the principles of the MathTASK programme. Now, I invite in-service teachers to participate in a number of workshops. The workshops are designed to provide the teachers a secure place to collaborate, exchange ideas and broaden their understanding of the mathematical and pedagogical value of such opportunities.

You have been invited to participate in the **second** phase of this study (participation in workshops and interview) because you are a mathematics teacher. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the study. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

Participation in this research study is voluntary. By giving consent to take part in this study you are telling me that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined.
- ✓ Agree to the use of your personal information as described.

(2) Who is running the study?

The study is being carried out by Paraskevi Papadaki, who is conducting this study as the basis for the degree of Doctor of Philosophy (PhD) at The University of East Anglia. This study is supported by an UEA SSF studentship and will take place under the supervision of Dr Irene Biza at the University of East Anglia.

(3) What will the study involve for me?

You will be invited to engage with a small number of Tasks in the context of online professional development workshops. These Tasks are based on a specific mathematical teaching situation and you will be asked to reflect and respond to a list of questions related to this situation. With your consent, your written response on the tasks and other (online) materials during the workshops will be collected and your engagement with the task will be video-recorded. Then, you will have the chance to discuss about the situation described in the Task and your responses to the questions with your peers and me as the facilitator of the workshop, this discussion will be video-recorded. During the workshops you will also have access to online interactive materials and discussion boards that will also be part of the data. Finally, you will be given a link to a shared online pin board that you can use anonymously as a journal during or in-between the workshops. With your consent, I will use these journal entries in my analysis. Another person of the MathTASK team might be present in the sessions as a moderator, in that case you will be notified in advance.

Following the workshops, you might be invited to an audio-recorded interview with me to discuss about your experience with the workshops and the Tasks.

(4) **How much of my time will the study take?**

The workshops will last 1 hour each.

If you decide to take part in the interview, this will last approximately 40-45 minutes.

(5) Do I have to be in the study? Can I withdraw from the study once I've started?

Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with the researcher or anyone else at the University of East Anglia and/or your school.

If you decide to take part in the study and then change your mind later, you are free to withdraw at any time. You can do this by simply telling me either via email, phone or in person.

If at the time of your withdrawal the group discussion is in progress you will need to leave the virtual room. I will do my best to edit out your contributions from the recordings. However, it might not be possible to withdraw completely your individual comments from the records once the group has started and up to the point of your withdrawal.

You are free to stop taking part at the interview at any time. Unless you say that you want me to keep them, any recordings will be erased and the information you have provided will not be included in the study results. You may also refuse to answer any questions that you do not wish to answer during the interview.

If you decide at a later time to withdraw from the study, your information will be removed from our records and will not be included in any results, up to the point we have analysed and published the results.

(6) Are there any risks or costs associated with being in the study?

Aside from giving up your time, I do not expect that there will be any risks or costs associated with taking part in this study.

(7) Are there any benefits associated with being in the study?

The Tasks are designed for professional development and mathematics education purposes, participation in this study will give you the opportunity to reflect on your practice, discuss issues related to the teaching and learning of mathematics and engage with mathematical and pedagogical discourses. In this case, the focus of the Tasks and the workshops will be based on critical situations that have been observed in your or your colleagues' classrooms. Your participation will contribute to the further elaboration of the Tasks and provide some insights on how discourses beyond the school curriculum might be beneficial to the teaching and learning of mathematics in your classroom.

(8) What will happen to information about me that is collected during the study?

Data from your written responses, interactions with the materials and video-recordings will be analysed. Outcomes of the analysis will be reported in academic and professional publications. Your information will be stored securely and your identity/information will be kept strictly confidential, except as required by law. Study findings may be published. Although every effort will be made to protect your identity, there is a risk that you might be identifiable due to the nature of the study and/or results. In this instance, data will be stored for a period of 10 years and then destroyed.

By providing your consent, you are agreeing to me collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 2018 General Data Protection Regulation Act and the University of East Anglia Research Data Management Policy (2015).

(9) What if I would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me directly via email at p.papadaki@uea.ac.uk. If you would like to know more at any stage during the study, please feel free to contact Dr Irene Biza at i.biza@uea.ac.uk.

(10) Will I be told the results of the study?

You have a right to receive feedback about the overall results at the end of the study. You can indicate to me that you wish to receive feedback by ticking the relevant box on the consent form. This feedback will be in the form of an anonymised up to three-page summary on the completion of the data analysis. Moreover, after the successful completion of my studies, my thesis will be available.

(11) What if I have a complaint or any concerns about the study?

The ethical aspects of this study have been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

If there is a problem please let me know at the following address:

Paraskevi Papadaki

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0)1603 591741

Email: p.papadaki@uea.ac.uk.

If you would like to speak to someone else you can contact my supervisor: Dr Irene Biza email: i.biza@uea.ac.uk Tel: +44 (0)1603 59 1741

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact the Head of the School of Education and Lifelong Learning, Professor Nalini Boodhoo, at N.Boodhoo@uea.ac.uk.

(12) OK, I want to take part – what do I do next?

You need to fill in one copy of the consent form and contact me via email or phone so that we can arrange the collection of the consent form and for further exchange of information. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARTICIPANT CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researcher if I wished to do so.

✓ The researcher has answered any questions that I had about the study and I am happy with the answers.

✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia, the school and/or the school trust, now or in the future.

✓ I understand that I can withdraw from the study at any time.

✓ I understand that I can refuse giving my written response of any kind.

✓ I understand that I may stop the observation of my engagement with the activity.

✓ I understand that I may leave the discussion group at any time if I do not wish to continue. I also understand that it will not be possible to withdraw my comments once the group has started as it is a group discussion

✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published. Although every effort will be made to protect my identity, I may be identifiable in these publications due to the nature of the study or results.

I consent to:

- **Video-recording** YES ☐ NO ☐
- **Observations** YES ☐ NO ☐
- **Photographs of produced work** YES ☐ NO ☐
- **Share my online contributions**

  YES ☐ NO ☐

- **Share my notes** YES ☐ NO ☐
- **Audio-recording of the interview (if applicable)** YES ☐ NO ☐
- **Reviewing transcripts of the interview (if applicable)** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?**
  YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

………………………………………………………………

Signature

………………………………………………………………

PRINT name

………………………………………………………………

Date

PARTICIPANT CONSENT FORM (2nd Copy to Participant)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

- ✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

- ✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researcher if I wished to do so.

- ✓ The researcher has answered any questions that I had about the study and I am happy with the answers.

- ✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia, the school and/or the school trust, now or in the future.

- ✓ I understand that I can withdraw from the study at any time.

- ✓ I understand that I can refuse giving my written response of any kind.

- ✓ I understand that I may stop the observation of my engagement with the activity.

- ✓ I understand that I may leave the discussion group at any time if I do not wish to continue. I also understand that it will not be possible to withdraw my comments once the group has started as it is a group discussion

- ✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published. Although every effort will be made to protect my identity, I may be identifiable in these publications due to the nature of the study or results.

I consent to:

- **Video-recording** YES ☐ NO ☐
- **Observations** YES ☐ NO ☐
- **Photographs of produced work** YES ☐ NO ☐
- **Share my online contributions**

  YES ☐ NO ☐
- **Share my notes** YES ☐ NO ☐
- **Audio-recording of the interview (if applicable)** YES ☐ NO ☐
- **Reviewing transcripts of the interview (if applicable)** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?**
  YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

……………………………………………………………

Signature

……………………………………………………………

PRINT name

……………………………………………………………

Date

4. Opt-in form for parents

Paraskevi Papadaki

PhD Researcher
[Insert date]

Faculty of Social Science

School of Education

University of East Anglia

Norwich Research Park

Implementing Tasks to explore teachers' discourse at the mathematical horizon

PARENTAL INFORMATION STATEMENT

(1)What is this study about?
The aim of this study is to identify, and support teachers to make the most of, classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. I call this content "mathematical horizon".

My research will begin with classroom observations and brief interviews with the teachers to look for such classroom opportunities. I will transform these opportunities into fictional classroom situations to be included in a series of professional development activities. The design of the activities will be based on the principles of the MathTASK programme. Then, in-service teachers will be invited to participate in a number of workshops. The workshops will be designed to provide the teachers a secure place to collaborate, exchange ideas and broaden their understanding of the mathematical and pedagogical value of such opportunities.

Your child has been invited to participate in the first stage of this study (classroom observation) because *s/he* is a student attending a classroom where the teacher is participating in the study. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to let your child take part in the research. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

Participation in this research study is voluntary. By giving your consent you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree for your child to take part in the research study as outlined below.
- ✓ Agree to the use of your child's personal information as described.
- ✓ You have received a copy of this Parental Information Statement to keep.

(2)Who is running the study?

The study is being carried out by Paraskevi Papadaki, who is conducting this study as the basis for the degree of Doctor of Philosophy (PhD) at The University of East Anglia. This study is supported by an UEA SSF studentship and will take place under the supervision of Dr Irene Biza at the University of East Anglia.

(3)What will the study involve?

Your child's participation in this study will involve them completing their day to day mathematics classes as normal. I will be present in around 2 of your child's classes (I will let you know which ones) and I will be taking notes about how the interactions of the students and the teacher.

[If the teacher agreed to audio-recording] With your permission I would also like to audio-record (please indicate your preference in the consent form) these lessons so that I can recall specific dialogues between the teacher and the students if necessary. You can get to see any notes I take that are specifically about your child. If you do not want your child to be recorded as part of the general classroom observations it might be necessary to move them to a position so what they say is not recorded.

[If the teacher agreed to video-recording] With your permission I would also like to video-record these lessons so that I can recall specific dialogues between the teacher and the students if necessary. You can get to see any notes I take that are specifically about your child. If you do not want your child to be video-recorded and you prefer an audio recording instead please indicate it in the consent form. I will consider your preference and if necessary I will only use audio-recording. If you do not want your child to be recorded in any way as part of the general classroom observations I will position any recording device away of your child but it might be necessary to move them to a different position.

(4)How much of my child's time will the study take?
I will be in about 2 mathematics classes but this will be part of your child's normal everyday activities and so no additional time is required.

(5)Does my child have to be in the study? Can they withdraw from the study once they've started?
Being in this study is completely voluntary and your child does not have to take part. Your decision whether to let them participate will not affect your/their relationship with the researchers or anyone else at the University of East Anglia, the teacher or the school, now or in the future. If you decide to let your child take part in the study and then change your mind later (or they no longer wish to take part), they are free to withdraw from the study at any time. You can do this by simply telling me either via email, phone or in person.

If your child takes part in an observation session, they are free to stop participating at any stage. If they (or you) decide at a later time to withdraw from the study, excluding their data will only be possible if the observation was videoed or your child was individually identified at the time of the observation. In this case, their information will be removed from my records and will not be included in any results, up to the point I have analysed and published the results.

(6)Are there any risks or costs associated with being in the study?
Some children may not want to say anything in front of others or may find it difficult to talk about schooling if they have had a negative experience. I will be mindful of anything that might cause concern and no child will be required to speak if they don't feel like it. I will also position myself in a place where I am least visible to the students to minimize the risk of destructing them. If anyone does get upset, I have been giving information about the correct process to follow.

I do not expect that there will be any other risks or costs associated with taking part in this study for your child.

(7)Are there any benefits associated with being in the study?

I hope that your child's participation in these observations will contribute in identifying classroom opportunities for connections to mathematical content that is important for the coherence of the curriculum across educational levels. This information will also inform a series of professional development workshops that will aid teacher in developing knowledge and skills to deliver coherent lessons in the future.

(8)What will happen to information that is collected during the study?

Data from my notes and audio- / video-recordings will be analysed. Outcomes of the analysis will be reported in academic and professional publications. Your child's information will be stored securely and their identity/information will only be disclosed with your permission, except as required by law. Study findings may be published, but your child will **not** be identified in these publications. In this instance, data will be stored for a period of 10 years and then destroyed.

By providing your consent, you are agreeing to me collecting personal information about your child for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 2018 General Data Protection Regulation Act and the University of East Anglia Research Data Management Policy (2015).

(9)What if we would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me directly via email at p.papadaki@uea.ac.uk. If you would like to know more at any stage during the study, please feel free to contact Dr Irene Biza at i.biza@uea.ac.uk.

(10) Will I be told the results of the study?

You and your child have a right to receive feedback about the overall results of this study. You can indicate to me that you wish to receive feedback by ticking the relevant box on the consent form. This feedback will be in the form of an anonymised up to three-page summary on the completion of the data analysis. Finally my thesis will be available online after its successful completion.

(11) What if we have a complaint or any concerns about the study?

The ethical aspects of this study have been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

If there is a problem please let me know at the following address:

Paraskevi Papadaki

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0)1603 591741

Email: p.papadaki@uea.ac.uk.

If you would like to speak to someone else you can contact my supervisor: Dr Irene Biza email: i.biza@uea.ac.uk Tel: +44 (0)1603 59 1741

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact please contact the Head of the School of Education and Lifelong Learning, Professor Richard Andrews, at Richard.Andrews@uea.ac.uk.

(12) OK, I'm happy for my child to take part – what do I do next?

You need to fill in one copy of the consent form and ask your child to return this to [name of the teacher] by [date]. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARENT/CARER CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT PARENT'S/CARER'S NAME], consent to my child ……………………………………………………………………………………[PRINT CHILD'S NAME] participating in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what my child will be asked to do, and any risks/benefits involved.
✓ I have read the Information Statement and have been able to discuss my child's involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and my child does not have to take part. My decision whether to let them take part in the study will not affect our relationship with the researchers or anyone else at the University of East Anglia, the teacher and/or the school, now or in the future.
✓ I understand that my child can withdraw from the study at any time.
✓ I understand that my child may stop participating in an observation at any time if they do not wish to continue. I also understand that it will not be possible to remove my data unless they are individually identified in some way.
✓ I understand that personal information about my child that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about my child will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my child's name or any identifiable information about my child.

I consent to:

[*adjust accordingly]

- **Audio-recording of my child** YES ☐ NO ☐

- **Video-recording of my child*** YES ☐ NO ☐

- **Observations of my child*** YES ☐ NO ☐

- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

____

☐ Email: ____

....

Signature

....

PRINT name

....

Date

PARENT/CARER CONSENT FORM (2nd Copy to Parent/Carer)

I, ... [PRINT PARENT'S/CARER'S NAME], consent to my child ………………………………………………………………………….[PRINT CHILD'S NAME] participating in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what my child will be asked to do, and any risks/benefits involved.
✓ I have read the Information Statement and have been able to discuss my child's involvement in the study with the researchers if I wished to do so.
✓ The researchers have answered any questions that I had about the study and I am happy with the answers.
✓ I understand that being in this study is completely voluntary and my child does not have to take part. My decision whether to let them take part in the study will not affect our relationship with the researchers or anyone else at the University of East Anglia, the teacher and/or the school, now or in the future.
✓ I understand that my child can withdraw from the study at any time.
✓ I understand that my child may stop participating in an observation at any time if they do not wish to continue. I also understand that it will not be possible to remove my data unless they are individually identified in some way.
✓ I understand that personal information about my child that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about my child will only be told to others with my permission, except as required by law.
✓ I understand that the results of this study may be published, and that publications will not contain my child's name or any identifiable information about my child.

I consent to:

[*adjust accordingly]

- **Audio-recording of my child** YES ☐ NO ☐

- **Video-recording of my child*** YES ☐ NO ☐

- **Observations of my child*** YES ☐ NO ☐

- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

____

☐ Email: ____

....

Signature

....

PRINT name

....

Date

5. Participant Information Statement for the students

Paraskevi Papadaki

PhD Researcher

Faculty of Social Science

School of Education

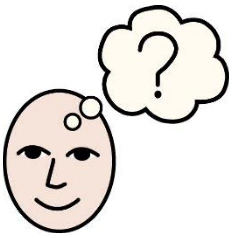
University of East Anglia

Norwich Research Park

[Insert date]

Implementing Tasks to explore teachers' discourse at the mathematical horizon

Study Information Sheet



Hello. My name is Paraskevi Papadaki. I am doing a research study to find out more about how mathematics teachers and students discuss about ideas in maths that might help the students in the future. Part of my study will be to go into different classrooms and find out more about opportunities for discussion during lessons.

I am asking you to be in my study because you are a student in *[name of the teacher]* maths class.

You can decide if you want to take part in the study or not. You don't have to - it's up to you.

This sheet tells you what I will ask you to do if you decide to take part in the study. Please read it carefully so that you can make up your mind about whether you want to take part.

If you decide you want to be in the study and then you change your mind later, that's ok. All you need to do is tell me that you don't want to be in the study anymore.

If you have any questions, you can ask me or your family or someone else who looks after you. If you want to, you can call me any time on 07706268987.

What will happen if I say that I want to be in the study?

If you decide that you want to be in our study, I will ask you to:

- Let me come and sit in your classroom for [number] lessons and make some notes about what happens. This could include things that you say or do with your teacher. If you agree, I may also want to take some video of your lessons so I can look at them later on [if the teachers does not agree to video recording, this sentence to be removed].
- Say and do what you normally do in your maths lessons.

[INSERT – if applicable to the study]:

[AUDIO RECORDING:] If you say it's ok, I will record what you say with an audio recorder.

[VIDEO RECORDING:] If you say it's ok, I will make a video of you with a video recorder.

Will anyone else know what I say in the study?

I won't tell anyone else what you say to us, except if you talk about someone hurting you or about you hurting yourself or someone else. Then we might need to tell someone to keep you and other people safe.

All of the information that I have about you from the study will be stored in a safe place and we will look after it very carefully. I will write a report about the study and show it to other people but I won't say your name in the report and no one will know that you were in the study.

How long will the study take?

I will be in 2 or 3 [to be specified at a later date] of your maths lessons this term.

Are there any good things about being in the study?

You won't get anything for being in the study, but you will be helping me do my research.

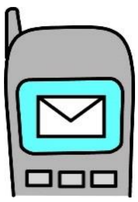
Are there any bad things about being in the study?

This study will take place while you continue your lesson as usual, but I don't think it will be bad for you or cost you anything.

Will you tell me what you learnt in the study at the end?

Yes, I will if you want me to. There is a question on the next page that asks you if you want me to tell you what I learnt in the study. If you circle Yes, when I finish the study, I will tell you what I learnt.

What if I am not happy with the study or the people doing the study?



If you are not happy with how I am doing the study or how I treat you, then you or the person who looks after you can:

- **Call** the university on +44 (0)1603 59 1741
- Write an **email** to i.biza@uea.ac.uk

OK, I want to take part – what do I do next?

If you're happy to fill in the 2 forms below and give number 1 to your teacher in the next class, I will get the form from him/her. You can keep this letter and the form 2 to remind you about the study.

This sheet is for you to keep.

Consent Form 1

If you are happy to be in the study, please

- **write** your **name** in the space below
- **sign** your **name** at the bottom of the next page
- put the **date** at the bottom of the next page.

You should only say 'yes' to being in the study if you know what it is about and you want to be in it.

If you don't want to be in the study, don't sign the form.

I, ...[PRINT NAME], am happy to be in this research study.

In saying yes to being in the study, I am saying that:

- ✓ I know what the study is about.
- ✓ I know what I will be asked to do.
- ✓ Someone has talked to me about the study.
- ✓ My questions have been answered.
- ✓ I know that I don't have to be in the study if I don't want to.
- ✓ I know that I can pull out of the study at any time if I don't want to do it anymore.
- ✓ I know that I don't have to answer any questions that I don't want to answer.

✓ I know that the researchers won't tell anyone what I say when we talk to each other, unless I talk about being hurt by someone or hurting myself or someone else.

Now we are going to ask you if you are happy to do a few other things in the study. Please circle 'Yes' or 'No' to tell us what you would like.

*** If applicable**

Are you happy for me to make **videos** of you while you work in the class?*****

Yes **No**

Are you happy for me to **audio record** your voice while you work in the class? *****

Yes **No**

Are you happy for me to **observe** you in class? **Yes** **No**

Do you want me to tell you what I **learnt** in the study? **Yes** **No**

………………………………………………………… ……………………………………………………….

Signature **Date**

Consent Form 2

If you are happy to be in the study, please

- **write** your **name** in the space below
- **sign** your **name** at the bottom of the next page
- put the **date** at the bottom of the next page.

You should only say 'yes' to being in the study if you know what it is about and you want to be in it.

If you don't want to be in the study, don't sign the form.

I, ...[PRINT NAME], am happy to be in this research study.

In saying yes to being in the study, I am saying that:

✓ I know what the study is about.

✓ I know what I will be asked to do.

✓ Someone has talked to me about the study.

✓ My questions have been answered.

✓ I know that I don't have to be in the study if I don't want to.

✓ I know that I can pull out of the study at any time if I don't want to do it anymore.

✓ I know that I don’t have to answer any questions that I don’t want to answer.

✓ I know that the researchers won’t tell anyone what I say when we talk to each other, unless I talk about being hurt by someone or hurting myself or someone else.

Now we are going to ask you if you are happy to do a few other things in the study. Please circle ‘Yes’ or ‘No’ to tell us what you would like.

*** If applicable**

Are you happy for me to make **videos** of you while you work in the class?*****

Yes **No**

Are you happy for me to **audio record** your voice while you work in the class? *****

Yes **No**

Are you happy for me to **observe** you in class? **Yes** **No**

Do you want me to tell you what I **learnt** in the study? **Yes** **No**

……………………………………………………… ……………………………………………………

Signature **Date**

Appendix IV: Letter of approval from the EDU Research Ethics Committee

EDU ETHICS APPROVAL LETTER 2018-2019

APPLICANT DETAILS	
Name:	Paraskevi Papadaki
School:	EDU
Current Status:	PGR
UEA Email address:	p.papadaki@uea.ac.uk
EDU REC IDENTIFIER:	2019/06/PP_IB

Approval details	
Approval start date:	**02/07/2019**
Approval end date:	30/09/2022
Specific requirements of approval:	
Please note that your project is only given ethical approval for the length of time identified above. Any extension to a project must obtain ethical approval by the EDU REC before continuing. Any amendments to your project in terms of design, sample, data collection, focus etc. should be notified to the EDU REC Chair as soon as possible to ensure ethical compliance. If the amendments are substantial a new application may be required.	

Yann Lebeau, EDU Chair, Research Ethics Committee

Appendix V: Indicative plan for the semi-structured interviews

1. **Brief interview after classroom observations**
   - In your view, how did the lesson go?
   - Was there anything in particular that caught your attention or made you change the layout of the lesson plan during the lesson?

2. **Main interview**
   - Introductions
   - Reminder about recording/technical issues/answer any questions about the process

<u>Introductory Questions</u>

- How many years are you teaching mathematics?
- Can you tell me a little bit about your journey becoming a mathematics teacher/teacher educator?

<u>Main part key-questions</u>

- Which mathematics topics do you remember from your undergraduate degree?
- Which topics do you think were the most relevant to your teaching?
- When you are teaching do you recall times where you bring content from your university studies?
- When I say "connections in mathematics" what comes to your mind?

<u>MathTASK</u> (the participants will be given a MathTASK to read and share their thoughts the main focus of the part is to introduce the participant to the idea of MathTASK activities)

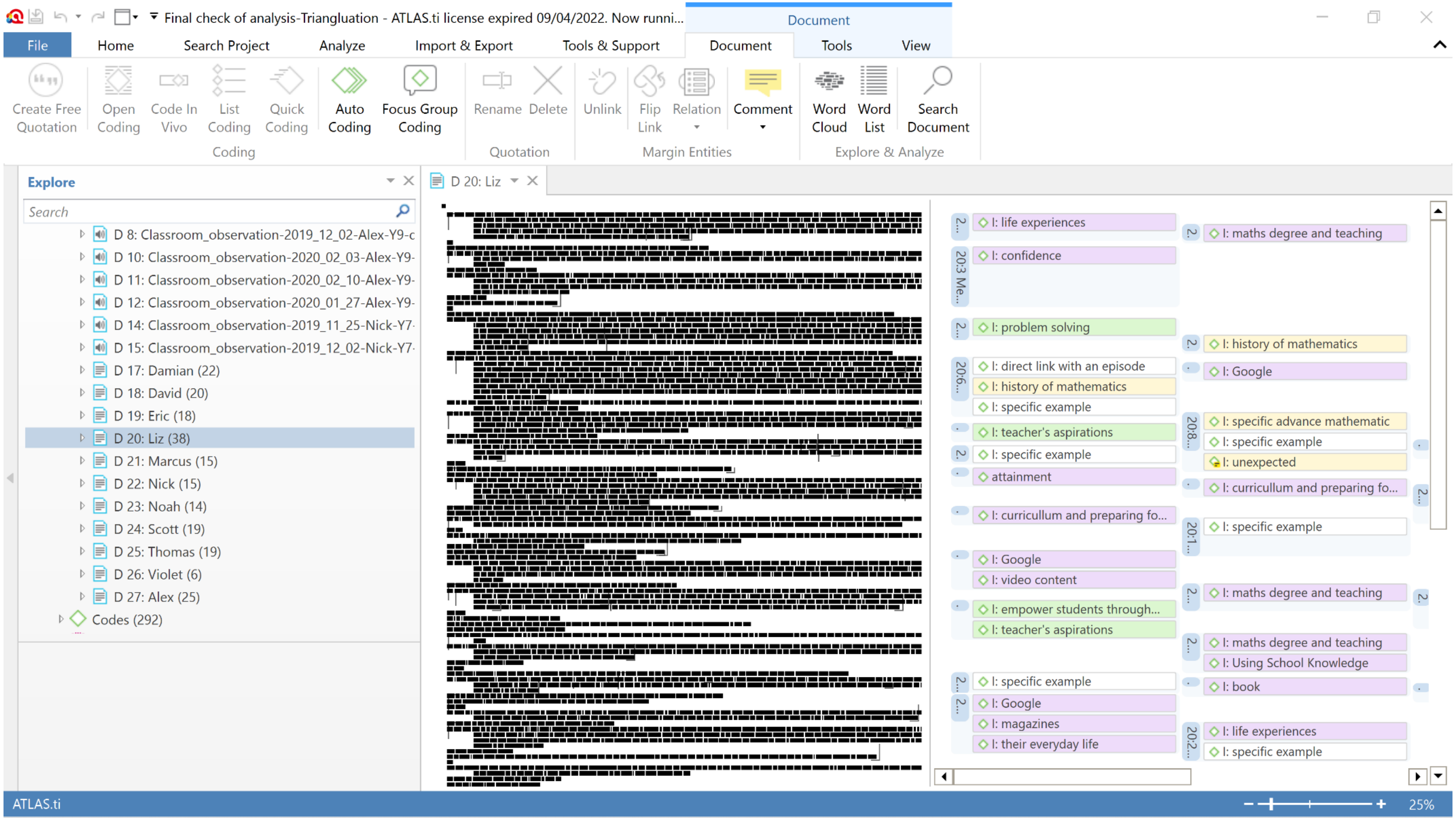
- How did you find the activity?
- If you knew that the activity was used in a professional development workshop would you be interested to attend?

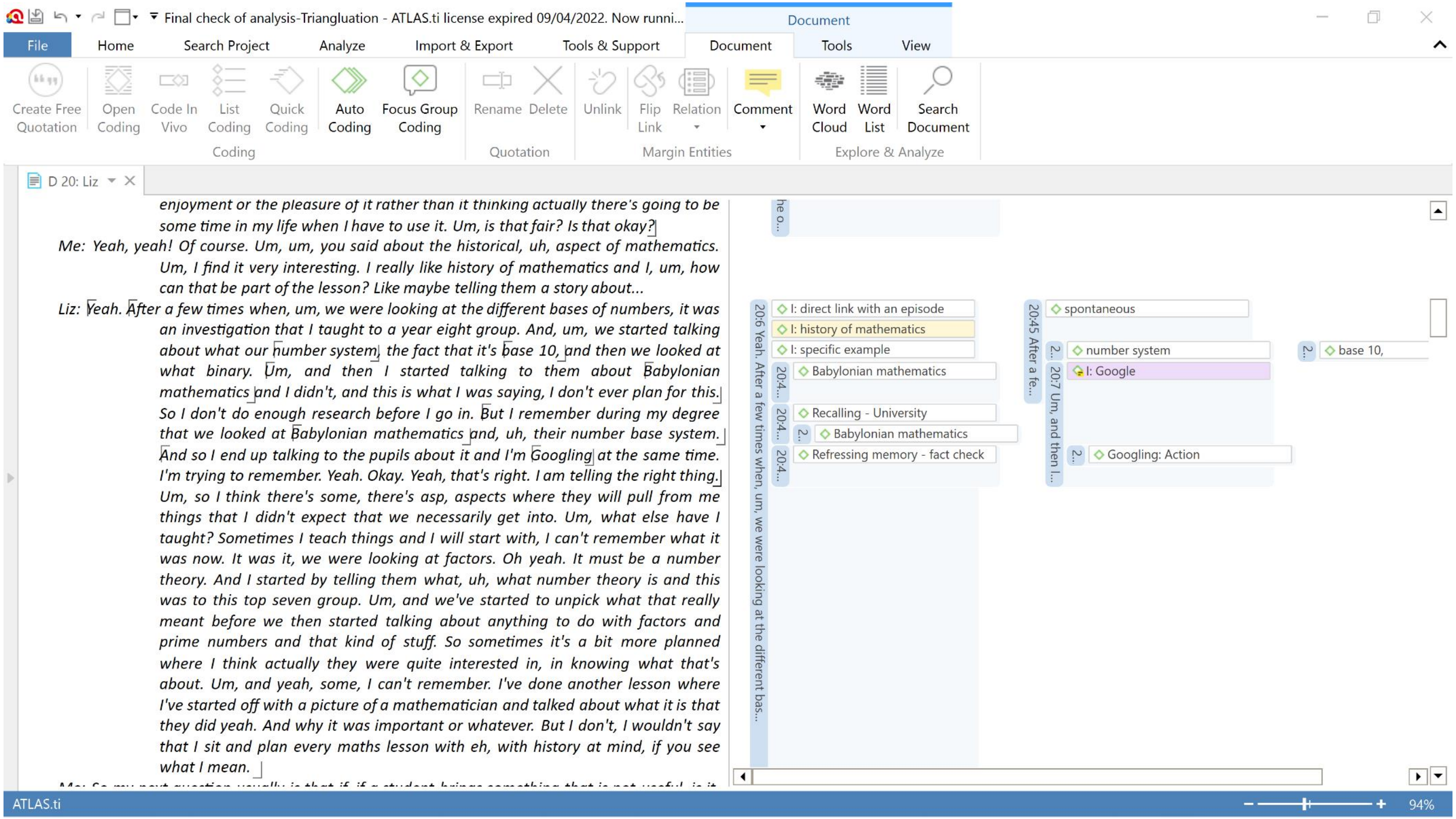
<u>Concluding question</u>

- If you could give one advice to your students regarding learning maths, what would that be?

Appendix VI: Snapshots from the analysis

1. Coding of Interviews using ATLAS.ti

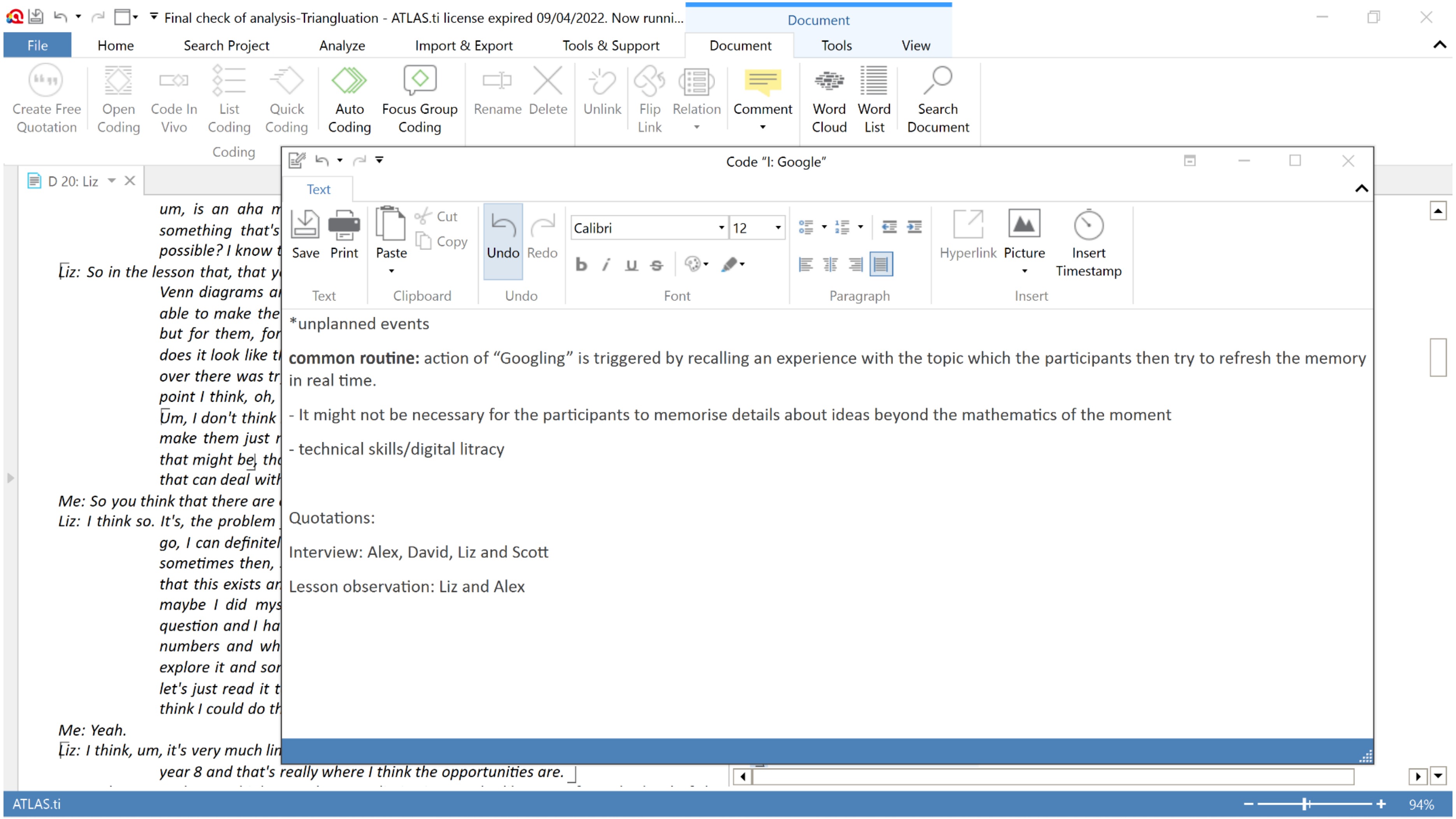




enjoyment or the pleasure of it rather than it thinking actually there's going to be some time in my life when I have to use it. Um, is that fair? Is that okay?

Me: Yeah, yeah! Of course. Um, um, you said about the historical, uh, aspect of mathematics. Um, I find it very interesting. I really like history of mathematics and I, um, how can that be part of the lesson? Like maybe telling them a story about...

Liz: Yeah. After a few times when, um, we were looking at the different bases of numbers, it was an investigation that I taught to a year eight group. And, um, we started talking about what our number system, the fact that it's base 10, and then we looked at what binary. Um, and then I started talking to them about Babylonian mathematics and I didn't, and this is what I was saying, I don't ever plan for this. So I don't do enough research before I go in. But I remember during my degree that we looked at Babylonian mathematics and, uh, their number base system. And so I end up talking to the pupils about it and I'm Googling at the same time. I'm trying to remember. Yeah. Okay. Yeah, that's right. I am telling the right thing. Um, so I think there's some, there's asp, aspects where they will pull from me things that I didn't expect that we necessarily get into. Um, what else have I taught? Sometimes I teach things and I will start with, I can't remember what it was now. It was it, we were looking at factors. Oh yeah. It must be a number theory. And I started by telling them what, uh, what number theory is and this was to this top seven group. Um, and we've started to unpick what that really meant before we then started talking about anything to do with factors and prime numbers and that kind of stuff. So sometimes it's a bit more planned where I think actually they were quite interested in, in knowing what that's about. Um, and yeah, some, I can't remember. I've done another lesson where I've started off with a picture of a mathematician and talked about what it is that they did yeah. And why it was important or whatever. But I don't, I wouldn't say that I sit and plan every maths lesson with eh, with history at mind, if you see what I mean.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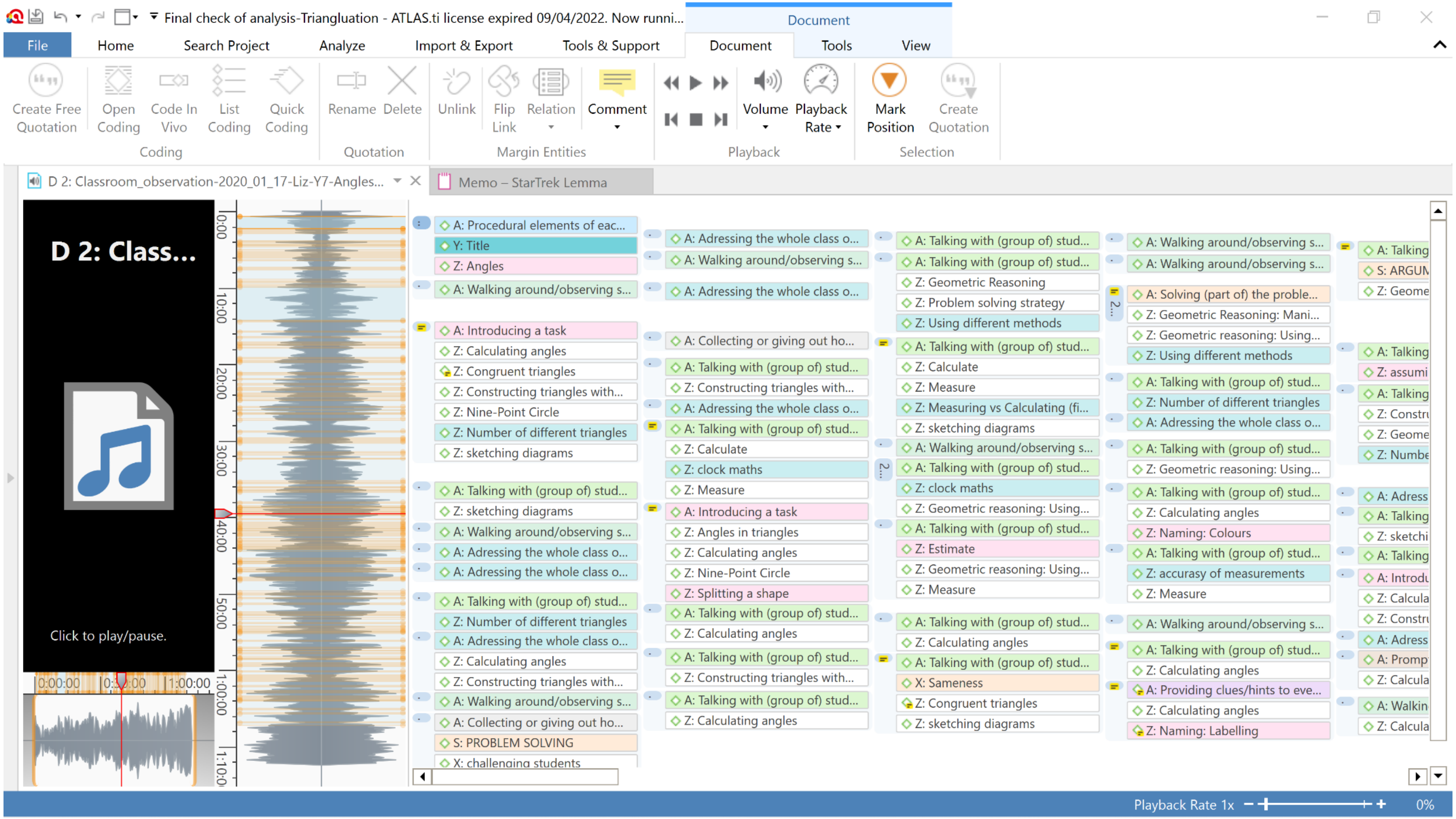
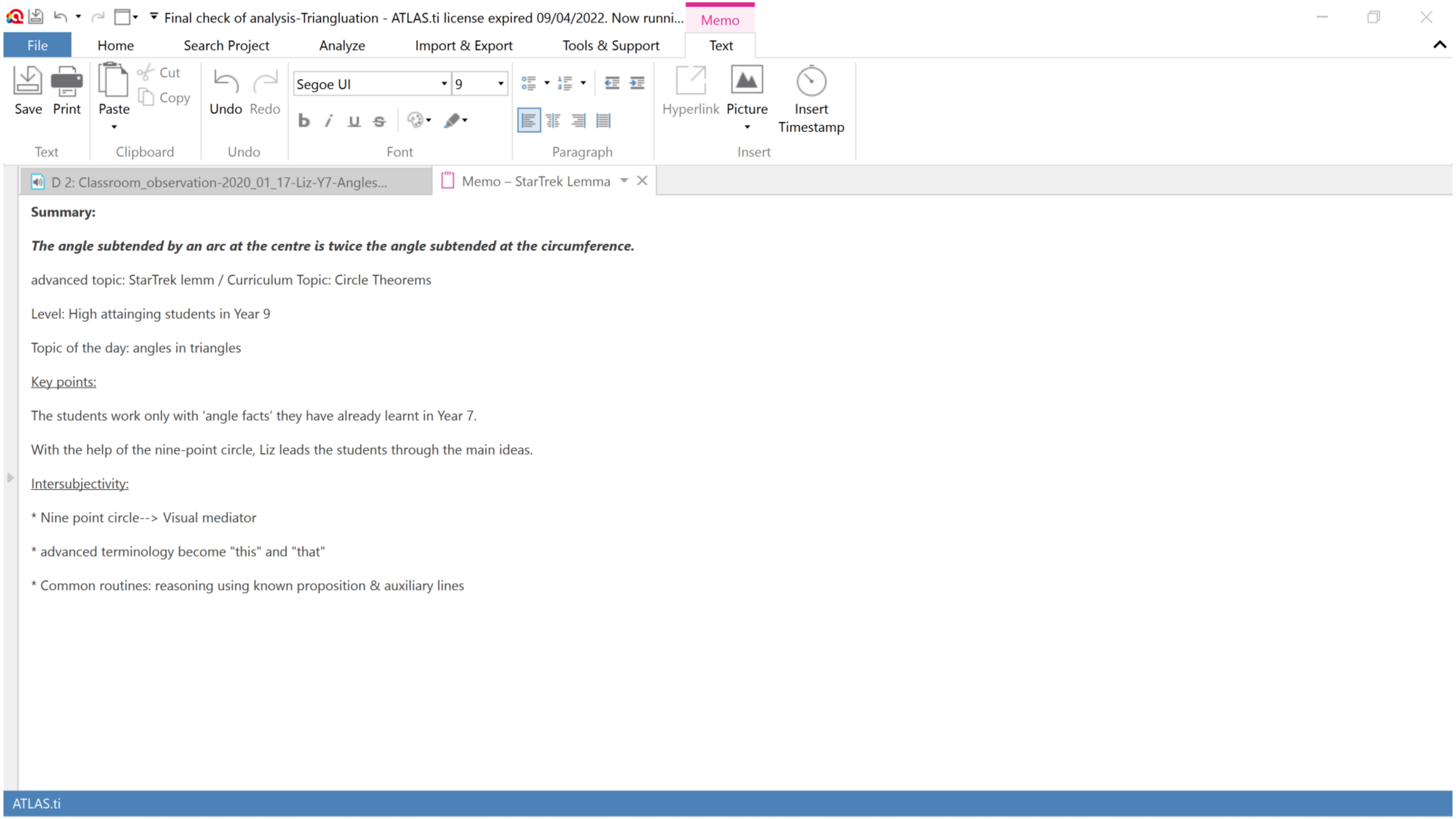
Code "I: Google"
*unplanned events
common routine: action of "Googling" is triggered by recalling an experience with the topic which the participants then try to refresh the memory in real time.
- It might not be necessary for the participants to memorise details about ideas beyond the mathematics of the moment
- technical skills/digital litracy
Quotations:
Interview: Alex, David, Liz and Scott
Lesson observation: Liz and Alex
Me: Yeah.
year 8 and that's really where I think the opportunities are.
ATLAS.ti
94%

2. Coding of Lessons using ATLAS.ti

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Memo
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D 2: Classroom_observation-2020_01_17-Liz-Y7-Angles...
Memo – StarTrek Lemma
Summary:
The angle subtended by an arc at the centre is twice the angle subtended at the circumference.
advanced topic: StarTrek lemm / Curriculum Topic: Circle Theorems
Level: High attainging students in Year 9
Topic of the day: angles in triangles
Key points:
The students work only with 'angle facts' they have already learnt in Year 7.
With the help of the nine-point circle, Liz leads the students through the main ideas.
Intersubjectivity:
* Nine point circle--> Visual mediator
* advanced terminology become "this" and "that"
* Common routines: reasoning using known proposition & auxiliary lines
ATLAS.ti

Appendix VII: Additional Pilot mathtasks (not used)

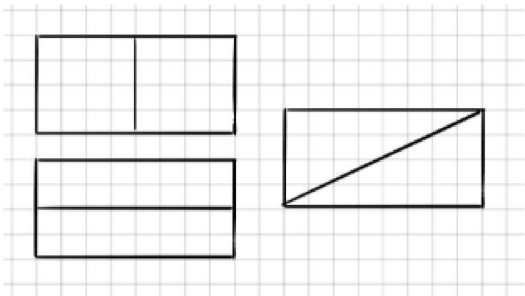
Splitting the rectangle in egual parts: Part 1

A Year 7 class is working on the following mathematical problem as a starter for a lesson on shapes and area*.

On a piece of squared paper, draw a rectangle.
Then, draw a straight line that will divide the rectangle into two parts of equal area.
How many different ways can you find to do this?

The following conversation takes place between two students, A and B.

Student A: How many ways did you find?
Student B: I found three.
Student A: Let me see!
[Student B shows her working]



Student A: Aah… I found those too! But, I think there is another way … look!
[Student A shows his working]

Student B: Hmm… how do you know that those two parts have the same area?
Student A: Well … they look a bit the same…
Student B: Oh… is there another way?
Student A: I didn't check. You think so?

Questions:

a. Solve the mathematical problem. Is there a common characteristic between the lines that divide the rectangles in two parts of equal area?
b. You are the teacher and you overhear the exchange between students A and B. What would you do?
c. Would you bring the whole class to this conversation? If so, how?

*We would like to thank Tim Rowland for the inspiring discussions and his support during the creation of the Task. This task was created and used as part of Evi Papadaki's PhD research project. Let us know whether it is useful and how we can improve it at @mathtask or email Evi Papadaki at P.Papadaki@uea.ac.uk. For more tasks, visit MathTASK.

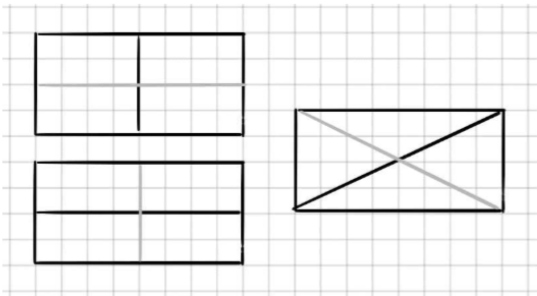
Splitting the rectangle in equal parts: Part 2

The following mathematical problem is given to the same Year 7 class as an extension of the "Splitting the rectangle in equal parts: Part 1" problem*.

> On a piece of squared paper, draw a rectangle.
> Then, draw *two* straight lines that divide the rectangle into four parts of equal area.
> How many different ways can you find to do this?

The following conversation takes place between the teacher and the students:

Teacher: How many ways did you find?
Student C: There! Only two. *[student C shows her working to the teacher]*



Teacher: How do you know that those four have the same area? *[the teacher points at the four triangles created by the diagonals in the shape on the right of the figure above]*
Student C: I… I just know…
Teacher: Shall we ask your classmates?
Student C: I guess so…
Teacher: Hey folks! *[The teacher shows student C's answer with the diagonals to the class]* Look what student C has done! What do you think?
Student D: She drew the di-diagonals.
Teacher: That's right! Do these four parts have the same area?
[None of the students answers]

Questions:

a. Is student C right? How would you explain to another teacher that student C's four triangles have the same area (or not)?
b. You are the teacher of this class, how would you explain to your class that student C's four triangles have the same area (or not)?
c. Would you use a similar activity with your students? If yes, what activity would you use and how? Justify your responses.

*We would like to thank Tim Rowland for the inspiring discussions and his support during the creation of the Task. This task was created and used as part of Evi Papadaki's PhD research project. Let us know whether it is useful and how we can improve it at @mathtask or email Evi Papadaki at P.Papadaki@uea.ac.uk. For more tasks, visit MathTASK.

Appendix VIII: Side-by-side comparison of the observed and the fictional dialogue created for the mathtask presented in Chapter 7

<table>
<tr><th colspan="2">Observed dialogue</th><th colspan="2">Fictional dialogue</th></tr>
<tr><th>Teacher</th><th>Students</th><th>Teacher</th><th>Student A</th></tr>
<tr><td></td><td>St 1: On the whiteboard, when you said, why is it only 180 degrees in the triangle?</td><td></td><td>Why is it only 180 degrees in the triangle?</td></tr>
<tr><td>Liz: Oh, there's 180 degrees in the triangle because.</td><td></td><td rowspan="5">Oh, there's 180 degrees in the triangle because all the corners of the triangle, when you put them together, will make a straight line. We'll do that, we'll do that one day.</td><td rowspan="5"></td></tr>
<tr><td></td><td>St 2: Of the equilateral triangle.</td></tr>
<tr><td>Liz: All the all the corners of the triangle, when you put them together, will make a straight line.</td><td></td></tr>
<tr><td></td><td>St 1: But why?</td></tr>
<tr><td>Liz: We'll do that, we'll do that one day.</td><td></td></tr>
<tr><td></td><td>St 1: Why is it, why is the straight line 180? [overlap].</td><td rowspan="2"></td><td rowspan="2">But why? Why is it, why is the straight line 180?</td></tr>
<tr><td></td><td>St 2: [inaudible]</td></tr>
</table>

Liz:	Why is it three hundred and sixty? I think it's to do with Babylonian maths.			I think it's to do with Babylonian maths.	
		St 1:	Why? What is that?		Why? What is that?
Liz:	So Babylonian, um, maths, many thousands of years ago used base 60. So, you know, we use base ten in our number system?			So, Babylonian maths, many thousands of years ago used base 60. So, you know, we use base 10 in our number system? So, it goes one, ten, a hundred, a thousand.	
		St 1:	Hmm.		Hmm…Yeah…
		St 3:	Yeah...		
Liz:	So it goes ten, one, ten, a hundred, a thousand. They used base 60.			They used base 60. Which is sort of linked with why we've got 12 numbers around the clock.	
		St 3:	Will we do that?		
Liz:	Which is, and it's short of linked with why we've got 12 numbers around the clock.				

	St 1: Sixty is a really bad number.		Will we do that? Sixty is a really bad number.
Liz: And they used 360 degrees. The 360 is, could have been any number.		And they used 360 degrees. The 360 is, could have been any number.	
	St 1: Exactly!		Exactly! Like sixty seconds in a clock.
	St 3: Like sixty seconds in a…		
Liz: Yes. Yes. Yeah. Yeah.		Yes. Yeah.	
	St 1: But why, why is a degree a degree? [gigles/laughs] Why is it degree? Like the amount of it.		But why, why is a degree a degree? Like the amount of it.
Liz: Why, why is it, the amount of.			
	St 3: Because, that's, that's just like the angle of the [overlaping]		
Liz: Because they took, because they took a circ-, they took a full point and just divided it up into a number that they wanted to divide up into.		Because they took a full point and just divided it up into a number that they wanted to divide up into. It could have been four	

	St 1: But why?	hundred. It could have been four hundred and fifty.	
Liz: It could have been four hundred. It could have been four hundred and fifty.			
	St 1: Exactly!		Exactly!
Liz: But because of that, there's all these, all these facts that we can use. So a straight line would be half of that.			