NOTES ON MODEL THEORY OF MODULES OVER DEDEKIND DOMAINS

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Abstract. We associate a formal power series to every pp-formula over a Dedekind domain and use it to study Ziegler spectra of Dedekind domains R and $\widetilde{R}$, where R a subring of $\widetilde{R}$, with particular interest in the case when $\widetilde{R}$ is the integral closure of R in a finite dimensional separable field extension of the field of fractions of R.

1. Introduction

Our long term interest regards the ring $\mathbb{A}$ of algebraic integers. This is a Bézout (hence Prüfer, equivalently arithmetical) domain of Krull dimension 1, but not a Dedekind domain. The decidability of the first order theory of modules over $\mathbb{A}$ was proved in [12, Theorem 3.7], see also [8], without any explicit description of its Ziegler spectrum, which is still lacking. Recall that this spectrum is the one-point union of the spectra of the localizations $\mathbb{A}_{\mathcal{M}}$ at the non-zero prime ideals $\mathcal{M}$ of $\mathbb{A}$, which are 1-dimensional valuation domains with value group isomorphic to the additive group of rationals; this implies [24, Lemma 8.3] that their Ziegler spectra have the continuum power. Finding the way these spectra are patched together could be a real difficulty towards a full description of the Ziegler spectrum of $\mathbb{A}$.

On the other hand, a pp-formula in the first order language of $\mathbb{A}$-modules contains only finitely many scalars of $\mathbb{A}$ and so is defined over the ring of integers of some finite dimensional Galois field extension of $\mathbb{Q}$, which is a Dedekind domain. This suggests as a possible way to analyse $\mathrm{Zg}(\mathbb{A})$

- first to consider the Ziegler spectrum of a Dedekind domain R, which is very well known (see § 2),
- but also to compare the spectra of two Dedekind domains $R \subseteq \widetilde{R}$, with particular emphasis on the case when both R and $\widetilde{R}$ are subrings of $\mathbb{A}$, or even the rings of algebraic integers of some finite dimensional field extension $\mathbb{Q} \subseteq Q \subseteq L$.

The latter will be one of the main topics of this paper, also devoted to a comparison of pp-formulas over R and $\widetilde{R}$.

Let us describe in the context of a discrete valuation domain V (with primitive generator π) the technique that we introduce in this paper to study pp-formulas.

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To every pair (φ, ψ) of pp-formulas over V we associate a formal power series

$$P_V(\varphi, \psi) := \sum_{n=1}^{\infty} \ell_V(\varphi, \psi, V/\pi^n V)\ t^n,$$

where $\ell_V(\varphi, \psi, V/\pi^n V)$ denotes the length of the V-module $\varphi/\psi(V/\pi^n V)$. We show (Proposition 4.3) that this power series in $\mathbb{Z}[[t]]$ is a rational function with a pole at $t = 1$ whose multiplicity is equal to the Krull-Gabriel dimension of φ/ψ, considered as a coherent functor on the category V-mod of finitely presented V-modules or, equivalently, the m-dimension of the pp-pair φ/ψ in the sense of Ziegler [24]. The map $(\varphi, \psi) \mapsto P_V(\varphi, \psi)$ respects the relations that define the Grothendieck group $G_0(V)$ (described in §2.5) and therefore induces a morphism $G_0(V) \to \mathbb{Z}[[t]]$, which we prove (Theorem 4.2) to be an embedding.

In the sequel, this technique is globalised to associate a Poincaré series $P_R(\varphi, \psi)$ to a pp-pair over *any* Dedekind domain R and used to study its Dedekind extensions $R \subseteq \widetilde{R}$ by determining $P_{\widetilde{R}}(\varphi, \psi)$.

Here is the plan of this article.

The background introductory section § 2 contains several important preliminaries both about model theory of modules (such as pp-formulas, pp-pairs, pp-types, pure-injective modules) and Dedekind domains (equivalent definitions, main examples and basic properties). We also recall a structure theorem of finitely generated modules over these domains. This leads to a representation theorem for pp-1-formulas over them. In the same section we will examine extensions of Dedekind domains $R \subseteq \widetilde{R}$ as described before, as well as the Grothendieck group of pp-pairs of a commutative ring R.

The first part of the paper is devoted to single Dedekind domains R. § 3 characterizes the pp-pairs over R such that the corresponding open set in the Ziegler topology has Cantor-Bendixson rank ≤ 1. In § 4 we equip every pp-pair over a discrete valuation domain with the Poincaré series. We show that the Cantor-Bendixson rank of a pp-pair is equal to the multiplicity of singularity at 1 of its Poincaré series. In § 5 we equip every pp-pair over a Dedekind domain R with a Poincaré series in $\mathbb{Z}[[t_{\mathcal{P}} : \mathcal{P}$ non-zero prime ideal of $R]]$. Here our main theorem (see 4.2 and 5.1) singles out a natural group homomorphism from the Grothendieck group of the category of pp-pairs of R to the additive group $\mathbb{Z}[[t_{\mathcal{P}} : \mathcal{P}$ non-zero prime ideal of $R]]$ and studies its main properties.

The second part of the paper deals with extensions of Dedekind domains $R \subseteq \widetilde{R}$ (as before). Now the main result (in § 6) describes the way an indecomposable pure-injective module over $\widetilde{R}$ decomposes over R, see 6.4 and 6.6. Then we compare the Poincaré series of the same pp-pair over R both over $\widetilde{R}$ and over R: this is the topic of § 7.

Finally, when $\widetilde{R}$ is the integral closure of R in a finite Galois extension of the field of fractions Q of R, we analyse how the automorphisms of the Galois group of $L \supseteq Q$ act on the pp-formulas over $\widetilde{R}$. The main result here is Theorem 8.7, providing an explicit isomorphism between the lattice of pp-1-formulas over $\widetilde{R}$ fixed by the Galois group and that of pp-1-formulas over R.

Our hope is that, in some future work, all these results may be of some help in the study of the Ziegler topology of $\mathbb{A}$.

For every ring R, R-Mod (respectively Mod-R) denotes the category of left (respectively right) R-modules, while R-mod is the category of finitely presented left R-modules. We mainly refer to [17] and [18] for model theory of modules, and to [11] for Dedekind domains.

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2. Background

2.1. **Dedekind domains.** An integral domain is a **Dedekind domain** if it satisfies any of the equivalent conditions of the following theorem.

Theorem 2.1. *For any integral domain R, the following are equivalent:*

(1) *R is Noetherian, integrally closed and has Krull dimension* 1 *(that is, each non-zero prime ideal is maximal);*
(2) *R is Noetherian and every localisation $R_{\mathcal{M}}$ at a maximal ideal $\mathcal{M}$ is a valuation domain;*
(3) *Every ideal of R can be written as a product of a finite number of prime ideals;*
(4) *R is Noetherian and all finitely generated torsion-free R-modules are projective.*

Dedekind domains include principal ideal domains PID, like the rings of integers and Gaussian integers. If R is a Dedekind domain with field of fractions Q and L is a finite dimensional field extension of Q then the integral closure of R in L is also a Dedekind domain. We are particularly interested in the case when R is the ring $\mathbb{Z}$ of integers, so that Q is the field $\mathbb{Q}$ of rationals. Then L is a number field, and the integral closure of $\mathbb{Z}$ in L is called the ring of algebraic integers of L. By the previous considerations, it is a Dedekind domain, even if sometimes not a PID.

A crucial property of Dedekind domains is unique factorization of ideals. According to Condition (3) in Theorem 2.1, every non-zero proper ideal $\mathcal{P}$ of a Dedekind domain R decomposes as a finite product $\prod_{j=1}^{m} \mathcal{P}_j^{h_j}$ where m and the h_j are positive integers, and the $\mathcal{P}_j$ are pairwise different non-zero prime (equivalently maximal) ideals of R.

This decomposition is also unique up to the order of the factors. The exponent h_j of the power $\mathcal{P}_j^{h_j}$ is the largest positive integer such that $\mathcal{P}_j^{h_j}$ contains $\mathcal{P}$. When

$\mathcal{M}$ is none of the $\mathcal{P}_j$ but is a non-zero prime ideal of R, then one agrees that its exponent in the decomposition above is 0.

Let us also recall the following fundamental result about finitely generated modules over Dedekind domains.

Theorem 2.2. [1, Theorems 6.3.20 and 6.3.23] *Let R be a Dedekind domain. Every finitely generated R-module is of the form*

$$R^n \oplus J \oplus \bigoplus_{i=1}^{l} R/\mathcal{P}_i^{k_i}$$

where $n, l \in \mathbb{N}$, J is an ideal of R and for $1 \leq i \leq l$, $\mathcal{P}_i$ is a non-zero prime ideal of R and k_i is a positive integer.

This confirms that all finitely generated torsion-free modules over a Dedekind domain are projective, so part of Condition (4) in Theorem 2.1, see also [1, Corollary 6.3.4], [2, 2.3.20, B and C]. In particular all ideals over a Dedekind domain are projective.

2.2. **pp-formulas and their special form over Dedekind domains.** For k a positive integer, a *pp-k-formula* is a formula in the language, $\mathcal{L}(R) = (0, +, (r\cdot)_{r\in R})$, of (left) R-modules of the form

$$\exists \overline{y}(A\overline{x} = B\overline{y})$$

where $\overline{x}$ is a k-tuple of variables and A, B are appropriately sized matrices with entries in R. If φ is a pp-k-formula and M is a left R-module then $\varphi(M)$ denotes the set of all elements $\overline{m} \in M^k$ such that $\varphi(\overline{m})$ holds. This is a subgroup of M^k, called pp-subgroup. When R is commutative, it is also a submodule.

Let pp_R^k denote the set of pp-k-formulas, more precisely of their equivalence classes modulo the first order theory T_R of R-modules. This set pp_R^k is a lattice under implication (equivalently under conjunction and sum of pp-formulas). For $M \in R$-Mod, write $\mathrm{pp}_R^k(M)$ for the set of pp-k-definable subsets of M or equivalently the quotient of pp_R^k after identifying pp-formulas which define the same set in M.

A *pp-k-pair* is an ordered pair of pp-formulas $\varphi, \psi \in \mathrm{pp}_R^k$ such that $\varphi \geq \psi$, that is, ψ implies φ in T_R.

For (φ, ψ) a pp-k-pair, we write $[\psi, \varphi]$ for the interval in pp_R^k, i.e. the set of $\sigma \in \mathrm{pp}_R^k$ such that $\psi \leq \sigma \leq \varphi$; if $M \in R$-Mod then we write $[\psi, \varphi]_M$ for the corresponding interval in $\mathrm{pp}_R^k(M)$.

Recall that a commutative ring is *arithmetical* if and only if its lattice of ideals is distributive. Equivalently, every localization of R at a maximal ideal is a valuation ring. Then an integral domain is arithmetical if and only if it is Prüfer, see [13, Theorem 6.6 p. 127].

Proposition 2.3. [5, 3.1] *Let R be a commutative ring. The lattice* pp_R^1 *is distributive if and only if R is an arithmetical ring. In particular* pp_R^1 *is distributive when R is a Dedekind domain.*

If M is finitely presented module and $\overline{m} \in M$ is a tuple of length k then there is a pp-k-formula φ which *generates the pp-type* of $\overline{m}$ in M, that is, for all pp-k-formulas ψ, $\psi \geq \varphi$ if and only if $\overline{m} \in \psi(M)$. Conversely, if φ is a pp-k-formula, then there exist a finitely presented module M and $\overline{m} \in M$ a tuple of length k such that φ generates the pp-type of $\overline{m}$ in M. We call M together with $\overline{m}$ a *free-realisation* of φ. For proofs of these assertions and more about free-realisations, see [18, Section 1.2.2].

Let $\varphi, \varphi' \in \mathrm{pp}_R^k$. If $\overline{m} \in M$ and $\overline{m'} \in M'$ are free-realisations of φ and φ' respectively then $\overline{m} + \overline{m'}$ in $M \oplus M'$ is a free-realisation of $\varphi + \varphi'$.

For every ordinal α one introduces a lattice $\mathrm{pp}_R(\alpha)$, starting from $\mathrm{pp}_R(0) = \mathrm{pp}_R^1$, collapsing at each (successor) step intervals of finite length and handling in the straightforward way limit ordinals. For instance, in the basic step, two pp-formulas $\varphi(x)$ and $\varphi'(x)$ are identified if and only if in pp_R^1 the closed interval $[\varphi \wedge \varphi', \varphi + \varphi']$ is of finite length. The m-dimension of pp_R, $\mathrm{mdim}(\mathrm{pp}_R)$, is

- the smallest ordinal α such that $\mathrm{pp}_R(\alpha)$ is a lattice of finite length, if such an ordinal exists,
- ∞ (or undefined) otherwise,

see [17, 10.2 pp. 203-208] or [18, 7.2 pp. 302-311] for the full proper definition. The same concept makes sense in every closed interval $[\psi, \varphi]$ with $\psi \leq \varphi$ pp-formulas. We will see later, mainly in Section 4, that $\mathrm{mdim}(\mathrm{pp}_R) = 2$ when R is a Dedekind domain which is not a field.

We now use Theorem 2.2 to deduce some special forms for pp-formulas over Dedekind domains. In the next statement and later, $=$ means equality in pp_R^k, that is, equivalence with respect to T_R.

Proposition 2.4. *Let φ be a pp-k-formula over a Dedekind domain R. Then φ decomposes as a finite sum*

$$\varphi = \varphi_0 + \sum_{\mathcal{P} \ prime} \varphi_{\mathcal{P}}$$

where φ_0 is freely realised in R^n, $\mathcal{P}$ ranges over non-zero prime ideals of R and $\varphi_{\mathcal{P}}$ is freely realised in a sum of modules $R/\mathcal{P}^n$, with n a positive integer.

Moreover φ_0 has the form $\exists \overline{y}\ \overline{x} = A_\varphi \overline{y}$ for some appropriately sized matrix A_φ over R.

Let φ, ψ be pp-k-formulas. Then $\varphi \leq \psi$ if and only if $\varphi_0 \leq \psi_0$ and for each non-zero prime ideal $\mathcal{P}$, $\varphi_{\mathcal{P}} \leq \psi_0 + \psi_{\mathcal{P}}$.

Proof. The first claim directly follows from the description of finitely generated modules over Dedekind domains in Theorem 2.2. In particular, since ideals are projective, any pp-formula realised in an ideal is also realised in a direct sum of copies of R.

The fact that pp-formulas freely realised in some R^n are of the form stated is [18, Lemma 1.2.29].

Next suppose $\varphi \leq \psi$. Then $\varphi_0 \leq \psi$ and $\varphi_{\mathcal{P}} \leq \psi$ for all $\mathcal{P}$. Since φ_0 is freely realised in R^k, $\varphi_0 \leq \psi$ if and only if $\varphi_0(R) \subseteq \psi(R) = \psi_0(R)$. In fact, for all $\mathcal{P}$, $\psi_{\mathcal{P}}(R) = 0$ by [18, Corollary 1.2.17], because $\mathrm{Hom}(R/\mathcal{P}^n, R) = 0$ for all n. Now $\varphi_0(R) \leq \psi_0(R)$ implies $\varphi_0 \leq \psi_0$ since φ_0 is freely realised in R^k.

On the other hand $\varphi_{\mathcal{P}} \leq \psi$ implies $\varphi_{\mathcal{P}}(R/\mathcal{P}^n) \subseteq \psi(R/\mathcal{P}^n) \subseteq \psi_0(R/\mathcal{P}^n) + \psi_{\mathcal{P}}(R/\mathcal{P}^n)$ for all positive integers n since $\psi_{\mathcal{Q}}(R/\mathcal{P}^n) = 0$ for $\mathcal{Q} \neq \mathcal{P}$ a non-zero prime ideal. Now $\varphi_{\mathcal{P}}(R/\mathcal{P}^n) \subseteq \psi_0(R/\mathcal{P}^n) + \psi_{\mathcal{P}}(R/\mathcal{P}^n)$ for all n implies $\varphi_{\mathcal{P}} \leq \psi_0 + \psi_{\mathcal{P}}$ because $\varphi_{\mathcal{P}}$ is freely realised in a sum of modules of the form $R/\mathcal{P}^n$. □

For R a commutative ring and J a finitely generated ideal of R, let $J \mid x$ denote the pp-formula which defines JM in all R-modules M. Equivalently, if $a_1, \ldots, a_n$ generate J, then $J \mid x := a_1|x + \ldots + a_n|x$.

Lemma 2.5. *Let R be a Dedekind domain. The map from the ideal lattice of R to pp_R^1 which sends any ideal I of R to $I \mid x \in \mathrm{pp}_R^1$ is a lattice homomorphism.*

Proof. The only thing that needs to be checked is that for all ideals I, J of R, $I \mid x \wedge J \mid x = I \cap J \mid x$.

Let $\mathcal{P}$ be a non-zero prime ideal of R. If N is an $R_{\mathcal{P}}$-module and K is an ideal of R then $KN = KR_{\mathcal{P}}N$. Moreover, $IR_{\mathcal{P}} \cap JR_{\mathcal{P}} = (I \cap J)R_{\mathcal{P}}$. So, since all indecomposable pure-injective R-modules are restrictions of (indecomposable pure-injective) $R_{\mathcal{P}}$-modules for some prime $\mathcal{P}$ (see the next subsection), it is enough to note that if R is a discrete valuation domain and I, J are ideals of R then $I \mid x \wedge J \mid x = I \cap J \mid x$. □

Lemma 2.6. *Let R be a Dedekind domain.*

(1) *If φ is a pp-1-formula freely realised in a finitely generated torsion-free module then φ has the form $J \mid x$ for some ideal J. Moreover, $J \mid x$ is equivalent to $\bigwedge_{j=1}^n \mathcal{P}_j^{h_j} \mid x$ where J decomposes in R as $\prod_{j=1}^n \mathcal{P}_j^{h_i}$, the $\mathcal{P}_j$ are pairwise distinct non-zero prime ideals of R and the h_j are positive integers.*

(2) *If φ is a pp-1-formula freely realised in $R/\mathcal{P}^n$ where $\mathcal{P}$ is a non-zero prime ideal of R and n is a positive integer, then φ has the form $\mathcal{P}^l \mid x \wedge x\mathcal{P}^r = 0$ where l, r are nonnegative integers, $l + r = n$ and $r > 0$.*

In particular, pp_R^1 is generated by formulas of the form $\mathcal{P}^h \mid x$ and $x\mathcal{P}^h = 0$ where $\mathcal{P}$ is a non-zero prime ideal and h is a positive integer.

Proof. (1) Since all finitely generated torsion-free modules are projective, if φ is freely realised in a finitely generated torsion-free module then φ is freely realised in R^n for some positive integer n. Therefore $\varphi = \sum_{i=1}^n \varphi_i$ where each φ_i is freely realised in R, whence has the form $a_i \mid x$ for some $a_i \in R$. Thus $\varphi = \sum_{i=1}^n (a_i \mid x) = (\sum_{i=1}^n a_i R) \mid x$.

The final part follows from Lemma 2.5, since $\prod_{j=1}^n \mathcal{P}_j^{h_j} = \bigcap_{j=1}^n \mathcal{P}_j^{h_j}$.

(2) Take $a \in R$ and look at $a + \mathcal{P}^n \in R/\mathcal{P}^n$. Suppose $a \in \mathcal{P}^h \setminus \mathcal{P}^{h+1}$ where $0 \leq h \leq n-1$. Then a satisfies $\mathcal{P}^h \mid x \wedge \mathcal{P}^{n-h} x = 0$. Now suppose that $b \in R$ satisfies the formula $\mathcal{P}^h \mid x \wedge \mathcal{P}^{n-h} x = 0$. So $b \in \mathcal{P}^h \cap \mathcal{P}^{h+(l-n)} = \mathcal{P}^h \cdot \mathcal{P}^{\max\{0,\, l-n\}}$. We need to show that there is a homomorphism $f : R/\mathcal{P}^n \to R/\mathcal{P}^l$ with $f(a+\mathcal{P}^n) = b + \mathcal{P}^l$. But such an f exists if and only if $b \in a\mathcal{P}^{\max\{0,\, l-n\}} R_{\mathcal{P}} = \mathcal{P}^h \mathcal{P}^{\max\{0, l-n\}} R_{\mathcal{P}}$. □

Corollary 2.7. *Let φ be a pp-1-formula over a Dedekind domain R. Then*

$$\varphi = \varphi(R)|x + \sum_{\mathcal{P} \in \Omega} \varphi_{\mathcal{P}}$$

where Ω is a finite set of non-zero prime ideals of R and, for all $\mathcal{P} \in \Omega$, $\varphi_{\mathcal{P}}$ is a pp-1-formula freely realised in a sum of modules of the form $R/\mathcal{P}^n$, n a positive integer. Moreover, if $\varphi(R) \neq 0$ we can suppose that $\mathcal{P} \in \Omega$ implies $\mathcal{P} \mid \varphi(R)$.

Proof. We know from Lemma 2.6 that $\varphi_0 := J \mid x$ for some ideal J of R. Now $J = \varphi_0(R) = \varphi(R)$ as required.

The "moreover" claim is true because if $\mathcal{P}$ does not divide $\varphi(R)$ then $\varphi(R) \mid x$ is equivalent to $x = x$ in $R/\mathcal{P}^n$ for any positive integer n. □

2.3. **Irreducible pp-types and indecomposable pure-injective modules.** Let R be a ring, $M \in R$-Mod and $\overline{m}$ a k-tuple of elements from M. The *pp-type* of $\overline{m}$ in M, denoted by $\mathrm{pp}^M(\overline{m})$, is the set of pp-$k$-formulas φ such that $M \models \varphi(\overline{m})$. For any filter p in the lattice of pp-k-formulas there exist an R-module M and $\overline{m}$ a k-tuple of elements from M such that $p = \mathrm{pp}^M(\overline{m})$.

A pp-k-type p is *irreducible* if for any $\psi_1, \psi_2 \in \mathrm{pp}_R^k$, if $\psi_1, \psi_2 \notin p$ then there exists $\sigma \in p$ such that $\psi_1 \wedge \sigma + \psi_2 \wedge \sigma \notin p$. When pp_R^1 is distributive, in particular when R is a Dedekind domain, a pp-1-type p is irreducible if and only if for all $\psi_1, \psi_2 \in \mathrm{pp}_R^1$, $\psi_1 + \psi_2 \in p$ implies $\psi_1 \in p$ or $\psi_2 \in p$, i.e. the pp-1-types are exactly the prime filters of the distributive lattice pp_R^1.

A *pure-embedding* between two modules is an embedding which preserves the solution sets of pp-formulas. We say a module U is *pure-injective* if for every pure-embedding $g : U \to M$, the image of U in M is a direct summand of M. A pure-injective module is *indecomposable* if it admits no non-trivial direct summands. Each pure-injective module is the *pure-injective envelope* (a minimal pure-injective extension) of a direct sum of indecomposable pure-injectives, up to a possible further pure-injective summand, which is *superdecomposable*, that is, with no indecomposable non-trivial direct summand.

Lemma 2.8. [24, Theorem 5.4] *Let R be a commutative ring and U an indecomposable pure-injective R-module. The set $\mathcal{P}(U)$ of the scalars $r \in R$ such that the* *endomorphism of U defined by $m \mapsto rm$ is not an automorphism is a maximal ideal of R (called the maximal ideal attached to U).*

Theorem 2.9. [18, Theorem 5.2.2] *Let R be a Dedekind domain. The indecomposable pure-injective R-modules are:*

(1) *For each non-zero prime ideal $\mathcal{P}$ of R,*
 (i) *$R/\mathcal{P}^n$ for every positive integer n,*
 (ii) *the completion, $\overline{R_{\mathcal{P}}} = \varprojlim R/\mathcal{P}^n$, of R in the $\mathcal{P}$-adic topology,*
 (iii) *the injective hull $E(R/\mathcal{P}) = \varinjlim R/\mathcal{P}^n$, of $R/\mathcal{P}$, and*

(2) *the field of fractions of R.*

Moreover over R there is no superdecomposable pure-injective module.

2.4. **The Ziegler spectrum.** The *Ziegler spectrum* $\mathrm{Zg}(R)$ of a ring R is the following topological space.

- The points are the (isomorphism classes of) indecomposable pure-injective R-modules.
- A basis of open sets for the topology is given by

$$(\varphi/\psi) := \{U \in \mathrm{Zg}(R) : \varphi(U) \supset \psi(U)\}$$

 where (φ, ψ) is a pp-pair, so that $\varphi(M) \supseteq \psi(M)$ for every R-module M. Here $\supset$ denotes proper inclusion. Indeed pp-1-pairs are enough to induce the topology.

For φ and ψ arbitrary, we put $(\varphi/\psi) = (\varphi/\psi \wedge \varphi)$. The Ziegler spectrum was introduced in [24], see also [17] and [18]. Over a Dedekind domain R (which is not a field) the Ziegler spectrum is well understood, see [17, 4.7 and Corollary 2.$\mathbb{Z}$11]. The isolated points are the indecomposable modules of finite length $R/\mathcal{P}^n \simeq R_{\mathcal{P}}/\mathcal{P}^n R_{\mathcal{P}}$ where $\mathcal{P}$ is a non-zero prime ideal and n is a positive integer. The points of Cantor-Bendixson rank (CB-rank from now on) 1 are the $\overline{R_{\mathcal{P}}}$ and the $E(R/\mathcal{P})$, for $\mathcal{P}$ as before. Finally, the field of fractions of R, viewed as an R-module, is the unique point of CB-rank 2.

2.5. **The Grothendieck group of pp-pairs.** For more detailed information about categories of pp-pairs see [18, 3.2.2] and [9, §1].

The objects of the *category* $\mathbb{L}_R^{\mathrm{eq}+}$ *of pp-pairs* are pairs of pp-k-formulas (φ, ψ) where $\varphi \geq \psi$ in pp_R^k and k is a positive integer. We identify $(\varphi(\overline{x}), \psi(\overline{x}))$ with $(\varphi(\overline{y}), \psi(\overline{y}))$ whenever $\overline{x}$ and $\overline{y}$ are tuples of variables of the same length.

Let (φ, ψ) and (σ, τ) be pp-pairs, with $\varphi, \psi \in \mathrm{pp}_R^k$ and $\sigma, \tau \in \mathrm{pp}_R^m$, and let $\overline{x}, \overline{y}$ be disjoint tuples of variables with length $|\overline{x}| = k$ and $|\overline{y}| = m$. The morphisms $\rho : (\varphi, \psi) \to (\sigma, \tau)$ are given by pp-formulas $\rho(\overline{x}; \overline{y})$ such that

(i) $\varphi(\overline{x}) \leq \exists \overline{y} \rho(\overline{x}; \overline{y})$,
(ii) $\psi(\overline{x}) \leq \rho(\overline{x}; 0)$,
(iii) $\exists \overline{x} \rho(\overline{x}; \overline{y}) \leq \sigma(\overline{y})$, and,

(iv) $\rho(0, \overline{y}) \leq \tau(\overline{y})$.

Recall that R-mod denotes the category of finitely presented R-modules. We write (R-mod, Ab) for the category of additive functors from R-mod to the category Ab of abelian groups and $(R\text{-mod, Ab})^{\text{fp}}$ for the full subcategory of the finitely presented functors in (R-mod, Ab). For any $F \in (R\text{-mod, Ab})^{\text{fp}}$, there exist $A, B, C \in R$-mod and a right exact sequence $A \to B \to C \to 0$ such that

$$0 \to (C, -) \to (B, -) \to (A, -) \to F \to 0 \tag{1}$$

is exact (see [18, 10.2]). Here, for $M \in R$-mod, $(M, -) := \mathrm{Hom}_R(M, -)$. The representable functors $(M, -)$ with $M \in R$-mod are exactly the projective objects in $(R\text{-mod}, \text{Ab})^{\text{fp}}$. Therefore every functor F in $(R\text{-mod}, \text{Ab})^{\text{fp}}$ has a projective resolution of length ≤ 2.

Theorem 2.10. ([18, Theorem 10.2.30]) *Let R be a ring. The category $\mathbb{L}_R^{\text{eq}+}$ is equivalent to* $(R\text{-mod, Ab})^{\text{fp}}$.

It will be useful for us to have description of the equivalence, at least on objects (for full details see [18, Theorem 10.2.30]). Suppose that (φ, ψ) is a pp-pair. Let $F_{\varphi/\psi} : R\text{-mod} \to \text{Ab}$ be the functor defined on objects by $F_{\varphi/\psi}(M) = \varphi(M)/\psi(M)$ and on morphisms $f : M \to N$ by $F_{\varphi/\psi}(f)(a + \psi(M)) = f(a) + \psi(N)$ for every $a \in \varphi(M)$. Then $F_{\varphi/\psi} \in (R\text{-mod, Ab})^{\text{fp}}$.

The equivalence functor from $\mathbb{L}_R^{\text{eq}+}$ to $(R\text{-mod, Ab})^{\text{fp}}$ is given on objects by sending (φ, ψ) to $F_{\varphi/\psi}$.

Now suppose that $F \in (R\text{-mod, Ab})^{\text{fp}}$. Take $A, B \in R$-mod and $f : A \to B$ such that

$$(B, -) \to (A, -) \to F \to 0$$

is exact. Take $\overline{a}$ a generating tuple for A. Let φ generate the pp-type of $\overline{a}$ in A and let ψ generate the pp-type of $f(\overline{a})$ in B. Then $F \cong F_{\varphi/\psi}$.

For example, if $F = (A, -)$, that is, if $f = 0$, then the pp-type of $\overline{a}$ in A is generated by any quantifier free formula $U\overline{x} = 0$, where U is a matrix of presentation for A. The projective objects of the category are therefore of the form $(A, -) \cong F_{\varphi/\psi}$, where φ is quantifier free and $\psi = 0$.

Let $\mathcal{A}$ be an abelian category and suppose that $\mathcal{C}$ is a (skeletally) small additive subcategory, closed under extensions in $\mathcal{A}$. The *Grothendieck group* $\mathrm{Gr}(\mathcal{C}; \mathcal{A})$ of such an inclusion $\mathcal{C} \subseteq \mathcal{A}$ is defined to be the abelian group with generators $[C]$, indexed by the isomorphism classes of $\mathcal{C}$, modulo the relations $[A] - [B] + [C]$, whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2}$$

is exact in $\mathcal{A}$. The (class) function $\Omega \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}; \mathcal{A})$, $C \mapsto [C]$, is additive in the sense that $\Omega(B) = \Omega(A) + \Omega(C)$, for every short exact sequence (2). It is universal with respect to this property, in the sense that every additive function

$\mathrm{Ob}(\mathcal{C}) \to G$ to an abelian group G factors uniquely through Ω. In case, $\mathcal{C} = \mathcal{A}$, the Grothendieck group is plainly denoted by $\mathrm{Gr}(\mathcal{A})$.

Let $\mathrm{K}_0(R\text{-mod}, \oplus)$ denote the free abelian group on the objects of R-mod modulo the subgroup generated by $A + B - M$ whenever M is isomorphic to $A \oplus B$. It may happen that some non-zero A in R-mod is sent to 0 in $\mathrm{K}_0(R\text{-mod}, \oplus)$ and that non-isomorphic $A, A' \in R$-mod have the same image in $\mathrm{K}_0(R\text{-mod}, \oplus)$ (see [23, Theorem 1.11 p. 74]). However, when R is commutative, if the image of $A \in R$-mod is zero then $A = 0$.

The defining relations on $\mathrm{K}_0(R\text{-mod}, \oplus)$ ensure that there is a unique map $\mathrm{K}_0(R\text{-mod}, \oplus) \to \mathrm{Gr}(\mathrm{proj}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+}); \mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+})$ induced by the assignment $A \mapsto (A, -)$; it is clearly surjective. By [23, Theorem 4.4 p. 102] or [20, Theorem 3.1.13], the composition

$$\mathrm{K}_0(R\text{-mod}, \oplus) \to \mathrm{Gr}(\mathrm{proj}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+}); \mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+}) \to \mathrm{Gr}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+})$$

has an inverse $F \mapsto [(A, -)] - [(B, -)] + [(C, -)]$ defined in terms of the projective resolution (1). This implies that both of the maps in the composition are isomorphisms. We document this as follows.

Remark 2.11. *For any ring R, the map from $K_0(R\text{-mod}, \oplus)$ to $\mathrm{Gr}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+})$ induced by sending $[M] \in K_0(R\text{-mod}, \oplus)$ to $[(M, -)] \in \mathrm{Gr}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+})$ is an isomorphism.*

In the remainder of this paper we put for simplicity $G_0(R) := \mathrm{Gr}(\mathbb{L}_{\mathrm{R}}^{\mathrm{eq}+})$ (so isomorphic to $\mathrm{K}_0(R\text{-mod}, \oplus)$) and we call it the *Grothendieck group of pp-pairs* of R. Just to summarize, we can view it, in terms of pp-formulas, as built in the following way.

- We consider the (additive) free abelian group generated by pp-k-pairs (φ, ψ) where k ranges over positive integers.
- Let (φ, ψ), (φ', ψ') and (φ'', ψ'') be pp-pairs with corresponding numbers of free variables k, k', k'', and assume that there are pp-formulas ι and π, with $k' + k$, $k + k''$ free variables respectively, defining in each R-module N a short exact sequence

  $$0 \to \varphi'(N)/\psi'(N) \xrightarrow{\iota(N)} \varphi(N)/\psi(N) \xrightarrow{\pi(N)} \varphi''(N)/\psi''(N) \to 0.$$

  Factor the free abelian group built before by the relations

  $$(\varphi, \psi) = (\varphi', \psi') + (\varphi'', \psi'')$$

  for every choice of (φ, ψ), (φ', ψ') and (φ'', ψ'') with this property.

The quotient group is just the Grothendieck group $G_0(R)$. We will denote by $[\varphi, \psi]_{G_0(R)}$ the class of a pp-pair (φ, ψ) in this group.

An R-module M is of *finite endolength* if it is of finite length as a module over its endomorphism ring. By [18, Proposition 4.4.25], $M \in R$-Mod is of finite endolength if and only if $\mathrm{pp}_R^1(M)$ is of finite length. Again, by [18, Proposition 4.4.25], when

M is of finite endolength every $\mathrm{End}(M)$-submodule L of M is pp-definable, i.e. there exists $\varphi \in \mathrm{pp}^1_R$ such that $L = \varphi(M)$. Viewing M^k as an $\mathrm{End}(M)$-module also of finite endolength [18, Lemma 4.4.26], the same argument shows that if L is an $\mathrm{End}(M)$-submodule of M^k then there exists $\varphi \in \mathrm{pp}^k_R$ such that $L = \varphi(M)$.

Given pp-formulas φ, ψ where $\varphi \geq \psi$ and $M \in R$-Mod, define the pp-length $l_R(\varphi, \psi, M)$ of (φ, ψ) at M to be the length of $[\psi, \varphi]_M$ as a lattice or equivalently (see [18, Proposition 4.4.25]) the endolength of $\varphi(M)/\psi(M)$, that is its length as an $\mathrm{End}(M)$-module. Note that if (φ, ψ) and (φ', ψ') are isomorphic in $\mathbb{L}^{\mathrm{eq}+}_R$ then $l_R(\varphi, \psi, M) = l_R(\varphi', \psi', M)$, because $\varphi(M)/\psi(M)$ and $\varphi'(M)/\psi'(M)$ are isomorphic as $\mathrm{End}(M)$-modules.

We can give an explicit description of $K_0(R\text{-mod}, \oplus)$ when R is a Dedekind domain based on 2.2.

Proposition 2.12. *Let R be a Dedekind domain. Then $K_0(R\text{-mod}, \oplus)$ is isomorphic to $\mathbb{Z} \oplus \mathrm{Cl}(R) \oplus \mathbb{Z}^{(\kappa)}$ where $\mathrm{Cl}(R)$ is the ideal class group of R and $\kappa :=$* $\sup\{|\mathrm{Spec\ R}|, \aleph_0\}$.

Proof. Let G' be the free abelian group on the isomorphism types of the finitely presented indecomposable torsion R-modules, i.e. modules of the form $R/\mathcal{P}^l$ where $\mathcal{P}$ is a maximal ideal of R and $l \in \mathbb{N}$. Let $G := \mathbb{Z} \oplus \mathrm{Cl}(R) \oplus G'$. We will define an isomorphism $\pi : G \to K_0(R\text{-mod}, \oplus)$. This is enough to prove the proposition because κ is equal to the size of the set of finitely presented indecomposable torsion R-modules. Every element of $\mathrm{Cl}(R)$ is the class of an ideal. So elements of G are of the form $(n, J, \sum_{i=1}^m M_i - \sum_{j=1}^l L_j)$ where $n \in \mathbb{Z}$, J is an ideal of R and M_i, L_j are finitely presented indecomposable torsion R-modules. Define

$$\pi(n, J, \sum_{i=1}^m M_i - \sum_{j=1}^l L_j) := (n-1)[R] + [J] + \sum_{i=1}^m [M_i] - \sum_{j=1}^l [L_j].$$

It follows from [1, 6.1.4] that π is a group homomorphism. By Theorem 2.2, π is surjective. By [23, Theorem 1.10 p. 73], $[A] = [B]$ in $K_0(R, \oplus)$ if and only if $A \oplus C \cong B \oplus C$ for some $C \in R$-mod. With a bit of work it follows from [1, 6.3.23], which describes the isomorphism types of finitely presented modules over Dedekind domains, that π is injective. □

2.6. **Extensions of Dedekind domains.** We recall some basic facts on this topic, see [11] and [16] for much more on it.

Let R be a Dedekind domain but not a field, $\widetilde{R}$ its integral closure in some finite dimensional extension L of its field of fractions Q.

Let $\mathcal{P}$ be a non-zero prime ideal of R. Then $\mathcal{P}\widetilde{R}$ is a non-zero proper ideal of $\widetilde{R}$ and so decomposes in $\widetilde{R}$ as

$$\mathcal{P}\widetilde{R} = \prod_{j=1}^{g} \mathcal{M}_j^{e_j}$$

where the $\mathcal{M}_j$ are the distinct prime ideals of $\widetilde{R}$ containing $\mathcal{P}\widetilde{R}$, that is, satisfying $\mathcal{M}_j \cap R = \mathcal{P}$. For all $j = 1, \ldots, g$ there is a ring embedding of $R/\mathcal{P}$ into $\widetilde{R}/\mathcal{M}_j$, given by $a + \mathcal{P} \mapsto a + \mathcal{M}_j$ for every $a \in R$.

The positive integer e_j is called the *ramification index* of $\mathcal{M}_j$ in $\widetilde{R}$ over R (with respect to $\mathcal{P}$).

The degree of the field extension $[\widetilde{R}/\mathcal{M}_j : R/\mathcal{P}]$ (denoted from now on by f_j) is called the *inertial degree* of $\mathcal{M}_j$ in $\widetilde{R}$ (with respect to $\mathcal{P}$).

When L is separable over Q (in particular, in the characteristic 0 case), the degree $[L : Q]$ coincides with $\sum_{j=1}^{g} e_j f_j$ (see [11, Corollary 6.7 p. 31]).

If L is a (finite) Galois extension of Q, then $e_j = e$, $f_j = f$ are constant for all j, and so $[L : Q] = efg$ ([11, Theorem 6.8 p. 32]).

The ideal $\mathcal{P}$ is said to *split completely* if $e_j = f_j = 1$ for all j, whence $[L : Q] = g$, and to *totally ramify* if $g = 1 = f_1$ (then there is a unique non-zero prime ideal of $\widetilde{R}$ extending it, and $e_1 = [L : Q]$).

The following very simple and familiar example will be useful later.

Example 2.13. The ring $\mathbb{Z}[i]$ of Gaussian integers is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$.

- Let $\mathcal{P} = 2\mathbb{Z}$. Then $\mathcal{P}\mathbb{Z}[i] = 2\mathbb{Z}[i]$ is the square of the prime ideal generated by $1 + i$. Therefore $g = 1$, $e_1 = 2$ and $\mathcal{P}$ totally ramifies. Moreover $\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, whence $f_1 = 1$.
- Next let $\mathcal{P} = p\mathbb{Z}$ with p prime, $p \equiv 3 \pmod 4$. Then $p\mathbb{Z}[i]$ is also prime, whence $g = 1$, $e_1 = 1$. Moreover it is easily seen that $f_1 = 2$.
- Finally let $\mathcal{P} = p\mathbb{Z}$ with p prime, $p \equiv 1 \pmod 4$. Then p can be expressed in $\mathbb{Z}$ as a sum $a^2 + b^2 = (a + ib)(a - ib)$ of two squares and $p\mathbb{Z}[i]$ decomposes in $\mathbb{Z}[i]$ as the product of the prime ideals generated by $a \pm ib$ (both irreducible since their common norm is prime). These ideals are different from each other. Therefore $g = 2$, $e_1 = e_2 = 1$, $f_1 = f_2 = 1$ and $\mathcal{P}$ splits completely.

Part 1. SINGLE DEDEKIND DOMAINS

In this part we deal with a single Dedekind domain R which is not a field and we denote by Q its field of fractions.

3. CB-rank and locally bounded pp-pairs

We give two equivalent characterizations of the pp-pairs (φ, ψ) over R such that the corresponding open set $(\varphi \,/\, \psi)$ of $\mathrm{Zg}(R)$ has CB-rank at most 1.

First let us put, for every commutative ring R and pp-pair (φ, ψ) over R, $(\varphi : \psi)_R = \{r \in R : r\varphi(N) \subseteq \psi(N) \ \forall N \in R\text{-Mod}\}$. Note that, if $r\varphi(U) \subseteq \psi(U)$ for all $U \in \mathrm{Zg}(R)$, then $r \in (\varphi : \psi)_R$. It is straightforward to prove:

Lemma 3.1. *For every pp-pair (φ, ψ), the set $(\varphi : \psi)_R$ is an ideal of R, and it is proper if and only if $\varphi > \psi$. Moreover, for every $N \in R$-Mod, $\varphi(N)/\psi(N)$ is naturally equipped with the structure of a module over $R/(\varphi : \psi)_R$.*

Indeed $r\varphi(N)$ itself can be regarded as a pp-subgroup of a given R-module N. Just define, for any pp-k-formula $\varphi = \varphi(\overline{x})$ and $r \in R \backslash \{0_R\}$,

- $r^{-1}\varphi(\overline{x})$ to be the pp-formula $\exists \overline{w}\ (r\overline{x} = \overline{w} \wedge \varphi(\overline{w}))$,
- $r\varphi(\overline{x})$ to be $\exists \overline{z}\ (\overline{x} = r\overline{z} \wedge \varphi(\overline{z}))$.

Similar notions $\varphi(\overline{x})r^{-1}$, $\varphi(\overline{x})r$ can be introduced among pp-formulas over right R-modules. However, as R is commutative, left modules can be naturally regarded as right, and conversely. Therefore we freely view modules from both sides.

For all R-modules N, $r \in R$ and pp-formulas φ, $\varphi(N) \supseteq r(r^{-1}\varphi(N))$. However $\varphi(N)$ is not necessarily equal to $r(r^{-1}\varphi(N))$. For example, take $R := \mathbb{Z}$, $r := 2$ and $\varphi(x)$ to be $x = x$. Then $2^{-1}\varphi(x)$ is $x = x$, but $2(2^{-1}\varphi(x))$ is $2 \mid x$.

Remark 3.2. *Let R be an integral domain, $r \in R \backslash \{0\}$ and φ a pp-formula. If N is a divisible R-module then $\varphi(N) = r(r^{-1}\varphi(N))$.*

Proof. Take $\overline{m} \in \varphi(N)$. Since N is divisible, $\overline{m} = r \cdot \overline{m}_1$ for some $\overline{m}_1 \in N$. So $\overline{m}_1 \in r^{-1}\varphi(N)$. Therefore $\overline{m} = r \cdot \overline{m}_1 \in r(r^{-1}\varphi(N))$. □

A pp-pair (φ, ψ) over R is said to be *locally bounded* if and only if there is a positive integer n such that for every $U \in \mathrm{Zg}(R)$, the pp-length of (φ, ψ) at U is $\leq n$. Let $n_R(\varphi, \psi)$ denote the minimal positive integer n with this property.

The main result of this section is the following.

Proposition 3.3. *Let (φ, ψ) be a pp-pair over a Dedekind domain R. Then the following are equivalent.*

1. *$Q \notin (\varphi/\psi)$, equivalently, the basic open set (φ/ψ) has* CB-*rank ≤ 1 in the Ziegler topology.*
2. $(\varphi : \psi)_R \neq \{0_R\}$.
3. *(φ, ψ) is locally bounded.*

The proof of Proposition 3.3 needs some preparatory work.

Let D denote elementary (Prest) duality, see [18, 1.3.1, pp. 30-32]. In particular recall that D determines an anti-isomorphism between the lattices of left and right pp-formulas ([18, Proposition 1.3.1 p. 31]) and exchanges a divisibility formula like $r \mid x$ with the annihilator formula $xr = 0$, and vice versa.

Lemma 3.4. *Let $\varphi(\overline{x})$ be a (right) pp-formula and $r \in R \backslash \{0_R\}$. Then $D(\varphi r^{-1})$ is equivalent to $rD\varphi$ (where both $D(\varphi r^{-1})$ and $D\varphi$ are left pp-formulas).*

Proof. Suppose φ is $\exists \overline{y}\ (\overline{x}A = \overline{y}B)$ where A and B are matrices with entries in R and suitable sizes. Then φr^{-1} is equivalent to $\exists \overline{y}\ (\overline{x}(r \cdot A) = \overline{y}B)$, whence $D(\varphi r^{-1})$

is equivalent to $\exists \overline{z}\ (\overline{x} = (r \cdot A)\overline{z} \wedge B\overline{z} = 0)$. On the other hand $D\varphi$ is equivalent to $\exists \overline{z}\ (\overline{x} = A\overline{z} \wedge B\overline{z} = 0)$. Therefore $rD\varphi$ is equivalent to $\exists \overline{w} \exists \overline{z}\ (\overline{x} = r\overline{w} \wedge \overline{w} = A\overline{z} \wedge B\overline{z} = 0)$, and consequently to $\exists \overline{z}\ (\overline{x} = (r \cdot A)\overline{z} \wedge B\overline{z} = 0)$ as required. □

A *definable subcategory* $\mathcal{D}$ of R-Mod is a full subcategory of R-Mod such that there exists a set of pp-pairs Ω such that $M \in \mathcal{D}$ if and only if $\varphi(M) = \psi(M)$ for all $(\varphi, \psi) \in \Omega$. The *dual* of the definable subcategory $\mathcal{D}$ is the full subcategory of Mod-R exactly those $M \in$ Mod-R with $D\varphi(M) = D\psi(M)$ for all $(\varphi, \psi) \in \Omega$. Note that, an arbitrary intersection of definable subcategories is a definable subcategory. For $M \in R$-Mod, the *definable subcategory generated by* M is the smallest definable subcategory containing M.

Lemma 3.5. *Let R be a coherent integral domain and Q its field of fractions. Let $\psi \leq \varphi$ be a pair of pp-formulas over R. The following are equivalent:*

(1) *$\varphi(Q) = \psi(Q)$;*
(2) *There exists $r \in R \backslash \{0\}$ such that $r\varphi(R) \subseteq \psi(R)$;*
(3) *There exists $r \in R \backslash \{0\}$ such that for all indecomposable pure-injective modules U in the definable subcategory generated by ${}_RR$, $r\varphi(U) \subseteq \psi(U)$.*

Moreover all these propositions imply:

(4) *There exists $r \in R \backslash \{0\}$ such that for all indecomposable pure-injective modules U in the dual of the definable subcategory generated by ${}_RR$, $r\varphi(U) \subseteq \psi(U)$.*

Proof. (1) $\Leftrightarrow$ (2) For any pp-formula α, $Q\alpha(R) = \alpha(Q)$. Suppose $r\varphi(R) \subseteq \psi(R)$. Then $\varphi(Q) = Q\varphi(R) \subseteq Q\psi(R) = \psi(Q)$.

Suppose $\varphi(Q) = \psi(Q)$. Since R is coherent, by [17, Theorem 14.16] $\varphi(R)$ is a finitely generated ideal of R. Let $a_1, \ldots, a_n$ generate $\varphi(R)$. Then each a_i is in $\varphi(Q) = \psi(Q)$. Hence there is $r_i \in R \backslash \{0\}$ such that $a_i r_i \in \psi(R)$. Set $r = \prod_{i=1}^{n} r_i$. Then $r \neq 0$ and $r\varphi(R) \subseteq \psi(R)$.

(2) $\Leftrightarrow$ (3) Obvious.

(1) $\Rightarrow$ (4) Since R is a domain, for all $r \in R \backslash \{0\}$, $|\,(rx = 0\,/\,x = 0)\,(R)\,| = 1$. Therefore, if U is in the dual of the definable subcategory generated by R, then $|\,(x = x\,/\,r \mid x)\,(U)\,| = 1$ for all $r \in R \backslash \{0\}$, i.e. U is a divisible module.

Note that $\varphi(Q) = \psi(Q)$ if and only if $D\varphi(Q) = D\psi(Q)$. As in the first equivalence, this is true if and only if $D\varphi(R) \supseteq rD\psi(R)$ for some $r \in R \backslash \{0\}$. By Lemma 3.4, $D(\psi r^{-1})$ is equivalent to $rD\psi$. So $\varphi(Q) = \psi(Q)$ if and only if $\psi(U)r^{-1} \supseteq \varphi(U)$ for all indecomposable pure-injective U in the dual of the definable subcategory generated by R (as a right, or also left module). Since U is divisible, $\psi(U)r^{-1} \supseteq \varphi(U)$ implies $\psi(U) \supseteq \varphi(U)r$. So we have proved that (1) implies (4).

□

Remark 3.6. *Let R be a Dedekind domain. Then $(\varphi : \psi)_R \neq 0$ implies (φ, ψ) locally bounded. In this case $n_R(\varphi, \psi)$ is less than or equal to the highest exponent* *in the decomposition of* $(\varphi : \psi)_R$ *as a product of powers of pairwise different non-zero prime ideals in* R.

Proof. If $(\varphi : \psi)_R = R$ then $\varphi = \psi$ and so clearly (φ, ψ) is locally bounded. Therefore suppose that $(\varphi : \psi)_R$ is a non-zero proper ideal. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be non-zero prime ideals of R and $h_1, \ldots, h_m$ positive integers such that $(\varphi : \psi)_R = \prod_{j=1}^m \mathcal{P}_j^{h_j}$. So for all indecomposable pure injective R-modules U, $\varphi(U)/\psi(U)$ is a module over $R/(\varphi : \psi)_R \cong \prod_{j=1}^m R/\mathcal{P}_j^{h_j}$. Therefore, if $\mathcal{P}(U)$ is the attached maximal ideal of U (see Lemma 2.8), and $\mathcal{P}(U)$ is not among $\mathcal{P}_1, \ldots, \mathcal{P}_m$, then $\varphi(U)/\psi(U) = 0$ while, if $\mathcal{P}(U) = \mathcal{P}_j$ for some j, then $\varphi(U)/\psi(U)$ is a uniserial $R/\mathcal{P}_i^{h_j}$-module and hence has finite length.

The final claim is straightforward. □

The *support* of a pp-pair (φ, ψ) over R is the (finite!) set of non-zero prime ideals of R factoring the ideal $(\varphi : \psi)_R$.

Therefore the support of (φ, ψ) is $\{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ according to the notation of Remark 3.6. Note that (φ/ψ) is closed on all indecomposable pure-injective modules U with attached maximal ideal $\mathcal{P}(U) \notin \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$. If (φ, ψ) is locally bounded, then for every $U \in \mathrm{Zg}(R) \setminus \{Q\}$ such that $\mathcal{P}(U)$ is in the support of (φ, ψ), the chain of the pp-subgroups between $\varphi(U)$ and $\psi(U)$ is of the form

$$\varphi(U) \supset \mathcal{P}\,\varphi(U) \supset \ldots \supset \mathcal{P}^n\,\varphi(U) = \psi(U)$$

for some natural $n \leq n_R(\varphi, \psi)$.

Remark 3.7. *Let* S *be a ring. Suppose* $U, U' \in \mathrm{Zg}(S)$ *are topologically distinguishable and* U *is in the closure of* U'. *Then for all pp-pairs* (φ, ψ), *if* $\varphi(U)/\psi(U)$ *is open then* $\varphi(U')/\psi(U')$ *has infinite pp-length.*

Proof. It follows from [24, 8.12]. □

We are finally able to show Proposition 3.3.

Proof. (1) ⇒ (2) Let $r \in R\backslash\{0_R\}$ be such that $r\varphi(R) \subseteq \psi(R)$. Since R is commutative noetherian, the pure-injective hull of R is $\prod \overline{R_\mathcal{P}}$ where $\mathcal{P}$ ranges over non-zero prime ideals of R. Therefore $r\varphi(\overline{R_\mathcal{P}}) \subseteq \psi(\overline{R_\mathcal{P}})$. The Prüfer modules over R are the duals of the adics, so by Lemma 3.5, (1) ⇒ (4), there exists $s \in R\backslash\{0_R\}$ such that $s\varphi(E(R/\mathcal{P})) \subseteq \psi(E(R/\mathcal{P}))$ for all non-zero prime ideals $\mathcal{P}$. Now $(rs\varphi/\psi)$ is a compact subset of $\mathrm{Zg}(R)$ and contains only finite length points which are isolated points. Hence it is finite. Take $t \neq 0_R$ in the intersection of the annihilators of the modules in $(rs\varphi/\psi)$. Then $rst\varphi(U) \subseteq \psi(U)$ for all indecomposable pure-injective R-modules U.
(3) ⇒ (1) Since Q is in the closure of all infinite length indecomposable pure-injective R-modules, by Remark 3.7, $\varphi(Q) = \psi(Q)$.
(2) ⇒ (3) This is Remark 3.6.

□

4. The Poincaré series: the local case

Throughout, let V be a discrete valuation domain, π a generator of its unique maximal ideal and Q its field of fractions. We assign to every pp-pair (φ, ψ) of V a series in $\mathbb{Z}[[t]]$ with constant term 0, denoted by $P_V(\varphi, \psi)(t)$, called the *Poincaré series* of the pp-pair (φ, ψ) with respect to V. We put

$$P_V(\varphi, \psi)(t) = \sum_{n=1}^{\infty} \ell_V(\varphi, \psi, V/\pi^n V)\ t^n.$$

Note that, according to the classification of indecomposable pure-injective modules over V given in Theorem 2.9, if U is such a module and has finite length, then the pp-length of $[\psi, \varphi]_U$, that is, the endolength of $\varphi(U)/\psi(U)$, is also equal to the length of $\varphi(U)/\psi(U)$ as a V-module. For this reason we will often write in the remainder of the paper "pp-length of $\varphi(U)/\psi(U)$" instead of "pp-length of $[\psi, \varphi]_U$".

Example 4.1. (1) $P_V(x = x,\, x = 0)\,(t) = \sum_{n=1}^{\infty} n t^n = t \cdot \sum_{n=1}^{\infty} n = \frac{t}{(t-1)^2}$. In view of future applications, we put for simplicity $\mathcal{W} := \frac{t}{(t-1)^2}$.

(2) $P_V(\pi x = 0,\, x = 0)\,(t) = \sum_{n=1}^{\infty} t^n = \sum_{n=0}^{\infty} t^n - 1 = \frac{1}{t-1} - 1 = \frac{-t}{t-1}$. Similarly $P_V(x = x,\, \pi \mid x)\,(t) = \frac{-t}{t-1}$. As before we put for simplicity $\mathcal{U}_1 := \frac{-t}{t-1}$.

(3) $P_V(\pi \mid x,\, x = 0)\,(t) = \sum_{n=1}^{\infty}(n-1)\,t^n = t^2 \cdot (\sum_{n=2}^{\infty}(n-1)\,t^{n-2}) = \frac{t^2}{(t-1)^2} = t^2 + \frac{2t-1}{(t-1)^2}$.

(4) For every positive integer K, $P_V(\pi^{K-1} \mid x \wedge \pi x = 0,\, \pi^K \mid x \wedge \pi x = 0)\,(t) = t^K$. In fact it is straightforward to see that the open set $(\pi^{K-1} \mid x \wedge \pi x = 0 \;/\; \pi^K \mid x \wedge \pi x = 0)$ isolates $V/\pi^K V$ in $\mathrm{Zg}(V)$.

(5) Similarly, for every positive integer K, $P_V(\pi^K x = 0, \pi^K x = 0 \wedge \pi \mid x)\,(t) = t + t^2 + \ldots + t^K$.

(6) Finally let us extend (2) and prove that, for every positive integer K,

$$P_V(\pi^K x = 0,\, x = 0)(t) = (1 + t + \ldots + t^{K-1})\,\frac{-t}{t-1}.$$

This will be used, together with (1) and (2), in the proof of one of the main results of this section. Let us put for simplicity, for every K, $\mathcal{U}_K = P_V(\pi^K x = 0,\, x = 0)(t)$. We proceed by induction on K. The case $K = 1$ is just (2), saying $\mathcal{U}_1 = \frac{-t}{t-1}$. Next we prove for all K that $\mathcal{U}_{K+1} = \mathcal{U}_K + t^K\, U_1$, which implies $\mathcal{U}_{K+1} = (1 + t + \ldots + t^K)\,\mathcal{U}_1$. By the definition of P_V,

$$\mathcal{U}_{K+1} = P_V(\pi^K x = 0,\, x = 0)(t) + P_V(\pi^{K+1} x = 0,\, \pi^K x = 0)(t) =$$

$$= U_K + P_V(\pi^{K+1} x = 0,\, \pi^K x = 0)(t).$$

Now the quotient group of the pp-subgroups defined by $\pi^{K+1} x = 0$ and $\pi^K x = 0$ in $V/\pi^l V$ is 0 for $l \leq K$ and isomorphic to $V/\pi V$ for $l > K$. So

$$P_V(\pi^{K+1} x = 0,\, \pi^K x = 0)(t) = \sum_{n=K+1}^{\infty} t^n = t^K \cdot \sum_{n=1}^{\infty} t^n = t^K\, \mathcal{U}_1$$

as required.

The main results of this section are the following.

- First we see that the Poincaré series define an injective group homomorphism from the Grothendieck group $G_0(V)$ to the additive group $\mathbb{Z}[[t]]$.
- Then we provide a description of the Poincaré series $P_V(\varphi,\psi)(t)$ based on the CB-rank of (φ/ψ), for (φ,ψ) a pp-pair.

Theorem 4.2. *Let V be as before. The function mapping, for every pp-pair (φ,ψ) over V, the class $[\varphi,\psi]_{G_0(V)}$ to $P_V(\varphi,\psi)(t)$ induces an injective group homomorphism of the Grothendieck group $G_0(V)$ into the additive group $\mathbb{Z}[[t]]$.*

Proof. First of all, the function sending any pp-pair (φ,ψ) to its Poincaré series defines a group homomorphism from the free abelian group of pp-pairs to $\mathbb{Z}[[t]]$. In fact, for every choice of pp-pairs $(\varphi(\overline{x}),\psi(\overline{x}))$ and $(\varphi'(\overline{y}),\psi'(\overline{y}))$ of V (with $\overline{x}$, $\overline{y}$ disjoint tuples of length k, k' respectively) and for every positive integer n,

$$\ell_V(\varphi(\overline{x})\wedge\varphi'(\overline{y}),\psi(\overline{x})\wedge\psi'(\overline{y})),V/\pi^nV)=\ell_V(\varphi,\psi,V/\pi^nV)+\ell_V(\varphi',\psi',V/\pi^nV).$$

Next take pp-formulas (φ,ψ), (φ',ψ'), (φ'',ψ'') forming in each V-module N a short exact sequence as described in § 2. Then, for N a V-module of finite pp-length, in particular for $N=V/\pi^nV$ with n a positive integer,

$$\ell_V(\varphi,\psi,N)=\ell_V(\varphi',\psi',N)+\ell_V(\varphi'',\psi'',N).$$

We get in this way the required homomorphism of $G_0(V)$ to $\mathbb{Z}[[t]]$.

Now let us deal with injectivity. We view pp-pairs as objects of the category $(V\text{-mod},\mathrm{Ab})^{\mathrm{fp}}$ (as in § 2). Finitely presented modules over V are finite direct sums of V and V/π^nV where n ranges over positive integers. Since $(M,-)$ preserves direct sum up to isomorphism, it follows from the result about projective resolutions that $G_0(V)$ is generated by $(V,-)$, $(V/\pi^nV,-)$, again for n a positive integer. Note that $(V,-)$ corresponds to $(x=x,\ x=0)$ and $(V/\pi^nV,-)$ to $(\pi^nx=0,\ x=0)$. For $F\in(V\text{-mod},\ \mathrm{Ab})^{\mathrm{fp}}$, let $P_V(F)(t)$ denote the Poincaré series of the corresponding pp-pair.

Now in order to obtain injectivity it is enough to prove that

$$\{P_V((V,-))(t),\ P_V((V/\pi^nV,-))(t)\ :\ n\in\mathbb{N},\ n\neq 0\}$$

is linearly independent over $\mathbb{Z}$ in $\mathbb{Z}[[t]]$. Using notation from Example 4.1, we have to show that $\{\mathcal{W},\mathcal{U}_n\ :\ n\in\mathbb{N}, n\neq 0\}$ is linearly independent over $\mathbb{Z}$. Let h be a positive integer and $a_0,a_1,a_2,\dots,a_h\in\mathbb{Z}$. First observe that

$$a_0\,\mathcal{W}+a_1\,\mathcal{U}_1+a_2\,\mathcal{U}_2+\ldots+a_h\,\mathcal{U}_h=0$$

if and only if

$$a_0\,\frac{t}{(t-1)^2}+a_1\,\frac{-t}{t-1}+a_2\,(1+t)\,\frac{-t}{t-1}+\ldots+a_h\,(1+t+\ldots+t^{h-1})\,\frac{-t}{t-1}=0,$$

that is (after multiplying by $(t-1)^2$), if and only if

$$a_0\, t - a_1\, t\,(t-1) - a_2\, t\,(t^2-1) - \ldots - a_h\, t\,(t^h-1) = 0.$$

Suppose $a_0, a_1, \ldots, a_h \in \mathbb{Z}$ satisfy the above equation. Comparing the coefficients of the highest degree power of t gives $a_h = 0$. Inductively, this implies $a_i = 0$ for $1 \le i \le h$. So $a_0\, t = 0$, and hence $a_0 = 0$. □

Now recall Ziegler's result [24, Theorem 8.6] that, for every pp-pair (φ, ψ) over V, the CB-rank of (φ/φ), viewed as an open subset of $\mathrm{Zg}(V)$, equals the m-dimension of (φ, ψ) (that is, of the interval $[\varphi, \psi]$ in the lattice of pp-formulas). Note that Ziegler just says "dimension". The m-dimension of (φ, ψ) coincides also with its Krull-Gabriel dimension, $\mathrm{KG}(\varphi/\psi)$, where (φ, ψ) is viewed as an object of the functor category (V-mod, Ab): see [6] for an introduction to the Krull-Gabriel dimension and [18, Proposition 13.2.1] for a proof of the equality of the two dimensions. Over a discrete valuation domain V, the m-dimension of a pp-pair is ≤ 2, as a consequence of the description of $\mathrm{Zg}(V)$ provided by [24] and recalled in § 2. Indeed this is true over any Dedekind domain (for the same reasons).

Proposition 4.3. *For every pp-pair (φ, ψ) over V, the Poincaré series $P_V(\varphi, \psi)(t)$ is a rational function $\frac{f(t)}{(t-1)^m}$, where $f(t)$ is a polynomial over the integers whose only pole is at $t = 1$ and has multiplicity $m = \mathrm{KG}(\varphi/\psi) \le 2$. Furthermore,*

1. *if $m = 0$ then (φ, ψ) is of finite length given by $f(1)$,*
2. *if $m = 1$ then (φ, ψ) is locally bounded and $\ell_V(\varphi, \psi, U) \le f(1)$ for all but finitely many $U \in \mathrm{Zg}(V)$,*
3. *if $m = 2$ then $Q \in (\varphi/\psi)$ and*

$$f(1) = \ell_V(\varphi, \psi, Q) = \dim_Q \varphi(Q)/\psi(Q).$$

Proof. Recall that the Poincaré series of (φ, ψ) is a $\mathbb{Z}$-linear combination of the Poincaré series denoted

$$\mathcal{W} := \frac{t}{(t-1)^2} = P_V(x = x, x = 0)(t) = P_V((V, -))(t),$$

$$\mathcal{U}_1 := \frac{-t}{t-1} = P_V(\pi x = 0, x = 0)(t) = P_V(V/V\pi, -)(t) \quad \text{and}$$

$$\mathcal{U}_{n+1} := \mathcal{U}_n + t^n \mathcal{U}_1 = P_V(\pi^{n+1} x = 0, x = 0)(t) = P_V((V/V\pi^{n+1}, -))(t)$$

for n a positive integer.

(1) The isolated points in $\mathrm{Zg}(V)$, which are exactly the finite length indecomposable pure-injective V-modules, are dense in $\mathrm{Zg}(V)$. Suppose $m = 0$. Then there is a positive integer n such that $V/\pi^i V \notin (\varphi/\psi)$ for all $i > n$. Take n minimal. Therefore the Poincaré series $P_V(\varphi, \psi)(t)$ is a polynomial $f(t)$ of degree n with integer coefficients. Moreover the pp-length of (φ, ψ) is equal to the pp-length of $\varphi(M)/\psi(M)$ where $M := \oplus_{i=1}^n V/\pi^i V$. The pp-length of $\varphi(M)/\psi(M)$ is finite since M is of finite length as a V-module.

For the claim about $f(1)$, we need to show that the pp-length of (φ, ψ) is equal to the sum of the pp-lengths of $\varphi(V/\pi^i V)/\psi(V/\pi^i V)$ for $1 \leq i \leq n$. It follows from [18, Lemma 4.4.31] that the pp-length of $\varphi(\oplus_{i=1}^n V/\pi^i V)/\psi(\oplus_{i=1}^n V/\pi^i V)$ is equal to the sum of the pp-lengths of $\varphi(V/\pi^i V)/\psi(V/\pi^i V)$ for $1 \leq i \leq \deg f$.

Next, in order to prove (2) and (3), suppose that (φ, ψ) has a projective resolution

$$0 \to (M_2 \oplus V^{r_2}, -) \longrightarrow (M_1 \oplus V^{r_1}, -) \longrightarrow (M_0 \oplus V^{r_0}, -) \longrightarrow \varphi/\psi \to 0$$

where M_0, M_1 and M_2 are finite length modules and $r_0, r_1, r_2 \in \mathbb{N}$. Now $P_V(\varphi, \psi)(t)$ equals $a_0 \mathcal{W} + \sum_{i=1}^n a_i \mathcal{U}_i$ where $a_0 = r_0 - r_1 + r_2$ and $a_i \in \mathbb{Z}$ for $i \geq 1$.

(3) The pp-length of $\varphi(Q)/\psi(Q)$ is equal to its dimension as a Q-vector space, which is equal to $a_0 = r_0 - r_1 + r_2$ since $(M, Q) = 0$ for all finite length modules M. Now $a_0 \neq 0$ if and only if $m = 2$. Moreover, if $m = 2$ then $f(1) = a_0$. So $Q \in (\varphi/\psi)$ if and only if $m = 2$; furthermore $f(1) = \ell_V(\varphi, \psi, Q)$.

(2) If $m = 1$ then $a_0 = 0$ and hence $Q \notin (\varphi/\psi)$. By Proposition 3.3, (φ, ψ) is locally bounded. For the final part, write $f(t) = q(t)(t-1) + r$ where $q \in \mathbb{Z}[t]$ and $r = f(1) \in \mathbb{Z}$ (note this can be done since the leading coefficient of $t-1$ is 1). Then

$$\frac{f(t)}{t-1} = q(t) + \frac{r}{t-1} = q(t) - r \cdot \sum_{i=1}^{\infty} t^i.$$

□

5. The Poincaré series: the global case

We extend the definition of the Poincaré series to pp-pairs over arbitrary Dedekind domains R. For every pp-pair (φ, ψ) of R we define

$$P_R(\varphi, \psi) = \sum_{\mathcal{P}} P_{R_{\mathcal{P}}}(\varphi, \psi)(t_{\mathcal{P}}) = \sum_{\mathcal{P}} \sum_{n=1}^{\infty} l_{R_{\mathcal{P}}}(\varphi, \psi, R_{\mathcal{P}}/\pi_{\mathcal{P}}^n R_{\mathcal{P}})\ t_{\mathcal{P}}^n$$

where $\mathcal{P}$ ranges over the non-zero prime ideals of R and, for all $\mathcal{P}$, $t_{\mathcal{P}}$ is a corresponding variable and $\pi_{\mathcal{P}}$ is a generator of the maximal ideal of the localization of R at $\mathcal{P}$. Thus $P_R(\varphi, \psi)$ is in the additive group $\mathbb{Z}[[(t_{\mathcal{P}})_{\mathcal{P}}]]$ (where the $\mathcal{P}$ are the non-zero prime ideals of R), and indeed in its subgroup formed by the series with only summands corresponding to single powers $t_{\mathcal{P}}^n$ with $\mathcal{P}$ as before and n a positive integer, so having constant term 0 and excluding monomials like $t_{\mathcal{P}} t_{\mathcal{P}'}$ with $\mathcal{P}$, $\mathcal{P}'$ different non-zero prime ideals. Let us denote by $\mathbb{Z}_0[[(t_{\mathcal{P}})_{\mathcal{P}}]]$ this subgroup.

When $\mathcal{P}$ is principal, generated by p say, we also write t_p instead of $t_{\mathcal{P}}$.

Recall that, if (φ, ψ) is a locally bounded pp-pair in L_R, then there are only finitely many non-zero prime ideals $\mathcal{P}$ of R such that the associated Poincaré series (over the localization $R_{\mathcal{P}}$) is not zero (see the proof of Remark 3.6). The collection of these ideals – the ones factoring $(\varphi : \psi)_R$ – is the support of the pp-pair (φ, ψ). So in this case $P_{R_{\mathcal{P}}}(\varphi, \psi)$ is 0 for almost all $\mathcal{P}$.

Theorem 5.1. *Let R be a Dedekind domain that is not a field. Then the function mapping, for every pp-pair (φ, ψ) of R, the class $[\varphi, \psi]_{G_0(R)}$ to $P_R(\varphi, \psi)$ induces* *a group homomorphism of the Grothendieck group $G_0(R)$ into the additive group $\mathbb{Z}[[(t_{\mathcal{P}})_{\mathcal{P}}]]$.*

Proof. The family of additive homomorphisms $G_0(R) \to G_0(R_{\mathcal{P}}) \to \mathbb{Z}[[t_{\mathcal{P}}]]$ coming from Theorem 4.2 sums into a homomorphism $G_0(R) \to \oplus_{\mathcal{P}}\mathbb{Z}[[t_{\mathcal{P}}]]$, which naturally maps into $\mathbb{Z}[[(t_{\mathcal{P}})_{\mathcal{P}}]]$. □

Since the modules $R/\mathcal{P}^n$ are pp-uniserial (that is the lattice of pp-subgroups is totally ordered [5, § 3]), for pp-1-formulas φ, ψ, $\varphi \geq \psi$ if and only if $\ell_R(\varphi, x = 0, R/\mathcal{P}^n) \geq \ell_R(\psi, x = 0, R/\mathcal{P}^n)$ for all non-zero prime ideals $\mathcal{P}$ and positive integers n. Therefore, whether $\varphi \geq \psi$ or not can be read off the Poincaré series. Moreover φ and ψ are equivalent as pp-formulas if and only if $\psi \cong \psi$ in $(R\text{-mod}, \mathrm{Ab})^{\mathrm{fp}}$, hence if and only if φ and ψ coincide in $G_0(R)$.

Notably this is not true for general pp-formulas. Moreover, for Dedekind domains, the homomorphism of $G_0(R)$ into the Poincaré series is not necessarily injective.

Proposition 5.2. *Let R be a Dedekind domain. If the homomorphism from the Grothendieck group of R to the Poincaré series is an embedding then R is a PID.*

Proof. Suppose J is a non-principal ideal of R. For each non-zero prime ideal $\mathcal{P}$ and positive integer n, the length of $\mathrm{Hom}_R(J, R/\mathcal{P}^n)$ is equal to the length of $J \otimes R/\mathcal{P}^n$ because $J \otimes_R -$ is the Auslander-Gruson-Jensen dual of $\mathrm{Hom}_R(J, -)$ (see [18, 10.3]; in terms of pairs of pp-formulas taking the Auslander-Gruson-Jensen dual is just Prest's duality). Now, $J \otimes R/\mathcal{P}^n \cong J/J\mathcal{P}^n$, which has length n as an R-module.

On the other hand, the length of $\mathrm{Hom}_R(R, R/\mathcal{P}^n)$ as an R-module is also n for all non-zero prime ideals $\mathcal{P}$ and positive integers n, but $\mathrm{Hom}_R(J, -)$ is not isomorphic to $\mathrm{Hom}_R(R, -)$ since J is not isomorphic to R.

In terms of pp-formulas, $\mathrm{Hom}_R(J, -)$ is (isomorphic to) the pp-2-formula freely realized by (a, b) where a, b generate J (recall that each non-principal ideal of a Dedekind domain R is 2-generated, see [11, Proposition 3.19 p. 15]) and $\mathrm{Hom}_R(R, -)$ is (isomorphic to) the pp-2-formula $x = x \wedge y = 0$. □

Part 2. EXTENSIONS OF DEDEKIND DOMAINS

In this part we deal with pairs of Dedekind domains $R \subseteq \widetilde{R}$ that are not fields, with R a subring of $\widetilde{R}$. Unless otherwise stated we assume throughout that R is a Dedekind domain (and not a field) and $\widetilde{R}$ is the integral closure of R in a finite dimensional (proper) *separable* field extension L of the field of fractions Q of R, which ensures that $\widetilde{R}$ is a Dedekind domain, too. Under the separability assumption, $\widetilde{R}$ is finitely generated as a module over R (see [11, proofs of Theorem 6.1 p. 26 and Corollary 6.7 p. 31]).

6. Restriction of scalars

First of all, a useful premise.

Remark 6.1. *Let R be an arithmetical ring and S a subring of R. If U is an indecomposable pure-injective R-module, then the reduct S-module ${}_SU$ realises only irreducible pp-1-types, and in particular is the pure-injective envelope of a direct sum of indecomposable pure-injective S-modules (with no superdecomposable summands).*

Note that in general the reduct ${}_SU$ of a pure-injective R-module U is also pure-injective (over S), but is not necessarily indecomposable when U is indecomposable pure-injective. Observe also that the previous remark becomes trivial when S is a Dedekind domain, because then S possesses no superdecomposable pure-injective modules. However recall that the domain of algebraic integers, which is arithmetical but not Dedekind, admits superdecomposable pure-injective modules, see for example [19, Proposition 6.2 and Example 6.3].

Proof. An indecomposable pure-injective module U over an arithmetical ring is pp-uniserial and remains so when restricted to S. But then all pp-1-types realised in ${}_SU$ are irreducible, and consequently ${}_SU$ cannot admit any superdecomposable direct summand (see for instance [17, Theorem 10.2 p. 202] and [24, § 7]). □

Remark 6.2. *As a module, L is indecomposable over $\widetilde{R}$ but decomposes as Q^n over R where $n = [L : Q]$.*

Recall (see Lemma 2.6) that all pp-1-formulas over R are a lattice combination of formulas of the form $x\mathcal{P}^i = 0$ and $\mathcal{P}^j | x$ with i, j positive integers, $\mathcal{P}$ a non-zero prime ideal of R.

Lemma 6.3. *Let $\mathcal{P}$ be a non-zero prime ideal of R, $i > j$ positive integers. If $R/\mathcal{P}^i \oplus R/\mathcal{P}^j$ is pp-uniserial then $i = j + 1$.*[1]

Proof. Note that $\mathcal{P}|x < x\mathcal{P}^j = 0$ in $R/\mathcal{P}^j$. So if $R/\mathcal{P}^i \oplus R/\mathcal{P}^j$ is pp-uniserial then $\mathcal{P}|x \leq x\mathcal{P}^j = 0$ in $R/\mathcal{P}^i$. This happens if and only if $\mathcal{P}^{j+1}$ annihilates $R/\mathcal{P}^i$, i.e. $i \leq j + 1$. □

Proposition 6.4. *Let $\mathcal{M}$ be a non-zero prime ideal of $\widetilde{R}$ and let $\mathcal{P} = R \cap \mathcal{M}$. Let e denote the ramification index of $\mathcal{M}$ and f be the inertial degree of $\mathcal{M}$. Let $\lambda, \mu, s \in \mathbb{N}$, $s > 0$, $0 \leq \mu < e$, $s = e\lambda + \mu$. Then, if viewed as an R-module, the indecomposable pure-injective $\widetilde{R}$-module $\widetilde{R}/\mathcal{M}^s$ decomposes as*

- $(R/\mathcal{P}^\lambda)^{ef-\mu f} \oplus (R/\mathcal{P}^{\lambda+1})^{\mu f}$ *when* $\lambda \geq 1$ *and*
- $(R/\mathcal{P})^{sf}$ *when* $\lambda = 0$.

[1] The next Proposition 6.4 implies that the converse is also true.

Proof. The annihilator of $\widetilde{R}/\mathcal{M}^s$ as an R-module is $\mathcal{M}^s \cap R = \mathcal{M}^{e\lambda+\mu} \cap R = (\mathcal{M} \cap R)^{\lambda+1} = \mathcal{P}^{\lambda+1}$. Since $\widetilde{R}/\mathcal{M}^s$ is pp-uniserial as an $\widetilde{R}$-module it is pp-uniserial also as an R-module. So, by Lemma 6.3, $\widetilde{R}/\mathcal{M}^s$ is of the form $(R/\mathcal{P}^i)^a \oplus (R/\mathcal{P}^{i+1})^b$ for some non-negative integers a, b and $i = \lambda$. In the case $\lambda = 0$ we may set $a = 0$.

As an $\widetilde{R}$-module, $\widetilde{R}/\mathcal{M}^s$ has a composition series of length s with factors isomorphic to $\widetilde{R}/\mathcal{M}$. Since $\widetilde{R}/\mathcal{M}$ has composition series of length f as an R-module, $\widetilde{R}/\mathcal{M}^s$ has a composition series of length sf as an R-module. Therefore $a\lambda + b(\lambda + 1) = sf$. So, if $\lambda = 0$ then $b = sf$ as required. Now assume $\lambda \geq 1$.

Let $\mathcal{M} = \mathcal{M}_1, \dots, \mathcal{M}_g$ be the distinct non-zero prime ideals of $\widetilde{R}$ lying over $\mathcal{P}$, with ramification indexes $e_1, \dots, e_g$ respectively. Then $\mathcal{P}^\lambda \widetilde{R} = \prod_{j=1}^g \mathcal{M}_j^{e_j\lambda}$. In any $\widetilde{R}$-module, $\prod_{i=1}^g \mathcal{M}_i^{e_i\lambda}|x$ is equivalent to $\mathcal{P}^\lambda|x$. The length of $\mathcal{M}^{e\lambda} \cdot (\widetilde{R}/\mathcal{M}^s) = \prod_{i=1}^g \mathcal{M}_i^{e_i\lambda} \cdot (\widetilde{R}/\mathcal{M}^s)$ is μ as an $\widetilde{R}$-module and hence μf as an R-module. The length of $\mathcal{P}^\lambda \cdot [(R/\mathcal{P}^\lambda)^a \oplus (R/\mathcal{P}^{\lambda+1})^b]$ is b. Therefore $b = \mu f$. It now follows from $a\lambda + b(\lambda + 1) = sf$ that $a = ef - \mu f$. □

Example 6.5. (See Example 2.13). Take $R = \mathbb{Z}$, $L = \mathbb{Q}(i)$, so that $\widetilde{R} = \mathbb{Z}[i]$ is the ring of Gaussian integers. A non-zero prime ideal $\mathcal{M}$ of $\mathbb{Z}[i]$ is either

- $\mathcal{M} = p\,\mathbb{Z}[i]$ where $p \in \mathbb{Z}$ is a prime $\equiv 3 \pmod 4$, or
- $\mathcal{M} = (a + ib)\,\mathbb{Z}[i]$ where a, b are integers and $(a + ib) \cdot (a - ib) = a^2 + b^2$ is a prime p (hence either $p = 2 = (1 + i) \cdot (1 - i)$ or $p \equiv 1 \pmod 4$).

First let us assume $s = 1$. In the former case $\mathbb{Z}[i]/\mathcal{M}$ is decomposable over $\mathbb{Z}$, as isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, in fact the inertial degree f of $\mathcal{M}$ is 2. In the latter case $\mathbb{Z}[i]/\mathcal{M} \simeq \mathbb{Z}/p\mathbb{Z}$ is indecomposable over $\mathbb{Z}$, in fact $f = 1$.

On the other hand, if $p = 2$ and $\mathcal{M} = (1 + i)\,\mathbb{Z}[i]$, then $\mathbb{Z}[i]/\mathcal{M}^2$ has order 4 but no element of period 4, so is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and is decomposable over $\mathbb{Z}$ (in fact $s = e = 2$, so that $\lambda = 1$ and $\mu = 0$). Note that $\mathbb{Z}[i]/\mathcal{M}^3$ is also decomposable over $\mathbb{Z}$ but this time as $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so as the direct sum of two non-isomorphic summands (as now $s = 3$, whence $e = 2$ implies $\lambda = \mu = 1$).

Proposition 6.6. *Let $\mathcal{M}$ be a non-zero prime ideal of $\widetilde{R}$ and let $\mathcal{P} = R \cap \mathcal{M}$. Let e denote the ramification index of $\mathcal{M}$ and f be the inertial degree of $\mathcal{M}$. Then, viewed as an R-module, $E(\widetilde{R}/\mathcal{M})$ decomposes as $E(R/\mathcal{P})^{ef}$ and $\overline{\widetilde{R}_{\mathcal{M}}}$ decomposes as $\overline{R_{\mathcal{P}}}^{ef}$.*

Recall that $E(-)$ denotes injective hull, see Theorem 2.9.

Proof. Since $E(\widetilde{R}/\mathcal{M})$ is a divisible $\widetilde{R}$-module, $E(\widetilde{R}/\mathcal{M})$ is a divisible R-module and hence injective because R is Dedekind [21, Theorem 4.24]. So, since R is noetherian, it decomposes as a direct sum of indecomposable injective R-modules [7, 5.24]. Since $\mathcal{P} \subseteq \mathcal{M}$, every element of $E(\widetilde{R}/\mathcal{M})$ is annihilated by some power of $\mathcal{P}$. Therefore, as an R-module, $E(\widetilde{R}/\mathcal{M})$ is a direct sum of copies of $E(R/\mathcal{P})$. It is now enough to compute the dimension, as an $R/\mathcal{P}$-vector space, of the socle of $E(\widetilde{R}/\mathcal{M})$ as an R-module. The socle of $E(\widetilde{R}/\mathcal{M})$ is equal to the union of the socles of $\widetilde{R}/\mathcal{M}^s$ for all $s \in \mathbb{N}$. It follows from Proposition 6.4 that the socle has dimension ef and hence $E(\widetilde{R}/\mathcal{M})$ is isomorphic to $E(R/\mathcal{P})^{ef}$.

If we complete the field L at the valuation induced by $\widetilde{R}_{\mathcal{M}}$ on L to get $L_{\mathcal{M}}$ and similarly Q at the valuation induced by $R_{\mathcal{P}}$ then $L_{\mathcal{M}}$ is a finite dimensional separable extension of $Q_{\mathcal{P}}$ but $\overline{\widetilde{R}_{\mathcal{M}}}$ may not be the integral closure of $\overline{R_{\mathcal{P}}}$. The ramification index of $\overline{\mathcal{M}}$ is e and the inertial degree of $\overline{\mathcal{M}}$ is f [11, Chapter II, Theorem 3.8]. Now $\overline{\widetilde{R}_{\mathcal{M}}}$ is equipped in a unique way with the structure of a $\overline{\widetilde{R}_{\mathcal{M}}}$-module. As an $\overline{R_{\mathcal{P}}}$-module, $\overline{\widetilde{R}_{\mathcal{M}}}$ is torsion-free. We claim that it has a minimal generating set of size ef. In fact, let π generate the maximal ideal of $\overline{\widetilde{R}_{\mathcal{M}}}$. Then π^e generates the maximal ideal of $\overline{R_{\mathcal{P}}}$. Let $u_1, \ldots, u_f \in \overline{\widetilde{R}_{\mathcal{M}}}$ be such that the residues of $u_1, \ldots, u_f$ are linearly independent over the residue field of $\overline{R_{\mathcal{P}}}$. Then $\{u_j\pi^i \mid 1 \le j \le f,\ 0 \le i \le e-1\}$ is a basis for $L_{\mathcal{M}}$ over $Q_{\mathcal{P}}$. If we denote the valuation on $L_{\mathcal{M}}$ by v and identify its value group with $\mathbb{Z}$ then for all $\alpha \in Q_{\mathcal{P}}$, $v(\alpha) \in e\mathbb{Z}$. By [4, proof of Proposition 3.19],

$$v\Big(\sum_{1\le j\le f,\ 0\le i\le e-1} u_j\pi^i\alpha_{ij}\Big) = \min_{i,j}\{i + v(\alpha_{ij})\}.$$

So $\sum u_j\pi^i\alpha_{ij} \in \overline{\widetilde{R}_{\mathcal{M}}}$ if and only if $i + v(\alpha_{ij}) \ge 0$ for $0 \le i \le e-1$ and $1 \le j \le f$. Since $v(\alpha_{ij}) \in e\mathbb{Z}$, this implies $\alpha_{ij} \in \overline{R_{\mathcal{P}}}$. Then $\overline{\widetilde{R}_{\mathcal{M}}}$ is generated by $\{u_j\pi^i \mid 1 \le j \le f \ \&\ 0 \le i \le e-1\}$.

Therefore $\overline{\widetilde{R}_{\mathcal{M}}}$ is isomorphic to $\overline{R_{\mathcal{P}}}^{ef}$ as an R-module. □

7. Comparing Poincaré series, and more

For every pp-pair (φ, ψ) of $\mathcal{L}(R)$, we compare its behavior over R and $\widetilde{R}$ in light of § 3. In fact (φ, ψ) can be viewed as a pp-pair also of $\mathcal{L}(\widetilde{R})$.

Proposition 7.1. *Let $R \subseteq \widetilde{R}$ be Dedekind domains that are not fields, $Q \subseteq L$ denote their fields of fractions, with L a finite dimensional separable extension of Q. Let (φ, ψ) be a pp-pair of $\mathcal{L}(R)$. Then the following statements hold:*

1. *(φ, ψ) is locally bounded over R if and only if it is over $\widetilde{R}$.*
2. *Under this assumption the support of (φ, ψ) over $\widetilde{R}$ consists of the non-zero prime ideals $\mathcal{M}$ of $\tilde{R}$ such that $\mathcal{M} \cap R$ is in the support of (φ, ψ) over R.*
3. *Assume again (φ, ψ) locally bounded. Let $\mathcal{M}$ be a non-zero prime ideal in the support of (φ, ψ) over $\widetilde{R}$ with ramification index e over $\mathcal{P} = \mathcal{M} \cap R$ (a non-zero prime ideal in the support of (φ, ψ) over R). Let s be a positive integer such that $n_{\overline{R}}(\varphi, \psi) \le s$, $\lambda, \mu \in \mathbb{N}$ such that $\lambda e < s \le (\lambda+1)e$. Then $n_R(\varphi, \psi) \le \lambda + 1$.*

Proof. (1) As a vector space over Q, L decomposes as $L \simeq Q^t$ for some finite cardinal t, which implies that $\varphi(L) = \varphi(Q)^t$ and $\psi(L) = \psi(Q)^t$. Then Condition (1) in Proposition 3.3 is true over R if and only if it is true over $\widetilde{R}$, whence (φ, ψ) is locally bounded over R if and only if it is over $\widetilde{R}$.

(2) Assume now (φ, ψ) locally bounded.

Clearly $(\varphi, \psi)_R \subseteq (\varphi, \psi)_{\widetilde{R}}$. For, let $r \in R$ satisfy $r\varphi(N) \subseteq \varphi(N)$ for every R-module N. Then the same is true for $\widetilde{R}$-modules (when restricted to R).

Moreover $(\varphi, \psi)_{\widetilde{R}} \cap R = (\varphi, \psi)_R$. The inclusion $\supseteq$ is clear. Conversely, let $r \in R$ be such that $r\varphi(U) \subseteq \psi(U)$ in every indecomposable pure-injective $\widetilde{R}$-module U. Remark 6.2 and Propositions 6.4 and 6.6 transfer this property to indecomposable pure-injective R-modules.

Now let $\mathcal{M}$ be a non-zero prime ideal containing $(\varphi, \psi)_{\widetilde{R}}$ in $\widetilde{R}$. Then $\mathcal{P} = \mathcal{M} \cap R$ is a non-zero prime ideal of R and contains $(\varphi, \psi)_R = (\varphi, \psi)_{\widetilde{R}} \cap R$.

(3) Use again Proposition 6.4. □

Note that (still keeping the notation in Statement (3) of Proposition 7.1) Proposition 6.4 also relates at least in principle $\ell_{\widetilde{R}}(\varphi, \psi, \widetilde{R}/\mathcal{M}^s)$ and $\ell_R(\varphi, \psi, R/\mathcal{P}^s)$ when s is a positive integer. For a more precise connection we have to specify φ and ψ.

Remark 7.2. *(φ, ψ) is of finite length over R if and only if it is over $\widetilde{R}$ (as it is straightforward to check).*

Now let $\mathcal{P}$ be a non-zero prime ideal of R. Then every power $t_{\mathcal{P}}^K$, with K a positive integer, can be expressed as the Poincaré series of a suitable pp-pair over R, see Example 4.1, (4). We wonder which is the Poincaré series of the same pp-pair over $\widetilde{R}$. So our goal reduces to find the representation of $t_{\mathcal{P}}^K$ over $\widetilde{R}$.

We denote by $\widetilde{t}_{\mathcal{M}}$ the variables over $\widetilde{R}$, when $\mathcal{M}$ ranges over non-zero prime ideals of $\widetilde{R}$.

Coming back to our $\mathcal{P}$, let $\mathcal{P}\widetilde{R} = \prod_{j=1}^{g} \mathcal{M}_j^{e_j}$ where g is a positive integer, the $\mathcal{M}_j$ are the (pairwise distinct) maximal ideals of $\widetilde{R}$ containing $\mathcal{P}\widetilde{R}$ and the positive integers e_j are their ramification indexes. We will see that each power $t_{\mathcal{P}}^K$ can be represented as a formal sum, with suitable coefficients, of powers of the $\widetilde{t}_{\mathcal{M}_j}$.

Example 7.3. (See Example 2.13.) Let $R = \mathbb{Z}$, $\widetilde{R} = \mathbb{Z}[i]$.

(1) Let $\mathcal{P} = 2\mathbb{Z}$. Then $2\mathbb{Z}[i]$ is in $\mathbb{Z}[i]$ the square of the prime ideal generated by $1 + i$. The variable t_2 equals $P_{\mathbb{Z}_2}(2x = 0, 2 \mid x \wedge 2x = 0)$ and even $P_{\mathbb{Z}}(2x = 0, 2 \mid x \wedge 2x = 0)$. Over the Gaussian integers the latter pp-pair is equivalent to $((1+i)^2 x = 0, (1+i)^2 \mid x \wedge (1+i)^2 x = 0)$, which is mapped by $P_{\mathbb{Z}[i]}$ to $\widetilde{t}_{1+i} + 2\widetilde{t}_{1+i}^{\,2} + \widetilde{t}_{1+i}^{\,3}$.

(2) Next let $\mathcal{P} = p\mathbb{Z}$ with p prime, $p \equiv 3 \pmod 4$. Then $p\mathbb{Z}[i]$ is still prime. In this case t_p coincides with $P_{\mathbb{Z}}(px = 0, p \mid x \wedge px = 0)$ and just corresponds to $\widetilde{t}_p$ when passing to Gaussian integers.

(3) Finally let $\mathcal{P} = p\mathbb{Z}$ with p prime, $p \equiv 1 \pmod 4$ and so $p = a^2 + b^2$ for some suitable integers a, b. Then $p\mathbb{Z}[i]$ is in $\mathbb{Z}[i]$ the product of the prime ideals generated by $a \pm ib$. Recall $t_p = P_{\mathbb{Z}}(px = 0, p \mid x \wedge px = 0)$. Over the Gaussian integers the latter pp-pair is equivalent to $((a+ib)(a-ib)x =$

$0, (a+ib)(a-ib) \mid x \wedge (a+ib)(a-ib)x = 0)$, which is mapped by $P_{\mathbb{Z}[i]}$ to $\widetilde{t}_{a+ib} + \widetilde{t}_{a-ib}$.

Now we generalize the preceding example, in particular its item (1).

Proposition 7.4. *Each power $t_{\mathcal{P}}^K$, K a positive integer, is expressed over $\widetilde{R}$ as*

$$\sum_{j=1}^{g}(\sum_{i=1}^{e_j-1} i\, \widetilde{t}_{\mathcal{M}_j}^{\,e_j(K-1)+i} + e_j\, \widetilde{t}_{\mathcal{M}_j}^{\,e_jK} + \sum_{i=1}^{e_j-1}(e_j - i)\, \widetilde{t}_{\mathcal{M}_j}^{\,e_jK+i}),$$

in more detail as

$$\sum_{j=1}^{g}(\widetilde{t}_{\mathcal{M}_j}^{\,e_j(K-1)+1} + 2\widetilde{t}_{\mathcal{M}_j}^{\,e_j(K-1)+2} + \ldots + (e_j-1)\widetilde{t}_{\mathcal{M}_j}^{\,e_j(K-1)+e_j-1} +$$

$$+ e_j\widetilde{t}_{\mathcal{M}_j}^{\,e_jK} + (e_j-1)\widetilde{t}_{\mathcal{M}_j}^{\,e_jK+1} + \ldots + \widetilde{t}_{\mathcal{M}_j}^{\,e_jK+e_j-1}).$$

In particular $t_{\mathcal{P}}$ itself is given by

$$\sum_{j=1}^{g}(\sum_{i=1}^{e_j-1} i\,\widetilde{t}_{\mathcal{M}_j}^{\,i} + e_j\,\widetilde{t}_{\mathcal{M}_j}^{\,e_j} + \sum_{i=1}^{e_j-1}(e_j-i)\,\widetilde{t}_{\mathcal{M}_j}^{\,e_j+i}),$$

that is

$$= \sum_{j=1}^{g}(\widetilde{t}_{\mathcal{M}_j} + 2\widetilde{t}_{\mathcal{M}_j}^{\,2} + \ldots + (e_j-1)\widetilde{t}_{\mathcal{M}_j}^{\,e_j-1} +$$

$$+ e_j\widetilde{t}_{\mathcal{M}_j}^{\,e_j} + (e_j-1)\widetilde{t}_{\mathcal{M}_j}^{\,e_j+1} + \ldots + \widetilde{t}_{\mathcal{M}_j}^{\,2e_j-1}).$$

Note that Proposition 7.4 defines a function from the $t_{\mathcal{P}}$, with $\mathcal{P}$ a non-zero prime ideal of R, to the additive group $\mathbb{Z}_0[[(\widetilde{t}_{\mathcal{M}})_{\mathcal{M}}]]$ where $\mathcal{M}$ ranges over the non-zero prime ideals of $\widetilde{R}$. When extended by linearity to the additive group $\mathbb{Z}_0[[(t_{\mathcal{P}})_{\mathcal{P}}]]$, this function determines a group homomorphism from it to $\mathbb{Z}_0[[(\widetilde{t}_{\mathcal{M}})_{\mathcal{M}}]]$. Recall that $\mathbb{Z}_0[[-]]$ was introduced at the beginning of Section 5.

Proof. Let π be a generator of the (principal) non-zero prime ideal $\mathcal{P}R_{\mathcal{P}}$ of $R_{\mathcal{P}}$, and similarly, for every $j = 1, \ldots, g$, let π_j denote a generator of the non-zero prime ideal $\mathcal{M}_j\widetilde{R}_{\mathcal{M}_j}$ of $\widetilde{R}_{\mathcal{M}_j}$. We can assume $\pi \in R$ and $\pi_j \in \widetilde{R}$ for all j.

For every $j = 1, \ldots, g$, the embedding of $R_{\mathcal{P}}$ into $\widetilde{R}_{\mathcal{M}_j}$ sends $\mathcal{P}R_{\mathcal{P}}$ into $\mathcal{P}\widetilde{R}_{\mathcal{M}_j} = \mathcal{M}_j^{e_j}\widetilde{R}_{\mathcal{M}_j}$. Therefore π is associated to $\pi_j^{e_j}$ in $\widetilde{R}_{\mathcal{M}_j}$.

Now recall that $t_{\mathcal{P}}^K$ equals $P_{R_{\mathcal{P}}}(\pi^{K-1} \mid x \wedge \pi x = 0, \pi^K \mid x \wedge \pi x = 0)$. Passing to $\widetilde{R}_{\mathcal{M}_j}$ we are led to consider the pp-pair $(\pi_j^{e_j(K-1)} \mid x \wedge \pi_j^{e_j}x = 0, \pi_j^{e_jK} \mid x \wedge \pi_j^{e_j}x = 0)$ and the corresponding lengths $l_n = l_{\widetilde{R}_{\mathcal{M}_j}}(\pi_j^{e_j(K-1)} \mid x \wedge \pi_j^{e_j}x = 0, \pi_j^{e_jK} \mid x \wedge \pi_j^{e_j}x = 0, \widetilde{R}_{\mathcal{M}_j}/\pi_j^n\widetilde{R}_{\mathcal{M}_j})$ when n ranges over positive integers.

- If $n \le e_j(K-1)$, then this pp-pair is equivalent to $(x = 0, x = 0)$ in $\widetilde{R}_{\mathcal{M}_j}/\pi^n\widetilde{R}_{\mathcal{M}_j}$, whence $l_n = 0$.
- Similarly, if $n \ge e_j(K+1)$, then the pp-pair is equivalent to $(\pi_j^{e_j}x = 0, \pi_j^{e_j}x = 0)$ in $\widetilde{R}_{\mathcal{M}_j}/\pi^n\widetilde{R}_{\mathcal{M}_j}$ and l_n is again 0.
- If $n = e_j(K-1)+i$ with $i = 1, \ldots, e_j - 1$, then similar computations prove $l_n = i$.

- Also, if $n = e_j K + i$ with again $i = 1, \ldots, e_j - 1$, then one gets $l_n = e_j - i$.
- Finally it turns out $l_{e_j K} = e_j$.

The equality stated in the theorem is now straightforward to prove. □

8. Galois groups and pp-formulas

Throughout, let R be a Dedekind domain with field of fractions Q, L a finite dimensional *Galois* extension of Q and $\widetilde{R}$ the integral closure of R in L. Let $G = \mathrm{Gal}(L, Q)$ be the Galois group of the extension $L \supseteq Q$. Then G acts on $\widetilde{R}$, and indeed there is a one-to-one correspondence between G and the group $\mathrm{Aut}(\widetilde{R})$ of automorphisms of $\widetilde{R}$, given by the restriction of any $\sigma \in G$ to $\widetilde{R}$ (see [14, Proposition 2.19 p. 15]). Every $\sigma \in G$ fixes R pointwise, whence, for every non-zero prime ideal $\mathcal{P}$ of R, G acts on the set of non-zero prime ideals of $\widetilde{R}$ that extend $\mathcal{P}$. Moreover G acts transitively on these ideals, that is, for any choice of two of them $\mathcal{M} \neq \mathcal{M}'$, there is some $\sigma \in G$ such that $\sigma(\mathcal{M}) = \mathcal{M}'$, see [11, Theorem 6.8 p. 32] or [16, §1, 9.1]. Let us say that two such ideals $\mathcal{M}, \mathcal{M}'$ are *conjugate* if and only if there exists $\sigma \in G$ such that $\sigma(\mathcal{M}) = \mathcal{M}'$.

The *decomposition group* of a maximal ideal $\mathcal{M}$ is the subgroup

$$G_{\mathcal{M}} := \{\sigma \in G \ : \ \sigma(\mathcal{M}) = \mathcal{M}\},$$

so the stabilizer of $\mathcal{M}$.

Define $\overline{\mathcal{M}}$ to be the product of the distinct non-zero prime ideals which are conjugate to $\mathcal{M}$. Written another way $\overline{\mathcal{M}} := \prod_{\sigma \in \Gamma(\mathcal{M})} \sigma(\mathcal{M})$ where $\Gamma(\mathcal{M})$ is a set of coset representatives of the decomposition group of $\mathcal{M}$ in G.

Then (see [16, p. 55]), for $\mathcal{P}$ a non-zero prime ideal of R and $\mathcal{M} \supseteq \mathcal{P}$ a non-zero prime ideal of $\widetilde{R}$ with ramification index e,

$$\mathcal{P}\widetilde{R} = \prod_{\sigma \in \Gamma(\mathcal{M})} \sigma(\mathcal{M})^e = \overline{\mathcal{M}}^e.$$

Let again $\sigma \in G$. For every pp-formula $\varphi(\overline{x})$ of $\mathcal{L}(\widetilde{R})$, σ defines a new pp-formula over $\widetilde{R}$, denoted $\sigma(\varphi)(\overline{x})$, where the scalars of $\widetilde{R}$ occurring in $\varphi(x)$ are replaced by their images under σ.

In this section we wish to examine how the automorphisms $\sigma \in G$ act on the pp-formulas $\varphi(\overline{x})$ of $\mathcal{L}(\widetilde{R})$ (up to logical equivalence with respect to $T_{\widetilde{R}}$). We focus on pp-1-formulas $\varphi(x)$. It is easy to see that the ones over $\widetilde{R}$ fixed by G are a lattice. We want to determine

- this lattice, so that of pp-1-formulas over $\widetilde{R}$ fixed by G,
- the subgroup of the automorphisms of G fixing every pp-1-formula over $\widetilde{R}$.

First a straightforward premise (valid not only for pp-1-formulas). Let $\sigma \in G$, φ and φ' pp-formulas of $\mathcal{L}(\tilde{R})$. Then φ and φ' are logically equivalent (in $T_{\widetilde{R}}$) if and only if their images $\sigma(\varphi)$ and $\sigma(\varphi')$ are.

Let $\mathrm{pp}^{1,G}_{\widetilde{R}}$ denote the lattice of (logical equivalence classes) of pp-1-formulas fixed by every $\sigma \in G$. Clearly $\mathrm{pp}^{1,G}_{\widetilde{R}}$ contains the lattice pp^1_R of pp-1-formulas over R. But this inclusion could also be proper as illustrated by the following example.

Example 8.1. Let $R = \mathbb{Z}$, so $Q = \mathbb{Q}$. Take $L = \mathbb{Q}(i)$, whence $\widetilde{R} = \mathbb{Z}[i]$. Then G consists of two elements, that is the identity map and the restriction of complex conjugation to L. Both preserve $(1+i) \mid x$ up to logical equivalence. In particular this is true of complex conjugation, because $1-i = -i \cdot (1+i)$ is associate with $1+i$ (i.e. they mutually divide each other), so that $(1-i) \mid x$ is equivalent to $(1+i) \mid x$. However there is no way to represent $(1+i) \mid x$ as a pp-formula over $\mathbb{Z}$. Note also that $(2+i) \mid x$ is not equivalent to $(2-i) \mid x$ even if $2+i$, $2-i$ are conjugate, because they are not associate in $\mathbb{Z}[i]$.

The following remark provides a generalization of this example, valid for every L and $\widetilde{R}$.

Remark 8.2. *Let J be an ideal of $\widetilde{R}$. Then, for every $\sigma \in G$,*

- *σ fixes the pp-1-formula $J \mid x$ if and only if $\sigma(J) = J$,*
- *similarly σ fixes the pp-1-formula $Jx = 0$ if and only if $\sigma(J) = J$.*

Consequently $J|x$ (respectively $Jx = 0$) is fixed by G if and only if J is fixed by G as an element of the lattice of ideals of $\widetilde{R}$.

Lemma 8.3. *Let S be any Dedekind domain. If I, J are non-zero coprime ideals of S, h, h', l, l' are non-negative integers, $l, l' \neq 0$, then*

$$(I^h \mid x \wedge I^l x = 0) + (J^{h'} \mid x \wedge J^{l'} x = 0) \text{ is equivalent to } I^h J^{h'} \mid x \wedge I^l J^{l'} x = 0$$

and

$$(I^h x = 0 + I^l \mid x) \wedge (J^{h'} x = 0 + J^{l'} \mid x) \text{ is equivalent to } I^h J^{h'} x = 0 + I^l J^{l'} \mid x.$$

Proof. It is enough to check that these pp-formulas define the same set on modules of the form $S/\mathcal{P}^n$ for $\mathcal{P}$ a non-zero prime ideal and n a positive integer.

Since I and J are coprime, for all non-zero prime ideals $\mathcal{P}$ either $\mathcal{P}$ does not divide I or $\mathcal{P}$ does not divide J. Without loss of generality, suppose $\mathcal{P}$ does not divide I. Then $(I^h \mid x \wedge I^l x = 0)(S/\mathcal{P}^n) = 0$, $(I^h J^{h'} \mid x)(S/\mathcal{P}^n) = (J^{h'} \mid x)(S/\mathcal{P}^n)$ and $(I^l J^{l'} x = 0)(S/\mathcal{P}^n) = (J^{l'} x = 0)(S/\mathcal{P}^n)$ because $(S/\mathcal{P}^n) \cdot I = S/\mathcal{P}^n$ and $\mathrm{ann}_{S/\mathcal{P}^n} I = 0$. So the two pp-formulas define the same sets in $S/\mathcal{P}^n$ as required.

The second statement follows by using Prest's duality. □

Lemma 8.4. *A non-zero proper ideal I of $\widetilde{R}$ is fixed by G if and only if it is a product of ideals of the form $\overline{\mathcal{M}}$ for some non-zero prime ideal $\mathcal{M}$.*

Proof. The reverse direction is clear since each ideal $\overline{\mathcal{M}}$ is fixed by all $\sigma \in G$.

Conversely, suppose that $\sigma(I) = I$. Let X be a set of representatives of the conjugacy classes of non-zero prime ideals $\mathcal{M}$ such that $\mathcal{M} \supseteq I$. For every non-zero prime ideal $\mathcal{M}$, let $k_{\mathcal{M}}(I)$ be the maximal non-negative integer such that

$\mathcal{M}^{k_{\mathcal{M}}(I)} \supseteq I$. Recall that $I = \prod_{\mathcal{M}} \mathcal{M}^{k_{\mathcal{M}}(I)}$. Now observe that, for every non-negative integer k, $\mathcal{M}^k \supseteq I$ if and only if $\sigma(\mathcal{M})^k \supseteq \sigma(I) = I$. So $k_{\mathcal{M}}(I) = k_{\sigma(\mathcal{M})}(I)$. Therefore $I = \prod_{\mathcal{M}\in X} \overline{\mathcal{M}}^{k_{\mathcal{M}}(I)}$. □

Proposition 8.5. *The lattice* $\mathrm{pp}^{1,G}_{\widetilde{R}}$ *of pp-1-formulas fixed by the Galois group* G *is the lattice generated by the formulas of the form* $I \mid x$ *and* $Ix = 0$ *where* I *ranges over the ideals of* $\widetilde{R}$ *such that* $\sigma(I) = I$ *for all* $\sigma \in G$.

Proof. Remark 8.2 implies that the lattice generated by formulas of the form $I|x$ and $Ix = 0$ where I is an ideal of $\widetilde{R}$ such that $\sigma(I) = I$ for all $\sigma \in G$ is a subset of $\mathrm{pp}^{1,G}_{\widetilde{R}}$.

We now show that if $\varphi \in \mathrm{pp}^{1,G}_{\widetilde{R}}$ then φ is equal to a lattice combination of formulas of the form $I|x$ and $Ix = 0$ where I ranges over the ideals of $\widetilde{R}$ such that $\sigma(I) = I$ for all $\sigma \in G$. Note that if φ is fixed by G then φ is equal to $\sum_{\sigma\in G} \sigma(\varphi)$.

By Lemma 2.6 and Corollary 2.7 ,

$$\varphi = \varphi(\widetilde{R})|x + \sum_{\mathcal{M}\in\Omega} \varphi_{\mathcal{M}}$$

for some finite subset Ω of non-zero prime ideals of $\widetilde{R}$ and $\varphi_{\mathcal{M}}$ a sum of formulas of the form $\mathcal{M}^h|x \wedge \mathcal{M}^l x = 0$ (with h, l nonnegative integers, $l > 0$).

Fix a non-zero prime ideal $\mathcal{M}$ of $\widetilde{R}$, h, l nonnegative integers, $l > 0$. Let $\Gamma(\mathcal{M})$ be a set of coset representatives of $G_{\mathcal{M}}$. By Lemma 8.3,

$$\begin{aligned}\sum_{\sigma\in G} \sigma(\mathcal{M}^h \mid x \wedge \mathcal{M}^l x = 0) &= \sum_{\sigma\in\Gamma(\mathcal{M})} (\sigma(\mathcal{M})^h \mid x \wedge \sigma(\mathcal{M})^l x = 0)\\ &= \overline{\mathcal{M}}^h \mid x \wedge \overline{\mathcal{M}}^l x = 0.\end{aligned}$$

If $\sigma(\varphi)$ and φ are equivalent, then $\sigma(\varphi(\widetilde{R})) = \sigma(\varphi)(\widetilde{R}) = \varphi(\widetilde{R})$. Therefore

$$\sum_{\sigma\in G} \sigma(\varphi(\widetilde{R})|x) = \varphi(\widetilde{R})|x.$$

So $\varphi = \sum_{\sigma\in G} \sigma(\varphi)$ is a lattice combination of formulas of the required form. □

Remark 8.6. *Let* $\mathcal{P}$ *be a non-zero prime ideal of* R *and* $\mathcal{M} \supseteq \mathcal{P}$ *be a non-zero prime ideal of* $\widetilde{R}$ *with ramification index* e *(so* $\mathcal{M} \cap R = \mathcal{P}$*). Then the following hold:*

(1) $\mathcal{P}\widetilde{R} = \overline{\mathcal{M}}^e$;
(2) $\overline{\mathcal{M}} = \mathrm{rad}(\mathcal{P}\widetilde{R})$.

Proof. (1) Recall that G acts transitively on the set of non-zero prime ideals $\mathcal{M}$ of $\widetilde{R}$ such that $\mathcal{M} \cap R = \mathcal{P}$ and that $\mathcal{P}\widetilde{R} = \prod_{\sigma\in\Gamma(\mathcal{M})} \sigma(\mathcal{M})^e = \overline{\mathcal{M}}^e$ where $\mathcal{M}$ is a non-zero prime ideal of $\widetilde{R}$ such that $\mathcal{M}\cap R = \mathcal{P}$ and $\Gamma(\mathcal{M})$ is a set of representatives of the cosets of $G_{\mathcal{M}}$ in G.
(2) Since $\mathcal{P}\widetilde{R} = \overline{\mathcal{M}}^e$, the non-zero prime ideals containing $\mathcal{P}\widetilde{R}$ are exactly those conjugate to $\mathcal{M}$. Therefore $\mathrm{rad}(\mathcal{P}\widetilde{R}) = \overline{\mathcal{M}}$. □

Theorem 8.7. *The lattice* $\mathrm{pp}^{1,G}_{\widetilde{R}}$ *of pp-1-formulas fixed by the Galois group* **is isomorphic to** pp^1_R *via the function induced by sending* $\mathcal{P}^k|x$ *to* $\mathrm{rad}(\mathcal{P}\widetilde{R})^k|x$ *and* $\mathcal{P}^l x = 0$ *to* $\mathrm{rad}(\mathcal{P}\widetilde{R})^l\, x = 0$ *when* $\mathcal{P}$ *ranges over non-zero prime ideals of* R, k *over non-negative integers and* l *over positive integers.*

It is often conceptually difficult to prove directly that lattice homomorphisms defined on generators are well-defined or injective. For this reason, we instead define a surjective spectral map from Spec $\mathrm{pp}^1_{\widetilde{R}}$ to Spec pp^1_R and check that the embedding from pp^1_R to $\mathrm{pp}^1_{\widetilde{R}}$ given by Stone duality indeed does what we claim in Theorem 8.7 on generators.

Recall (see [3] for more on these topics) that the *spectrum*, Spec L, of a bounded distributive lattice L is defined as the set of prime filters of L with the topology given by the basis of (compact) open sets

$$\mathcal{O}(a) := \{\mathcal{F} \in \operatorname{Spec} L \mid a \in \mathcal{F}\}, \text{ where } a \in L.$$

The space Spec L is spectral and all spectral spaces occur in this way. Recall that a *spectral space* is simply a (quasi-)compact T_0-space which is sober and has a basis of compact open sets which is closed under finite intersections. In particular, the set of compact open sets, $\mathring{K}(T,\tau)$, of a spectral space (T,τ), ordered by inclusion, is a bounded distributive lattice.

Moreover a *spectral map* $f : X \to Y$ between spectral spaces X, Y is a continuous map such that the preimage of every compact open subset is compact. Note that, in order to see whether a map is spectral, it is enough to check this condition on a subbasis.

Stone duality is an anti-equivalence between the category of bounded distributive lattices Dist with bounded lattice homomorphisms and the category of spectral spaces $\mathsf{Spectral}$ with spectral maps. The anti-equivalence is given by functors Spec : $\mathsf{Dist} \to \mathsf{Spectral}$ and $\mathring{K}$: $\mathsf{Spectral} \to \mathsf{Dist}$, as defined before, and natural isomorphisms $\nu : \mathrm{Id}_{\mathsf{Dist}} \to \mathring{K}\,\mathrm{Spec}$ and $\epsilon : \mathrm{Id}_{\mathsf{Spectral}} \to \mathrm{Spec}\,\mathring{K}$ which are defined as follows.

Let L_1, L_2 be bounded distributive lattices and $f : L_1 \to L_2$ be a bounded lattice homomorphism. Then Spec f : Spec $L_2 \to$ Spec L_1 denotes the function sending any $p \in$ Spec L_2 to $f^{-1}(p) \in$ Spec L_1.

Let $(T_1, \tau_1), (T_2, \tau_2)$ be spectral spaces and let $g : (T_1, \tau_1) \to (T_2, \tau_2)$ be a spectral map. Then it is given $\mathring{K}(g) : \mathring{K}(T_2, \tau_2) \to \mathring{K}(T_1, \tau_1)$ sending any $\mathcal{O} \in \mathring{K}(T_2, \tau_2)$ to $g^{-1}(\mathcal{O}) \in \mathring{K}(T_1, \tau_1)$.

The natural isomorphism $\nu : \mathrm{Id}_{\mathsf{Dist}} \to \mathring{K}$ Spec is defined by $\nu_L(a) := \mathcal{O}(a)$ and the natural isomorphism $\epsilon : \mathrm{Id}_{\mathsf{Spectral}} \to$ Spec $\mathring{K}$ is defined by $\epsilon_{(T,\tau)}(x) := \{\mathcal{U} \in \mathring{K}(T,\tau) \mid x \in U\}$.

Coming back to a Dedekind domain R, we are in the lucky position of already knowing the prime filters of pp^1_R because they are exactly the irreducible pp-1-types, as listed in the following definition.

Definition 8.8. *Let R be a Dedekind domain with field of fractions Q.*

- *For each maximal ideal $\mathcal{P}$ of R, $l, m \in \mathbb{N}$, $l > 0$, let $p^R_{l,m}(\mathcal{P})$ denote the pp-type of $a + \mathcal{P}^{l+m} \in R/\mathcal{P}^{l+m}$ where $a \in \mathcal{P}^m \backslash \mathcal{P}^{m+1}$.*
- *For each maximal ideal $\mathcal{P}$ of R and $l \in \mathbb{N}$, $l > 0$, let $p^R_{l,\infty}(\mathcal{P})$ denote the pp-type of $a + R_{\mathcal{P}} \in Q/R_{\mathcal{P}}$ such that $a \in \mathcal{P}^{-l} R_{\mathcal{P}} \backslash \mathcal{P}^{-l+1} R_{\mathcal{P}}$.*
- *For each maximal ideal $\mathcal{P}$ of R and $m \in \mathbb{N}$, let $p^R_{\infty,m}(\mathcal{P})$ denote the pp-type of $a \in R_{\mathcal{P}}$ such that $a \in \mathcal{P}^m R_{\mathcal{P}} \backslash \mathcal{P}^{m+1} R_{\mathcal{P}}$.*
- *Let $p^R_{\infty,\infty}$ be the pp-type of a non-zero element of Q.*

Remark 8.9. *Let R be a Dedekind domain. For each maximal ideal $\mathcal{P}$,*

$$\mathcal{O}(\mathcal{P}^k x = 0) := \{p^R_{l,m}(\mathcal{P}) \mid k \geq l\} \cup \{p^R_{l,\infty}(\mathcal{P}) \mid k \geq l\}$$

and

$$\mathcal{O}(\mathcal{P}^k | x) := \{p^R_{l,m}(\mathcal{P}) \mid m \geq k\} \cup \{p^R_{\infty,m}(\mathcal{P}) \mid m \geq k\} \cup \bigcup_{\mathcal{Q} \neq \mathcal{P}} \{p^R_{l,m}(\mathcal{Q}) \mid 1 \leq l \leq \infty \text{ and } 1 \leq m \leq \infty\} \cup \{p^R_{\infty,\infty}\}.$$

That being said, let us prove now Proposition 8.7.

Proof. Define $\Omega : \mathrm{Spec}(\mathrm{pp}^1_{\widetilde{R}}) \to \mathrm{Spec}(\mathrm{pp}^1_R)$ by

$$\begin{aligned}
\Omega(p^{\widetilde{R}}_{l,m}(\mathcal{M})) &:= p^R_{l,m}(\mathcal{M} \cap R) \\
\Omega(p^{\widetilde{R}}_{l,\infty}(\mathcal{M})) &:= p^R_{l,\infty}(\mathcal{M} \cap R) \\
\Omega(p^{\widetilde{R}}_{\infty,m}(\mathcal{M})) &:= p^R_{\infty,m}(\mathcal{M} \cap R) \\
\Omega(p^{\widetilde{R}}_{\infty,\infty}) &:= p^R_{\infty,\infty}
\end{aligned}$$

for $\mathcal{M}$ a maximal ideal of $\widetilde{R}$, $l, m \in \mathbb{N}$ and $m > 0$.

Let $\mathcal{P}$ a maximal ideal of R and $k \in \mathbb{N}, k > 0$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_g$ be the pairwise distinct prime ideals of $\widetilde{R}$ such that $\mathcal{M}_i \cap R = \mathcal{P}$ for $i = 1, \ldots, g$. Then

$$\Omega^{-1}(\mathcal{O}(\mathcal{P}^k | x)) = \bigcap_{i=1}^{g} \mathcal{O}(\mathcal{M}_i^k | x) =$$

$$= \mathcal{O}(\bigwedge_{i=1}^{g} \mathcal{M}_i^k | x) = \mathcal{O}(\prod_{i=1}^{g} \mathcal{M}_i^k | x) = \mathcal{O}(\mathrm{rad}(\mathcal{P}\widetilde{R})^k | x)$$

and

$$\Omega^{-1}(\mathcal{O}(\mathcal{P}^k x = 0)) = \bigcup_{i=1}^{g} \mathcal{O}(\mathcal{M}_i^k x = 0) = \mathcal{O}(\sum_{i=1}^{g} \mathcal{M}_i^k x = 0) =$$

$$= \mathcal{O}(\prod_{i=1}^{g} \mathcal{M}_i^k \, x = 0) = \mathcal{O}(\mathrm{rad}(\mathcal{P}\widetilde{R})^k \, x = 0).$$

In both sequences of equations, the first equalities are simple observations using Remark 8.9 and the second equalities follow from the definition of the spectrum of a distributive lattice. The penultimate equalities follow from Lemma 8.3. The final equalities are Remark 8.6.

Because the open sets of the form $\mathcal{O}(\mathcal{P}^k x = 0)$ and $\mathcal{O}(\mathcal{P}^k|x)$ are a subbasis of Spec pp^1_R these equations imply that Ω is a spectral map.

Let $\mathcal{P}$ be a non-zero prime ideal of R. Since $\Omega^{-1}(\mathcal{O}(\mathcal{P}^k|x)) = \mathcal{O}(\mathrm{rad}(\mathcal{P}\widetilde{R})^k|x)$,

$$\nu^{-1}_{\mathrm{pp}^1_{\widetilde{R}}} \circ \mathring{K}\Omega \circ \nu_{\mathrm{pp}^1_R}(\mathcal{P}^k|x) \text{ is } \mathrm{rad}(\mathcal{P}\widetilde{R})^k|x$$

and, since $\Omega^{-1}(\mathcal{O}(\mathcal{P}^k x = 0)) = \mathcal{O}(\mathrm{rad}(\mathcal{P}\widetilde{R})^k x = 0)$,

$$\nu^{-1}_{\mathrm{pp}^1_{\widetilde{R}}} \circ \mathring{K}\Omega \circ \nu_{\mathrm{pp}^1_R}(\mathcal{P}^k x = 0) \text{ is } \mathrm{rad}(\mathcal{P}\widetilde{R})^k x = 0.$$

So the lattice homomorphism

$$\nu^{-1}_{\mathrm{pp}^1_{\widetilde{R}}} \circ \mathring{K}\Omega \circ \nu_{\mathrm{pp}^1_R} : \mathrm{pp}^1_R \to \mathrm{pp}^1_{\widetilde{R}}$$

is induced by sending $\mathcal{P}^k|x$ to $\mathrm{rad}(\mathcal{P}\widetilde{R})^k|x$ and $\mathcal{P}^k x = 0$ to $\mathrm{rad}(\mathcal{P}\widetilde{R})^k x = 0$ as required. Moreover, it is injective, since Ω is surjective. □

Now let us deal with the subgroup $G^{\mathrm{pp}^1_{\widetilde{R}}}$ consisting of the automorphisms $\sigma \in G$ preserving every pp-1-formula of $\mathcal{L}(\widetilde{R})$ up to logical equivalence.

Indeed, for every pp-formula $\varphi(x)$ of $\mathcal{L}(\widetilde{R})$ we can introduce the subgroup G^{φ} of the $\sigma \in G$ preserving $\varphi(x)$. For instance, when $L = \mathbb{Q}(i)$ and $\widetilde{R} = \mathbb{Z}[i]$, we have already implicitly seen that $G^{(1+i)|x} = G$ while $G^{(2+i)|x}$ includes only the identity function. When we consider the whole $G^{\mathrm{pp}^1_{\widetilde{R}}}$ the following holds.

Proposition 8.10. *Let $\sigma \in G$. Then $\sigma \in G^{\mathrm{pp}^1_{\widetilde{R}}}$ if and only if σ fixes (setwise) every non-zero prime ideal of $\widetilde{R}$. In particular, if there is some non-zero prime ideal $\mathcal{P}$ of R that completely splits over $\widetilde{R}$, then $G^{\mathrm{pp}^1_{\widetilde{R}}}$ is the trivial group.*

Note that the latter statement applies to $R = \mathbb{Z}$, or also when Q is a number field, see for example [15, Exercise 30(d) p. 63].

Proof. The first claim follows easily from Lemma 2.6 and Remark 8.2.

So let us deal with the second claim. Let $\mathcal{P}$ be a non-zero prime ideal of R that completely splits over $\widetilde{R}$. Then $\mathcal{P}\widetilde{R}$ decomposes in $\widetilde{R}$ as $\prod_{j=1}^{g} \mathcal{M}_j$, where each $\mathcal{M}_j$ is a non-zero prime ideal with both ramification index and inertial degree 1. Hence $g = [L : K] = |G|$ and, by transitivity, for every j there is exactly one $\sigma_j \in G$ sending $\mathcal{M}_1$ to $\mathcal{M}_j$. So the only $\sigma \in G$ fixing $\mathcal{M}_1$ is the identity. Any σ different from the identity moves $\mathcal{M}_1$ and so corresponds to the first case. □

We provide an example of a Galois field extension $L \supseteq Q$ such that $G = G^{\mathrm{pp}^1_{\widetilde{R}}}$, that is every σ in the Galois group $G = \mathrm{Gal}(L, Q)$ fixes every pp-formula over $\widetilde{R}$.

Example 8.11. Let $Q = \mathbb{Q}_3$ the 3-adic completion of $\mathbb{Q}$. So R is a complete discrete valuation ring with a unique maximal ideal $\mathcal{P}$. Let $L = Q(\sqrt{3})$, or also $Q(\sqrt{6})$. Then L is a quadratic extension of Q defined by an Eisenstein polynomial, $x^2 - 3$ and $x^2 - 6$ respectively. Therefore $\mathrm{Gal}(L, Q)$ has order 2. Moreover L totally ramifies (see [22, Lecture 11, Example 11.6 p. 2]), the unique maximal ideal $\mathcal{M}$ of $\tilde{R}$ extends $\mathcal{P}$ and $\mathcal{P}\tilde{R}$ is a power of $\mathcal{M}$. Therefore even the non-identity $\sigma \in G$ fixes $\mathcal{M}$.

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