Decomposition numbers of Ariki-Koike algebras

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Abstract

This thesis is concerned with the representation theory of the symmetric groups and related algebras, in particular the combinatorics underlying the representations of Ariki-Koike algebras. The Ariki-Koike algebras generalise Iwahori-Hecke algebras of the symmetric group, and so in turn generalise the symmetric groups themselves.

The representation theory of these algebras is the subject of a great deal of research, with the most important outstanding problem being the determination of the decomposition numbers, i.e. the composition multiplicities of the simple modules $D^{\boldsymbol{\mu}}$ in the Specht modules $S^{\boldsymbol{\lambda}}$. The aim of this thesis is to contribute and make progress on the decomposition number problem.

We shall first develop some combinatorial lemmas related to the abacus display of multipartitions. Then, we will use these to examine blocks of the Ariki-Koike algebras. In particular, we prove a sufficient condition such that restriction of modules leads to a natural correspondence between the multipartitions of n whose Specht modules belong to a block B and those of $n - \delta_i(B)$ whose Specht modules belong to the block B', obtained from B applying a Scopes' equivalence. This bijection gives us an equivalence for the decomposition numbers of the corresponding Ariki-Koike algebras.

We will then define the addition of a runner full of beads for the abacus display of a multipartition and investigate some combinatorial properties of this operation. We focus our attention on the q-decomposition numbers, i.e. the polynomials arising from the Fock space representation of the quantun group $U_q(\widehat{\mathfrak{sl}}_e)$ that coincide with the decomposition numbers for $q = 1$. Using an LLT-type algorithm for Ariki-Koike algebras, we relate q-decomposition numbers for different values of e for the class of e-multiregular multipartitions, by adding a full runner of beads to each component of the abacus displays for the labelling multipartitions.

Contents

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Introduction

0.1 Background

The representation theory of the symmetric group is a fascinating study in its own right, as well as being intrinsically linked to that of other fundamental objects, including a wealth of diagram algebras and semigroups.

Representations of the symmetric group $\mathfrak{S}_n$ on n letters over the complex field are well understood since the algebra $\mathbb{C}\mathfrak{S}_n$ is semisimple. Their study can be traced back to the work of Young [You00], Frobenius [Fro03] and Specht [Spe35], whose ideas are still present today. On the other hand, representations of the symmetric group over fields of positive characteristic are more difficult.

A constructive approach to the topic over an arbitrary field, not just over the complex numbers, was given by James [Jam78] who developed the use of combinatorial tools, such as diagrams, tableaux and abacuses. James's approach generalises in a straightforward way to give techniques for studying representations of algebras which include the symmetric group algebra as special case. Some examples of these related algebras are the Iwahori-Hecke algebras $H_{\mathbb{F},q}(\mathfrak{S}_n)$ of the symmetric group and the Ariki-Koike algebras $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$.

The Iwahori-Hecke algebra $H_{\mathbb{F},q}(\mathfrak{S}_n)$ was introduced by Dipper and James in [DJ86] as deformation of the symmetric group algebra. Hence, results in the representation theory of $\mathfrak{S}_n$ can be recovered from the corresponding results in the representation theory of $H_{\mathbb{F},q}(\mathfrak{S}_n)$. Subsequently, a new approach for studying the representations of the Iwahori-Hecke algebra took hold thanks to Murphy [Mur92, Mur95] that discovered a basis for $H_{\mathbb{F},q}(\mathfrak{S}_n)$. The Murphy basis is an example of a cellular basis, as defined later by Graham and Lehrer [GL96] and so cellular theory can be used to study the representations of these algebras.

The Ariki-Koike algebra $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$ was introduced by Ariki and Koike in [AK94]. Its representation theory has many similarities with the one of Iwahori-Hecke algebras, and in many aspects it can be seen as a generalisation of it. For example, a cellular basis for Ariki-Koike algebras was constructed by Dipper, James and Mathas in [DJM98] and thus cellular theory can be used for such algebras as well. Furthermore, the indexing of $H_{\mathbb{F},q}(\mathfrak{S}_n)$-modules by partitions generalises to an indexing of $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$-modules by multipartitions, i.e. tuples of partitions.

Similarly to Iwahori-Hecke algebras, the main problem of interest in the representation theory of Ariki-Koike algebras is the *decomposition number problem*, which asks for the composition multiplicities of simple modules in the so called Specht modules. The Specht modules arise as cell modules of the cellular algebra. The *decomposition matrix* records these multiplicities.

It is known that computing the decomposition numbers in the case $\mathbb{F} = \mathbb{C}$ is an important first step in working out the decomposition numbers over any field (see [Gec92, Gec98]). In fact, the following result holds. Let $\boldsymbol{D}_p$ be the decomposition matrix of $\mathcal{H}_{\mathbb{F}_p,q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$ with $\mathbb{F}_p$ a field of characteristic $p > 0$ and $\boldsymbol{D}$ be the decomposition matrix of $\mathcal{H}_{\mathbb{C},\zeta,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$. Then there exists a square unitriangular matrix $\boldsymbol{A}$, called *adjustment matrix*, such that

$$\boldsymbol{D}_p = \boldsymbol{D}\boldsymbol{A}.$$

Fortunately, the decomposition numbers $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ can be computed when $\mathbb{F} = \mathbb{C}$; they are the values at $q = 1$ of certain polynomials $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$, which have accordingly become known as 'q-decomposition numbers'. This result has first been conjectured for Iwahori–Hecke algebras by Lascoux, Leclerc and Thibon [LLT96] and proved for the wider class of Ariki-Koike algebras by Ariki [Ari96]. It is by far the most significant theorem in this regard. The q-decomposition numbers arise from the Fock space representation of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$. This has a natural basis indexed by the set of partitions for $H_{\mathbb{F},q}(\mathfrak{S}_n)$ (respectively, of multipartitions for $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$), and a 'canonical basis' which is invariant under the bar involution. The q-decomposition numbers are the entries of the transition matrix between these two bases.

For Iwahori-Hecke algebras of $\mathfrak{S}_n$, there is a fast algorithm due to Lascoux, Leclerc and Thibon [LLT96] for computing the canonical basis and so the q-decomposition numbers.

For Ariki-Koike algebras, there are different generalisations of this algorithm due to Jacon [Jac05], Yvonne [Yvo07a] and Fayers [Fay10]. We will use the one presented in [Fay10] because it adapts better to our purposes.

So, since the decomposition numbers in characteristic 0 can be computed by the LLT algorithm and its generalisations, in effect the problem of determining the decomposition matrices in arbitrary characteristic is equivalent to computing adjustment matrices. However, not a great deal is known about adjustment matrices; the most general statement we have about adjustment matrices is James's Conjecture.

Conjecture 0.1.1 (James's Conjecture)**.** Let $\mathbb{F}$ be a field of characteristic $p > 0$ and suppose that $\boldsymbol{D}_p = \boldsymbol{D}\boldsymbol{A}$. If $n < pe$, then the adjustment matrix $\boldsymbol{A}$ is the identity matrix.

For Iwahori–Hecke algebras, this conjecture has been verified for blocks of weight at most four, thanks to the work of Richards [Ric96] and Fayers [Fay07c, Fay08a]. However, after being a central focus of research in representation theory for thirty years, this conjecture was finally shown to be false by Williamson in [WKM17].

0.2 Overview

In Chapter 1, we define the algebras that we will work with, along with giving an overview of any background material that we will need in order to study their representation theory. This will include both the algebraic setup we require and some combinatorial definitions such as partitions, tableaux and abacuses together with their generalisations for Ariki-Koike algebras. We also consider some recent work by Fayers [Fay06, Fay07b] concerning the weight of a multipartition and the core blocks of Ariki-Koike algebras.

Once the necessary background is set, in Chapter 2 we generalise what Scopes proved in [Sco91] about the blocks of symmetric groups to the blocks of Ariki-Koike algebras. Scopes gives a combinatorial description of two blocks of symmetric groups that are Morita equivalent, using the abacus display of a partition. In particular, Scopes establishes a natural correspondence between Specht modules and simple modules in the blocks B and $\phi_i(B)$ of the symmetric groups, where ϕ_i is the map swapping the runners $i-1$ and i of the abacus display of each partition in the block B. This leads to the blocks B and $\phi_i(B)$ having the same decomposition matrices. This result was generalised to Iwahori-Hecke algebras by Jost [Jos99]. So, taking inspiration from [Sco91] and [Jos99], we find an analogous combinatorial way to establish in which cases two blocks B and $\Phi_i(B)$ of Ariki-Koike algebras have the same decomposition matrices. Here, Φ_i is the map ϕ_i acting componentwise, i.e., it swaps the runners $i-1$ and i in each component of the abacus display of a multipartition.

In particular, we find a sufficient condition such that the following result holds for Ariki-Koike algebras.

Theorem 0.2.1 (Proposition 2.2.8)**.** Fix $i \in \{0, 1, \dots, e-1\}$. Suppose that in each component of every r-multipartition that belongs to the block B of $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$ there is no abacus configuration of the type ● ┼ in runners $i-1$ and i. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in B$ then,

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}} = d_{\Phi_i(\boldsymbol{\lambda})\Phi_i(\boldsymbol{\mu})}.$$

In order to prove this, we consider the multicore $\boldsymbol{m}$ obtained from the abacus display of $\boldsymbol{\lambda} \in B$ by sliding all its beads as high as possible. Then, we give a lower bound for the minimal difference between the positions of the lowest beads of two consecutive runners of $\boldsymbol{m}$. To get this lower bound, we show the existence of a particular sequence of multicores with non-increasing weights from $\boldsymbol{m}$ to a multicore in its core block. Thus, we use this lower bound to get a condition on the weight of the block B so that no configuration $\begin{smallmatrix} i-1 & i \\ \bullet & + \end{smallmatrix}$ appears in the abacus display of any multipartition in B. Finally, we show that there is a natural correspondence between Specht modules and simple modules in the blocks B and $\Phi_i(B)$ of Ariki-Koike algebras.

In Chapter 3, we introduce the Fock space representation of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$ and present the LLT-type algorithm for Ariki-Koike algebras given in [Fay10]. This algorithm allows us to generalise the 'full' runner removal theorem of Iwahori-Hecke algebras in [Fay07a] to Ariki-Koike algebras.

In the attempt of tackling the decomposition number problem, in [JM02] James and Mathas proved the so called 'empty' runner removal theorem in which they relate q-decomposition numbers of Iwahori-Hecke algebras for different values of e, by adding empty runners to the abacus displays for the labelling partitions. After that, in [Fay07a] Fayers proves a similar theorem, which involves adding full runners to these abacus displays. For a class of multipartitions, called e-multiregular, we generalise Fayers' theorem to the Ariki-Koike algebras showing that the q-decomposition numbers $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ and $d_{\boldsymbol{\lambda}^+\boldsymbol{\mu}^+}(q)$ coincide, where $\boldsymbol{\lambda}^+$ and $\boldsymbol{\mu}^+$ are the multipartitions obtained from the e-abacus display of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by adding a runner full of beads in each of their components.

Theorem 0.2.2 (Theorem 3.2.32)**.** Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be r-multipartitions in a block B of $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}((\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n)$ with $\boldsymbol{\mu}$ e-multiregular. If the new inserted runners defining $\boldsymbol{\lambda}^+$ and $\boldsymbol{\mu}^+$ are 'long enough', then

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = d_{\boldsymbol{\lambda}^+\boldsymbol{\mu}^+}(q).$$

We first present this result for $r = 2$ and then we generalise it for any $r \geq 2$. Thus, we define the addition of a runner full of beads in each component of an abacus display of a multipartition. We show that adding a runner full of beads to an abacus display of the empty partition corresponds to a precise sequence of induction operators. We then prove some results that describe how the addition of a full runner interacts with the induction operators. Finally, we use all these properties together with the Fayers' LLT-type algorithm for Ariki-Koike algebras [Fay10] to show that the coefficients of the canonical basis element corresponding to the multipartition $\boldsymbol{\mu}$ coincide with the coefficients of the canonical basis element corresponding to the multipartition $\boldsymbol{\mu}^+$.

Recent work has given us a new line of attack. The cyclotomic quiver Hecke algebras of type A, known as KLR algebras (defined independently by Khovanov and Lauda and by Rouquier [KL09, Rou08]), have been shown to be isomorphic to Ariki-Koike algebras by Brundan and Kleshchev in [BK09]. Via this isomorphism, the $\mathbb{Z}$-grading of the KLR algebras can be used in the setting of Ariki-Koike algebras, and thus graded Specht modules [BKW11] and graded decomposition numbers can be studied. However, this is beyond the scope of what we are going to consider in this thesis.

1

Preliminaries

In this chapter we will state the necessary background information related to the algebras that we will work with, and the relevant combinatorial ideas that we will need. In particular, we will detail the definition of decomposition numbers for an Ariki-Koike algebra and describe the combinatorics involved.

1.1 Combinatorics

1.1.1 Partitions and tableaux

Let n be a positive integer.

Definition 1.1.1. A **composition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $|\lambda| := \sum_{b \geq 1} \lambda_b = n$. The integers λ_b, for $b \geq 1$, are called the **parts** of λ.
A composition λ of n is a **partition** if $\lambda_b \geq \lambda_{b+1}$ for all $b \geq 1$.

Since $n < \infty$, there is a k such that $\lambda_b = 0$ for $b > k$ and we write $\lambda = (\lambda_1, \dots, \lambda_k)$. We write $\varnothing$ for the empty partition $(0, 0, \dots, 0)$. If a partition has repeated parts, for convenience we group them together with an index. For example,

$$(4, 4, 2, 1, 0, 0, \dots) = (4, 4, 2, 1) = (4^2, 2, 1)$$

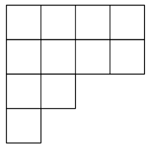
Definition 1.1.2. If λ is a partition, we define the **conjugate partition** λ' of λ to be the partition with b^{th} part $\lambda'_b = \#\{c \geq 1 \mid \lambda_c \geq b\}$.

Definition 1.1.3. The **Young diagram** of a partition λ is the subset

$$[\lambda] = \{(b, c) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0} \mid c \leq \lambda_b\}.$$

The elements of $[\lambda]$ are called **nodes** of λ. The k^{th} **row** (resp. **column**) of a diagram consists of those nodes whose first (resp. second) coordinate is k.

It is useful to represent the Young diagram of a partition λ as an array of boxes in the plane. For example, the partition $\lambda = (4^2, 2, 1)$ can be represented as follows.



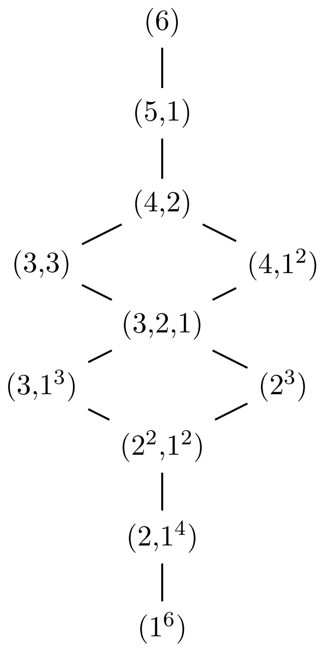
The set of partitions of n is partially ordered by the so-called dominance order defined in the following way.

Definition 1.1.4. If λ and μ are partitions of n, we say that λ **dominates** μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{b=1}^{i} \lambda_b \geq \sum_{b=1}^{i} \mu_b$$

for all $1 \leq i \leq n$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, we write $\lambda \triangleright \mu$.

Example 1.1.5. The dominance relation on the set of partitions of 6 is shown by the tree:



Definition 1.1.6. Let λ be a partition of n. A **λ-tableau** is a bijection $\mathfrak{t}\colon [\lambda] \to \{1, 2, \dots, n\}$. We say that $\mathfrak{t}$ has **shape** λ and write $\text{shape}(\mathfrak{t}) = \lambda$.
Equivalently, a **λ-tableau** is one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integer $1, 2, \dots, n$ allowing no repeats.

Example 1.1.7. Consider $\lambda = (4, 3, 1)$ a partition of 8. Then

1	2	4	5
3	6	7	
8			

and

4	5	7	3
2	1	8	
6			

are λ-tableaux.

We distinguish two particular types of λ-tableaux.

Definition 1.1.8. We say a λ-tableau $\mathfrak{t}$ is **row standard** if its entries increase from left to right in each row of the Young diagram, and we say $\mathfrak{t}$ is **standard** if, in addition to being row standard, its entries increase down the columns of the Young diagram. We write $\mathrm{Std}(\lambda)$ for the set of standard λ-tableaux.

Example 1.1.9. In the same notation of Example 1.1.7, we have that the first tableau is standard, while the second one is neither row standard nor standard.

Definition 1.1.10. Suppose λ is a partition of n and (b, c) is a node of $[\lambda]$.

1. The (b, c)-**hook** of λ is defined to be the set $H_{bc}(\lambda)$ of nodes in $[\lambda]$ directly to the right of or below (b, c), including the node (b, c) itself. The (b, c)-**hook length** $h_{bc}(\lambda)$ is the total number of nodes in $H_{bc}(\lambda)$. Then if an (b, c)-hook has length e, we call it an e-**hook**.

2. The **rim** of $[\lambda]$ is defined to be the set of nodes

$$\{(b, c) \in [\lambda] \mid (b + 1, c + 1) \notin [\lambda]\}.$$

3. Define an e-**rim hook** to be a connected subset R of the rim containing exactly e nodes such that $[\lambda] \setminus R$ is the Young diagram of a partition.

Note that there is a one-to-one correspondence between hooks and rim hooks.

Let $e \in \{2, 3, \ldots\} \cup \{\infty\}$ and set $I = \mathbb{Z}/e\mathbb{Z}$ (which we identify with $\{0, 1, \ldots, e - 1\}$) unless $e = \infty$, in which case set $I = \mathbb{Z}$.

Definition 1.1.11. Let λ be a partition of n.

1. Define the e-**residue** of a node (b, c) to be

$$\mathrm{res}(b, c) = \begin{cases} c - b \quad \mod e & \text{if } e = \{2, 3, \ldots\}, \\ c - b & \text{if } e = \infty. \end{cases}$$

   Define the **residue diagram** of λ to be the diagram formed by filling in the box of $[\lambda]$ at node (b, c) with $\mathrm{res}(b, c)$.

2. Let $i \in I$, let $c_i(\lambda)$ be the number of nodes in $[\lambda]$ of residue i. We define the **residue content** of λ to be

$$\mathrm{cont}(\lambda) = \begin{cases} (c_0(\lambda), c_1(\lambda), \ldots, c_{e-1}(\lambda)) & \text{if } e = \{2, 3, \ldots\}, \\ (\ldots, c_{-1}(\lambda), c_0(\lambda), c_1(\lambda), \ldots) & \text{if } e = \infty. \end{cases}$$

3. If λ has no e-rim hooks, or $e = \infty$, then we say that λ is an e-**core**.

4. We say that μ is the **e-core of** λ if μ is the e-core obtained from $[\lambda]$ removing all the possible e-rim hooks.

5. If we can remove w e-rim hooks from $[\lambda]$ to produce an e-core, then we say that λ has **e-weight** w and we write weight$(\lambda) = w$. In particular, an e-core has weight 0.

Example 1.1.12. Let $e = 4$. Consider the partition $\lambda = (3, 2)$. Then the residue diagram is

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 0 \\ \cline{1-2}\end{array}.$$

We can remove a 4-rim hook, that is the yellow shaded one in the diagram. So, λ has 4-weight 1 and its 4-core is (1).

Let $e < \infty$. For each $l \geq 1$, we define the l^{th} **ladder** in $\mathbb{N}^2$ to be the set

$$\mathcal{L}_l = \{(b, c) \in \mathbb{N}^2_{>0} \mid b + (e-1)(c-1) = l\}.$$

All the nodes in $\mathcal{L}_l$ have the same residue (namely, $1 - l \mod e$), and we define the residue of $\mathcal{L}_l$ to be this residue. If λ is a partition, we define the l^{th} ladder $\mathcal{L}_l(\lambda)$ of λ to be the intersection of $\mathcal{L}_l$ with the Young diagram of λ.

Example 1.1.13. Suppose $e = 3$, and $\lambda = (4, 3, 1)$. Consider the Young diagram of λ. Then in first diagram we label each node of $[\lambda]$ with the number of the ladder in which it lies, while in the second one we filled the nodes with their residues:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}.$$

1.1.2 β-numbers and abacus

Here we introduce a new way for representing partitions. It is clear that a diagram $[\lambda]$ is uniquely determined by its first column hook lengths $h_{k1}(\lambda), \ldots, h_{21}(\lambda), h_{11}(\lambda)$. It is useful to extend this idea to the case where λ has some zero parts at the end. Therefore, we define the set of β-numbers as follows.

Definition 1.1.14. Let $e \in \{2, 3, \ldots\} \cup \{\infty\}$. Let λ be a partition of n and let a be an integer. For every $i \geq 1$, we define the **β-number** β_i to be

$$\beta_i := \lambda_i + a - i$$

and we call the **set of β-numbers** associated to λ with respect to a to be

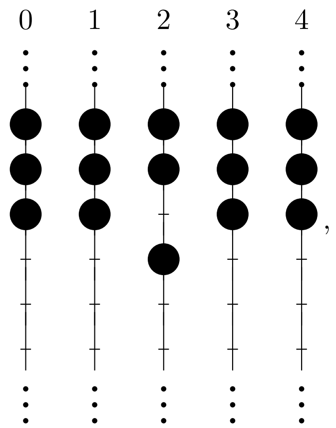
$$B_a(\lambda) = \{\beta_i \mid i \geq 1\}.$$

Given a set of β-numbers for a partition λ, we can create an abacus display. We take an abacus with e infinite vertical runners, which we label $0, 1, \dots, e-1$ from left to right (or $\dots, -1, 0, 1, \dots$ from left to right, if $e = \infty$), and we mark positions on runner l and label them with the integers congruent to l modulo e, so that (if $e < \infty$) then position $(x+1)e + l$ lies immediately below position $xe+l$, for each x. Then the **e-abacus display**, or the **e-abacus configuration**, associated to λ with respect to a is the abacus display with a bead placed at position β_i for each $i \geq 1$ and it is denoted by $\mathrm{Ab}_e(\lambda)$. If it is clear which e we are referring to, we simply say abacus configuration. When we draw abacus configurations we will draw only a finite part of the runners and we will assume that above the drawn part the runners are full of beads and below the drawn part there are no beads.

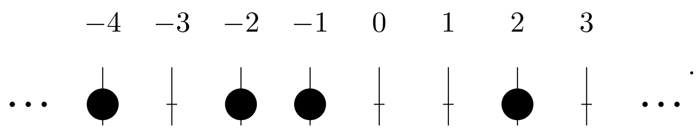
Example 1.1.15. Suppose $\lambda = (3, 1^2)$, and $a = 0$. Then we have

$$B_0(\lambda) = \{2, -1, -2, -4, -5, -6, \dots\}.$$

So the abacus display with $e = 5$ is



while the abacus display with $e = \infty$ is



If $e < \infty$, an abacus display for a partition is useful for visualising the removal of e-rim hooks. If we are given an abacus display for λ with β-numbers in a set B, then $[\lambda]$ has a e-rim hook if and only if there is a β-number $\beta_i \in B$ such that $\beta_i - e \notin B$. Furthermore, removing a e-rim hook corresponds to reducing such a β-number by e. On the e-abacus, this corresponds to sliding a bead up one position on its runner. So, λ is an e-core if and only if every bead in the abacus display has a bead immediately above it. Using this, we can see that the definition of e-weight and e-core of λ are well defined.

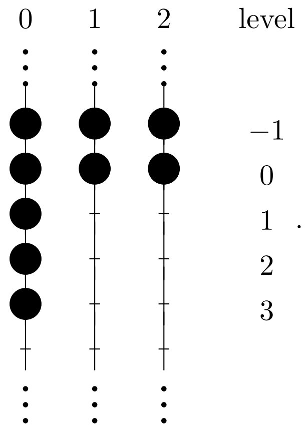
Lemma 1.1.16. Let λ be a partition. Then the e-core and e-weight of λ depend only on λ (and e).

Moreover, if $e < \infty$ and $a \in \mathbb{Z}$ we say that the bead corresponding to the β-number $xe + l$ with $0 \leq l < e$ is at **level** $\ell^a(\lambda) = x$ for $x \in \mathbb{Z}$.

Example 1.1.17. Consider $\lambda = (4, 2)$ and $a = 6$. Then we have

$$B_6(\lambda) = \{9, 6, 3, 2, 1, 0, -1, -2, -3, \ldots\}.$$

So, an abacus display for λ when $e = 3$ is



Finally, we can also notice that each bead corresponds to the end of a row of the diagram of λ (or to a row of length 0).

1.1.3 Multipartitions

Let n and r be positive integers.

Definition 1.1.18. A **multicomposition** of n with r components is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions such that

$$|\boldsymbol{\lambda}| := |\lambda^{(1)}| + \ldots + |\lambda^{(r)}| = n.$$

If, in addition, each $\lambda^{(j)}$ is a partition, then we say that $\boldsymbol{\lambda}$ is a r-**multipartition** of n. We write the unique multipartition of 0 as $\boldsymbol{\varnothing}$.

If r is understood, we shall just call this a multipartition of n. Note that a partition of n is essentially a multipartition of n with one component, that is, $r = 1$.

Also the set of r-multipartitions of n, as the set of partitions, is partially ordered by the so-called dominance order defined in the following way.

Definition 1.1.19. Given two r-multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of n, we say that $\boldsymbol{\lambda}$ **dominates** $\boldsymbol{\mu}$, and write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, if

$$\sum_{a=1}^{j-1} |\lambda^{(a)}| + \sum_{b=1}^{i} \lambda_b^{(j)} \geq \sum_{a=1}^{j-1} |\mu^{(a)}| + \sum_{b=1}^{i} \mu_b^{(j)}$$

for $j = 1, 2, \ldots, r$ and for all $i \geq 1$.

The dominance order is certainly the 'correct' order to use for multipartitions, but it is sometimes useful to have a total order, $>$, on the set of multipartitions. The one we use is given as follows.

Definition 1.1.20. Given two r-multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of n, we write $\boldsymbol{\lambda} > \boldsymbol{\mu}$ if and only if the minimal $j \in \{1, \ldots, r\}$ for which $\lambda^{(j)} \neq \mu^{(j)}$ and the minimal $i \geq 1$ such that $\lambda_i^{(j)} \neq \mu_i^{(j)}$ satisfy $\lambda_i^{(j)} > \mu_i^{(j)}$. This is called the **lexicographic order** on multipartitions.

It is simple to verify that $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ implies $\boldsymbol{\lambda} > \boldsymbol{\mu}$. But the reverse implication is false in general. For instance, $\boldsymbol{\lambda} = ((6, 3, 1^3), (2^2, 1^3)) > ((3^2, 2^3), (5, 2)) = \boldsymbol{\mu}$, but $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are not comparable with the dominance order.

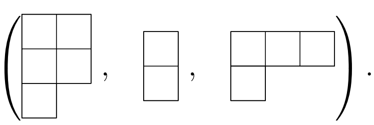
Definition 1.1.21. Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of n, we define its **Young diagram** to be the subset

$$[\boldsymbol{\lambda}] := \{(b, c, j) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0} \times \{1, \ldots, r\} \mid c \leq \lambda_b^{(j)}\}.$$

The elements of $[\boldsymbol{\lambda}]$ are called **nodes**.

Definition 1.1.22. We say that a node $\mathfrak{n} \in [\boldsymbol{\lambda}]$ is **removable** if $[\boldsymbol{\lambda}] \backslash \{\mathfrak{n}\}$ is also the Young diagram of a multipartition. We say that an element $\mathfrak{n} \in \mathbb{N}_{>0}^2 \times \{1, \ldots, r\}$ is an **addable node** if $\mathfrak{n} \notin [\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}] \cup \{\mathfrak{n}\}$ is the Young diagram of a multipartition.

As for partitions, we can draw the Young diagram of an r-multipartition as an r-tuple of the Young diagrams of its component partitions. For example, the diagram of $((2^2, 1), (1^2), (3, 1))$ is drawn as



Define a bijection $'$ from $\mathbb{N}_{>0}^2 \times \{1, \ldots, r\}$ to itself by

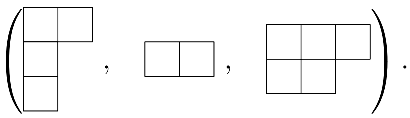
$$(b, c, j)' = (c, b, r + 1 - j).$$

Definition 1.1.23. Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, define the **conjugate multipartition** to have Young diagram

$$[\boldsymbol{\lambda}'] = \{\mathfrak{n}' \mid \mathfrak{n} \in \boldsymbol{\lambda}\};$$

that is, $\boldsymbol{\lambda}' = (\lambda^{(r)\prime}, \ldots, \lambda^{(1)\prime})$, where $\lambda^{(j)\prime}$ is the usual conjugate partition to $\lambda^{(j)}$.

Example 1.1.24. Consider $\boldsymbol{\lambda} = ((2^2, 1), (1^2), (3, 1))$ as above. Then $\boldsymbol{\lambda}' = ((2, 1^2), (2), (3, 2))$ and its Young diagram is



From the definition of conjugate multipartition, we have the following.

Lemma 1.1.25. [Fay08b, Lemma 1.2] If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions, then $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda}' \trianglelefteq \boldsymbol{\mu}'$.

Again, as for partitions, we can give the following definitions.

Definition 1.1.26. Let $\boldsymbol{\lambda}$ be an r-multipartition of n.

- A **$\boldsymbol{\lambda}$-tableau** is a bijection $\mathfrak{t}\colon [\boldsymbol{\lambda}] \to \{1, \ldots, n\}$. We can represent a $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ by drawing $[\boldsymbol{\lambda}]$ and then filling in the box at position (b, c, j) with its image under $\mathfrak{t}$.
- We say a $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ is **(row) standard** if each of its components $\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}, \ldots, \mathfrak{t}^{(r)}$ are (row) standard. We write $\mathrm{Std}(\boldsymbol{\lambda})$ for the set of standard $\boldsymbol{\lambda}$-tableaux.

Example 1.1.27. Let $\boldsymbol{\lambda} = ((3, 1), (2^2, 1))$. Then

$$\left(\begin{matrix}1&3&4\\2&&\end{matrix},\ \begin{matrix}5&7\\6&9\\8&\end{matrix}\right),\ \left(\begin{matrix}1&2&3\\4&&\end{matrix},\ \begin{matrix}5&6\\8&9\\7&\end{matrix}\right),\ \left(\begin{matrix}4&3&1\\2&&\end{matrix},\ \begin{matrix}5&7\\6&9\\8&\end{matrix}\right)$$

are $\boldsymbol{\lambda}$-tableaux. The first one is standard, the second one is row standard, but not standard and the last one is neither.

Finally, we may also generalise to multipartitions the definition of β-numbers and the construction of an abacus display as follows.

Definition 1.1.28. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a r-multipartition of n and let $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. For every $i \geq 1$ and for every $j \in \{1, \ldots, r\}$, we define the **β-number** β_i^j to be

$$\beta_i^j := \lambda_i^{(j)} + a_j - i.$$

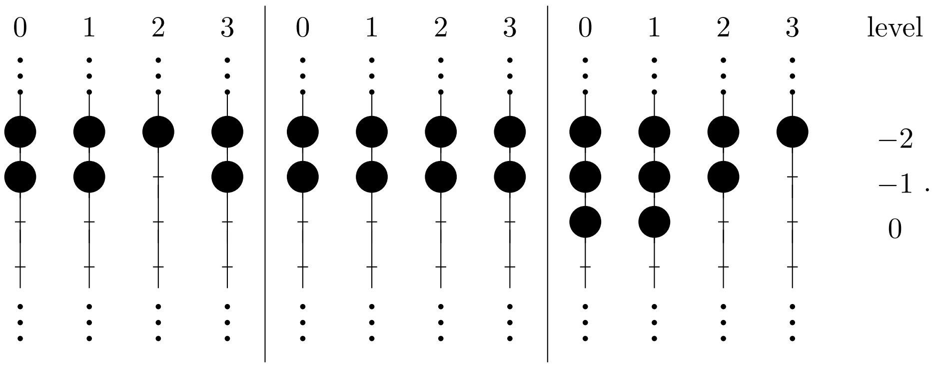
We refer to any r-tuple of integers $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ as a **multicharge**. The set $B_{a_j}^j = \{\beta_1^j, \beta_2^j, \ldots\}$ is the set of β-numbers (defined using the integer a_j) of partition $\lambda^{(j)}$. It is easy to see that any set $B_{a_j}^j = \{\beta_1^j, \beta_2^j, \ldots\}$ is a set containing exactly $a_j + N$ integers greater than or equal to $-N$, for sufficiently large N.

For each set $B^j_{a_j}$ we have a corresponding abacus display. Hence, we can define $\mathrm{Ab}_e(\boldsymbol{\lambda})$ the e-**abacus display**, or e-**abacus configuration**, for an r-multipartition $\boldsymbol{\lambda}$ with respect to $\boldsymbol{a}$ to be the r-tuple of e-abacus displays associated to each component $\lambda^{(j)}$ with respect to a_j. Again, as for partitions, we can say that the bead corresponding to the β-number $xe + i$ with $0 \leq i < e$ in the β-set $B^j_{a_j}$ is at **level** $\ell^{\boldsymbol{a}}_j(\boldsymbol{\lambda}) = x$ for $x \in \mathbb{Z}$.

Example 1.1.29. Suppose that $r = 3$, $\boldsymbol{a} = (-1, 0, 1)$ and $\boldsymbol{\lambda} = ((1), \varnothing, (1^2))$. Then we have

$$\begin{aligned} B^1_{-1} &= \{-1, -3, -4, -5, \ldots\}; \\ B^2_0 &= \{-1, -2, -3, \ldots\}; \\ B^3_1 &= \{1, 0, -2, -3, -4, \ldots\}. \end{aligned}$$

So, the abacus display with respect to the multicharge $\boldsymbol{a}$ for $\boldsymbol{\lambda}$ when $e = 4$ is



1.2 The Symmetric Group

Let n be a positive integer. A **permutation** of the set $\{1, 2, \ldots, n\}$ is a bijection

$$\pi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

A permutation π that interchanges a and b with $a \neq b$ and fixes all the other elements is called a **transposition** and it is written as $\pi = (a, b)$. Following [Jam78], any permutation can be written as a product of transpositions. Hence there is a well-defined function

$$\mathrm{sgn} : \mathfrak{S}_n \to \{\pm 1\}$$

such that $\mathrm{sgn}\pi = (-1)^j$ if π is a product of j transpositions.

Definition 1.2.1. For $n \geq 1$, the set of all permutations of $\{1, 2, \ldots, n\}$ together with the product operation given by composition is called the **symmetric group of degree** n and it is denoted by $\mathfrak{S}_n$.

Remark 1.2.2. For $1 \leq i \leq n$, let s_i be the transposition $(i, i+1)$. Then $\mathfrak{S}_n$ is generated by the elements $s_1, \dots, s_{n-1}$ subject to the relations:

$$\begin{aligned} s_i^2 &= 1, && \text{for } 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, && \text{for } 1 \leq i < j-1 \leq n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && \text{for } 1 \leq i \leq n-2. \end{aligned}$$

The second and the third relations are called **braid relations** of $\mathfrak{S}_n$.

Definition 1.2.3. Let $w \in \mathfrak{S}_n$ such that $w = s_{i_1} \cdot \ldots \cdot s_{i_k}$. If k is minimal then we say that $s_{i_1} \cdot \ldots \cdot s_{i_k}$ is a **reduced expression** for w and we say that w has **length** k and write $\ell(w) = k$. The identity element 1 has length 0.

The representation theory of the symmetric group algebra $\mathbb{F}\mathfrak{S}_n$ for any field $\mathbb{F}$ was studied by James in [Jam78], and this is where we take most of our definitions and results from. The approach is largely combinatorial, and although we will be interested in more complicated algebras later on, many of the definitions and ideas here will be important.

Let λ be a partition of n and let $\mathfrak{t}$ be a λ-tableau. The symmetric group $\mathfrak{S}_n$ acts naturally on λ-tableaux on the right by permuting the entries.

Example 1.2.4. Let $\lambda = (3, 2)$ be a partition of 5 and $\sigma = (1, 2, 4)(3, 5) \in \mathfrak{S}_5$. Then σ acts on the λ-tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$ as follows.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\ \sigma = \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}.$$

Definition 1.2.5. Let λ be a partition of n. The **initial tableau** $\mathfrak{t}^\lambda$ is defined to be the tableau obtained by writing the numbers $1, 2, \dots, n$ in order from left to right, going down the rows of each successive part of λ. Given a λ-tableau $\mathfrak{t}$ we define the permutation $d(\mathfrak{t}) \in \mathfrak{S}_n$ by $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$.

Example 1.2.6. In the same notation of Example 1.2.4, let $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}$. Then

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \text{and} \quad d(\mathfrak{t}) = (1, 2, 4)(3, 5).$$

Definition 1.2.7. Define the **row stabilizer** of $\mathfrak{t}$ by

$$R_\mathfrak{t} = \{\pi \in \mathfrak{S}_n \mid \text{for all } i \in \{1, \dots, n\},\ i \text{ and } i\pi \text{ are in the same row of } \mathfrak{t}\},$$

and similiarly define the **column stabilizer** of $\mathfrak{t}$ by

$$C_\mathfrak{t} = \{\pi \in \mathfrak{S}_n \mid \text{for all } i \in \{1, \dots, n\},\ i \text{ and } i\pi \text{ are in the same column of } \mathfrak{t}\}.$$

Definition 1.2.8. For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, we associate a **Young subgroup** $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ defined by

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\{1,2,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{\lambda_1+\ldots+\lambda_{k-1}+1,\ldots,n\}}.$$

Remark 1.2.9. Note that:

1. for any partition λ, the row stabilizer of $\mathfrak{t}^\lambda$ is $R_{\mathfrak{t}^\lambda} = \mathfrak{S}_\lambda$;
2. $R_{\mathfrak{t}\pi} = \pi^{-1} R_{\mathfrak{t}} \pi$ and $C_{\mathfrak{t}\pi} = \pi^{-1} C_{\mathfrak{t}} \pi$. Indeed, for any permutation σ:

$$\begin{aligned} \sigma \in \pi^{-1} R_{\mathfrak{t}} \pi &\iff \pi\sigma\pi^{-1} \in R_{\mathfrak{t}} \\ &\iff \{\mathfrak{t}\}\pi\sigma\pi^{-1} = \{\mathfrak{t}\} \\ &\iff \{\mathfrak{t}\}\pi\sigma = \{\mathfrak{t}\}\pi \\ &\iff \{\mathfrak{t}\pi\}\sigma = \{\mathfrak{t}\pi\} \\ &\iff \sigma \in R_{\mathfrak{t}\pi}. \end{aligned}$$

Example 1.2.10. Let $\lambda = (4, 3, 1)$ and let $\mathfrak{t} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}$ be a λ-tableau, then $R_{\mathfrak{t}} = \mathfrak{S}_{\{1,2,4,5\}} \times \mathfrak{S}_{\{3,6,7\}} \times \mathfrak{S}_{\{8\}}$ and $C_{\mathfrak{t}} = \mathfrak{S}_{\{1,3,8\}} \times \mathfrak{S}_{\{2,6\}} \times \mathfrak{S}_{\{4,7\}} \times \mathfrak{S}_{\{5\}}$.

We define an equivalence relation $\sim$ on the set of λ-tableaux by $\mathfrak{t} \sim \mathfrak{s}$ if and only if $\mathfrak{t}\pi = \mathfrak{s}$ for some $\pi \in R_{\mathfrak{t}}$.

Definition 1.2.11. We call an equivalence class under $\sim$ a **λ-tabloid** and denote it by $\{\mathfrak{t}\}$, where $\mathfrak{t}$ is one of the tableaux contained in the equivalence class.

We draw a diagram for $\{\mathfrak{t}\}$ by writing out the entries of $\mathfrak{t}$ in the layout of their diagram and then adding lines between the rows.

Example 1.2.12. If $\mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$, then $\{\mathfrak{t}\} = \begin{array}{ccc} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} = \begin{array}{ccc} \hline 3 & 1 & 5 \\ \hline 4 & 2 \\ \cline{1-2} \end{array}$.

Definition 1.2.13. $\mathfrak{S}_n$ acts on the set of λ-tabloids by $\{\mathfrak{t}\}\pi = \{\mathfrak{t}\pi\}$ for $\pi \in \mathfrak{S}_n$.

Remark 1.2.14. This action is well-defined, since $\{\mathfrak{t}_1\} = \{\mathfrak{t}_2\}$ implies $\mathfrak{t}_2 = \mathfrak{t}_1\sigma$ for some $\sigma \in R_{\mathfrak{t}_1}$. Then $\pi^{-1}\sigma\pi \in \pi^{-1} R_{\mathfrak{t}_1} \pi = R_{\mathfrak{t}_1\pi}$ by Remark 1.2.9, so $\{\mathfrak{t}_1\pi\} = \{\mathfrak{t}_1\pi\pi^{-1}\sigma\pi\} = \{\mathfrak{t}_1\sigma\pi\} = \{\mathfrak{t}_2\pi\}$.

Definition 1.2.15. Let $\mathbb{F}$ be an arbitrary field and let λ be a partition of n. Define $M(\lambda)$ to be the vector space over $\mathbb{F}$ whose basis elements are the λ-tabloids.

The action of $\mathfrak{S}_n$ on tabloids has just been defined, by $\{\mathfrak{t}\}\pi = \{\mathfrak{t}\pi\}$ for $\pi \in \mathfrak{S}_n$. Extending this action linearly on $M(\lambda)$, this turns $M(\lambda)$ into an $\mathbb{F}\mathfrak{S}_n$-module.

Definition 1.2.16. Let $\mathfrak{t}$ be a tableau. Then the **signed column sum** $\kappa_{\mathfrak{t}}$ is the element of $\mathbb{F}\mathfrak{S}_n$ obtained by summing the elements of the column stabilizer of $\mathfrak{t}$, attaching the signature of each permutation. That is,

$$\kappa_{\mathfrak{t}} = \sum_{\pi \in C_{\mathfrak{t}}} (\mathrm{sgn}\pi)\pi.$$

The **polytabloid** $e_{\mathfrak{t}}$ associated with the tableau $\mathfrak{t}$ is given by

$$e_{\mathfrak{t}} = \{\mathfrak{t}\}\kappa_{\mathfrak{t}}.$$

Example 1.2.17. If $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 2 & 5 & 1 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$, then $\kappa_{\mathfrak{t}} = 1 - (2,3) - (4,5) + (2,3)(4,5)$ and

$$e_{\mathfrak{t}} = \begin{array}{ccc}\hline 2 & 5 & 1 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 3 & 5 & 1 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 2 & 4 & 1 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3 & 4 & 1 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}.$$

Now, we may define the main modules of interest that are the so-called Specht modules.

Definition 1.2.18. Given a partition λ of n, the **Specht module** $S(\lambda)$ is the submodule of $M(\lambda)$ spanned by the polytabloids $e_{\mathfrak{t}}$.

Examples 1.2.19.

- If $\lambda = (n)$, there is a unique λ-tabloid that is $\begin{array}{c}\hline 1\ 2 \cdots n \\ \hline\end{array}$. So $M(\lambda)$ and $S(\lambda)$ are equal and, since $\begin{array}{c}\hline 1\ 2 \cdots n \\ \hline\end{array}\,\pi = \begin{array}{c}\hline 1\ 2 \cdots n \\ \hline\end{array}$ for all $\pi \in \mathfrak{S}_n$, we get that $M(\lambda) = S(\lambda)$ is the trivial $\mathbb{F}\mathfrak{S}_n$-module.

- If $\lambda = (1^n)$, each equivalence class $\{\mathfrak{t}\}$ consists of a single λ-tableau, and this tableau can be identified with a permutation. Since the action of $\mathfrak{S}_n$ is preserved,
$$M(\lambda) \cong \mathbb{F}\mathfrak{S}_n,$$
that is, $M(\lambda)$ is isomorphic to the regular $\mathbb{F}\mathfrak{S}_n$-module. Note that $e_{\mathfrak{t}}$ is the signed sum of all $n!$ permutations regarded as tabloids and, for any permutation π, we have $e_{\mathfrak{t}\pi} = e_{\mathfrak{t}}\pi = (\mathrm{sgn}\pi)e_{\mathfrak{t}}$. Thus, $S(\lambda)$ is the alternating $\mathbb{F}\mathfrak{S}_n$-module.

Moreover, we can find a basis for these fundamental modules, as stated in the following result.

Theorem 1.2.20. [Jam78, Theorem 8.4] $\{e_{\mathfrak{t}} \mid \mathfrak{t} \text{ is a standard } \lambda\text{-tableau}\}$ is a basis for $S(\lambda)$.

Now, in order to determine all the irreducible modules of $\mathbb{F}\mathfrak{S}_n$, we define a bilinear form $\langle\ ,\ \rangle$ on $M(\lambda)$ by

$$\langle\{\mathfrak{t}_1\},\{\mathfrak{t}_2\}\rangle = \begin{cases} 1 & \text{if } \{\mathfrak{t}_1\} = \{\mathfrak{t}_2\}, \\ 0 & \text{if } \{\mathfrak{t}_1\} \neq \{\mathfrak{t}_2\}. \end{cases}$$

This bilinear form together with the next two results will be crucial for classifying all the irreducible $\mathbb{F}\mathfrak{S}_n$-modules, up to isomorphism.

Theorem 1.2.21 (Submodule Theorem)**.** [Jam76] If U is a submodule of $M(\lambda)$, then $U \supseteq S(\lambda)$ or $U \subseteq S(\lambda)^{\perp}$.

Theorem 1.2.22. [Jam78, Theorem 4.9] The quotient module $S(\lambda)/(S(\lambda) \cap S(\lambda)^{\perp})$ is zero or absolutely irreducible. Furthermore, if this is non-zero, then $S(\lambda) \cap S(\lambda)^{\perp}$ is the unique maximal submodule of $S(\lambda)$ and $S(\lambda)/(S(\lambda) \cap S(\lambda)^{\perp})$ is self-dual.

Using Maschke's Theorem, we see that if char $\mathbb{F} = 0$ we obtain $S(\lambda) \cap S(\lambda)^{\perp} = 0$ and so $M(\lambda) = S(\lambda) \oplus S(\lambda)^{\perp}$. This leads to the following theorems.

Theorem 1.2.23. [Jam78, Theorem 4.12] Let $\mathbb{F}$ be a field with char $\mathbb{F} = 0$. The Specht modules over $\mathbb{F}$ are self-dual and absolutely irreducible, and give all the ordinary irreducible $\mathbb{F}\mathfrak{S}_n$-modules, up to isomorphism.

Theorem 1.2.24. [Jam78, Theorem 4.13] If char $\mathbb{F} = 0$, the composition factors of $M(\lambda)$ are $S(\lambda)$ (once) and some of $\{S(\mu) : \mu \rhd \lambda\}$ (possibly with repeats).

Theorem 1.2.23 gives us the irreducible modules of $\mathfrak{S}_n$ over a field $\mathbb{F}$ with characteristic 0. However, if our field $\mathbb{F}$ has characteristic $p > 0$, then we cannot use Maschke's Theorem as above since the characteristic of $\mathbb{F}$ may divide $|\mathfrak{S}_n|$. So, the next definition is needed in order to fully determine the irreducible $\mathbb{F}\mathfrak{S}_n$-modules over a field of characteristic p.

Definition 1.2.25. A partition λ is ***p*-singular** if $\lambda_{i+1} = \lambda_{i+2} = \ldots = \lambda_{i+p} > 0$ for some i. Otherwise, λ is ***p*-regular**.

Theorem 1.2.26. [Jam78, Theorem 11.1] Suppose that $S(\lambda)$ is a Specht module defined over a field of characteristic p. Then the quotient module $S(\lambda)/(S(\lambda) \cap S(\lambda)^{\perp})$ is non-zero if and only if λ is p-regular.

Suppose that $\mathbb{F}$ has characteristic p and that λ is p-regular. We define

$$D(\lambda) = S(\lambda)/(S(\lambda) \cap S(\lambda)^{\perp}).$$

We have a counterpart to Theorem 1.2.23 for $\mathbb{F}$ with characteristic $p > 0$.

Theorem 1.2.27. [Jam76, Theorem 6] $\{D(\lambda) \mid \lambda \text{ is } p\text{-regular}\}$ is a complete set of non-isomorphic irreducible $\mathbb{F}\mathfrak{S}_n$-modules over a field $\mathbb{F}$ of characteristic p.

Theorem 1.2.28. [Jam78, Corollary 12.2]

1. If λ is p-regular, then $S(\lambda)$ has a unique top composition factor $D(\lambda) = S(\lambda)/(S(\lambda) \cap S(\lambda)^{\perp})$. If D is a composition factor of $S(\lambda) \cap S(\lambda)^{\perp}$, then D is isomorphic to $D(\mu)$ for some $\mu \rhd \lambda$.

2. If λ is p-singular, then all the composition factors of $S(\lambda)$ have the form $D(\mu)$ with $\mu \rhd \lambda$.

It is useful to record the composition multiplicities of the irreducible $\mathbb{F}\mathfrak{S}_n$-modules in the following way.

Definition 1.2.29. Let λ and μ be partitions of n with μ p-regular. Define the **decomposition numbers** of $\mathbb{F}\mathfrak{S}_n$

$$d_{\lambda\mu} := [S(\lambda) : D(\mu)]$$

to be the composition multiplicity of $D(\mu)$ in $S(\lambda)$. We call the matrix $\boldsymbol{D} = (d_{\lambda\mu})$ the **decomposition matrix** of $\mathfrak{S}_n$.

By Theorem 1.2.28 we have that

- $d_{\lambda\lambda} = 1$ for every p-regular partition λ.
- $d_{\lambda\mu} \neq 0$ only if $\mu \rhd \lambda$.

So, we can conclude the following.

Corollary 1.2.30. $\boldsymbol{D}$ becomes lower unitriangular when the p-regular partitions are placed in lexicographic order before all the p-singular partitions.

1.3 The Iwahori-Hecke algebra of $\mathfrak{S}_n$

This section introduces the Iwahori-Hecke algebras of the symmetric group and deals with their representation theory.

1.3.1 The Iwahori-Hecke algebra

Definition 1.3.1. Let $\mathbb{F}$ be a field and let q be an arbitrary non-zero element of $\mathbb{F}$. The **Iwahori-Hecke algebra** $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is the unital associative $\mathbb{F}$-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations:

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, && \text{for } 1 \leq i \leq n-1, \\ T_i T_j &= T_j T_i, && \text{for } 1 \leq i < j-1 \leq n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } 1 \leq i \leq n-2. \end{aligned}$$

For brevity, we may write $\mathcal{H}_n$ for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. Note that when $q = 1$, the first relation becomes $T_i^2 = 1$, and so in this case $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is isomorphic to the group algebra $\mathbb{F}\mathfrak{S}_n$.

Definition 1.3.2. Define e to be the **quantum characteristic** of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, that is, the minimal integer e such that $1 + q + q^2 + \ldots + q^{e-1} = 0$. If no such integer exists, let $e = \infty$.

Note that if $q = 1$ (as in the case of $\mathfrak{S}_n$) then $e = \operatorname{char} \mathbb{F}$.

Suppose $w \in \mathfrak{S}_n$ and let $w = s_{i_1} \ldots s_{i_k}$ be a reduced expression for w. We define

$$T_w = T_{i_1} \ldots T_{i_k}.$$

By Matsumoto's Theorem ([Mat99], Theorem 1.8) for reduced expressions, we have that T_w is independent of the choice of reduced expression for w and hence is well defined. If w is the identity element of $\mathfrak{S}_n$, then we identify T_w with the identity element of $\mathbb{F}$. The following result tells us how we perform right multiplication in $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.

Proposition 1.3.3. [Mat99, Lemma 1.12] Let $w \in \mathfrak{S}_n$. Then

$$T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) > \ell(w), \\ qT_{ws_i} + (q-1)T_w & \text{if } \ell(ws_i) < \ell(w). \end{cases}$$

Example 1.3.4. Let $w = (1,3,2) = (1,2)(2,3)$ and consider $s_2 = (2,3)$. Then $ws_2 = (1,2)$, so $\ell(ws_2) = 1 < 2 = \ell(w)$. Hence

$$T_{(1,3,2)}T_{(2,3)} = qT_{(1,2)} + (q-1)T_{(1,3,2)}.$$

If, instead, we consider $s_1 = (1,2)$, then $ws_1 = (1,2)(2,3)(1,2) = (1,3)$, so $\ell(ws_1) = 3 > 2 = \ell(w)$. Hence

$$T_{(1,3,2)}T_{(1,2)} = T_{(1,3)}.$$

By Proposition 1.3.3, the elements $\{T_w \mid w \in \mathfrak{S}_n\}$ certainly span $\mathcal{H}_n$. In fact, it can also be shown that they are linearly independent to obtain the following theorem.

Theorem 1.3.5. [Mat99, Thoerem 1.13] The Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is free as an $\mathbb{F}$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$.

1.3.2 Representation theory of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$

There are different approches to study the representation theory of the Iwahori-Hecke algebra of $\mathfrak{S}_n$. The one we chose uses the theory of cellular algebras (due to

Graham and Lehrer [GL96]). The basic strategy is to construct a cellular basis for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ and then apply the theory of cellular algebras to produce the irreducible $\mathcal{H}_n$-modules. There are different cellular bases for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. The one we use is a very natural basis indexed by pairs of standard tableaux; it was discovered by Murphy [Mur92, Mur95]. We decided to adopt this strategy because it allows us to use only a small amount of work to obtain many useful properties of the representation theory of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, and moreover will set us up with a method to use when considering the Ariki-Koike algebras later on.

We shall describe the main points which demonstrate how we apply the cellular theory to the Iwahori-Hecke algebra, but many of the details will be omitted. We take our definitions and results from [Mat99] and as such the missing details can be found there. We begin by defining a cellular basis.

Definition 1.3.6. Let R be a commutative domain with 1 and let A be an associative unital R-algebra that is free as an R-module. Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $\mathcal{T}(\lambda)$ and elements $c^\lambda_{\mathfrak{st}} \in A$ for every $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$ such that

$$\mathscr{C} = \{c^\lambda_{\mathfrak{st}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$$

is a (free) basis of A. For each $\lambda \in \Lambda$, let $\check{A}^\lambda$ be the R-submodule of A with basis $\{c^\mu_{\mathfrak{uv}} \mid \mu \in \Lambda,\ \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)\}$. Then the pair $(\mathscr{C}, \Lambda)$ is a **cellular basis** of A if

(i) the R-linear map $*\colon A \to A$ determined by $(c^\lambda_{\mathfrak{st}})^* = c^\lambda_{\mathfrak{ts}}$, for all $\lambda \in \Lambda$ and all $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathcal{T}(\lambda)$, is an algebra anti-isomorphism of A; and,

(ii) for any $\lambda \in \Lambda$, $\mathfrak{t} \in \mathcal{T}(\lambda)$ and $a \in A$ there exist elements $r_{\mathfrak{v}} \in R$ such that for each $\mathfrak{s} \in \mathcal{T}(\lambda)$

$$c^\lambda_{\mathfrak{st}} a \equiv \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_{\mathfrak{v}} c^\lambda_{\mathfrak{sv}} \mod \check{A}^\lambda.$$

If A has a cellular basis then we say that A is a **cellular algebra**.

Note that each $r_{\mathfrak{v}}$ depends on $\mathfrak{v}$, $\mathfrak{t}$ and a, but *not* on $\mathfrak{s}$. By applying $*$ to part (ii), we get a similar formula for multiplying on the left: for $\mathfrak{s} \in \mathcal{T}(\lambda)$, $a \in A$, we have that for every $\mathfrak{t} \in \mathcal{T}(\lambda)$

$$a^* c^\lambda_{\mathfrak{st}} \equiv \sum_{\mathfrak{u} \in \mathcal{T}(\lambda)} r_{\mathfrak{u}} c^\lambda_{\mathfrak{ut}} \mod \check{A}^\lambda \qquad (1.3.1)$$

where the $r_{\mathfrak{u}}$ are the same as those determined in part (ii) above. Note that (1.3.1) and (1.3.6)(ii) show that $\check{A}^\lambda$ is a two-sided ideal.

Example 1.3.7. Let $A = R[x]$ be the polynomial ring in an indeterminate x over R. Let Λ be the set of non-negative integers with their natural order. For each $n \in \Lambda$ set $\mathcal{T}(n) = \{n\}$ and set $c^n_{nn} = x^n$. Then $\{x^n \mid n \in \mathbb{N}\}$ is a cellular basis of A, because

- $*\colon A \to A$ is just the identity on A; and,
- for any $n \in \mathbb{N}$ and $a = \sum_{i\geq 0} a_i x^i \in A$ we can take $r_n = a_0$ so that

  $$c^n_{nn} a = x^n a = \sum_{i\geq 0} a_i x^{n+i} \equiv a_0 x^n = r_n c^n_{nn} = \sum_{\mathfrak{v}\in\{n\}} r_{\mathfrak{v}} c^n_{n\mathfrak{v}} \mod \check{A}^n,$$

  since here $\check{A}^n$ is the R-submodule of A with basis $\{x^k \mid k > n\}$.

So $A = R[x]$ is a cellular algebra.

Let A be an arbitrary cellular algebra and fix $\lambda \in \Lambda$. Let A^λ be the R-submodule of A with basis $\{c^\mu_{\mathfrak{uv}} \mid \mu \in \Lambda,\ \mu \geq \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)\}$. Thus, $\check{A}^\lambda \subset A^\lambda$ and $A^\lambda/\check{A}^\lambda$ has basis $c^\lambda_{\mathfrak{st}} + \check{A}^\lambda$ where $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$.

If $\mathfrak{s} \in \mathcal{T}(\lambda)$ define $C^\lambda_{\mathfrak{s}}$ to be the R-submodule of $A^\lambda/\check{A}^\lambda$ with basis $\{c^\lambda_{\mathfrak{st}} + \check{A}^\lambda \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$. Then $C^\lambda_{\mathfrak{s}}$ is a right A-module by Definition 1.3.6(ii) that determines the action of A on $c^\lambda_{\mathfrak{st}}$. However, this action is independent of $\mathfrak{s}$; that is, $C^\lambda_{\mathfrak{s}} \cong C^\lambda_{\mathfrak{t}}$ for any $\mathfrak{s}$, $\mathfrak{t} \in \mathcal{T}(\lambda)$. This motivates us to define the following modules.

Definition 1.3.8. Define the **(right) cell module** C^λ to be the right A-module with basis $\{c^\lambda_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}(\lambda)\}$ and action determined by

$$c^\lambda_{\mathfrak{t}} a = \sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r_{\mathfrak{v}} c^\lambda_{\mathfrak{v}} \tag{1.3.2}$$

where $r_{\mathfrak{v}}$ is the element of R determined by Definition 1.3.6(ii).

Then, $C^\lambda \cong C^\lambda_{\mathfrak{s}}$ for any $\mathfrak{s} \in \mathcal{T}(\lambda)$ via the canonical R-linear map which sends $c^\lambda_{\mathfrak{t}}$ to $c^\lambda_{\mathfrak{st}} + \check{A}^\lambda$ for all $\mathfrak{t} \in \mathcal{T}(\lambda)$.

Note that, by Definition 1.3.6 and by (1.3.1), for any $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\lambda)$ we have that there exists an element $r_{\mathfrak{st}} \in R$ such that

$$c^\lambda_{\mathfrak{us}} c^\lambda_{\mathfrak{tv}} \equiv r_{\mathfrak{st}} c^\lambda_{\mathfrak{uv}} \mod \check{A}^\lambda.$$

Thanks to this result, we get that there is a unique symmetric, associative, bilinear map $\langle\ ,\ \rangle\colon C^\lambda \times C^\lambda \to R$ given by

$$\langle c^\lambda_{\mathfrak{s}}, c^\lambda_{\mathfrak{t}} \rangle = r_{\mathfrak{st}}.$$

Let $\mathrm{rad} C^\lambda = \{x \in C^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in C^\lambda\}$. This is an A-submodule of C^λ, so we define $D^\lambda = C^\lambda/\mathrm{rad} C^\lambda$.

Define $\Lambda_0 = \{\mu \in \Lambda \mid D^\mu \neq 0\}$. Then we have the following results that allow us to classify all the simple A-modules for A cellular algebra.

Proposition 1.3.9. [Mat99, Proposition 2.11(i)] Suppose that R is a field and let $\lambda \in \Lambda_0$. Then the right A-module D^λ is absolutely irreducible.

Theorem 1.3.10. [Mat99, Theorem 2.16] Suppose that R is a field and that Λ is finite. Then $\{D^\mu \mid \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.

Now, we can define and describe the decomposition matrix of A.

Definition 1.3.11. Let $\mu \in \Lambda_0$ and $\lambda \in \Lambda$. Define $d_{\lambda\mu} = [C^\lambda : D^\mu]$ to be the composition multiplicity of the irreducible module D^μ in C^λ. The matrix $\boldsymbol{D} = (d_{\lambda\mu})$ is the so called **decomposition matrix** of A.

The next results follows from the proof of Theorem 1.3.10.

Corollary 1.3.12. Suppose that R is a field. Then the decomposition matrix D of A is unitriangular, that is

- $d_{\lambda\lambda} = 1$ if $\lambda \in \Lambda_0$;
- $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.

Another interesting fact is how cell modules and blocks of a cellular algebra are related. So, now, we give some definitions for a finite dimensional algebra in order to explain the next results.

Definition 1.3.13. Let A be a finite dimensional algebra over a field $\mathbb{F}$. Suppose

$$A = B_1 \oplus \ldots \oplus B_c$$

is a decomposition of A into a direct sum of indecomposable two-sided ideals $B_1, \ldots, B_c$, called the **blocks** of A. An A-module is said to **belong to the block** B_i if all of its composition factors lie in B_i.

Definition 1.3.14. Let A be a cellular algebra with cellular basis $(\mathscr{C}, \Lambda)$. We say that $\lambda, \mu \in \Lambda$ are **cell-linked** if there exists a sequence

$$\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$$

of elements of Λ such that the cell modules $C^{\lambda_{i-1}}$ and C^{λ_i} have a common irreducible composition factor for all $i = 1, \ldots, k$.

Remark 1.3.15. This defines an equivalence relation on Λ and, if $\mu \in \Lambda_0$ with $d_{\lambda\mu} \neq 0$, then λ and μ are cell-linked.

Hence we have the following results.

Proposition 1.3.16. [Mat99, Corollary 2.22] Suppose that $\mathbb{F}$ is a field and Λ is finite. Let λ and μ be elements of Λ. Then λ and μ are cell-linked if and only if C^λ and C^μ are in the same block. In particular, all the irreducible constituents of a cell module belong to the same block.

Now we can show how the cellular theory can be applied to the Iwahori-Hecke algebra. We aim to construct a cellular basis for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, so that we can exhibit cell modules and irreducible modules for it. We will consider the results above when $R = \mathbb{F}$ that is a field, because we are working with an $\mathbb{F}$-algebra.

Let λ be a partition of n and $\mathcal{H}_n = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. Let $m_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w$. Define M^λ to be the right $\mathcal{H}_n$-module generated by m_λ.

Example 1.3.17. Let $\lambda = (2,2)$. Then $\mathfrak{t}^\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ and $\mathfrak{S}_\lambda = \langle (1,2), (3,4) \rangle \cong \mathfrak{S}_2 \times \mathfrak{S}_2$. So

$$m_\lambda = 1 + T_{(1,2)} + T_{(3,4)} + T_{(1,2)}T_{(3,4)}.$$

Proposition 1.3.18. [Mat99, Proposition 3.3] Suppose that λ is a composition of n and let

$$\mathscr{D}_\lambda = \{d \in \mathfrak{S}_n \mid \mathfrak{t}^\lambda d \text{ is row standard}\}$$

Then $\mathscr{D}_\lambda$ is a complete set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Moreover, if $w \in \mathfrak{S}_n$ and $d \in \mathscr{D}_\lambda$, then $\ell(wd) = \ell(w) + \ell(d)$ and $T_{wd} = T_w T_d$.

Thus, each row standard λ-tableau corresponds to a right coset of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. By Proposition 1.3.18, $d(\mathfrak{t})$ is the unique element of minimal length in the coset $\mathfrak{S}_\lambda d(\mathfrak{t})$. Such coset representatives are called **distinguished coset representatives**.

Corollary 1.3.19. [Mat99, Corollary 3.4] Let λ be a composition of n. Then M^λ is a free $\mathbb{F}$-module with basis

$$\{m_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \text{ is a row standard } \lambda\text{-tableau}\}.$$

We call this basis the **row standard basis** of M^λ.

Hence, we know that $\mathcal{H}_n = M^{(1^n)}$ has a basis indexed by row standard (1^n)-tableaux. Now, we want to transform this basis of $\mathcal{H}_n$ into a cellular basis indexed by pairs of standard tableaux of the same shape.

First of all, we need to introduce an involution on $\mathcal{H}_n$. Let $*$ be the anti-automorphism of $\mathcal{H}_n$ determined by $T_i^* = T_i$ for $i = 1, 2, \ldots, n-1$. Then $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$. Now we define what will become the basis elements in our cellular basis.

Let λ be a partition of n and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ-tableaux. Define

$$m_{\mathfrak{s}\mathfrak{t}} := T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}.$$

Remark 1.3.20. Note, in particular, that

$$m^*_{\mathfrak{s}\mathfrak{t}} = \left(T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}\right)^* = T^*_{d(\mathfrak{t})} m_\lambda T_{d(\mathfrak{s})} = m_{\mathfrak{t}\mathfrak{s}}.$$

Proposition 1.3.21. [Mat99, Proposition 3.16] The Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is free as an $\mathbb{F}$-module with basis

$$\mathcal{M} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some partition } \lambda \text{ of } n\}.$$

We call $\mathcal{M}$ the **Murphy basis** of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.
Let Λ^+ be the partially ordered set of partitions with the dominance order $\trianglerighteq$. Define $\check{\mathcal{H}_n}^\lambda$ to be the $\mathbb{F}$-module with basis

$$\{m_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\mu) \text{ for some partition } \mu \text{ of } n \text{ such that } \mu \triangleright \lambda\}.$$

We can now state the most important result of this section.

Theorem 1.3.22. [Mur92, Mur95] The Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is free as an $\mathbb{F}$-module with basis

$$\mathcal{M} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some partition } \lambda \text{ of } n\}.$$

Moreover, the following hold.

(i) The R-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$, for all $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{M}$, is an anti-isomorphism of $\mathcal{H}_n$.

(ii) Suppose that $h \in \mathcal{H}_n$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then there exist elements $r_\mathfrak{v} \in R$ such that for all $\mathfrak{s} \in \operatorname{Std}(\lambda)$

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \mod \check{\mathcal{H}_n}^\lambda.$$

Consequently, $(\mathcal{M}, \Lambda^+)$ is a cellular basis of $\mathcal{H}_n$.

Looking at the definition of cellular algebra, in our case we have taken $(\Lambda, \geq)$ to be Λ^+, i.e., the set of partitions with the dominance order $\trianglerighteq$, and for each $\lambda \in \Lambda$ we have chosen $\mathcal{T}(\lambda) = \operatorname{Std}(\lambda)$.

Example 1.3.23. Consider $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_3)$. Let $\Lambda^+ = \{(3), (2, 1), (1^3)\}$ be the ordered set of partitions of 3 and let

$$\mathfrak{s} = \boxed{1\,|\,2\,|\,3}, \quad \mathfrak{t} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \mathfrak{u} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \mathfrak{v} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

be the corresponding standard tableaux. Set $\mathcal{T}((3)) = \{\mathfrak{s}\}$, $\mathcal{T}((2,1)) = \{\mathfrak{t},\mathfrak{u}\}$, $\mathcal{T}((1^3)) = \{\mathfrak{v}\}$ and $d(\mathfrak{s}) = d(\mathfrak{t}) = d(\mathfrak{v}) = 1$, $d(\mathfrak{u}) = (2,3)$. Then, by Theorem 1.3.22, $\{m_{\mathfrak{ss}}, m_{\mathfrak{tt}}, m_{\mathfrak{tu}}, m_{\mathfrak{ut}}, m_{\mathfrak{uu}}, m_{\mathfrak{vv}}\}$ is a cellular basis of $\mathcal{H}_3$, where

$$\begin{aligned}
m_{\mathfrak{ss}} &= T^*_{d(\mathfrak{s})} m_{(3)} T_{d(\mathfrak{s})} = 1m_{(3)}1 = 1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1,\\
m_{\mathfrak{tt}} &= T^*_{d(\mathfrak{t})} m_{(2,1)} T_{d(\mathfrak{t})} = 1m_{(2,1)}1 = 1 + T_1,\\
m_{\mathfrak{tu}} &= T^*_{d(\mathfrak{t})} m_{(2,1)} T_{d(\mathfrak{u})} = 1m_{(2,1)}T_2 = (1+T_1)T_2,\\
m_{\mathfrak{ut}} &= T^*_{d(\mathfrak{u})} m_{(2,1)} T_{d(\mathfrak{t})} = T_2m_{(2,1)}1 = T_2(1+T_1),\\
m_{\mathfrak{uu}} &= T^*_{d(\mathfrak{u})} m_{(2,1)} T_{d(\mathfrak{u})} = T_2m_{(2,1)}T_2 = T_2(1+T_1)T_2,\\
m_{\mathfrak{vv}} &= T^*_{d(\mathfrak{v})} m_{(1^3)} T_{d(\mathfrak{v})} = 1m_{(1^3)}1 = 1.
\end{aligned}$$

To exhibit part (ii) of Theorem 1.3.22, we note that:

$$\begin{aligned}
m_{\mathfrak{tt}}T_2 &= (1+T_1)T_2 = m_{\mathfrak{tu}},\\
m_{\mathfrak{ut}}T_2 &= T_2(1+T_1)T_2 = m_{\mathfrak{uu}},
\end{aligned}$$

and more interestingly,

$$\begin{aligned}
m_{\mathfrak{tu}}T_2 &= (1+T_1)T_2T_2\\
&= (1+T_1)(q+(q-1)T_2)\\
&= q(1+T_1) + (q-1)(1+T_1)T_2\\
&= qm_{\mathfrak{tt}} + (q-1)m_{\mathfrak{tu}}
\end{aligned}$$

and,

$$\begin{aligned}
m_{\mathfrak{uu}}T_2 &= T_2(1+T_1)T_2T_2\\
&= T_2(1+T_1)(q+(q-1)T_2)\\
&= qT_2(1+T_1) + (q-1)T_2(1+T_1)T_2\\
&= qm_{\mathfrak{ut}} + (q-1)m_{\mathfrak{uu}}
\end{aligned}$$

As an example of when we require the definition of $\check{\mathcal{H}_n}^{\lambda}$, observe that

$$m_{\mathfrak{vv}}T_1 = T_1 = (1+T_1) - 1 = m_{\mathfrak{tt}} - m_{\mathfrak{vv}} \equiv -m_{\mathfrak{vv}} \mod \check{\mathcal{H}_3}^{(1^3)}.$$

Now that we have a cellular basis for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, we can use the theory of cellular algebras to construct all its irreducible modules. The cell modules of $\mathcal{H}_n$ are called Specht modules.

Definition 1.3.24. Let λ be a partition of n. We define the **Specht module** S^λ to be the right $\mathcal{H}_n$-module generated by $\check{\mathcal{H}_n}^\lambda + m_\lambda$.

Given a standard λ-tableau $\mathfrak{t}$, let $m_\mathfrak{t} := \check{\mathcal{H}_n}^\lambda + m_{\mathfrak{t}^\lambda\mathfrak{t}}$. Then Theorem 1.3.22 gives us the following basis of S^λ and shows that S^λ is isomorphic to the cell module of $\mathcal{H}_n$ indexed by λ.

Proposition 1.3.25. [Mat99, Proposition 3.22] Let λ be a partition of n. Then the Specht module S^λ is free as an $\mathbb{F}$-module with basis $\{m_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$.

An important note is that the Specht module S^λ defined above is isomorphic to the dual of the Dipper and James Specht module $S(\lambda)$ (see [DJ86]) defined in Section 1.2; that is $S^\lambda \cong S(\lambda)^\diamond$ where $^\diamond$ denote the dual of an $\mathcal{H}_n$-module (see [Mat99, Exercise 2.7*(ii)*]). One can check that $S(\lambda)^\diamond \cong S(\lambda')$, so it is necessary to replace λ with λ' when comparing the previous results with those of Dipper and James. In particular, this must be done when comparing the results here for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ with those for $\mathfrak{S}_n$ given in Section 1.2. We will go into more details about this in the case of the Ariki-Koike algebras.

Now translating the notation of the general cellular case, we have a unique symmetric, associative bilinear form $\langle\ ,\ \rangle$ on S^λ which allows us to define $\mathrm{rad}S^\lambda$ and so the right $\mathcal{H}_n$-module

$$D^\lambda := S^\lambda/\mathrm{rad}S^\lambda.$$

By Theorem 1.3.10 we obtain the following result.

Theorem 1.3.26. $\{D^\lambda \mid \lambda$ is a partition of n such that $D^\lambda \neq 0\}$ is a complete set of non-isomorphic irreducible $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$-modules over $\mathbb{F}$.

We want to classify those partitions λ for which $D^\lambda \neq 0$. For clarity, we will provide the two equivalent classification of simple $\mathcal{H}_n$-modules both for the definition of Specht module introduced above and for the one given in [DJ86]. Define the **decomposition number** of $\mathcal{H}_n$

$$d_{\lambda\mu} := [S^\lambda : D^\mu]$$

to be the composition multiplicity of D^μ in S^λ.

For our definition of Specht modules, the partitions λ indexing the $\mathcal{H}_n$-modules $D^\lambda \neq 0$, called e-restricted partitions, are defined as follows.

Definition 1.3.27. A partition λ is e**-restricted** if $\lambda_i - \lambda_{i+1} < e$ for every $i \geq 1$.

Theorem 1.3.28. [Mat99, Theorem 3.43]

$$\{D^\mu \mid \mu \text{ is an } e\text{-restricted partition of } n\}$$

is a complete set of non-isomorphic irreducible $\mathcal{H}_n$-modules. Moreover, if μ is an e-restricted partition of n and λ is a partition of n, then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.

Now, we translate this result in terms of the Dipper and James' definition of Specht modules. Notice that a partition λ is e-regular if and only if its conjugate λ' is e-restricted. Hence, the Dipper and James' classification of the simple $\mathcal{H}_n$-modules can be stated as follows.

Theorem 1.3.29. [DJ86, Theorem 7.6]

$$\{D(\mu) \mid \mu \text{ is an } e\text{-regular partition of } n\}$$

is a complete set of non-isomorphic irreducible $\mathcal{H}_n$-modules. Moreover, if μ is an e-regular partition of n and λ is a partition of n, then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglelefteq \mu$.

Notice that $e = \operatorname{char} \mathbb{F}$ in the case where $q = 1$; consequently, these results do indeed agree with the corresponding results from the representation theory of $\mathfrak{S}_n$ In general, e is taking the place of $p = \operatorname{char} \mathbb{F}$ in the $\mathfrak{S}_n$ case, and the irreducible $\mathcal{H}_n$-modules depend only on e and not on the choice of $\mathbb{F}$ or q.

Finally, recall the notion of a block from the theory of cellular algebras above and Lemma 1.1.16. Then we can state when two Specht modules lie in the same block of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.

Theorem 1.3.30 (The Nakayama conjecture)**.** [DJ87, JM97] Suppose that λ and μ are partitions of n. Then the Specht modules S^λ and S^μ belong to the same block of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ if and only if λ and μ have the same e-core.

The Nakayama conjecture can be stated also in the following way using Theorem 2.7.41 in [JK81] that tells us two partitions of the same integer have the same p-core if and only if they have the same residue content.

Theorem 1.3.31 (The Nakayama conjecture)**.** Suppose that λ and μ are partitions of n. Then the Specht modules S^λ and S^μ belong to the same block of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ if and only if $\operatorname{cont}(\lambda) = \operatorname{cont}(\mu)$.

1.4 The Ariki-Koike algebras

In this section, we introduce the Ariki-Koike algebras and see that most of the previous results about representation theory of the Iwahori-Hecke algebra can be extended to these algebras.

1.4.1 The Ariki-Koike algebras

Let $r \geq 1$ and let $W_{r,n}$ be the complex reflection group $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$. This has a 'Coxeter-like' presentation with generators $s_0, \dots, s_{n-1}$ and relations

$$\begin{aligned} s_0^r &= 1, & \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0, & \\ s_i^2 &= 1, & \text{for } 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, & \text{for } 0 \leq i < j-1 \leq n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & \text{for } 1 \leq i \leq n-2. \end{aligned}$$

Now, we can define the Ariki-Koike algebra as a deformation of the group algebra $\mathbb{F}W_{r,n}$.

Definition 1.4.1. Let $\mathbb{F}$ be a field, and suppose $q, Q_1, \dots, Q_r$ are elements of $\mathbb{F}$, with q non-zero. Let $\boldsymbol{Q} = \{Q_1, \dots, Q_r\}$. The **Ariki-Koike algebra** $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}(W_{r,n})$ of $W_{r,n}$ is defined to be the unital associative $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_0 - Q_1) \cdots (T_0 - Q_r) &= 0, & \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, & \\ (T_i + 1)(T_i - q) &= 0, & \text{for } 1 \leq i \leq n-1, \\ T_i T_j &= T_j T_i, & \text{for } 0 \leq i < j-1 \leq n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq n-2. \end{aligned}$$

For brevity, we may write $\mathcal{H}_{r,n}$ for $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}(W_{r,n})$. Notice that the subalgebra of $\mathcal{H}_{r,n}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.

We define the **quantum characteristic** e of the Ariki-Koike algebra identically to that of the Iwahori-Hecke algebra. Hence, $e \in \{2, 3, 4, \dots\} \cup \{\infty\}$ and so set $I = \mathbb{Z}/e\mathbb{Z}$ (which we identify with $\{0, 1, \dots, e-1\}$) unless $e = \infty$, in which case set $I = \mathbb{Z}$.
Similarly, for $w \in \mathfrak{S}_n$ we set $T_w = T_{i_1} \dots T_{i_k}$ where $s_{i_1} \dots s_{i_k}$ is a reduced expressions for w, and we have the same multiplication formula as given by Proposition 1.3.3.

Definition 1.4.2. For every $k \in \{1, \dots, n\}$, define the elements $L_k \in \mathcal{H}_{r,n}$ by

$$L_k = q^{1-k} T_{k-1} \dots T_1 T_0 T_1 \dots T_{k-1}.$$

Hence we get the following fact.

Theorem 1.4.3. [AK94] The Ariki-Koike algebra $\mathcal{H}_{r,n}$ is free as an $\mathbb{F}$-module with basis

$$\{L_1^{c_1}L_2^{c_2}\dots L_n^{c_n}T_w \mid w \in \mathfrak{S}_n \text{ and } 0 \le c_i < r \text{ for } i = 1, 2, \dots, n\}.$$

We say $\boldsymbol{Q}$ is **q-connected** if, for each $j \in \{1, \dots, r\}$, $Q_j = q^{a_j}$ for some $a_j \in \mathbb{Z}$.

In [DM02], Dipper and Mathas prove that any Ariki-Koike algebra is Morita equivalent to a direct sum of tensor products of smaller Ariki-Koike algebras, each of which has q-connected parameters. Thus, we may assume that we are always working with a Ariki-Koike algebra with each Q_j being an integral power of q. So we assume that we can find an r-tuple of integers $\kappa = (\kappa_1, \dots, \kappa_r)$ such that $Q_j = q^{\kappa_j}$ for each j. We call such κ a **multicharge** of $\mathcal{H}_{r,n}$. If e is finite then we may change any of the κ_j by adding a multiple of e, and we shall still have $Q_j = q^{\kappa_j}$. Thus, for e finite we will consider a multicharge of $\mathcal{H}_{r,n}$ to be

$$(\kappa_1 \mod e, \dots, \kappa_r \mod e).$$

If $e = \infty$, then we have only one possible choice of multicharge κ.

1.4.2 The representation theory of $\mathcal{H}_{\mathbb{F},q,Q}(W_{r,n})$

Here, we give some of the main ideas of the representation theory of the Ariki-Koike algebra $\mathcal{H}_{\mathbb{F},q,\boldsymbol{Q}}(W_{r,n})$. In particular, as for the Iwahori-Hecke algebra the cellular theory helps us again in our aim. Indeed, we exhibit a cellular basis for the Ariki-Koike algebras and then apply the theory of cellular algebras to obtain cell modules and irreducible modules. Most of the results are 'generalised versions' of those for the Iwahori-Hecke algebra $\mathcal{H}_n$. For example, we use multipartitions instead of partitions, and have irreducible modules indexed by Kleshchev multipartitions as opposed to e-restricted partitions.

As for $\mathcal{H}_n$, there is more than one cellular basis for $\mathcal{H}_{r,n}$. The first such basis was given by Graham and Lehrer [GL96]. However, the cellular basis that we use was constructed by Dipper, James and Mathas [DJM98] and, as with the Murphy basis, its basis elements are indexed by pairs of standard tableaux - only now our tableaux correspond to multipartitions. Our definitions and results are taken from [Mat99].

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a multipartition of n. Let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the (row standard) $\boldsymbol{\lambda}$-tableau with the numbers $\{1, 2, \dots, n\}$ entered in order along the rows of first $\mathfrak{t}^{\lambda^{(1)}}$, and then $\mathfrak{t}^{\lambda^{(2)}}$ and so on. Then the row stabilizer of $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$. For each row standard $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$, let $d(\mathfrak{t})$ be the element of $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{t})$. An argument similar to Proposition 1.3.18 shows that $\{d(\mathfrak{t}) \in \mathfrak{S}_n \mid \mathfrak{t} \text{ is row standard}\}$ is a complete set of (distinguished) right coset representatives of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$.

Let $\boldsymbol{\lambda}$ be a multicomposition of n. Let $m_{\boldsymbol{\lambda}} := x_{\boldsymbol{\lambda}} u^+_{\boldsymbol{\lambda}}$ where

$$x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w \text{ and } u^+_{\boldsymbol{\lambda}} = \prod_{s=1}^{r} \prod_{i=1}^{a_s} (L_i - Q_s)$$

with $a_s = |\lambda^{(1)}| + \ldots + |\lambda^{(s-1)}|$ for $s = 1, \ldots, r$. Let $*$ be the anti-automorphism of $\mathcal{H}_{r,n}$ determined by $T_i^* = T_i$ for $0 \leq i \leq n-1$. Then $*$ is an involution and $T_w^* = T_{w^{-1}}$ for every $w \in \mathfrak{S}_n$, $L_k^* = L_k$ and $(h_1 h_2)^* = h_2^* h_1^*$ for every h_1, $h_2 \in \mathcal{H}_{r,n}$.

Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard $\boldsymbol{\lambda}$-tableaux. Define

$$m_{\mathfrak{s}\mathfrak{t}} := T^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}.$$

Note that, again, this agrees with our earlier definition in Subsection 1.3.2 of $m_{\mathfrak{s}\mathfrak{t}}$ when $r = 1$ and λ partition and that $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$. Define $\check{\mathcal{H}}^{\boldsymbol{\lambda}}_{r,n}$ to be the $\mathbb{F}$-module with basis

$$\{m_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\mu}) \text{ for some multipartition } \boldsymbol{\mu} \text{ of } n \text{ such that } \boldsymbol{\mu} \rhd \boldsymbol{\lambda}\}.$$

Theorem 1.4.4. [DJM98] The Ariki-Koike algebra $\mathcal{H}_{r,n}$ is free as an $\mathbb{F}$-module with basis

$$\mathscr{M} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for some multipartition } \boldsymbol{\lambda} \text{ of } n\}.$$

Moreover, the following hold.

(i) The $\mathbb{F}$-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$, for all $m_{\mathfrak{s}\mathfrak{t}} \in \mathscr{M}$, is an anti-isomorphism of $\mathcal{H}_{r,n}$.

(ii) Suppose that $h \in \mathcal{H}_{r,n}$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then there exist elements $r_{\mathfrak{v}} \in \mathbb{F}$ such that for all $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \check{\mathcal{H}}^{\boldsymbol{\lambda}}_{r,n}.$$

Consequently, if $\boldsymbol{\Lambda}^+$ is the set of of multipartitions of n ordered by dominance, then $(\mathscr{M}, \boldsymbol{\Lambda}^+)$ is a cellular basis of $\mathcal{H}_{r,n}$.

The basis $\mathscr{M}$ is often called the **Murphy basis** of $\mathcal{H}_{r,n}$ in the literature.

Now that we have a cellular basis for $\mathcal{H}_{r,n}$, we can use the theory of cellular algebras to construct all its irreducible modules in exactly the same way we did for $\mathcal{H}_n$.

Definition 1.4.5. Let $\boldsymbol{\lambda}$ be a multipartition of n. We define the **Specht module** $S^{\boldsymbol{\lambda}}$ to be the right $\mathcal{H}_{r,n}$-module generated by $\check{\mathcal{H}}^{\boldsymbol{\lambda}}_{r,n} + m_{\boldsymbol{\lambda}}$.

Given a standard $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$, let $m_{\mathfrak{t}} := \check{\mathcal{H}}^{\boldsymbol{\lambda}}_{r,n} + m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}}$. Then Theorem 1.4.4 gives us the following basis of $S^{\boldsymbol{\lambda}}$ and shows that $S^{\boldsymbol{\lambda}}$ is isomorphic to the cell module of $\mathcal{H}_{r,n}$ indexed by $\boldsymbol{\lambda}$.

Proposition 1.4.6. Let $\boldsymbol{\lambda}$ be a multipartition of n. Then the Specht module $S^{\boldsymbol{\lambda}}$ is free as an $\mathbb{F}$-module with basis $\{m_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$.

Now translating again the notation of the general cellular case, we have a unique symmetric, associative bilinear form $\langle\ ,\ \rangle$ on $S^{\boldsymbol{\lambda}}$ which allows us to define rad $S^{\boldsymbol{\lambda}}$ and so the right $\mathcal{H}_{r,n}$-module

$$D^{\boldsymbol{\lambda}} := S^{\boldsymbol{\lambda}}/\mathrm{rad}S^{\boldsymbol{\lambda}}.$$

By Theorem 1.3.10, without too much work, the cellular theory produces the following result.

Theorem 1.4.7. Let $\mathbb{F}$ be a field. Then

1. For each multipartition $\boldsymbol{\lambda}$, $D^{\boldsymbol{\lambda}}$ is either zero or absolutely irreducible.

2. $\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda}$ is a multipartition of n such that $D^{\boldsymbol{\lambda}} \neq 0\}$ is a complete set of non-isomorphic irreducible $\mathcal{H}_{r,n}$-modules over $\mathbb{F}$.

3. If $D^{\boldsymbol{\mu}} \neq 0$, then the composition multiplicity $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] \neq 0$ only if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$; further, $[S^{\boldsymbol{\mu}} : D^{\boldsymbol{\mu}}] = 1$.

In particular, note that every field is a splitting field for $\mathcal{H}_{r,n}$. Define, once again as in Section 1.3.2, the **decomposition numbers** of $\mathcal{H}_{r,n}$ to be

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}} := [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}].$$

Therefore, we have that, by part 3. of Theorem 1.4.7, the decomposition matrix $(d_{\boldsymbol{\lambda}\boldsymbol{\mu}})$ of $\mathcal{H}_{r,n}$ is unitriangular when its rows and column are ordered in a way that is compatible with the dominance order.

1.4.3 Induction and restriction

If $n > 1$, then $\mathcal{H}_{r,n-1}$ is naturally a submodule of $\mathcal{H}_{r,n}$, and in fact $\mathcal{H}_{r,n}$ is free as an $\mathcal{H}_{r,n-1}$-module. So there are well-behaved induction and restriction functors between the module categories of $\mathcal{H}_{r,n-1}$ and $\mathcal{H}_{r,n}$. Given modules M, N for $\mathcal{H}_{r,n-1}$ and $\mathcal{H}_{r,n}$, respectively, we write $M \uparrow^{\mathcal{H}_{r,n}}$ and $N \downarrow_{\mathcal{H}_{r,n-1}}$ for the induced and restricted modules. If B and C are blocks of $\mathcal{H}_{r,n-1}$ and $\mathcal{H}_{r,n}$, respectively, then we may write $M \uparrow^{C}$ and $N \downarrow_{B}$ for the projections of the induced and restricted modules onto B and C.

Theorem 1.4.8. [Mat09, Corollary 3.7; Ari96, Lemma 2.1]

- Suppose $\boldsymbol{\lambda}$ is a multipartition of $n-1$, and let $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$ be the addable nodes of $[\boldsymbol{\lambda}]$. For each $i = 1, \ldots, s$, let $\boldsymbol{\lambda}^{+i}$ be the multipartition of n with $[\boldsymbol{\lambda}^{+i}] = [\boldsymbol{\lambda}] \cup \{\mathfrak{n}_i\}$. Then $S^{\boldsymbol{\lambda}} \uparrow^{\mathcal{H}_{r,n}}$ has a filtration in which the factors are $S^{\boldsymbol{\lambda}^{+1}}, \ldots, S^{\boldsymbol{\lambda}^{+s}}$.

- Suppose $\boldsymbol{\lambda}$ is a multipartition of n, and let $\mathfrak{n}_1, \ldots, \mathfrak{n}_t$ be the removable nodes of $[\boldsymbol{\lambda}]$. For each $i = 1, \ldots, t$, let $\boldsymbol{\lambda}^{-i}$ be the multipartition of $n-1$ with $[\boldsymbol{\lambda}^{-i}] = [\boldsymbol{\lambda}] \setminus \{\mathfrak{n}_i\}$. Then $S^{\boldsymbol{\lambda}} \downarrow_{\mathcal{H}_{r,n-1}}$ has a filtration in which the factors are $S^{\boldsymbol{\lambda}^{-1}}, \ldots, S^{\boldsymbol{\lambda}^{-t}}$.

1.4.4 Weight and hub of multipartitions

Now, we turn to some combinatorial facts about mutipartitions that will be useful in the study of representation theory of Ariki-Koike algebras. In particular, we follow the work of Fayers in [Fay06], generalising the notion of weight and core to multipartitions. As seen at the end of Subsection 1.3.2, the weight and core of a partition λ play an important role in determining the block that S^λ belongs to and its properties. In particular, we saw (Theorem 1.3.30) that two Specht modules S^λ and S^μ belong to the same block if and only if λ and μ have the same core. However, for $r > 1$ the natural generalisation of this is not necessarily true.

In Subsection 1.1.2, we defined the notion of weight and core for partitions of n. We wish to generalise these notions to multipartitions of n. In order to do this we need to introduce the notion of residue also for the multipartitions.

Definition 1.4.9. Let $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ be a multicharge. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a multipartition of n and (b, c, j) be a node of $[\boldsymbol{\lambda}]$. To each node $(b, c, j) \in [\boldsymbol{\lambda}]$ we associate its **residue**

$$\operatorname{res}_{\boldsymbol{a}}(b, c, j) = \begin{cases} c - b + a_j \mod e & \text{if } e = \{2, 3, \ldots\}, \\ c - b + a_j & \text{if } e = \infty. \end{cases}$$

We refer to a node of residue i as an i-node.

Moreover, for e finite, by the definition of the β-numbers in Subsection 1.1.3, the node at the end of the row (if it exists) has residue i if and only if the corresponding bead is on runner i of the abacus. Thus, in each component $\lambda^{(j)}$ of a multipartition $\boldsymbol{\lambda}$, if we increase any β-number by one, this is equivalent to moving a bead from runner i to runner $i+1 \mod e$ which is equivalent to adding a node of residue $i+1$ to the Young diagram of $\lambda^{(j)}$. Similarly, decreasing a β-number by one is equivalent to moving a bead from runner i to runner $i-1 \mod e$ which is equivalent to removing a node of residue i from the Young diagram of $\lambda^{(j)}$.

We give now the definition of weight of a multipartition. Fix $\boldsymbol{a} = (a_1, \ldots, a_r) \in I^r$ a multicharge for $\mathcal{H}_{r,n}$.

Definition 1.4.10. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a multipartition of n. Let $c_i(\boldsymbol{\lambda})$ denote the number of nodes in $[\boldsymbol{\lambda}]$ of residue $i \in I$. Define the **weight** $w(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ to be the integer

$$w(\boldsymbol{\lambda}) = \left(\sum_{j=1}^{r} c_{a_j}(\boldsymbol{\lambda}) \right) - \frac{1}{2} \sum_{i \in I} (c_i(\boldsymbol{\lambda}) - c_{i+1}(\boldsymbol{\lambda}))^2.$$

We shall see later that $w(\boldsymbol{\lambda})$ is a non-negative integer.

Example 1.4.11. Suppose $r = 2$, $(a_1, a_2) = (0, 1)$ and $\boldsymbol{\lambda} = ((2^2), (2, 1))$. If $e = 2$, then the residues of the nodes in $[\boldsymbol{\lambda}]$ are

$$\left(\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 \\ \cline{1-1} \end{array} \right).$$

So we have

$$\begin{aligned} w(\boldsymbol{\lambda}) &= c_{a_1}(\boldsymbol{\lambda}) + c_{a_2}(\boldsymbol{\lambda}) - \frac{1}{2}(c_0(\boldsymbol{\lambda}) - c_1(\boldsymbol{\lambda}))^2 - \frac{1}{2}(c_1(\boldsymbol{\lambda}) - c_0(\boldsymbol{\lambda}))^2 \\ &= 4 + 3 - \frac{1}{2}(1+1) \\ &= 6. \end{aligned}$$

The definition of weight given above generalises the definition of the weight of a partition. In order to justify this assertion, we must show first that it really is a generalisation. Indeed, the following result holds.

Proposition 1.4.12. [Fay06, Proposition 2.1] Suppose $r = 1$. Let λ be a partition, and let $\boldsymbol{\lambda}$ be the multipartition (λ). Then $w(\boldsymbol{\lambda}) = \mathrm{weight}(\lambda)$.

Given a multipartition $\boldsymbol{\lambda}$, it is also useful to define the hub of it. We will see better in the following the importance of this definition. For each $i \in I$ and $j \in \{1, \ldots, r\}$, define

$$\begin{aligned} \delta_i^j(\boldsymbol{\lambda}) =& (\text{the number of removable } i\text{-nodes of } [\lambda^{(j)}]) \\ &- (\text{the number of addable } i\text{-nodes of } [\lambda^{(j)}]), \end{aligned}$$

and put $\delta_i(\boldsymbol{\lambda}) = \sum_{j=1}^{r} \delta_i^j(\boldsymbol{\lambda})$. The collection $(\delta_i(\boldsymbol{\lambda}) \mid i \in I)$ of integers is called the **hub** of $\boldsymbol{\lambda}$.

We summarise now some useful results from [Fay06], mostly concerning weight and hub of a multipartition.

Recall that if λ is a partition, removing an e-rim hook from λ corresponds to reducing the weight of λ by 1. The following result generalises this to the case of multipartitions.

Proposition 1.4.13. [Fay06, Corollary 3.4] Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multipartition, and that $\boldsymbol{\lambda}^-$ is a multipartition obtained from $\boldsymbol{\lambda}$ by removing an e-rim hook from some $\lambda^{(j)}$. Then $w(\boldsymbol{\lambda}) = w(\boldsymbol{\lambda}^-) + r$.

So, using Proposition 1.4.13 we can calculate the difference in weight between a multipartition and the multipartition formed by removing all e-rim hooks from each of its components.

Now, recall the definition of an e-core from Subsection 1.1.2 and define its generalisation to multipartitions.

Definition 1.4.14. An multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a e-**multicore** if $\lambda^{(j)}$ is an e-core for each $j \in \{1, \dots, r\}$.

Note that when $e = \infty$, every multipartition is an e-multicore. Of course, if $r = 1$ an e-multicore is an e-core, and has weight 0. But when $r \geq 2$, calculating the weight of a multicore is non-trivial. The next result shows us how to reduce the calculation of weight for a multicore to the case $r = 2$.

Proposition 1.4.15. [Fay06, Proposition 3.5] Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multicore. Then

$$w(\boldsymbol{\lambda}) = \sum_{1 \leq j < k \leq r} w((\lambda^{(j)}, \lambda^{(k)})).$$

Example 1.4.16. Suppose $r = 3$, $(a_1, a_2, a_3) = (1, 0, 2)$, and $\boldsymbol{\lambda} = ((1^2), (2), (2, 1))$. If $e = 4$, then the 4-residue diagram of $[\boldsymbol{\lambda}]$ is

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} \right).$$

We can calculate, applying Definition 1.4.10,

$$\begin{aligned} w((\lambda^{(1)}, \lambda^{(2)})) &= 0, \\ w((\lambda^{(1)}, \lambda^{(3)})) &= 2, \\ w((\lambda^{(2)}, \lambda^{(3)})) &= 1. \end{aligned}$$

Thus, by Proposition 1.4.15, $w(\boldsymbol{\lambda}) = 3$.

Suppose e is finite, $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multicore and $\boldsymbol{a} = (a_1, \dots, a_r)$ is a multicharge of $\mathcal{H}_{r,n}$. We construct the corresponding abacus display for $\boldsymbol{\lambda}$ as in Section 1.1.3, and then for each $i \in I$ and $1 \leq j \leq r$ we define $\mathfrak{b}^{\boldsymbol{a}}_{ij}(\boldsymbol{\lambda})$ to be the position of the lowest bead on runner i of the abacus for $\lambda^{(j)}$; that is, the largest element of $B^j_{a_j}$ congruent to i modulo e. When $e = \infty$, we define

$$\mathfrak{B}_{ij}(\boldsymbol{\lambda}) = \begin{cases} 1 & \text{if } i \in B^j_{a_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, fix $e \in \{2,3,4,\ldots\} \cup \{\infty\}$. Let $i \in I$ and $j,k \in \{1,\ldots,r\}$, we define

$$\gamma_i^{jk}(\boldsymbol{\lambda}) = \begin{cases} \frac{1}{e}(\mathfrak{b}_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda}) - \mathfrak{b}_{ik}^{\boldsymbol{a}}(\boldsymbol{\lambda})) & \text{if } e < \infty, \\ \mathfrak{B}_{ij}(\boldsymbol{\lambda}) - \mathfrak{B}_{ik}(\boldsymbol{\lambda}) & \text{if } e = \infty. \end{cases}$$

$\gamma_i^{jk}(\boldsymbol{\lambda})$ may then be regarded as the difference in height between the lowest bead on runner i of the abacus display for $\lambda^{(j)}$ and the lowest bead on runner i of the abacus display for $\lambda^{(k)}$. If e is finite, then the integers $\gamma_i^{jk}(\boldsymbol{\lambda})$ depend on the choice of $\boldsymbol{a}$ if we change any a_j by a multiple of e, but the differences

$$\gamma_{il}^{jk}(\boldsymbol{\lambda}) := \gamma_i^{jk}(\boldsymbol{\lambda}) - \gamma_l^{jk}(\boldsymbol{\lambda})$$

do not.

Now suppose $\boldsymbol{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(r)})$ is a multicore, $i,l \in I$ and $j,k \in \{1,\ldots,r\}$. If $e = \infty$ suppose also that $\gamma_{il}^{jk}(\boldsymbol{\lambda}) = 2$. We define $s_{il}^{jk}(\boldsymbol{\lambda})$ to be the multicore whose abacus configuration is obtained from that of $\boldsymbol{\lambda}$ by moving a bead from runner i to runner l on the abacus for $\lambda^{(j)}$, and moving a bead from runner l to runner i on the abacus for $\lambda^{(k)}$. It is worth noting that $s_{il}^{jk}(\boldsymbol{\lambda}) = s_{li}^{kj}(\boldsymbol{\lambda})$ for all i,l,j,k. Moreover, the following holds.

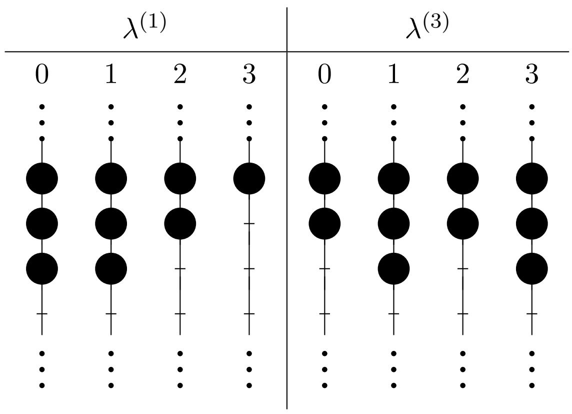
Proposition 1.4.17. [Fay07b, Proposition 1.6] Let $\boldsymbol{\lambda}$ be a multicore and let $s_{il}^{jk}(\boldsymbol{\lambda})$ be defined as above. Then

1. $s_{il}^{jk}(\boldsymbol{\lambda})$ has the same hub as $\boldsymbol{\lambda}$, and,
2. $w(s_{il}^{jk}(\boldsymbol{\lambda})) = w(\boldsymbol{\lambda}) - r(\gamma_{il}^{jk}(\boldsymbol{\lambda}) - 2)$.

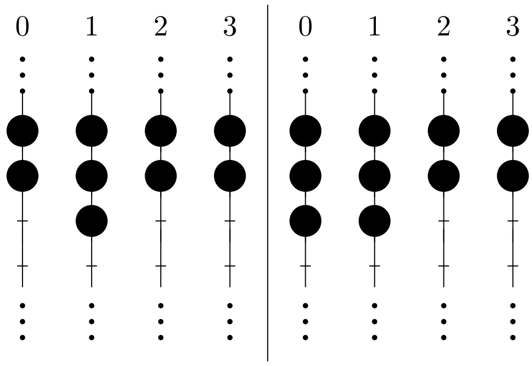
Example 1.4.18. Recalling the multipartition $\boldsymbol{\lambda}$ from last example, we examine the multicore $(\lambda^{(1)},\lambda^{(3)}) = ((1^2),(2,1))$. So we are in the case $r = 2$, $(a_1,a_2) = (1,2)$. If $e = 4$, then the residue diagram of $[(\lambda^{(1)},\lambda^{(3)})]$ is

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right).$$

An abacus display for $(\lambda^{(1)},\lambda^{(3)})$ is



We may read off $\gamma_0^{12} = 1$, $\gamma_1^{12} = 0$, $\gamma_2^{12} = 0$ and $\gamma_3^{12} = -2$; in particular $\gamma_{03}^{12} = 3$. Then the abacus display for $s_{03}^{12}((\lambda^{(1)}, \lambda^{(3)}))$ is



and we have

$$\begin{aligned} w(s_{03}^{12}((\lambda^{(1)}, \lambda^{(3)}))) &= w((\lambda^{(1)}, \lambda^{(3)})) - 2(\gamma_{03}^{12}((\lambda^{(1)}, \lambda^{(3)})) - 2) \\ &= w((\lambda^{(1)}, \lambda^{(3)})) - 2(3 - 2) \\ &= w((\lambda^{(1)}, \lambda^{(3)})) - 2. \end{aligned}$$

Using Proposition 1.4.17, in the case $r = 2$ we may reduce the calculation of the weight of a multicore $\boldsymbol{\lambda}$ to the case where we have $\gamma_{il}^{jk}(\boldsymbol{\lambda}) \leq 2$ because if $\gamma_{il}^{jk}(\boldsymbol{\lambda}) \geq 3$, we can obtain $w(\boldsymbol{\lambda})$ from $w(s_{il}^{jk}(\boldsymbol{\lambda}))$ inductively. The following result tells us how to find the weight in this case for $r = 2$.

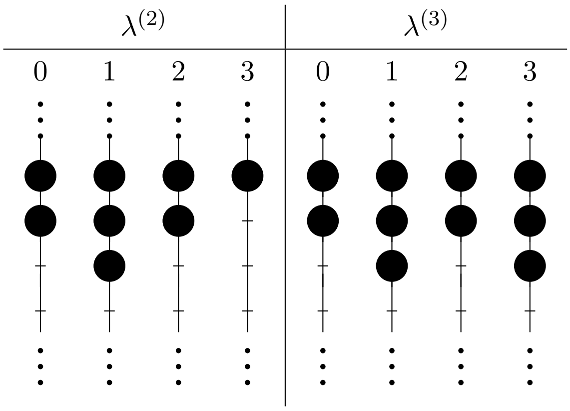
Proposition 1.4.19. [Fay07b, Proposition 1.7] Suppose that $r = 2$ and λ is a multicore.

1. If $\gamma_{il}^{12}(\boldsymbol{\lambda}) \leq 2$ for all i, l, then $w(\boldsymbol{\lambda})$ is the smaller of the two integers

$$\#\{i \mid \gamma_{il}^{12}(\boldsymbol{\lambda}) = 2 \text{ for some } l\} \text{ and } \#\{l \mid \gamma_{il}^{12}(\boldsymbol{\lambda}) = 2 \text{ for some } i\}.$$

2. $w(\boldsymbol{\lambda}) = 0$ if and only if $\gamma_{il}^{12}(\boldsymbol{\lambda}) \leq 1$ for all i, l.

Example 1.4.20. Returning to the example $\boldsymbol{\lambda} = ((1^2), (2), (2, 1))$, we calculate the weight of $(\lambda^{(2)}, \lambda^{(3)})$ when $e = 4$. An abacus display for $(\lambda^{(2)}, \lambda^{(3)})$ is



We may read off $\gamma_0^{12} = 0$, $\gamma_1^{12} = 0$, $\gamma_2^{12} = 0$ and $\gamma_3^{12} = -2$. So $\gamma_{il}^{12}((\lambda^{(2)}, \lambda^{(3)})) \leq 2$ for all i, l. Then, using Proposition 1.4.19, we have

$$\#\{i \mid \gamma_{il}^{12} = 2 \text{ for some } l\} = \#\{0, 1, 2\} = 3$$

and

$$\#\{i \mid \gamma_{il}^{12} = 2 \text{ for some } l\} = \#\{3\} = 1.$$

Hence, $w((\lambda^{(2)}, \lambda^{(3)})) = 1$.

So now we have an algorithm allowing us to calculate the weight of a multipartition given its abacus configuration.

1. If necessary, slide all beads up their runners as far as they will go, and use Proposition 1.4.13 to calculate the change in weight.
2. For $j < k$, calculate the weight of $\boldsymbol{\lambda}_{jk} = (\lambda^{(j)}, \lambda^{(k)})$:
   (a) Calculate $\gamma_i^{12}(\boldsymbol{\lambda}_{jk})$;
   (b) If there is a choice of i and l such that $\gamma_{il}^{12}(\boldsymbol{\lambda}_{jk}) \geq 3$, replace $\boldsymbol{\lambda}_{jk}$ with $s_{il}^{12}(\boldsymbol{\lambda}_{jk})$ and use Proposition 1.4.17 to calculate the change of weight. Repeat this step until $\gamma_{il}^{12}(s_{il}^{12}(\boldsymbol{\lambda}_{jk})) \leq 2$ for all i and l.
   (c) Use Proposition 1.4.19 to calculate $w(\boldsymbol{\lambda}_{jk})$.
3. Finally, add together all the $w(\boldsymbol{\lambda}_{jk})$ and use Proposition 1.4.15.

This enables us to prove the following result, which gives us further reassurance that our definition of weight is an appropriate generalisation of the weight of a partition.

Corollary 1.4.21. [Fay06, Corollary 3.9] Let $\boldsymbol{\lambda}$ be a multipartition. Then $w(\boldsymbol{\lambda})$ is a non-negative integer.

1.4.5 Kleshchev multipartitions

Residues of nodes are also useful in classifying the simple $\mathcal{H}_{r,n}$-modules. Indeed, the notion of residue helps us to describe a certain subset $\mathcal{K}$ of the set of all multipartitions, which index the simple modules for $\mathcal{H}_{r,n}$. We impose a partial order $>$ on the set of nodes of residue $i \in I$ of a multipartition by saying that (b, c, j) is **above** (b', c', j') (or (b', c', j') is **below** (b, c, j)) if either $j < j'$ or ($j = j'$ and $b < b'$). In this case we write $(b, c, j) > (b', c', j')$. Note this order restricts to a total order on the set of all addable and removable nodes of residue $i \in I$ of a multipartition.

Suppose $\boldsymbol{\lambda}$ is a multipartition, and given $i \in I$ define the ***i*-signature of $\boldsymbol{\lambda}$ with respect to** $>$ by examining all the addable and removable i-nodes of $\boldsymbol{\lambda}$

in turn from higher to lower, and writing a $+$ for each addable node of residue i and a $-$ for each removable node of residue i. Now construct the **reduced** i**-signature** by successively deleting all adjacent pairs $-+$. If there are any $-$ signs in the reduced i-signature of $\boldsymbol{\lambda}$, the corresponding removable nodes are called **normal nodes** of $[\boldsymbol{\lambda}]$. The leftmost normal node is called a **good node of** $[\boldsymbol{\lambda}]$ **with respect to** $>$.

Definition 1.4.22. We say that $\boldsymbol{\lambda}$ is **Kleshchev** if and only if there is a sequence

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(n), \boldsymbol{\lambda}(n-1), \dots, \boldsymbol{\lambda}(0) = \boldsymbol{\varnothing}$$

of multipartitions such that for each k, $[\boldsymbol{\lambda}(k-1)]$ is obtained from $[\boldsymbol{\lambda}(k)]$ by removing a good node with respect to the order $>$.

This definition depends on the multicharge $\boldsymbol{a} = (a_1, \dots, a_r)$ of $\mathcal{H}_{r,n}$, and we may use the term '$(a_1, \dots, a_r)$-Kleshchev' if there is danger of ambiguity. We write $\mathcal{K}(a_1, \dots, a_r)$ for the set of $(a_1, \dots, a_r)$-Kleshchev multipartitions.

Example 1.4.23. Suppose $r = 2$ and $e = 4$. Consider the multicharge $\boldsymbol{a} = (1, 0)$ of $\mathcal{H}_{2,6}$. Then the multipartition $\boldsymbol{\lambda} = (\varnothing, (1^6))$ is Kleshchev. Indeed, we have the following sequence of multipartitions obtained from $\boldsymbol{\lambda}$ by removing each time a good node - we write $[\boldsymbol{\lambda}(k)] \overset{i}{\leftarrow}_{>} [\boldsymbol{\lambda}(k-1)]$ to denote that $[\boldsymbol{\lambda}(k-1)]$ is obtained from $[\boldsymbol{\lambda}(k)]$ by removing a good i-node for $i \in I$ with respect to the total order $>$:

$$\left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} 1 & \\ \cline{1-1} 0 & \\ \cline{1-1} 3 & \\ \cline{1-1} \multicolumn{1}{c}{2} & \end{array}\right) \overset{3}{\leftarrow}_{>} \left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} 1 & \\ \cline{1-1} 0 & \\ \cline{1-1} \multicolumn{1}{c}{3} & \end{array}\right) \overset{0}{\leftarrow}_{>} \left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} 1 & \\ \cline{1-1} \multicolumn{1}{c}{0} & \end{array}\right) \overset{1}{\leftarrow}_{>}$$

$$\left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \multicolumn{1}{c}{1} & \end{array}\right) \overset{2}{\leftarrow}_{>} \left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} 3 & \\ \cline{1-1} \multicolumn{1}{c}{2} & \end{array}\right) \overset{3}{\leftarrow}_{>} \left(1,\ \begin{array}{|c|l} \cline{1-1} 0 & 1 \\ \cline{1-1} \multicolumn{1}{c}{3} & \end{array}\right) \overset{0}{\leftarrow}_{>} (\varnothing, \varnothing).$$

Remark 1.4.24. [Fay08b] If $r = 1$, then the multipartition (λ) is Kleshchev if and only if λ is e-restricted.

The importance of Kleshchev multipartitions lies in the fact (proved by Ariki [Ari01, Theorem 4.2]) that if $\boldsymbol{\lambda}$ is Kleshchev, then $S^{\boldsymbol{\lambda}}$ has an irreducible cosocle $D^{\boldsymbol{\lambda}}$, and the set $\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda}$ is a Kleshchev multipartition$\}$ is a complete set of non-isomorphic simple $\mathcal{H}_{r,n}$-modules.

As for Iwahori-Hecke algebras, also for the Ariki-Koike algebras there are two different but equivalent classifications of the simple modules of $\mathcal{H}_{r,n}$. One is the classification we presented here following [Mat04] for which a complete set of non-isomorphic simple $\mathcal{H}_{r,n}$-modules is given by

$$\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \text{ is a Kleshchev multipartition}\}.$$

In this case, as stated in Remark 1.4.24, for $r = 1$ a Kleshchev multipartition is an e-restricted partition.

This will be the setting in which we will work in Chapter 2.

The other classification is the one arising when we define as Specht module for the Ariki-Koike algebra $\mathcal{H}_{r,n}$ the so called **dual Specht module** $S'(\boldsymbol{\lambda})$. The module $S'(\boldsymbol{\lambda})$ is defined in [Mat03] to be the cell module arising from a cellular basis $\{n_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_{r,n}$. The details of this definition can be found in [Mat03, §4].

This will be the setting in which we will work in Chapter 3.

In order to give a classification of the simple modules for $\mathcal{H}_{r,n}$ with the above definition of Specht modules we need to introduce some definitions.

We can define another partial order $\succ$ on the set of nodes of residue $i \in I$ of a multipartition by saying that (b, c, j) is **above** (b', c', j') (or (b', c', j') is **below** (b, c, j)) if either $j > j'$ or ($j = j'$ and $b > b'$). In this case we write $(b, c, j) \succ (b', c', j')$. Note again that also this order restricts to a total order on the set of all addable and removable nodes of residue $i \in I$ of a multipartition.

Suppose $\boldsymbol{\lambda}$ is a multipartition, and given $i \in I$ define the **i-signature of $\boldsymbol{\lambda}$ with respect to $\succ$** by examining all the addable and removable i-nodes of $\boldsymbol{\lambda}$ in turn from higher to lower, and writing a $+$ for each addable node of residue f and a $-$ for each removable node of residue f. Now construct the **reduced i-signature with respect to $\succ$** by successively deleting all adjacent pairs $-+$. If there are any $-$ signs in the reduced f-signature of $\boldsymbol{\lambda}$, the leftmost of these nodes is called a **good node of $[\boldsymbol{\lambda}]$ with respect to $\succ$**.

Definition 1.4.25. We say that $\boldsymbol{\lambda}$ is a **dual Kleshchev multipartition** if and only if there is a sequence

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(n), \boldsymbol{\lambda}(n-1), \ldots, \boldsymbol{\lambda}(0) = \boldsymbol{\varnothing}$$

of multipartitions such that for each k, $[\boldsymbol{\lambda}(k-1)]$ is obtained from $[\boldsymbol{\lambda}(k)]$ by removing a good node with respect to $\succ$.

We write $\mathcal{K}'(a_1, \ldots, a_r)$ for the set of dual Kleshchev multipartitions with multicharge $(a_1, \ldots, a_r)$.

Hence, with the same notation of [Mat03], when we define the Specht modules as $S'(\boldsymbol{\lambda})$, we set $D'(\boldsymbol{\lambda}) = S'(\boldsymbol{\lambda})/\text{rad } S'(\boldsymbol{\lambda})$ with rad $S'(\boldsymbol{\lambda})$ the radical of the bilinear form of $S'(\boldsymbol{\lambda})$ (the form is defined in terms of the structural constant of the basis $\{n_{\mathfrak{s}\mathfrak{t}}\}$). In this case, a complete set of non-isomorphic simple $\mathcal{H}_{r,n}$-modules is given by

$$\{D'(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \text{ is a dual Kleshchev multipartition}\}.$$

Now, we want to clarify the link between these two classifications of simple modules of $\mathcal{H}_{r,n}$.

Following [Fay08b], we consider the Ariki-Koike algebra $\mathcal{H}'_{r,n}$ with standard generators $T'_0, \ldots, T'_{n-1}$ and parameters $(q^{-1}, Q_r, \ldots, Q_1)$. Hence, if $\boldsymbol{a} = (a_1, \ldots, a_r)$ is a multicharge for $\mathcal{H}_{r,n}$, then $-\boldsymbol{a} = (-a_r, \ldots, -a_1)$ is a multicharge for $\mathcal{H}'_{r,n}$. If $\boldsymbol{\lambda}$ is a Kleshchev multipartition of $\mathcal{H}'_{r,n}$, we write $\boldsymbol{\lambda} \in \mathcal{K}(-a_r, \ldots, -a_1)$. Moreover, we have a bijection $^\diamond$ from $\mathcal{K}(a_1, \ldots, a_r)$ to $\mathcal{K}(-a_r, \ldots, -a_1)$, with the properties that

- $\boldsymbol{\varnothing}^\diamond = \boldsymbol{\varnothing}$, and
- if $\boldsymbol{\lambda}$ is a multipartition with a good i-node $\mathfrak{n}$, then $\boldsymbol{\lambda}^\diamond$ has a good $-i$-node $\mathfrak{m}$, and $(\boldsymbol{\lambda} \setminus \{\mathfrak{n}\})^\diamond = \boldsymbol{\lambda}^\diamond \setminus \{\mathfrak{m}\}$.

This is, indeed, a bijection between the two crystal graphs with vertices the sets $\mathcal{K}(a_1, \ldots, a_r)$ and $\mathcal{K}(-a_r, \ldots, -a_1)$, under which the label of arrows are negated. The bijection $^\diamond$ may be viewed as a generalisation of the Mullineux involution [Mul79]. See [Fay08b] for details.

Example 1.4.26. Suppose $r = 2$ and $e = 4$. Consider the multicharge $\boldsymbol{a} = (1, 0)$ of $\mathcal{H}_{2,6}$ and the $(1, 0)$-Kleshchev multipartition $\boldsymbol{\lambda} = (\varnothing, (1^6))$. Then - see Example 1.4.23 - there is a sequence

$$\boldsymbol{\lambda} \overset{3}{\leftarrow\!\!>} (\varnothing, (1^5)) \overset{0}{\leftarrow\!\!>} (\varnothing, (1^4)) \overset{1}{\leftarrow\!\!>} (\varnothing, (1^3)) \overset{2}{\leftarrow\!\!>} (\varnothing, (1^2)) \overset{3}{\leftarrow\!\!>} (\varnothing, (1)) \overset{0}{\leftarrow\!\!>} (\varnothing, \varnothing).$$

We want to apply $^\diamond$ to $\boldsymbol{\lambda}$. By definition of $^\diamond$, we have that $\boldsymbol{\lambda}^\diamond$ is the $(0, 3)$-Kleshchev multipartition of $\mathcal{H}'_{2,6}$ obtained from the empty multipartition $\boldsymbol{\varnothing}$ by adding a $-i$-node if $[\boldsymbol{\lambda}(k)] \overset{i}{\leftarrow\!\!>} [\boldsymbol{\lambda}(k-1)]$ starting from the last good i-node removed in the above sequence. Thus, we get

$$\boldsymbol{\lambda}^\diamond = ((3), (3)) = \left(\boxed{0}\boxed{1}\boxed{2}, \quad \boxed{3}\boxed{0}\boxed{1}\right).$$

We collect together some basic facts on conjugation in the following lemmas. We will use these properties in order to give a combinatorial description of the dual Kleshchev multipartitions.

Lemma 1.4.27. [Fay08b, Lemma 1.2]

1. If $\mathfrak{m}$ and $\mathfrak{n}$ are two nodes with $\mathfrak{m}$ above $\mathfrak{n}$ with respect to $>$, then $\mathfrak{m}'$ lies below $\mathfrak{n}'$ with respect to $>$.
2. If $\mathfrak{n}$ is a node of $\boldsymbol{\lambda}$ with residue i, then $\mathfrak{n}'$ is a node of $\boldsymbol{\lambda}'$ with residue $-i$.

3. If $\mathfrak{n}$ is an addable (respectively, a removable) node of $\boldsymbol{\lambda}$ with residue i, then $\mathfrak{n}'$ is an addable (respectively, a removable) node of $\boldsymbol{\lambda}'$ with residue $-i$.

Lemma 1.4.28. If $\mathfrak{m}$ and $\mathfrak{n}$ are two nodes with $\mathfrak{m}$ above $\mathfrak{n}$ with respect to $>$, then $\mathfrak{m}'$ lies above $\mathfrak{n}'$ with respect to $\succ$.

Proof. This follows directly from the definition of conjugate of a multipartition and the definition of the two orders $>$ and $\succ$. □

Proposition 1.4.29. The map

$$\xi : \mathcal{K}(a_1, \ldots, a_r) \to \mathcal{K}'(a_1, \ldots, a_r)$$
$$\boldsymbol{\lambda} \mapsto (\boldsymbol{\lambda}^\diamond)'$$

is a bijection such that for any two multipartitions $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{K}(a_1, \ldots, a_r)$,

$$\boldsymbol{\mu} \overset{i}{\longleftarrow}_{>} \boldsymbol{\nu} \iff (\boldsymbol{\mu}^\diamond)' \overset{i}{\longleftarrow}_{\succ} (\boldsymbol{\nu}^\diamond)'. \tag{1.4.1}$$

Proof. Consider a multipartition $\boldsymbol{\mu} \in \mathcal{K}(a_1, \ldots, a_r)$. Then, by definition of $^\diamond$, we have that

$$\boldsymbol{\mu} \overset{i}{\longleftarrow}_{>} \boldsymbol{\nu} \iff \boldsymbol{\mu}^\diamond \overset{-i}{\longleftarrow}_{>} \boldsymbol{\nu}^\diamond$$

for some $i \in I$. So, we want to prove that

$$\boldsymbol{\mu}^\diamond \overset{-i}{\longleftarrow}_{>} \boldsymbol{\nu}^\diamond \iff (\boldsymbol{\mu}^\diamond)' \overset{i}{\longleftarrow}_{\succ} (\boldsymbol{\nu}^\diamond)'. \tag{1.4.2}$$

By Lemma 1.4.27, we know that $\gamma = (b, c, j)$ is a removable (respectively, addable) i-node if and only if $\gamma' = (c, b, r + 1 - j)$ is a removable (respectively, addable) $(-i)$-node. Let w_i be the i-signature of $\boldsymbol{\mu}^\diamond$ with respect to $>$. Similarly, let w'_{-i} be the $(-i)$-signature of $(\boldsymbol{\mu}^\diamond)'$ with respect to $\succ$. Write $w_i = \gamma_1 \cdots \gamma_s$ where for any $m = 1, \ldots, s$, γ_m is an i-node that is addable or removable. Then by definition of the orders $>$ and $\succ$, we have

$$w'_{-i} = \gamma'_1 \cdots \gamma'_s.$$

Write $\widetilde{w}_i = (+)^p(-)^t$ for the reduced i-signature of $\boldsymbol{\mu}^\diamond$ with respect to $>$. We deduce from Lemma 1.4.28 that $\widetilde{w}'_{-i} = (+')^p(-')^t$ coincides with the reduced $(-i)$-signature with respect to $\succ$. In particular, γ is the good i-node for $\boldsymbol{\mu}^\diamond$ with respect to $>$ if and only if γ' is the good $(-i)$-node for $(\boldsymbol{\mu}^\diamond)'$ with respect to $\succ$. Hence, the bijection ξ satisfies (1.4.1). □

Remark 1.4.30. Notice that Proposition 1.4.29 implies that a dual Kleshchev multipartition for $\mathcal{H}_{r,n}$ is a multipartition that lies in the set

$$\{(\boldsymbol{\mu}^\diamond)' \mid \boldsymbol{\mu} \in \mathcal{K}(a_1, \ldots, a_r)\}.$$

In this case, for $r = 1$ a dual Kleshchev multipartition is an e-regular partition. Indeed, if $r = 1$ then $\boldsymbol{\mu} = (\mu) \in \mathcal{K}(a_1)$ is an e-restricted partition by Remark 1.4.24. By definition of $^\diamond$, $(\mu)^\diamond = \mu^\diamond \in \mathcal{K}(-a_1)$ and so it is e-restricted as well and $\mu^\diamond$ is e-restricted if and only if $(\mu^\diamond)'$ is e-regular.

In particular, we have the following result that explains why in the literature (see [Fay10]), we often find stated that the dual Kleshchev multipartitions are the conjugate of the Kleshchev multipartions.

Lemma 1.4.31. If $\boldsymbol{\lambda} \in \mathcal{K}(a_1, \dots, a_r)$, then $\boldsymbol{\lambda}' \in \mathcal{K}'(-a_r, \dots, -a_1)$.

Proof. Let $\boldsymbol{\lambda} \in \mathcal{K}(a_1, \dots, a_r)$. As in the proof of Proposition 1.4.29, γ is the good i-node for $\boldsymbol{\lambda}$ with respect to $>$ if and only if γ' is the good $(-i)$-node for $\boldsymbol{\lambda}'$ with respect to $\succ$. Hence, $\boldsymbol{\lambda}' \in \mathcal{K}'(-a_r, \dots, -a_1)$. □

1.4.6 Blocks of Ariki-Koike algebras

It follows from the cellularity of $\mathcal{H}_{r,n}$ that each Specht module $S^{\boldsymbol{\lambda}}$ lies in one block of $\mathcal{H}_{r,n}$, and we abuse notation by saying that a multipartition $\boldsymbol{\lambda}$ lies in a block B if $S^{\boldsymbol{\lambda}}$ lies in B. On the other hand, each block contains at least one Specht module, so in order to classify the blocks of $\mathcal{H}_{r,n}$, it suffices to describe the corresponding partition of the set of multipartitions.

Recall the concept of a block from the theory of cellular algebras as defined in 1.3.13. Hence, we have the following classification of the blocks of $\mathcal{H}_{r,n}$.

Theorem 1.4.32. [LM07, Theorem 2.11] Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions of n. Then, $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ lie in the same block of $\mathcal{H}_{r,n}$ if and only if $c_i(\boldsymbol{\lambda}) = c_i(\boldsymbol{\mu})$ for all $i \in I$.

Moreover, we can notice that an important feature of the weight and hub of a multipartition is that they are invariants of the block containing $\boldsymbol{\lambda}$, and in fact determine this block.

Proposition 1.4.33. [Fay06, Proposition 3.2 & Lemma 3.3] Suppose $\boldsymbol{\lambda}$ is a multipartition of n and $\boldsymbol{\mu}$ is a multipartition of m. Then:

1. if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub, then $m \equiv n \mod e$, and
$$w(\boldsymbol{\lambda}) - w(\boldsymbol{\mu}) = \frac{r(n-m)}{e};$$
2. if $n = m$, then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same block of $\mathcal{H}_{r,n}$ if and only if they have the same hub.

In view of this result, we may define the *hub of a block* B to be the hub of any multipartition $\boldsymbol{\lambda}$ in B, and we write $\delta_i(B) = \delta_i(\boldsymbol{\lambda})$.

1.4.7 Scopes isometries

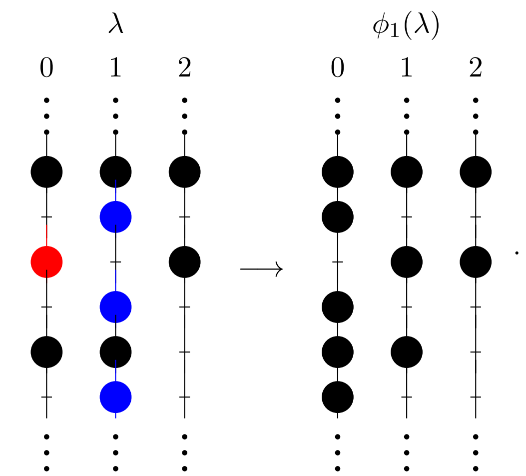
Here we introduce maps between blocks of Ariki-Koike algebras analogous to those defined by Scopes [Sco91] between blocks of symmetric groups. Suppose $i \in \mathbb{Z}/e\mathbb{Z}$, and let $\phi_i \colon \mathbb{Z} \to \mathbb{Z}$ be the map given by

$$\phi_i(x) = \begin{cases} x+1 & x \equiv i-1 \mod e \\ x-1 & x \equiv i \mod e \\ x & \text{otherwise.} \end{cases}$$

If e is finite, then ϕ_i descends to give a bijection from I to I; we abuse notation by referring to this map as ϕ_i also.

Now suppose $\boldsymbol{\lambda}$ is a multipartition, and that we have chosen an abacus display for $\boldsymbol{\lambda}$. For each j, we define a partition $\phi_i(\lambda^{(j)})$ by replacing each β-number β with $\phi_i(\beta)$. Equivalently, we simultaneously remove all removable i-nodes from $[\lambda^{(j)}]$ and add all addable i-nodes of $[\lambda^{(j)}]$, or in terms of abacus configuration we swap the runners $(i-1)$ and i of each abacus in the abacus display of $\boldsymbol{\lambda}$. If $i = 0$, we rearrange the order of the runners in $\boldsymbol{\lambda}$ so that each runner $e-1$ is to the left of each runner 0 and we add one bead to each runner $e-1$ so that the abacus display still represents the same multipartition $\boldsymbol{\lambda}$. We define $\Phi_i(\boldsymbol{\lambda})$ to be the multipartition $(\phi_i(\lambda^{(1)}), \dots, \phi_i(\lambda^{(r)}))$.

Example 1.4.34. Here we give an example of the definition of ϕ_i for a partition. For $e = 3$, consider the abacus configuration of the partition λ given below and apply to λ the map ϕ_1. Then,



So, it can be seen that the removal of removable 1-nodes (represented by beads in blue) and the addition of addable 1-nodes (represented by beads in red) for λ due to the application of ϕ_1 is simultaneous.

Proposition 1.4.35. [Fay06, Proposition 4.6] Suppose B is a block of $\mathcal{H}_{r,n}$ and $i \in I$. Then there is a block $\bar{B}$ of $\mathcal{H}_{r,n-\delta_i(B)}$ with the same weight as B.

Moreover, Φ_i gives a bijection between the set of multipartitions in B and the set of multipartitions in $\bar{B}$.

We write $\Phi_i(B)$ for the block $\bar{B}$ described in Proposition 1.4.35.

1.4.8 Core blocks of Ariki-Koike algebras

Following the work of Fayers in [Fay07b], we want to generalise the notion of core to multipartitions and so we introduce core blocks of Ariki-Koike algebras, giving several equivalent definitions.

In order to introduce core blocks, we need to consider separately the case $e = \infty$, in this case, every block of $\mathcal{H}_{r,n}$ will be a core block. For the case where e is finite, the definition is given by the equivalent statements in the following theorem. It is straightforward to check that these statements, appropriately rephrased, all hold for every block of $\mathcal{H}_{r,n}$ when $e = \infty$, with property (4) following from Proposition 1.4.33.

Theorem 1.4.36. Suppose that e is finite, and that $\boldsymbol{\lambda}$ is a multipartition lying in a block B of $\mathcal{H}_{r,n}$. Let κ be a multicharge for $\mathcal{H}_{r,n}$. The following are equivalent.

1. $\boldsymbol{\lambda}$ is a multicore, and there exist a multicharge $\boldsymbol{a} = (a_1, \dots, a_r)$ such that $a_j \equiv \kappa_j \bmod e$ for all j and integers $\alpha_0, \dots, \alpha_{e-1}$ such that for each i, j, $\mathfrak{b}^{\boldsymbol{a}}_{ij}(\boldsymbol{\lambda})$ equals either α_i or $\alpha_i + e$.

2. $\boldsymbol{\lambda}$ is a multicore, and there exist a multicharge $\boldsymbol{a} = (a_1, \dots, a_r)$ such that $a_j \equiv \kappa_j \bmod e$ for all j and integers $s_1, \dots, s_r$ such that
$$\frac{\mathfrak{b}^{\boldsymbol{a}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\boldsymbol{a}}_{ik}(\boldsymbol{\lambda})}{e} \leq s_j - s_k + 1$$
for all $i \in \{0, \dots, e-1\}$, $j, k \in \{1, \dots, r\}$.

3. $\boldsymbol{\lambda}$ is a multicore, and for any multicharge $\boldsymbol{a} = (a_1, \dots, a_r)$ such that $a_j \equiv \kappa_j \bmod e$ for all j there exist integers $s_1, \dots, s_r$ such that
$$\frac{\mathfrak{b}^{\boldsymbol{a}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\boldsymbol{a}}_{ik}(\boldsymbol{\lambda})}{e} \leq s_j - s_k + 1$$
for all $i \in \{0, \dots, e-1\}$, $j, k \in \{1, \dots, r\}$.

4. There is no block of any $\mathcal{H}_{r,m}$ with the same hub as B and smaller weight.

5. Every multipartition in B is a multicore.

Now we can make the definition of a core block for the Ariki-Koike algebra $\mathcal{H}_{r,n}$.

Definition 1.4.37. Suppose B is a block of $\mathcal{H}_{r,n}$. Then we say that B is a *core block* if and only if either

- e is finite and the equivalent conditions of Theorem 1.4.36 are satisfied for any multipartition $\boldsymbol{\lambda}$ in B, or
- $e = \infty$.

Theorem 1.4.36 gives us several equivalent conditions for a multipartition to lie in a core block. Moreover, condition (2) of Theorem 1.4.36 together with point (2) of Proposition 1.4.33 implies that, of the blocks with a given hub, only the one with the smallest weight is a core block. So, if $\boldsymbol{\lambda}$ is a multipartition with this hub, then we may speak of this core block as the **core block** of $\boldsymbol{\lambda}$.

Now, let $e < \infty$. Let $\boldsymbol{\lambda}$ be a multipartition and κ be a multicharge for $\mathcal{H}_{r,n}$. Recalling the definition of level for an e-core given in Subsection 1.1.2, define $\ell_{ij}^{\kappa}(\boldsymbol{\lambda})$ to be the level of the last bead on runner i of the abacus display for $\lambda^{(j)}$ with respect to κ. Note that $\mathfrak{b}_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda}) = \ell_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda})e + i$ and $\gamma_i^{jk}(\boldsymbol{\lambda}) = \ell_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda}) - \ell_{ik}^{\boldsymbol{a}}(\boldsymbol{\lambda})$. Using Theorem 1.4.36, we see that for $\boldsymbol{\lambda}$ corresponding to $S^{\boldsymbol{\lambda}}$ in a core block, we have that there exist a multicharge $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ such that $a_j \equiv \kappa_j \mod e$ and integers $b_0, b_1, \ldots, b_{e-1}$ such that for each $i \in I$ where I is defined as in Subsection 1.4.1 and $j \in \{1, \ldots, r\}$, $\ell_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda})$ equals either b_i or $b_i + 1$. We call such an e-tuple $(b_0, b_1, \ldots, b_{e-1})$ a *base tuple* for $\boldsymbol{\lambda}$. Adapting Theorem 1.4.36 we have the following result.

Proposition 1.4.38. Suppose $e < \infty$, $\boldsymbol{\lambda}$ is a multicore and $\kappa = (\kappa_1, \ldots, \kappa_r)$ is a multicharge for $\mathcal{H}_{r,n}$. Then $S^{\boldsymbol{\lambda}}$ lies in a core block of $\mathcal{H}_{r,n}$ if and only if there is $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ such that $a_j \equiv \kappa_j \mod e$ and an abacus configuration for $\boldsymbol{\lambda}$ such that

$$|\gamma_i^{jk}(\boldsymbol{\lambda})| = |\ell_{ij}^{\boldsymbol{a}}(\boldsymbol{\lambda}) - \ell_{ik}^{\boldsymbol{a}}(\boldsymbol{\lambda})| \leq 1 \text{ for each } i \in I \text{ and } j, k \in \{1, \ldots, r\}.$$

Corollary 1.4.39. In the same setting of Proposition 1.4.38. If $S^{\boldsymbol{\lambda}}$ lies in a core block of $\mathcal{H}_{r,n}$, then there exists $\boldsymbol{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ such that $a_j \equiv \kappa_j \mod e$ and an abacus configuration for $\boldsymbol{\lambda}$ such that

$$|\delta_i^j(\boldsymbol{\lambda}) - \delta_i^k(\boldsymbol{\lambda})| \leq 2 \text{ for each } i \in I \text{ and } j, k \in \{1, \ldots, r\}.$$

Proof. By Proposition 1.4.38, for each $i \in I$ and $j, k \in \{1, \ldots, r\}$ we have

$$\begin{aligned}
|\delta_i^j(\boldsymbol{\lambda}) - \delta_i^k(\boldsymbol{\lambda})| &= |\ell_{ij}^{\boldsymbol{a}} - \ell_{i-1,j}^{\boldsymbol{a}} - (\ell_{ik}^{\boldsymbol{a}} - \ell_{i-1,k}^{\boldsymbol{a}})| \\
&= |\ell_{ij}^{\boldsymbol{a}} - \ell_{ik}^{\boldsymbol{a}} + \ell_{i-1,j}^{\boldsymbol{a}} - \ell_{i-1,k}^{\boldsymbol{a}}| \\
&\leq |\ell_{ij}^{\boldsymbol{a}} - \ell_{ik}^{\boldsymbol{a}}| + |\ell_{i-1,j}^{\boldsymbol{a}} - \ell_{i-1,k}^{\boldsymbol{a}}| \\
&\leq 1 + 1 = 2.
\end{aligned}$$

□

2

Equivalence of decomposition matrices for blocks of Ariki-Koike algebras

In [Sco91], under some conditions, Scopes establishes a natural correspondence between Specht modules and simple modules in the blocks B and $\phi_i(B)$ of the symmetric groups where ϕ_i is the map swapping the runners $i-1$ and i of each partition in the block B. This leads to an equivalence of decomposition matrices for these two blocks, meaning that the blocks B and $\phi_i(B)$ have the same decomposition matrices. In the last part of her paper Scopes proves Donovan's conjecture for blocks of the symmetric groups. In particular, given a partition λ of n, the bijection ϕ_i gives a Morita equivalence between the blocks B of λ and the block $\phi_i(B)$ of $\phi_i(\lambda)$ for the symmetric group algebra $\mathbb{F}\mathfrak{S}_n$.

This chapter is intended as the Ariki-Koike algebra version of Scopes' paper about decomposition numbers. In particular, we prove a sufficient condition such that two blocks of the Ariki-Koike algebras have the same decomposition matrices.

The generalisation of Donovan's conjecture and hence of the Morita equivalence between the blocks B and $\Phi_i(B)$ of Ariki-Koike algebras (with Φ_i the map ϕ_i acting componentwise) can be seen as a special case of Theorem 3.3 in [Web23] where Webster proves this using t-exact Chuang-Rouquier equivalences in the more general setting of highest weight categorifications. This result shows as well that our generalisation of Scopes equivalence is the natural one.

We would like to underline the fact that there are example of Morita equivalences that do not imply the equivalence of decomposition matrices. Indeed, if we consider $n = 8$ and $p = 3$, we have that the block of the partition (8) and the block of the partition (1^8) for the symmetric group algebra $\mathbb{F}_3\mathfrak{S}_8$ are Morita equivalent, but they have different decomposition matrices. In the sense that we cannot reorder the rows and the columns to get from one matrix to the other because in the block of (1^8) there are two Specht modules indexed by p-restricted partitions that are simple, i.e. $S^{(1^8)}$ and $S^{(2^3,1^2)}$; while in the block of $S^{(8)}$ there is only $S^{(8)}$ that is simple and indexed by a p-regular partition.

2.1 Results about multicores

In this section, we give some results concerning properties of multicores that play a fundamental role in the proof of the main result of this chapter. We fix some notation.

Let $\mathbb{F}$ be a field, and let $q, Q_1, \ldots, Q_r$ be non-zero elements of $\mathbb{F}$. Assume that $(Q_1, \ldots, Q_r)$ are q-connected parameters. Let $e \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ be the quantum characteristic of $\mathcal{H}_{r,n}$. Set $I = \mathbb{Z}/e\mathbb{Z}$ (which we identify with $\{0, 1, \ldots, e-1\}$) unless $e = \infty$, in which case set $I = \mathbb{Z}$. For $i \in I$ and a multicore $\boldsymbol{m}$, denote by:

$$d_i(\boldsymbol{m}) = \min\{\delta_i^j(\boldsymbol{m}) \mid j \in \{1, \ldots, r\}\},$$

where $\delta_i^j(\boldsymbol{m})$ is defined in Subsection 1.4.4. If the value of i is clear, we will write $d(\boldsymbol{m})$ instead of $d_i(\boldsymbol{m})$.

Firstly, we notice an important and useful property of the abacus display of a multicore $\boldsymbol{\mu}$ lying in a core block C of $\mathcal{H}_{r,n}$ and then we give some results about multicores not necessarily in a core block.

If $\boldsymbol{\mu}$ is a multipartition lying in a core block C of $\mathcal{H}_{r,n}$ then, by the definition of a base tuple, there exists a multicharge $\boldsymbol{a} = (a_1, \ldots, a_r)$ of $\mathcal{H}_{r,n}$ and at least one base tuple $(b_0, \ldots, b_{e-1})$ such that $\boldsymbol{\mu}$ has abacus display where for $i \in I$, $i \geq 1$ and $j \in \{1, \ldots, r\}$

$$\delta_i^j(\boldsymbol{\mu}) \in \{b_i - b_{i-1} - 1, b_i - b_{i-1}, b_i - b_{i-1} + 1\}. \tag{2.1.1}$$

Notice that (2.1.1) holds for all the multicores in the core block C since the base tuple is an invariant of a core block.

Fix $i \in I$, $i \geq 1$ and a multicharge $\mathbf{a} = (a_1, \ldots, a_r)$ such that (2.1.1) holds. Let $(b_0, \ldots, b_{e-1})$ be the corresponding base tuple such that b_i is as big as possible and b_{i-1} is as small as possible between all the possible choices of base tuples corresponding to $\boldsymbol{a}$. Then we can define

$$K_i := b_i - b_{i-1} - 1 \tag{2.1.2}$$

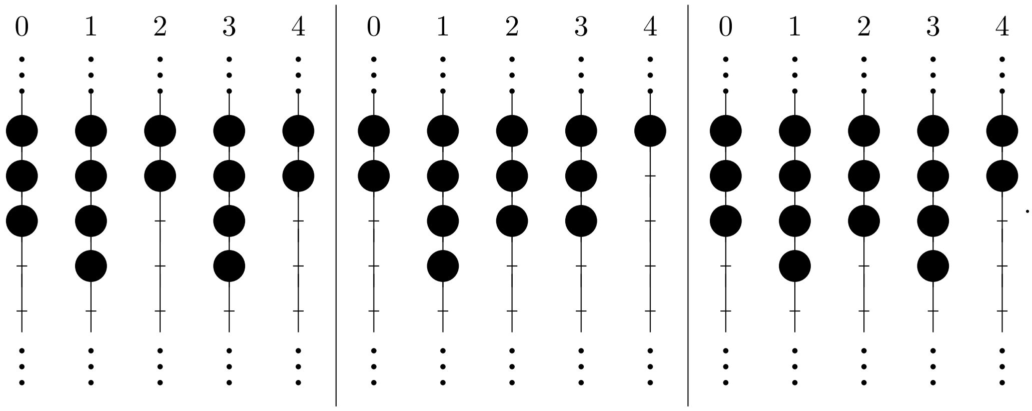
If it is clear what i we are referring to, we will simply write K instead of K_i. Note that

$$K \leq d(\boldsymbol{\mu}) \tag{2.1.3}$$

for all $\boldsymbol{\mu} \in C$.

Example 2.1.1. Suppose $e = 5$, $r = 3$ and $(Q_1, Q_2, Q_3) = (1, q^3, q)$. So, $\kappa =$

$(0, 3, 1)$. Let $\boldsymbol{\mu}$ be the multicore $((4, 3, 1), (4, 2^3), (3, 2))$ which has abacus display with respect to the multicharge $\boldsymbol{a} = (0, -2, 1)$



Since $|\gamma_i^{jk}(\boldsymbol{\mu})| \leq 1$ for all $j, k \in \{1, 2, 3\}$ and $i \in \{0, \ldots, 4\}$, $\boldsymbol{\mu}$ lies in a core block C by Proposition 1.4.38. Then we can define a base tuple for $\boldsymbol{\mu}$ and thus for C. In particular, we can consider the following two base tuples:

1. $(b_0, b_1, b_2, b_3, b_4) = (2, 4, 2, 3, 1)$;
2. $(b'_0, b'_1, b'_2, b'_3, b'_4) = (2, 3, 2, 3, 1)$.

Take $i = 1$. In order to define K_1 we need to choose $(b_0, b_1, b_2, b_3, b_4) = (2, 4, 2, 3, 1)$ because $b_1 > b'_1$ and $b_0 = b'_0$, so we get $K_1 = 1$. Take $i = 3$. In order to define K_3 we can choose either $(b_0, b_1, b_2, b_3, b_4)$ or $(b'_0, b'_1, b'_2, b'_3, b'_4)$ because $b_3 = b'_3$ and $b_2 = b'_2$, so we get $K_3 = 0$.

We now want to exhibit, for all multicores of $\mathcal{H}_{r,n}$, a sequence of multicores of non-increasing weight ending in a multicore lying in a core block. In order to do this, we need a preliminary lemma adapted from the proof of Proposition 3.7 in [Fay07b] to the case of multicores.

Lemma 2.1.2. If $\boldsymbol{\lambda}$ is a multicore not lying in a core block of $\mathcal{H}_{r,n}$, then there is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_u$ of multicores such that, for all $t = 0, \ldots, u-1$, we have $\boldsymbol{\lambda}_{t+1} = s_{il}^{jk}(\boldsymbol{\lambda}_t)$ for some i, l, j, k and $\boldsymbol{\lambda}_u$ lies in a core block. Furthermore, $w(\boldsymbol{\lambda}_{t+1}) \leq w(\boldsymbol{\lambda}_t)$ for all $t = 0, \ldots, u-1$.

Proposition 2.1.3. Let $\boldsymbol{m}$ be a multicore of $\mathcal{H}_{r,n}$. Then there exist a multicore $\boldsymbol{\mu}$ in the core block C of $\boldsymbol{m}$ and a sequence of multicores

$$\boldsymbol{m} = \boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_{s-1}, \boldsymbol{m}_s = \boldsymbol{\mu}$$

such that:

1) the core block of $\boldsymbol{m}_t$ is C for all $t = 0, \ldots, s$;

2) $\boldsymbol{m}_{t+1} = s_{il}^{jk}(\boldsymbol{m}_t)$ for some $j,k \in \{1,\dots,r\}$, $i,l \in \{1,\dots,e-1\}$ for all $t = 0,\dots,s-1$;

3) there exists $0 \le v \le s$ such that

   i) $w(\boldsymbol{m}_{t+1}) < w(\boldsymbol{m}_t)$ for all $t = 0,\dots,v-1$ and $w(\boldsymbol{m}_{t+1}) \le w(\boldsymbol{m}_t)$ for all $t = v,\dots,s-1$;

   ii) $|\gamma_{il}^{jk}(\boldsymbol{m}_v)| \le 2$ for all i,l and j,k.

Proof. Let $\boldsymbol{m}_0 := \boldsymbol{m}$ be a multicore of $\mathcal{H}_{r,n}$. Then, apply the following procedure for $t \ge 0$.

1. Calculate $\gamma_i^{jk}(\boldsymbol{m}_t)$ for all $i \in \{0,\dots,e-1\}$ and $j,k \in \{1,\dots,r\}$;

2. If there is a choice of i,l and j,k such that $\gamma_{il}^{jk}(\boldsymbol{m}_t) \ge 3$, set $\boldsymbol{m}_{t+1} = s_{il}^{jk}(\boldsymbol{m}_t)$. By Proposition 1.4.17 we have

$$w(\boldsymbol{m}_{t+1}) < w(\boldsymbol{m}_t).$$

3. Repeat this step until we have $\boldsymbol{m}_{t+1}$ with $\gamma_{il}^{jk}(\boldsymbol{m}_{t+1}) \le 2$ for all i,l and j,k.

Suppose that we stop for $t+1 = v$. Notice that $\gamma_{il}^{jk}(\boldsymbol{m}_v) \le 2$ for all i,l and j,k implies $|\gamma_{il}^{jk}(\boldsymbol{m}_v)| \le 2$ for all i,l and j,k. Indeed,

$$\gamma_{il}^{jk}(\boldsymbol{m}_v) = -\gamma_{li}^{jk}(\boldsymbol{m}_v) = -\gamma_{il}^{kj}(\boldsymbol{m}_v).$$

Now, if $\boldsymbol{m}_v$ is not in the core block C, apply Lemma 2.1.2 until we get a multicore $\boldsymbol{\mu}$ in the core block C. □

Before stating the main result, we need some preliminary lemmas.

Lemma 2.1.4. Suppose that $\boldsymbol{m}$ is a multicore such that $|\gamma_{il}^{jk}(\boldsymbol{m})| \le 2$ for all $i,l \in I$ and $j,k \in \{1,\dots r\}$. Fix $\bar{i} \in I$, and let d be the integer such that $d(\boldsymbol{m}) = d-1$. Then

$$\delta_{\bar{i}}^{j}(\boldsymbol{m}) \in \{d-1, d, d+1\} \text{ for all } j \in \{1,\dots,r\}.$$

Proof. Consider $\boldsymbol{m}$ a multicore such that $|\gamma_{il}^{jk}(\boldsymbol{m})| \le 2$ for all $i,l \in I$ and $j,k \in \{1,\dots r\}$. Fix $\bar{i} \in I$. Then, since $|\gamma_{il}^{jk}(\boldsymbol{m})| \le 2$ for all $i,l \in I$ and $j,k \in \{1,\dots r\}$, we have that

$$|\gamma_{(\bar{i}-1)\bar{i}}^{jk}(\boldsymbol{m})| = |\delta_{\bar{i}}^{k}(\boldsymbol{m}) - \delta_{\bar{i}}^{j}(\boldsymbol{m})| \le 2.$$

Hence, since $d(\boldsymbol{m}) = d-1$, we get $\delta_{\bar{i}}^{j}(\boldsymbol{m}) \in \{d-1, d, d+1\}$ for all $j \in \{1,\dots,r\}$. □

Lemma 2.1.5. Let $\boldsymbol{m}$ be a multicore with core block C and such that $|\gamma_{il}^{jk}(\boldsymbol{m})| \leq 2$ for all $i, l \in I$ and $j, k \in \{1, \dots r\}$. Suppose that $\boldsymbol{\mu}$ is a multicore in the core block C. Fix $\bar{i} \in I$. Then $d(\boldsymbol{\mu}) \leq d(\boldsymbol{m}) + 1$.

Proof. By Lemma 2.1.4, we know that $\delta_{\bar{i}}^{j}(\boldsymbol{m}) \in \{d-1, d, d+1\}$ and $\delta_{\bar{i}}^{j}(\boldsymbol{\mu}) \in \{d'-1, d', d'+1\}$ for some integers d and d' for all $j \in \{1, \dots, r\}$. Let a, b, and c be the number of $\delta_{\bar{i}}^{j}(\boldsymbol{m})$ equal respectively to $d-1$, d, and $d+1$. Let a', b', and c' be the number of $\delta_{\bar{i}}^{j}(\boldsymbol{\mu})$ equal respectively to $d'-1$, d', and $d'+1$. Notice that $a > 0$ and $a' > 0$ by definition of $d-1$ and $d'-1$. By Proposition 2.1.3, we can go from the multicore $\boldsymbol{m}$ to the multicore $\boldsymbol{\mu}$ in the core block C via a sequence of multicores $\boldsymbol{m}_t$ such that $\boldsymbol{m}_{t+1} = s_{il}^{jk}(\boldsymbol{m}_t)$ for some $i, l \in I$ and $j, k \in \{1, \dots, r\}$. By point (1) of Proposition 1.4.17, we know that each multicore $\boldsymbol{m}_t$ occurring in this sequence has the same hub of $\boldsymbol{m}$, then $\boldsymbol{m}$ and $\boldsymbol{\mu}$ have the same hub. So,

$$a(d-1) + b(d) + c(d+1) = a'(d'-1) + b'(d') + c'(d'+1), \tag{2.1.4}$$

where $a+b+c = a'+b'+c' = r$. Suppose by contradiction that $d(\boldsymbol{\mu}) > d(\boldsymbol{m})+1$. Then $d' > d+1$, and so looking at (2.1.4) we have

$$\begin{aligned}
&\text{LHS} \leq r(d+1) \text{ with equality if and only if } a = b = 0;\\
&\text{RHS} \geq a'(d+1) + b'(d+2) + c'(d+3) \geq r(d+1)\\
&\qquad \text{with equality if and only if } b' = c' = 0;
\end{aligned}$$

We must have equality in both terms, but this is a contradiction since $a > 0$. □

Lemma 2.1.6. Let $\boldsymbol{m}$ be a multicore of $\mathcal{H}_{r,n}$ and $\boldsymbol{m}' = s_{il}^{jk}(\boldsymbol{m})$ for some i, l, j, k. Fix $\bar{i} \in I$. Then $d(\boldsymbol{m}') \geq d(\boldsymbol{m}) - 2$. Moreover, if $\gamma_{il}^{jk}(\boldsymbol{m}) = 1$, then $d(\boldsymbol{m}') \geq d(\boldsymbol{m}) - 1$.

Proof. The fact that $d(\boldsymbol{m}') \geq d(\boldsymbol{m}) - 2$ follows from the definition of s_{il}^{jk} and that $|\delta_{\bar{i}}^{j}(\boldsymbol{m}) - \delta_{\bar{i}}^{j}(\boldsymbol{m}')| = 2$ if and only if $\{i, l\} = \{\bar{i}-1, \bar{i}\}$.

Suppose $\gamma_{il}^{jk}(\boldsymbol{m}) = 1$. Then,

- if $\{i, l\} \neq \{\bar{i}-1, \bar{i}\}$, then $d(\boldsymbol{m}') \geq d(\boldsymbol{m}) - 1$ since the only case in which $\delta_{\bar{i}}^{j}(\boldsymbol{m}')$ decreases by 2 with respect to $\delta_{\bar{i}}^{j}(\boldsymbol{m})$ is when $\{i, l\} = \{\bar{i}-1, \bar{i}\}$;
- if $\{i, l\} = \{\bar{i}-1, \bar{i}\}$, then

$$\gamma_{\bar{i}-1,\bar{i}}^{jk}(\boldsymbol{m}) = 1 \Leftrightarrow \delta_{\bar{i}}^{k}(\boldsymbol{m}) - \delta_{\bar{i}}^{j}(\boldsymbol{m}) = 1 \Leftrightarrow \delta_{\bar{i}}^{k}(\boldsymbol{m}) = \delta_{\bar{i}}^{j}(\boldsymbol{m}) + 1.$$

Thus,

$$\begin{aligned}
&\delta_{\bar{i}}^{j}(\boldsymbol{m}') = \delta_{\bar{i}}^{j}(\boldsymbol{m}) + 2 \text{ and,}\\
&\delta_{\bar{i}}^{k}(\boldsymbol{m}') = \delta_{\bar{i}}^{k}(\boldsymbol{m}) - 2 = \delta_{\bar{i}}^{j}(\boldsymbol{m}) + 1 - 2 = \delta_{\bar{i}}^{j}(\boldsymbol{m}) - 1.
\end{aligned}$$

Hence, $d(\boldsymbol{m}') \geq d(\boldsymbol{m}) - 1$.

□

Lemma 2.1.7. Let $\boldsymbol{m}$ be a multicore of $\mathcal{H}_{r,n}$. Suppose that $\boldsymbol{m} = \boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_v$ is a sequence of multicores such that $\boldsymbol{m}_{t+1} = s_{il}^{jk}(\boldsymbol{m}_t)$ for some i, l, j, k and $w(\boldsymbol{m}_{t+1}) < w(\boldsymbol{m}_t)$ for all $t = 0, \ldots, v-1$. Let $w(\boldsymbol{m}) = w(\boldsymbol{m}_v) + hr$ with $h > 0$. Then $d(\boldsymbol{m}) \geq d(\boldsymbol{m}_v) - h$.

Proof. We proceed by induction on v. If $v = 1$, the sequence of multicores consists of $\boldsymbol{m}_0 = \boldsymbol{m}$ and $\boldsymbol{m}_1 = s_{il}^{jk}(\boldsymbol{m}_0)$ for some i, l, j, k with $w(\boldsymbol{m}_0) = w(\boldsymbol{m}_1) + hr$ for $h > 0$. Note that $\boldsymbol{m}_0 = s_{li}^{jk}(\boldsymbol{m}_1)$. By Proposition 1.4.17(2), the weight of $\boldsymbol{m}_0$ is $w(\boldsymbol{m}_0) = w(\boldsymbol{m}_1) - r(\gamma_{li}^{jk}(\boldsymbol{m}_1) - 2)$, so $h = -\gamma_{li}^{jk}(\boldsymbol{m}_1) + 2$. Moreover, $h > 0$ and so we have that $\gamma_{li}^{jk}(\boldsymbol{m}_1) \leq 1$. Hence, we just need to check the following two cases.

- If $\gamma_{li}^{jk}(\boldsymbol{m}_1) \leq 0$, then $d(\boldsymbol{m}_0) \geq d(\boldsymbol{m}_1) - 2$ by Lemma 2.1.6 and $h = -\gamma_{li}^{jk}(\boldsymbol{m}_1) + 2 \geq 2$. Thus,

$$d(\boldsymbol{m}_0) \geq d(\boldsymbol{m}_1) - 2 \geq d(\boldsymbol{m}_1) + \gamma_{li}^{jk}(\boldsymbol{m}_1) - 2 = d(\boldsymbol{m}_1) - h.$$

- If $\gamma_{li}^{jk}(\boldsymbol{m}_1) = 1$, then $h = 1$ and $d(\boldsymbol{m}_0) \geq d(\boldsymbol{m}_1) - 1$ by Lemma 2.1.6. Thus,

$$d(\boldsymbol{m}_0) \geq d(\boldsymbol{m}_1) - 1 = d(\boldsymbol{m}_1) - h.$$

Suppose $v > 1$. Let $w(\boldsymbol{m}) = w(\boldsymbol{m}_{v-1}) + h'r$ with $0 \leq h' < h$ and $w(\boldsymbol{m}_{v-1}) = w(\boldsymbol{m}_v) + h''r$ with $h'' > 0$ so that $h = h'' + h'$. By induction hypothesis we know that $d(\boldsymbol{m}) \geq d(\boldsymbol{m}_{v-1}) - h'$. In order to get the result we want to show that $d(\boldsymbol{m}) \geq d(\boldsymbol{m}_v) - h$. We know from the base step that that $d(\boldsymbol{m}_{v-1}) \geq d(\boldsymbol{m}_v) - h''$. Thus,

$$d(\boldsymbol{m}) \geq d(\boldsymbol{m}_{v-1}) - h' \geq d(\boldsymbol{m}_v) - h'' - h' = d(\boldsymbol{m}_v) - h.$$

□

Proposition 2.1.8. Fix $\bar{i} \in \{1, \ldots, e-1\}$. Let $K = K_{\bar{i}}$ be the integer defined in (2.1.2) for a core block C. Suppose that $\boldsymbol{m}$ is a multicore with core block C and weight

$$w(\boldsymbol{m}) = w(C) + hr$$

with $0 \leq h \leq K$. Then $d(\boldsymbol{m}) \geq K - h$.

Proof. Let

$$\boldsymbol{m} = \boldsymbol{m}_0, \ldots, \boldsymbol{m}_v, \boldsymbol{m}_{v+1}, \ldots, \boldsymbol{m}_s = \boldsymbol{\mu}$$

be the sequence defined in Proposition 2.1.3, where v is such that $|\gamma_{il}^{jk}(\boldsymbol{m}_v)| \leq 2$ for all i, l, j, k, and $\boldsymbol{\mu} \in C$.

By Lemma 2.1.7, we have that

$$d(\boldsymbol{m}) \geq d(\boldsymbol{m}_v) - h', \tag{2.1.5}$$

where $0 \leq h' \leq h$ is such that $w(\boldsymbol{m}) = w(\boldsymbol{m}_v) + h'r$.

If $h' = h$, then $\boldsymbol{m}_v = \boldsymbol{m}_s = \boldsymbol{\mu} \in C$; therefore $d(\boldsymbol{m}) \geq d(\boldsymbol{\mu}) - h \geq K - h$ by (2.1.3).

Otherwise $h' < h$, and Lemma 2.1.5 can be applied to get that

$$d(\boldsymbol{m}_v) \geq d(\boldsymbol{\mu}) - 1 \geq d(\boldsymbol{\mu}) - (h - h').$$

Combining this with (2.1.5), we have:

$$d(\boldsymbol{m}) \geq d(\boldsymbol{m}_v) - h' \geq d(\boldsymbol{\mu}) - (h - h') - h' = d(\boldsymbol{\mu}) - h \geq K - h.$$

□

2.2 Decomposition numbers for blocks of $\mathcal{H}_{r,n}$

Now, we want to generalise Lemma 2.1 of Scopes' paper [Sco91] to the Ariki-Koike algebras $\mathcal{H}_{r,n}$. Thus, we want to show that, for $i \in I$, the decomposition matrices of the blocks B and $\Phi_i(B)$ are the same, provided that

$$w(B) \leq w(C) + K_i r \tag{2.2.1}$$

where

- C is the core block of B,
- K_i is the integer defined in (2.1.2).

Notice that this condition is a block condition, i.e., it is satisfied by all the multipartitions in the block.

Remark 2.2.1. If $r = 1$, condition (2.2.1) is equivalent to Scopes' condition for the symmetric group in [Sco91, Section 2].

Moreover, note that Proposition 1.4.13 implies the following.

Corollary 2.2.2. Let $\boldsymbol{\mu}$ be a multipartition in a core block and let $0 \leq h \leq K_i$. Let B be the block containing the multipartitions obtained by adding h e-rim hooks to $\boldsymbol{\mu}$. Then B satisfies condition (2.2.1).

Lemma 2.2.3. Fix $i \in I$. Let B be a block of $\mathcal{H}_{r,n}$ such that (2.2.1) holds and $\delta_i(B) \geq 0$. Then, in each component of every r-multipartition $\boldsymbol{\lambda}$ of n such that $S^{\boldsymbol{\lambda}}$ belongs to the block B, there is no abacus configuration of the type $\bullet \quad |$ in runners $i-1$ and i.

Proof. First of all, notice that

$$w(B) - w(C) = \ell r, \quad \text{for some } \ell \in \{0, \dots, K_i\} \tag{2.2.2}$$

since $w(B) - w(C)$ can take as values only integral multiples of r.

Now, consider the multicore $\boldsymbol{m}$ associated to $\boldsymbol{\lambda}$, that is the multicore whose abacus display is obtained by the one of $\boldsymbol{\lambda}$ sliding all the beads up as high as possible. Then $\boldsymbol{m}$ has the same core block C of $\boldsymbol{\lambda}$ and weight $w(\boldsymbol{m}) = w(C) + sr$ with $0 \leq s \leq \ell$. Thus, by Proposition 2.1.8 we can say that

$$d(\boldsymbol{m}) \geq K_i - s \geq \ell - s,$$

since $\ell \leq K_i$ by (2.2.2). Moreover, in order to get $\boldsymbol{\lambda}$ from $\boldsymbol{m}$ we just need to slide beads down of a total number of $\ell - s$ spaces. This implies that in each component $m^{(j)}$ of $\boldsymbol{m}$ we need to slide beads down of at most $\ell - s$ spaces. Since $d(\boldsymbol{m}) \geq \ell - s$, similarly to the proof of [Sco91, Lemma 2.1] we can conclude that each component $\lambda^{(j)}$ of $\boldsymbol{\lambda}$ has no configuration $\bullet\ \mid$ in runners $i-1$ and i. □

Proposition 2.2.4. Fix $i \in I$. Let B be a block of $\mathcal{H}_{r,n}$ such that (2.2.1) holds and $\delta_i(B) \geq 0$. Suppose that $\boldsymbol{\lambda}$ is an r-multipartition of n such that $S^{\boldsymbol{\lambda}}$ belongs to the block B. Then the multipartition $\Phi_i(\boldsymbol{\lambda})$ of $n - \delta_i(B)$ is such that $S^{\Phi_i(\boldsymbol{\lambda})}$ belongs to $\Phi_i(B)$ and

$$S^{\boldsymbol{\lambda}} \downarrow_{\Phi_i(B)} \sim \delta_i(B)! S^{\Phi_i(\boldsymbol{\lambda})},$$
$$S^{\Phi_i(\boldsymbol{\lambda})} \uparrow^{B} \sim \delta_i(B)! S^{\boldsymbol{\lambda}}.$$

Proof. Suppose that $\boldsymbol{\mu}$ is a multipartition of $n - \delta_i(B)$ and $S^{\boldsymbol{\mu}}$ is a factor of $S^{\boldsymbol{\lambda}} \downarrow_{\mathcal{H}_{r,n-\delta_i(B)}}$ using the Specht filtration given in Theorem 1.4.8. The diagram $[\boldsymbol{\mu}]$ can be obtained from $[\boldsymbol{\lambda}]$ by removing $\delta_i(B)$ nodes. The multiplicity of $S^{\boldsymbol{\mu}}$ as a factor is the number of ways in which the node removal can be affected. The abacus of $\boldsymbol{\mu}$ is obtained from that of $\boldsymbol{\lambda}$ by successively moving $\delta_i(B)$ beads one place to the left. The module $S^{\boldsymbol{\mu}}$ belongs to $\Phi_i(B)$ if and only if $\boldsymbol{\mu}$ has the same hub as $\Phi_i(B)$ by point (2) of Proposition 1.4.33. This is equivalent to the fact that $\boldsymbol{\mu}$ is obtained from the abacus display of $\boldsymbol{\lambda}$ by moving a total of $\delta_i(B)$ beads from runners i to runners $i-1$.

By Lemma 2.2.3, in each component of the multipartition $\boldsymbol{\lambda}$, we have no abacus configuration of the type $\bullet\ \mid$ in runners $i-1$ and i. This implies that $\boldsymbol{\lambda}$ has no addable i-nodes and so Φ_i consists only of removing i-nodes. Hence, the number of ways in which the node removal can be effected is $\delta_i(B)!$ because they have all the same residue i and so we can remove these nodes in any order. This gives the first result about restriction. This also shows that there is exactly one $\boldsymbol{\mu}$ that can be found by removing $\delta_i(B)$ i-nodes.

Similarly, the second result about induction can be proved by adding i-nodes instead of removing i-nodes. □

Definition 2.2.5. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$ and $M = (M_1, \ldots, M_r)$ be two r-tuple of integers with $\Lambda_j = \{x_1^j, x_2^j, \ldots\}$ and $M_j = \{y_1^j, y_2^j, \ldots\}$ for all j. We write $\Lambda > M$ if and only if the minimal $j \in \{1, \ldots, r\}$ for which $\Lambda_j \neq M_j$ and the minimal $i \geq 1$ such that $x_i^j \neq y_i^j$ satisfy $x_i^j > y_i^j$.

Lemma 2.2.6. Let $i \in I$. If condition (2.2.1) holds, then Φ_i preserves the lexicographic order of multipartitions.

Proof. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions whose corresponding Specht modules belong to block B. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$ and $M = (M_1, \ldots, M_r)$ be the associated sets of β-numbers.

Let $\bar{\boldsymbol{\lambda}} = \Phi_i(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\mu}} = \Phi_i(\boldsymbol{\mu})$, and let their corresponding sets of β-numbers be $\bar{\Lambda} = (\bar{\Lambda}_1, \ldots, \bar{\Lambda}_r)$ and $\bar{M} = (\bar{M}_1, \ldots, \bar{M}_r)$.

Now if $\boldsymbol{\lambda} > \boldsymbol{\mu}$, then by Definition 2.2.5 we have $\Lambda > M$. Similarly $\Lambda > M$ implies that $\boldsymbol{\lambda} > \boldsymbol{\mu}$ and we obtain

$$\boldsymbol{\lambda} > \boldsymbol{\mu} \Leftrightarrow \Lambda > M \Leftrightarrow \Lambda \setminus (\Lambda \cap M) > M \setminus (\Lambda \cap M).$$

Assume $\boldsymbol{\lambda} > \boldsymbol{\mu}$. Let j_0 be the minimal $j \in \{1, \ldots, r\}$ such that

$$\Lambda_j \setminus (\Lambda_j \cap M_j) > M_j \setminus (\Lambda_j \cap M_j).$$

Let $\Lambda_{j_0} \setminus (\Lambda_{j_0} \cap M_{j_0}) = \{x_1, \ldots, x_t\}$, with $x_l > x_{l+1}$ for $l = 1, \ldots, t-1$, and let $M_{j_0} \setminus (\Lambda_{j_0} \cap M_{j_0}) = \{y_1, \ldots, y_s\}$, with $y_m > y_{m+1}$ for $m = 1, \ldots, s-1$. Then,

$$\bar{\Lambda}_{j_0} \setminus (\bar{\Lambda}_{j_0} \cap \bar{M}_{j_0}) = \{\phi_i(x_1), \ldots, \phi_i(x_t)\}$$

and

$$\bar{M}_{j_0} \setminus (\bar{\Lambda}_{j_0} \cap \bar{M}_{j_0}) = \{\phi_i(y_1), \ldots, \phi_i(y_s)\}.$$

Since $\boldsymbol{\lambda} > \boldsymbol{\mu}$, it follows that $x_1 > y_1$. We have three cases to consider.

Case 1 *x_1 belongs to column $i-1$.* In this case $\phi_i(x_1) = x_1+1 > y_m+1 \geq \phi_i(y_m)$ for all m. Hence $\bar{\Lambda} \setminus (\bar{\Lambda} \cap \bar{M}) > \bar{M} \setminus (\bar{\Lambda} \cap \bar{M})$, so $\bar{\boldsymbol{\lambda}} > \bar{\boldsymbol{\mu}}$.

Case 2 *x_1 belongs to column i.* Clearly $\phi_i(x_1) = x_1 - 1 \geq y_m + m - 1 > y_m + 1 \geq \phi_i(y_m)$ for all $m \geq 3$ as $x_1 > y_1 > y_2 > y_3 > \cdots$.

If y_2 does not belong to column $i-1$, then $\phi_i(y_2) \leq y_2$ and $x_1 - 1 > y_2$ as $x_1 > y_1 > y_2$. So $\phi_i(x_1) = x_1 - 1 > y_2 \geq \phi_i(y_2)$. If y_2 belongs to column $i-1$ then $\phi_i(y_2) = y_2 + 1$. Since $x_1 - 1$ and y_2 belongs to column $i-1$ and $y_2 < x_1 - 1$ as above, so $y_2 \leq x_1 - 1 - e$. Hence $\phi_i(x_1) = x_1 - 1 > y_2 + 1 = \phi_i(y_2)$.

If y_1 lies in column i, then $\phi_i(x_1) = x_1 - 1 > y_1 - 1 = \phi_i(y_1)$. If y_1 lies in column k, where $k \neq i, i-1$, then $y_1 \leq x_1 - 2$, so $\phi_i(x_1) = x_1 - 1 > x_1 - 2 \geq y_1 = \phi_i(y_1)$. Suppose y_1 lies in column $i-1$. If $y_1 = x_1 - 1$, then the abacus of $\mu^{(j_0)}$ presents a configuration $\bullet \quad |$ in runners $i-1$ and i where the bead corresponds to y_1. However, by Lemma 2.2.3 a multipartition in the block B cannot have this configuration in runners $i-1$ and i. Hence $y_1 \neq x_1 - 1$, so $y_1 + e < x_1$ and $\phi_i(y_1) = y_1 + 1 < x_1 - 1 = \phi_i(x_1)$. Thus $\phi_i(x_1) > \phi_i(y_m)$ for all m, and $\bar{\Lambda} \setminus (\bar{\Lambda} \cap \bar{M}) > \bar{M} \setminus (\bar{\Lambda} \cap \bar{M})$, so $\bar{\boldsymbol{\lambda}} > \bar{\boldsymbol{\mu}}$.

Case 3 x_1 *does not belong to column* i *nor to column* $i-1$. By similar arguments we see that $\phi_i(x_1) = x_1 > \phi_i(y_m)$ for all $m \geq 2$.

If y_1 does not lie in column $i-1$ then $\phi_i(y_1) = y_1$ or $\phi_i(y_1) = y_1 - 1$. So $\phi_i(x_1) = x_1 > y_1 \geq \phi_i(y_m)$. If y_1 lies in column $i-1$, then $\phi_i(y_1) = y_1 + 1$. Since x_1 is not in runner $i-1$ or in runner i, then $y_1 < x_1 - 1$. So again $\phi_i(x_1) > \phi_i(y_1)$. Thus $\phi_i(x_1) > \phi_i(y_l)$ for all l, and $\bar{\Lambda} \setminus (\bar{\Lambda} \cap \bar{M}) > \bar{M} \setminus (\bar{\Lambda} \cap \bar{M})$, so $\bar{\boldsymbol{\lambda}} > \bar{\boldsymbol{\mu}}$.

□

Now we note that if condition (2.2.1) holds, Φ_i preserves the Kleshchev property. Indeed, we have the following result.

Lemma 2.2.7. [Fay07b, Lemma 1.9] Suppose $\boldsymbol{\lambda}$ is a multipartition, and that $[\boldsymbol{\lambda}]$ has no addable nodes of residue i for $i \in I$. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if $\Phi_i(\boldsymbol{\lambda})$ is.

Given that, we just need to notice that condition (2.2.1) implies that $[\boldsymbol{\lambda}]$ has no addable nodes of residue i. Hence, we can conclude that if condition (2.2.1) holds, then Kleshchev multipartitions are preserved by Φ_i.

Similarly to Lemma 2.4 in [Sco91], we have the following proposition. For this, let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be Kleshchev multipartitions of n and define the **Cartan matrix** $\boldsymbol{C} = (c_{\boldsymbol{\lambda}\boldsymbol{\mu}})$ of $\mathcal{H}_{r,n}$ where $c_{\boldsymbol{\lambda}\boldsymbol{\mu}} = [P^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ is the composition multiplicity of the simple module $D^{\boldsymbol{\mu}}$ in the principal indecomposable module $P^{\boldsymbol{\lambda}}$. Recall that if we denote by $\boldsymbol{D}$ the decomposition matrix of $\mathcal{H}_{r,n}$, Graham and Lehrer in [GL96] proved that

$$\boldsymbol{C} = \boldsymbol{D}^t \boldsymbol{D} \tag{2.2.3}$$

where $\boldsymbol{D}^t$ is the transpose of the matrix $\boldsymbol{D}$.

Proposition 2.2.8. Fix $i \in \{1, \ldots, e-1\}$. Let B be a block of $\mathcal{H}_{r,n}$ such that (2.2.1) holds and $\delta_i(B) \geq 0$. Suppose that $\boldsymbol{\lambda}$ is a Kleshchev r-multipartition of n such that $S^{\boldsymbol{\lambda}}$ belongs to the block B. Then

1. $D^{\boldsymbol{\lambda}} \downarrow_{\Phi_i(B)} \sim \delta_i(B)! D^{\Phi_i(\boldsymbol{\lambda})}$.

2. $D^{\Phi_i(\boldsymbol{\lambda})}\uparrow^B \sim \delta_i(B)!D^{\boldsymbol{\lambda}}$.

3. The blocks B and $\Phi_i(B)$ have the same decomposition matrix.

4. The blocks B and $\Phi_i(B)$ have the same Cartan matrix.

Proof. Let $\boldsymbol{\lambda}_1 > \boldsymbol{\lambda}_2 > \ldots > \boldsymbol{\lambda}_y$ be the Kleshchev multipartitions whose Specht modules belong to B; then $\Phi_i(\boldsymbol{\lambda}_1) > \Phi_i(\boldsymbol{\lambda}_2) > \ldots > \Phi_i(\boldsymbol{\lambda}_y)$ are the Kleshchev multipartitions whose Specht modules belong to $\Phi_i(B)$ by Lemmas 2.2.6 and 2.2.7. Suppose

$$S^{\boldsymbol{\lambda}} \sim \sum_{j=1}^{y} d_{\boldsymbol{\lambda}\boldsymbol{\lambda}_j} D^{\boldsymbol{\lambda}_j}, \quad d_{\boldsymbol{\lambda}\boldsymbol{\lambda}_j} \in \mathbb{N} \tag{2.2.4}$$

If $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, then $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$, that is equivalent to say if $\boldsymbol{\lambda} < \boldsymbol{\mu}$, then $\boldsymbol{\lambda} \ntrianglerighteq \boldsymbol{\mu}$. Hence, by point (2) of Theorem 1.4.7 we have that if $\boldsymbol{\lambda} < \boldsymbol{\mu}$, then $d_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0$.

Hence, (2.2.4) becomes

$$S^{\boldsymbol{\lambda}} \sim \sum_{j=1}^{y} d_{\boldsymbol{\lambda}\boldsymbol{\lambda}_j} D^{\boldsymbol{\lambda}_j}, \quad d_{\boldsymbol{\lambda}\boldsymbol{\lambda}_j} = \begin{cases} 1 & \text{if } \boldsymbol{\lambda} = \boldsymbol{\lambda}_j \\ 0 & \text{if } \boldsymbol{\lambda} < \boldsymbol{\lambda}_j \end{cases}. \tag{2.2.5}$$

In particular, $S^{\boldsymbol{\lambda}_y} = D^{\boldsymbol{\lambda}_y}$. We want to prove points 1. and 2. for $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_y$.

By Proposition 2.2.4 we have

$$S^{\boldsymbol{\lambda}_y}\downarrow_{\Phi_i(B)} \sim \delta_i(B)!S^{\Phi_i(\boldsymbol{\lambda}_y)} \quad \text{and} \quad S^{\Phi_i(\boldsymbol{\lambda}_y)}\uparrow^B \sim \delta_i(B)!S^{\boldsymbol{\lambda}_y},$$

so, since $S^{\boldsymbol{\lambda}_y} = D^{\boldsymbol{\lambda}_y}$ we get the result:

$$D^{\boldsymbol{\lambda}_y}\downarrow_{\Phi_i(B)} \sim \delta_i(B)!D^{\Phi_i(\boldsymbol{\lambda}_y)} \quad \text{and} \quad D^{\Phi_i(\boldsymbol{\lambda}_y)}\uparrow^B \sim \delta_i(B)!D^{\boldsymbol{\lambda}_y}.$$

Now, suppose that points 1. and 2. holds for $\boldsymbol{\lambda}_l, \ldots, \boldsymbol{\lambda}_y$ with $1 < l \leq y$, that is for $l \leq j \leq y$ we have

$$D^{\boldsymbol{\lambda}_j}\downarrow_{\Phi_i(B)} \sim \delta_i(B)!D^{\Phi_i(\boldsymbol{\lambda}_j)} \quad \text{and} \quad D^{\Phi_i(\boldsymbol{\lambda}_j)}\uparrow^B \sim \delta_i(B)!D^{\boldsymbol{\lambda}_j}.$$

Thus, we want to prove points 1. and 2. for $\boldsymbol{\lambda}_{l-1}$. Then

$$(S^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\uparrow^B \underset{\substack{\uparrow \\ \text{by Prop. 2.2.4}}}{\sim} (\delta_i(B)!)^2 S^{\boldsymbol{\lambda}_{l-1}} \underset{\substack{\uparrow \\ \text{by (2.2.5)}}}{\sim} (\delta_i(B)!)^2 \Big(\sum_{j=l}^{y} d_{\boldsymbol{\lambda}_{l-1}\boldsymbol{\lambda}_j} D^{\boldsymbol{\lambda}_j} + D^{\boldsymbol{\lambda}_{l-1}}\Big),$$

and, applying first (2.2.5) and then Proposition 2.2.4 together with the hypothesis on $j \geq l$

$$(S^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\uparrow^B \sim \sum_{j=l}^{y} (\delta_i(B)!)^2 d_{\boldsymbol{\lambda}_{l-1}\boldsymbol{\lambda}_j} D^{\boldsymbol{\lambda}_j} + (D^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\uparrow^B .$$

So

$$D^{\boldsymbol{\lambda}_{l-1}}\downarrow_{\Phi_i(B)}\uparrow^{B}\sim(\delta_i(B)!)^2 D^{\boldsymbol{\lambda}_{l-1}}. \tag{2.2.6}$$

Now notice that for some $\alpha_j, \beta_j \in \mathbb{N}$

$$D^{\boldsymbol{\lambda}_{l-1}}\downarrow_{\Phi_i(B)}\sim\sum_{j=l-1}^{y}\alpha_j D^{\Phi_i(\boldsymbol{\lambda}_j)}, \tag{2.2.7}$$

$$D^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\uparrow^{B}\sim\sum_{j=l-1}^{y}\beta_j D^{\boldsymbol{\lambda}_j}. \tag{2.2.8}$$

Then, by the hypothesis on $j \geq l$ and using (2.2.7), (2.2.8)

$$(D^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\uparrow^{B}\sim\sum_{j=l}^{y}(\delta_i(B)!\alpha_j+\alpha_{l-1}\beta_j)D^{\boldsymbol{\lambda}_j}+\alpha_{l-1}\beta_{l-1}D^{\boldsymbol{\lambda}_{l-1}}. \tag{2.2.9}$$

Now combining (2.2.6) and (2.2.9), we get

$$(\delta_i(B)!)^2 D^{\boldsymbol{\lambda}_{l-1}}\sim\sum_{j=l}^{y}(\delta_i(B)!\alpha_j+\alpha_{l-1}\beta_j)D^{\boldsymbol{\lambda}_j}+\alpha_{l-1}\beta_{l-1}D^{\boldsymbol{\lambda}_{l-1}}.$$

and so by the uniqueness of the composition series of $(D^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\uparrow^{B}$ we have $\alpha_{l-1}\beta_{l-1}=(\delta_i(B)!)^2$ and $\alpha_j = 0 = \beta_j$ for all $l \leq j \leq y$. Thus, by (2.2.7) and (2.2.8) we obtain

$$D^{\boldsymbol{\lambda}_{l-1}}\downarrow_{\Phi_i(B)}\sim\alpha_{l-1}D^{\Phi_i(\boldsymbol{\lambda}_{l-1})} \quad \text{and} \quad D^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\uparrow^{B}\sim\beta_{l-1}D^{\boldsymbol{\lambda}_{l-1}}$$

with $\alpha_{l-1}\beta_{l-1}=(\delta_i(B)!)^2$. Hence, using Proposition 2.2.4 and (2.2.5) we have that

$$(S^{\boldsymbol{\lambda}_{l-1}})\downarrow_{\Phi_i(B)}\sim\sum_{j=l}^{y}\delta_i(B)!d_{\boldsymbol{\lambda}_{l-1}\boldsymbol{\lambda}_j}D^{\Phi_i(\boldsymbol{\lambda}_j)}+\alpha_{l-1}D^{\Phi_i(\boldsymbol{\lambda}_{l-1})}$$

and,

$$\delta_i(B)!S^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\sim\delta_i(B)!\left(\sum_{j=l}^{y}d_{\Phi_i(\boldsymbol{\lambda}_{l-1})\Phi_i(\boldsymbol{\lambda}_j)}D^{\Phi_i(\boldsymbol{\lambda}_j)}+D^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\right)$$

and so we can conclude that $d_{\boldsymbol{\lambda}_{l-1}\boldsymbol{\lambda}_j}=d_{\Phi_i(\boldsymbol{\lambda}_{l-1})\Phi_i(\boldsymbol{\lambda}_j)}$ for all $l \leq j \leq y$, $\alpha_{l-1} = \delta_i(B)!$ and so $\beta_{l-1}=\delta_i(B)!$. Therefore,

$$D^{\boldsymbol{\lambda}_{l-1}}\downarrow_{\Phi_i(B)}\sim\delta_i(B)!D^{\Phi_i(\boldsymbol{\lambda}_{l-1})} \quad \text{and} \quad D^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\uparrow^{B}\sim\delta_i(B)!D^{\boldsymbol{\lambda}_{l-1}}$$

and

$$S^{\Phi_i(\boldsymbol{\lambda}_{l-1})}\sim\sum_{j=l-1}^{y}d_{\boldsymbol{\lambda}_{l-1}\boldsymbol{\lambda}_j}D^{\Phi_i(\boldsymbol{\lambda}_j)}. \tag{2.2.10}$$

Thus, points 1. and 2. are proved for any $\boldsymbol{\lambda}_j$ with $1 \leq j \leq y$, point 3. follows immediately from (2.2.10), and point 4. follows from (2.2.3). $\square$

3

Full runner removal theorem for Ariki-Koike algebras

One of the most important outstanding problems in the representation theory of symmetric groups and related algebras is the determination of the decomposition numbers.

Ariki's theorem in [Ari96] tells us that the decomposition numbers for Iwahori-Hecke algebras and Ariki–Koike algebras in characteristic 0 are the *q-decomposition numbers* $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ evaluated at $q = 1$. The q-decomposition numbers arise from the Fock space representation of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$. This has a natural basis indexed by the set of partitions for $\mathcal{H}_n$ (or multipartitions for $\mathcal{H}_{r,n}$), and a *canonical basis* which is invariant under the bar involution. The q-decomposition numbers are the entries of the transition matrix between these two bases and therefore can be deduced from the computation of the canonical basis.

For Iwahori-Hecke algebras of $\mathfrak{S}_n$, there is a fast algorithm due to Lascoux, Leclerc and Thibon [LLT96] for computing the canonical basis.

For Ariki-Koike algebras, there are different generalisations of this algorithm due to Jacon [Jac05], Yvonne [Yvo07a] and Fayers [Fay10]. Yvonne's algorithm is very slow compared to the others, since it computes the canonical basis for the whole of the Fock space. Jacon's algorithm is faster, however it works in a particular type of twisted Fock space. Fayers' algorithm remains in the more natural setting of the untwisted Fock space; although the twisted and untwisted Fock spaces are isomorphic, so that in principle one canonical basis determines the other, it is in practice very difficult to give an explicit isomorphism.

In this chapter we introduce the Fock space representation of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$ and present the Fayers' LLT-type algorithm for Ariki-Koike algebras. They are the background of the second main result of this thesis, i.e. a 'full' runner removal theorem for Ariki-Koike algebras. As we outlined in the introduction, James and Mathas in [JM02] and Fayers in [Fay07a] proved an empty runner removal theorem and a full runner removal theorem for Iwahori-Hecke algebras, respectively.

We generalise Fayers' result to the case of Ariki-Koike algebras. For the class of e-multiregular multipartitions, we show that the q-decomposition numbers $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ and $d_{\boldsymbol{\lambda}^+\boldsymbol{\mu}^+}(q)$ coincide, where $\boldsymbol{\lambda}^+$ and $\boldsymbol{\mu}^+$ are the multipartitions obtained from the e-abacus display of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by adding a 'long enough' runner full of beads to each of their components.

Throughout this chapter, we let $e \geq 2$ be an integer and we identify I with the set $\mathbb{Z}/e\mathbb{Z}$.

3.1 An LLT-type algorithm for Ariki-Koike algebras

In this section we consider the integrable representation theory of the quantised enveloping algebra $\mathcal{U} = U_q(\widehat{\mathfrak{sl}}_e)$. For any dominant integral weight Λ for $\mathcal{U}$, the irreducible highest-weight module $V(\Lambda)$ for $\mathcal{U}$ can be constructed as a submodule $M^{\boldsymbol{s}}$ of a *Fock space* $\mathcal{F}^{\boldsymbol{s}}$ (which depends not just on Λ but on an ordering of the fundamental weights involved in Λ). Using the standard basis of the Fock space, one can define a *canonical basis* for $M^{\boldsymbol{s}}$. There is considerable interest in computing this canonical basis (that is, computing the transition coefficients from the canonical basis to the standard basis) because of Ariki's theorem, which says that these coefficients, evaluated at $q = 1$, yield decomposition numbers for certain cyclotomic Hecke algebras. In the case where Λ is of level 1, there is a fast algorithm due to Lascoux, Leclerc and Thibon [LLT96] for computing the canonical basis. The purpose of this section is to present the generalisation of this algorithm to higher levels given by Fayers in [Fay10]. The way Fayers does this is to compute the canonical basis for an intermediate module $M^{\otimes \boldsymbol{s}}$, which is defined to be the tensor product of level 1 highest-weight irreducibles. It is then straightforward to discard unwanted vectors to get the canonical basis for $M^{\boldsymbol{s}}$.

3.1.1 The quantum algebra $U_q(\widehat{\mathfrak{sl}}_e)$ and the Fock space

We use the following notation for multipartitions. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is an r-multipartition for $r > 1$, then we write $\boldsymbol{\lambda}_-$ for the $(r-1)$-multipartition $(\lambda^{(2)}, \ldots, \lambda^{(r)})$. If $\boldsymbol{\nu}$ is an $(r-1)$-multipartition, we write $\boldsymbol{\nu}_+$ for the r-multipartition $(\varnothing, \nu^{(1)}, \ldots, \nu^{(r-1)})$. Finally, if $\boldsymbol{\mu}$ is an r-multipartition, we write $\boldsymbol{\mu}_0$ for the r-multipartition $(\boldsymbol{\mu}_-)_+ = (\varnothing, \mu^{(2)}, \ldots, \mu^{(r)})$. We write $\mathcal{P}^r$ for the set of r-multipartitions.

Definition 3.1.1. We say that a multipartition $\boldsymbol{\lambda}$ is **e-multiregular** if $\lambda^{(j)}$ is e-regular for each j. We write $\mathcal{R}$ for the set of e-regular partitions and $\mathcal{R}^r$ for the set of all e-multiregular r-multipartitions, if e is understood.

We let $\mathcal{U}$ denote the quantised enveloping algebra $U_q(\widehat{\mathfrak{sl}}_e)$. This is a $\mathbb{Q}(q)$-algebra with generators e_i, f_i for $i \in I$ and q^h for $h \in P^\vee$, where $P^\vee$ is a free $\mathbb{Z}$-module with basis $\{h_i \mid i \in I\} \cup \{d\}$. Denote the dual basis of $(P^\vee)^*$ by

$\{\Lambda_0, \ldots, \Lambda_{e-1}, \delta\}$. These generators are subjected to the following relations

$$q^h q^{h'} = q^{h+h'}, \quad q^0 = 1,$$

$$q^h e_j q^{-h} = q^{\langle \alpha_j, h\rangle} e_j,$$

$$q^h f_j q^{-h} = q^{-\langle \alpha_j, h\rangle} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j\rangle \\ k \end{bmatrix} e_i^{1-\langle \alpha_i, h_j\rangle - k} e_j e_i^k = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j\rangle \\ k \end{bmatrix} f_i^{1-\langle \alpha_i, h_j\rangle - k} f_j f_i^k = 0 \quad (i \neq j),$$

where $\alpha_j = a\Lambda_i - \lambda_{i-1} - \Lambda_{i+1} + \delta_{i0}\delta$ for $i = 0, 1, \ldots, e-1$. Here we follow the usual notation for q-integers, q-factorials and q-binomial coefficients:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k][k-1]\cdots[1], \quad \begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]![k]!}.$$

For any integer $m > 0$, we write $f_i^{(m)}$ to denote the quantum divided power $f_i^m/[m]!$.

There are various choices for a comultiplication which makes $\mathcal{U}$ into a Hopf algebra (and hence allows us to regard the tensor product of two $\mathcal{U}$-modules as a $\mathcal{U}$-module). We use the comultiplication denoted Δ in [Kas02], which is defined by

$$\begin{aligned} \Delta : e_i &\longmapsto e_i \otimes q^{-h_i} \ + \ 1 \otimes e_i, \\ f_i &\longmapsto f_i \otimes 1 \ + \ q^{h_i} \otimes f_i, \\ q^h &\longmapsto q^h \otimes q^h \end{aligned}$$

for all $i \in I$ and all $h \in P^\vee$.

The $\mathbb{Q}$-linear ring automorphism $^- : \mathcal{U} \to \mathcal{U}$ defined by

$$\overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h}$$

for $i \in I$ and $h \in P^\vee$ is called the **bar involution**.

Now we fix $\boldsymbol{s} \in I^r$ for some $r \geq 1$, and define the **Fock space** $\mathcal{F}^{\boldsymbol{s}}$ to be the $\mathbb{Q}(q)$-vector space with a basis $\{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in \mathcal{P}^r\}$, which we call the **standard basis**. This has the structure of a $\mathcal{U}$-module: for a full description of the module action, we refer to [Fay10]. Here, we describe the action of the generators $f_0, \ldots, f_{e-1}$.

Given $\boldsymbol{\lambda}, \boldsymbol{\xi} \in \mathcal{P}^r$, then we write $\boldsymbol{\lambda} \xrightarrow{m:i} \boldsymbol{\xi}$ to indicate that $\boldsymbol{\xi}$ is obtained from $\boldsymbol{\lambda}$

by adding m addable i-nodes. If this is the case, then we consider the total order $>$ on addable and removable nodes and so we define the integer

$$N_i(\boldsymbol{\lambda}, \boldsymbol{\xi}) = \sum_{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}} ((\text{number of addable } i\text{-nodes of } \boldsymbol{\xi} \text{ above } \mathfrak{n}) - (\text{number of removable } i\text{-nodes of } \boldsymbol{\lambda} \text{ above } \mathfrak{n})). \tag{3.1.1}$$

Now the action of $f_i^{(m)}$ is given by

$$f_i^{(m)} \boldsymbol{\lambda} = \sum_{\boldsymbol{\lambda} \xrightarrow{m:i} \boldsymbol{\xi}} q^{N_i(\boldsymbol{\lambda}, \boldsymbol{\xi})} \boldsymbol{\xi}.$$

Proposition 3.1.2. Let $i \in I$. Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}$ are r-multipartitions such that $\boldsymbol{\lambda} \xrightarrow{m:i} \boldsymbol{\xi}$. Then

$$N_i(\boldsymbol{\lambda}, \boldsymbol{\xi}) = \sum_{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}} \left(N_i(\lambda^{(J_\mathfrak{n})}, \xi^{(J_\mathfrak{n})}) + \sum_{j=1}^{J_\mathfrak{n}-1} N_i(\lambda^{(j)}, \xi^{(j)}) \right),$$

where $J_\mathfrak{n}$ is the component of $\mathfrak{n}$ in $\boldsymbol{\xi}$.

Proof. This follows from the definition of $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi})$ and from the total order $>$ on the set of all addable and removable nodes in a multipartition (Subsection 1.4.5). Indeed, for each $\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}$, a term of $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi})$ consists of

$$\#\{\text{addable } i\text{-nodes of } \boldsymbol{\xi} \text{ above } \mathfrak{n}\} - \#\{\text{removable } i\text{-nodes of } \boldsymbol{\lambda} \text{ above } \mathfrak{n}\},$$

and the nodes above $\mathfrak{n}$ are exactly those above $\mathfrak{n}$ in the component $J_\mathfrak{n}$ and all the nodes in components j with $j < J_\mathfrak{n}$. So, for each $\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}$, a term of $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi})$ consists of

$$N_i(\lambda^{(J_\mathfrak{n})}, \xi^{(J_\mathfrak{n})}) + \sum_{j=1}^{J_\mathfrak{n}-1} N_i(\lambda^{(j)}, \xi^{(j)})$$

where $N_i(\lambda^{(J_\mathfrak{n})}, \xi^{(J_\mathfrak{n})})$ is given by (3.1.1), and for $j < J_\mathfrak{n}$

$$N_i(\lambda^{(j)}, \xi^{(j)}) = \#\{\text{addable } i\text{-nodes of } \xi^{(j)}\} - \#\{\text{removable } i\text{-nodes of } \lambda^{(j)}\}.$$

□

The Fock space is of interest because the submodule $M^{\boldsymbol{s}}$ generated by $\boldsymbol{\varnothing}$ is isomorphic to the irreducible highest-weight module $V(\Lambda_{s_1} + \cdots + \Lambda_{s_r})$. This submodule inherits a bar involution from $\mathcal{U}$: this is defined by $\overline{\boldsymbol{\varnothing}} = \boldsymbol{\varnothing}$ and $\overline{um} = \overline{u}\,\overline{m}$ for all $u \in \mathcal{U}$ and $m \in M^{\boldsymbol{s}}$. This bar involution allows one to define a **canonical basis** for $M^{\boldsymbol{s}}$; this consists of vectors $G^{\boldsymbol{s}}(\boldsymbol{\mu})$, for $\boldsymbol{\mu}$ lying in some subset of $\mathcal{P}^r$ (with our conventions, this is what Brundan and Kleshchev [BK09] call the set of *regular multipartitions*). These canonical basis vectors are characterised by the following properties:

- $\overline{G^{\boldsymbol{s}}(\boldsymbol{\mu})} = G^{\boldsymbol{s}}(\boldsymbol{\mu})$;
- if we write $G^{\boldsymbol{s}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}^r} d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\boldsymbol{\lambda}$ with $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{Q}(q)$, then $d^{\boldsymbol{s}}_{\boldsymbol{\mu}\boldsymbol{\mu}}(q) = 1$, while $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in q\mathbb{Z}[q]$ if $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$; in particular, $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = 0$ unless $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$.

A lot of effort has been put in computing the canonical basis elements (i.e. computing the transition coefficients $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$), because of the following theorem.

Theorem 3.1.3. [Ari96, Theorem 4.4] Let $\mathbb{F}$ be a field of characteristic 0 and $\boldsymbol{s} \in I^r$ be a multicharge. Suppose $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ are r-multipartitions of n with $\boldsymbol{\mu}$ dual Kleshchev. Then

$$[S'(\boldsymbol{\lambda})\colon D'(\boldsymbol{\mu})] = d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1).$$

This theorem, indeed, says that the coefficients $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ specialised at $q = 1$ give the decomposition numbers of the corresponding cyclotomic Hecke algebras. In fact, thanks to the work of Brundan and Kleshchev [BK09], the coefficients $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ (with q still indeterminate) can be regarded as *graded decomposition numbers* where 'graded' is referred to the $\mathbb{Z}$-grading that Ariki-Koike algebras inherits by being isomorphic to certain quotients of KLR algebras.

In [BK09], they also extend the bar involution on $M^{\boldsymbol{s}}$ to the whole of $\mathcal{F}^{\boldsymbol{s}}$. The way of showing this involves using Uglov's construction [Ugl99] of twisted Fock spaces, and then taking a limit via Yvonne's theorem [Yvo07b]. The extension of the bar involution to $\mathcal{F}^{\boldsymbol{s}}$ yields a canonical basis for the whole of $\mathcal{F}^{\boldsymbol{s}}$, indexed by the set of all r-multipartitions. In particular we get the following result.

Theorem 3.1.4. [Fay10] For each multipartition $\boldsymbol{\mu}$, there is a unique vector

$$G^{\boldsymbol{s}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}^r} d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\boldsymbol{\lambda} \in \mathcal{F}^{\boldsymbol{s}} \text{ with } d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{Q}(q)$$

such that

- $\overline{G^{\boldsymbol{s}}(\boldsymbol{\mu})} = G^{\boldsymbol{s}}(\boldsymbol{\mu})$;
- $d^{\boldsymbol{s}}_{\boldsymbol{\mu}\boldsymbol{\mu}}(q) = 1$, while $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in q\mathbb{Z}[q]$ if $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$;
- $d^{\boldsymbol{s}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = 0$ unless $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$.

In principle, Uglov's construction gives an algorithm for computing the canonical basis of $M^{\boldsymbol{s}}$. However, in practice this algorithm is extremely slow. So, we introduce and use Fayers' algorithm that is much faster.

Fayers' approach is to compute the canonical basis for a module lying in between $M^{\boldsymbol{s}}$ and $\mathcal{F}^{\boldsymbol{s}}$. The way $\mathcal{F}^{\boldsymbol{s}}$ is defined and the choice of coproduct on $\mathcal{U}$ mean that there is an isomorphism

$$\mathcal{F}^{\boldsymbol{s}} \xrightarrow{\sim} \mathcal{F}^{(s_1)} \otimes \cdots \otimes \mathcal{F}^{(s_r)}$$

defined by linear extension of

$$\boldsymbol{\lambda} \longmapsto (\lambda^{(1)}) \otimes \cdots \otimes (\lambda^{(r)}).$$

We will henceforth identify $\mathcal{F}^{\boldsymbol{s}}$ and $\mathcal{F}^{(s_1)} \otimes \cdots \otimes \mathcal{F}^{(s_r)}$ via this isomorphism. Since each $\mathcal{F}^{(s_k)}$ contains a submodule $M^{(s_k)}$ isomorphic to $V(\Lambda_{s_k})$, $\mathcal{F}^{\boldsymbol{s}}$ contains a submodule $M^{\otimes \boldsymbol{s}} = M^{(s_1)} \otimes \cdots \otimes M^{(s_r)}$ isomorphic to $V(\Lambda_{s_1}) \otimes \ldots \otimes V(\Lambda_{s_r})$. This algorithm will compute the canonical basis of $M^{\otimes \boldsymbol{s}}$.

For presenting Fayers' LLT-type algorithm, we need the following result on canonical basis coefficients, since it will allow us to apply one of the steps of the algorithm. Recall that for any r-multipartition $\boldsymbol{\lambda}$ we define $\boldsymbol{\lambda}_- = (\lambda^{(2)}, \ldots, \lambda^{(r)})$; we also define $\boldsymbol{s}_- = (s_2, \ldots, s_r)$ for $\boldsymbol{s} \in I^r$.

Corollary 3.1.5. [Fay10, Corollary 3.2] Suppose $\boldsymbol{s} \in I^r$ for $r > 1$ and $\boldsymbol{\mu} \in \mathcal{P}^r$ with $\mu^{(1)} = \varnothing$. If we write

$$G^{\boldsymbol{s}_-}(\boldsymbol{\mu}_-) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^{r-1}} d^{\boldsymbol{s}_-}_{\boldsymbol{\nu}\boldsymbol{\mu}_-} \boldsymbol{\nu},$$

then

$$G^{\boldsymbol{s}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^{r-1}} d^{\boldsymbol{s}_-}_{\boldsymbol{\nu}\boldsymbol{\mu}_-} \boldsymbol{\nu}_+.$$

3.1.2 The LLT algorithm for $r = 1$

Before presenting the Fayers' LLT-type algorithm for Ariki-Koike algebras, we restrict attention to the case $r = 1$, and explain the LLT algorithm for computing canonical basis elements $G^{(s_1)}(\mu)$. (In fact, the superscript $^{(s_1)}$ is unnecessary here, because $G^{(s_1)}(\mu)$ is independent of s_1; in general, $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ should be unchanged if a fixed element of I is added to $s_1, \ldots, s_r$ simultaneously.) The LLT algorithm was first described in the paper [LLT96], to which we refer for more details and examples.

The canonical basis elements for $M^{(s_1)}$ are indexed by the e-regular partitions. To construct $G^{(s_1)}(\mu)$ when μ is e-regular, we begin by constructing an auxiliary vector $A(\mu)$. Let $l_1 < \cdots < l_t$ be the values of l for which $\mathcal{L}_l(\mu)$ is non-empty. For each k, let m_k denote the number of nodes in $\mathcal{L}_{l_k}(\mu)$, and let i_k denote the residue of $\mathcal{L}_{l_k}$. Then the vector $A(\mu)$ is defined by

$$A(\mu) = f_{i_t}^{(m_t)} \dots f_{i_1}^{(m_1)} \cdot \varnothing.$$

$A(\mu)$ is obviously bar-invariant, and a lemma due to James [JK81, 6.3.54 & 6.3.55] implies that when we expand $A(\mu)$ as

$$A(\mu) = \sum_{\nu \in \mathcal{P}} a_\nu \nu,$$

we have $a_\mu = 1$, while $a_\nu = 0$ unless $\mu \trianglerighteq \nu$. This means that $A(\mu)$ must equal $G^{(s_1)}(\mu)$ plus a $\mathbb{Q}(q + q^{-1})$-linear combination of canonical basis vectors $G^{(s_1)}(\nu)$ with $\mu \triangleright \nu$. Assuming (by induction on the dominance order) that these $G^{(s_1)}(\nu)$ have been computed, it is straightforward to subtract the appropriate multiples of these vectors from $A(\mu)$ to recover $G^{(s_1)}(\mu)$. Moreover, the fact that the coefficients of the standard basis elements in $A(\mu)$ all lie in $\mathbb{Z}[q, q^{-1}]$ means that the coefficients of the canonical basis elements in $A(\mu)$ lie in $\mathbb{Z}[q + q^{-1}]$. A more precise description of the procedure to strip off these canonical basis elements is given in the algorithm in Section 3.1.3.

3.1.3 An LLT-type algorithm for $r \geq 1$

Now, following [Fay10] we give an algorithm for Ariki-Koike algebras which generalises the LLT algorithm for $r = 1$. As mentioned above, this algorithm actually computes the canonical basis of $M^{\otimes \boldsymbol{s}} \cong M^{(s_1)} \otimes \dots \otimes M^{(s_r)}$.

Since the canonical basis elements $G^{(s_k)}(\mu)$ indexed by e-regular partitions μ form a basis for $M^{(s_k)}$, the tensor product $M^{(s_1)} \otimes \dots \otimes M^{(s_r)}$ has a basis consisting of all vectors $G^{(s_1)}(\mu^{(1)}) \otimes \dots \otimes G^{(s_r)}(\mu^{(r)})$, where $\mu^{(1)}, \dots, \mu^{(r)}$ are e-regular partitions. Translating this to the Fock space $\mathcal{F}^{\boldsymbol{s}}$, we find that $M^{\otimes \boldsymbol{s}}$ has a basis consisting of vectors

$$H^{\boldsymbol{s}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}^r} d^{(s_1)}_{\lambda^{(1)} \mu^{(1)}} \dots d^{(s_r)}_{\lambda^{(r)} \mu^{(r)}} \boldsymbol{\lambda}$$

for all e-multiregular multipartitions $\boldsymbol{\mu}$. In fact, Fayers shows that

Proposition 3.1.6. [Fay07b, Proposition 4.2] The canonical basis vectors $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ indexed by e-multiregular r-multipartitions $\boldsymbol{\mu}$ form a basis for the module $M^{\otimes \boldsymbol{s}}$.

This implies in particular that the span of these vectors is a $\mathcal{U}$-submodule of $\mathcal{F}^{\boldsymbol{s}}$, which will enable our recursive algorithm to work. Proposition 3.1.6 enables us to construct canonical basis vectors labelled by e-multiregular multipartitions recursively. As in the LLT algorithm, the idea is that to construct the canonical basis vector $G^{\boldsymbol{s}}(\boldsymbol{\mu})$, we construct an auxiliary vector $A(\boldsymbol{\mu})$ which is bar-invariant, and which we know equals $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ plus a linear combination of 'lower' canonical basis vectors; the bar-invariance of $A(\boldsymbol{\mu})$, together with dominance properties, allows these lower terms to be stripped off. In our algorithm, we take additional care to make sure that $A(\boldsymbol{\mu})$ lies in $M^{\otimes \boldsymbol{s}}$; then we know by Proposition 3.1.6 that all the canonical basis vectors occurring in $A(\boldsymbol{\mu})$ are labelled by e-multiregular multipartitions, and therefore we can assume that these have already been constructed.

In fact, the proof of Proposition 3.1.6, combined with the construction in the LLT algorithm for partitions, gives us an LLT-type algorithm for Ariki-Koike algebras. We formalise this as follows.

Our algorithm is recursive, using a partial order on multipartitions which is finer than the dominance order: define $\boldsymbol{\mu} \succcurlyeq \boldsymbol{\nu}$ if either $|\mu^{(1)}| > |\nu^{(1)}|$ or $\mu^{(1)} \trianglerighteq \nu^{(1)}$. If $r > 1$, we assume when computing $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathcal{R}^r$ that we have already computed the vector $G^{\boldsymbol{s}-}(\boldsymbol{\mu}_-)$, and that we have computed $G^{\boldsymbol{s}}(\boldsymbol{\nu})$ for all $\boldsymbol{\nu} \in \mathcal{R}^r$ with $\boldsymbol{\mu} \succ \boldsymbol{\nu}$.

1. If $\boldsymbol{\mu} = \boldsymbol{\varnothing}$, then $G^{\boldsymbol{s}}(\boldsymbol{\mu}) = \boldsymbol{\varnothing}$.

2. If $\boldsymbol{\mu} \neq \boldsymbol{\varnothing}$ but $\mu^{(1)} = \varnothing$, then compute the canonical basis vector $G^{\boldsymbol{s}-}(\boldsymbol{\mu}_-)$. Then $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ is given by

$$G^{\boldsymbol{s}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^{r-1}} d^{\boldsymbol{s}-}_{\boldsymbol{\nu}\boldsymbol{\mu}_-}(q)\boldsymbol{\nu}_+.$$

3. If $\mu^{(1)} \neq \varnothing$, then apply the following procedure.

   (a) Let $\boldsymbol{\mu}_0 = (\varnothing, \mu^{(2)}, \dots, \mu^{(r)})$, and compute $G^{\boldsymbol{s}}(\boldsymbol{\mu}_0)$.

   (b) Let $m_1, \dots, m_t$ be the sizes of the non-empty ladders of $\mu^{(1)}$ in increasing order, and $i_1, \dots, i_t$ be their residues. Define $A(\boldsymbol{\mu}) = f_{i_t}^{(m_t)} \dots f_{i_1}^{(m_1)} G^{\boldsymbol{s}}(\boldsymbol{\mu}_0)$. Write $A(\boldsymbol{\mu}) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} a_{\boldsymbol{\nu}}\boldsymbol{\nu}$.

   (c) If there is no $\boldsymbol{\nu} \neq \boldsymbol{\mu}$ for which $a_{\boldsymbol{\nu}} \notin q\mathbb{Z}[q]$, then stop. Otherwise, take such a $\boldsymbol{\nu}$ which is maximal with respect to the dominance order, let α be the unique element of $\mathbb{Z}[q+q^{-1}]$ for which $a_{\boldsymbol{\nu}} - \alpha \in q\mathbb{Z}[q]$, replace $A(\boldsymbol{\mu})$ by $A(\boldsymbol{\mu}) - \alpha G^{\boldsymbol{s}}(\boldsymbol{\nu})$, and repeat. The remaining vector will be $G^{\boldsymbol{s}}(\boldsymbol{\mu})$.

The vector $A(\boldsymbol{\mu})$ computed in step 3 is a bar-invariant element of $M^{\boldsymbol{s}}$, because $G^{\boldsymbol{s}}(\boldsymbol{\mu}_0)$ is. Hence by Proposition 3.1.6 $A(\boldsymbol{\mu})$ is a $\mathbb{Q}(q+q^{-1})$-linear combination of canonical basis vectors $G^{\boldsymbol{s}}(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \mathcal{R}^r$. Furthermore, the rule for applying f_i to a multipartition and the combinatorial results used in the LLT algorithm imply that $a_{\boldsymbol{\mu}} = 1$, and that if $a_{\boldsymbol{\lambda}} \neq 0$, then $\boldsymbol{\mu} \succcurlyeq \boldsymbol{\lambda}$. In particular, the partition $\boldsymbol{\nu}$ appearing in step 3(c) satisfies $\boldsymbol{\mu} \succ \boldsymbol{\nu}$; moreover, when $\alpha G^{\boldsymbol{s}}(\boldsymbol{\nu})$ is subtracted from $A(\boldsymbol{\mu})$, the condition that $a_{\boldsymbol{\mu}} = 1$ and $a_{\boldsymbol{\lambda}}$ is non-zero only for $\boldsymbol{\mu} \succcurlyeq \boldsymbol{\lambda}$ remains true (because of Proposition 3.1.4 and the fact that the order $\succcurlyeq$ refines the dominance order). So we can repeat, and complete step 3(c).

Example 3.1.7. Let us take $e = r = 2$, and write the set $I = \mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$. Take $\boldsymbol{s} = (0, 0)$.

- First let us compute the canonical basis element $G^{\boldsymbol{s}}(((2,1),(1)))$. In the level 1 Fock space $\mathcal{F}^{(0)}$, we have $G^{(0)}((1)) = (1)$, where the partition (1) really stands for the 1-multipartition $((1))$. The non-empty ladders of the partition $(2,1)$ are $\mathcal{L}_1$ and $\mathcal{L}_2$, of lengths $1, 2$ and residues $0, 1$ respectively. So we compute

$$
\begin{aligned}
A(((2,1),(1))) =& f_1^{(2)} f_0(\varnothing,(1)) \\
=&((2,1),(1)) + q((2),(2)) + q^2((2),(1^2)) + q^2((1^2),(2)) \\
&+ q^3((1^2),(1^2)) + q^4((1),(2,1)).
\end{aligned}
$$

  Since the coefficients in $A(((2,1),(1)))$ (apart from the leading one) are divisible by q, we have $A(((2,1),(1))) = G^{\boldsymbol{s}}(((2,1),(1)))$.

- Next we compute $G^{\boldsymbol{s}}(((4),\varnothing))$. This time our auxiliary vector is

$$
\begin{aligned}
A(((4),\varnothing)) =& f_1 f_0 f_1 f_0(\varnothing,\varnothing) \\
=& ((4),\varnothing) + q((3,1),\varnothing) + q((2,1^2),\varnothing) + q^2((1^4),\varnothing) \\
&+ (1+q^2)((2,1),(1)) + 2q((2),(2)) + 2q^2((2),(1^2)) \\
&+ 2q^2((1^2),(2)) + 2q^3((1^2),(1^2)) + (q^2+q^4)((1),(2,1)) \\
&+ q^2(\varnothing,(4)) + q^3(\varnothing,(3,1)) + q^3(\varnothing,(2,1^2)) + q^4(\varnothing,(1^4)).
\end{aligned}
$$

  And so we have

$$
\begin{aligned}
G^{\boldsymbol{s}}(((4),\varnothing)) =& A(((4),\varnothing)) - G^{\boldsymbol{s}}(((2,1),(1))) \\
=& ((4),\varnothing) + q((3,1),\varnothing) + q((2,1^2),\varnothing) + q^2((1^4),\varnothing) \\
&+ q^2((2,1),(1)) + q((2),(2)) + q^2((2),(1^2)) + q^2((1^2),(2)) \\
&+ q^3((1^2),(1^2)) + q^2((1),(2,1)) + q^2(\varnothing,(4)) + q^3(\varnothing,(3,1)) \\
&+ q^3(\varnothing,(2,1^2)) + q^4(\varnothing,(1^4)).
\end{aligned}
$$

3.2 Addition of a full runner

Here, we recall the definition presented in [Fay07a] of adding a runner 'full' of beads to the abacus display of a partition. This will provide us the setting to generalise this definition to the case of a multipartition.

3.2.1 Truncated abacus configuration

To begin, consider the abacus configuration for a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$. The set $B_a(\lambda)$ of β-numbers used to define the abacus is an infinite set, and as such we have an infinite amount of beads in the abacus configuration, in particular there is a point where every row to the north of this point is completely full of beads. Instead we can consider a *truncated* abacus configuration which has only finitely many beads on each runner, which we associate to a partition by filling in all the rows north of the highest beads with other beads. Conversely, if we are given a partition λ we can fix a truncated abacus configuration associated to it. Let N be an integer so that $x \in B_a(\lambda)$ whenever $x < Ne$. Then we define the truncated abacus configuration for λ to be the one corresponding to the set $B_a(\lambda) \cap \{Ne, Ne+1, \ldots\}$. Notice that in terms of truncated abacus configuration, it makes sense to talk about the number of beads in the abacus. In particular, the truncated abacus configuration of λ constructed in this way consists of $a - Ne$ beads. Indeed,

$$B_a(\lambda) \cap \{Ne, Ne+1, \ldots\} = \{\lambda_1 + a - 1, \ldots, \lambda_t + a - t, 0 + a - t - 1, \ldots, Ne\}.$$

Hence, the number of beads of the truncated abacus configuration is equal to

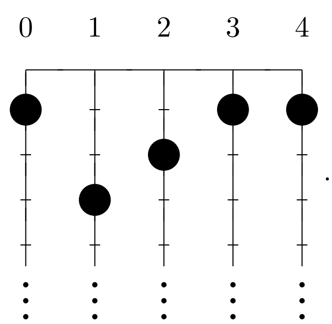
$$|\{\lambda_1 + a - 1, \ldots, \lambda_t + a - t, 0 + a - t - 1, \ldots, Ne\}|,$$

that is $t + a - t - 1 - Ne + 1 = \lambda'_1 + a - \lambda'_1 - 1 - Ne + 1 = a - Ne$. Thus, we will write $\mathrm{Ab}_e(\lambda)_{a-Ne}$ for the truncated e-abacus configuration of λ with $a - Ne$ beads.

Example 3.2.1. Suppose $\lambda = (7, 4, 2^2)$, $e = 5$ and $a = 0$. Then we take $N = -1$, so that

$$B_0(\lambda) \cap \{-5, -4, -3, \ldots\} = \{6, 2, -1, -2, -5\}.$$

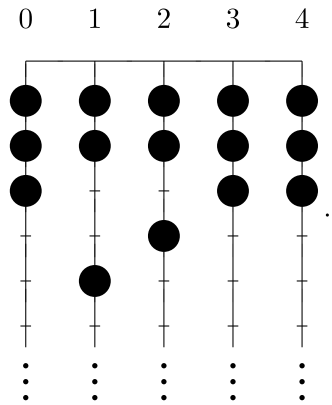
So, the truncated abacus display is



If we take $N = -3$, then we have

$$B_0(\lambda) \cap \{-15, -14, -13, \ldots\} \\ = \{6, 2, -1, -2, -5, -6, -7, -8, -9, -10, -11, -12, -13, -14, -15\}.$$

This choice of N gives the following truncated abacus configuration



Notice that for partitions the label of a runner does not correspond in general to the residue of the nodes represented by beads in that runner. Indeed, take $i, j \in I$, the beads on runner j correspond to i-nodes of a partition λ with $j \equiv i+b \mod e$ where b is the number of beads of the truncated abacus configuration of λ.

Now, we want to have a closer look at the truncated e-abacus configuration of the empty partition $\varnothing$.

Remark 3.2.2. Let $e \geq 2$. Any truncated e-abacus configuration of $\varnothing$ has all the beads as high as possible with the runners from 0 to i consisting of $h+1$ beads and the runners from $i+1$ to $e-1$ consisting of h beads for some $i \in I$ and some $h \geq 0$.

Proof. If $\mu = \varnothing$ and a is an integer, then its set of β-numbers is

$$B_a(\mu) = \{a-1, a-2, a-3, \ldots\}.$$

Write $a - 1 = Me + i$ with $M \in \mathbb{Z}$ and $0 \leq i \leq e-1$. Choosing $N \leq M$, we get

$$B_a(\mu) \cap \{Ne, Ne+1, \ldots\} = \{a-1, \ldots, Ne\},$$

that gives a truncated abacus configuration with $a - Ne = (M-N)e + i + 1$ beads and with all the positions in the abacus lower than $a-1$ filled with beads. The positions in the abacus lower than $a-1$ are all the positions in the rows above the one of $a-1$ and all the positions in the same row and to the left of $a-1$. In particular, setting $h := M - N$, there will be $h+1$ beads in the first

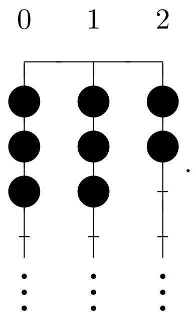
$i+1$ runners and h beads in the remaining $e-i-1$ runners. □

Example 3.2.3. Suppose $\mu = \varnothing$ and $e = 3$.

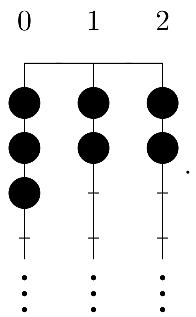
1. We choose $a = 2$. Then we take $N = -2$, so that

$$B_2(\mu) \cap \{-6, -5, -4, -3, \ldots\} = \{1, 0, -1, -2, -3, -4, -5, -6\}.$$

So, the truncated abacus display is



2. If we choose $a = 1$, we get the following truncated abacus display



3.2.2 Addition of a full runner for $r = 1$ and empty partition

Following [Fay07a], we define the addition of a 'full' runner for the abacus display of a partition.

Given a partition λ and a non-negative integer k, we construct a new partition λ^{+k} as follows. Let $a, N \in \mathbb{Z}$ such that $a \geq Ne$. Construct the truncated abacus configuration for λ with $b := a - Ne$ beads as in Section 3.2.1. Write $b+k = ce+d$, with $0 \leq d \leq e-1$, and add a runner to the abacus display immediately to the left of runner d; now put c beads on this new runner, in the top c positions, i.e. the position labelled $d, d+e+1, \ldots, d+(c-1)(e+1)$ in the usual labelling for an abacus with $e+1$ runners. The partition whose abacus display is obtained is λ^{+k}.

Remark 3.2.4. The beads in the new inserted runner of λ^{+k} correspond to nodes of residue k mod $(e+1)$.

Proof. Construct the truncated e-abacus configuration for λ with b beads as above. Write $b + k = ce + d$ with $0 \leq d \leq e - 1$. So the new inserted runner is labelled by d. We want to show that $d \equiv k + (b + c) \bmod (e + 1)$, since $b + c$ is the number of beads in the truncated $(e + 1)$-abacus configuration of λ^{+k}. This is true because $k + b + c = ce + d + c = c(e + 1) + d$. □

We extend the operator $^{+k}$ linearly to the whole of the Fock space $\mathcal{F}$. We can now state the full runner removal theorem for the Iwahori-Hecke algebras of $\mathfrak{S}_n$ in terms of canonical bases.

Theorem 3.2.5. [Fay07a, Theorem 3.1] Suppose μ is an e-regular partition and $k \geq \mu_1$. Then $G_{e+1}(\mu^{+k}) = G_e(\mu)^{+k}$.

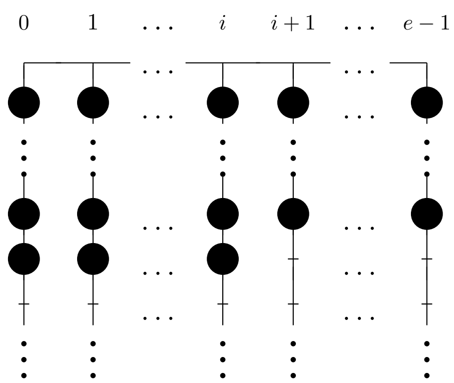
We want to focus our attention on the addition of a 'full' runner for the empty partition $\varnothing$.

Remark 3.2.6. Let k be a non-negative integer. Notice that $\varnothing^{+k}$ is an $(e+1)$-core. This follows by construction because in its abacus configuration all the beads are as high as possible.

Proposition 3.2.7. Let k be a non-negative integer.

1. If $k \in \{0, \dots, e\}$, then $\varnothing^{+k} = \varnothing$.
2. Let $k > e$ and $k = k_1 e + k_2$ with $k_1 \geq 1$ and $0 \leq k_2 \leq e - 1$.
   (a) If $k_2 = 0$, then $\varnothing^{+k} = ((k_1 - 1)e, (k_1 - 2)e, \dots, e)$.
   (b) If $k_2 \neq 0$, then $\varnothing^{+k} = ((k_1 - 1)e + k_2, (k_1 - 2)e + k_2, \dots, k_2)$.

Proof. Take a, $N \in \mathbb{Z}$ such that $a \geq Ne$ and construct the truncated abacus configuration of $\varnothing$ consisting of $b := a - Ne$ beads. Let i be the label of the runner of the last bead in $\mathrm{Ab}_e(\varnothing)$, we have $a - 1 = Me + i$ for some $M \in \mathbb{Z}$. Then $b = he + i + 1$ with $h := M - N$. By Remark 3.2.2, the truncated abacus display of $\varnothing$ looks like the following:



with $h+1$ beads in runners from 0 to i and h beads in runners from $i+1$ to $e-1$.

1. If $k = \{0, \dots, e\}$, then it is enough to show that the new inserted runner has either h or $h+1$ beads because by Remark 3.2.2 the resulting truncated $(e+1)$-abacus configuration of $\varnothing^{+k}$ represents $\varnothing$.

   - If $k \in \{0, \dots, e-i-2\}$, then $b+k = he+i+1+k$ with $i+1+k \in \{i+1, \dots, e-1\}$. Hence, the new inserted runner consists of h beads and it is on the left of runner $i+1+k$. By Remark 3.2.2, the runner i of $\mathrm{Ab}_e(\varnothing)$ has $h+1$ beads, and all the runners of $\mathrm{Ab}_e(\varnothing)$ from $i+1$ to $e-1$ has exactly h beads. So, $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ represents $\varnothing$.
   - If $k \in \{e-i-1, \dots, e\}$, then $b+k = (h+1)e+i+1+k-e$ with $i+1+k-e \in \{0, \dots, i+1\}$. Hence, the new inserted runner consists of $h+1$ beads and it is on the left of runner $i+1+k-e$. By Remark 3.2.2, all the runners of $\mathrm{Ab}_e(\varnothing)$ from 0 to i has exactly $h+1$ beads. So, $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ represents $\varnothing$.

2. If $k > e$, write $k = k_1 e + k_2$ with $k_1 \geq 1$ and $0 \leq k_2 \leq e-1$.

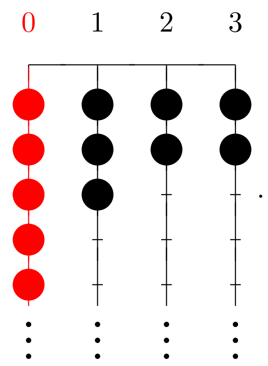
   (a) If $k_2 = 0$ and $i \neq e-1$, then $b+k = (h+k_1)e+i+1$. Hence, we add the new runner to the left of runner $i+1$ with $h+k_1$ beads in the top positions. Reading off the partition from $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ we get the partition $((k_1-1)e, (k_1-2)e, \dots, e)$. Indeed, there are k_1-1 beads after the first empty position in $\mathrm{Ab}_{e+1}(\varnothing^{+k})$, and all of them are beads of the new inserted runner, so between two consecutive beads there are e empty positions.
   If $k_2 = 0$ and $i = e-1$, then $b+k = (h+k_1+1)e$. Hence, we add the new runner to the left of runner 0 with $h+k_1+1$ beads in the top positions. As above, reading off the partition from $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ we get the partition $((k_1-1)e, (k_1-2)e, \dots, e)$.

   (b) If $k_2 \neq 0$ and $k_2 \in \{0, \dots, e-i-2\}$, then $b+k = (h+k_1)e+i+1+k_2$. Hence, we add the new runner to the left of runner $i+1+k_2$ with $h+k_1$ beads in the top positions. Reading off the partition from $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ we get the partition $((k_1-1)e+k_2, (k_1-2)e+k_2, \dots, k_2)$. Indeed, the first bead from the first empty position occurs after $(b+k)-b$ mod $e = k_2$ empty spaces. Also, there are k_1 beads after the first empty position in $\mathrm{Ab}_{e+1}(\varnothing^{+k})$, and all of them are beads of the new inserted runner, so between two consecutive beads there are e empty positions.
   If $k_2 \neq 0$ and $k_2 \in \{e-i-1, \dots, e-1\}$, then $b+k = (h+k_1+1)e + i+1+k_2-e$. As above, reading off the partition from $\mathrm{Ab}_{e+1}(\varnothing^{+k})$ we get the partition $((k_1-1)e+k_2, (k_1-2)e+k_2, \dots, k_2)$.

□

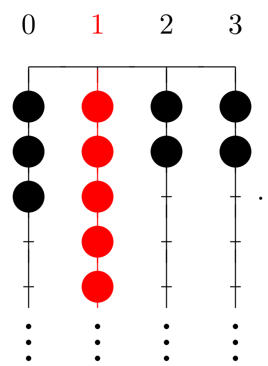
Example 3.2.8. Consider the empty partition and $e = 3$. Consider the abacus display in Example 3.2.3 with $b = 7$.

1. Take $k = 8 = 2 \cdot 3 + 2$. Then $\varnothing^{+8}$ has the following abacus configuration



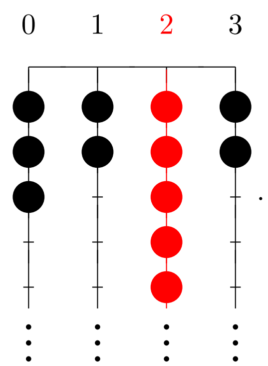
So, we have that $\varnothing^{+8} = (5, 2)$.

2. Take $k = 9 = 3 \cdot 3 + 0$. Then $\varnothing^{+9}$ has the following abacus configuration



So, we have that $\varnothing^{+9} = (6, 3)$.

3. Take $k = 10 = 3 \cdot 3 + 1$. Then $\varnothing^{+10}$ has the following abacus configuration



So, we have that $\varnothing^{+10} = (7, 4, 1)$.

Remark 3.2.9. All the removable nodes of $\varnothing^{+k}$ have the same residue because if there are any removable nodes in $\varnothing^{+k}$, they are the nodes corresponding to the beads in the new inserted runner and they have all the same residue $k \mod (e+1)$ as shown in Remark 3.2.4.

This remark is helpful because it allows us to find an induction sequence from $\varnothing$ to $\varnothing^{+k}$, as shown in the following. From now on, given an integer j we denote by $\overline{j}$ the residue of j modulo $e+1$ and by $\mathfrak{F}_{\overline{i}}$ for $i \in \mathbb{Z}$ the generator $f_{\overline{i}}$ of the quantised enveloping algebra $U_q(\widehat{\mathfrak{sl}}_{e+1})$. For $\ell \geq 1$ and $i \in \mathbb{Z}$, set

$$\mathcal{G}_{\overline{i}}^{(\ell)} := \mathfrak{F}_{\overline{i}}^{(\ell)} \mathfrak{F}_{\overline{i-1}}^{(\ell)} \dots \mathfrak{F}_{\overline{i-(e-1)}}^{(\ell)}. \tag{3.2.1}$$

Proposition 3.2.10. Let k be a non-negative integer. Write $k = k_1 e + k_2$ with $k_1 \geq 0$ and $0 \leq k_2 \leq e-1$. Then

$$\mathfrak{F}_{\overline{k}}^{(k_1)} \mathfrak{F}_{\overline{k-1}}^{(k_1)} \dots \mathfrak{F}_{\overline{k-k_2+1}}^{(k_1)} \mathcal{G}_{\overline{k-k_2}}^{(k_1-1)} \mathcal{G}_{\overline{k-k_2+1}}^{(k_1-2)} \dots \mathcal{G}_{\overline{k-k_2+k_1-2}}^{(1)}(\varnothing) = \varnothing^{+k}.$$

where $\mathfrak{F}_{\overline{k}}^{(k_1)} \mathfrak{F}_{\overline{k-1}}^{(k_1)} \dots \mathfrak{F}_{\overline{k-k_2+1}}^{(k_1)}$ only occur if $k_2 \neq 0$.

Proof. We prove this by induction on k.

- If $k < e$, then $\varnothing^{+k} = \varnothing$ by Proposition 3.2.7, so there is nothing to prove.
- Suppose $k \geq e$ and write $k = k_1 e + k_2$ with $k_1 \geq 1$ and $0 \leq k_2 \leq e-1$. Then by Lemma 3.3 in [Fay07a] we have $\varnothing^{+k} = \mathfrak{F}_{\overline{k}}^{(m)}(\varnothing^{+(k-1)})$ for some constant m. In particular, by the proof of Proposition 3.2.7 we know that
  - if $k_2 = 0$, then $m = k_1 - 1$;
  - if $k_2 \neq 0$, then $m = k_1$.

  We study the two cases separately.

  - If $k_2 = 0$, then $k - 1 = (k_1 - 1)e + e - 1$ and so by the induction hypothesis we have

$$\begin{aligned}\varnothing^{+(k-1)} &= \mathfrak{F}_{\overline{k-1}}^{(k_1-1)} \mathfrak{F}_{\overline{k-2}}^{(k_1-1)} \dots \mathfrak{F}_{\overline{k-1-e+1+1}}^{(k_1-1)} \mathcal{G}_{\overline{k-1-e+1}}^{(k_1-1-1)} \\ &\quad \mathcal{G}_{\overline{k-1-e+1+1}}^{(k_1-1-2)} \dots \mathcal{G}_{\overline{k-1-e+1+k_1-1-2}}^{(1)}(\varnothing) \\ &= \mathfrak{F}_{\overline{k-1}}^{(k_1-1)} \mathfrak{F}_{\overline{k-2}}^{(k_1-1)} \dots \mathfrak{F}_{\overline{k-e+1}}^{(k_1-1)} \mathcal{G}_{\overline{k+1}}^{(k_1-2)} \mathcal{G}_{\overline{k+2}}^{(k_1-3)} \dots \mathcal{G}_{\overline{k+k_1-2}}^{(1)}(\varnothing).\end{aligned}$$

    Hence,

$$\begin{aligned}\varnothing^{+k} &= \mathfrak{F}_{\overline{k}}^{(k_1-1)} \mathfrak{F}_{\overline{k-1}}^{(k_1-1)} \mathfrak{F}_{\overline{k-2}}^{(k_1-1)} \dots \mathfrak{F}_{\overline{k-e+1}}^{(k_1-1)} \mathcal{G}_{\overline{k+1}}^{(k_1-2)} \mathcal{G}_{\overline{k+2}}^{(k_1-3)} \dots \mathcal{G}_{\overline{k+k_1-2}}^{(1)}(\varnothing) \\ &= \mathcal{G}_{\overline{k}}^{(k_1-1)} \mathcal{G}_{\overline{k+1}}^{(k_1-2)} \mathcal{G}_{\overline{k+2}}^{(k_1-3)} \dots \mathcal{G}_{\overline{k+k_1-2}}^{(1)}(\varnothing).\end{aligned}$$

– If $k_2 \neq 0$, then $k-1 = k_1 e + k_2 - 1$ and so by induction hypothesis we have

$$\begin{aligned}\varnothing^{+(k-1)} &= \mathfrak{F}^{(k_1)}_{k-1}\mathfrak{F}^{(k_1)}_{k-1-1}\cdots\mathfrak{F}^{(k_1)}_{k-1-k_2+1+1}\mathcal{G}^{(k_1-1)}_{k-1-k_2+1}\\ &\quad\mathcal{G}^{(k_1-2)}_{k-1-k_2+1+1}\cdots\mathcal{G}^{(1)}_{k-1-k_2+1+k_1-2}(\varnothing)\\ &= \mathfrak{F}^{(k_1)}_{k-1}\mathfrak{F}^{(k_1)}_{k-2}\cdots\mathfrak{F}^{(k_1)}_{k-k_2+1}\mathcal{G}^{(k_1-1)}_{k-k_2}\mathcal{G}^{(k_1-2)}_{k-k_2+1}\cdots\mathcal{G}^{(1)}_{k-k_2+k_1-2}(\varnothing).\end{aligned}$$

Hence,

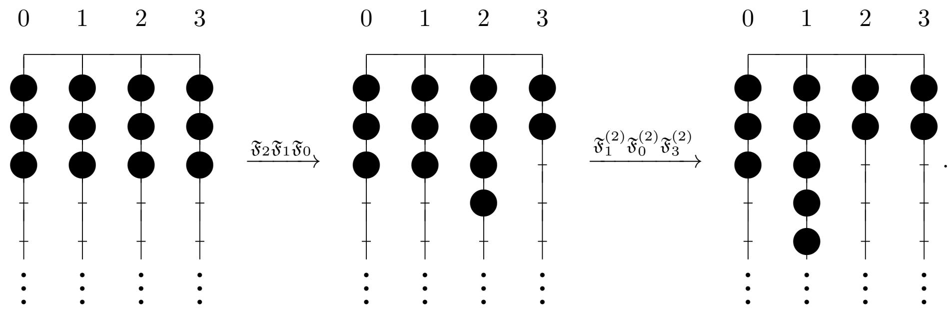
$$\varnothing^{+k} = \mathfrak{F}^{(k_1)}_{k}\mathfrak{F}^{(k_1)}_{k-1}\mathfrak{F}^{(k_1)}_{k-2}\cdots\mathfrak{F}^{(k_1)}_{k-k_2+1}\mathcal{G}^{(k_1-1)}_{k-k_2}\mathcal{G}^{(k_1-2)}_{k-k_2+1}\cdots\mathcal{G}^{(1)}_{k-k_2+k_1-2}(\varnothing).$$

□

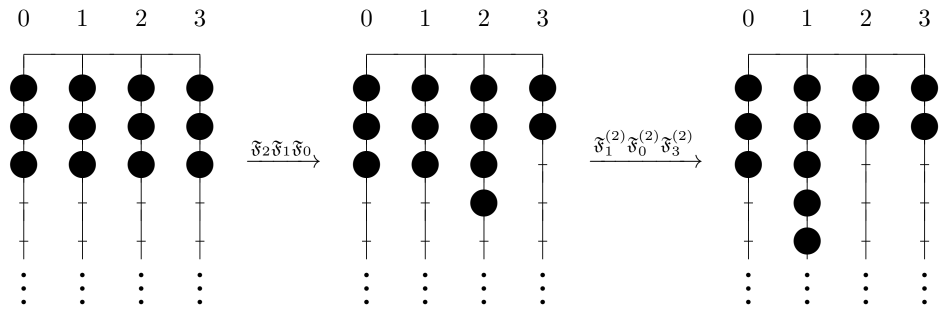
We give some examples of the induction sequence we want to deal with.

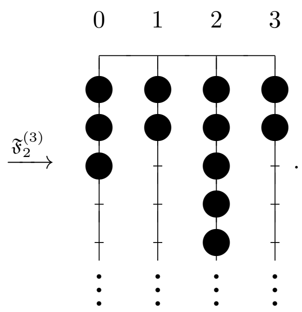
Example 3.2.11. With the same notation and choices of Example 3.2.8, the induction sequence given in Proposition 3.2.10 from $\varnothing$ to $\varnothing^{+k}$ acts as follows. Notice that at each step we apply the induction operators starting from the rightmost one.

2. For $k = 9$, in terms of abacus configuration we have



3. For $k = 10$, in terms of abacus configuration we have





Another useful result will be the next lemma that gives a way to establish when a sequence of induction operators $\mathfrak{F}_{\bar{i}}^{(m)}$ for $m, i \in \mathbb{Z}$ with $m \geq 0$ acts non-zero.

Lemma 3.2.12. Let $k \geq e+1$. Write $k = k_1 e + k_2$ with $k_1 \geq 1$ and $0 \leq k_2 \leq e-1$. Consider

$$\mathfrak{F}_{\overline{k+e+1}}^{(q_{e+1})} \mathfrak{F}_{\overline{k+e}}^{(q_e)} \dots \mathfrak{F}_{\overline{k+2}}^{(q_2)} \mathfrak{F}_{\overline{k+1}}^{(q_1)} (\varnothing^{+(k-e-1)}), \tag{3.2.2}$$

for some $q_j \geq 0$ for all $1 \leq j \leq e+1$. Then (3.2.2) $\neq 0$ if and only if

1. $k_1 - 1 \geq q_1 \geq q_2 \geq \ldots \geq q_{e+1-k_2}$, and $q_{e+2-k_2} \geq q_{e+3-k_2} \geq \ldots \geq q_{e+1}$, and,
2. if $q_{e+1-k_2} = k_1 - 1$, then $q_{e+1-k_2} \geq q_{e+2-k_2} - 1$;
   if $q_{e+1-k_2} < k_1 - 1$, then $q_{e+1-k_2} \geq q_{e+2-k_2}$.

Proof. We want to prove that (3.2.2) $\neq 0$ if and only if conditions 1. and 2. hold. This is equivalent to proving that

$$\mathfrak{F}_{\overline{k+j}}^{(q_j)} \dots \mathfrak{F}_{\overline{k+1}}^{(q_1)} (\varnothing^{+(k-e-1)}) \neq 0 \text{ for all } 1 \leq j \leq e+1 \tag{3.2.3}$$

if and only if conditions 1. and 2. hold. Recall that if s is the runner corresponding to nodes of residue $\overline{k+j}$, then $\mathfrak{F}_{\overline{k+j}}^{(q_j)}$ moves q_j beads from runner $s-1$ to runner s. Moreover, by Proposition 3.2.7, the addable nodes of residue $\overline{k+1}$ of $\varnothing^{+(k-e-1)}$ are the nodes corresponding to beads in the new inserted runner and there are $k_1 - 1$ of them.

In particular, given the abacus configuration of $\varnothing^{+(k-e-1)}$, for each $j \neq e+2-k_2$ the number of addable nodes of residue $\overline{k+j}$ of any term in $\mathfrak{F}_{\overline{k+j-1}}^{(q_{j-1})} \dots \mathfrak{F}_{\overline{k+1}}^{(q_1)} (\varnothing^{+(k-e-1)})$ is q_{j-1}. For $j = e+2-k_2$, the number of addable nodes of residue $\overline{k+j} = \overline{k+e+2-k_2} = \overline{k-k_2+1}$ of any term in $\mathfrak{F}_{\overline{k-k_2}}^{(q_{e+1-k_2})} \dots \mathfrak{F}_{\overline{k+1}}^{(q_1)} (\varnothing^{+(k-e-1)})$ is

- $q_{e+1-k_2} + 1$ if $q_{e+1-k_2} = k_1 - 1$,
- q_{e+1-k_2} if $q_{e+1-k_2} < k_1 - 1$.

We proceed by induction on j.

- If $j = 1$, then $\mathfrak{F}_{\overline{k+1}}^{(q_1)} (\varnothing^{+(k-e-1)}) \neq 0$ if and only if $q_1 \leq k_1 - 1$ because the number of addable nodes of residue $\overline{k+1}$ of $\varnothing^{+(k-e-1)}$ is $k_1 - 1$.

- If $j > 1$, then by induction hypothesis we know

$$\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)}) := \mathfrak{F}^{(q_{j-1})}_{k+j-1} \cdots \mathfrak{F}^{(q_1)}_{k+1}(\varnothing^{+(k-e-1)}) \neq 0$$

  if and only if conditions 1. and 2. hold for $q_1, \ldots, q_{j-1}$. We want to prove that

$$\mathfrak{F}^{(q_j)}_{k+j}\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)}) \neq 0$$

  if and only if the conditions 1. and 2. hold also for q_j.

  - If $j \leq e+1-k_2$, then the number of addable nodes of residue $\overline{k+j}$ of any term in $\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)})$ is q_{j-1}. So, $\mathfrak{F}^{(q_j)}_{k+j}\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)}) \neq 0$ if and only if $q_{j-1} \geq q_j$.
  - If $j = e+2-k_2$, then the number of addable nodes of residue $\overline{k+j} = \overline{k-k_2+1}$ of any term in $\mathfrak{F}_{e+1-k_2}(\varnothing^{+(k-e-1)})$ is

$$\begin{cases} q_{e+1-k_2}+1 & \text{if } q_{e+1-k_2} = k_1 - 1, \\ q_{e+1-k_2} & \text{if } q_{e+1-k_2} < k_1 - 1. \end{cases}$$

    So, if $q_{e+1-k_2} = k_1 - 1$ then $\mathfrak{F}^{(q_{e+2-k_2})}_{k-k_2+1}\mathfrak{F}_{e+1-k_2}(\varnothing^{+(k-e-1)}) \neq 0$ if and only if $q_{e+1-k_2} \geq q_{e+2-k_2} - 1$. If $q_{e+1-k_2} < k_1 - 1$ then $\mathfrak{F}^{(q_{e+2-k_2})}_{k-k_2+1}\mathfrak{F}_{e+1-k_2}(\varnothing^{+(k-e-1)}) \neq 0$ if and only if $q_{e+1-k_2} \geq q_{e+2-k_2}$.
  - If $j > e+2-k_2$, then the number of addable nodes of residue $\overline{k+j}$ of any term of $\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)})$ is q_{j-1}. So, $\mathfrak{F}^{(q_j)}_{k+j}\mathfrak{F}_{j-1}(\varnothing^{+(k-e-1)}) \neq 0$ if and only if $q_{j-1} \geq q_j$.

□

Corollary 3.2.13. In the same notation of Lemma 3.2.12, if Equation (3.2.2) is non-zero then it holds that

- if $q_{e+1-k_2} < k_1 - 1$, then $q_{e+1} \leq q_1$;
- if $q_{e+1-k_2} = k_1 - 1$, then $q_{e+1} \leq q_1 + 1$.

Proof. It follows directly from writing the inequalities of conditions 1. and 2. of Lemma 3.2.12 one next to the other for the two cases. □

3.2.3 Addition of a full runner for $r \geq 2$

Let $\boldsymbol{\lambda}$ be a r-multipartition of n and $\kappa = (\kappa_1, \ldots, \kappa_r)$ be a multicharge for $\mathcal{H}_{r,n}$. For each $j \in \{1, \ldots r\}$, we represent each component $\lambda^{(j)}$ with a truncated abacus consisting of n_j beads such that $n_j \equiv \kappa_j \mod e$.

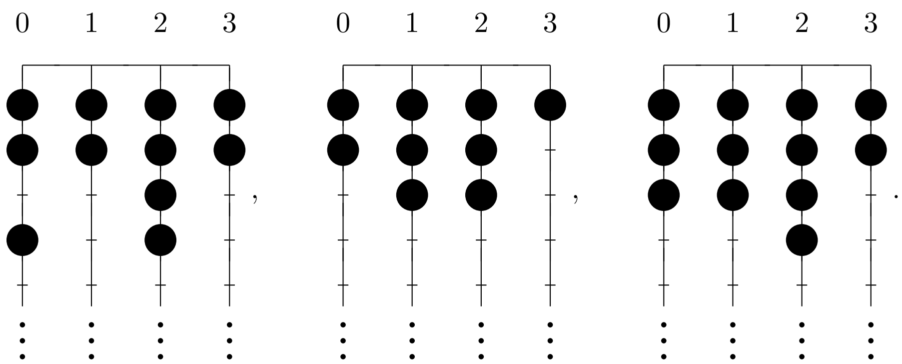
Given $\boldsymbol{\lambda}$ as above, we construct a new r-multipartition as follows. Let $0 \leq d \leq e-1$. For each $j \in \{1, \ldots, r\}$,

- take n_j determined as above and construct the abacus display for $\lambda^{(j)}$ with n_j beads;
- set $k^{(j)}$ a non-negative integer such that $n_j + k^{(j)} \equiv d \mod e$ for all $j \in \{1, \ldots, r\}$;
- write $n_j + k^{(j)} = c_j e + d$ for all $j \in \{1, \ldots, r\}$;
- add a runner to each component of the abacus display immediately to the left of runner d;
- for each j, put c_j beads on the new inserted runner of each component j, in the top c_j positions, i.e. the positions labelled by $d, d+e+1, \ldots, d+(c_j-1)(e+1)$ in the usual labelling for an abacus with $e+1$ runners.

The r-multipartition whose abacus is obtained is $\boldsymbol{\lambda}^{+\boldsymbol{k}} := \boldsymbol{\lambda}^{+(k^{(1)},\ldots,k^{(r)})}$.

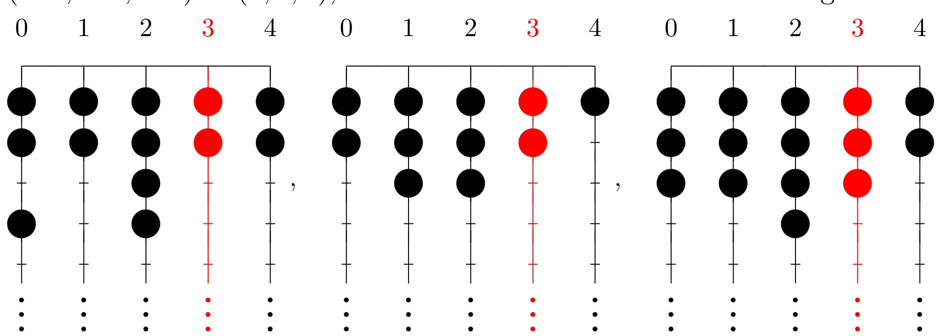
Remark 3.2.14. Notice that choosing the number of beads $(n_1, \ldots, n_r)$ in order to construct the abacus display of a multipartition $\boldsymbol{\lambda}$ corresponds to the choice of the multicharge $\boldsymbol{a} = (n_1, \ldots, n_r)$.

Examples 3.2.15. Suppose $\boldsymbol{\lambda} = ((4,3,2),(2^2),(3))$ and $e = 4$. Choose $\boldsymbol{n} = (n_1, n_2, n_3) = (11, 9, 12)$, that corresponds to choose $\boldsymbol{n}$ as multicharge. So we get the following abacus display:



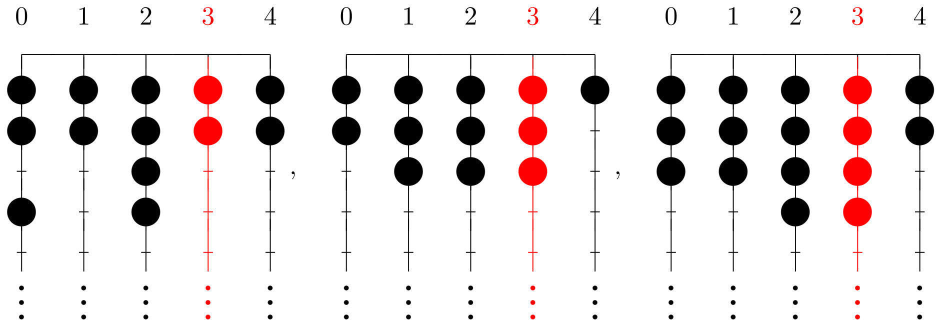
Then,

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (0, 2, 3)$, we obtain $\boldsymbol{\lambda}^{+(0,2,3)}$ with abacus configuration



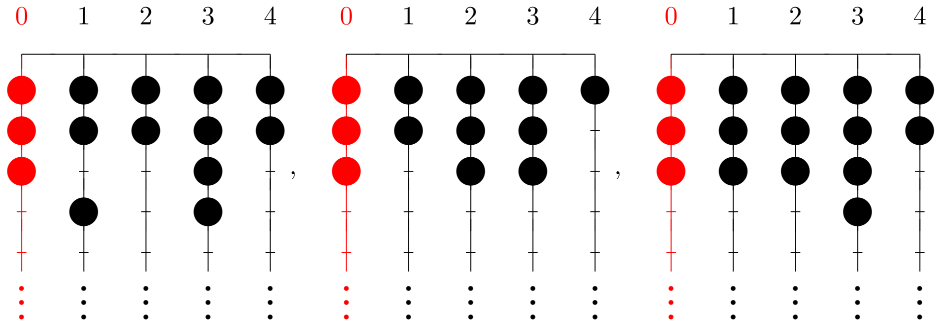
and multicharge equals to $(13, 11, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (0, 6, 7)$, we obtain $\boldsymbol{\lambda}^{+(0,6,7)}$ with abacus configuration



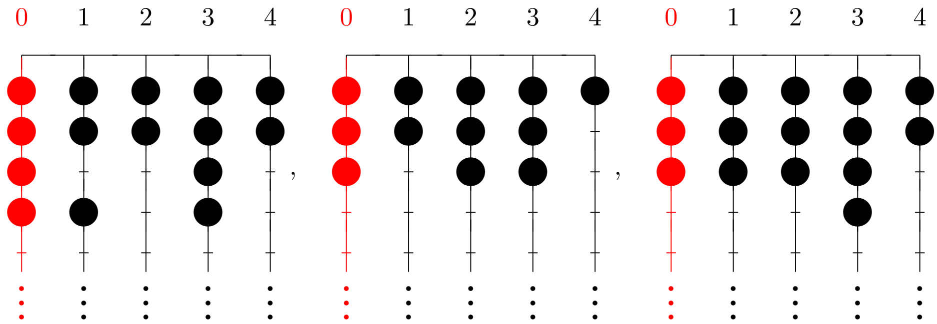
and multicharge equals to $(13, 12, 16)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (1, 3, 0)$, we obtain $\boldsymbol{\lambda}^{+(1,3,0)}$ with abacus configuration



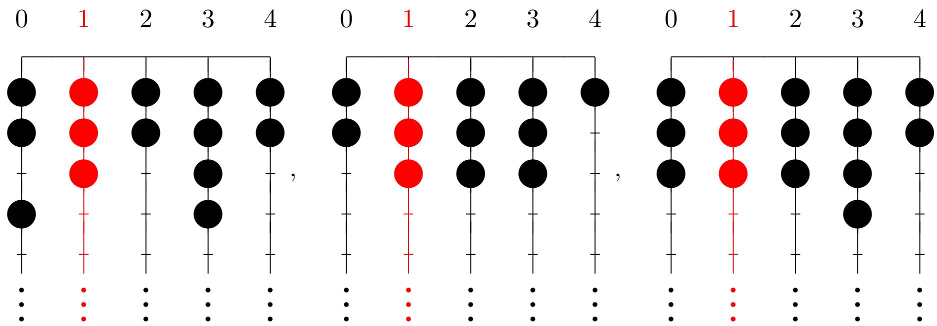
and multicharge equals to $(14, 12, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (5, 3, 0)$, we obtain $\boldsymbol{\lambda}^{+(5,3,0)}$ with abacus configuration



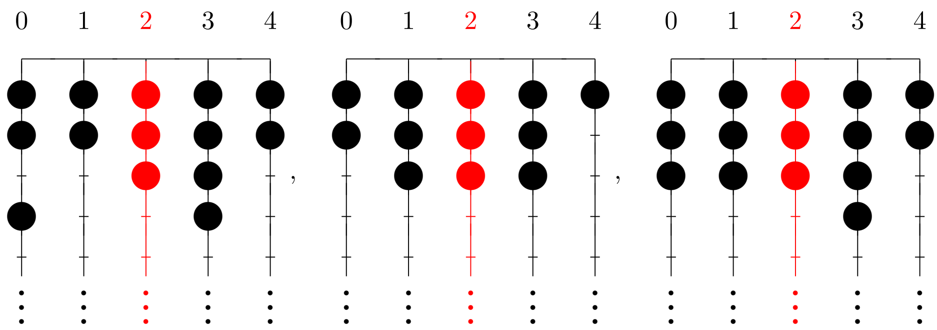
and multicharge equals to $(15, 12, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (2, 4, 1)$, we obtain $\boldsymbol{\lambda}^{+(2,4,1)}$ with abacus configuration



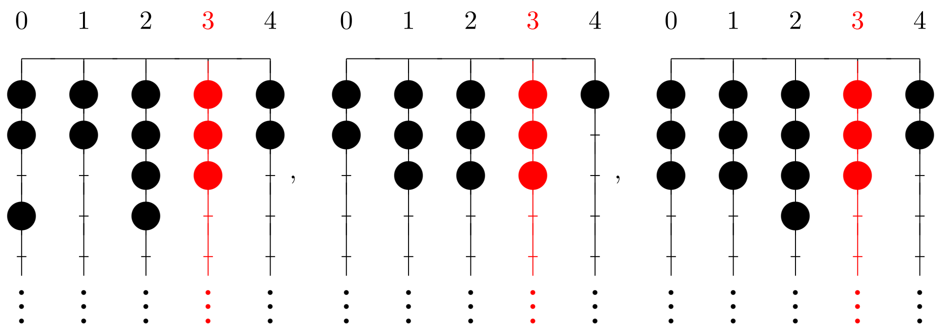
and multicharge equals to $(14, 12, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (3, 5, 2)$, we obtain $\boldsymbol{\lambda}^{+(3,5,2)}$ with abacus configuration



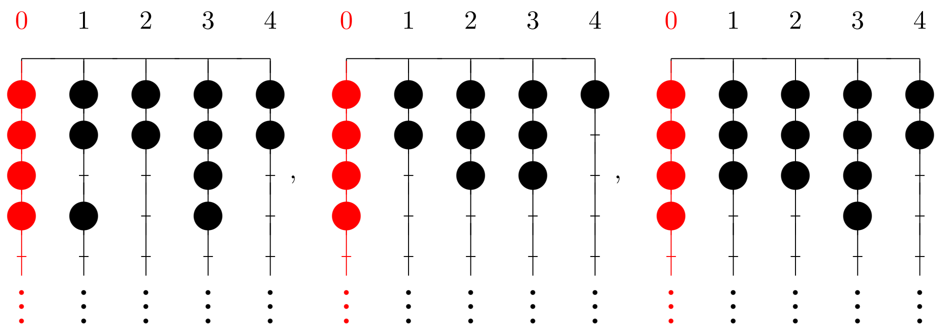
and multicharge equals to $(14, 12, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (4, 6, 3)$, we obtain $\boldsymbol{\lambda}^{+(4,6,3)}$ with abacus configuration



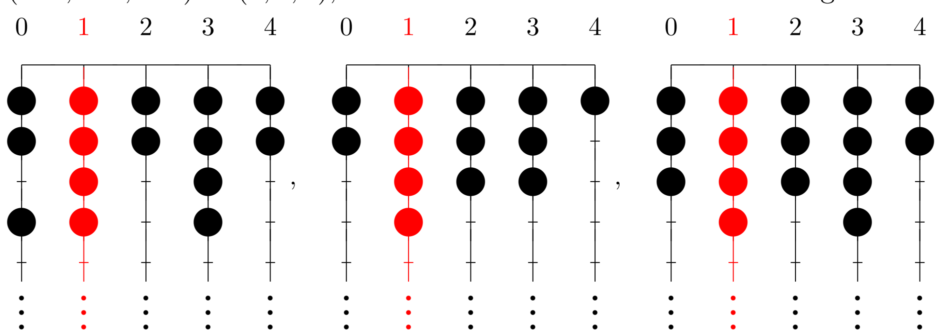
and multicharge equals to $(14, 12, 15)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (5, 7, 4)$, we obtain $\boldsymbol{\lambda}^{+(5,7,4)}$ with abacus configuration



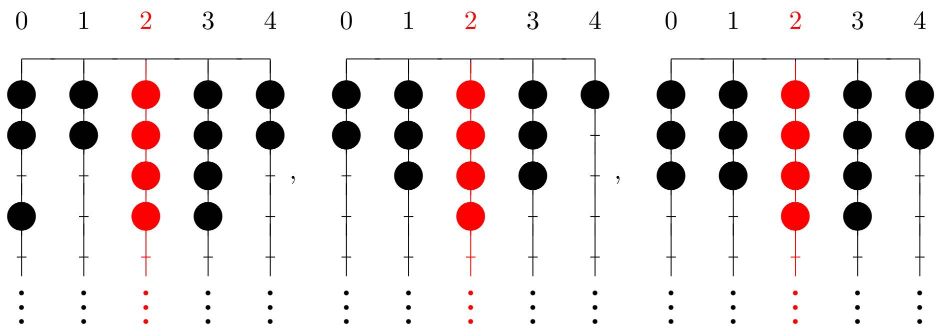
and multicharge equals to $(15, 13, 16)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (6, 8, 5)$, we obtain $\boldsymbol{\lambda}^{+(6,8,5)}$ with abacus configuration



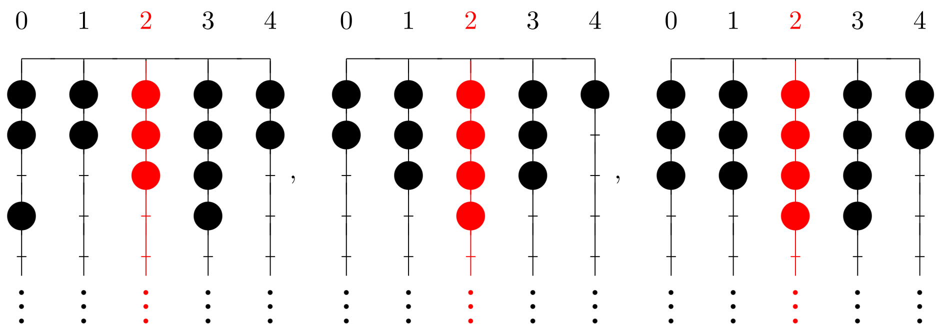
and multicharge equals to $(15, 13, 16)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (7, 9, 6)$, we obtain $\boldsymbol{\lambda}^{+(7,9,6)}$ with abacus configuration



and multicharge equals to $(15, 13, 16)$;

- if $(k^{(1)}, k^{(2)}, k^{(3)}) = (3, 9, 6)$, we obtain $\boldsymbol{\lambda}^{+(3,9,6)}$ with abacus configuration



and multicharge equals to $(14, 13, 16)$.

Notice that since we are considering another multicharge for $\boldsymbol{\lambda}^{+\boldsymbol{k}}$, namely the multicharge given by $(n_1 + c_1, \ldots, n_r + c_r)$ and we are labelling the runners in the usual way for an abacus display with $e + 1$ runners, the label of the each runner corresponds to the residue of the nodes corresponding to the beads in each runner.

Lemma 3.2.16. Let $e \geq 2$, and let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \ldots, \tilde{n}_r)$ be two choices of number of beads for the e-abacus display of $\boldsymbol{\lambda}$. Denote by $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ (respectively, $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$) the multipartition obtained applying $^{+\boldsymbol{k}}$ to $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\boldsymbol{n}}$ (respectively, $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\tilde{\boldsymbol{n}}}$). If $(\tilde{n}_1, \ldots, \tilde{n}_r) = (n_1 + h_1, \ldots, n_r + h_r)$ with $h_j \in \mathbb{Z}$ and $h_j \equiv h_1 \mod e$ for all $j \in \{1, \ldots, r\}$, then

1. $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}} = \boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$;

2. the underlying multicharges are the same up to a shift, that is, if $\boldsymbol{a} = (a_1, \ldots, a_r)$ is the multicharge corresponding to $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_r)$ is the multicharge corresponding to $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$, then there exists $s \in \{0, \ldots, e-1\}$ such that
$$a_j \equiv \tilde{a}_j + s \mod (e + 1).$$

   Therefore, there is an isomorphism between the Ariki-Koike algebra with multicharge $\boldsymbol{a}$ and the Ariki-Koike algebra with multicharge $\tilde{\boldsymbol{a}}$.

Proof. Consider an r-multipartition $\boldsymbol{\lambda}$ and construct two abacus displays for $\boldsymbol{\lambda}$, one with $\boldsymbol{n} = (n_1, \ldots, n_r)$ beads and the other with $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \ldots, \tilde{n}_r)$ beads where $\tilde{n}_j = n_j + h_j$ and $h_j \in \mathbb{Z}$ for each $j \in \{1, \ldots, r\}$.

Suppose that $h_1 \equiv \ldots \equiv h_r \bmod e$. Thus, write $h_j = c_{h_j}e + d_h$ with $0 \leq d_h \leq e-1$ for $j \in \{1, \ldots, r\}$. We want to show that $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$ represent the same multipartition. From the construction of $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$, we have for $j \in \{1, \ldots, r\}$

$$n_j + k_j = c_j e + d;$$
$$\tilde{n}_j + k_j = \tilde{c}_j e + \tilde{d}.$$

Hence, using $\tilde{n}_j = n_j + h_j$, we get

$$n_j + h_j + k_j = c_j e + d + h_j = \tilde{c}_j e + \tilde{d} \tag{3.2.4}$$

for each $j \in \{1, \ldots, r\}$. Now, we distinguish two cases:

- if $h_j \equiv 0 \bmod e$, i.e. $h_j = c_{h_j}e$ for some $c_{h_j} \in \mathbb{Z}$, the abacus display $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\tilde{\boldsymbol{n}}}$ is obtained from the abacus display $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\boldsymbol{n}}$ by adding or removing c_{h_j} entire rows of beads at the top of the abacus. Thus, by (3.2.4) we get

  $$\tilde{n}_j + k_j = c_j e + d + c_{h_j}e = (c_j + c_{h_j})e + d = \tilde{c}_j e + \tilde{d}.$$

  This implies $\tilde{c}_j = c_j + c_{h_j}$ and $\tilde{d} = d$, so we add a runner to the abacus display $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\tilde{\boldsymbol{n}}}$ immediately to the left of runner d and put $c_j + c_{h_j}$ beads on this new runner, in the top $c_j + c_{h_j}$ positions. Hence, $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$ represent the same multipartition because we add the runner in the same position of each abacus and the level of the last bead of the inserted runner is the same in both abacuses.

- if $h_j = c_{h_j}e + d_h$ for some $c_{h_j} \in \mathbb{Z}$ and $d_h \in \{1, \ldots, e-1\}$, the abacus display $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\tilde{\boldsymbol{n}}}$ is obtained from the abacus $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\boldsymbol{n}}$ by adding or removing c_{h_j} entire rows of beads and a row with d_h beads at the top of the abacus. Thus, by (3.2.4) we get

  $$\tilde{n}_j + k_j = c_j e + d + c_{h_j}e + d_h = (c_j + c_{h_j})e + (d + d_h) = \tilde{c}_j e + \tilde{d}.$$

  This implies $\tilde{c}_j = c_j + c_{h_j}$ and $\tilde{d} = d + d_h$, so we add a runner to the abacus display $\mathrm{Ab}_e(\boldsymbol{\lambda})_{\tilde{\boldsymbol{n}}}$ immediately to the left of runner $d + d_h$ and put $c_j + c_{h_j}$ beads on this new runner, in the top $c_j + c_{h_j}$ positions. Hence, $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$ represent the same multipartition because, in order to get $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$, we add the runner in a position translated of d_h compared to the one in $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ and the level of the last bead of the inserted runner is the same in both abacuses.

To conclude, the multicharge corresponding to $\boldsymbol{\lambda}_{\boldsymbol{n}}^{+\boldsymbol{k}}$ is $\boldsymbol{a} = (n_1 + c_1, \ldots, n_r + c_r)$ and the multicharge corresponding to $\boldsymbol{\lambda}_{\tilde{\boldsymbol{n}}}^{+\boldsymbol{k}}$ is $\tilde{\boldsymbol{a}} = (n_1 + h_1 + c_1 + c_{h_1}, \ldots, n_r +$

$h_r + c_r + c_{h_r})$. For all $j \in \{1, \dots, r\}$, we can write $h_j = c_{h_j} e + s$ for some $c_{h_j} \in \mathbb{Z}$ and $s \in \{0, \dots, e-1\}$ since $h_j \equiv h_1 \mod e$. Hence, for all $j \in \{1, \dots, r\}$

$$\begin{aligned}(n_j + h_j + c_j + c_{h_j}) - (n_j + c_j) &= h_j + c_{h_j} \\ &= c_{h_j} e + s + c_{h_j} \\ &= c_{h_j}(e+1) + s \\ &\equiv s \mod (e+1).\end{aligned}$$

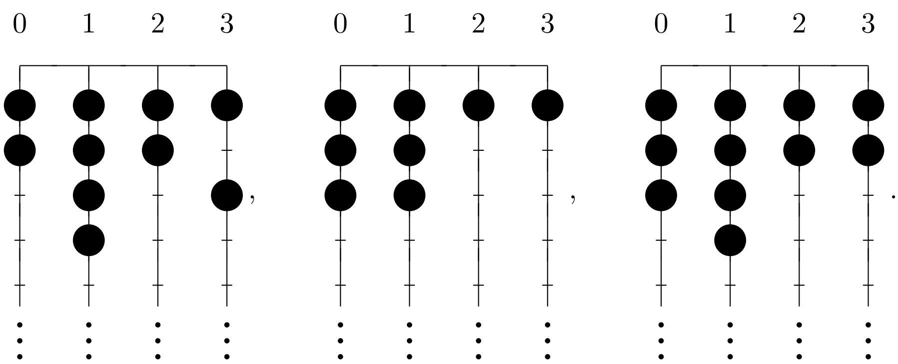
Thus, the isomorphism between the Ariki-Koike algebra with multicharge $\boldsymbol{a}$ and the one with multicharge $\tilde{\boldsymbol{a}}$ is given by

$$\begin{aligned}&T_0 \mapsto q^{-s} T_0 \\ &T_i \mapsto T_i \text{ for all } i = 1, \dots, n-1.\end{aligned}$$

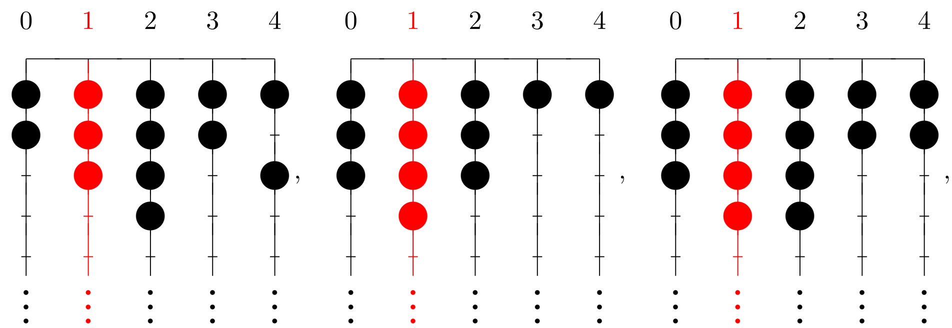
□

Example 3.2.17. Let $\boldsymbol{\lambda} = ((4,3,2),(2^2),(3))$ and $e = 4$, as in Example 3.2.15. For $\boldsymbol{k} = (3,9,6)$,

- if we choose $\boldsymbol{n} = (n_1, n_2, n_3) = (11, 9, 12)$, we get the abacus display of $\boldsymbol{\lambda}^{+(3,9,6)}$ as in Example 3.2.15 that represents the multipartition $((5,3,2^2),(5,2^3),(3^2))$.
- if we choose $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) = (10, 8, 11)$, we have the following truncated abacus for $\boldsymbol{\lambda}$:



Hence, $\boldsymbol{\lambda}^{+(3,9,6)}$ has the following abacus configuration



that represents again the multipartition $((5,3,2^2),(5,2^3),(3^2))$.

We extend the operator $^{+\boldsymbol{k}}$ linearly to the whole Fock space.

Lemma 3.2.18. Let $\boldsymbol{k}$ be an r-tuple of non-negative integers. If $\boldsymbol{\lambda}$ is an e-multiregular r-multipartition, then $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ is an $(e+1)$-multiregular r-multipartition.

Proof. Suppose that $\boldsymbol{\lambda}$ is an e-multiregular r-multipartition. Then $\lambda^{(j)}$ is e-regular for each $j \in \{1,\dots,r\}$, that is $\lambda^{(j)}$ has at most $e-1$ equal parts, equivalently it has at most $e-1$ consecutive beads in its abacus display. Hence, by construction each $\lambda^{(j)^{+k^{(j)}}}$ has at most e consecutive beads in its abacus display and so $\lambda^{(j)^{+k^{(j)}}}$ is $(e+1)$-regular for each $j \in \{1,\dots r\}$. □

3.2.4 Induction operators and addition of a full runner

In this section, we give some results that will be really helpful in the proof of our main theorem (Theorem 3.2.32) in which we prove a runner removal-type theorem for Ariki-Koike algebras.

We will work in the following setting. Let $\boldsymbol{\lambda}$ be an r-multipartition. Consider the truncated abacus configuration of $\boldsymbol{\lambda}$ with $\boldsymbol{n} = (n_1,\dots,n_r)$ beads. Set $\boldsymbol{k} = (k^{(1)},\dots,k^{(r)})$ to be an r-tuple of non-negative integers such that for all $j \in \{1,\dots,r\}$

$$n_j + k^{(j)} = c_j e + d \tag{3.2.5}$$

with $0 \le d \le e-1$. Then $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ is the r-multipartition obtained from $\boldsymbol{\lambda}$ by adding a new runner with c_j beads (as described in Section 3.2.3) to the left of runner d in each component of $\boldsymbol{\lambda}$. This determines that the new inserted runner is labelled by d mod $(e+1)$. For our first lemma, following the proof of Lemma 3.5 in [Fay07a], we need to define the following function:

$$\begin{array}{rccc} g\colon & (\mathbb{Z}/e\mathbb{Z}) \setminus \{d\} & \to & (\mathbb{Z}/(e+1)\mathbb{Z}) \setminus \{d, d+1\} \\ & d-i \mod e & \mapsto & d-i \mod (e+1) \end{array}$$

for any $d \in I$ and $i = 1,\dots,e-1$.

Lemma 3.2.19. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}$ be r-multipartitions. Let $\boldsymbol{k} = (k^{(1)}, \ldots, k^{(r)})$ be an r-tuple of non-negative integers and $d \in \{0, \ldots, e-1\}$ be the label of the new inserted runner of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ such that (3.2.5) holds. Suppose $i \in I \setminus \{d\}$. Then $\boldsymbol{\lambda} \xrightarrow{m:i} \boldsymbol{\xi}$ if and only if $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:g(i)} \boldsymbol{\xi}^{+\boldsymbol{k}}$, and if this happens then we have $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi}) = N_{g(i)}(\boldsymbol{\lambda}^{+\boldsymbol{k}}, \boldsymbol{\xi}^{+\boldsymbol{k}})$.

Proof. We have $\boldsymbol{\lambda} \xrightarrow{m:i} \boldsymbol{\xi}$ if and only if the abacus display for $\boldsymbol{\xi}$ may be obtained by moving m beads from runner $i-1$ to runner i, and in this case $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi})$ is determined by the configurations of these two runners in the two abacus displays. The fact that $i \neq d$ means that in constructing the abacus displays for $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ and $\boldsymbol{\xi}^{+\boldsymbol{k}}$ the new runner is not added in between these two runners, and so the condition $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:g(i)} \boldsymbol{\xi}^{+\boldsymbol{k}}$ and the coefficient $N_i(\boldsymbol{\lambda}, \boldsymbol{\xi}) = N_{g(i)}(\boldsymbol{\lambda}^{+\boldsymbol{k}}, \boldsymbol{\xi}^{+\boldsymbol{k}})$ are determined from these two runners in exactly the same way. □

Corollary 3.2.20. Suppose $i \in I \setminus \{d\}$, $m \geq 1$ and $\boldsymbol{\lambda}$ is any multipartition. Then $\left(f_i^{(m)}(\boldsymbol{\lambda})\right)^{+\boldsymbol{k}} = \mathfrak{F}_{g(i)}^{(m)}(\boldsymbol{\lambda}^{+\boldsymbol{k}})$.

Proof. This is immediate from Lemma 3.2.19 and the description of the action of $f_i^{(m)}$ in Section 3.1.3. □

For the next lemma, we introduce the following notation. If $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}$ are r-multipartitions, then we write $\boldsymbol{\lambda} \xrightarrow[m:i]{m:i+1} \boldsymbol{\xi}$ to indicate that $\boldsymbol{\xi}$ is obtained from $\boldsymbol{\lambda}$ by adding first m addable $(i+1)$-nodes and then m addable i-nodes. This notation is just a shorter version of the following one:

$$\boldsymbol{\lambda} \xrightarrow{m:i+1} \boldsymbol{\nu} \xrightarrow{m:i} \boldsymbol{\xi}$$

where $\boldsymbol{\nu}$ is an r-multipartition.

Lemma 3.2.21. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}$ be r-multipartitions. Let $\boldsymbol{k} = (k^{(1)}, \ldots, k^{(r)})$ be an r-tuple of non-negative integers and $d \in \{0, \ldots, e-1\}$ be the label of the new inserted runner of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ such that (3.2.5) holds. If the last bead in runner $d-1$ of each component $\lambda^{(j)}$ is at most in position $(c_j - 1)e + d - 1$ for all $j = 1, \ldots, r$, then $\boldsymbol{\lambda} \xrightarrow{m:d} \boldsymbol{\xi}$ if and only if $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow[m:d]{m:d+1} \boldsymbol{\xi}^{+\boldsymbol{k}}$.

Before proving this lemma, it is worth noticing the following.

Remark 3.2.22. In the assumptions of Lemma 3.2.21, every move of a bead $\mathfrak{b}$ at level ℓ from runner d to runner $d+1$ in a component J of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ determines uniquely a move of the bead $\bar{\mathfrak{b}}$ at level ℓ from runner $d-1$ to runner d in the component J of $\boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is an r-multipartition such that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:d+1} \boldsymbol{\nu}$.

Proof of Lemma 3.2.21. Suppose that $\boldsymbol{\lambda} \xrightarrow{m:d} \boldsymbol{\xi}$. This means that the abacus display for $\boldsymbol{\xi}$ may be obtained by moving m beads from runner $d-1$ to runner d of $\boldsymbol{\lambda}$. Notice that this moving of beads from runner $d-1$ to runner d can occur simultaneously in different components of $\boldsymbol{\lambda}$, say $\lambda^{(j_1)}, \dots, \lambda^{(j_s)}$ are the components of $\boldsymbol{\lambda}$ involved to get $\boldsymbol{\xi}$. Let $m^{(j_t)}$ be the number of beads moved in $\lambda^{(j_t)}$, for $t = 1, \dots, s$. So, in each of these components there are at least $m^{(j_t)}$ levels $\ell_1, \dots, \ell_{m^{(j_t)}}$ of the abacus that present a configuration of the type $\bullet \quad |$ in runners $d-1$ and d.

When we apply the operator $^{+\boldsymbol{k}}$ to $\boldsymbol{\lambda}$, by assumption we are going to add a new runner in between runner $d-1$ and runner d in each component of $\boldsymbol{\lambda}$. This new runner by hypothesis has the last bead that is at least at the same level of the last bead in runner $d-1$ of every component of $\boldsymbol{\lambda}$. This implies that in each component $(\lambda^{(j_1)})^{+k^{(j_1)}}, \dots, (\lambda^{(j_s)})^{+k^{(j_t)}}$, at each level $\ell_1, \dots, \ell_{m^{(j_t)}}$ for $t = 1, \dots, s$ we have an abacus configuration of the type

$$\begin{array}{ccc} d-1 & d & d+1 \\ \bullet & \bullet & | \end{array}.$$

Now, consider the r-multipartition $\boldsymbol{\mu}$ such that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow[m:d]{m:d+1} \boldsymbol{\mu}$ where the a beads that we move from runner d to runner $d+1$ are in the components $j_1, \dots, j_s$ and at levels $\ell_1, \dots, \ell_{m^{(j_t)}}$ for $t = 1, \dots, s$. Then $\boldsymbol{\mu}$ is exactly the multipartition $\boldsymbol{\xi}^{+\boldsymbol{k}}$. Indeed, given our assumption on the position of the last bead in the new runner, after moving a beads from runner d to runner $d+1$ as above, the only beads that we can move from runner $d-1$ to runner d are the beads in the components $j_1, \dots, j_s$ and at levels $\ell_1, \dots, \ell_{m^{(j_t)}}$ for $t = 1, \dots, s$. This last move fills the empty spaces that we create with the first move in some runners d, giving then an abacus display with all runners d full of beads and with the last bead in position $(c_j - 1)(e+1) + d$ for $j = 1, \dots, r$.

Conversely, suppose that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow[m:d]{m:d+1} \boldsymbol{\xi}^{+\boldsymbol{k}}$. We want to show that $\boldsymbol{\lambda} \xrightarrow{m:d} \boldsymbol{\xi}$. This, again, follows by our assumption on the position of the last bead in each runner $d-1$ of $\boldsymbol{\lambda}$. Indeed, this hypothesis implies that if we look at the abacus configuration of an addable $(d+1)$-node in any component of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$, it will be one of the following:

1. $$\begin{array}{ccc} d-1 & d & d+1 \\ \bullet & \bullet & | \end{array},$$

2. $$\begin{array}{ccc} d-1 & d & d+1 \\ | & \bullet & | \end{array}.$$

We do not need to consider the second type of abacus configuration, because moving a bead from runner d to runner $d+1$ cannot be followed by moving a bead from runner $d-1$ to runner d. Thus, there is no chance of getting $\boldsymbol{\xi}^{+\boldsymbol{k}}$ from this abacus configuration since every move of a bead from runner d to runner $d+1$ determines uniquely a move of a bead from runner $d-1$ to runner d. Hence, suppose that the abacus configuration of all addable $(d+1)$-nodes of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ we move to get $\boldsymbol{\xi}^{+\boldsymbol{k}}$ is of type 1. In this case, if we move a bead $\mathfrak{b}$ from runner d to runner $d+1$ in $\boldsymbol{\lambda}^{+\boldsymbol{k}}$, then we can only move the bead from runner $d-1$ to runner d in the same component and at the same level of the bead $\mathfrak{b}$. Hence, if $j_1, \ldots, j_s$ are the components of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ involved to get $\boldsymbol{\xi}^{+\boldsymbol{k}}$ and $m^{(j_t)}$ is the number of beads moved in the component j_t, for $t = 1, \ldots, s$, then $\boldsymbol{\xi}$ can be obtained from $\boldsymbol{\lambda}$ moving $m^{(j_t)}$ beads from runner $d-1$ to runner d in the components j_t, for $t = 1, \ldots, s$. □

Lemma 3.2.23. With the same assumption of Lemma 3.2.21. We have

$$N_d(\boldsymbol{\lambda}, \boldsymbol{\xi}) = N_{d+1}(\boldsymbol{\lambda}^{+\boldsymbol{k}}, \boldsymbol{\nu}) + N_d(\boldsymbol{\nu}, \boldsymbol{\xi}^{+\boldsymbol{k}})$$

where $\boldsymbol{\nu}$ is the unique r-multipartition such that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:d+1} \boldsymbol{\nu} \xrightarrow{m:d} \boldsymbol{\xi}^{+\boldsymbol{k}}$.

Proof. For $j = 1, \ldots, r$, set $g^{(j)}e + d - 1$ to be the position of the last bead of $\boldsymbol{\lambda}$ in runner $d-1$ of the component $\lambda^{(j)}$. Notice that, instead of considering all the r-multipartitions $\boldsymbol{\nu}$ such that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:d+1} \boldsymbol{\nu}$, we can restrict to consider only those r-multipartitions $\boldsymbol{\nu}$ where no one of the m addable $(d+1)$-nodes of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$, added to obtain $\boldsymbol{\nu}$, corresponds to a bead in position $x(e+1)+d$ with $g^{(j)} < x \leq c_j - 1$ for each $j = 1, \ldots, r$. Indeed, if we consider $\boldsymbol{\nu}$ such that $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:d+1} \boldsymbol{\nu}$ where at least one of the m addable $(d+1)$-nodes added to $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ is in position $x(e+1)+d$ with $g^{(J)} < x \leq c_J - 1$ for a component J, then there is no multipartition $\boldsymbol{\pi}$ for which $\boldsymbol{\nu} \xrightarrow{m:d} \boldsymbol{\pi}$ because by Remark 3.2.22 we should move the bead in position $x(e+1)+d-1$ in the component J of $\boldsymbol{\nu}$, but there is no bead in that position. This can be seen in terms of abacus display in the following way: the abacus configuration at the level x of the component $\nu^{(J)}$ in runners $d-1$, d, $d+1$ is

$$\begin{array}{ccc} d-1 & d & d+1 \\ + & + & \bullet \end{array},$$

from which is clear that we have no chance of moving an addable d-node at level x from runner $d-1$ to runner d. Thus, when we restrict to such r-multipartition $\boldsymbol{\nu}$, then we have

1. $\#\{\mathfrak{n} \in \boldsymbol{\nu} \setminus \boldsymbol{\lambda}^{+\boldsymbol{k}}\} = \#\{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}\}$;

2. $\#\{\mathfrak{n} \in \boldsymbol{\xi}^{+\boldsymbol{k}} \setminus \boldsymbol{\nu}\} = \#\{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}\}$ that is equivalent to

$$\#\{\text{addable } d\text{-nodes of } \boldsymbol{\nu}\} = \#\{\text{addable } d\text{-nodes of } \boldsymbol{\lambda}\}.$$

Indeed, provided that we are considering the r-multipartitions $\boldsymbol{\nu}$ satisfying the condition above we can conclude that these equalities holds for the following reasons.

1. Since the new inserted runner of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ is full of beads and with the last bead in a higher position than any last bead in every runner $d-1$ of $\boldsymbol{\lambda}$, the addable $(d+1)$-nodes of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ consists of the addable d-nodes of $\boldsymbol{\lambda}$ and the addable $(d+1)$-nodes of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ at level x with $g^{(j)} < x \leq c_j - 1$ for each $j \in \{1, \dots r\}$, that is

$$\begin{aligned}
&\#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}}\} \\
&\quad = \#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}} \text{ at level } x \leq g^{(j)}\} \\
&\qquad + \#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}} \text{ at level } g^{(j)} < x \leq c_j - 1\} \\
&\quad = \#\{\text{addable } d\text{-nodes of } \boldsymbol{\lambda}\} \\
&\qquad + \#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}} \text{ at level } g^{(j)} < x \leq c_j - 1\}.
\end{aligned}$$

   Anyway, restricting to the r-multipartitions $\boldsymbol{\nu}$ as above means that we are excluding the r-multipartitions obtained from $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ by adding addable $(d+1)$-nodes at level x with $g^{(j)} < x \leq c_j - 1$. So, in this case

$$\#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}}\} = \#\{\text{addable } d\text{-nodes of } \boldsymbol{\lambda}\}.$$

   Hence, $\#\{\mathfrak{n} \in \boldsymbol{\nu} \setminus \boldsymbol{\lambda}^{+\boldsymbol{k}}\} = \#\{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}\}$.

2. By Remark 3.2.22, there is a correspondence between the addable $(d+1)$-nodes of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ and the addable d-nodes of $\boldsymbol{\nu}$ and so we can state that $\#\{\mathfrak{n} \in \boldsymbol{\xi}^{+\boldsymbol{k}} \setminus \boldsymbol{\nu}\} = \#\{\mathfrak{n} \in \boldsymbol{\nu} \setminus \boldsymbol{\lambda}^{+\boldsymbol{k}}\} = \#\{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}\}$.

Recall that by definition

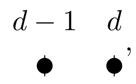
$$\begin{aligned}
N_d(\boldsymbol{\lambda}, \boldsymbol{\xi}) = \sum_{\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}} (&\#\{\text{addable } d\text{-nodes of } \boldsymbol{\xi} \text{ above } \mathfrak{n}\} \\
&- \#\{\text{removable } d\text{-nodes of } \boldsymbol{\lambda} \text{ above } \mathfrak{n}\}),
\end{aligned}$$

$$\begin{aligned}
N_{d+1}(\boldsymbol{\lambda}^{+\boldsymbol{k}}, \boldsymbol{\nu}) = \sum_{\mathfrak{n} \in \boldsymbol{\nu} \setminus \boldsymbol{\lambda}^{+\boldsymbol{k}}} (&\#\{\text{addable } (d+1)\text{-nodes of } \boldsymbol{\nu} \text{ above } \mathfrak{n}\} \\
&- \#\{\text{removable } (d+1)\text{-nodes of } \boldsymbol{\lambda}^{+\boldsymbol{k}} \text{ above } \mathfrak{n}\}),
\end{aligned}$$

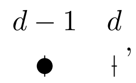
$$\begin{aligned}
N_d(\boldsymbol{\nu}, \boldsymbol{\xi}^{+\boldsymbol{k}}) = \sum_{\mathfrak{n} \in \boldsymbol{\xi}^{+\boldsymbol{k}} \setminus \boldsymbol{\nu}} (&\#\{\text{addable } d\text{-nodes of } \boldsymbol{\xi}^{+\boldsymbol{k}} \text{ above } \mathfrak{n}\} \\
&- \#\{\text{removable } d\text{-nodes of } \boldsymbol{\nu} \text{ above } \mathfrak{n}\}).
\end{aligned}$$

Now, consider $\boldsymbol{\lambda}$ and $\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}$. Let J be the component of $\boldsymbol{\xi}$ of the node $\mathfrak{n}$. Set

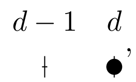
- r_1 to be the number of rows of the abacus of $\lambda^{(J)}$ above $\mathfrak{n}$ with a configuration of the following type in runners $d-1$ and d:



- r_2 to be the number of rows of the abacus of $\lambda^{(J)}$ above $\mathfrak{n}$ with a configuration of the following type in runners $d-1$ and d:

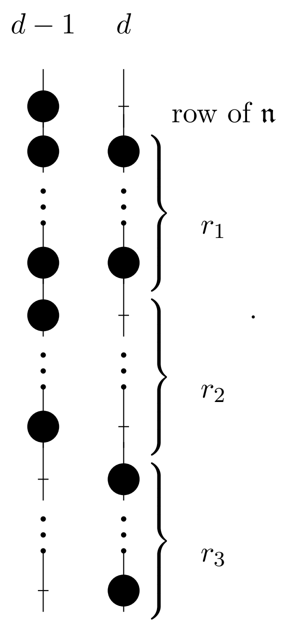


- r_3 to be the number of rows of the abacus of $\lambda^{(J)}$ above $\mathfrak{n}$ with a configuration of the following type in runners $d-1$ and d:

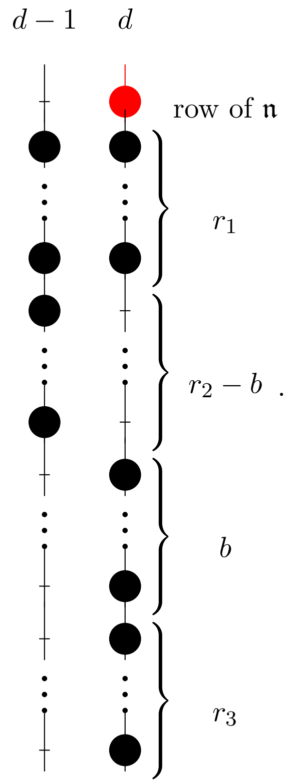


- b to be the number of removable d-nodes in $\boldsymbol{\xi} \setminus \boldsymbol{\lambda}$ above $\mathfrak{n}$ in the component J.

In order to make easier to visualise the abacus display, we can assume that the rows of the abacus of $\lambda^{(J)}$ above $\mathfrak{n}$ with the same configuration between the runner $d-1$ and d occurs as shown below:



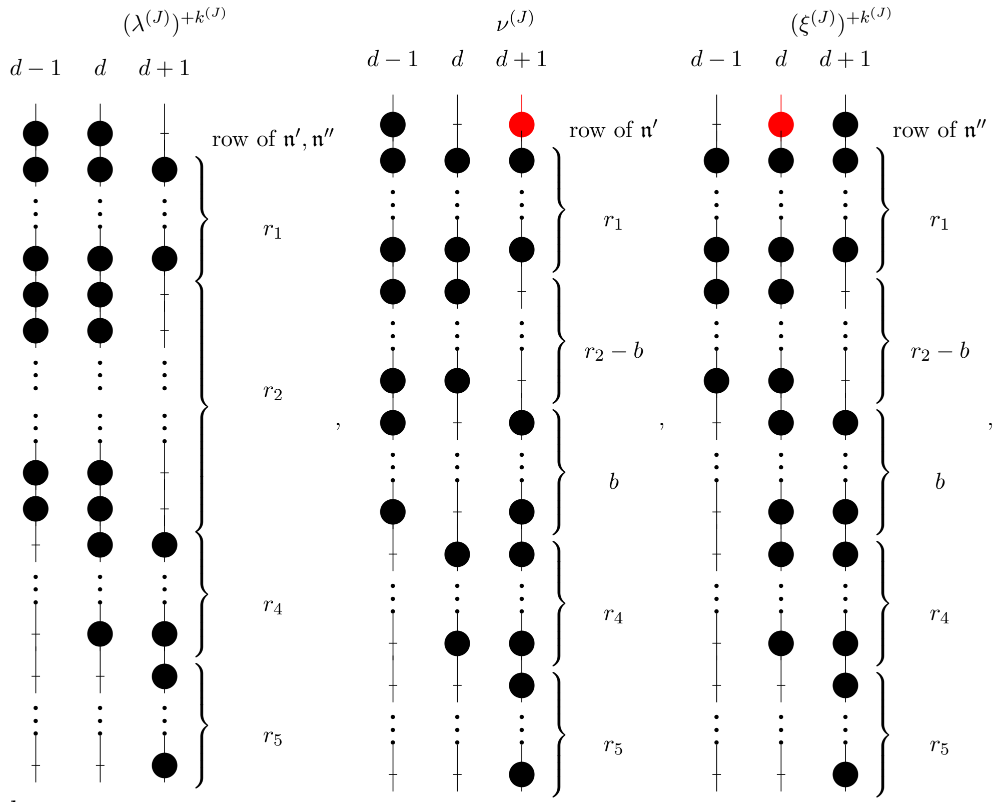
Then we can assume that the abacus display of component J of $\boldsymbol{\xi}$ has the following configuration in the row of $\mathfrak{n}$ (with $\mathfrak{n}$ in red) and those above $\mathfrak{n}$:



Notice that the assumption on the order of the rows is not necessary for the scope of the proof, but it makes easier to represent the abacus display. Thus, for $\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}$ we have that

$$N_d(\lambda^{(J)}, \xi^{(J)}) = (r_2 - b) - r_3.$$

Thus, when we consider the component J of the corresponding r-multipartitions $\boldsymbol{\lambda}^{+\boldsymbol{k}} \xrightarrow{m:d+1} \boldsymbol{\nu} \xrightarrow{m:d} \boldsymbol{\xi}^{+\boldsymbol{k}}$ in terms of abacus display above the nodes $\mathfrak{n}'$ and $\mathfrak{n}''$ (in red) that are the nodes corresponding to the node $\mathfrak{n}$, we have the following configurations between runners $d-1$, d, $d+1$:



where $r_4 + r_5 = r_3$.

Thus, for $\mathfrak{n}' \in \boldsymbol{\nu} \setminus \boldsymbol{\lambda}^{+\boldsymbol{k}}$ and $\mathfrak{n}'' \in \boldsymbol{\xi}^{+\boldsymbol{k}} \setminus \boldsymbol{\nu}$ we have that

$$N_{d+1}((\lambda^{(J)})^{+k^{(J)}}, \nu^{(J)}) = (r_2 - b) - r_5,$$
$$N_d(\nu^{(J)}, (\xi^{(J)})^{+k^{(J)}}) = 0 - r_4.$$

Hence, for the component J we have

$$N_d(\lambda^{(J)}, \xi^{(J)}) = N_{d+1}((\lambda^{(J)})^{+k^{(J)}}, \nu^{(J)}) + N_d(\nu^{(J)}, (\xi^{(J)})^{+k^{(J)}}).$$

By Proposition 3.1.2, to conclude we just need to show that for each $\mathfrak{n} \in \boldsymbol{\xi} \setminus \boldsymbol{\lambda}$ and for each component $j < J$

$$N_d(\lambda^{(j)}, \xi^{(j)}) = N_{d+1}((\lambda^{(j)})^{+k^{(j)}}, \nu^{(j)}) + N_d(\nu^{(j)}, (\xi^{(j)})^{+k^{(j)}}). \tag{3.2.6}$$

We can extend easily the previous argument to every component $j < J$ of $\boldsymbol{\xi}$. Indeed, if $j < J$ is a component of $\boldsymbol{\xi}$ then, by the total order of all addable and removable nodes of a multipartition, all the nodes in such a component j of $\boldsymbol{\xi}$ are above $\mathfrak{n}$ and so in order to prove (3.2.6) we can use exactly the same argument explained for component J, considering all the nodes in the component j, instead of just the ones above $\mathfrak{n}$. Thus, we can conclude. □

Corollary 3.2.24. Let $\boldsymbol{\lambda}$ be an r-multipartition. Let $\boldsymbol{k} = (k^{(1)}, \ldots, k^{(r)})$ be an

r-tuple of non-negative integers and $d \in \{0, \dots, e-1\}$ be the label of the new inserted runner of $\boldsymbol{\lambda}^{+\boldsymbol{k}}$ such that (3.2.5) holds. If the last bead in runner $d-1$ of each component $\lambda^{(j)}$ is at most in position $(c_j-1)e+d-1$ for all $j = 1, \dots, r$, then $\left(f_d^{(m)}(\boldsymbol{\lambda})\right)^{+\boldsymbol{k}} = \mathfrak{F}_d^{(m)}\mathfrak{F}_{d+1}^{(m)}(\boldsymbol{\lambda}^{+\boldsymbol{k}})$.

Proof. This is immediate from Lemmas 3.2.21 and 3.2.23 and the description of the action of $f_i^{(m)}$ in Section 3.1.3. □

3.2.5 Case $r = 2$

In this section we prove the so called 'runner removal' theorem for Ariki-Koike algebras for $r = 2$. The reason why we study the case for $r = 2$ is that dealing with multipartitions with only two components is easier, but at the same time it gives a good idea of the strategy that will be followed in the next session where we will deal with the general case ($r \geq 2$).

In particular, we will compute canonical basis vectors $G^{\boldsymbol{s}}(\boldsymbol{\mu})$ and, respectively, $G^{\boldsymbol{s}^+}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ in the Fock spaces $\mathcal{F}^{\boldsymbol{s}}$ and, respectively $\mathcal{F}^{\boldsymbol{s}^+}$, rather than working with the Ariki-Koike algebras directly.

From now on, we assume that $r = 2$.

Proposition 3.2.25. Let $\boldsymbol{s} = (s_1, s_2) \in I^2$ be a multicharge and $\boldsymbol{\mu} = (\varnothing, \mu) \in \mathcal{P}^2$. Let $k^{(1)}, k^{(2)}$ be non-negative integers such that $k^{(2)} - k^{(1)} \geq \mu_1 + e - 1$. Write $k^{(1)} = k_1^{(1)}e + k_2^{(1)}$ with $k_1^{(1)} \geq 0$ and $0 \leq k_2^{(1)} \leq e-1$. Denote by $\boldsymbol{a}^{+\boldsymbol{k}} = (a_1, a_2)$ the multicharge associated to the multipartition $(\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$. Then, setting $\alpha = k^{(1)} + a_1$ and $\beta = k^{(1)} - k_2^{(1)} + a_1$

$$\mathfrak{F}_{\alpha}^{(k_1^{(1)})}\mathfrak{F}_{\alpha-1}^{(k_1^{(1)})}\dots\mathfrak{F}_{\beta+1}^{(k_1^{(1)})}\mathcal{G}_{\beta}^{(k_1^{(1)}-1)}\mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)}\dots\mathcal{G}_{\beta+k_1^{(1)}-2}^{(1)}((\varnothing, \mu^{+k^{(2)}})) = (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}}).$$

where $\mathfrak{F}_{\alpha}^{(k_1^{(1)})}\mathfrak{F}_{\alpha-1}^{(k_1^{(1)})}\dots\mathfrak{F}_{\beta+1}^{(k_1^{(1)})}$ only occur if $\alpha \neq \beta$.

Proof. Construct the truncated e-abacus configuration of the 2-multipartition $(\varnothing, \mu)$ consisting of $\boldsymbol{n} = (n_1, n_2)$ beads with $n_j \equiv s_j \mod e$ for $j = 1, 2$. Write

- $n_1 + k^{(1)} = c_1e + d$
- $n_2 + k^{(2)} = c_2e + d$

with $0 \leq d \leq e-1$. Construct now the truncated $(e+1)$-abacus display of the 2-multipartition $(\varnothing, \mu^{+k^{(2)}})$ with $(n_1 + c_1, n_2 + c_2)$ beads. Notice that the multicharge associated to this multipartition is $\boldsymbol{a}^{+\boldsymbol{k}} = (a_1, a_2) = (n_1+c_1, n_2+c_2)$. We proceed now by induction on $k^{(1)}$. The induction hypothesis is the following. If $k = k_1e + k_2$ with $k_1 \geq 0$ and $0 \leq k_2 \leq e-1$ is a non-negative integer such that $k < k^{(1)}$, then it holds that

$$\mathfrak{F}_{\bar{a}}^{(k_1)}\dots\mathfrak{F}_{b+1}^{(k_1)}\mathcal{G}_{b}^{(k_1-1)}\dots\mathcal{G}_{b+k_1-3}^{(1)}((\varnothing, \mu^{+k^{(2)}})) = (\varnothing^{+k}, \mu^{+k^{(2)}}),$$

where $a = k + a_1$, $b = k - k_2 + a_1$, and $\mathfrak{F}_{\overline{a}}^{(k_1)} \dots \mathfrak{F}_{\overline{b+1}}^{(k_1)}$ only occur if $a \neq b$.

If $k^{(1)} < e$, then $\varnothing^{+k^{(1)}} = \varnothing$ by Proposition 3.2.7. So,

$$(\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}}) = (\varnothing, \mu^{+k^{(2)}}).$$

Suppose $k^{(1)} \geq e$ and write $k^{(1)} = k_1^{(1)} e + k_2^{(1)}$ with $k_1^{(1)} \geq 1$ and $0 \leq k_2^{(1)} \leq e - 1$. Then by Proposition 3.2.10 we have

$$\mathfrak{F}_{\overline{k^{(1)}}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{k^{(1)}-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)})} \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}}}^{(k_1^{(1)}-1)} \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}+k_1^{(1)}-2}}^{(1)} (\varnothing) = \varnothing^{+k^{(1)}},$$

where $\mathfrak{F}_{\overline{k^{(1)}}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{k^{(1)}-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)})}$ only occur if $k_2^{(1)} \neq 0$. Hence, we want to apply this induction sequence to $(\varnothing, \mu^{+k^{(2)}})$ and show that the only resulting multipartition is $(\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$. The first fact that we need to consider when we deal with multipartitions is the multicharge. Therefore, the residues involved in the induction sequence need to be translated by the multicharge. So, the induction sequence that we want to apply to $(\varnothing, \mu^{+k^{(2)}})$ is the following:

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{\beta+1}}^{(k_1^{(1)})} \mathcal{G}_{\overline{\beta}}^{(k_1^{(1)}-1)} \mathcal{G}_{\overline{\beta+1}}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\overline{\beta+k_1^{(1)}-2}}^{(1)}$$

with $\alpha = k^{(1)} + a_1$ and $\beta = k^{(1)} - k_2^{(1)} + a_1$.

If $k^{(1)} = e$, then by Proposition 3.2.7 $\varnothing^{+e} = \varnothing$. So,

$$(\varnothing^{+e}, \mu^{+k^{(2)}}) = (\varnothing, \mu^{+k^{(2)}}).$$

If $k^{(1)} > e$, then for $\alpha \neq \beta$ consider

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{\beta+1}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\beta}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\beta-1}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(k_1^{(1)}-1)} (\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}}); \quad (3.2.7)$$

while for $\alpha = \beta$ consider

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\overline{\alpha-e-1}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\alpha+1}}^{(k_1^{(1)}-2)} (\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}}). \quad (3.2.8)$$

Notice that in (3.2.7) and (3.2.8) the number of induction operators $\mathfrak{F}$ is exactly $e+1$. Moreover, we have

$$\overline{\alpha} = \overline{k^{(1)} + a_1} = \overline{k^{(1)} + n_1 + c_1} = \overline{c_1 e + d + c_1} = \overline{c_1(e+1) + d}.$$

Thus $\alpha \equiv d \bmod (e+1)$. We now want to show that

$$(3.2.7) = (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}});$$

$$(3.2.8) = (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}}).$$

However, note that in (3.2.7) and (3.2.8)

- the residues involved in the induction sequences are the same and in the same order;
- the first operator is applied one time less than the last operator.

Hence, the argument we give in the following works exactly in the same way in the two cases. So, we will show only that (3.2.7) $= (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$ because a similar argument applies to prove that (3.2.8) $= (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$.

Showing that (3.2.7) $= (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$ is equivalent to showing that this induction sequence is applied only to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}})$. Indeed, if this sequence acts only on the first component we have (3.2.7) $= (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$ by Proposition 3.2.10. Recall that the action of an induction operator $\mathfrak{F}_i^{(m)}$ for $m \geq 1$ on the abacus display of a multipartition consists of moving m beads from runner $i-1$ to an empty position in runner i in some components. Suppose that this induction sequence acts on both components of the multipartition $(\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}})$ and it does not give 0. In particular, suppose that

- $\mathfrak{F}_{\alpha}^{(p_{e+1})} \dots \mathfrak{F}_{\alpha+1}^{(p_1)}$ is the induction subsequence acting on $\mu^{+k^{(2)}}$,
- $\mathfrak{F}_{\alpha}^{(q_{e+1})} \dots \mathfrak{F}_{\alpha+1}^{(q_1)}$ is the induction subsequence acting on $\varnothing^{+(k^{(1)}-e-1)}$,

with $q_{e+1} = k_1^{(1)} - p_{e+1}$, $q_1 = k_1^{(1)} - 1 - p_1$ and at least one $p_i \neq 0$. Since we apply $\mathfrak{F}_{\alpha+1}^{(p_1)}$ to $\mu^{+k^{(2)}}$, this means that we are moving p_1 beads from runner d to runner $d+1$. By construction of $\mu^{+k^{(2)}}$, this corresponds in terms of abacus display in having p_1 empty spaces in the runner d of $\mu^{+k^{(2)}}$. The action of the terms $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)}$ involves moving beads from runner $d+1$ to runner $d+2$, and then from runner $d+2$ to runner $d+3$, and so forth until we move beads from runner $d-2$ to runner $d-1$. Let l be the level of the last bead of the new inserted runner of $\mu^{+k^{(2)}}$. There are now two cases to consider:

1. the beads moved by this induction in runner $d-1$ are at most at level l;
2. one of the beads moved by this induction in runner $d-1$ is at level $l+1$.

Case 1. If the moved beads are at most at the same level of the last bead of the new inserted runner of $\mu^{+k^{(2)}}$, then the number of beads that the last induction operator $\mathfrak{F}_{\alpha}^{(p_{e+1})}$ can move from runner $d-1$ to runner d can be at most p_1. If not (i.e., $p_{e+1} > p_1$), then the action of the induction subsequence of $\mu^{+k^{(2)}}$ is 0 because there are no enough gaps in runner d of $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)} \mathfrak{F}_{\alpha+1}^{(p_1)}(\mu^{+k^{(2)}})$ where to move beads from runner $d-1$ to runner d. In fact, the gaps in runner d of $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)} \mathfrak{F}_{\alpha+1}^{(p_1)}(\mu^{+k^{(2)}})$ are p_1. Indeed, the first operator $\mathfrak{F}_{\alpha+1}^{(p_1)}$ moves p_1 beads from runner d to runner $d+1$ of $\mu^{+k^{(2)}}$. Also, the beads moved by $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)}$

in runner $d-1$ are at most at level l and this is the level of the last bead in the new inserted runner of $\mu^{+k^{(2)}}$. Thus, we have $p_{e+1} \leq p_1$. Hence,

$$q_1 = k_1^{(1)} - 1 - p_1 \leq k_1^{(1)} - 1,$$
$$q_{e+1} = k_1^{(1)} - p_{e+1} \geq k_1^{(1)} - p_1 = q_1 + 1.$$

We claim that the action of this induction sequence on $\varnothing^{+(k^{(1)}-e-1)}$ is 0. If $q_{e+1} > q_1 + 1$, then we can conclude that $\mathfrak{F}_{\overline{\alpha}}^{(q_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(q_1)}(\varnothing^{+(k^{(1)}-e-1)}) = 0$ by Lemma 3.2.12. If $q_{e+1} = q_1 + 1$, by contradiction we assume that $\mathfrak{F}_{\overline{\alpha}}^{(q_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(q_1)}(\varnothing^{+(k^{(1)}-e-1)})$ is non-zero. By Lemma 3.2.12 this means that $q_{e+1} = \dots = q_{e+2-k_2} = k_1$ and $q_{e+1-k_2} = \dots = q_1 = k_1 - 1$. However, this is a contradiction because this would imply that $p_i = 0$ for all i, while we are assuming that at least one of the p_i's is non-zero. Therefore, we proved that the induction sequence acts non-zero when all the induction operators are applied to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}})$. Hence, we can conclude in this case.

Case 2. If one of the moved beads is at a higher level than the last bead of the new inserted runner of $\mu^{+k^{(2)}}$, then the number of beads that the last induction operator $\mathfrak{F}_{\overline{\alpha}}^{(p_{e+1})}$ can move from runner $d-1$ to runner d can be at most p_1+1. If not (i.e. $p_{e+1} > p_1+1$), then the action of the induction subsequence of $\mu^{+k^{(2)}}$ is 0 because there are no enough gaps in runner d of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\mu^{+k^{(2)}})$ where to move beads from runner $d-1$ to runner d. In fact, the gaps in runner d until level l of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\mu^{+k^{(2)}})$ are p_1 because the first operator $\mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}$ moves p_1 beads from runner d to runner $d+1$ of $\mu^{+k^{(2)}}$. Also, by the assumption of Case 2., there is a bead in runner $d-1$ at level $l+1$ of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\mu^{+k^{(2)}})$ with a gap on its right. This is because l is the level of the last bead in the new inserted runner of $\mu^{+k^{(2)}}$. Thus, in total, there are at most p_1+1 possible positions where to move beads from runner $d-1$ to runner d in $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\mu^{+k^{(2)}})$. Thus, we have $p_{e+1} \leq p_1 + 1$. Hence,

$$q_1 = k_1^{(1)} - 1 - p_1 \leq k_1^{(1)} - 1,$$
$$q_{e+1} = k_1^{(1)} - p_{e+1} \geq k_1^{(1)} - p_1 - 1 = q_1.$$

We claim that the action of this induction sequence on $\varnothing^{+(k^{(1)}-e-1)}$ is 0. In order to prove this, we deal with the following three cases separately:

a. $q_{e+1} = q_1$;

b. $q_{e+1} = q_1 + 1$;

c. $q_{e+1} > q_1 + 1$.

For cases b. and c. we can conclude as in Case 1. with analogous arguments. As for case a., we need to be more cautious. In this case we have $q_{e+1} = q_1$ and so by Lemma 3.2.12 the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$ is non-zero if and only if

$$q_1 = q_2 = \ldots = q_{e+1-k_2^{(1)}} = q_{e+2-k_2^{(1)}} = \ldots = q_e = q_{e+1}.$$

We want to show that the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$ acts as 0. If $q_1 = q_2 = \ldots = q_{e+1-k_2^{(1)}} = q_{e+2-k_2^{(1)}} = \ldots = q_e = q_{e+1}$, then

$$\begin{aligned} p_1 &= \ldots = p_{e+1-k_2} = k_1^{(1)} - q_1 - 1, \\ p_{e+2-k_2} &= \ldots = p_{e+1} = k_1^{(1)} - q_{e+1} = k_1^{(1)} - q_1. \end{aligned}$$

In this case we have no problem with the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$, but the induction subsequence on $\mu^{+k^{(2)}}$ is the one that acts as 0. We need again to distinguish two cases.

- If the first p_1 beads moved from runner d to runner $d+1$ involve at least one bead not in the last p_1 positions of the new inserted runner, then the action of the induction subsequence on $\mu^{+k^{(2)}}$ is 0. Indeed, the last induction operator requires to move p_{e+1} beads from runner $d-1$ to runner d, but in runner d we have at most $p_{e+1} - 1 = p_1$ empty positions.

- If the first p_1 beads moved from runner d to runner $d+1$ are exactly in the last p_1 positions of the new inserted runner, then the action of the induction subsequence on $\mu^{+k^{(2)}}$ is 0. Notice that our assumption $k^{(2)} - k^{(1)} \geq \mu_1 + e - 1$ implies that the last bead of μ is at most at level $c_2 - k_1^{(1)}$ of the abacus of μ. Indeed, the difference in height between the last bead of the new inserted runner and the last bead of μ is given by $c_2 - 1 - x$ where $\mu_1 + n_1 - 1 = xe + i$ for $x \geq 0$ and $0 \leq i \leq e-1$ and so we get that

$$\begin{aligned} c_2 - 1 - x &= \frac{1}{e}(k^{(2)} + n_2 - d) - 1 - \frac{1}{e}(\mu_1 + n_2 - 1 - i) \\ &= \frac{1}{e}(k^{(2)} - \mu_1) + \frac{1}{e}(-d - e + 1 + i) \\ &\geq \frac{1}{e}(k^{(1)} + e - 1) + \frac{1}{e}(-d - e + 1 + i) \\ &= \frac{1}{e}(k_1^{(1)} e + k_2^{(1)}) + \frac{1}{e}(-d - e + 1 + i + e - 1) \\ &= k_1^{(1)} + \frac{1}{e}(k_2^{(1)} - d + i) \\ &> k_1^{(1)} + \frac{1}{e}(-e) \\ &= k_1^{(1)} - 1. \end{aligned}$$

If the induction subsequence on $\mu^{+k^{(2)}}$ is 0 at some step before the last induction operator then we are done. Suppose that we can apply all the induction operators until the last one $\mathfrak{F}_{\overline{\alpha}}^{(p_{e+1})}$, and that we get something non-zero. Then the action of this last operator is 0 because we do not have enough beads in runner $d-1$ to move to in runner d in the abacus display of $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)}\mathfrak{F}_{\alpha+1}^{(p_1)}(\mu^{+k^{(2)}})$. In fact, the position $(c_2-1)(e+1)+(d-1)$ is empty since $x < c_2 - k_1^{(1)} \leq c_2 - 1$ and so there are only $p_1 = p_{e+1} - 1$ addable nodes of residue d.

Therefore, we proved that the induction sequence acts non-zero when all the induction operators are applied to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}})$ also in this case.

So, we get that (3.2.7) $= (\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$. By induction hypothesis we know also that

$$(\varnothing^{+(k^{(1)}-e-1)}, \mu^{+k^{(2)}}) = \mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\beta+1}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\beta+k_1^{(1)}-3}^{(1)}((\varnothing, \mu^{+k^{(2)}})).$$

Hence, we can conclude. □

The above proposition gives the following result about the canonical basis coefficients of $(\varnothing, \mu^{+k^{(2)}})$ and $(\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$.

Proposition 3.2.26. Let $\boldsymbol{\mu} = (\varnothing, \mu) \in \mathcal{P}^2$. Let $k^{(1)}, k^{(2)}$ be non-negative integers such that $k^{(2)} - k^{(1)} \geq \mu_1 + e - 1$. Let $\boldsymbol{s}^+ \in (\mathbb{Z}/((e+1)\mathbb{Z}))^2$. Suppose that

$$G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \mu^{+k^{(2)}})) = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)(\varnothing, \lambda^{+k^{(2)}})$$

where $d_{\lambda\mu}(q) \in q\mathbb{N}[q]$. Then

$$G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})) = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)(\varnothing^{+k^{(1)}}, \lambda^{+k^{(2)}}).$$

Proof. Let $\boldsymbol{s}^+ = (s_1, s_2)$ be a multicharge. Suppose that $G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \mu^{+k^{(2)}})) = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)(\varnothing, \lambda^{+k^{(2)}})$ with $d_{\lambda\mu}(q) \in q\mathbb{N}[q]$ for $\lambda \neq \mu$. We want to apply the induction sequence from $\varnothing$ to $\varnothing^{+k^{(1)}}$ given by Proposition 3.2.10 to $G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \mu^{+k^{(2)}}))$. In order to do this, as in Proposition 3.2.25, we need to translate the residues involved in the induction by the corresponding multicharge. So, writing $k^{(1)} = k_1^{(1)}e + k_2^{(1)}$ with $k_1^{(1)} \geq 0$ and $0 \leq k_2^{(1)} \leq e-1$, we want to apply the following induction sequence

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \dots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})} \mathcal{G}_{\beta}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\beta+k_1^{(1)}-2}^{(1)}(G_{e+1}^{\boldsymbol{s}}(\varnothing, \mu^{+k^{(2)}})), \qquad (3.2.9)$$

where

- $\alpha = k^{(1)} + s_1$ and $\beta = k^{(1)} - k_2^{(1)} + s_1$;
- $\mathfrak{F}_{\alpha}^{(k_1^{(1)})}\mathfrak{F}_{\alpha-1}^{(k_1^{(1)})}\dots\mathfrak{F}_{\beta+1}^{(k_1^{(1)})}$ occur if $\alpha \neq \beta$.

In order to compute this action we note the following: $\mu_1 \geq \lambda_1$ for all the partitions λ such that $\mu \trianglerighteq \lambda$. This implies that the hypotheses of Proposition 3.2.25 are satisfied by all the partitions λ such that $\mu \trianglerighteq \lambda$. Thus, we have

$$\begin{aligned}(3.2.9) &= \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)\mathfrak{F}_{\alpha}^{(k_1^{(1)})}\mathfrak{F}_{\alpha-1}^{(k_1^{(1)})}\dots\mathfrak{F}_{\beta+1}^{(k_1^{(1)})}\mathcal{G}_{\beta}^{(k_1^{(1)}-1)}\mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)}\dots\mathcal{G}_{\beta+k_1^{(1)}-2}^{(1)}((\varnothing,\lambda^{+k^{(2)}}))\\ &= \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)(\varnothing^{+k^{(1)}},\lambda^{+k^{(2)}}),\end{aligned}$$

which is of the form $(\varnothing^{+k^{(1)}},\mu^{+k^{(2)}}) + \sum_{\mu\neq\lambda} d_{\lambda\mu}(q)(\varnothing^{+k^{(1)}},\lambda^{+k^{(2)}})$. Hence, this shows that $G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}})) = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu}(q)(\varnothing^{+k^{(1)}},\lambda^{+k^{(2)}})$ by the uniqueness of the canonical basis. □

Write $f_{i_l}^{(m_l)}\cdots f_{i_1}^{(m_1)}$ with $m_1,\dots,m_l$ non-negative integers and $i_1,\dots,i_l \in I$. Fix $d \in \{0,\dots,e-1\}$. Then define $\mathfrak{F}$ to be the induction sequence obtained in the following way: for all $j = 1,\dots,l$

- if $i_j \neq d$, replace $f_{i_j}^{(m_j)}$ with $\mathfrak{F}_{g(i_j)}^{(m_j)}$;
- if $i_j = d$, replace $f_{d}^{(m_j)}$ with $\mathfrak{F}_{d}^{(m_j)}\mathfrak{F}_{d+1}^{(m_j)}$.

Proposition 3.2.27. Let $\boldsymbol{\mu} = (\mu^{(1)},\mu^{(2)})$ be an e-multiregular 2-multipartition and let $\boldsymbol{k} = (k^{(1)},k^{(2)})$ be a 2-tuple of non-negative integers such that $k^{(2)} - k^{(1)} \geq \mu_1^{(2)} + e - 1$. Let $d \in \{0,\dots,e-1\}$ be the label of the new inserted runner of $\boldsymbol{\mu}^{+\boldsymbol{k}}$ such that (3.2.5) holds. Suppose that

$$\mathfrak{f}\cdot G_e^{\boldsymbol{s}}((\varnothing,\mu^{(2)})) = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}\boldsymbol{\nu},$$

where $g_{\boldsymbol{\nu}} \in \mathbb{Z}[q,q^{-1}]$ and $\mathfrak{f} = f_{i_l}^{(m_l)}\cdots f_{i_1}^{(m_1)}$ for some $m_1,\dots,m_l \in \mathbb{Z}_{\geq 0}$ and $i_1,\dots,i_l \in I$ is such that $\mathfrak{f}\cdot\varnothing = \mu^{(1)} + \sum_{\mu^{(1)}\triangleright\tau} t_\tau \tau$ for $t_\tau \in \mathbb{Z}[q,q^{-1}]$. Then

$$\mathfrak{F}\cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}},\mu^{(2)+k^{(2)}})) = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}\boldsymbol{\nu}^{+\boldsymbol{k}},$$

where $\mathfrak{F}$ is defined as above.

Proof. By definition of $^{+\boldsymbol{k}}$ we have that

$$\left(\mathfrak{f}\cdot G_e^{\boldsymbol{s}}((\varnothing,\mu^{(2)}))\right)^{+\boldsymbol{k}} = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}\boldsymbol{\nu}^{+\boldsymbol{k}}.$$

Moreover, we get that

$$\begin{aligned}\left(\mathfrak{f}\cdot G_e^{\boldsymbol{s}}((\varnothing,\mu^{(2)}))\right)^{+\boldsymbol{k}} &= \left(\mathfrak{f}\cdot \sum_{\mu^{(2)}\trianglerighteq\lambda} d_{\lambda\mu^{(2)}}(q)(\varnothing,\lambda)\right)^{+\boldsymbol{k}} \\ &= \mathfrak{F}\cdot \sum_{\mu^{(2)}\trianglerighteq\lambda} d_{\lambda\mu^{(2)}}(q)(\varnothing^{+k^{(1)}},\lambda^{+k^{(2)}}) \quad \text{by Cor. 3.2.20, 3.2.24} \\ &= \mathfrak{F}\cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}},\mu^{(2)+k^{(2)}})) \qquad \text{by Prop. 3.2.26}\end{aligned}$$

Hence,

$$\left(\mathfrak{f}\cdot G_e^{\boldsymbol{s}}((\varnothing,\mu^{(2)}))\right)^{+\boldsymbol{k}} = \mathfrak{F}\cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing,\mu^{(2)})^{+\boldsymbol{k}}).$$

Thus, we can conclude

$$\mathfrak{F}\cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}},\mu^{(2)+k^{(2)}}) = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}\boldsymbol{\nu}^{+\boldsymbol{k}}.$$

□

Theorem 3.2.28. Let $\boldsymbol{\mu} = (\mu^{(1)},\mu^{(2)}) \in \mathcal{P}^2$ is an e-multiregular multipartition of n and $\boldsymbol{s} \in I^2$. Let $\boldsymbol{k} = (k^{(1)},k^{(2)})$ with $k^{(j)}$ a non-negative integer for $j = 1,2$ such that $k^{(2)} - k^{(1)} \geq \mu_1^{(2)} + e - 1$. Then $G_{e+1}^{\boldsymbol{s}^+}(\boldsymbol{\mu}^{+\boldsymbol{k}}) = G_e^{\boldsymbol{s}}(\boldsymbol{\mu})^{+\boldsymbol{k}}$.

Proof. Suppose that $\boldsymbol{\mu} = (\mu^{(1)},\mu^{(2)})$ is an e-multiregular 2-multipartition. In order to simplify the following notation we write μ instead of $\mu^{(2)}$. Then by Theorem 3.2.5 we have

$$G_{e+1}(\mu^{+k^{(2)}}) = G_e(\mu)^{+k^{(2)}}, \tag{3.2.10}$$

since the condition $k^{(2)} - k^{(1)} \geq \mu_1 + e - 1$ implies that $k^{(2)} \geq \mu_1$. More explicitly, we have that

$$G_e(\mu) = \mu + \sum_{\mu\triangleright\lambda} d_{\lambda\mu}(q)\lambda,$$

then (3.2.10) implies that

$$G_{e+1}(\mu^{+k^{(2)}}) = \mu^{+k^{(2)}} + \sum_{\mu\triangleright\lambda} d_{\lambda\mu}(q)\lambda^{+k^{(2)}}.$$

Moreover, by Corollary 3.1.5 it holds that

$$G_{e+1}^{\boldsymbol{s}^+}((\varnothing,\mu^{+k^{(2)}})) = (\varnothing,\mu^{+k^{(2)}}) + \sum_{\mu\triangleright\lambda} d_{\lambda\mu}(q)(\varnothing,\lambda^{+k^{(2)}}).$$

By Proposition 3.2.26 we have that

$$G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}})) = (\varnothing^{+k^{(1)}},\mu^{+k^{(2)}}) + \sum_{\mu\rhd\lambda} d_{\lambda\mu}(q)(\varnothing^{+k^{(1)}},\lambda^{+k^{(2)}}).$$

Using the LLT algorithm on partitions, we can write $G^{(s_1)}(\mu^{(1)})$ as $\mathfrak{f}\cdot\varnothing$ in the Fock space $\mathcal{F}^{(s_1)}$, for some $\mathfrak{f}\in\mathcal{U}$. Applying the induction sequence $\mathfrak{f}$ to $G^{\boldsymbol{s}}_e((\varnothing,\mu))$ we can write

$$\mathfrak{f}\cdot G^{\boldsymbol{s}}_e((\varnothing,\mu)) = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}(\nu^{(1)},\nu^{(2)}). \tag{3.2.11}$$

where $g_{\boldsymbol{\nu}}\in\mathbb{Z}[q,q^{-1}]$ because $\mathfrak{f}\cdot G^{\boldsymbol{s}}_e((\varnothing,\mu))\in M^{\otimes\boldsymbol{s}}$ and $M^{\otimes\boldsymbol{s}}$ is a $\mathcal{U}$-submodule of $\mathcal{F}^{\boldsymbol{s}}$. Performing step (c) of the LLT algorithm for multipartitions in [Fay10] we get

$$\mathfrak{f}\cdot G^{\boldsymbol{s}}_e((\varnothing,\mu)) - \sum_{\boldsymbol{\mu}\rhd\boldsymbol{\sigma}} a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q)G^{\boldsymbol{s}}_e(\boldsymbol{\sigma}) = G^{\boldsymbol{s}}_e((\mu^{(1)},\mu)) \tag{3.2.12}$$

where $a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q)\in\mathbb{Z}[q+q^{-1}]$.

Now consider the induction sequence $\mathfrak{F}$ that is obtained translating the induction sequence $\mathfrak{f}$ from e to $e+1$. This means that for each $i\in I$

- if $i\neq d$, we replace $f_i^{(m)}$ with $\mathfrak{F}^{(m)}_{g(i)}$,
- if $i=d$, we replace $f_d^{(m)}$ with $\mathfrak{F}^{(m)}_d\mathfrak{F}^{(m)}_{d+1}$.

Then apply $\mathfrak{F}$ to $G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}}))$. By Proposition 3.2.27 we have

$$\mathfrak{F}\cdot G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}})) = \sum_{\boldsymbol{\nu}\in\mathcal{P}^2} g_{\boldsymbol{\nu}}(\nu^{(1)+k^{(1)}},\nu^{(2)+k^{(2)}}). \tag{3.2.13}$$

We now consider (3.2.13) and since the coefficients occurring in the sum are exactly the same of (3.2.11) we perform the following subtraction of terms

$$\mathfrak{F}\cdot G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}})) - \sum_{\boldsymbol{\mu}\rhd\boldsymbol{\sigma}} a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q)G^{\boldsymbol{s}^+}_{e+1}(\boldsymbol{\sigma}^{+\boldsymbol{k}}). \tag{3.2.14}$$

We proceed by induction on the dominance order. Suppose that $G^{\boldsymbol{s}^+}_{e+1}(\boldsymbol{\sigma}^{+\boldsymbol{k}}) = (G^{\boldsymbol{s}}_e(\boldsymbol{\sigma}))^{+\boldsymbol{k}}$ for all $\boldsymbol{\sigma}=(\sigma^{(1)},\sigma^{(2)})$ e-multiregular multipartition of n such that $\boldsymbol{\sigma}\lhd\boldsymbol{\mu}$. Then

$$(3.2.14) = \mathfrak{F}\cdot G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}},\mu^{+k^{(2)}})) - \sum_{\boldsymbol{\mu}\rhd\boldsymbol{\sigma}} a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q)(G^{\boldsymbol{s}}_e(\boldsymbol{\sigma}))^{+\boldsymbol{k}}. \tag{3.2.15}$$

Since we are performing exactly the same operations of (3.2.12) and the starting coefficients are the same in (3.2.11) and (3.2.13), by definition of $^{+\boldsymbol{k}}$ we have

$$(3.2.15) = G^{\boldsymbol{s}}_e(\boldsymbol{\mu})^{+\boldsymbol{k}}.$$

Moreover, by uniqueness of the canonical basis of $M^{\otimes \boldsymbol{s}^+}$ we can state

$$(3.2.15) = G_{e+1}^{\boldsymbol{s}^+}((\mu^{(1)^{+k^{(1)}}}, \mu^{+k^{(2)}})).$$

Hence,

$$G_{e+1}^{\boldsymbol{s}^+}(\boldsymbol{\mu}^{+\boldsymbol{k}}) = G_e^{\boldsymbol{s}}(\boldsymbol{\mu})^{+\boldsymbol{k}}.$$

□

3.2.6 Case $r \geq 2$

In this section we generalise Theorem 3.2.28 for any $r \geq 2$. The proof is essentially the same, but we need to take in to account that now we have $r \geq 2$ components in each multipartition playing a role when we apply an induction operator $\mathfrak{f}$ or $\mathfrak{F}$.

We start generalising the conditions on the non-negative integers $k^{(1)}, \dots, k^{(r)}$ as follows. Let $r \geq 2$. Let $k^{(1)}, \dots, k^{(r)}$ be non-negative integers such that

$$k^{(r)} \geq \mu_1^{(r)}, \text{ and;}$$
$$k^{(j)} - k^{(h)} \geq \mu_1^{(j)} + e - 1 + \sum_{t=h+1}^{j-1} |\mu^{(t)}| \text{ for all } 1 \leq h < j \leq r. \tag{3.2.16}$$

Proposition 3.2.29. Let $\boldsymbol{s} = (s_1, \dots, s_r) \in I^r$ be a multicharge and $(\varnothing, \boldsymbol{\mu}) = (\varnothing, \mu^{(2)}, \dots, \mu^{(r)})$ be a r-multipartition. Let $(k^{(1)}, \boldsymbol{k}) = (k^{(1)}, k^{(2)} \dots, k^{(r)})$ be an r-tuple of non-negative integers such that conditions (3.2.16) hold. Write $k^{(1)} = k_1^{(1)} e + k_2^{(1)}$ with $k_1^{(1)} \geq 0$ and $0 \leq k_2^{(1)} \leq e-1$. Denote by $\boldsymbol{a}^{+(k^{(1)}, \boldsymbol{k})} = (a_1, \dots, a_r)$ the multicharge associated to the multipartition $(\varnothing^{+k^{(1)}}, \mu^{(2)^{+k^{(2)}}}, \dots, \mu^{(r)^{+k^{(r)}}})$. Then, setting $\alpha = k^{(1)} + a_1$ and $\beta = k^{(1)} - k_2^{(1)} + a_1$,

$$\mathfrak{F}_{\alpha}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \dots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})} \mathcal{G}_{\beta}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)} \dots \mathcal{G}^{(1)}_{\beta+k_1^{(1)}-2}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})) = (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}}).$$

where $\mathfrak{F}_{\alpha}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \dots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})}$ occur if $\alpha \neq \beta$.

Proof. Construct the truncated e-abacus configuration of the r-multipartition $(\varnothing, \boldsymbol{\mu})$ consisting of $\boldsymbol{n} = (n_1, \dots, n_r)$ beads with $n_j \equiv s_j \mod e$ for all j. For $j = 1, \dots, r$, write

$$n_j + k^{(j)} = c_j e + d$$

with $0 \leq d \leq e-1$. Construct now the truncated $(e+1)$-abacus display of the r-multipartition $(\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})$ with $(n_1 + c_1, \dots, n_r + c_r)$ beads. Notice that the multicharge associated to this multipartition is $\boldsymbol{a}^{+(k^{(1)}, \boldsymbol{k})} = (n_1 + c_1, \dots, n_r + c_r)$. Thus $a_j = n_j + c_j$ for $1 \leq j \leq r$. We proceed now by induction on $k^{(1)}$ with the following induction hypothesis. If $k = k_1 e + k_2$ with $k_1 \geq 0$ and $0 \leq k_2 \leq e-1$

is a non-negative integer such that $k < k^{(1)}$, then it holds that

$$\mathfrak{F}_{\overline{a}}^{(k_1)} \dots \mathfrak{F}_{\overline{b+1}}^{(k_1)} \mathcal{G}_{\overline{b}}^{(k_1-1)} \dots \mathcal{G}_{\overline{b+k_1-3}}^{(1)}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})) = (\varnothing^{+k}, \boldsymbol{\mu}^{+\boldsymbol{k}}),$$

where $a = k + a_1$, $b = k - k_2 + a_1$, and $\mathfrak{F}_{\overline{a}}^{(k_1)} \dots \mathfrak{F}_{\overline{b+1}}^{(k_1)}$ only occur if $a \neq b$.

If $k^{(1)} < e$, then $\varnothing^{+k^{(1)}} = \varnothing$ by Proposition 3.2.7. So,

$$(\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}}) = (\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}}).$$

Suppose $k^{(1)} \geq e$ and write $k^{(1)} = k_1^{(1)} e + k_2^{(1)}$ with $k_1^{(1)} \geq 1$ and $0 \leq k_2^{(1)} \leq e - 1$. Then by Proposition 3.2.10 we have

$$\mathfrak{F}_{\overline{k^{(1)}}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{k^{(1)}-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)})} \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}}}^{(k_1^{(1)}-1)} \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\overline{k^{(1)}-k_2^{(1)}+k_1^{(1)}-2}}^{(1)}(\varnothing) = \varnothing^{+k^{(1)}},$$

where $\mathfrak{F}_{\overline{k^{(1)}}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{k^{(1)}-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{k^{(1)}-k_2^{(1)}+1}}^{(k_1^{(1)})}$ only occur if $k_2^{(1)} \neq 0$. Hence, we want to apply this induction sequence to $(\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})$ and show that the only resulting multipartition is $(\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$. The first fact that we need to consider when we deal with multipartitions is the multicharge. Therefore, the residues involved in the induction sequence need to be translated by the multicharge. So, the induction sequence that we want to apply to $(\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})$ is the following:

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{\beta+1}}^{(k_1^{(1)})} \mathcal{G}_{\overline{\beta}}^{(k_1^{(1)}-1)} \mathcal{G}_{\overline{\beta+1}}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\overline{\beta+k_1^{(1)}-2}}^{(1)}$$

with $\alpha = k^{(1)} + a_1$ and $\beta = k^{(1)} - k_2^{(1)} + a_1$.

If $k^{(1)} = e$, then by Proposition 3.2.7 $\varnothing^{+e} = \varnothing$. So,

$$(\varnothing^{+e}, \mu^{(2)+k^{(2)}}, \dots, \mu^{(r)+k^{(r)}}) = (\varnothing, \mu^{(2)+k^{(2)}}, \dots, \mu^{(r)+k^{(r)}}).$$

If $k^{(1)} > e$, then for $\alpha \neq \beta$ consider

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)})} \dots \mathfrak{F}_{\overline{\beta+1}}^{(k_1^{(1)})} \mathfrak{F}_{\overline{\beta}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\beta-1}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(k_1^{(1)}-1)} (\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}}) \quad (3.2.17)$$

while for $\alpha = \beta$ consider

$$\mathfrak{F}_{\overline{\alpha}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\alpha-1}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\overline{\alpha-e-1}}^{(k_1^{(1)}-1)} \mathfrak{F}_{\overline{\alpha+1}}^{(k_1^{(1)}-2)} (\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}}). \quad (3.2.18)$$

Notice that in (3.2.17) and in (3.2.18) the number of induction operators $\mathfrak{F}_i^{(a)}$ is exactly $e + 1$. Moreover, we have

$$\overline{\alpha} = \overline{k^{(1)} + a_1} = \overline{k^{(1)} + n_1 + c_1} = \overline{c_1 e + d + c_1} = \overline{c_1(e+1) + d}.$$

Thus $\alpha \equiv d \bmod (e+1)$. We want to show now that

$$(3.2.17) = (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}});$$

$$(3.2.18) = (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}}).$$

However, note that in (3.2.17) and (3.2.18)

- the residues involved in the induction sequences are the same and in the same order;
- the first operator is applied one time less than the last operator.

Hence, the argument we give in the following works exactly in the same way in the two cases. So, we will show only that (3.2.17) $= (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$ because a similar argument applies to prove that (3.2.18) $= (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$.

Showing that (3.2.17) $= (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$ is equivalent to showing that this induction sequence is applied only to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}})$. Indeed, if this sequence acts only on the first component we have (3.2.17) $= (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$ by Proposition 3.2.10. Recall that the action of an induction operator $\mathfrak{F}_i^{(a)}$ for $a \geq 1$ on the abacus display of a multipartition consists of moving a beads from runner $i-1$ to an empty position in runner i in some components. Suppose that this induction sequence acts not only on the first component of the multipartition $(\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}})$ and it does not give 0. In particular, suppose that

- $\mathfrak{F}_{\overline{\alpha}}^{(p_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}$ is the induction subsequence acting on $\boldsymbol{\mu}^{+\boldsymbol{k}}$,
- $\mathfrak{F}_{\overline{\alpha}}^{(q_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(q_1)}$ is the induction subsequence acting on $\varnothing^{+(k^{(1)}-e-1)}$,

with $q_{e+1} = k_1^{(1)} - p_{e+1}$, $q_1 = k_1^{(1)} - 1 - p_1$ and at least one $p_i \neq 0$. Since we apply $\mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}$ to $\boldsymbol{\mu}^{+\boldsymbol{k}}$, this means that we are moving p_1 beads from runner d to runner $d+1$ in some components of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. Call such components $j_1, \dots, j_b$. By construction of $\boldsymbol{\mu}^{+\boldsymbol{k}}$, this corresponds in terms of abacus display to having p_1 empty spaces in the runners d of components $j_1, \dots, j_b$ of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. The action of the terms $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)}$ involves moving beads from runner $d+1$ to runner $d+2$, and then from runner $d+2$ to runner $d+3$, and so forth until we move beads from runner $d-2$ to runner $d-1$. For $j \in \{2, \dots, r\}$, let l_j be the level of the last bead of the new inserted runner in the component j of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. There are now two cases to consider:

1. the beads moved by this induction in runner $d-1$ are at most at level l_j for all components j;
2. one of the beads moved by this induction in runner $d-1$ is at level $l_j + 1$ for some j.

Case 1. If the moved beads are at most at the same level of the last bead of each new inserted runner of $\boldsymbol{\mu}^{+\boldsymbol{k}}$, then the number of beads that the last induction operator $\mathfrak{F}_{\overline{\alpha}}^{(p_{e+1})}$ can move from runner $d-1$ to runner d can be at most p_1. If not (i.e., $p_{e+1} > p_1$), then the action of the induction subsequence of $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is 0 because there are no enough gaps in runner d of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ where to move beads from runner $d-1$ to runner d. In fact, the gaps in runner d of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ are p_1. Indeed, the first operator $\mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}$ moves p_1 beads from runner d to runner $d+1$ of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. Also, the beads moved by $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)}$ in runner $d-1$ are, for all j, at most at level l_j, that is the level of the last bead in the new inserted runner of component j of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. Thus, we have $p_{e+1} \leq p_1$. Hence,

$$q_1 = k_1^{(1)} - 1 - p_1 \leq k_1^{(1)} - 1,$$
$$q_{e+1} = k_1^{(1)} - p_{e+1} \geq k_1^{(1)} - p_1 = q_1 + 1.$$

We claim that the action of this induction sequence on $\varnothing^{+(k^{(1)}-e-1)}$ is 0. If $q_{e+1} > q_1 + 1$, then we can conclude that $\mathfrak{F}_{\overline{\alpha}}^{(q_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(q_1)}(\varnothing^{+(k^{(1)}-e-1)}) = 0$ by Lemma 3.2.12. If $q_{e+1} = q_1 + 1$, by contradiction we assume that $\mathfrak{F}_{\overline{\alpha}}^{(q_{e+1})} \dots \mathfrak{F}_{\overline{\alpha+1}}^{(q_1)}(\varnothing^{+(k^{(1)}-e-1)})$ is non-zero. By Lemma 3.2.12 this means that $q_{e+1} = \dots = q_{e+2-k_2} = k_1$ and $q_{e+1-k_2} = \dots = q_1 = k_1 - 1$. However, this is a contradiction because this would imply that $p_i = 0$ for all i, while we are assuming that at least one of the p_i's is non-zero. Therefore, we proved that the induction sequence acts non-zero when all the induction operators are applied to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}})$. Hence we conclude in this case.

Case 2. If one of the moved beads is at a higher level than the last bead of one of the new inserted runners of $\boldsymbol{\mu}^{+\boldsymbol{k}}$, then the number of beads that the last induction operator $\mathfrak{F}_{\overline{\alpha}}^{(p_{e+1})}$ can move from runner $d-1$ to runner d can be at most p_1+1. If not (i.e. $p_{e+1} > p_1+1$), then the action of the induction subsequence of $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is 0 because there are no enough gaps in runners d of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ where to move $p_{e+1} > p_1 + 1$ beads from runner $d-1$ to runner d. In fact, the total number of gaps in runner d of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ created by the first operator $\mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}$ is p_1. Also, by the assumption of Case 2., there is a component J of $\mathfrak{F}_{\overline{\alpha-1}}^{(p_e)} \dots \mathfrak{F}_{\overline{\alpha+2}}^{(p_2)} \mathfrak{F}_{\overline{\alpha+1}}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$ with a bead in runner $d-1$ at level l_J+1 and with a gap on its right. This is because l_J is the level of the last bead in the new inserted runner of component J of $\boldsymbol{\mu}^{+\boldsymbol{k}}$. Thus, in total, there are at most $p_1 + 1$ possible positions where to move beads from runner $d-1$ to runner d in

$\mathfrak{F}^{(p_e)}_{\alpha-1} \dots \mathfrak{F}^{(p_2)}_{\alpha+2}\mathfrak{F}^{(p_1)}_{\alpha+1}(\boldsymbol{\mu}^{+\boldsymbol{k}})$. Thus, we have $p_{e+1} \leq p_1 + 1$. Hence,

$$q_1 = k_1^{(1)} - 1 - p_1 \leq k_1^{(1)} - 1,$$
$$q_{e+1} = k_1^{(1)} - p_{e+1} \geq k_1^{(1)} - p_1 - 1 = q_1.$$

We claim that the action of this induction sequence on $\varnothing^{+(k^{(1)}-e-1)}$ is 0. In order to prove this, we deal with the following three cases separately:

a. $q_{e+1} = q_1$;

b. $q_{e+1} = q_1 + 1$;

c. $q_{e+1} > q_1 + 1$.

For cases b. and c. we can conclude as in Case 1. with analogous arguments. As for case a., we need to be more cautious. In this case we have $q_{e+1} = q_1$ and so by Lemma 3.2.12 the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$ is non-zero if and only if

$$q_1 = q_2 = \ldots = q_x = q_{x+1} = \ldots = q_e = q_{e+1}.$$

We want to show that the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$ acts as 0. If $q_1 = q_2 = \ldots = q_{e+1-k_2^{(1)}} = q_{e+2-k_2^{(1)}} = \ldots = q_e = q_{e+1}$, then

$$p_1 = \ldots = p_{e+1-k_2^{(1)}} = k_1^{(1)} - q_1 - 1,$$
$$p_{e+2-k_2^{(1)}} = \ldots = p_{e+1} = k_1^{(1)} - q_{e+1} = k_1^{(1)} - q_1.$$

In this case we have no problem with the induction subsequence acting on $\varnothing^{+k^{(1)}-e-1}$, but the induction subsequence on $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is the one that acts as 0. We need again to distinguish two cases.

- If the first p_1 beads moved from runner d to runner $d+1$ involve at least one bead not in the last p_1 positions of the new inserted runner, then the action of the induction subsequence on $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is 0. Indeed, the last induction operator requires to move p_{e+1} beads from runner $d-1$ to runner d, but in runner d we have at most $p_{e+1} - 1 = p_1$ empty positions.

- If the first p_1 beads moved from runner d to runner $d+1$ are exactly in the last p_1 positions of the new inserted runners of component J of $\boldsymbol{\mu}^{+\boldsymbol{k}}$, then the action of the induction subsequence on $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is 0. Notice that our assumption

$$k^{(j)} - k^{(h)} \geq \mu_1^{(j)} + \sum_{t=h+1}^{j-1} |\mu^{(t)}| + e - 1 \text{ for all } 1 \leq h < j \leq r$$

implies that the last bead of $\mu^{(J)}$ is at most at level $c_J - k_1^{(1)}$ of the abacus of $\mu^{(J)}$. Indeed, the difference in height between the last bead of the new inserted runner and the last bead of $\mu^{(J)}$ is given by $c_J - 1 - x$ where $\mu_1^{(J)} + n_J - 1 = xe + i$ for $x \geq 0$ and $0 \leq i \leq e-1$ and so we get that

$$\begin{aligned} c_J - 1 - x &= \frac{1}{e}(k^{(J)} + n_J - d) - 1 - \frac{1}{e}(\mu_1^{(J)} + n_J - 1 - i) \\ &= \frac{1}{e}(k^{(J)} - \mu_1^{(J)}) + \frac{1}{e}(-d - e + 1 + i) \\ &\geq \frac{1}{e}(k^{(1)} + \sum_{t=2}^{J-1} |\mu^{(t)}| + e - 1) + \frac{1}{e}(-d - e + 1 + i) \\ &\geq \frac{1}{e}(k^{(1)} + e - 1) + \frac{1}{e}(-d - e + 1 + i) \\ &= \frac{1}{e}(k_1^{(1)} e + k_2^{(1)}) + \frac{1}{e}(-d - e + 1 + i + e - 1) \\ &= k_1^{(1)} + \frac{1}{e}(k_2^{(1)} - d + i) \\ &> k_1^{(1)} + \frac{1}{e}(-e) \\ &= k_1^{(1)} - 1. \end{aligned}$$

If the induction subsequence on $\boldsymbol{\mu}^{+\boldsymbol{k}}$ is 0 at some step before the last induction operator then we are done. Suppose that we can apply all the induction operators until the last one $\mathfrak{F}_{\bar{\alpha}}^{(p_{e+1})}$, and that we get something non-zero. Then the action of this last operator is 0 because we do not have enough beads in runner $d-1$ to move in runner d in the abacus display of $\mathfrak{F}_{\alpha-1}^{(p_e)} \dots \mathfrak{F}_{\alpha+2}^{(p_2)} \mathfrak{F}_{\alpha+1}^{(p_1)}(\boldsymbol{\mu}^{+\boldsymbol{k}})$. In fact, the position $(c_J - 1)(e+1) + (d-1)$ is empty since $x < c_J - k_1^{(1)} \leq c_J - 1$ and so there are only $p_{e+1} - 1$ addable nodes of residue d.

Therefore, we proved that the induction sequence acts non-zero when all the induction operators are applied to the first component of $(\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}})$ also in this case. So, we get that (3.2.17) $= (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$. By induction hypothesis we know also that

$$(\varnothing^{+(k^{(1)}-e-1)}, \boldsymbol{\mu}^{+\boldsymbol{k}}) = \mathfrak{F}_{\bar{\alpha}}^{(k_1^{(1)}-1)} \dots \mathfrak{F}_{\beta+1}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta}^{(k_1^{(1)}-2)} \dots \mathcal{G}_{\overline{\beta+k_1^{(1)}-3}}^{(1)}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})).$$

Hence, we can conclude. □

The above proposition gives the following result about the canonical basis coefficients of $(\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})$ and $(\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})$.

Proposition 3.2.30. Let $(\varnothing, \boldsymbol{\mu}) = (\varnothing, \mu^{(2)}, \dots, \mu^{(r)})$ be a r-multipartition and $(k^{(1)}, \boldsymbol{k}) = (k^{(1)}, k^{(2)} \dots, k^{(r)})$ be an r-tuple of non-negative integers such that conditions (3.2.16) hold. Let $\boldsymbol{s}^+ \in (\mathbb{Z}/(e+1)\mathbb{Z})^r$. Suppose that

$$G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})) = \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing, \boldsymbol{\lambda}^{+\boldsymbol{k}})$$

where $d_{\boldsymbol{\lambda\mu}}(q) \in q\mathbb{N}[q]$ for $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. Then

$$G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) = \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing^{+k^{(1)}}, \boldsymbol{\lambda}^{+\boldsymbol{k}}).$$

Proof. Let $\boldsymbol{s}^+ = (s_1, \ldots, s_r)$ be a multicharge. Suppose that $G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})) = \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing, \boldsymbol{\lambda}^{+\boldsymbol{k}})$ with $d_{\boldsymbol{\lambda\mu}}(q) \in q\mathbb{N}[q]$ for $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. We want to apply the induction sequence from $\varnothing$ to $\varnothing^{+k^{(1)}}$ given by Proposition 3.2.10 to $G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}}))$. Hence, we want to use Proposition 3.2.29. So, writing $k^{(1)} = k_1^{(1)} e + k_2^{(1)}$ with $k_1^{(1)} \geq 0$ and $0 \leq k_2^{(1)} \leq e-1$, we want to apply the following induction sequence

$$\mathfrak{F}_{\alpha}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \ldots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})} \mathcal{G}_{\beta}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)} \ldots \mathcal{G}^{(1)}_{\beta+k_1^{(1)}-2}(G_{e+1}^{\boldsymbol{s}^+}(\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})), \qquad (3.2.19)$$

where

- $\alpha = k^{(1)} + s_1$ and $\beta = k^{(1)} - k_2^{(1)} + s_1$;
- $\mathfrak{F}_{\alpha}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \ldots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})}$ occur if $\alpha \neq \beta$.

Our assumptions are exactly the hypotheses of Proposition 3.2.29. Thus, we have

$$\begin{aligned}(3.2.19) &= \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q) \mathfrak{F}_{\alpha}^{(k_1^{(1)})} \mathfrak{F}_{\alpha-1}^{(k_1^{(1)})} \ldots \mathfrak{F}_{\beta+1}^{(k_1^{(1)})} \mathcal{G}_{\beta}^{(k_1^{(1)}-1)} \mathcal{G}_{\beta+1}^{(k_1^{(1)}-2)} \ldots \mathcal{G}^{(1)}_{\beta+k_1^{(1)}-2}((\varnothing, \boldsymbol{\lambda}^{+\boldsymbol{k}})) \\ &= \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing^{+k^{(1)}}, \boldsymbol{\lambda}^{+\boldsymbol{k}}),\end{aligned}$$

which is of the form $(\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}}) + \sum_{\boldsymbol{\mu} \neq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing^{+k^{(1)}}, \boldsymbol{\lambda}^{+\boldsymbol{k}})$. Hence, this shows that $G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) = \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing^{+k^{(1)}}, \boldsymbol{\lambda}^{+\boldsymbol{k}})$ by the uniqueness of the canonical basis. □

Write $\mathfrak{f}_{i_l}^{(h_l)} \cdots \mathfrak{f}_{i_1}^{(h_1)}$ with $h_1, \ldots, h_l$ non-negative integers and $i_1, \ldots, i_l \in I$. Fix $d \in \{0, \ldots, e-1\}$. Then define $\mathfrak{F}$ to be the induction sequence obtained in the following way: for all $j = 1, \ldots, l$

- if $i_j \neq d$, replace $\mathfrak{f}_{i_j}^{(h_j)}$ with $\mathfrak{F}_{g(i_j)}^{(h_j)}$;
- if $i_j = d$, replace $\mathfrak{f}_{d}^{(h_j)}$ with $\mathfrak{F}_{d}^{(h_j)} \mathfrak{F}_{d+1}^{(h_j)}$.

Proposition 3.2.31. Let $(\varnothing, \boldsymbol{\mu}) = (\varnothing, \mu^{(2)}, \dots, \mu^{(r)})$ be an e-multiregular r-multipartition and $(k^{(1)}, \boldsymbol{k}) = (k^{(1)}, k^{(2)} \dots, k^{(r)})$ be an r-tuple of non-negative integers such that conditions (3.2.16) hold. Let $d \in \{0, \dots, e-1\}$ be the label of the new inserted runner of $(\varnothing, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}$ such that (3.2.5) holds. Suppose that

$$\mathfrak{f} \cdot G_e^{\boldsymbol{s}}((\varnothing, \boldsymbol{\mu})) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}} \boldsymbol{\nu},$$

where $g_{\boldsymbol{\nu}} \in \mathbb{Z}[q, q^{-1}]$ and $\mathfrak{f} = f_{i_l}^{(h_l)} \cdots f_{i_1}^{(h_1)}$ for some $h_1, \dots, h_l \in \mathbb{Z}_{\geq 0}$ and $i_1, \dots, i_l \in I$ is such that $\mathfrak{f} \cdot \varnothing = \mu^{(1)} + \sum_{\mu^{(1)} \rhd \tau} t_\tau \tau$ for $t_\tau \in \mathbb{Z}[q, q^{-1}]$. Then

$$\mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}} \boldsymbol{\nu}^{+(k^{(1)}, \boldsymbol{k})},$$

where $\mathfrak{F}$ is defined as above.

Proof. By definition of $^{+\boldsymbol{k}}$ we have that

$$(\mathfrak{f} \cdot G_e^{\boldsymbol{s}}((\varnothing, \boldsymbol{\mu})))^{+(k^{(1)}, \boldsymbol{k})} = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}} \boldsymbol{\nu}^{+(k^{(1)}, \boldsymbol{k})}.$$

Moreover, we get that

$$\begin{aligned}
(\mathfrak{f} \cdot G_e^{\boldsymbol{s}}((\varnothing, \boldsymbol{\mu})))^{+(k^{(1)}, \boldsymbol{k})} &= \left(\mathfrak{f} \cdot \sum_{\mu \trianglerighteq \lambda} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing, \boldsymbol{\lambda}) \right)^{+(k^{(1)}, \boldsymbol{k})} \\
&= \mathfrak{F} \cdot \sum_{\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}} d_{\boldsymbol{\lambda\mu}}(q)(\varnothing, \boldsymbol{\lambda})^{+(k^{(1)}, \boldsymbol{k})} && \text{by Cor. 3.2.20, 3.2.24} \\
&= \mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}) && \text{by Prop. 3.2.30}
\end{aligned}$$

Hence,

$$(\mathfrak{f} \cdot G_e^{\boldsymbol{s}}((\varnothing, \boldsymbol{\mu})))^{+(k^{(1)}, \boldsymbol{k})} = \mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}).$$

Thus, we can conclude

$$\mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}} \boldsymbol{\nu}^{+(k^{(1)}, \boldsymbol{k})}.$$

□

Theorem 3.2.32. Let $(\mu^{(1)}, \boldsymbol{\mu}) = (\mu^{(1)}, \mu^{(2)} \dots, \mu^{(r)}) \in \mathcal{P}^r$ is an e-multiregular multipartition of n and $\boldsymbol{s} = (s_1, \dots, s_r) \in I^r$. Let $(k^{(1)}, \boldsymbol{k}) = (k^{(1)}, k^{(2)} \dots, k^{(r)})$ with $k^{(j)}$ a non-negative integer for $j = 1, \dots, r$ such that conditions (3.2.16) hold. Then

$$G_{e+1}^{\boldsymbol{s}^+}((\mu^{(1)}, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}) = G_e^{\boldsymbol{s}}((\mu^{(1)}, \boldsymbol{\mu}))^{+(k^{(1)}, \boldsymbol{k})}.$$

Proof. Suppose that $(\mu^{(1)}, \boldsymbol{\mu}) = (\mu^{(1)}, \dots, \mu^{(r)})$ is an e-multiregular

r-multipartition. We proceed by induction on the number r of components.

If $r = 1$, then $(\mu^{(1)}, \boldsymbol{\mu}) = \mu$ and $(k^{(1)}, \boldsymbol{k}) = k$, i.e., we are in the partition case. Thus, by Theorem 3.2.5 we have

$$G_{e+1}(\mu^{+k}) = G_e(\mu)^{+k},$$

since $k \geq \mu_1$.

Suppose $r > 1$. By induction on r, we know that for an e-multiregular $(r-1)$-multipartition $\boldsymbol{\mu}$

$$G^{\boldsymbol{s}^+}_{e+1}(\boldsymbol{\mu}^{+\boldsymbol{k}}) = G^{\boldsymbol{s}}_e(\boldsymbol{\mu})^{+\boldsymbol{k}}. \tag{3.2.20}$$

More explicitly, we have that

$$G^{\boldsymbol{s}}_e(\boldsymbol{\mu}) = \boldsymbol{\mu} + \sum_{\boldsymbol{\mu} \rhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\boldsymbol{\lambda},$$

then (3.2.20) implies that

$$G^{\boldsymbol{s}^+}_{e+1}(\boldsymbol{\mu}^{+\boldsymbol{k}}) = \boldsymbol{\mu}^{+\boldsymbol{k}} + \sum_{\boldsymbol{\mu} \rhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\boldsymbol{\lambda}^{+\boldsymbol{k}}.$$

Now, we want to show that this is also true for the r-multipartition $(\mu^{(1)}, \boldsymbol{\mu})$. By Corollary 3.1.5 it holds that

$$G^{\boldsymbol{s}^+}_{e+1}((\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}})) = (\varnothing, \boldsymbol{\mu}^{+\boldsymbol{k}}) + \sum_{\boldsymbol{\mu} \rhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)(\varnothing, \boldsymbol{\lambda}^{+\boldsymbol{k}}).$$

By Proposition 3.2.30 we have that

$$G^{\boldsymbol{s}^+}_{e+1}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) = (\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}}) + \sum_{\boldsymbol{\mu} \rhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)(\varnothing^{+k^{(1)}}, \boldsymbol{\lambda}^{+\boldsymbol{k}}).$$

Using the LLT algorithm on partitions, we can write $G^{(s_1)}(\mu^{(1)})$ as $\mathfrak{f}\cdot\varnothing$ in the Fock space $\mathcal{F}^{(s_1)}$, for some $\mathfrak{f} \in \mathcal{U}$. Applying the induction sequence $\mathfrak{f}$ to $G^{\boldsymbol{s}}_e((\varnothing, \boldsymbol{\mu}))$ we can write

$$\mathfrak{f} \cdot G^{\boldsymbol{s}}_e((\varnothing, \boldsymbol{\mu})) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}}\boldsymbol{\nu}, \tag{3.2.21}$$

where $g_{\boldsymbol{\nu}} \in \mathbb{Z}[q, q^{-1}]$ because $\mathfrak{f} \cdot G^{\boldsymbol{s}}_e((\varnothing, \boldsymbol{\mu})) \in M^{\otimes \boldsymbol{s}}$ and $M^{\otimes \boldsymbol{s}}$ is a $\mathcal{U}$-submodule of $\mathcal{F}^{\boldsymbol{s}}$. Performing step (c) of the LLT algorithm for multipartitions in [Fay10] we get

$$\mathfrak{f} \cdot G^{\boldsymbol{s}}_e((\varnothing, \boldsymbol{\mu})) - \sum_{(\mu^{(1)}, \boldsymbol{\mu}) \rhd \boldsymbol{\sigma}} a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q) G^{\boldsymbol{s}}_e(\boldsymbol{\sigma}) = G^{\boldsymbol{s}}_e((\mu^{(1)}, \boldsymbol{\mu})) \tag{3.2.22}$$

where $a_{\boldsymbol{\sigma}\boldsymbol{\mu}}(q) \in \mathbb{Z}[q + q^{-1}]$.

Now consider the induction sequence $\mathfrak{F}$ that is obtained translating the induction sequence $\mathfrak{f}$ from e to $e+1$. This means that for each $i \in I$

- if $i \neq d$, we replace $f_i^{(h)}$ with $\mathfrak{F}_{g(i)}^{(h)}$,
- if $i = d$, we replace $f_d^{(h)}$ with $\mathfrak{F}_d^{(h)}\mathfrak{F}_{d+1}^{(h)}$.

Then apply $\mathfrak{F}$ to $G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}}))$. By Proposition 3.2.31 we have

$$\mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) = \sum_{\boldsymbol{\nu} \in \mathcal{P}^r} g_{\boldsymbol{\nu}} \boldsymbol{\nu}^{+(k^{(1)}, \boldsymbol{k})}. \tag{3.2.23}$$

We now consider (3.2.23). Since the coefficients occurring in the sum are exactly the same of (3.2.21), we perform the following subtraction of terms

$$\mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) - \sum_{(\mu^{(1)}, \boldsymbol{\mu}) \rhd \boldsymbol{\sigma}} a_{\boldsymbol{\sigma}(\mu^{(1)}, \boldsymbol{\mu})}(q) G_{e+1}^{\boldsymbol{s}^+}(\boldsymbol{\sigma}^{+(k^{(1)}, \boldsymbol{k})}). \tag{3.2.24}$$

We proceed by induction on the dominance order. Suppose that $G_{e+1}^{\boldsymbol{s}^+}(\boldsymbol{\sigma}^{+(k^{(1)}, \boldsymbol{k})}) = (G_e^{\boldsymbol{s}}(\boldsymbol{\sigma}))^{+(k^{(1)}, \boldsymbol{k})}$ for all $\boldsymbol{\sigma} \lhd (\mu^{(1)}, \boldsymbol{\mu})$. Then

$$(3.2.24) = \mathfrak{F} \cdot G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \boldsymbol{\mu}^{+\boldsymbol{k}})) - \sum_{(\mu^{(1)}, \boldsymbol{\mu}) \rhd \boldsymbol{\sigma}} a_{\boldsymbol{\sigma}(\mu^{(1)}, \boldsymbol{\mu})}(q) (G_e^{\boldsymbol{s}}(\boldsymbol{\sigma}))^{+(k^{(1)}, \boldsymbol{k})}. \tag{3.2.25}$$

Since we are performing exactly the same operations of (3.2.22) and the starting coefficients are the same in (3.2.21) and (3.2.23), by definition of $^{+\boldsymbol{k}}$ we have

$$(3.2.25) = G_e^{\boldsymbol{s}}((\mu^{(1)}, \boldsymbol{\mu}))^{+(k^{(1)}, \boldsymbol{k})}.$$

Moreover, by uniqueness of the canonical basis of $M^{\otimes \boldsymbol{s}^+}$ we can state

$$(3.2.25) = G_{e+1}^{\boldsymbol{s}^+}((\mu^{(1)^{+k^{(1)}}}, \boldsymbol{\mu}^{+\boldsymbol{k}})).$$

Hence,

$$G_{e+1}^{\boldsymbol{s}^+}((\mu^{(1)}, \boldsymbol{\mu})^{+(k^{(1)}, \boldsymbol{k})}) = G_e^{\boldsymbol{s}}((\mu^{(1)}, \boldsymbol{\mu}))^{+(k^{(1)}, \boldsymbol{k})}.$$

□

4

Final remarks

Three natural questions arise from this thesis.

A question one could ask after reading Chapter 2 is the following.

Question 4.1. Is there a graded version of Proposition 2.2.8?

As we stated in different paragraphs of this thesis without ever going into much detail, in [BK09] Brundan and Kleshchev provide an explicit isomorphism between certain cyclotomic quotients of KLR algebras and Ariki-Koike algebras. This isomorphism allows us to use the $\mathbb{Z}$-grading of KLR algebras on Ariki-Koike algebras and then define and study the graded decomposition numbers (see [BKW11]). Therefore, since Brundan and Kleshchev in [BK09] proved a graded version of Ariki's theorem (Theorem 3.1.3), we believe a graded version of Proposition 2.2.8, as stated in the following, can be proved.

Conjecture 4.0.1. Fix $i \in \{0, 1, \dots, e-1\}$. Suppose that in each component of every r-multipartition that belongs to a block B of $\mathcal{H}_{r,n}$ there is no abacus configuration of the type $\bullet \quad |$ in runners $i-1$ and i. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in B$ then,

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = d_{\Phi_i(\boldsymbol{\lambda})\Phi_i(\boldsymbol{\mu})}(q)$$

where Φ_i swaps the runners $i-1$ and i in each component of the abacus display of a multipartition.

The next two questions arise from Chapter 3.

Question 4.2. Can Theorem 3.2.32 be extended to any multipartition $\boldsymbol{\mu}$?

We can express Theorem 3.2.32 in terms of q-decomposition numbers as follows.

Theorem 4.0.2. Let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)}) \in \mathcal{P}^r$ be an e-multiregular multipartition of n. Let $\boldsymbol{k} = (k^{(1)}, \dots, k^{(r)})$ with $k^{(j)}$ non-negative integer for $j = 1, \dots, r$ such that conditions (3.2.16) hold. Then

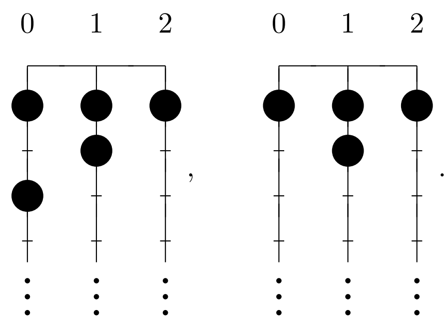
$$d^{e+1}_{\boldsymbol{\lambda}^{+\boldsymbol{k}}\boldsymbol{\mu}^{+\boldsymbol{k}}}(q) = d^{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q).$$

Since the q-decomposition number $d^e_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ is defined even when $\boldsymbol{\mu}$ is not e-multiregular, it comes natural to wonder whether Theorem 4.0.2 holds also in this more general case. However, the proof of a theorem including the non e-multiregular case seems to require an 'empty' runner removal theorem for Ariki-Koike algebras. This is for example the case for Iwahori-Hecke algebras (see [Fay07a]). Therefore, the next question emerges naturally.

Question 4.3. Is a runner removal theorem with an empty runner instead of a runner full of beads true for Ariki-Koike algebras?

There are different reasons why we think that this question should have a positive answer. First of all, an empty runner removal theorem is established for the Iwahori-Hecke algebras of the symmetric groups and the q-Schur algebras by James and Mathas in [JM02]. Also, we have seen so far how much the representation theory of Ariki-Koike algebras resambles the one of the symmetric groups and how often results for the symmetric groups (or, the Iwahori-Hecke algebras of the symmetric groups) can be extended to Ariki-Koike algebras. Furthermore, all the examples we have examined with the help of GAP confirmed our belief. An instance of them is the following.

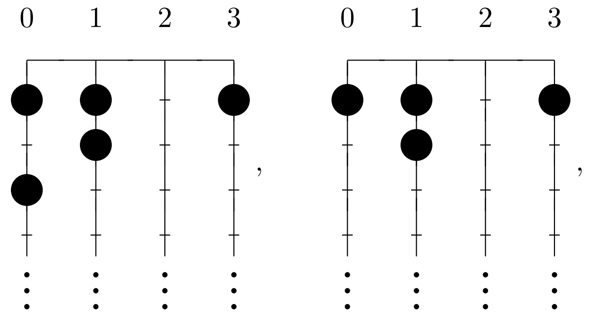
Example 4.0.3. Let $r = 2$, $e = 3$, and write the set $I = \mathbb{Z}/3\mathbb{Z}$ as $\{0, 1, 2\}$. Take $\boldsymbol{s} = (2, 1)$. Consider $\boldsymbol{\mu} = ((2, 1), (1))$ and its 3-abacus display



The canonical basis element $G^{\boldsymbol{s}}_3(((2, 1), (1)))$ is given by

$$\begin{aligned} G^{\boldsymbol{s}}_3(((2,1),(1))) &= ((2,1),(1)) + q((1^3),(1)) + q^2((1^2),(1^2)) \\ &= \boldsymbol{\mu} + q\boldsymbol{\lambda}_1 + q^2\boldsymbol{\lambda}_2. \end{aligned}$$

Now, consider the multipartition with 4-abacus display



that corresponds to the multipartition $((4,2,1),(2,1))$. Notice that the abacus display of $((4,2,1),(2,1))$ is obtained from the 3-abacus display of $\boldsymbol{\mu}$ by adding a runner with no beads to the left of each runner 2 and relabelling the runners in the usual way. Write $\boldsymbol{\mu}^{+\varnothing}$ for $((4,2,1),(2,1))$ and $\boldsymbol{s}^{+\varnothing}$ for the corresponding multicharge, that is $(1,0)$.

Now, we compute the canonical basis element for $\boldsymbol{\mu}^{+\varnothing}$. We get

$$G_4^{\boldsymbol{s}^{+\varnothing}}(((4,2,1),(2,1))) = ((4,2,1),(2,1)) + q((3,2^2),(2,1)) + q^2((3,2,1),(2^2)).$$

Hence, we notice that

$$G_4^{\boldsymbol{s}^{+\varnothing}}(\boldsymbol{\mu}^{+\varnothing}) = \boldsymbol{\mu}^{+\varnothing} + q\boldsymbol{\lambda}_1^{+\varnothing} + q^2\boldsymbol{\lambda}_2^{+\varnothing},$$

where $\boldsymbol{\lambda}_t^{+\varnothing}$ for $t \in \{1,2\}$ is obtained from the 3-abacus display of $\boldsymbol{\lambda}_t$ by adding a runner with no beads to the left of runner 2 in each component. Thus, in this case we have that the q-decomposition numbers match up, that is for $t \in \{1,2\}$

$$d^e_{\boldsymbol{\lambda}_t\boldsymbol{\mu}}(q) = d^{e+1}_{\boldsymbol{\lambda}_t^{+\varnothing}\boldsymbol{\mu}^{+\varnothing}}(q).$$

Therefore, we conjecture that the following claim should hold.

Conjecture 4.0.4. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^r$ be in a block B of $\mathcal{H}_{r,n}$ with $\boldsymbol{\mu}$ e-multiregular. Suppose that $\boldsymbol{\lambda}^{+\varnothing}$ and $\boldsymbol{\mu}^{+\varnothing}$ are the multipartitions obtained from the e-abacus display of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by adding an 'empty' runner in each of their components. Then

$$d^e_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = d^{e+1}_{\boldsymbol{\lambda}^{+\varnothing}\boldsymbol{\mu}^{+\varnothing}}(q).$$

However, the argument we used to prove the 'full' runner removal theorem is most likely not adaptable in a straightforward way to the proof of the above conjecture, because it heavily relies on the fact that the runner we add to each component is long enough.

Moreover, for Iwahori-Hecke algebras (where one usually looks to take inspiration to prove results for Ariki-Koike algebras), the proof of the 'empty' runner removal theorem provided by James and Mathas in [JM02] makes use of q-Schur algebras. They first show that certain decomposition numbers of the q-Schur algebras $\mathcal{S}_{\mathbb{C},q}(n)$ and $\mathcal{S}_{\mathbb{C},q'}(m)$ are equal for specified $m > n$. Then, since the decomposition matrix for the Iwahori-Hecke algebras $H_{\mathbb{C},q}(\mathfrak{S}_n)$ is a submatrix of the decomposition matrix of $\mathcal{S}_{\mathbb{C},q}(n)$, they deduce the analogous result for the q-decomposition numbers for Iwahori-Hecke algebras.

Although q-Schur algebras are defined also for Ariki-Koike algebras, a proper checking on how much of the proof for Iwahori-Hecke algebras can be extended to the Ariki-Koike algebras case needs to be done. Some further work is therefore required to adapt this proof for Ariki-Koike algebras.

Another interesting research direction is to interpret these equalities between certain q-decomposition numbers of Ariki-Koike algebras as consequences of Morita equivalences a la Chuang-Miyachi (see [CM10]). The first step in this direction would be looking for "candidate" Morita equivalences. Anyway, we haven't explored this in depth enough to be able to provide any suitable candidate.

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