Hochschild and cyclic homology of 3-preprojective algebras of type A

Davide Morigi

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University of East Anglia

School of Mathematics

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Abstract

The subjects of study of this thesis are 3-preprojective algebras Π of type A, a higher generalisation of classical preprojective algebras of type A in the sense of Iyama. These algebras enjoy many nice homological properties: they are Frobenius, almost Koszul and (twisted) periodic. They can be described in terms of quivers with relations, which makes computations easier to perform.

In this thesis, we use the aforementioned properties to give a conjectural description of the Hochschild homology, cohomology and cyclic homology groups of these algebras. The conjectural part depends on the truthfulness of a formula that allows us to compute the cyclic homology in terms of the determinant of the Hilbert series of Π, and on the computation of the determinant of a matrix with polynomial entries, that we verify with the use of GAP [32] for the first 30 cases.

Contents

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1

Introduction

1.1 Motivation

Preprojective algebras were first introduced by Gelfand and Pomonarev in [33] (see also [17]) via quivers with relations: the preprojective algebra Π_Q of a quiver Q is the path algebra of the double quiver $\overline{Q}$ with relations given by $\sum_{a \in Q_1}(aa^* - a^*a)$. The name is justified: the preprojective algebra is isomorphic (as a module over the path algebra of Q) to the sum of one copy of each preprojective module. The authors also show when the preprojective algebra Π_Q is finite dimensional: this happens exactly when the path algebra kQ is of finite representation type or equivalently (by Gabriel's theorem, see [30]), when Q is of Dynkin type A, D or E.

In [5], Baer, Geigle and Lenzing gave an equivalent definition (see [57], [12] for a proof of the equivalence), that highlights their importance in the representation theory of quivers: for a quiver Q and $\Lambda = kQ$ its path algebra, the preprojective algebra of Q is the direct sum of the spaces $\mathrm{Hom}_\Lambda(\Lambda, \tau^{-m}(\Lambda))$, where τ^{-1} is the inverse Auslander-Reiten translate, and the multiplication is defined in a natural way.

Since their introduction, preprojective algebras turned out to be ubiquitous objects in mathematics. For example, they appeared as:

(i) a non-commutative resolution of Klenian singularities, in the extended Dynkin case (see [14]);

(ii) a characterisation of graded 2-Calabi-Yau algebras, in the non-Dynkin case (see [6]);

(iii) algebras whose irreducible representations are in correspondence with Lusztig's semicanonical basis for some quantum enveloping algebras (see [51]).

A limitation in Auslander-Reiten theory is that it works best for hereditary algebras, i.e., for algebras of global dimension at most one. To overcome this issue, Iyama (see [42]) introduced higher Auslander-Reiten theory, a generalisation of classical AR theory that can be more useful for algebras of higher global dimension (see also the survey [47] by Jasso and Sondre). In this setting, Iyama and Oppermann generalised the concepts of algebra of finite representation type and preprojective algebra, introducing d-representation finite algebras and $(d+1)$-preprojective algebras (see [44] and [45]). The name is justified: as a module, a $(d+1)$-preprojective algebra is isomorphic to the sum of one copy of each $(d+1)$-preprojective module, that is, a module of the form $\tau_d^{-1}P$ for some projective P, where τ_d^{-1} is the inverse d-Auslander-Reiten translate, an higher analogue of the classical inverse AR translate.

Higher preprojective algebras appear in conformal field theory (see [26] and [27]) and in commutative and non-commutative geometry (see [8], [39] and [53]).

An example of d-representation finite algebras is the family of d-representation finite algebras of type A. These were introduced in [43] and [44] as a generalisation of path algebras of quivers of Dynkin type A. They depend on two parameters d, s, and enjoy a very nice combinatorial description given in terms of quivers with relations. Also their preprojective algebras, called $(d+1)$-preprojective algebras of type A, can be described in terms of quivers with relations (see [44]) and enjoy good homological properties: for example, they are Frobenius and almost Koszul.

The main actors of the thesis, alongside $(d+1)$-preprojective algebras of type A, are some homological invariants, namely Hochschild homology, Hochschild cohomology, and (reduced) cyclic homology.

Hochschild homology and cohomology were introduced by Hochschild in [40] to classify square zero extensions of an associative algebra, and can be interpreted as a generalisation of the modules of differential forms to non-commutative algebras by the Hochschild-Kostant-Rosenberg theorem (see [41]). They can be used to understand the structure and deformations of an algebra, and to identify essential information about their representations.

Cyclic homology can be seen both as a variant of Hochschild homology and as a generalisation of de Rham homology for manifolds. It was introduced independently by A. Connes in [10], B. Tsygan in [61] and D. Quillen and L-J. Loday in [50]. Between many others, cyclic homology has applications in differential geometry, where it is used to prove index theorems based on spectral triples (see [11]) and deformation quantization of Poisson structures (see [55]).

1.2 Main results

Graded Hochschild (co)homology and cyclic homology of preprojective algebras were computed in [13] and in [22] for non-Dynkin quivers, and in [21] in the ADE case. The first case is simpler: the global dimension of such an algebra is 2, so the n-th Hochschild (co)homology and (reduced) cyclic homology groups vanish for $n \geq 3$.

The Dynkin case is less trivial: the global dimension of a preprojective algebras of type ADE is infinite, so there are nonzero homology groups also in higher degrees.

In this thesis, inspired by the method used in [21] for preprojective algebras of type ADE, we compute Hochschild (co)homology of 3-preprojective algebras $\Pi = \Pi^{(2,s)}$ of type A.

We start by exhibiting an explicit graded projective Π-bimodule resolution of Π, that is a generalisation of the Schofield resolution given in [58] for preprojective algebras of type ADE. Using a result from [35], we know that the algebra Π is almost Koszul, and precisely $(s-1,3)$-Koszul. Therefore the first 4 terms of the Koszul bimodule complex of Π lie in a minimal projective Π-bimodule resolution of Π by a result in [7]. A partial description of the fourth syzygy $\Omega^4_{\Pi^e}(\Pi)$ is given in [7, Thm. 3.15]; using this we are able to get that:

$$\Omega^4_{\Pi^e}(\Pi) \cong {}_1\Pi_{\eta^{-1}},$$

where η is the Nakayama automorphism of Π. Therefore the algebra Π is twisted periodic of period 4. Furthermore, an explicit formula given in [38] shows that η has order 3, and therefore the algebra Π is periodic of period 12.

We use the (twisted) periodicity to give some relations between the Hochschild homology and cohomology. In particular, using similar arguments to [25], we prove the following.

Proposition 1. Denote by $HH_n(\Pi)$ (resp. $HH^n(\Pi)$) the n-th Hochschild homology (resp. Hochschild cohomology) group of $\Pi = \Pi^{(2,s)}$, and let $h = s+2$. Then there are isomorphisms of graded vector spaces:

$$\begin{aligned} HH_{i+12}(\Pi) &\cong HH_i(\Pi)[3h], && i \geq 1 \\ HH_i(\Pi) &\cong HH_{23-i}(\Pi)^*[6h], && i = 1, \ldots, 12 \\ HH^i(\Pi) &\cong HH_{15-i}(\Pi)[-3h-3], && i = 1, \ldots, 12. \end{aligned}$$

Therefore, the computation of the whole Hochschild homology and cohomology can be deduced from $HH^0(\Pi), HH_0(\Pi), HH_1(\Pi), \ldots, HH_5(\Pi)$ and $HH_{11}(\Pi)$.

Afterwards, we explicitly write down the graded Hochschild homology complex arising from the periodic resolution of Π described above, and find the possible range of (weight) degrees in which each Hochschild homology group can live in.

Then we consider the Connes' exact sequence (see [49]). This is a degree-preserving long exact sequence:

$$0 \to \overline{HH}_0(\Pi) \overset{B_0}{\to} HH_1(\Pi) \overset{B_1}{\to} HH_2(\Pi) \to \ldots, \tag{1}$$

where the i-th reduced cyclic homology group $\overline{HC}_i(\Pi)$ can be computed as $\overline{HC}_i(\Pi) \cong \ker(B_{i+1}) = \mathrm{Im}(B_i)$. This sequence shows that the Hochschild homology can be obtained in terms of the reduced cyclic homology. Furthermore, (1) allows us to give the possible range of weight degrees in which each reduced cyclic homology group can live in. Unlike for standard preprojective algebras of type ADE, for higher preprojective algebras of type A one finds that there are overlaps in the degrees different groups sit.

In order to compute the reduced cyclic homology, its Euler characteristic is very useful. This is a formal series defined by:

$$\chi_{\overline{HC}_*(\Pi)}(t) = \sum_{i=0}^{\infty}(-1)^i h_{\overline{HC}_i(\Pi)}(t),$$

where $h_{\overline{HC}_*(\Pi)}(t)$ is the Hilbert series of the i-th reduced cyclic homology group of Π.

Using a degree argument involving the sequence (1) we show that, in order to get all the Hochschild homology groups $HH_i(\Pi)$, $i \geq 1$, it is enough to compute $\chi_{\overline{HC}_*(\Pi)}(t)$, together with $HH_0(\Pi)$, $HH_1(\Pi)$ and $HH_4(\Pi)$ (for standard preprojective algebras of type ADE, the knowledge of the Euler characteristic of the reduced cyclic homology together with the zeroth Hochschild homology group was enough, since different reduced cyclic homology groups sit in different weight degrees).

Now, based on a private communication with Pavel Etingof, the following result should hold.

Conjecture 2. Let $\chi_{\overline{HC}_*(\Pi)}(t) = \sum_{k\geq 0} a_k t^k$. Then:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{s=1}^{\infty}\det H_\Pi(t^s),$$

where $H_\Pi(t)$ is the Hilbert series of Π as a $\Pi(0)$-bimodule.

Using the fact that Π is almost Koszul, we are able to give a formula for $H_\Pi(t)$.

Proposition 3. Let $\Pi = k\overline{Q}/(R)$, and let C (resp. D) be the adjacency matrix of $\overline{Q}$ (resp. of the opposite quiver $\overline{Q}^*$ of $\overline{Q}$). Denote by P the permutation matrix associated to the inverse of the Nakayama permutation ν^{-1}. Then:

$$H_\Pi(t) = (1-Pt^h)(1-Ct+Dt^2-t^3)^{-1},$$

where $h = s+2$.

Theorem 2 and Proposition 3 imply that, in order to compute $\chi_{\overline{HC}_*(\Pi)}(t)$ it is enough to compute the determinants of the matrices $1-Pt^h$ and $1-Ct+Dt^2-t^3$.

Therefore:

1. we prove an explicit formula for $\det(1 - Pt^h)$.

2. we give a conjectural formula for $\det(1 - Ct + Dt^2 - t^3)$. We support the conjecture with examples and GAP code, that confirms the formula for $s \leq 30$.

In turn, provided the formula for $\det(1 - Ct + Dt^2 - t^3)$ is correct, we get a formula for the Euler characteristic $\chi_{\overline{HC}_*(\Pi)}(t)$.

Afterwards, we explicitly compute the first and fourth Hochschild homology groups of Π. For both of them, the strategy is the following:

1. we consider the 3-term subcomplex of the Hochschild homology complex whose homology computes $HH_1(\Pi)$ (resp. $HH_4(\Pi)$).

2. using the presentation of Π in terms of quiver with relations, we give a combinatorial description of the spaces and of the maps in the 3-terms subcomplex.

3. using such description, we explicitly compute $HH_1(\Pi)$ and $HH_4(\Pi)$.

After calculating explicitly also $HH_0(\Pi)$ and $HH^0(\Pi)$, we trace back our steps to get a description of all the reduced cyclic homology groups of Π. In turn, this also gives a description of the Hochschild homology and cohomology groups of Π.

The main result of the thesis is the following.

Theorem 4. Let $h = s + 2$, and let L, U, W, Z be graded vector spaces (with Z concentrated in degree 0) whose Hilbert series are given as follows:

$$h_L(t) = \sum_{k=0}^{\lfloor \frac{h-3}{3} \rfloor} t^{3k},$$

$$h_U(t) = \sum_{k=0}^{\lfloor \frac{h-1}{3} \rfloor} t^{3k},$$

$$h_W(t) = \sum_{k=\lfloor \frac{h}{2} \rfloor + 1}^{\lceil \frac{2}{3} h \rceil - 1} t^{3k},$$

$$h_Z(t) = \begin{cases} \frac{(h-2)(h-4)}{6}, & \text{if } s \equiv 0, 2 \pmod 6 \\ \frac{(h-2)(h-4)+1}{6}, & \text{if } s \equiv 1 \pmod 6 \\ \frac{(h-2)(h-4)-3}{6}, & \text{if } s \equiv 3, 5 (\text{mod } 6) \\ \frac{(h-2)(h-4)+4}{6}, & \text{if } s \equiv 4 \pmod 6. \end{cases}$$

Provided Conjecture 2 and the conjectural formula for $\det(1 - Ct + Dt^2 - t^3)$ hold, the following are isomorphisms of graded vector spaces:

$$
\begin{aligned}
HH_0(\Pi) &\cong \Pi(0), & \overline{HC}_0(\Pi) &= 0, & HH^0(\Pi) &\cong L \\
HH_1(\Pi) &= 0, & \overline{HC}_1(\Pi) &= 0, & HH^1(\Pi) &\cong U[-3] \\
HH_2(\Pi) &\cong U, & \overline{HC}_2(\Pi) &\cong U, & HH^2(\Pi) &= 0 \\
HH_3(\Pi) &\cong U, & \overline{HC}_3(\Pi) &= 0, & HH^3(\Pi) &\cong Z^*[-3] \\
HH_4(\Pi) &\cong W, & \overline{HC}_4(\Pi) &\cong W, & HH^4(\Pi) &\cong Z[-3] \\
HH_5(\Pi) &\cong W, & \overline{HC}_5(\Pi) &= 0, & HH^5(\Pi) &= 0 \\
HH_6(\Pi) &\cong W^*[3h], & \overline{HC}_6(\Pi) &\cong W^*[3h], & HH^6(\Pi) &\cong U^*[-3] \\
HH_7[\Pi] &\cong W^*[3h], & \overline{HC}_7(\Pi) &= 0, & HH^7(\Pi) &\cong U^*[-3] \\
HH_8(\Pi) &\cong U^*[3h], & \overline{HC}_8(\Pi) &\cong U^*[3h], & HH^8(\Pi) &\cong W^*[-3] \\
HH_9(\Pi) &\cong U^*[3h], & \overline{HC}_9(\Pi) &= 0, & HH^9(\Pi) &\cong W^*[-3] \\
HH_{10}(\Pi) &= 0, & \overline{HC}_{10}(\Pi) &= 0, & HH^{10}(\Pi) &\cong W[-3h-3] \\
HH_{11}(\Pi) &\cong Z[3h], & \overline{HC}_{11}(\Pi) &\cong Z[3h], & HH^{11}(\Pi) &\cong W[-3h-3] \\
HH_{12}(\Pi) &\cong Z^*[3h], & \overline{HC}_{12}(\Pi) &= 0, & HH^{12}(\Pi) &\cong U[-3h-3] \\
HH_{12+i}(\Pi) &\cong HH_i(\Pi)[3h], & \overline{HC}_{12+i}(\Pi) &\cong \overline{HC}_i(\Pi)[3h], & HH^{12+i}(\Pi) &\cong HH^i(\Pi)[-3h],
\end{aligned}
$$

for $i \geq i$.

2

Preliminaries

In this chapter we set most of the notation used throughout the thesis and recall classical results from the literature.

My only contribution in this chapter is in subsection 2.3.2, where results from [25] are adapted to out setting.

2.1 Quivers

Quivers provide a very convenient tool in the study of the representation theory of finite dimensional associative algebras. Indeed, they allow us to visualize the modules over any such algebra as a collection of matrices, each of which is associated to an arrow in the quiver.

In this chapter we introduce quivers and their representations, and we give some results that will be useful in the following.

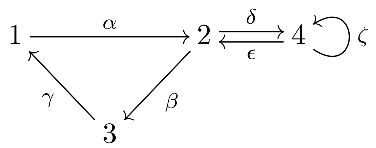
For convenience, we assume that k is an algebraically closed field. All the definitions and results in this section can be found in [60].

2.1.1 Basic definitions

Definition 2.1.1.1. A **quiver** Q is a quadruple (Q_0, Q_1, s, t), where:

- Q_0 is a set, called **set of vertices**;
- Q_1 is a set, called **set of arrows**;
- $s : Q_1 \to Q_0$ associates to each arrow its starting point;
- $t : Q_1 \to Q_0$ associates to each arrow its terminal point.

Example 2.1.1.2. Let $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ and $s(\alpha) = 1$, $t(\alpha) = 2$, $s(\beta) = 2$, $t(\beta) = 3$, $s(\gamma) = 3$, $t(\gamma) = 1$, $s(\delta) = 2$, $t(\delta) = 4$, $s(\epsilon) = 4$, $t(\epsilon) = 2$, $s(\zeta) = t(\zeta) = 4$. Then we can represent the quiver $Q = (Q_0, Q_1, s, t)$ as follows.



If Q_0, Q_1 are both finite, we call Q a **finite quiver**. Throughout the thesis, all quivers will be assumed to be finite.

Definition 2.1.1.3. A representation of a quiver Q is a collection $(M_i, \varphi_a)_{i \in Q_0, a \in Q_1}$, where:

- M_i is a k-vector space for all $i \in Q_0$;
- $\varphi_a : M_{s(a)} \to M_{t(a)}$ is a k-linear map, for all $a \in Q_1$.

A representation is called **finite dimensional** if each M_i is finite dimensional.

We now introduce the natural concept of morphisms of representations.

Definition 2.1.1.4. Let Q be a quiver, and $M = (M_i, \varphi_a)_{i \in Q_0, a \in Q_1}$, $N = (N_i, \psi_a)_{i \in Q_0, a \in Q_1}$ two representations of Q. A **morphism of representations** $f : M \to N$ is a collection $(f_i)_{i \in Q_0}$ of linear maps

$$f_i : M_i \to N_i$$

where, for all arrows $a : i \to j$ in Q_1, the following diagram commutes.

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_a} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ N_i & \xrightarrow{\psi_a} & N_j \end{array}$$

For any quiver Q, we define the category **rep** Q, having as objects finite dimensional representations of Q and as arrows morphisms of representations of Q.

If we define the direct sum of representations of Q and indecomposable representations of Q in the obvious way (see [60, §1.2]), then we get the following.

Theorem 2.1.1.5. [60, §1.2] For every quiver Q, the category rep Q is Krull-Schmidt.

This means that, for every $M \in$ rep Q, there exist indecomposable $M_1, \ldots, M_n \in$ rep Q such that:

$$M \cong M_1 \oplus M_2 \oplus \ldots \oplus M_n.$$

2.1.2 Path algebras and bound quiver algebras

Throughout this subsection, let Q be a quiver.

The concepts of representation of Q (resp. morphism of representations of Q) can be interpreted in the more convenient language of modules (resp. morphism of modules) over an algebra, called **path algebra** of Q.

In this subsection we introduce such concepts and start studying the category of modules over the path algebra of a quiver.

Definition 2.1.2.1. Let $i, j \in Q_0$. A **path** $c = a_1 \dots a_n$ from i to j in Q is a sequence of arrows $a_1, \dots, a_n \in Q_1$ such that

$$s(a_1) = i, \qquad s(a_{i+1}) = t(a_i) \quad \text{for all } i = 1, \dots n-1, \qquad t(a_n) = j.$$

We will call e_i the trivial path at vertex i.

Definition 2.1.2.2. The **path algebra** kQ of Q is the k-algebra with basis the set of paths in the quiver Q and multiplication defined on basis elements $c = a_1 \dots a_n, c' = b_1 \dots b_m$ by:

$$cc' = \begin{cases} a_1 \dots a_n b_1 \dots b_m, & \text{if } s(b_1) = t(a_n) \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1.2.3. Consider the quiver:

$$Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \ .$$

Then a basis for kQ is given by:

$$\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}.$$

For example, the product $e_1 \cdot \alpha$ gives α, while $\beta \cdot \alpha = 0$.

One can introduce relations on the path algebra of Q, in order to identify some paths in kQ. This gives rise to bound quiver algebras, a concept that plays a central role in representation theory of (finite dimensional) algebras.

Definition 2.1.2.4. Define the **arrow ideal** R_Q of A as the two-sided ideal in kQ generated by all arrows in Q.

We can decompose the arrow ideal R_Q as:

$$R_Q = \bigoplus_{\ell \geq 1} kQ_\ell, \tag{2.1.2.1}$$

where kQ_ℓ is the vector space generated by all paths of length ℓ in Q.

Definition 2.1.2.5. A two sided ideal $I \subset kQ$ is called **admissible** if there is an integer $m \geq 2$ such that:

$$R_Q^m \subset I \subset R_Q^2. \tag{2.1.2.2}$$

If such condition holds, the pair (Q, I) is called a **bound quiver**, and the quotient kQ/I is called a **bound quiver algebra**.

The condition $R_Q^m \subset I$ on the ideal I is given so that R_Q is a nilpotent ideal inside kQ/I (making kQ/I is finite-dimensional), while the condition $I \subset R_Q^2$ is given so that no arrows of Q are cut when taking the quotient (making the bound quiver algebra connected, provided Q is connected).

Example 2.1.2.6. Consider the quiver:

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \circlearrowleft \gamma$$

and let $I = (\alpha\beta, \gamma^3)$. Then the algebra kQ/I is finite dimensional, and has basis given by:

$$\{e_1, e_2, e_3, \alpha, \beta, \gamma, \beta\gamma, \gamma^2, \beta\gamma^2\}.$$

For an ideal $I \subset kQ$, define the category $\mathbf{rep}(Q, I)$ of finite dimensional representations of kQ/I, i.e., representations of Q which satisfy the relations induced by the ideal I.

The next result shows that this category is equivalent to the category of finitely generated (left) kQ/I-modules, and will thus allow us to work with the more familiar language coming from algebras and modules.

Theorem 2.1.2.7. [60, §5.2] Let $I \subset kQ$ an admissible ideal, and $A = kQ/I$. Then there is an equivalence of categories between the category mod A of finitely generated left A-modules and the category rep(Q, I) of finite-dimensional bound quiver representations:

$$\text{mod } A \cong \text{rep } (Q, I).$$

The following theorem shows why the study of the category mod kQ/I of finite dimensional (left) modules over a bound quiver algebra kQ/I is important.

Theorem 2.1.2.8. [60, §5.1] Let A be a finite dimensional associative k-algebra. Then there is a finite quiver Q, an admissible ideal $I \subset kQ$ and an equivalence of categories between the category of finitely generated (left) A-modules and the category of finitely generated (left) kQ/I-modules:

$$\text{mod } A \cong \text{mod } kQ/I.$$

Therefore, understanding the representation theory of quivers with relations implies the understanding of the representation theory of all finite-dimensional associative algebras.

Modules that play a prominent role in the representation theory of quivers are simples, projectives and injectives.

Definition 2.1.2.9. Let A be an algebra.

- $S \in \text{mod } A$ is called **simple** if its only submodules are 0 and S itself.

- $P \in \text{mod } A$ is called **projective** if the functor $\text{Hom}(P, -)$ is exact.
- $I \in \text{mod } A$ is called **injective** if the functor $\text{Hom}(-, I)$ is exact.

Remark 2.1.2.10. Given a (left) A-module M, its dual $M^* = \text{Hom}_k(M, k)$ has a structure of (right) A-module, given by:

$$(f.a)(b) = f(ab)$$

for all $f \in M^*$, $a, b \in M$.

We now define three classes of kQ/I-modules, that turn out to be a complete set of simple, projective and injective indecomposable kQ/I-modules.

Definition 2.1.2.11. Let $I \subset kQ$ be an admissible ideal, $A = kQ/I$. Fix an element $i \in Q_0$. Define:

- $S(i)$ to be the one dimensional module generated by an element s_i, where the action of A on $S(i)$ is induced by the following action of kQ on $S(i)$:

$$c \cdot s_i = \begin{cases} s_i, & \text{if } c = e_i \\ 0, & \text{otherwise} \end{cases}, \qquad \text{for all paths } c \in kQ;$$

- $P(i) = Ae_i$;
- $I(i) = Hom_k(e_i A, k)$.

Proposition 2.1.2.12. [60, §2.2, §5.3] Let $I \subset kQ$ be an admissible ideal, and $A = kQ/I$. Then:

- The set $\{S(i), i \in Q_0\}$ is a complete set of simple modules of A;
- The set $\{P(i), i \in Q_0\}$ is a complete set of indecomposable projective modules of A;
- The set $\{I(i), i \in Q_0\}$ is a complete set of indecomposable injective modules of A.

The importance of projective and injective modules lies in the fact that they can be used to produce approximations of arbitrary modules, as stated by the following proposition.

Proposition 2.1.2.13. Let A be an associative algebra, and let $M \in \text{mod } A$. Then there exist projective (resp. injective) A-modules P_i (resp. I_i), $i \geq 0$ and long exact sequences

$$\ldots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0 \qquad (2.1.2.3)$$

$$0 \to M \to I_0 \to I_1 \to I_2 \to I_3 \to \dots \tag{2.1.2.4}$$

Any exact sequence as in (2.1.2.3) (resp. (2.1.2.4)) is called **projective resolution** (resp. **injective resolution**) of M.

Example 2.1.2.14. Let

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3,$$

and set $A = kQ$. Then we can write the modules $S(i), P(i), I(i)$ as follows:

$$S(1) = 1 \qquad P(1) = 1 \qquad I(1) = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$S(2) = 2 \qquad P(2) = \begin{matrix} 1 \\ 2 \end{matrix} \qquad I(2) = \begin{matrix} 2 \\ 3 \end{matrix}$$

$$S(3) = 3 \qquad P(3) = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad I(3) = 3$$

For example, we denote the projective module $P(3) = Ae_3$ as $\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$, since a basis of $P(3)$ is given by $\{\alpha\beta, \beta, e_3\}$, and the start point of such paths is 1,2,3, respectively.

Definition 2.1.2.15. Let A be an associative finite dimensional k-algebra.

- For $M \in \text{mod } A$ define the **projective dimension** pd M of M as the smallest non-negative integer d such that there exists a projective resolution of the form:

  $$0 \to P_d \to P_{d-1} \to \dots \to P_1 \to P_0 \to M \to 0.$$

  If M does not admit a finite projective resolution, we set pd $M = \infty$.

- The **global dimension** of A is defined as:

  $$\text{gldim } A = \sup\{\text{pd } M | M \in \text{mod } A\}.$$

If gldim $A \leq 1$, the algebra A is called **hereditary**.

If $A = kQ$, then the indecomposable projective and injective modules defined above allow us to give standard projective and injective resolutions of length one of any $M \in \text{mod } A$.

Theorem 2.1.2.16. [60, §2.2] Suppose that Q is a connected and acyclic quiver. Let $A = kQ$, $M \in \text{mod } A$. Then there exist projective (rep. injective) modules

$P_0, P_1 \in \text{mod } A$ (resp. $I_0, I_1 \in \text{mod } A$) and A-module homomorphisms $p_1 : P_1 \to P_0$, $p_0 : P_0 \to M$ (resp. $i_0 : M \to I_0$, $i_1 : I_0 \to I_1$) such that the following sequence is exact:

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0 \quad (\text{resp. } 0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \longrightarrow 0).$$

Therefore kQ is hereditary.

More is true: if an hereditary algebra A also satisfies two additional conditions, then it arises as the path algebra of a quiver. We define the required conditions, and then state the theorem.

First of all, we introduce the concept of basic algebra.

Recall that an **idempotent** in A is an element $e \in A$ such that $e^2 = e$.

Definition 2.1.2.17.

- Two idempotent elements $e, f \in A$ are called **orthogonal** if $ef = 0 = fe$.
- An idempotent element $e \in A$ is called **primitive** if it cannot be written as $e = e_1 + e_2$, with $e_1, e_2 \in A$ nonzero orthogonal idempotents.
- The algebra A is called **basic** if for every set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ in A such that $e_1 + \ldots + e_n = 1$, we have:

$$Ae_i \cong Ae_j \iff i = j.$$

Definition 2.1.2.18. Let S be a ring. A **connected** S-algebra is a $\mathbb{Z}_{\geq 0}$-algebra $A = \oplus_{n \geq 0} A(n)$ such that $A(0) \cong S$.

The next result is the content of [3, §7, Thm. 1.7].

Theorem 2.1.2.19. Let A be a basic, connected, hereditary finite dimensional algebra. Then there exists a finite, acyclic, connected quiver Q such that:

$$A \cong kQ.$$

2.1.3 Auslander-Reiten theory and Gabriel's theorem

Let Q be a finite quiver without oriented cycles, and $A = kQ$. We want to study the category mod A of left A-modules. In order to do this, it is useful to introduce some concepts of Auslander-Reiten theory.

Definition 2.1.3.1. Define the following functors.

- The contravariant functor $D : \text{Hom}_k(-, k) : \text{mod } A \to \text{mod } A^{\text{op}}$ is called **duality functor**.

- The covariant functor $\nu = D\mathrm{Hom}_A(-, A) : \mathrm{mod}\ A \to \mathrm{mod}\ A$ is called **Nakayama functor**.

Proposition 2.1.3.2. [60, §2.3.2] Let proj A (resp. inj A) be the category of finitetly generated projective (resp. injective) left A-modules. Then:

$$\nu : \mathrm{proj}\ A \to \mathrm{inj}\ A$$

is an equivalence of categories, with quasi-inverse given by:

$$\nu^{-1} = \mathrm{Hom}(DA^{\mathrm{op}}, -) : \mathrm{inj}\ A \to \mathrm{proj}\ A.$$

Furthermore,

$$\nu P(i) = I(i), \qquad \nu^{-1} I(i) = P(i) \qquad \text{for all } i \in Q_0.$$

Proposition 2.1.3.3. [60, §2.3.2] The functor ν is right exact, that is, for every exact sequence of A-modules

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

then the sequence:

$$\nu L \xrightarrow{\nu f} \nu M \xrightarrow{\nu g} \nu N \to 0$$

is exact. Dually, ν^{-1} is left exact.

Proposition 2.1.3.3 leads to the construction of the Auslander-Reiten translate and inverse Auslander-Reiten translate.

Definition 2.1.3.4. Let $M \in \mathrm{mod}\ A$ be indecomposable. Let:

$$0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

be a minimal projective resolution of M. Define the **Auslander-Reiten translate** of M as:

$$\tau M = \ker \nu p_1.$$

Dually, if

$$0 \to M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \to 0$$

is a minimal injective resolution of M, define the **inverse Auslander-Reiten translate** of M as:

$$\tau^{-1} M = \mathrm{coker}\ \nu^{-1} i_1.$$

Remark 2.1.3.5. It follows directly from the definition that, if M is a projective (resp. injective) module, then $\tau M = 0$ (resp. $\tau^{-1} M = 0$).

Example 2.1.3.6. Let

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3,$$

and consider the simple module $S(3) = 3$ at vertex 3. It is easy to check that a projective resolution of length 1 of $S(3)$ is given as follows:

$$0 \to \underbrace{\begin{matrix} 1 \\ 2 \end{matrix}}_{P(2)} \to \underbrace{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}}_{P(3)} \to 3 \to 0.$$

Applying ν to this sequence yields the exact sequence:

$$\underbrace{\begin{matrix} 2 \\ 3 \end{matrix}}_{I(2)} \to \underbrace{3}_{I(3)} \to 0 \to 0,$$

and this can be completed to a short exact sequence as follows.

$$0 \to 2 \to \underbrace{\begin{matrix} 2 \\ 3 \end{matrix}}_{I(2)} \to \underbrace{3}_{I(3)} \to 0$$

Therefore $\tau S(3) = 2$.

Also, since $S(3) = 3 = I(1)$, then it is injective, and thus $\tau^{-1} S(3) = 0$.

Definition 2.1.3.7. Let $M, N \in \operatorname{mod} A$. A morphism $f : M \to N$ is called **left minimal almost split** if:

- f is not a section, i.e., there is no morphism $h : N \to M$ such that $hf = \mathrm{id}_M$;
- for every A-module morphism $l : M \to L$ that is not a section there is a unique A-module morphism $l' : N \to L$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle l}\downarrow & \swarrow {\scriptstyle \exists l'} & \\ L & & \end{array}$$

- if $h : M \to M$ is such that $hf = f$, then h is an automorpshism.

  Right minimal almost split morphisms are defined dually.

Definition 2.1.3.8. A short exact sequence in $\operatorname{mod} A$

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called **almost split** (or **Auslander-Reiten sequence**) if f is left minimal almost split and g is right minimal almost split.

Definition 2.1.3.9. A morphism $f : M \to N$ in mod A is called **irreducible** if:

- f is not a section;
- f is not a retraction;
- if $f = gh$ for some A-module morphisms $h : M \to L$ and $g : L \to N$, then either g is a retraction or h is a section.

The importance of almost split sequences and irreducible morphisms is revealed by the following proposition.

Proposition 2.1.3.10. [60, §7.1] A short exact sequence in mod A

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is almost split if and only if L, N are indecomposable and f, g are irreducible morphisms. Moreover, in this situation, $N \cong \tau^{-1}L$.

Definition 2.1.3.11. For $M, N \in \text{mod } A$, define:

$$\text{rad}(M, N) = \{f \in \text{Hom}(M, N) | f \text{ is neither a section nor a retraction}\}.$$

Denote by $\text{rad}^2(M, N)$ the set of morphisms $f : M \to N$ such that $f = gh$, with $g \in \text{rad}(L, N)$, $h \in \text{rad}(M, L)$ for some $L \in \text{mod } A$.

Proposition 2.1.3.12. [60, §7.1] Let $M, N \in \text{mod } A$ be indecomposable. Then $f : M \to N$ is irreducible if and only if $f \in \text{rad}(M, N) \setminus \text{rad}^2(M, N)$.

We are now able to give the definition of Auslander-Reiten quiver.

Definition 2.1.3.13. Let $A = kQ$, where Q is connected and without oriented cycles. The **Auslander-Reiten quiver** Γ_Q of A is the quiver with:

- set of vertices given by the isoclasses of indecomposable A-modules.
- if M, N are indecomposable A-modules, then the number of arrows $M \to N$ is given by $\dim \big(\text{rad}(M, N)/\text{rad}^2(M, N)\big)$.

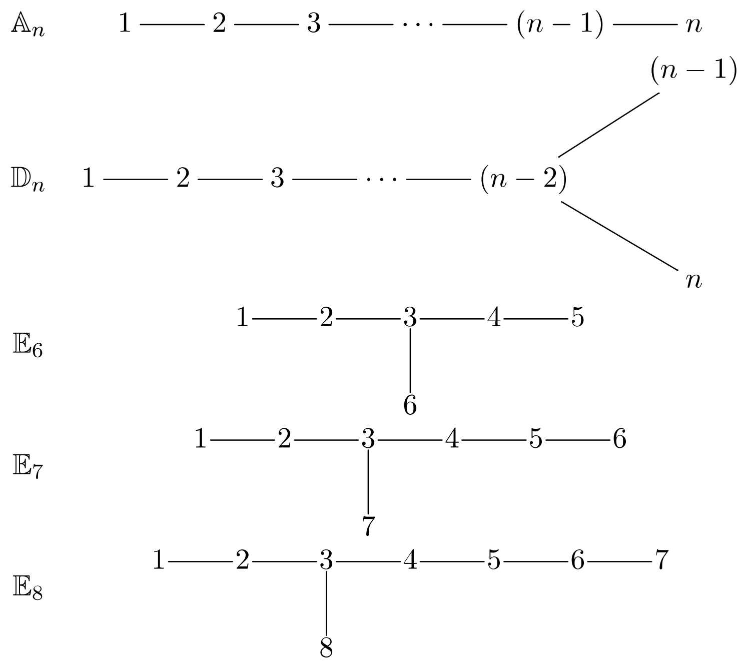
The Auslander-Reiten quiver of $A = kQ$ therefore encodes the representation theory of A; indeed, it contains all the information about the "building blocks" of modules (indecomposable A-modules), and "building blocks" of morphisms between modules (irreducible morphisms).

A natural question to ask is when the Auslander-Reiten quiver of A is finite. This depends just on the shape of Q, and a complete classification is given by Gabriel's theorem.

Before stating it, we give the definition of quiver of finite representation type.

Definition 2.1.3.14. Let Q be connected and without oriented cycles, $A = kQ$. Then Q is called of **finite representation type** if its Auslander-Reiten quiver is finite or, equivalently, if the number of isoclasses of indecomposable A-modules is finite.

Gabriel's Theorem 2.1.3.15. [60, §8.4] Let Q be a connected quiver. Then Q is of finite representation type if and only if its underlying graph is one of the following.



In each one of these cases, there is an algorithm that allows to compute the Auslander-Reiten quiver.

Definition 2.1.3.16. Let $Q_0 = \{1, \ldots, n\}$, $A = kQ$. The **dimension vector** of $M \in \operatorname{mod} A$ is given by:

$$\underline{\dim}\, M = \begin{pmatrix} \dim Me_1 \\ \dim Me_2 \\ \vdots \\ \dim Me_n \end{pmatrix}.$$

Theorem 2.1.3.17. [2, Thm. 5.2.2] Let Q be a quiver of finite representation type, $A = kQ$. Let $M, N \in \operatorname{mod} A$ be indecomposable such that $\underline{\dim}\, M = \underline{\dim}\, N$. Then $M \cong N$.

Therefore, if Q is of finite representation type, the dimension vector identifies uniquely the corresponding indecomposable kQ-module.

Now, let Q be a quiver of type $\mathbb{A}_n$, that is, the underlying graph of Q is given by:

$$1 \text{———} 2 \text{———} \cdots \text{———} (n-1) \text{———} n.$$

We give an algorithm that computes the Auslander-Reiten quiver of Q.

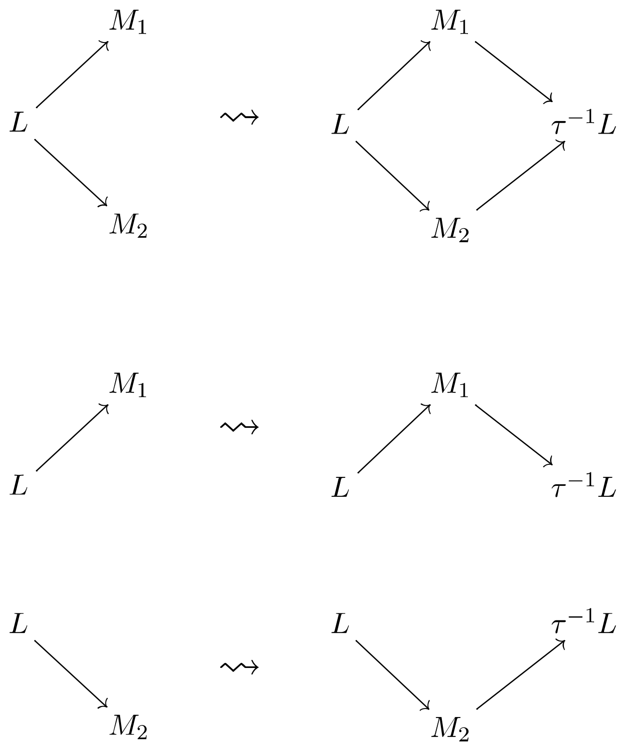
Starting from Q, we define a new quiver Γ_Q through the following algorithm, called **knitting algorithm** (see [60, §3.1.1] for a more exhaustive treatment).

1. Compute the indecomposable projective modules

$$P(1), P(2), \ldots, P(n).$$

2. For all arrows $i \to j$ in Q, add an arrow $P(i) \to P(j)$ in Γ_Q.

3. (Knitting) There are three types of mesh.



Complete each one of them as shown above in a way such that:

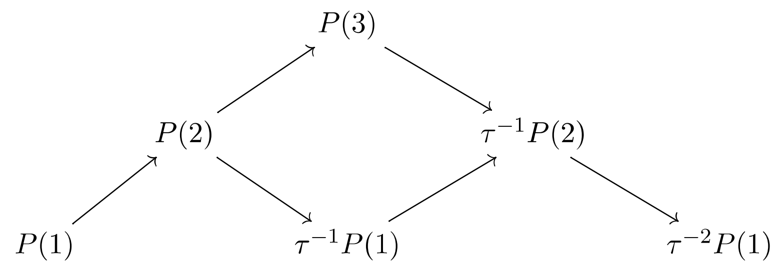
$$\underline{\dim}\ L + \underline{\dim}\ \tau^{-1}L = \sum_{i=1}^{2} \underline{\dim}\ M_i$$

4. Repeat step 3 until you get negative integers in the dimension vectors.

The inverse Auslander-Reiten translate of an A-module M can be computed as stated in Definition 2.1.3.4.

Proposition 2.1.3.18. Let Q be a quiver of type $\mathbb{A}_n$. Then the knitting algorithm computes the Auslander-Reiten quiver Γ_Q of Q.

Example 2.1.3.19. Let $Q = 1 \longrightarrow 2 \longrightarrow 3$. Following the knitting algorithm one gets that the Auslander-Reiten quiver Γ_Q of Q is given by:



Explicitly, the indecomposable A-modules that are vertices of Γ_Q are given by:

$$P(1) = 1, \quad P(2) = \begin{array}{c} 1 \\ 2 \end{array}, \quad P(3) = I(1) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array},$$

$$\tau^{-1}P(1) = 2, \quad \tau^{-2}P(1) = I(3) = 3, \ \tau^{-2}P(2) = I(2) = \begin{array}{c} 2 \\ 3. \end{array}$$

2.1.4 Preprojective algebras

We are now almost ready to present two equivalent definitions of preprojective algebras. The first one should make it clear why these algebras are relevant from a representation theory point of view, while the second one allows to make explicit computations on preprojective algebras.

Throughout this subsection, Q will be a connected quiver without oriented cycles, and $A = kQ$ its path algebra.

Definition 2.1.4.1. Let $M \in \text{mod } A$. Then M is called **preprojective** if there exists a projective A-module P such that:

$$M \cong \tau^{-n}P$$

for some $n \geq 0$.

Remark 2.1.4.2. Suppose Q is of finite representation type.

In subsection 2.1.3 we said that there is an algorithm that computes the Auslander-Reiten quiver Γ_Q of Q. In the algorithm, all the vertices are obtained by repeatedly applying τ^{-1} to indecomposable projective modules. Hence, in this case all indecomposable A-modules are preprojective. The converse also holds: if $A = kQ$ is such that every indecomposable A-module is preprojective, then Q is of finite representation type (see [56, Thm 1.7], [4, Prop. VIII.1.13] or [31, Prop. 6.4]).

We now give the first definition of preprojective algebra (see [7, §4.1]).

Definition 2.1.4.3. The **preprojective algebra** of A is the graded algebra

$$B = B^{(0)} \oplus B^{(1)} \oplus B^{(2)} \oplus \dots,$$

where $B^{(n)} = \mathrm{Hom}_A(A, \tau^{-n}A)$ for all $n \geq 0$, and the multiplication $B^{(n)} \times B^{(m)} \to B^{(n+m)}$ is defined by $(u, v) \to uv$, where uv is the composite map given by:

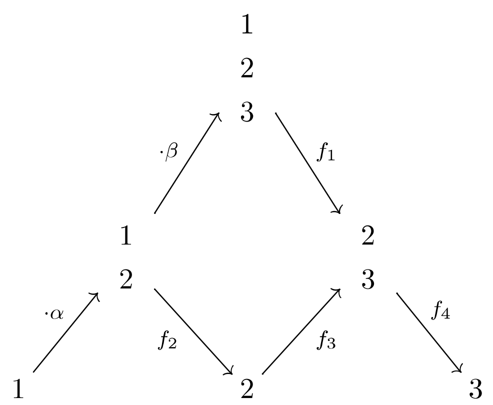
$$A \xrightarrow{u} \tau^{-n}A \xrightarrow{\tau^{-n}v} \tau^{-m-n}A.$$

Remark 2.1.4.4. The algebra B defined above is isomorphic as an A-module to the direct sum of one copy of each indecomposable preprojective A-module. Therefore, by Remark 2.1.4.2, if A is of finite representation type (meaning that Q is of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ by Gabriel's Theorem 2.1.3.15), then the associated preprojective algebra is finite dimensional.

Example 2.1.4.5. Let Q be the $\mathbb{A}_3$ quiver with arrows oriented from left to right, $A = kQ$.

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

If we explicitly write down the maps in the Auslander-Reiten quiver from Example 2.1.3.19, we get that Γ_Q is given by



where $\cdot\alpha$ and $\cdot\beta$ are left multiplication by α and β, f_i are left A-module morphisms such that $f_1(e_3) = e_3$, $f_2(e_2) = e_2$, $f_3(e_2) = \beta$ and $f_4(e_3) = e_3$. Now, one can show that:

$$f_2 \cdot \alpha = 0, \qquad f_1 \cdot \beta = f_3 f_2, \qquad f_4 f_3 = 0.$$

Therefore, a basis for the preprojective algebra B is given by

$$\{id_{P(1)}, id_{P(2)}, id_{P(3)}, \cdot\alpha, \cdot\beta, \cdot\beta \cdot \alpha, f_1, f_2, f_4 f_1, f_1 \cdot \beta = f_3 f_2\}.$$

We now give the second definition of preprojective algebra.

Definition 2.1.4.6. Let $\overline{Q}$ be the **double quiver** of Q, having vertex set Q_0

and arrows $y_{ij} : i \to j$ and $y^*_{ji} : j \to i$ corresponding to each arrow $i \to j$ in Q_1.

The **preprojective algebra** of Q is the bound quiver algebra

$$\Pi = k\overline{Q}/\rho,$$

where the ideal $\rho \subseteq k\overline{Q}$ is generated by the quadratic relations:

$$\rho_i = \sum_{j \in \mathcal{N}(i)} \varepsilon_{ij} y_{ij} y^*_{ij},$$

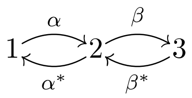
one for each $i \in Q_0$, where $\mathcal{N}(i)$ is the set of the neighbours of i in Q and

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i \to j \in Q_1 \\ -1 & \text{if } j \to i \in Q_1. \end{cases}$$

The following result can be found in [57, Cor. of Thm. A] and in [12, Thm. 0.1].

Proposition 2.1.4.7. The two definitions of preprojective algebra coincide.

Example 2.1.4.8. Consider the quiver Q of type $\mathbb{A}_3$ from Example 2.1.4.5. The quiver $\overline{Q}$ is the following.



The ideal ρ is generated by the paths $\alpha\alpha^*$, $\alpha^*\alpha - \beta\beta^*$ and $\beta^*\beta$. Hence the preprojective algebra

$$\Pi = k\overline{Q}/\rho$$

has basis:

$$\{e_1, e_2, e_3, \alpha, \beta, \alpha^*, \beta^*, \alpha\beta, \alpha^*\alpha = \beta\beta^*, \beta^*\beta\}.$$

The algebra isomorphism $\Pi \to B$ is given as follows.

$$\begin{array}{cccc} \Psi: & \Pi & \to & B \\ & e_i & \mapsto & id_{Ae_i} \\ & \alpha & \mapsto & \cdot\alpha \\ & \beta & \mapsto & \cdot\beta \\ & \alpha^* & \mapsto & f_2 \\ & \beta^* & \mapsto & f_1 \end{array}$$

2.2 Hochschild (co)homology and cyclic homology

In this chapter we introduce some homological invariants of associative algebras. In particular, we give definitions and properties that will be needed in the following about Hochschild homology, cohomology and cyclic homology of algebras.

Throughout this section, A will be assumed to be an associative (not necessarily finite-dimensional) algebra over k. Unless otherwise indicated, we will assume that tensor products are taken over k, that is, $\otimes = \otimes_k$.

2.2.1 Hochschild homology and cohomology

All the definitions and results stated in this subsection can be found in [63].

Definition 2.2.1.1. Let A^{op} be the opposite algebra of A. Define the **enveloping algebra** of A as

$$A^e = A \otimes A^{\mathrm{op}},$$

with multiplication given by:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

Definition 2.2.1.2. Let A, B be associative algebras. An A-B bimodule M is an abelian group $(M, +)$ such that:

- M is a left A-module and a right B-module.
- For all $a \in A$, $b \in B$ and $m \in M$ we have:

$$(a \cdot m) \cdot b = a \cdot (m \cdot b).$$

We also always assume that the ground field k acts centrally on M. If $A = B$, then M is called an A-bimodule.

The concepts of A^e-modules and A-bimodules are equivalent, as stated by the following remark.

Remark 2.2.1.3. Let A**-bimod** be the category of A-bimodules. Then we have an equivalence of categories:

$$A\text{-bimod} \xrightarrow{\sim} A^e\text{-mod},$$

obtained by giving to any A-bimodule M the structure of a left A^e-module as follows

$$(a \otimes b) \cdot m = amb$$

for all $a, b \in A$ and $m \in M$.

For $k \in \mathbb{Z}$, $k \geq 1$, define:

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}}.$$

It can be given the structure of an A-bimodule as follows:

$$b \cdot (a_1 \otimes \ldots \otimes a_k) \cdot c = ba_1 \otimes a_2 \otimes \ldots \otimes a_{k-1} \otimes a_k c$$

for all $b, a_1, \ldots, a_k, c \in A$.

Definition 2.2.1.4. The following complex of A-bimodules is called **bar complex**.

$$B(A): \qquad \cdots \stackrel{b'_3}{\to} A^{\otimes 4} \stackrel{b'_2}{\to} A^{\otimes 3} \stackrel{b'_1}{\to} A^{\otimes 2} \to 0, \tag{2.2.1.1}$$

where

$$b'_n(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$.

Proposition 2.2.1.5. [63, §1.1] The bar complex provides a projective A-bimodule resolution of A.

We are now ready to define the Hochschild homology and cohomology of A.

Definition 2.2.1.6. Consider the complexes:

$$C_\bullet(A) = A \otimes_{A^e} B(A),$$

$$C^\bullet(A) = \bigoplus_{n \geq 0} \operatorname{Hom}_{A^e}(B_n(A), A),$$

where the differentials for $C_\bullet(A)$ are given by $b_n = 1 \otimes_{A^e} b'_n$, and the differentials for $C^\bullet(A)$ are given by b'^*_n, where $b'^*_n(f) = f b'_n$.

The **Hochschild homology** $HH_*(A)$ and the **Hochschild cohomology** $HH^*(A)$ of A are defined by:

$$HH_n(A) = H_n(C_\bullet(A)),$$

$$HH^n(A) = H^n(C^\bullet(A)).$$

The complexes $C_\bullet(A)$ and $C^\bullet(A)$ computing the Hochschild homology and cohomology can be simplified. Before doing this we introduce some maps, that will provide a convenient way of writing the differentials for some complexes in the following.

Definition 2.2.1.7. Fix $n \geq 1$. Define the A-bimodule morphisms $d_0, \ldots, d_n : A^{\otimes n+1} \to A^{\otimes n}$ as follows:

$$\begin{aligned} d_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) =& a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \qquad i = 0, \ldots, n-1 \\ d_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) =& a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}. \end{aligned}$$

Remark 2.2.1.8. Fix $n \geq 1$. Using the maps d_i defined above, the differentials b'_n in the bar complex (2.2.1.1) can be rewritten as follows:

$$b'_n = \sum_{i=0}^{n-1} (-1)^i d_i. \tag{2.2.1.2}$$

Proposition 2.2.1.9. Fix $n \geq 1$. There is a k-module isomorphism:

$$\begin{array}{ccc} A \otimes_{A^e} B_n(A) & \xrightarrow{\sim} & A^{\otimes n+1} \\ b \otimes_{A^e} (a_0 \otimes \ldots \otimes a_{n+1}) & \mapsto & a_{n+1} b a_0 \otimes a_1 \otimes \ldots \otimes a_n \end{array}.$$

Therefore, the n-th vector space in $C_\bullet(A)$ can be identified with

$$C_n(A) \cong A^{\otimes n+1},$$

and the differential $A^{\otimes n+1} \to A^{\otimes n}$ can be expressed as:

$$\begin{aligned} b_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) =& \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &+ (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}. \end{aligned}$$

for all $a_0, \ldots, a_n \in A$.

Using the notation from Definition 2.2.1.7, the differential map b_n can be written as:

$$b_n = \sum_{i=0}^{n} (-1)^i d_i. \tag{2.2.1.3}$$

Definition 2.2.1.6 leads to specific information encoded by $HH_n(A)$ and $HH^n(A)$ when n is small. We include such information in the following proposition for $n = 0$.

Proposition 2.2.1.10. [63, §1.2] Let

$$Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$$

be the **center** of A, and

$$[A, A] = \mathrm{Span}_k\{[a, b] = ab - ba \mid a, b \in A\}$$

be the **commutator** of A.

Then we have isomorphisms:

$$HH^0(A) \cong Z(A),$$

$$HH_0(A) \cong A/[A, A].$$

However, for generic $n \geq 1$, Definition 2.2.1.6 becomes very hard to use, and rarely allows us to compute the Hochschild homology and cohomology of A.

A classical result in homological algebra (see [63, §A.3]) says that these groups can actually be computed starting from any projective A-bimodule resolution of A. Furthermore, Hochschild homology and cohomology are Morita invariant (see [62, Thm. 9.5.6] and [48, §5.11]).

Definition 2.2.1.11. Two algebras A and B are called **Morita equivalent** if there is an equivalence of additive categories between mod A and mod B.

Theorem 2.2.1.12. The following isomorphisms hold:

$$HH_n(A) \cong \mathrm{Tor}_n^{A^e}(A, A),$$

$$HH^n(A) \cong \mathrm{Ext}_{A^e}^n(A, A).$$

Furthermore, the Hochschild homology and cohomology are Morita invariant, i.e., if A and B are Morita equivalent k-algebras, then:

$$HH_n(A) \cong HH_n(B), \qquad HH^n(A) \cong HH^n(B)$$

for all $n \geq 0$.

Therefore, the strategy to compute the Hochschild homology and cohomology of an algebra A is the following:

- Find a convenient projective A-bimodule resolution $P_\bullet$ for A.
- Consider the complexes $A \otimes_{A^e} P_\bullet$ and $\mathrm{Hom}_{A^e}(P_\bullet, A)$.
- The homology of the complex $A \otimes_{A^e} P_\bullet$ gives $HH_*(A)$, while the cohomology of the complex $\mathrm{Hom}_{A^e}(P_\bullet, A)$ gives $HH^*(A)$.

2.2.2 Cyclic homology

A concept that is crucial in the computation of the Hochschild homology and cohomology of 3-preprojective algebras of type A is cyclic homology.

In this subsection we give its definition, while in the next one we show its connection with Hochschild homology. We refer the reader to [49, Chapter 2] for further details.

Definition 2.2.2.1. Let $n \geq 1$ be an integer. Define the action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on $A^{\otimes n+1}$ by letting its generator $t = t_n$ act by:

$$t_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1},$$

for all $a_0, \ldots, a_n \in A$.

Let $N = 1 + t + \ldots + t^n$ be the corresponding **norm operator** on $A^{\otimes n+1}$.

Recall that the maps b' and b are the differential maps of the bar complex and of the complex computing the Hochschild homology, respectively. Using their description given in terms of the maps d_n given in (2.2.1.2) and (2.2.1.3) one gets the following.

Lemma 2.2.2.2. [49, §2.1.1] Using Proposition 2.2.1.9, consider the Hochschild homology complex having the same underlying vector spaces as the bar complex $B(A)$. Then the operators t, N, b, b' satisfy the following identities.

$$(1-t)b' = b(1-t), \qquad b'N = Nb$$

Lemma 2.2.2.2 allows us to define a first quadrant bicomplex $CC(A)$, called the **cyclic bicomplex**.

$$\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & \\
{\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & \\
{\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle -b}\downarrow & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow{N} &
\end{array} \tag{2.2.2.1}$$

Definition 2.2.2.3. The **cyclic homology groups** $HC_n(A)$ of A, $n \geq 0$ are the homology groups of the total complex Tot $CC(A)$. Equivalently:

$$HC_n(A) := H_n(\text{Tot } CC(A)).$$

We now introduce an operator B that will prove to be crucial in the connection between cyclic and Hochschild homology.

Definition 2.2.2.4. Define the **Connes' differential** to be the map $B : A^{\otimes n+1} \to A^{\otimes n+2}$ given by:

$$\begin{aligned} B(a_0 \otimes \ldots \otimes a_n) = &\sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1} \\ &- (-1)^{ni} a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}. \end{aligned}$$

One can show that there is a contractible subcomplex in Tot $CC(A)$; using [49, Lem. 2.1.6] we can get rid of it to get a simplified version of the bicomplex (2.2.2.1), that still computes the cyclic homology of A (see [49, §2.1.7]).

Proposition 2.2.2.5. The cyclic homology of A can be obtained by computing the homology of the total complex of the following first quadrant complex:

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\ {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\ A^{\otimes 2} & \xleftarrow{B} & A^{\otimes 2} & & \\ {\scriptstyle b}\downarrow & & & & \\ A & & & & \end{array}$$

In particular, the operators B and b satisfy:

$$Bb + bB = 0. \tag{2.2.2.2}$$

Since the Connes' differential B and the Hochschild homology differential b commute by (2.2.2.2), we have the following.

Corollary 2.2.2.6. The Connes' differential B induces on the Hochschild homology a homomorphism:

$$B_n : HH_n(A) \to HH_{n+1}(A)$$

for all $n \geq 0$.

As a consequence of the definition of $CC(A)$ and of cyclic homology, one can show another connection between Hochschild and cyclic homology (see [49, Thm. 2.2.1]).

Theorem 2.2.2.7. There is a long exact sequence:

$$\ldots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{T} HC_{n-2}(A) \xrightarrow{D} HH_{n-1}(A) \xrightarrow{I} \ldots.$$

2.2.3 Reduced Hochschild and cyclic homology

The results of the previous subsection can all be translated to the setting of reduced Hochschild and cyclic homology.

Throughout this subsection, suppose that the algebra A is $\mathbb{Z}_{\geq 0}$-graded, that is,

$$A = \bigoplus_{n \geq 0} A(n),$$

with $A(n)A(m) \subset A(n+m)$ for all $n, m \geq 0$.

Definition 2.2.3.1. Define the **reduced Hochschild homology** (resp. **reduced cyclic homology**) of A as:

$$\overline{HH}_n(A) := \frac{HH_n(A)}{HH_n(A(0))} \qquad (\text{resp. } \overline{HC}_n(A) := \frac{HC_n(A)}{HC_n(A(0))}).$$

If the subalgebra $A(0)$ of A is semisimple, we can explicitly compute both its Hochschild and cyclic homology and, as a consequence, reduce the computation of $\overline{HH}_*(A)$ and $\overline{HC}_*(A)$ to the computation of $HH_*(A)$ and $HC_*(A)$, respectively.

Proposition 2.2.3.2. Let S be a semisimple algebra. Then:

$$HH_n(S) \cong \begin{cases} S, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}$$

$$HC_n(S) \cong \begin{cases} S, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By [49, Thm. 1.2.13], the Hochschild homology of an algebra A can be computed from a simplified version of the complex given in Proposition 2.2.1.9. This is obtained by substituting the tensor product $\otimes_k$ over the ground field k with the tensor product $\otimes_S$ over S, and the differential maps are defined in the same way. Therefore, using the isomorphisms

$$\begin{array}{ccc} S^{\otimes_S n} & \to & S \\ a_1 \otimes_S \ldots \otimes_S a_n & \mapsto & a_1 \ldots a_n \end{array}$$

for all $n \geq 1$, we have that the Hochschild homology of S can be deduced by computing the homology of the following complex:

$$\ldots \xrightarrow{0} S \xrightarrow{1} S \xrightarrow{0} S \xrightarrow{1} S \xrightarrow{0} S \to 0,$$

where 1 is the identity map and 0 is the zero map. This complex has zeroth homology group isomorphic to S, while all higher homology groups vanish. Therefore we get the statement for $HH_n(S)$.

Now, consider the exact sequence from Theorem 2.2.2.7. Then, if we insert the values of $HH_i(S)$ just obtained, we get that this sequence is given by:

$$\begin{aligned} \ldots \to \underbrace{HH_3}_{=0} \to HC_3 \to HC_1 \to \underbrace{HH_2}_{=0} \to HC_2 \to HC_0 \to \underbrace{HH_1}_{=0} \\ \to HC_1 \to 0 \to \underbrace{HH_0}_{=S} \to HC_0 \to 0. \end{aligned}$$

Therefore, we have $HC_0(S) \cong S$, $HC_1(S) = 0$ and $HC_n(S) \cong HC_{n-2}(S)$ for all $n \geq 2$. Hence we get the statement for $HC_n(S)$. □

An immediate consequence of Proposition 2.2.3.2 is the following.

Corollary 2.2.3.3. Suppose that $A(0)$ is a semisimple subalgebra of A. Then:

$$\overline{HH}_n(A) = \begin{cases} HH_0(A)/A(0), & \text{if } n = 0 \\ HH_n(A), & \text{if } n > 0 \end{cases}$$

$$\overline{HC}_n(A) = \begin{cases} HC_n(A)/A(0), & \text{if } n \text{ is even} \\ HC_n(A), & \text{if } n \text{ is odd.} \end{cases}$$

A fundamental reason for the introduction of reduced Hochschild and cyclic homology is that the reduced analogue of the exact sequence in Theorem 2.2.2.7 can be simplified. Indeed, by [49, Prop. 5.3.12] we have the following.

Proposition 2.2.3.4. Suppose that k contains $\mathbb{Q}$ and $A(0)$ is semisimple over k. Consider the reduced analogue of the exact sequence given in Theorem 2.2.2.7.

$$\ldots \to \overline{HH_n}(A) \xrightarrow{I} \overline{HC_n}(A) \xrightarrow{T} \overline{HC_{n-2}}(A) \xrightarrow{D} \overline{HH_{n-1}}(A) \xrightarrow{I} \ldots \quad (2.2.3.1)$$

Then the maps T are 0, and (2.2.3.1) breaks into the following short exact sequences:

$$0 \to \overline{HC}_{n-1}(A) \xrightarrow{D_{n-1}} \overline{HH}_n(A) \xrightarrow{I_n} \overline{HC}_n(A) \to 0, \qquad n \geq 0. \quad (2.2.3.2)$$

If we glue the short exact sequences (2.2.3.2) together, we get the following long exact sequence:

$$0 \to \overline{HH}_0(A) \xrightarrow{D_0 \circ I_0} \overline{HH}_1(A) \xrightarrow{D_1 \circ I_1} \overline{HH}_2(A) \xrightarrow{D_2 \circ I_2} \overline{HH}_3(A) \to \ldots, \quad (2.2.3.3)$$

where

$$\ker(D_{n+1} \circ I_{n+1}) = \operatorname{im}(D_n \circ I_n) \cong \overline{HC}_n(A).$$

Remark 2.2.3.5. Using an equivalent definition of cyclic homology we have not presented here, one can check (see [49, §2.2.5]) that the differential $D_n \circ I_n$ is equal to the Connes' differential we introduced in Definition 2.2.2.4. Therefore, in the following we will write B_n for $D_n \circ I_n$, and call it Connes' differential.

The formula for B_n given in Definition 2.2.2.4 also implies that it is degree-preserving.

Definition 2.2.3.6. The sequence (2.2.3.3) is called **Connes' exact sequence**.

2.3 Frobenius, selfinjective and almost Koszul algebras

In this section we describe some fundamental structural properties that the algebras we study in this thesis enjoy.

2.3.1 Frobenius and selfinjective algebras

In this subsection, we introduce Frobenius and selfinjective algebras, and highlight their connection. The definitions and results, unless otherwise stated, are taken from [28] and [29].

Throughout this subsection, A will be an associative finite dimensional k-algebra. Let $\mathcal{S}(A)$ be a complete set of isoclasses of simple left A-modules.

Definition 2.3.1.1. The algebra A is called **selfinjective** if it is injective in $\operatorname{mod} A$.

There is an equivalent definition of selfinjective algebra, that uses the concepts of projective cover and socle of a module.

Definition 2.3.1.2. Let $M \in \operatorname{mod} A$.

- A **projective cover** of M is a pair $(P(M), p)$ with $P(M) \in \operatorname{mod} A$ projective and $p : P(M) \to M$ a superfluous epimorphism, i.e., an epimorphism such that, if $\ker p + H = P(M)$ for some $H \in \operatorname{mod} A$, then $H = P(M)$.
- The **socle** $\operatorname{Soc}(M)$ of a module M is the largest submodule of M generated by simple modules or, equivalently, the largest semisimple submodule of M.

Theorem 2.3.1.3. The following are equivalent.

1. The algebra A is selfinjective.
2. The map
$$\begin{array}{rccc} \nu : & \mathcal{S}(A) & \to & \mathcal{S}(A) \\ & [S] & \mapsto & [\operatorname{Soc}(P(S))] \end{array}$$
defines a permutation of the set $\mathcal{S}(A)$.

The permutation ν from Theorem 2.3.1.3 is called **Nakayama permutation** of A.

Definition 2.3.1.4. The algebra A is called **Frobenius** if there exists a linear form $\pi : A \to k$ such that $\ker \pi$ does not contain any nonzero left ideals.

Given any linear form $\pi : A \to k$, consider the bilinear form:

$$\begin{array}{rccc} (.,.)_\pi : & A \times A & \to & k \\ & (a,b) & \mapsto & \pi(ab). \end{array}$$

The bilinear form $(.,.)_\pi$ is **associative**, i.e., $(ax,b)_\pi = (a,xb)_\pi$ for all $a,b,x \in A$.

We can rewrite the defining property of a Frobenius algebra A in terms of an associative bilinear form on A as follows.

Proposition 2.3.1.5. The algebra A is Frobenius if and only if there exists a non-degenerate associative bilinear form $(.,.) : A \times A \to k$.

Proposition 2.3.1.6. Let A be a Frobenius algebra, and $(.,.) : A \times A \to k$ a bilinear form as in Proposition 2.3.1.5. Then there exists an automorphism $\eta : A \to A$, called **Nakayama automorphism** of A such that:

$$(x,y) = (y, \eta(x))$$

for all $x, y \in A$.

Frobenius and selfinjective algebras are related as stated in the following theorem.

Theorem 2.3.1.7. Suppose that A is selfinjective and basic. Then A is Frobenius.

The Nakayama permutation and the Nakayama automorphism are deeply related. We examine this relation in the setting of bound quiver algebras.

Remark 2.3.1.8. Let Q be a connected quiver and $I \subset kQ$ an admissible ideal. Let $A = kQ/I$ be selfinjective, and $\nu : \mathcal{S}(A) \to \mathcal{S}(A)$ the associated Nakayama permutation.

By Proposition 2.1.2.12, the simple A-modules are in bijection with the vertices of Q. Therefore, the Nakayama permutation ν can be identified with a permutation

$$\nu : Q_0 \to Q_0.$$

Proposition 2.3.1.9. Let Q be a connected quiver and $I \subset kQ$ an admissible ideal. Let $A = kQ/I$ be selfinjective, and $\nu : Q_0 \to Q_0$ the associated Nakayama permutation.

By Theorem 2.3.1.7, A is Frobenius. Furthermore, the Nakayama automorphism $\eta : A \to A$ satisfies:

$$\eta(e_i) = e_{\nu(i)}$$

for all $i \in Q_0$. Equivalently, the Nakayama automorphism restricts to the Nakayama permutation on the set of vertices of Q.

2.3.2 (Twisted) periodic Frobenius algebras

In this subsection we study graded Frobenius algebras that have a periodic projective bimodule resolution.

Fix a finite-dimensional associative $\mathbb{Z}_{\geq 0}$-graded k-algebra

$$A = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A(i),$$

that is, $A(i)A(j) \subset A(i+j)$ for all $i, j \geq 0$.

Definition 2.3.2.1. Define the category **grmod** A of **left graded A-modules** as follows.

- The objects are left A-modules M such that there is a decomposition

$$M = \bigoplus_{j \in \mathbb{Z}} M(j),$$

  where $A(i)M(j) \subset M(i+j)$. for all $i \geq 0, j \in \mathbb{Z}$.

- The morphisms are left A-module morphisms $f : M \to N$ such that:

$$f(M(i)) \subset N(i)$$

  for all $i \in \mathbb{Z}$.

The category **grbimod** A of graded A-bimodules is defined analogously.

Given $M \in$ grmod A (resp. $M \in$ grbimod A) and $r \in \mathbb{Z}$, define $M[r] \in$ grmod A (resp. $M[r] \in$ grbimod A), called **shift** of M by r, by:

$$M[r](i) = M(i-r)$$

for all $i \in \mathbb{Z}$.

Proposition 2.3.2.2. Let $M \in$ grbimod A. Then the spaces $M^* = \mathrm{Hom}_k(M, k)$ (resp. $\mathrm{Hom}_{A^e}(M, A)$) have a structure of graded right A-module (resp. of graded A-bimodule) given by:

$$M^*(i) = M(-i)^*, \qquad (\text{resp. } \mathrm{Hom}_{A^e}(M, A)(i) = \mathrm{Hom}_{A^e}(M(-i), A))$$

for all $i \in \mathbb{Z}$.

Definition 2.3.2.3. Let $M \in$ mod A, and ϕ be an automorphism of A. Define ${}_\phi M$ to be the left A-module with underlying vector space M, and action twisted by ϕ, i.e.:

$$a \cdot m := \phi(a)m$$

for all $a \in A$ and $m \in M$.

If M is a right A-module, then M_ϕ is defined dually, with action twisted by ϕ on the right.

Definition 2.3.2.4. Let $n \geq 0$ be an integer, $r \in \mathbb{Z}$, and ϕ be an automorphism of A. The algebra A is called **twisted periodic** of **twisted period** n and **twisted shift** r with **twist** ϕ if there exist graded projective A-bimodules $P_0, \ldots, P_{n-1}$ and graded bimodule maps $\iota_0, f_1, \ldots, f_{n-1}, \iota_1$ such that the following sequence is exact:

$$0 \to A_\phi[r] \xrightarrow{\iota_1} P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{\iota_0} A \to 0. \tag{2.3.2.1}$$

If $\phi = \mathrm{id}_A$, we call the algebra A **periodic** of **period** n and **shift** r.

Remark 2.3.2.5. Using the notation from Definition 2.3.2.4, suppose that $\phi = \mathrm{id}_A$, i.e., that A is a periodic algebra of period n and twist r. Then, if we shift (2.3.2.1) by r, we get the following exact sequence:

$$0 \to A[2r] \to P_{n-1}[r] \to P_{n-2}[r] \to \ldots \to P_1[r] \to P_0[r] \to A[r] \to 0.$$

If we iterate this process, i.e., if we shift (2.3.2.1) by $2r, 3r, \ldots$, and glue these exact sequences together, then we get the following projective A-bimodule resolution of A:

$$\begin{aligned} \ldots \to P_0[2r] \xrightarrow{f_n[r]} P_{n-1}[r] \xrightarrow{f_{n-1}[r]} P_{n-2}[r] \to \ldots \to P_1[r] \xrightarrow{f_1[r]} P_0[r] \xrightarrow{f_n} P_{n-1} \\ \xrightarrow{f_{n-1}} P_{n-2} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{\iota_0} A \to 0, \end{aligned} \tag{2.3.2.2}$$

where $f_n = \iota_1 \iota_0[r]$.

The following proposition is an immediate consequence of Remark 2.3.2.5.

Proposition 2.3.2.6. Let A be a graded periodic algebra of period n and shift r. Then:

$$HH_{m+n}(A) \cong HH_m(A)[r]$$

$$HH^{n+m}(A) \cong HH^m(A)[-r]$$

$$\overline{HC}_{m+n}(A) \cong \overline{HC}_m(A)[r]$$

for all $m \geq 1$.

Proof. Let $P_\bullet$ be the periodic projective A-bimodule resolution given in (2.3.2.2). Then the ℓ-th Hochschild homology and cohomology groups can be computed as follows:

$$HH_\ell(\Pi) \cong H_\ell(P_\bullet \otimes_{A^e} A), \qquad HH^\ell(A) \cong H^\ell(\mathrm{Hom}_{A^e}(P_\bullet, A)).$$

Since $P_\bullet$ is periodic of period n and shift r, so are the complexes $P_\bullet \otimes_{A^e} A$ and $\mathrm{Hom}_{A^e}(P_\bullet, A)$. Also, by Proposition 2.3.2.2, we have that:

$$\mathrm{Hom}_{A^e}(P_i[j], A) = \mathrm{Hom}_{A^e}(P_i, A)[-j]$$

for all i, j. Therefore we get the statements for the Hochschild homology and cohomology.

As for $\overline{HC}_*(A)$, recall that the reduced cyclic homology groups can be computed as kernels of the Connes' exact sequence by Proposition 2.2.3.4. In particular, since:

$$\overline{HH}_{m+n}(A) = HH_{m+n}(A) \cong HH_m(A)[r] = \overline{HH}_m(A)[r]$$

for all $m \geq 1$ and the Connes' differentials are degree-preserving by Remark 2.2.3.5, we have that $\overline{HC}_{m+n}(A) \cong \overline{HC}_m(A)[r]$ for all $m \geq 1$. □

Therefore, if A is periodic of period n, in order to compute the Hochschild homology, cohomology and reduced cyclic homology of A, it is enough to compute the corresponding first $n+1$ homology groups.

We now want to present some periodicity results for the Hochschild homology and cohomology of graded Frobenius twisted periodic algebras. By a result of N. Hanihara (see [40, Cor. 2.4], and [36, Thm. 1.4] for the ungraded case), the Frobenius assumption can be dropped, as stated in the following theorem.

Theorem 2.3.2.7. Let A be a twisted periodic algebra. Then A is selfinjective. Furthermore, if A is also graded and basic, then it is graded Frobenius, i.e., there is an integer $t \geq 0$ such that the linear map π defining the Frobenius structure on A is graded, given by $\pi : A[t] \to k$, where k is concentrated in degree 0.

Definition 2.3.2.8. For an $M \in \mathrm{grbimod}\ A$, define:

$$M^\vee = \mathrm{Hom}_{A^e}(M, A^e).$$

The following results can be found in the ungraded case in [25, Lem. 2.1.35].

Lemma 2.3.2.9. Let $t \geq 0$ be an integer, and A be a graded Frobenius algebra with defining graded linear form $\pi : A[t] \to k$. Let η be the associated Nakayama automorphism, and ϕ, ψ automorphisms of A. There are graded isomorphisms of A-bimodules:

$$A_\eta \cong A^*[t], \qquad {}_\phi A_\psi \cong A_{\phi^{-1}\psi}, \qquad A^\vee \cong A_{\eta^{-1}}[t], \qquad A^\vee \otimes_A A^* \cong A \cong A^* \otimes_A A^\vee.$$

Lemma 2.3.2.10. [1, Prop. 1.1.4] Let $P, M \in \mathrm{grbimod}\ A$, with P projective.

Then there is a graded isomorphism:

$$\mathrm{Hom}_{A^e}(P, M) \cong P^* \otimes_{A^e} M$$

The next two results give isomorphisms between some Hochschild homology groups and Hochschild homology and cohomology groups of a graded periodic algebra A. Their proofs are inspired by [25], where the authors give similar results for the (ungraded) stable Hochschild homology and cohomology of a Frobenius algebra.

Proposition 2.3.2.11. Let A be a graded periodic algebra of period n and shift r. Then:

$$\begin{aligned} HH_i(A) \cong & HH_{n-1-i}(A)^*[r], \qquad i = 1, \dots, n-2 \\ HH_n(A) \cong & HH_{n-1}(A)^*[2r]. \end{aligned}$$

Proof. Using the notation from Definition 2.3.2.4, we have $\phi = \mathrm{id}_A$. Let $P_\bullet^{par}$ be the exact sequence (2.3.2.1) giving the first n terms of a periodic projective A-bimodule resolution of A.

Apply the exact functor $(-)^\vee \otimes_A A^*$ to $P_\bullet^{par}$. Since A is Frobenius, we have $A^\vee \otimes_A A^* \cong A$ as graded A-bimodules by Lemma 2.3.2.9. As a consequence, the complex $(P_\bullet^{par})^\vee \otimes_A A^*$ is given by:

$$0 \leftarrow A[-r] \leftarrow P_{n-1}^\vee \otimes_A A^* \leftarrow P_{n-2}^\vee \otimes_A A^* \leftarrow \dots \leftarrow P_1^\vee \otimes_A A^* \leftarrow P_0^\vee \otimes_A A^* \leftarrow A \leftarrow 0.$$

Hence $(P_{n-1-\bullet}^{par})^\vee \otimes_A A^*[r]$ gives the first n terms of another projective A-bimodule resolution of A.

Therefore, the Hochschild homology group $HH_i(A)$, $i = 1, \dots, n-2$, can be obtained computing the i-th homology of the complex $((P_{n-1-\bullet}^{par})^\vee \otimes_A A^*) \otimes_{A^e} A[r]$.

Notice that, for $i = 0, \dots, n-1$, we have the following isomorphisms of left A-modules:

$$(P_i^\vee \otimes_A A^*) \otimes_{A^e} A \cong P_i^\vee \otimes_{A^e} A^* \cong \mathrm{Hom}_{A^e}(P_i, A^*) \cong (A \otimes_{A^e} P_i)^*,$$

where the second isomorphisms follows from Lemma 2.3.2.10 and the last isomorphisms uses the standard tensor-hom adjunction.

Thus $((P_{n-1-\bullet}^{par})^\vee \otimes_A A^*) \otimes_{A^e} A[r] \cong (A \otimes_{A^e} P_{n-1-\bullet}^{par})^*[r]$. For $i = 1, \dots, n-2$, the i-th homology group of the second complex is given by $HH_{n-1-i}(A)^*[r]$. Hence we get the first part of the statement.

In order to prove the second part notice that, since A is periodic of period n and shift r, then it is also periodic of period $2n$ and shift $2r$. Therefore we just proved that

$$HH_i(A) \cong HH_{2n-1-i}(A)^*[2r]$$

for $i = 1, \dots 2n-2$. In particular, taking $i = n$, we get the second part of the statement. □

If the algebra A is twisted periodic with twist given by an automorphism of finite order, then one can show that the Hochschild cohomology of the algebra A can be deduced from the Hochschild homology, as stated by the following proposition.

Proposition 2.3.2.12. Let $q \geq 0$, $m \geq 1$ be integers, $h \in \mathbb{Z}$. Suppose that A is a graded twisted periodic algebra, with twisted period q, twisted shift h and twist of order m given by the inverse of the Nakayama automorphism η^{-1} of A. Let $t \geq 0$ be such that $A^\vee \cong A_{\eta^{-1}}[t]$.

Then there are graded isomorphisms:

$$HH_{q+i}(A) \cong HH^{2mq-i-1}[(2m+1)h - t],$$

for all $1 \leq i < (m+1)q$.

Proof. First of all, notice that the non-negative integer t in the statement always exists by Lemma 2.3.2.9, and is given by the degree of the linear form $\pi : A \to k$ defining the graded Frobenius algebra structure on A.

By Definition 2.3.2.4 we know that the sequence (2.3.2.1), with r replaced by h, n by q and ϕ by η^{-1}, is exact. This is equivalent to saying that there exist projective A-bimodules $P_0, \dots, P_{q-1}$ and a complex of A-bimodules:

$$P_\bullet^{(1)} = 0 \to P_{q-1} \to \dots P_0 \to 0 \tag{2.3.2.3}$$

whose homology is given by $H_0(P_\bullet^{(1)}) = A$, $H_i(P_\bullet^{(1)}) = 0$ for $i = 1, \dots, q-2$ and $H_{q-1}(P_\bullet^{(1)}) = A_{\eta^{-1}}[h]$.

By repeatedly applying the exact functor $(- \otimes_{A^e} A_{\eta^{-1}})[h]$, we get the following complex, for $j \geq 1$:

$$P_\bullet^{(j)} = 0 \to P_{jq-1} \to \dots \to P_{(j-1)q} \to 0,$$

whose homology is given by $H_0(P_\bullet^{(j)}) = A_{\eta^{-(j-1)}}[(j-1)h]$, $H_i(P_\bullet^{(j)}) = 0$ for $i = 1, \dots, q-2$ and $H_{q-1}(P_\bullet^{(j)}) = A_{\eta^{-j}}[jh]$.

Therefore, if we glue the complexes $P_\bullet^{(j)}$ $(j \geq 1)$ together, we get the following projective A-bimodule resolution of A:

$$\cdots \to P_\bullet^{(2m)} \to \cdots \to P_\bullet^{(m+2)} \to P_\bullet^{(m+1)} \to P_\bullet^{(m)} \to P_\bullet^{(m-1)} \to \cdots \to P_\bullet^{(2)} \to P_\bullet^{(1)} \to A \to 0 \tag{2.3.2.4}$$

where, for each $j \geq 2$, the map $P_\bullet^{(j)} \to P_\bullet^{(j-1)}$ is obtained by composition as follows.

$$\begin{array}{ccc} P_\bullet^{(j)} & \longrightarrow & P_\bullet^{(j-1)} \\ & \searrow \quad \nearrow & \\ & A_{\eta^{-(j-1)}}[(j-1)h] & \end{array}$$

Now, consider the subcomplex $Q_\bullet$ of (2.3.2.4) given by:

$$Q_\bullet = 0 \to P_\bullet^{(m+2)} \to P_\bullet^{(m+1)} \to \dots P_\bullet^{(2)} \to 0.$$

This complex gives the first $(m+1)q$ terms of a projective A-bimodule resolution of $A_{\eta^{-1}}[h] = A_{\eta^{-1}}[t][h-t] \cong A^\vee[h-t]$, where the isomorphism holds by assumption.

Now, consider the complex:

$$0 \to P_\bullet^{(2m)} \to P_\bullet^{(2m-1)} \to \dots \to P_\bullet^{(m+1)} \to P_\bullet^{(m)} \to 0,$$

and apply the exact functor $(-)^\vee$ to it. Then we obtain the following complex:

$$0 \to (P_\bullet^{(m)})^\vee \to (P_\bullet^{(m+1)})^\vee \to \dots \to (P_\bullet^{(2m-1)})^\vee \to (P_\bullet^{(2m)})^\vee \to 0. \quad (2.3.2.5)$$

This computes the first $(m+1)q$ terms of a projective A-bimodule resolution of $A^\vee_{\eta^{-2m}}[-2mh] = A^\vee[-2mh]$, where the equality holds since η has order m. Therefore, if we shift the complex (2.3.2.5) by $c := 2mh + (h-t)$, we get the following exact complex:

$$\widetilde{Q}_\bullet = 0 \to (P_\bullet^{(m)})^\vee[c] \to (P_\bullet^{(m+1)})^\vee[c] \to \dots \to (P_\bullet^{(2m-1)})^\vee[c] \to (P_\bullet^{(2m)})^\vee[c] \to 0.$$

This gives another projective A-bimodule resolution of $A^\vee[c][-2mh] = A^\vee[h-t]$.

Hence, we have that:

$$H_i(Q_\bullet \otimes_{A^e} A) \cong H_i(\widetilde{Q}_\bullet \otimes_{A^e} A) \quad (2.3.2.6)$$

for all $i = 1, \dots, (m+1)q - 1$.

Since (2.3.2.4) is a projective A-bimodule resolution of A then, by definition of $Q_\bullet$, we have:

$$H_i(Q_\bullet \otimes_{A^e} A) \cong HH_{q+i}(A).$$

Furthermore, we have that:

$$P_i^\vee \otimes_{A^e} A = \mathrm{Hom}_{A^e}(P_i, A^e) \otimes_{A^e} A \cong \mathrm{Hom}_{A^e}(P_i, A).$$

for all $i \geq 0$. Therefore, the homology of the complex $\widetilde{Q}_\bullet \otimes_{A^e} A$ gives some

Hochschild cohomology groups of A. In particular, we have:

$$\begin{aligned} H_i(\widetilde{Q}_\bullet \otimes_{A^e} A) =& H_i(\mathrm{Hom}_{A^e}(P_{2mh-1-\bullet}, A)[c]) \cong HH^{2mq-1-i}(A)[c] \\ =& HH^{2mq-i-1}(A)[2mh + (h-t)]. \end{aligned}$$

Therefore, using the isomorphism (2.3.2.6), we get the statement. □

Remark 2.3.2.13. In the setting of Proposition 2.3.2.12, the algebra A has period qm.

Therefore, in view of Proposition 2.3.2.6, the whole Hochschild cohomology $HH^*(A)$ of A can be deduced from the computation of $HH^0(A)$, together with qm consecutive Hochschild cohomology groups $HH^i(A), \dots, HH^{i+qm-1}(A)$ for any $i \geq 1$.

Hence, by Proposition 2.3.2.12, the Hochshcild cohomology of A can be obtained by computing the Hochschild homology $HH_*(A)$, together with $HH^0(A)$.

Finally, in view of Proposition 2.3.2.11, in order to get the Hochschild homology ring $HH_*(A)$ it is enough to compute $HH_0(A)$, $HH_1(A)$,..., $HH_{\lfloor \frac{mq-1}{2} \rfloor}(A)$ and $HH_{mq-1}(A)$.

To sum up, the computation of $HH_*(A)$ and $HH^*(A)$ follows from the knowledge of the following Hochschild homology and cohomology groups:

$$HH^0(A), HH_0(A), HH_1(A), \dots, HH_{\lfloor \frac{mq-1}{2} \rfloor}(A), HH_{mq-1}(A).$$

2.3.3 Almost Koszul algebras

The first step in the computation of the Hochschild homology groups $HH_*(A)$ and Hochschild cohomology groups $HH^*(A)$ of an algebra A is giving a projective A-bimodule resolution of A.

One of the many nice properties of almost Koszul algebras, that we introduce in this subsection, is that they provide the first few terms of a minimal projective A-bimodule resolution of A.

The definition and results presented here about almost Koszul algebras are taken from [7].

Throughout this subsection, fix a $\mathbb{Z}_{\geq 0}$-graded algebra

$$A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A(n), \tag{2.3.3.1}$$

and suppose the subalgebra $S = A(0)$ of A is semisimple. We set $\otimes = \otimes_S$.

Remark 2.3.3.1. The grading (2.3.3.1) will be referred to as the **weight grading** of A, and the component $A(n)$ will be called the component of A of **weight degree** n. This should not be confused with the **homological degree**

of a vector space in a complex, that is the position of that vector space in the complex.

Definition 2.3.3.2. An algebra A is called **left almost Koszul** if there exist integers $p, q \geq 1$ such that:

1. $A(n) = 0$ for all $n > p$;

2. there is a graded complex

$$P^\bullet : 0 \to P^q \to \ldots \to P^1 \to P^0 \to 0$$

   of projective left A-modules such that each P^i is generated by its component $P^i(i)$ of weight degree i, and the only nonzero homology groups are $H_0(P^\bullet) = A(0) = S$ in weight degree 0 and $H_q(P^\bullet) = W = A(p) \otimes P^q(q)$ in weight degree $p + q$.

If these two conditions hold, we say that A is **left** (p, q)**-Koszul**.

Right almost Koszul algebras are defined similarly.

Proposition 2.3.3.3. [7, Prop. 3.4] An algebra A is left (p, q)-Koszul if and only if it is right (p, q)-Koszul.

Definition 2.3.3.4. Let $V = A(1)$. The algebra A is called **quadratic** if there exists an S-bimodule $R \subset V^{\otimes 2}$ such that

$$A \cong V^{\otimes}/(R),$$

where $V^{\otimes} = S \oplus V \oplus V^{\otimes 2} \oplus \ldots$ is the tensor algebra of V over S

For convenience, for two S-bimodules M, N, we may write MN instead of $M \otimes_S N$.

Proposition 2.3.3.5. [7, Prop. 3.7] A (p, q)-Koszul algebra A is generated by $A(0)$ and $A(1)$. Furthermore, if $q \geq 2$, then A is quadratic.

From now on, we assume that A is quadratic, that is,

$$A = V^{\otimes}/(R),$$

where $R \subset V^{\otimes 2}$.

We define S-bimodules $K^i \subset V^{\otimes i}$ recursively as follows:

$$\begin{aligned} K^0 &= S \\ K^1 &= V \\ K^2 &= R \\ K^{i+1} &= VK^i \cap K^iV, \quad i \geq 2. \end{aligned}$$

Definition 2.3.3.6. The **left Koszul complex** of A is the complex of left projective A-modules given by:

$$\ldots \to A \otimes K^n \to \ldots A \otimes K^1 \to A \otimes K^0 \to 0,$$

where the differentials $d' : A \otimes K^n \to A \otimes K^{n-1}$ are obtained taking the following composition of natural maps:

$$A \otimes K^n \hookrightarrow A \otimes VK^{n-1} \twoheadrightarrow A \otimes K^{n-1},$$

where the first map is induced by the inclusion $K^n \hookrightarrow VK^{n-1}$ and the second one is induced by the projection $A \otimes V \twoheadrightarrow A$.

Analogously, one defines the **right Koszul complex** of A as the complex of right projective A-modules given by:

$$\ldots \to K^n \otimes A \to \ldots K^1 \otimes A \to K^0 \otimes A \to 0,$$

where the differentials $d'' : K^n \otimes A \to K^{n-1} \otimes A$ are obtained taking the following composition of natural maps:

$$K^n \otimes A \hookrightarrow K^{n-1}V \otimes A \twoheadrightarrow K^{n-1} \otimes A.$$

Proposition 2.3.3.7. [7, Prop. 3.9] Let $p, q \geq 2$. An algebra A is (p, q)-Koszul if and only if:

1. $A(n) = 0$ for all $n > p$;

2. $K^m = 0$ for all $m > q$;

3. the only nonzero homologies of the left (resp. right) Koszul complex are $A(0) = S$ in homological degree 0, and $A(p) \otimes K^q$ (resp. $K^q \otimes A(p)$) in homological degree q.

In this case the left (resp. right) Koszul complex of A gives the first $q + 1$ terms of a minimal projective left (resp. right) A-module resolution of S.

Starting from the left and right Koszul complexes, it is possible to construct an A-bimodule complex, that provides the first $q + 1$ terms of a projective A-bimodule resolution of A.

Definition 2.3.3.8. For $m \geq 0$, define the following A-bimodule:

$$P(K^m) = A \otimes K^m \otimes A,$$

whose component of weight degree t is given by:

$$P(K^m)(t) = \bigoplus_{i+m+j=t} A(i) \otimes K^m \otimes A(j), \qquad t \geq 0.$$

The **Koszul bimodule complex** $P(K^\bullet, d)$ is the following projective A-bimodule complex:

$$\ldots \to P(K^n) \to \ldots \to P(K^1) \to P(K^0) \to 0. \tag{2.3.3.2}$$

The differentials $P(K^n) \to P(K^{n-1})$ are defined by $d = d_l + d_r$, where $d_l, d_r : P(K^n) \to P(K^{n-1})$ are anticommuting differentials given by $d_l = d' \otimes \mathrm{id}_A$, $d_r = (-1)^n \mathrm{id}_A \otimes d''$, where d' and d'' are the differentials in the left and right Koszul complexes, respectively.

Theorem 2.3.3.9. [7, Thm. 3.15] Let $p, q \geq 2$. Suppose that A is (p,q)-Koszul. Then the Koszul bimodule complex $(P(K^\bullet), d)$ provides the first $q+1$ terms of a minimal projective A-bimodule resolution of A.

Furthermore, the $(q+1)$-st syzygy $\Sigma = \ker(P(K^q) \to P(K^{q-1}))$ is generated by its weight degree $p+q$ component Z. More precisely, the inclusion of Z in Σ induces a left A-module isomorphism $A \otimes Z \xrightarrow{\sim} \Sigma$ and a right A-module isomorphism $Z \otimes A \xrightarrow{\sim} \Sigma$. Moreover, the canonical S-bimodule projections:

$$\pi_l : Z \to A(p) \otimes K^q \otimes A(0), \qquad \pi_r : Z \to A(0) \otimes K^q \otimes A(p)$$

are isomorphisms.

An important concept in the computation of the homology groups of the algebras A we will study in this thesis is its Hilbert series. Let

$$A = kQ/I, \tag{2.3.3.3}$$

where $I \subset kQ$ is a homogeneous ideal, graded by path length. We give the definition of Hilbert series of A.

Definition 2.3.3.10. Let $W = \oplus_{d \geq 0} W(d)$ be a $\mathbb{Z}_{\geq 0}$-graded S-bimodule. The **Hilbert series** of W is defined as the $|Q_0| \times |Q_0|$ matrix $H_W(t)$ whose (i,j)-th entry is the formal series given by:

$$(H_W(t))_{i,j} = \sum_{n \geq 0} \dim(e_i W(n) e_j) t^n.$$

The next formula is a consequence of [7, Prop. 3.14], and can be proved directly by using the left Koszul complex of A and the Euler-Poincaré principle.

Proposition 2.3.3.11. Let $A = kQ/I$ be a (p,q)-Koszul algebra, where $I \subset kQ$ is a quadratic ideal. Let P be the $|Q_0| \times |Q_0|$ matrix with (i,j)-th entry equal to the multiplicity of the i-th simple $S(i)$ as a composition factor of the $(q+1)$-th syzygy of $S(j)$. Then the Hilbert series of A is given by:

$$H_A(t) = (1 + (-1)^q P t^{p+q})(1 + \sum_{i=1}^{q} (-1)^i H_{K_i}(t))^{-1},$$

where the $K_i's$ are the $A(0)$-bimodules defined before Definition 2.3.3.6.

3

Higher preprojective algebras

3.1 Definitions and higher preprojective algebras

The concept of algebra of finite representation type introduced in subsection 2.1.3 can be generalised to a higher setting.

In this section we give the definition of d-representation finite algebra and $(d+1)$-preprojective algebra, and start studying d-representation finite algebras of type A. All the definitions and results in this section are taken from [44].

Definition 3.1.0.1. Let A be a finite dimensional algebra over k. For $M \in \text{mod } A$, define the additive subcategory:

$$\textbf{add M} = \{X \in \text{mod } A | X \text{ is isomorphic to a direct summand of } M^n, n > 0\}.$$

Definition 3.1.0.2. A module $M \in \text{mod } A$ is called d**-cluster tilting** if

$$\begin{aligned}\text{add } M =& \{X \in \text{mod } A | \text{Ext}^i_A(M, X) = 0 \text{ for all } i \in \{1, \dots, d-1\}\} \\ =& \{X \in \text{mod } A | \text{Ext}^i_A(X, M) = 0 \text{ for all } i \in \{1, \dots, d-1\}\}.\end{aligned}$$

Definition 3.1.0.3. A finite dimensional algebra A is called d**-representation finite** if:

- gldim $A \leq d$;
- there exists a d-cluster tilting object in mod A.

Remark 3.1.0.4. Let $d = 1$.

- By Theorem 2.1.2.19, a finite dimensional algebra A has global dimension 1 if and only if it is the path algebra of a connected acyclic quiver Q.
- For $M, X \in \text{mod } A$, the condition $\text{Ext}^i_A(M, X) = 0$ for all $i \in \{1, \dots, d-1\}$ is empty. Therefore, a 1-cluster tilting module is a module $M \in \text{mod } A$ such that add $M = \text{mod } A$. Such a module exists if and only if Q is of finite representation type, and it is given by the sum of all indecomposable A-modules (see [44]).

Therefore, a finite dimensional algebra A is 1-representation finite if and only if it is of finite representation type or equivalently, by Gabriel's Theorem 2.1.3.15, if $A = kQ$ with Q of Dynkin type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$.

In order to appreciate the definition of $(d+1)$-preprojective algebra, one should introduce d-Auslander-Reiten translate τ_d and inverse d-Auslander-Reiten translate τ_d^{-1}. However, since we will mainly work with specific examples of $(d+1)$-preprojective algebras, we just give the definition, and refer the reader to [35] and [44] for more details.

Definition 3.1.0.5. Let A be a finite dimensional algebra of global dimension d. Consider the A^{en}-module

$$E = \mathrm{Ext}_A^d(A^*, A).$$

Then the $(d+1)$**-preprojective algebra** of A is the tensor algebra of the A^{e}-module E:

$$\Pi = \Pi_{d+1}(A) := T_A(E)$$

The following theorem justifies the name higher preprojective algebra.

Theorem 3.1.0.6. [35, Prop. 2.12] Let A be a finite dimensional algebra of global dimension d. Then, both as a left and a right A-module, the algebra $\Pi_{d+1}(A)$ is isomorphic to the sum of all indecomposable d-preprojective modules, that is, all modules of the form $\tau_d^{-1}(P)$ for $P \in \mathrm{mod}\ A$ indecomposable projective.

Just like preprojective algebras of Dynkin type, also $(d+1)$-preprojective algebras of d-representation finite algebras enjoy very nice homological properties.

Theorem 3.1.0.7. [35, Thm. B, Cor. 4.13] Let A be a d-representation finite algebra, and Π its $(d+1)$-preprojective algebra. Then:

1. Π is selfinjective.
2. if A is $\mathbb{Z}$-graded and $\Pi = \oplus_{i\geq 0}\Pi(i)$ is the induced $\mathbb{Z}$-grading induced on Π, then Π is $(p, d+1)$-Koszul, where $p = \max\{i \geq 0 | \Pi(i) \neq 0\}$.
3. Π is twisted periodic of twisted period $d+2$.

The twisted periodicity of Π is deeply related to the stably Calabi-Yau property (see [46, Thm. 1.8]). Indeed, Π is stably $(d+1)$-Calabi-Yau by [45, Thm. 1.1].

We now start constructing the main object of this thesis, that is d-preprojective algebras of type A. We start by introducing two classes of quivers.

Definition 3.1.0.8. Fix two integers $d \geq 1$, $s \geq 2$. Define the quivers $Q^{(d,s)}$ and $\overline{Q}^{(d,s)}$ as follows.

- The vertex set of both $Q^{(d,s)}$ and $\overline{Q}^{(d,s)}$ is given by:

$$\{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} | \sum_{i=1}^{d+1} x_i = s - 1\}$$

- The arrows are given by:

$$Q_1^{(d,s)} = \{e_x \alpha_i : x \to x + f_i | x \in Q_0^{(d,s)} \text{ for } 1 \leq i \leq d \text{ whenever } x_i \geq 1\},$$

$$\overline{Q}_1^{(d,s)} = \{e_x \alpha_i : x \to x + f_i | x \in \overline{Q}_0^{(d,s)} \text{ for } 0 \leq i \leq d \text{ whenever } x_i \geq 1\},$$

where

$$\begin{aligned} f_0 =&(1, 0, \dots, 0, -1), \\ f_i =&(0, \dots, 0, \underset{i}{-1}, \underset{i+1}{1}, 0, \dots, 0) \quad \text{for } i = 1, \dots, d. \end{aligned}$$

We now define relations on the path algebras of the quivers $Q^{(d,s)}$ and $\overline{Q}^{(d,s)}$.

Definition 3.1.0.9. For $d \geq 1$ and $s \geq 2$, let $I^{(d,s)} \subset kQ^{(d,s)}$ (resp. $\overline{I}^{(d,s)} \subset k\overline{Q}^{(d,s)}$) be the ideal generated by the following elements:

$$\begin{aligned} e_x \alpha_i \alpha_j =& e_x \alpha_j \alpha_i, \qquad && \text{if } x_i, x_j \geq 1 \\ e_x \alpha_i \alpha_{i+1} =& 0, && \text{if } x_i \geq 1 \text{ and } x_{i+1} = 0 \end{aligned}$$

where $x \in Q_0^{(d,s)}$ and $1 \leq i < j \leq d$ (resp. $0 \leq i < j \leq d$).

Define the algebras:

$$\Lambda^{(d,s)} = kQ^{(d,s)}/I^{(d,s)}, \qquad \Pi^{(d,s)} = k\overline{Q}^{(d,s)}/\overline{I}^{(d,s)}.$$

The following two results explain why we are interested in these algebras.

Proposition 3.1.0.10. [43, Thm. 1.18, Thm. 6.12] Fix $d \geq 1$, $s \geq 2$. Then the algebra $\Lambda^{(d,s)}$ is d-representation finite.

Therefore, we can consider its $(d+1)$-preprojective algebra $\Pi_{d+1}(\Lambda^{(d,s)})$.

Theorem 3.1.0.11. [35, Thm. 5.12] Let $d \geq 1$, $s \geq 2$. Then:

$$\Pi_{d+1}(\Lambda^{(d,s)}) \cong \Pi^{(d,s)} = k\overline{Q}^{(d,s)}/\overline{I}^{(d,s)}.$$

Example 3.1.0.12. Let $d = 1$ and $s \geq 2$. The quivers $Q^{(1,s)}$ and $\overline{Q}^{(1,s)}$ are given as follows.

$$Q^{(1,s)} = \; (s-1,0) \xrightarrow{\alpha_0} (s-2,1) \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_0} (1,s-2) \xrightarrow{\alpha_0} (0,s-1)$$

$$\overline{Q}^{(1,s)} = (s-1,0) \underset{\alpha_0}{\overset{\alpha_1}{\rightleftarrows}} (s-2,1) \underset{\alpha_0}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\alpha_0}{\overset{\alpha_1}{\rightleftarrows}} (1,s-2) \underset{\alpha_0}{\overset{\alpha_1}{\rightleftarrows}} (0,s-1)$$

The set of relations $I^{(1,s)}$ is empty; therefore, the algebra $\Lambda^{(1,s)}$ is the path algebra of the Dynkin quiver of type $\mathbb{A}_{s-1}$ with rightwise orientation.

The set or relations $\overline{I}^{(d,s)}$ is generated by two kinds of relations:

- commutativity relations

$$e_{(i,j)}\alpha_1\alpha_0 = e_{(i,j)}\alpha_0\alpha_1$$

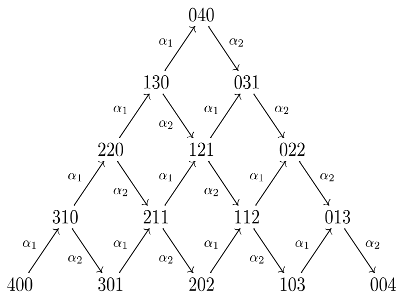
  if i, j are both nonzero;

- zero relations $e_{(s-1,0)}\alpha_1\alpha_0 = 0$ and $e_{(0,s-1)}\alpha_0\alpha_1 = 0$.

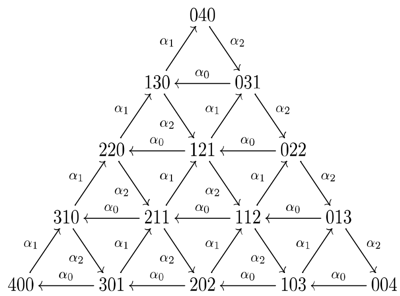
Therefore, the 2-preprojective algebra $\Pi^{(1,s)}$ of $\Lambda^{(1,s)}$ is the classical preprojective algebra of a quiver of Dynkin type $\mathbb{A}_{s-1}$ (see Definition 2.1.4.6).

Example 3.1.0.13. Let $d = 2$ and $s = 5$. The quivers $Q^{(2,5)}$ and $\overline{Q}^{(2,5)}$ are given as follows.

$Q^{(2,5)}$



$\overline{Q}^{(2,5)}$



The algebra $\Lambda^{(2,5)}$ and its 3-preprojective algebra $\Pi^{(2,5)}$ are respectively given by the path algebra of $Q^{(2,5)}$ and $\overline{Q}^{(2,5)}$, modulo the following types of relations:

- commutativity of all the squares in the quivers. For example, both in $\Lambda^{(2,5)}$ and in $\Pi^{(2,5)}$ the relation $e_{211}\alpha_1\alpha_2 = e_{211}\alpha_2\alpha_1$ holds, while in $\Pi^{(2,5)}$ also the relation $e_{103}\alpha_0\alpha_1 = e_{103}\alpha_1\alpha_0$ holds.

- zero relations for all paths of length 2 having start and end point on the rim of the quiver such that the midpoint is not on the rim. For example, the relation $e_{301}\alpha_1\alpha_2 = 0$ holds both in $\Lambda^{(2,5)}$ and in $\Pi^{(2,5)}$. In $\Pi^{(2,5)}$ also the relation $e_{220}\alpha_2\alpha_0 = 0$ holds.

3.2 Some structural properties of d-preprojective algebras of type A

Fix $d \geq 1$, $s \geq 2$. Throughout this section, let $\Pi = \Pi^{(d,s)}$ be the $(d+1)$-preprojective algebra of $\Lambda^{(d,s)}$, and let $\overline{Q} = \overline{Q}^{(d,s)}$.

In this section we collect some results about Π.

Lemma 3.2.0.1. Any nonzero path $c \in \Pi$ can be written in the form:

$$c = e_x \alpha_0^{b_0} \alpha_1^{b_1} \ldots \alpha_d^{b_d}$$

for some $x \in \overline{Q}_0$ and some integers $b_0, \ldots, b_d \geq 0$.

Proof. Let $x = s(c)$. Write

$$c = e_x \alpha_0^{b_{10}} \alpha_1^{b_{11}} \ldots \alpha_d^{b_{1d}} \ldots \alpha_0^{b_{1m}} \ldots \alpha_d^{b_{md}} \tag{3.2.0.1}$$

for some integers $b_{ij} \geq 0$. Since $c \neq 0$, then the relations $\alpha_i \alpha_j = \alpha_j \alpha_i$ hold for all $i, j \in \{0, \ldots, d\}$. Therefore we can rearrange the arrows α_i in (3.2.0.1) in any order. Hence we get the statement, where $b_i = b_{1i} + b_{2i} + \ldots + b_{mi}$ for all $i = 0, \ldots, d$. □

Lemma 3.2.0.2. Let $x = (x_1, \ldots, x_{d+1}) \in \overline{Q}_0$. A path $c = e_x \alpha_0^{b_0} \alpha_1^{b_1} \ldots \alpha_d^{b_d} \in k\overline{Q}$ is a nonzero element of Π if and only if $x_i \geq b_i$ for all $i = 1, \ldots, d$ and $x_{d+1} \geq b_0$.

Proof. Suppose that $c \in \Pi$ is a nonzero path. The commutativity relations $\alpha_i \alpha_j = \alpha_j \alpha_i$, $i, j \in \{0, \ldots, d\}$, $i < j$ allow us to rearrange the writing of c in any order. In particular, since $c \neq 0$, we have $e_x \alpha_i^{b_i} \neq 0$ for all $i = 0, \ldots, d$. So all components of

$$t(e_x \alpha_i^{b_i}) = \begin{cases} (x_1 + b_0, x_2, \ldots, x_d, x_{d+1} - b_0), & \text{if } i = 0 \\ (x_1, \ldots, x_{i-1}, x_i - b_i, x_{i+1} + b_i, \ldots, x_{d+1}), & \text{if } i = 1, \ldots, d \end{cases}$$

must be positive. Therefore we get $x_i \geq b_i$ for all $i = 1, \ldots, d$ and $x_{d+1} \geq b_0$.

Conversely, suppose w.l.o.g. that $x_i < b_i$ for some $i = 1, \ldots, d$. Then $t(e_x \alpha_i^{b_i})$ has negative i-th component, so it is zero. This also implies that $c = 0$ in Π. □

Lemma 3.2.0.3. Let $x, y \in \overline{Q}_0$. Then, if $e_x \Pi e_y \neq 0$, every shortest nonzero path $x \to y$ in Π equals:

$$c = e_x \alpha_0^{b_0} \alpha_1^{b_1} \ldots \alpha_d^{b_d},$$

in the algebra Π, where at least one of the b_i's is zero.

Furthermore, if $\widetilde{c}$ is another nonzero path $x \to y$ in Π, then:

$$\widetilde{c} = c(\alpha_0 \ldots \alpha_d)^m,$$

for some $m \geq 0$.

Proof. By contradiction, suppose that all b_i's are positive. Then, due to the commutativity relations on Π, we can rearrange the arrows in c to write it as:

$$c = c'(\alpha_0 \dots \alpha_d),$$

where $c' = e_x \alpha_0^{b_0 - 1} \alpha_1^{b_1 - 1} \dots \alpha_d^{b_d - 1}$. Then c' is a path $x \to y$ that is shorter than c. Hence we have a contradiction.

Now, let $\widetilde{c} = e_x \alpha_0^{\widetilde{b_0}} \alpha_1^{\widetilde{b_1}} \dots \alpha_d^{\widetilde{b_d}}$ be a nonzero path in Π with $t(\widetilde{c}) = y$. Let $m = \min\{\widetilde{b_i} | i = 0, \dots, d\}$. Then we have:

$$\widetilde{c} = e_x \alpha_0^{\widetilde{b_0} - m} \alpha_1^{\widetilde{b_1} - m} \dots \alpha_d^{\widetilde{b_d} - m} (\alpha_0 \dots \alpha_d)^m,$$

where at least one of the exponents $\widetilde{b_i} - m$ is zero.

Hence $e_x \alpha_0^{\widetilde{b_0} - m} \dots \alpha_d^{\widetilde{b_d} - m} = c$, and so we get the statement. □

Definition 3.2.0.4. For an S-bimodule M define the S-**centraliser** of M by:

$$M^S = \{m \in M | sm = ms \text{ for all } s \in S\}.$$

Proposition 3.2.0.5. Let W be a homogeneous finite subset of Π consisting of linearly independent elements. Assume also that every $w \in W$ is a linear combination of paths, $w = \sum_{i=1}^n \lambda_i c_i$ with $s(c_1) = \dots = s(c_n)$, $t(c_1) = \dots = t(c_n)$ and let M be the S-bimodule generated by W. Let x_w be the shortest path $t(w) \to s(w)$. Then the set

$$\{w \otimes x_w (\alpha_0 \alpha_1 \dots \alpha_d)^r | w \in W, r \geq 0\} \tag{3.2.0.2}$$

spans $(M \otimes_S \Pi)^S$ as a vector space. In particular, the nonzero distinct elements of the form $w \otimes x_w (\alpha_0 \dots \alpha_d)^r$, $r \geq 0$, form a basis of $(M \otimes_S \Pi)^S$.
Furthermore:

$$(M \otimes_S \Pi)^S = \bigoplus_{\ell \geq 0} (M \otimes_S \Pi)^S ((d+1)\ell). \tag{3.2.0.3}$$

Proof. The space $(M \otimes_S \Pi)^S$ is spanned by all the elements of the form $w \otimes x$, where $w \in W$ and $x : t(w) \to s(w) \in \Pi$ is a nonzero path. Lemma 3.2.0.3 implies that the element $w \otimes x$ is of the form

$$w \otimes x_w (\alpha_0 \alpha_1 \dots \alpha_d)^r$$

for some $r \geq 0$. So the set (3.2.0.2) generates $(M \otimes \Pi)^S$, and thus the nonzero distinct elements of (3.2.0.2) form a basis of $(M \otimes \Pi)^S$.

Now, for fixed $w \in W$ and $r \geq 0$, consider the element:

$$c = wx_w(\alpha_0\alpha_1 \ldots \alpha_d)^r \in kQ,$$

and let b_i be the number of occurrences of α_i in c for $i = 0, \ldots, d$. Since c is a path with the same starting and ending point, we have that $b_0 = b_1 = \ldots = b_d$ by definition of α_i. So, if we let ℓ to be the length of the element c, we have $\ell = (d+1)b_0 \equiv 0 \pmod{d+1}$. Thus the length of $w \otimes x_w(\alpha_0\alpha_1 \ldots \alpha_d)^r$ is a multiple of $d+1$. Hence we get the decomposition (3.2.0.3). □

Using the results above we can easily derive the following.

Proposition 3.2.0.6. The following decomposition of Π holds:

$$\Pi = \bigoplus_{j=0}^{s-1} \Pi(j). \tag{3.2.0.4}$$

Furthermore, the space $e_x\Pi(s-1)$ is 1-dimensional for all $x = (x_1, \ldots, x_{d+1}) \in \overline{Q}_0$, generated by the path:

$$u_{x,\nu(x)} = e_x\alpha_0^{x_{d+1}}\alpha_1^{x_1} \ldots \alpha_d^{x_d}, \tag{3.2.0.5}$$

that has endpoint $\nu(x) := (x_{d+1}, x_0, \ldots, x_d)$.

Proof. Fix $x \in \overline{Q}_0$. A direct application of Lemma 3.2.0.2 shows that the path (3.2.0.5) is nonzero. Also, it has length $s-1$ since $x_1 + \ldots + x_{d+1} = s-1$.

Furthermore, the path $u_{x,\nu(x)}$ has endpoint $(x_{d+1}, x_1, \ldots, x_d)$ by definition of the α_i's.

Now, let:

$$c = e_x\alpha_0^{m_{d+1}}\alpha_1^{m_1} \ldots \alpha_d^{m_d}$$

be a path of length $s-1$ with starting point x, with $c \neq u_{x,\nu(x)}$. Then there is necessarily some $i \in \{1, \ldots, d+1\}$ such that $m_i > x_i$. But then $c = 0$ by Lemma 3.2.0.2. Therefore $e_x\Pi(s-1)$ is one-dimensional generated by $u_{x,\nu(x)}$.

Finally, if

$$c' = e_x\alpha_0^{b_{d+1}}\alpha_1^{b_1} \ldots \alpha_d^{b_d}$$

is a path of length $\ell > s-1$, it follows that $b_i > x_i$ for some $i \in \{1, \ldots, d+1\}$, and so we can conclude that $c' = 0$ by Lemma 3.2.0.2. Hence $\Pi(\ell) = 0$ for all $\ell > s-1$, and we get the decomposition (3.2.0.4). □

By Theorem 3.1.0.7, we know that Π is selfinjective. Therefore, by Theorem 2.3.1.7(1), the algebra Π is Frobenius. However, since $(d+1)$-preprojective algebras of type A are particularly nice, we can explicitly define a non-degenerate bilinear form of Π, that gives Π the structure of Frobenius algebra.

Definition 3.2.0.7. Let $f : \Pi \to k$ be the linear map defined on paths $c \in \Pi$ by

$$f(c) = \begin{cases} 1, & \text{if } c = u_{x,\nu(x)} \text{ for some } x \in \overline{Q}_0 \\ 0, & \text{otherwise.} \end{cases}$$

Define the bilinear form $(.,.) : \Pi \times \Pi \to k$ by $(a, b) = f(ab)$.

The following theorem is the content of [38, Thm. 3.5]. However, a direct proof using the results of this section is possible, so we present it.

Theorem 3.2.0.8. The bilinear form $(.,.)$ given in Definition 3.2.0.7 gives Π the structure of a Frobenius algebra. The associated Nakayama automorphism $\eta : \Pi \to \Pi$ is given by:

$$\eta(e_x \alpha_0^{b_0} \alpha_1^{b_1} \dots \alpha_d^{b_d}) = e_{\nu(x)} \alpha_0^{b_d} \alpha_1^{b_0} \alpha_2^{b_1} \dots \alpha_d^{b_{d-1}}. \tag{3.2.0.6}$$

In particular, the permutation $\nu : \overline{Q}_0 \to \overline{Q}_0$ defined in Proposition 3.2.0.6 is the Nakayama permutation of Π.

Proof. We want to use the characterisation of Frobenius algebra given in Proposition 2.3.1.5.

First of all, notice that:

$$(a, bc) = f(abc) = (ab, c)$$

for all $a, b, c \in \Pi$, and so $(.,.)$ is associative.

Now, fix $a \in \Pi$, $a \neq 0$. Write

$$a = \sum_{i \in I} \lambda_i c_i,$$

where I is a nonempty finite set, c_i is a nonzero path of Π and $\lambda_i \neq 0$ for all $i \in I$. Fix $i \in I$, and let $s(c_i) = x = (x_1, \dots, x_{d+1})$. By Lemma 3.2.0.2, the path c_i can be written as:

$$c_i = e_x \alpha_0^{b_0} \alpha_1^{b_1} \dots \alpha_d^{b_d},$$

where $x_i \geq b_i$ for all $i = 1, \dots, d$ and $x_{d+1} \geq b_0$. Now, define the path:

$$c_i' = e_{t(c_i)} \alpha_0^{x_{d+1} - b_0} \alpha_1^{x_1 - b_1} \dots \alpha_d^{x_d - b_d}.$$

Using notation from Proposition 3.2.0.6, we have that:

$$c_i c_i' = u_{x\nu(x)},$$

and therefore $(c_i, c_i') = 1$. Also, since c_i is the only path $\widetilde{c}$ in Π such that

$(\widetilde{c}, c'_i) \neq 0$, we have that $(c_j, c'_i) = 0$ for all $j \in I$, $j \neq i$. As a consequence:

$$(a, c'_i) = \lambda_i \neq 0,$$

and therefore $(.,.)$ is non-degenerate.

In order to prove that the Nakayama automorphism is given by (3.2.0.6), we need to show that $(a, b) = (b, \eta(a))$ for all paths $a, b \in \Pi$. To do this, it is enough to notice that, if a, b are paths in Π such that $ab = u_{x,\nu(x)}$ for some $x = (x_1, \ldots, x_{d+1}) \in \overline{Q}_0$, then $b\eta(a)$ is equal to $u_{t(x),\nu(t(x))}$. This can be done writing the paths a, b explicitly and using the explicit form of $u_{x\nu(x)}$ given by Proposition 3.2.0.6. □

Example 3.2.0.9. Let $d = 2$, $s = 5$ as in Example 3.1.0.13. Then the automorphism $\eta : \Pi \to \Pi$ sends any path $c \in \Pi$ to its clockwise rotation of 120 degrees.

For example, $\eta(e_{220}\alpha_1\alpha_2) = e_{022}\alpha_2\alpha_0$.

4

Strategy to compute Hochschild and cyclic homology of 3-preprojective algebras of type A

In this chapter we outline the strategy to compute the Hochschild homology, cohomology and cyclic homology of 3-preprojective algebras of type A. This is inspired by [21], where the authors compute these homological invariants for classical preprojective algebras of finite representation type.

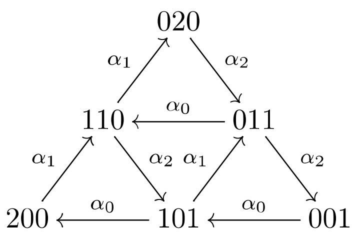
Throughout this chapter, $s \geq 2$ is a fixed integer. Let:

$$\Pi = \Pi^{(2,s)} = k\overline{Q}/(R),$$

where $\overline{Q} = \overline{Q}^{(2,s)}$, $R = \{e_x(\alpha_p\alpha_q - \alpha_q\alpha_p) | x \in \overline{Q}_0, p, q \in \{0,1,2\}, p < q\}$. Let $S = \Pi(0)$, $V = \Pi(1)$.

Notice that, for all $v \in \overline{Q}_1$ there is exactly one relation in R, with start point $t(v)$ and end point $s(v)$. Let r_v be such relation.

Example 4.0.0.1. Let $s = 3$. The quiver $\overline{Q}$ is given by:



For example, the relation in R associated to the arrow $v_1 = e_{200}\alpha_1$ is given by $r_{v_1} = -e_{110}\alpha_2\alpha_0$, while the one associated to $v_2 = e_{011}\alpha_0$ is given by
$r_{v_2} = e_{110}(\alpha_1\alpha_2 - \alpha_2\alpha_1)$.

If not otherwise stated, $\otimes = \otimes_S$.

4.1 Projective Π-bimodule resolution of Π

In this section we give a projective Π-bimodule resolution of Π.

By Proposition 3.2.0.6 we have $s-1=\max\{i\geq 0|\Pi(i)\neq 0\}$, and therefore Theorem 3.1.0.7(2) implies that the algebra Π is $(s-1,3)$-Koszul. Therefore, the Koszul bimodule complex (2.3.3.2) gives the first four terms of a projective Π-bimodule resolution of Π by Theorem 2.3.3.9. We use this result, together with the partial description of the fourth syzygy given in Theorem 2.3.3.9 to get the required projective resolution of Π.

Before doing this, we need a preliminary lemma. For convenience, let:

$$\widetilde{r}_v=\begin{cases} r_v, & \text{if } v=\alpha_0,\alpha_2 \\ -r_v, & \text{if } v=\alpha_1. \end{cases}$$

Lemma 4.1.0.1. The map

$$\begin{array}{rccc} \Psi: & \overline{Q}_0 & \to & VR\cap RV \\ & x & \mapsto & e_x(\alpha_0\widetilde{r}_{\alpha_0}+\alpha_1\widetilde{r}_{\alpha_1}+\alpha_2\widetilde{r}_{\alpha_2}) \end{array}$$

extends to a graded S-bimodule isomorphism $\Psi: S[3]\xrightarrow{\sim} VR\cap RV$.

Proof. First of all, notice that Ψ is well defined, i.e., $\Psi(x)\in VR\cap RV$ for all $x\in\overline{Q}_0$. Indeed, for $x\in\overline{Q}_0$, we have:

$$\begin{aligned} e_x(\alpha_0\widetilde{r}_{\alpha_0}+\alpha_1\widetilde{r}_{\alpha_1}+\alpha_2\widetilde{r}_{\alpha_2}) =& e_x(\alpha_0(\alpha_1\alpha_2-\alpha_2\alpha_1)+\alpha_1(\alpha_2\alpha_0-\alpha_0\alpha_2) \\ & +\alpha_2(\alpha_0\alpha_1-\alpha_1\alpha_0)) \\ =& e_x((\alpha_1\alpha_2-\alpha_2\alpha_1)\alpha_0+(\alpha_2\alpha_0-\alpha_0\alpha_2)\alpha_1 \\ & +(\alpha_0\alpha_1-\alpha_1\alpha_0)\alpha_2) \\ =& e_x(\widetilde{r}_{\alpha_0}\alpha_0+\widetilde{r}_{\alpha_1}\alpha_1+\widetilde{r}_{\alpha_2}\alpha_2)\in RV, \end{aligned}$$

and therefore $\Psi(x)\in VR\cap RV$.

Also, Ψ is injective; indeed, $\Psi(x)$ has starting point e_x for all $x\in\overline{Q}_0$. Thus, if $\Psi(x)=\Psi(y)$ for some $x,y\in\overline{Q}_0$, then $e_x=e_y$, i.e., $x=y$. Extending this by linearity yields that, if $t,t'\in S$ are such that $\Psi(t)=\Psi(t')$, then $t=t'$.

Therefore, we just need to show that Ψ is surjective. First of all, notice that:

$$VR\cap RV=(VR\cap RV)^S.$$

By contradiction, suppose there exists $a\in VR\cap RV$ such that $a\notin(VR\cap RV)^S$. In particular, there are $x,y\in\overline{Q}_0$, $x\neq y$, such that $0\neq e_xae_y\in VR\cap RV$. Now, it is easy to check that e_xVRe_y and e_xRVe_y are either 0 or one-dimensional for all $x,y\in\overline{Q}_0$. In particular, since $0\neq e_xae_y\in e_xVRe_y$, then it can be written as:

$$e_xae_y=e_x\lambda\alpha_pr_{\alpha_q}=e_x\lambda\alpha_p(\pm(\alpha_p\alpha_r-\alpha_r\alpha_p)),$$

where $\lambda\in k$ and $\{p,q,r\}=\{0,1,2\}$. But the right-hand side clearly does not belong to RV. Thus, we have a contradiction, and hence $VR \cap RV = (VR \cap RV)^S$.

Furthermore, we have:

$$(VR)^S = \mathrm{span}_k\{e_x\alpha_i\widetilde{r}_{\alpha_i} | x \in \overline{Q}_0, i = 0, 1, 2\},$$

$$(RV)^S = \mathrm{span}_k\{e_x\widetilde{r}_{\alpha_i}\alpha_i | x \in \overline{Q}_0, i = 0, 1, 2\}.$$

Hence, any element $a \in (VR \cap RV)^S$ can be written as:

$$a = \sum_{x\in\overline{Q}_0}\sum_{i=0,1,2} \lambda_{x,i}e_x\alpha_i\widetilde{r}_{\alpha_i} = \sum_{x\in\overline{Q}_0}\sum_{i=0,1,2} \mu_{x,i}e_x\widetilde{r}_{\alpha_i}\alpha_i$$

for some coefficients $\lambda_{x,i}, \mu_{x,i} \in k$. Expanding the sums in i and the relations $\widetilde{r}_{\alpha_i}$, $i = 0, 1, 2$, on both sides, we get the following equations for $\lambda_{x,i}$ and $\mu_{x,i}$:

$$\lambda_{x_0} = \lambda_{x_1} = \lambda_{x_2} = \mu_{x_0} = \mu_{x_1} = \mu_{x,2},$$

for all $x \in \overline{Q}_0$. Therefore:

$$a = \sum_{x\in\overline{Q}_0} e_x\lambda_{x,0} \sum_{i=0,1,2} \alpha_i\widetilde{r}_{\alpha_i} \in \mathrm{Im}(\Psi).$$

Hence Ψ is surjective, and we get the statement. □

From now on, let $h = s + 2$. The proof of the following proposition is an adaptation of [26, Thm. 5.1] to our setting.

Proposition 4.1.0.2. Let $\mathcal{N} = {}_1\Pi_{\eta^{-1}}$, where η is the Nakayama automorphism of Π, explicitly defined in Theorem 3.2.0.8. The following complex of graded Π-bimodules is exact:

$$0 \to \mathcal{N}[h] \overset{\iota_0}{\to} \Pi \otimes \Pi[3] \overset{\mu_3}{\to} \Pi \otimes R \otimes \Pi \overset{\mu_2}{\to} \Pi \otimes V \otimes \Pi \overset{\mu_1}{\to} \Pi \otimes \Pi \overset{\mu_0}{\to} \Pi \to 0.$$

The differentials are given by:

$$\begin{aligned}
\mu_0(1 \otimes 1) =& 1\\
\mu_1(1 \otimes v \otimes 1) =& v \otimes 1 - 1 \otimes v\\
\mu_2(1 \otimes e_x(\alpha_p\alpha_q - \alpha_q\alpha_p) \otimes 1) =& e_x \otimes \alpha_p \otimes \alpha_q + e_x\alpha_p \otimes \alpha_q \otimes 1\\
&- e_x \otimes \alpha_q \otimes \alpha_p - e_x\alpha_q \otimes \alpha_p \otimes 1\\
\mu_3(1 \otimes 1) =& \sum_{v\in\overline{Q}_1} v \otimes \widetilde{r}_v \otimes 1 - \sum_{v\in\overline{Q}_1} 1 \otimes \widetilde{r}_v \otimes v.\\
\iota_0(1) =& \sum x_i \otimes x_i^*,
\end{aligned}$$

where $\{x_i\}$ is a basis of Π consisting of all nonzero paths, and $\{x_i^*\}$ is its dual basis under the non-degenerate bilinear form $(.,.)$ given in Definition 3.2.0.7.

Proof. The exactness of the complex

$$\Pi \otimes \Pi[3] \xrightarrow{\mu_3} \Pi \otimes R \otimes \Pi \xrightarrow{\mu_2} \Pi \otimes V \otimes \Pi \xrightarrow{\mu_1} \Pi \otimes \Pi \xrightarrow{\mu_0} \Pi \to 0$$

follows directly from the almost Koszulness of Π, i.e., from Theorems 3.1.0.7(2) and 2.3.3.9. The only thing to notice is that, using the notation from Definition 2.3.3.8, we have $K_3 = VR \cap RV \cong S[3]$ as S-bimodules by Lemma 4.1.0.1, and therefore $\Pi \otimes K_3 \otimes \Pi \cong \Pi \otimes \Pi[3]$. Also, one can check that the map μ_3 given in the statement in such that the following diagram commutes:

$$\begin{array}{ccc} \Pi \otimes S \otimes \Pi[3] & \xrightarrow{\mu_3} & \Pi \otimes R \otimes \Pi \\ \downarrow{\scriptstyle 1\otimes\Psi\otimes 1} & & \downarrow{\scriptstyle \mathrm{id}} \\ \Pi \otimes (VR \cap RV) \otimes \Pi & \xrightarrow{d} & \Pi \otimes R \otimes \Pi, \end{array}$$

where Ψ is the S-bimodule isomorphism defined in Lemma 4.1.0.1 and d is the differential in the Koszul bicomplex from Definition 2.3.3.8.

We now want to give a description of $\Sigma = \ker \mu_3$, the fourth syzygy of Π. By Theorem 2.3.3.9 we know that Σ is generated both as a left and as a right Π-module by its component Z of total weight degree $s + 2 = h$, where:

$$Z \subset (\Pi \otimes \Pi[3])(h) = (\Pi \otimes \Pi)(s-1) = \bigoplus_{k=0}^{s-1} (\Pi(k) \otimes \Pi(s-1-k)).$$

Theorem 2.3.3.9 also tells us that the projection:

$$Z \to \Pi(0) \otimes \Pi(s-1) = S \otimes \Pi(s-1) \cong \Pi(s-1) \tag{4.1.0.1}$$

is an S-bimodule isomorphism. Now, by Proposition 3.2.0.6 and Theorem 3.2.0.8 we have $\Pi(s-1) \cong {}_1S_\nu$ as S-bimodules, where ν is the Nakayama permutation of Π. Hence:

$$Z \cong {}_1S_\nu$$

as S-bimodule.

We now want to exhibit a basis of $Z \subset \ker \mu_3$ given by some elements $\{\omega_{i\nu(i)} | i \in \overline{Q}_0\}$ such that for all $i \in \overline{Q}_0$, $\omega_{i,\nu(i)}$ is linear combinations of elements in $(\Pi \otimes \Pi[3])(s+2)$ with start point i and end point $\nu(i)$, and such that $e_i \otimes u_{i,\nu(i)}$ is one of its summands.

We claim that:

$$\omega_{i\nu(i)} = e_i \sum x_j \otimes x_j^*. \tag{4.1.0.2}$$

To prove this we show that $(\omega_{i\nu(i)})_k$, the component of $\omega_{i\nu(i)}$ that lies in

$\Pi(k) \otimes \Pi(s-1-k)$, is given by

$$(\omega_{i\nu(i)})_k = e_i \sum_{x_j \in B_k^i} x_j \otimes x_j^*,$$

for all $k = 0, \dots, s-1$, where B_k^i is a basis of $e_i\Pi(k)$ consisting of paths. We do this by induction on k.

- If $k = 0$ then, by (4.1.0.1) we have that $(\omega_{i\nu(i)})_0 = e_i \otimes u_{i,\nu(i)}$, where $u_{i,\nu(i)} = e_i^*$ is the path that generates $e_i\Pi(s-1)$ defined in Proposition 3.2.0.6.

- Let $k > 0$. Notice that, by definition of μ_3, we have:

  $$\mu_3 : \Pi(j) \otimes \Pi(s-1-j) \to (\Pi(j+1) \otimes R \otimes \Pi(s-1-j)) \oplus (\Pi(j) \otimes R \otimes \Pi(s-j)).$$

  for all $j = 0, \dots, s-1$.

  Therefore the components of $\omega_{i\nu(i)}$ that are sent to $\Pi(k) \otimes R \otimes \Pi(s-k)$ by μ_3 are $(\omega_{i\nu(i)})_{k-1}$ and $(\omega_{i\nu(i)})_k$. Since $\mu_3(\omega_{i\nu(i)}) = 0$, we have:

  $$\mu_3((\omega_{i\nu(i)})_{k-1} + (\omega_{i\nu(i)})_k)_{k,s-k} = 0 \tag{4.1.0.3}$$

  where, for $x \in \Pi \otimes R \otimes \Pi$, we denote by $x_{a,b}$ its component that lies in $\Pi(a) \otimes R \otimes \Pi(b)$.

  By induction hypothesis we know that:

  $$(\omega_{i\nu(i)})_{k-1} = e_i \sum_{x_j \in B_{k-1}^i} x_j \otimes x_j^*.$$

  Hence, applying μ_3 we get:

  $$\mu_3((\omega_{i\nu(i)})_{k-1})_{k,s-k} = e_i \sum_{\substack{x_j \in B_{k-1}^i \\ v \in \overline{Q}_1}} x_j v \otimes \widetilde{r}_v \otimes x_j^*.$$

  Now, let $x_k = e_i \sum_{y_\ell \in B_k^i} y_\ell \otimes y_\ell^* \in \Pi(k) \otimes \Pi(s-1-k)$. Then:

  $$\mu_3(x_k)_{k,s-k} = -e_i \sum_{\substack{y_\ell \in B_k^i \\ v' \in V}} y_\ell \otimes \widetilde{r}_{v'} \otimes v' y_\ell^*.$$

  Therefore:

  $$\mu_3((\omega_{i\nu(i)})_{k-1} + x_k)_{k,s-k} = e_i\Big(\sum_{\substack{x_j \in B_{k-1}^i \\ v \in V}} x_j v \otimes \widetilde{r}_v \otimes x_j^* - \sum_{\substack{y_\ell \in B_k^i \\ v' \in V}} y_\ell \otimes \widetilde{r}_{v'} \otimes v' y_\ell^*\Big).$$

Now, the terms in the second sum correspond to the term in the first sum. Indeed, if we take $y_\ell = x_j v$, we have $x_j^* = v y_\ell^*$. This can be seen using the associativity of the bilinear form $(.,.)$ as follows:

$$(x_j, v y_\ell^*) = (x_j v, y_\ell^*) = (y_\ell, y_\ell^*) = 1$$

and using the explicit form of $(.,.)$ given by Definition 3.2.0.7.

Therefore we have $\mu_3((\omega_{i\nu(i)})_{k-1} + x_k)_{k,s-k} = 0$; this implies $(\omega_{i\nu(i)})_k = x_k$ by (4.1.0.3), and thus we get the claim.

Now, let:

$$\omega = \sum_{i \in \overline{Q}_0} \omega_{i\nu(i)}.$$

The claim we proved implies that $Z = S\omega = \omega S$. Therefore, by Theorem 2.3.3.9, we have $\Sigma \cong \Pi\omega$ as left Π-module, and $\Sigma \cong \omega\Pi$ as right Π-module. Thus, for any element $a \in \Sigma$ there are unique elements $x, y \in \Pi$ such that:

$$a = x\omega = \omega y.$$

Hence, the map $\gamma : \Pi \to \Pi$ defined by $\gamma(x) = y$, is an automorphism of Π, and we have:

$$x\omega = \omega\gamma(x), \qquad \text{for all } x \in \Pi. \tag{4.1.0.4}$$

Since ω is homogeneous, we can assume γ to have degree 0.

We want to show that $\gamma = \eta$, the Nakayama automorphism of Π.

If we substitute $x = e_i$, $i \in \overline{Q}_0$ in (4.1.0.4), we get $e_i\omega = \omega\gamma(e_i)$. Hence

$$\gamma(e_i) = t(\omega) = \nu(e_i) = \eta(e_i),$$

where the last equality holds by Proposition 2.3.1.9. Therefore γ and η coincide on S.

Also, if we substitute $x = v \in \overline{Q}_1$ in (4.1.0.4), we see that $\gamma(v)$ is an element of $\Pi(1)$ that has $\eta(s(v))$ as start point and $\eta(t(v))$ as end point. Therefore γ and η differ by a constant on arrows, i.e., for all $v \in \overline{Q}_1$ there is a constant $c_v \in k \setminus \{0\}$ such that:

$$\gamma(v) = c_v \eta(v). \tag{4.1.0.5}$$

We want to show that $c_v = 1$. In order to do this, for fixed $v : i \to j \in \overline{Q}_1$, substitute $x = \eta^{-1}(v)$ in (4.1.0.4). Then we have:

$$\eta^{-1}(v)\omega_{\eta^{-1}(j)j} = \omega_{\eta^{-1}(i)i}\gamma(\eta^{-1}(v)) \overset{(4.1.0.5)}{=} \omega_{\eta^{-1}(i)i} c_{\eta^{-1}(v)} v. \tag{4.1.0.6}$$

If we substitute the expressions for $\omega_{\eta^{-1}(j)j}$ and $\omega_{\eta^{-1}(i)i}$ from (4.1.0.2) in (4.1.0.6) and consider the component living in $\Pi(1) \otimes \Pi(s-1)$ we have:

$$\eta^{-1}(v) \otimes u_{\eta^{-1}(j),j} = c_{\eta^{-1}(v)} \sum_{b \in B_1^{\eta^{-1}(i)}} b \otimes b^* v.$$

Notice that, for $b = \eta^{-1}(v) \in B_1^{\eta^{-1}(i)}$, we have:

$$\begin{aligned}(e_{\eta^{-1}(j)}, b^* v) =&(e_{\eta^{-1}(j)}, (\eta^{-1}(v))^* v) = ((\eta^{-1}(v))^*, v) = ((\eta^{-1}(v))^*, \eta(\eta^{-1}(v))) \\ =&(\eta^{-1}(v), (\eta^{-1}(v))^*) = 1,\end{aligned}$$

and therefore $b^* v = u_{\eta^{-1}(j)j}$. Hence $c_{\eta^{-1}(v)} = 1$ for all $v \in \overline{Q}_1$ and, since η is an automorphism, $c_v = 1$ for all $v \in \overline{Q}_1$.

Therefore, γ and η coincide on the set of vertices and arrows, and so they coincide on all of Π, i.e., $\gamma = \eta$. Hence, by (4.1.0.4), we have:

$$x\omega = \omega\eta(x) \qquad \text{for all } x \in \Pi. \tag{4.1.0.7}$$

This gives Σ the structure of a Π-bimodule. More precisely, we have a Π-bimodule isomorphism

$$\begin{aligned}\iota_0 : \ {}_1\Pi_{\eta^{-1}} &\to \Sigma \\ x &\mapsto x\omega.\end{aligned}$$

This is clearly an isomorphism of vector spaces since we know that $\Sigma = \Pi\omega$. To show that it is a Π-bimodule morphism, let $a, b, x \in \Pi$. Then:

$$\iota_0(a \cdot x \cdot b) = \iota_0(ax\eta^{-1}(b)) = ax\eta^{-1}(b)\omega = ax\omega b = a\iota_0(x)b,$$

where the second to last equality follows by substituting $x = \eta^{-1}(b)$ to (4.1.0.7). □

Remark 4.1.0.3. In the proof of Proposition 4.1.0.2, the fact that the automorphism γ satisfying (4.1.0.4) coincides on the arrows of $\overline{Q}$ with the Nakayama automorphism η of Π relies on the explicit formula for η given in Theorem 3.2.0.8. In particular, a similar argument could be applied to any Frobenius algebra whose Nakayama automorphism sends arrows to arrows.

For other Frobenius algebras this might not be the case, and one needs much more work and nasty computations to show that the constant c_v in the proof of Proposition 4.1.0.2 is equal to 1 (see for example [26, Sect. 5.1] and [7, Sect. 4.4]).

Remark 4.1.0.4. By Theorem 4.1.0.2, the fourth syzygy of Π as a Π-bimodule is given by:

$$\Omega^4_{\Pi^e}(\Pi) \cong \Pi_{\eta^{-1}}[h]. \tag{4.1.0.8}$$

Furthermore, the linear form $f : \Pi \to k$ given in Definition 3.2.0.7 that induces the graded Frobenius algebra structure on Π has degree $h-3$, since this is the length of the element $u_{x,\nu(x)}$ for all $x \in \overline{Q}_0$. Therefore, in view of Lemma 2.3.2.9, we also have the graded Π-bimodule isomorphism:

$$\Pi^\vee \cong \Pi_{\eta^{-1}}[h-3]. \qquad (4.1.0.9)$$

The isomorphisms (4.1.0.8) and (4.1.0.9) together give:

$$\Pi^\vee[3] \cong \Omega^4_{\Pi^e}(\Pi).$$

The above isomorphism is the defining property of what Eu and Schedler call graded Calabi-Yau Frobenius algebras with dimension 3 of shift 3 in [25, Defn. 2.3.6].

Remark 4.1.0.5. For each integer $k \geq 1$ define the following Π-bimodule:

$$\mathcal{N}^{(k)} = {}_1\Pi_{\eta^{-k}}.$$

Notice that, since the Nakayama automorphism has order 3 by Theorem 3.2.0.8, then $\mathcal{N}^{(k)} = \mathcal{N}^{(k \pmod 3)}$ for all integers $k \geq 1$. Furthermore, we can make the following identifications:

$$\begin{array}{rccc} \Phi_2 : & \mathcal{N} \otimes_\Pi \mathcal{N} & \to & \mathcal{N}^{(2)} \\ & x \otimes 1 & \mapsto & x \end{array} \qquad \begin{array}{rccc} \Phi_3 : & \mathcal{N} \otimes_\Pi \mathcal{N} \otimes_\Pi \mathcal{N} & \to & \Pi \\ & x \otimes 1 \otimes 1 & \mapsto & x \end{array}$$

These are clearly vector space isomorphisms. A straightforward computation also shows that they are Π-bimodule morphisms. For example, for Φ_3 we have:

$$\begin{aligned} \Phi_3(a \cdot (x \otimes 1 \otimes 1) \cdot b) =& \Phi_3(ax \otimes 1 \otimes \eta^{-1}(b)) = \Phi_3(ax \otimes 1 \cdot \eta^{-1}(b) \otimes 1) \\ =& \Phi_3(ax \otimes \eta^{-2}(b) \otimes 1) = \Phi_3(ax \cdot \eta^{-2}(b) \otimes 1 \otimes 1) \\ =& \Phi_3(ax\eta^{-3}(b) \otimes 1 \otimes 1) = \Phi_3(axb \otimes 1 \otimes 1) \\ =& axb = a \cdot \Phi_3(x \otimes 1 \otimes 1) \cdot b \end{aligned}$$

A similar argument shows that also Φ_2 is a Π-bimodule morphism.

Proposition 4.1.0.2 and Remark 4.1.0.5 allow us to give a projective Π-bimodule resolution of Π.

Theorem 4.1.0.6. The following sequence is exact, and gives a projective Π-

bimodule resolution of Π that is periodic of period 12.

$$\begin{aligned}\dots \to \Pi\otimes\Pi[3h] &\stackrel{\mu_{12}}{\to} \Pi\otimes\mathcal{N}^{(2)}[2h+3] \stackrel{\mu_{11}}{\to} \Pi\otimes R\otimes\mathcal{N}^{(2)}[2h] \stackrel{\mu_{10}}{\to} \Pi\otimes V\otimes\mathcal{N}^{(2)}[2h]\\ &\stackrel{\mu_9}{\to} \Pi\otimes\mathcal{N}^{(2)}[2h] \stackrel{\mu_8}{\to} \Pi\otimes\mathcal{N}[h+3] \stackrel{\mu_7}{\to} \Pi\otimes R\otimes\mathcal{N}[h] \stackrel{\mu_6}{\to} \Pi\otimes V\otimes\mathcal{N}[h]\\ &\stackrel{\mu_5}{\to} \Pi\otimes\mathcal{N}[h] \stackrel{\mu_4}{\to} \Pi\otimes\Pi[3] \stackrel{\mu_3}{\to} \Pi\otimes R\otimes\Pi \stackrel{\mu_2}{\to} \Pi\otimes V\otimes\Pi \stackrel{\mu_1}{\to} \Pi\otimes\Pi \stackrel{\mu_0}{\to} \Pi \to 0\end{aligned} \tag{4.1.0.10}$$

The differentials are given by:

$$\begin{aligned}\mu_0(1\otimes 1) =&1\\ \mu_1(1\otimes v\otimes 1) =&v\otimes 1 - 1\otimes v\\ \mu_2(1\otimes e_x(\alpha_p\alpha_q-\alpha_q\alpha_p)\otimes 1) =&e_x\otimes\alpha_p\otimes\alpha_q + e_x\alpha_p\otimes\alpha_q\otimes 1\\ &-e_x\otimes\alpha_q\otimes\alpha_p - e_x\alpha_q\otimes\alpha_p\otimes 1\\ \mu_3(1\otimes 1) =& \sum_{v\in\overline{Q}_1} v\otimes\widetilde{r}_v\otimes 1 - \sum_{v\in\overline{Q}_1} 1\otimes\widetilde{r}_v\otimes v\\ \mu_4(1\otimes 1) =& \sum x_i\otimes x_i^*\\ \mu_i =&\mu_{i-4} \qquad\qquad i\geq 5,\end{aligned}$$

where $\{x_i\}$ is a basis of Π consisting of all nonzero paths, and $\{x_i^*\}$ is its dual basis under the non-degenerate bilinear form $(.,.)$ given in Definition 3.2.0.7.

Proof. By Proposition 4.1.0.2, we know that the following sequence is exact:

$$0\to\mathcal{N}[h] \stackrel{\iota_0}{\to} \Pi\otimes\Pi[3] \stackrel{\mu_3}{\to} \Pi\otimes R\otimes\Pi \stackrel{\mu_2}{\to} \Pi\otimes V\otimes\Pi \stackrel{\mu_1}{\to} \Pi\otimes\Pi \stackrel{\mu_0}{\to} \Pi\to 0. \tag{4.1.0.11}$$

If we apply the exact functor $-\otimes_\Pi\mathcal{N}[h]$, in view of Remark 4.1.0.5 we get the following exact sequence:

$$\begin{aligned}0\to\mathcal{N}^{(2)}[2h] &\stackrel{\iota_2}{\to} \Pi\otimes\mathcal{N}[h+3] \stackrel{\mu_7}{\to} \Pi\otimes R\otimes\mathcal{N}[h] \stackrel{\mu_6}{\to} \Pi\otimes V\otimes\mathcal{N}[h]\\ &\stackrel{\mu_5}{\to} \Pi\otimes\mathcal{N}[h] \stackrel{\iota_1}{\to} \mathcal{N}[h]\to 0,\end{aligned}$$

where the differentials are defined as in (4.1.0.11).

Applying the functor $-\otimes_\Pi\mathcal{N}[h]$ a second time, in view of Remark 4.1.0.5 we get the following exact sequence:

$$\begin{aligned}0\to\Pi[3h] &\stackrel{\iota_4}{\to} \Pi\otimes\mathcal{N}^{(2)}[2h+3] \stackrel{\mu_{11}}{\to} \Pi\otimes R\otimes\mathcal{N}^{(2)}[2h]\\ &\stackrel{\mu_{10}}{\to} \Pi\otimes V\otimes\mathcal{N}^{(2)}[2h] \stackrel{\mu_9}{\to} \Pi\otimes\mathcal{N}^{(2)}[2h] \stackrel{\iota_3}{\to} \mathcal{N}^{(2)}[2h]\to 0\end{aligned}$$

If we glue these three exact sequences together and notice that:

$$\iota_0\iota_1(1\otimes 1) = \iota_0(1) = \sum x_i\otimes x_i^* = \mu_4(1\otimes 1),$$

we get the periodic projective Π-bimodule resolution of Π given in the statement. □

4.2 Some results on Hochschild homology and cohomology of Π

By Theorem 3.2.0.8 we know that Π is a graded Frobenius algebra with Nakayama automorphism η of period 3. Furthermore, Proposition 4.1.0.2 tells us that Π is twisted periodic with twisted period 4 and twisted shift $h = s + 2$. Therefore, as explicitly stated in Theorem 4.1.0.6, Π is periodic of period 12 and shift $3h$.

As a consequence, we can apply the three results stated in section 2.3 about Hochschild homology, cohomology and cyclic homology.

In particular, Proposition 2.3.2.6 gives the following.

Proposition 4.2.0.1. The following isomorphisms hold.

$$HH_{n+12}(\Pi) \cong HH_n(\Pi)[3h]$$

$$HH^{n+12}(\Pi) \cong HH^n(\Pi)[-3h]$$

$$\overline{HC}_{n+12}(\Pi) \cong \overline{HC}_n(\Pi)[3h]$$

for all $n \geq 1$.

Furthermore, Proposition 2.3.2.11 implies:

Proposition 4.2.0.2. The following isomorphisms hold.

$$\begin{aligned} HH_i(\Pi) \cong& HH_{11-i}(\Pi)^*[3h], \qquad i = 1, \ldots, 10 \\ HH_{12}(\Pi) \cong& HH_{11}(\Pi)^*[6h]. \end{aligned}$$

Furthermore, by Remark 4.1.0.4 we have the graded Π-bimodule isomorphism:

$$\Pi^\vee \cong \Pi_{\eta^{-1}}[h-3].$$

Hence, Proposition 2.3.2.12 gives the following connection between the Hochschild cohomology and homology of Π.

Proposition 4.2.0.3. The following isomorphisms hold.

$$\begin{aligned} HH^i(\Pi) \cong& HH_{3-i}(\Pi)[-3], \qquad i = 1, 2 \\ HH^i(\Pi) \cong& HH_{15-i}(\Pi)[-3h-3], \qquad i = 3, \ldots, 12. \end{aligned}$$

Proof. Using the notation of Proposition 2.3.2.12, we have that $m = 3$, $q = 4$,

$h = h$ and $t = h - 3$. Therefore, the statement of Proposition 2.3.2.12 implies:

$$HH_{4+j}(\Pi) \cong HH^{24-j-1}(\Pi)[6h+3] \tag{4.2.0.1}$$

for $j = 1, \ldots, 15$.

Now, fix $i \in \{1, \ldots, 10\}$ and let $j = 11 - i$. Then we have the following:

$$\begin{aligned} HH^i(\Pi) \cong & HH^{12+i}(\Pi)[3h] = HH^{24-j-1}(\Pi)[3h] = HH^{24-j-1}(\Pi)[6h+3][-3h-3] \\ \cong & HH_{4+j}(\Pi)[-3h-3] = HH_{15-i}(\Pi)[-3h-3], \end{aligned}$$

where the first isomorphism follows from Proposition 4.2.0.1, and the third to last follows from (4.2.0.1).

Notice that, for $i = 1, 2$ we have $15 - i > 12$; therefore, we can use the first isomorphism in Proposition 4.2.0.1 to get:

$$HH^i(\Pi) \cong HH_{15-i}(\Pi)[-3h-3] \cong HH_{3-i}(\Pi)[-3],$$

that gives the first statement.

Finally, for $i \in \{10, 11\}$, let $j = 23 - i$. Then, using Proposition 4.2.0.1 and (4.2.0.1) we get the following isomophisms:

$$\begin{aligned} HH^i(\Pi) = & HH^{24-j-1}(\Pi) \cong HH_{4+j}(\Pi)[-6h-3] \\ = & HH_{27-i}(\Pi)[-6h-3] \cong HH_{15-i}(\Pi)[-3h-3]. \end{aligned}$$

□

We now explicitly write the Hochschild homology complex obtained by applying the functor $- \otimes_{\Pi^e} \Pi$ to the periodic resolution of Π given in Theorem 4.1.0.6, since we will need it for explicit computations later in the thesis.

In order to simplify such computations, we make some between modules that appear in this complex.

Consider the functors:

$$(\Pi \otimes -) \otimes_{\Pi^e} \Pi, (-)^S : (S\text{-}\Pi)\text{-bimod} \to S\text{-bimod}$$

where, for an $(S\text{-}\Pi)$-bimodule M, M^S is the S-centraliser of M, defined in Definition 3.2.0.4.

Lemma 4.2.0.4. There is a natural isomorphism

$$\Psi : (\Pi \otimes -) \otimes_{\Pi^e} \Pi \xrightarrow{\sim} (-)^S :$$

given on (S-Π)-bimodules M by:

$$\begin{array}{rccc} \Psi_M : & (\Pi \otimes M) \otimes_{\Pi^e} \Pi & \to & M^S \\ & (a \otimes m) \otimes b & \mapsto & mba. \end{array}$$

Proof. First of all notice that, if $(a \otimes m) \otimes b \in (\Pi \otimes M) \otimes_{\Pi^e} \Pi$, then $mba \in M^S$. Indeed, since $\otimes = \otimes_S$, we have $t(a) = s(m)$.

Now, for all S-Π bimodules M, consider the S-bimodule morphism:

$$\begin{array}{rccc} \zeta_M : & M^S & \to & (\Pi \otimes M) \otimes_{\Pi^e} \Pi \\ & m & \mapsto & (1 \otimes m) \otimes_{\Pi^e} 1. \end{array}$$

Then we have:

- $\zeta_M \Psi_M((a \otimes m) \otimes_{\Pi^e} b) = \zeta_M(mba) = (1 \otimes mba) \otimes_{\Pi^e} 1 = (a \otimes m) \otimes_{\Pi^e} b$;
- $\Psi_M \zeta_M(m) = \Psi_M((1 \otimes m) \otimes_{\Pi^e} 1) = m$.

Therefore Ψ_M is an isomorphism for all S-Π bimodules M.

Also, Ψ is a natural transformation, that is, if $f : M \to N$ is an S-Π bimodule morphism, then the following diagram commutes.

$$\begin{array}{ccc} (\Pi \otimes M) \otimes_{\Pi^e} \Pi & \xrightarrow{\Psi_M} & M^S \\ \downarrow{\scriptstyle (1\otimes f)\otimes_{\Pi^e} 1} & & \downarrow{\scriptstyle f} \\ (\Pi \otimes N) \otimes_{\Pi^e} \Pi & \xrightarrow{\Psi_N} & N^S \end{array}$$

Indeed we have

$$\begin{aligned} f\Psi_M((a \otimes m) \otimes_{\Pi^e} b) &= f(mba) = f(m)ba = \Psi_N((a \otimes f(m)) \otimes_{\Pi^e} b) \\ &= \Psi_N((1 \otimes f) \otimes_{\Pi^e} 1)((a \otimes m) \otimes_{\Pi^e} b). \end{aligned}$$

□

Proposition 4.2.0.5. The homology of the following complex is the Hochschild homology $HH_*(\Pi)$ of Π.

$$\begin{aligned} \ldots \to \Pi^S[3h] &\overset{\mu'_{12}}{\to} (\mathcal{N}^{(2)})^S[2h+3] \overset{\mu'_{11}}{\to} (R \otimes \mathcal{N}^{(2)})^S[2h] \overset{\mu'_{10}}{\to} (V \otimes \mathcal{N}^{(2)})^S[2h] \\ &\overset{\mu'_9}{\to} (\mathcal{N}^{(2)})^S[2h] \overset{\mu'_8}{\to} \mathcal{N}^S[h+3] \overset{\mu'_7}{\to} (R \otimes \mathcal{N})^S[h] \overset{\mu'_6}{\to} (V \otimes \mathcal{N})^S[h] \\ &\overset{\mu'_5}{\to} \mathcal{N}^S[h] \overset{\mu'_4}{\to} \Pi^S[3] \overset{\mu'_3}{\to} (R \otimes \Pi)^S \overset{\mu'_2}{\to} (V \otimes \Pi)^S \overset{\mu'_1}{\to} \Pi^S \to 0. \end{aligned} \tag{4.2.0.2}$$

The differential maps μ'_i are given by:

$$\begin{aligned}\mu'_{4k+1}(v\otimes a) =& a\eta^{-k}(v) - va\\ \mu'_{4k+2}(e_x(\alpha_p\alpha_q-\alpha_q\alpha_p)\otimes a) =& e_x(\alpha_p\otimes\alpha_q a - \alpha_q\otimes\alpha_p a)\\ &+ e_{t(e_x)\alpha_p}\alpha_q\otimes a\eta^{-k}(\alpha_p) - e_{t(e_x)\alpha_q}\alpha_p\otimes a\eta^{-k}(\alpha_q)\\ \mu'_{4k+3}(a) =& \sum_{v\in\overline{Q}_1}\widetilde{r}_v(a\eta^{-k}(v)-va)\\ \mu'_{4k+4}(b) =& \sum_{i\in\overline{Q}_0} x_i^*\eta(a)\eta^{-k}(x_i),\end{aligned}$$

where $v\in V$, $a\in\mathcal{N}^{(k)}$, $b\in\mathcal{N}^{(k+1)}$, $p,q\in\{0,1,2\}$, $p<q$ and $\{x_i\}$ is a basis of Π consisting of all nonzero paths, with $\{x_i^*\}$ is its dual basis under the non-degenerate bilinear form $(.,.)$ given in Definition 3.2.0.7.

Proof. Let $S_\bullet$ be the projective Π-bimodule resolution of Π given in (4.1.0.10). Then the Hochschild homology $HH_*(\Pi)$ can be calculated by taking the homology of the complex $S_\bullet\otimes_{\Pi^e}\Pi$. By Lemma 4.2.0.4, we know that the S-Π bimodules $S_n\otimes_{\Pi^e}\Pi$ can be identified with the bimodules from (4.2.0.2). Therefore, we just need to check that the maps μ'_i correspond to the differentials $\mu_i\otimes_{\Pi^e}\mathrm{id}_\Pi$ under the identification given by Lemma 4.2.0.4. This can be done by direct computation.

For example, the formula for μ'_{4k+1} is obtained as follows:

$$\begin{aligned}\mu'_{4k+1}(v\otimes a) =& \Psi(\mu_{4k+1}\otimes 1_\Pi)\zeta(v\otimes a) = \Psi(\mu_{4k+1}\otimes_{\Pi^e} 1_\Pi)(1\otimes(v\otimes a)\otimes_{\Pi^e} 1)\\ =& \Psi(\mu_{4k+1}(1\otimes v\otimes a)\otimes_{\Pi^e} 1) = \Psi(\mu_{4k+1}(1\otimes v\otimes 1\cdot\eta^k(a))\otimes_{\Pi^e} 1)\\ =& \Psi(\mu_{4k+1}(1\otimes v\otimes 1)\otimes_{\Pi^e}\eta^k(a)) = \Psi((v\otimes 1 - 1\otimes v)\otimes_{\Pi^e}\eta^k(a))\\ =& 1\cdot\eta^k(a)v - v\cdot\eta^k(a) = a\eta^{-k}(v) - va.\end{aligned}$$

The differential maps μ'_{4k+2}, μ'_{4k+3} and μ'_{4k+4} can be obtained similarly. □

4.3 Connection between cyclic and Hochschild homology of Π

In this section we show that, if we are able to compute the (graded) reduced cyclic homology groups of Π, then we get the Hochschild homology and, in turn, the Hochschild cohomology of Π.

Definition 4.3.0.1. Let $M=\bigoplus_{i\in\mathbb{Z}_{\geq 0}} M(i)$ be a $\mathbb{Z}_{\geq 0}$-graded vector space, and $0\leq a\leq b$ two integers. We say that M **lives in degrees** $[a,b]$ if $M(i)=0$ for all $i<a$ and all $i>b$.

Recall that, by Proposition 2.2.3.4, there is an exact sequence, called Connes' exact sequence:

$$0 \to \overline{HH}_0(\Pi) \to \overline{HH}_1(\Pi) \to \overline{HH}_2(\Pi) \to \overline{HH}_3(\Pi) \to \dots \qquad (4.3.0.1)$$

where all the maps are (weight) degree-preserving and

$$\overline{HC}_n(\Pi) \cong \ker(\overline{HH}_{n+1}(\Pi) \to \overline{HH}_{n+2}(\Pi)) = \mathrm{im}(\overline{HH}_n(\Pi) \to \overline{HH}_{n+1}(\Pi)).$$

Also, $\overline{HH}_n(\Pi) = HH_n(\Pi)$ for $n \geq 1$ by Corollary 2.2.3.3.

Now, let $(\widetilde{S}_\bullet, \mu')$ be the complex (4.2.0.2) from Proposition 4.2.0.5. This computes the Hochschild homology of Π, and thus:

$$HH_i(\Pi) \cong \frac{\ker \mu'_i}{\mathrm{im}\ \mu'_{i+1}}$$

for all $i \geq 1$. Hence $HH_i(\Pi)$ is a subquotient of $\widetilde{S}_i$, and therefore it lives in the same weight degrees where $\widetilde{S}_i$ lives.

By Proposition 4.2.0.2 we know that, in order to determine in what weight degrees $\overline{HH}_0(\Pi), HH_1(\Pi), \dots, HH_{12}(\Pi)$ live in, it is enough to compute where $\overline{HH}_0(\Pi), HH_1(\Pi), \dots, HH_5(\Pi)$ and $HH_{11}(\Pi)$ live. Furthermore, Π lives in weight degrees 0 to $s - 1 = h - 3$ by Proposition 3.2.0.6.

As a consequence, we have:

- $\widetilde{S}_0 = \Pi^S$; therefore $\overline{HH}_0(\Pi)$ lives in degrees $[0, h-3]$.
- $\widetilde{S}_1 = (V \otimes \Pi)^S$; therefore $HH_1(\Pi)$ lives in degrees $[1, h-2]$.
- $\widetilde{S}_2 = (R \otimes \Pi)^S$; therefore $HH_2(\Pi)$ lives in degrees $[2, h-1]$.
- $\widetilde{S}_3 = \Pi^S[3]$; therefore $HH_3(\Pi)$ lives in degrees $[3, h]$.
- $\widetilde{S}_4 = \mathcal{N}^S[h]$; therefore $HH_4(\Pi)$ lives in degrees $[h, 2h-3]$.
- $\widetilde{S}_5 = (V \otimes \mathcal{N})^S[h]$; therefore $HH_5(\Pi)$ lives in degrees $[h+1, 2h-2]$.
- $\widetilde{S}_{11} = (\mathcal{N}^{(2)})^S[2h+3]$; therefore $HH_{11}(\Pi)$ lives in degrees $[2h+3, 3h]$.

Therefore, Proposition 4.2.0.2 implies that:

- $HH_6(\Pi)$ lives in degrees $[h+2, 2h-1]$.
- $HH_7(\Pi)$ lives in degrees $[h+3, 2h]$.
- $HH_8(\Pi)$ lives in degrees $[2h, 3h-3]$.
- $HH_9(\Pi)$ lives in degrees $[2h+1, 3h-2]$.
- $HH_{10}(\Pi)$ lives in degrees $[2h+2, 3h-1]$.

- $HH_{12}(\Pi)$ lives in degrees $[3h, 4h-3]$.

Now, using the exactness of the graded complex (4.3.0.1), together with the knowledge of the weight degrees the $HH_i(\Pi)$'s live in, we have the following diagram:

$$
\begin{array}{cccccc}
 & 0 & & & & \\
\text{degrees} & \downarrow & & & & \\
[0,h-3] & \overline{HH}_0(\Pi) & \cong & & C & \overline{HC}_0(\Pi)\cong C \\
 & {\scriptstyle B_0}\downarrow & & & \cong\downarrow & \\
[1,h-2] & HH_1(\Pi) & \cong & X_1 & \oplus C & \overline{HC}_1(\Pi)\cong X_1 \\
 & {\scriptstyle B_1}\downarrow & & \cong\downarrow & & \\
[2,h-1] & HH_2(\Pi) & \cong & X_1 & \oplus X_2 & \overline{HC}_2(\Pi)\cong X_2 \\
 & {\scriptstyle B_2}\downarrow & & & \cong\downarrow & \\
[3,h] & HH_3(\Pi) & \cong & K_1[h] & \oplus X_2 & \overline{HC}_3(\Pi)\cong K_1[h] \\
 & {\scriptstyle B_3}\downarrow & & \cong\downarrow & & \\
[h,2h-3] & HH_4(\Pi) & \cong & K_1[h] & \oplus X_3 & \overline{HC}_4(\Pi)\cong X_3 \\
 & {\scriptstyle B_4}\downarrow & & & \cong\downarrow & \\
[h+1,2h-2] & HH_5(\Pi) & \cong & X_4 & \oplus X_3 & \overline{HC}_5(\Pi)\cong X_4 \\
 & {\scriptstyle B_5}\downarrow & & \cong\downarrow & \cong\downarrow & \\
[h+2,2h-1] & HH_6(\Pi) & \cong & X_4^*[3h] & \oplus X_3^*[3h] & \overline{HC}_6(\Pi)\cong X_3^*[3h] \\
 & {\scriptstyle B_6}\downarrow & & & \cong\downarrow & \\
[h+3,2h] & HH_7(\Pi) & \cong & K_1^*[2h] & \oplus X_3^*[3h] & \overline{HC}_7(\Pi)\cong K_1^*[2h] \\
 & {\scriptstyle B_7}\downarrow & & \cong\downarrow & & \\
[2h,3h-3] & HH_8(\Pi) & \cong & K_1^*[2h] & \oplus X_2^*[3h] & \overline{HC}_8(\Pi)\cong X_2^*[3h] \\
 & {\scriptstyle B_8}\downarrow & & & \cong\downarrow & \\
[2h+1,3h-2] & HH_9(\Pi) & \cong & X_1^*[3h] & \oplus X_2^*[3h] & \overline{HC}_9(\Pi)\cong X_1^*[3h] \\
 & {\scriptstyle B_9}\downarrow & & \cong\downarrow & & \\
[2h+2,3h-1] & HH_{10}(\Pi) & \cong & X_1^*[3h] & & \overline{HC}_{10}(\Pi)=0 \\
 & {\scriptstyle B_{10}}\downarrow & & & & \\
[2h+3,3h] & HH_{11}(\Pi) & \cong & K_2[3h] & & \overline{HC}_{11}(\Pi)\cong K_2[3h] \\
 & {\scriptstyle B_{11}}\downarrow & & \cong\downarrow & & \\
[3h,4h-3] & HH_{12}(\Pi) & \cong & K_2^*[3h] & & \overline{HC}_{12}(\Pi)=0 \\
 & \downarrow & & & & \\
 & \vdots & & & &
\end{array}
\tag{4.3.0.2}
$$

The left column contains the weight degrees the $HH_i(\Pi)$'s live in. The second column contains the Connes' exact sequence (4.3.0.1); furthermore, we can we can use the following three facts to establish an isomorphism of $HH_i(\Pi)$ with the vector spaces in the third column, for all $i = 1, \dots, 12$:

1. The sequence (4.3.0.1) is exact and degree-preserving.
2. We have isomorphisms:

$$HH_i(\Pi) \cong HH_{11-i}(\Pi)^*[3h], \qquad i = 6, \dots, 10.$$

3. We have an isomorphism:

$$HH_{12}(\Pi) \cong HH_{11}(\Pi)^*[6h].$$

In particular, we know that B_0 is injective by 1. Therefore, if we set $\overline{HH}_0(\Pi) = C$, then C is a direct summand of $HH_1(\Pi)$. Therefore, there must be a space X_1 such that $HH_1(\Pi) \cong C \oplus X_1$. By 1. we also know that X_1 embeds in $HH_2(\Pi)$. Therefore, there is some space Y such that $HH_3(\Pi) \cong X_2 \oplus Y$. But Y embeds in $HH_4(\Pi)$ by 1., and therefore it must live both in degrees $[3, h]$ and $[h, 2h-3]$. Therefore, we can say that $Y = K_1[h]$ for some vector space K_1 living in degree 0. Once again using 1., we know that $K_1[h]$ embeds in $HH_4(\Pi)$, and therefore there is a space X_3 such that $HH_4(\Pi) \cong K_1[h] \oplus X_3$. But X_3 embeds in $HH_5(\Pi)$ by 1, and therefore $HH_5(\Pi) \cong X_3 \oplus X_4$ for some space X_4. Now, using 2., we know that $HH_6(\Pi) \cong X_4^*[3h] \oplus X_3^*[3h]$. Similarly we get the isomorphisms for $HH_7(\Pi), \dots, HH_{10}(\Pi)$. Now, since B_9 is surjective, then B_{10} is the zero map and, in turn, B_{11} is injective. Furthermore, if we let $HH_{11}(\Pi) = Y$, then it lives both in degrees $[2h+3, 3h]$ and $[3h, 4h-3]$ since it embeds in $HH_{12}(\Pi)$. Thus we have that $HH_{11}(\Pi) = K_2[3h]$ for some space K_2 that lives in degree 0. Finally, using 3., we have $HH_{12}(\Pi) \cong K_2^*[3h]$.

The last column gives the reduced cyclic homology groups given in terms of X_1, X_2, X_3, X_4, K_1 and K_2, and is obtained using $\overline{HC}_i(\Pi) \cong \ker B_{i+1}$ for all $i \geq 0$.

We now introduce Hilbert series for vector spaces.

Definition 4.3.0.2. Let $W = \oplus_{n \in \mathbb{Z}_{\geq 0}} W(n)$ be a $\mathbb{Z}_{\geq 0}$-graded k-vector space. The **Hilbert series** of W is defined as the series:

$$h_W(t) = \sum_{n=0}^{\infty} \dim W(n) t^n,$$

Remark 4.3.0.3. Notice that this definition is different from the definition of Hilbert series for a graded S-bimodule given in Definition 2.3.3.10. Indeed, the

Hilbert series $H_W(t)$ of an S-bimodule W is a $|Q_0| \times |Q_0|$ matrix with polynomial entries, while the Hilbert series $h_W(t)$ of a graded vector space W is a formal series.

The two definitions agree if and only if $S = k$, i.e., if the quiver Q has only one vertex.

We now give the definition of a formal series obtained from the Hilbert series of the reduced cyclic homology groups of Π. Its computation will turn out to be crucial for our purposes.

Definition 4.3.0.4. Let A be a $\mathbb{Z}_{\geq 0}$-graded algebra. Define the **Euler characteristic** of the reduced cyclic homology of A as the following formal series:

$$\chi_{\overline{HC}_*(A)}(t) = \sum_{i=0}^{\infty} (-1)^i h_{\overline{HC}_i(A)}(t)$$

Proposition 4.3.0.5. The computation of the Hochschild homology, cohomology and reduced cyclic homology groups of Π follows from the computation of the Euler characteristic $\chi_{\overline{HC}_*(\Pi)}(t)$ of Π, together with $HH_0(\Pi)$, $HH_1(\Pi)$, $HH_4(\Pi)$ and $HH^0(\Pi)$.

Proof. By Proposition 4.2.0.3, the Hochschild cohomology of Π can be deduced from the Hochschild homology of Π. Furthermore, looking at diagram (4.3.0.2), it is clear that the computation of $HH_*(\Pi)$ follows from the reduced cyclic homology $\overline{HC}_*(\Pi)$ or, equivalently, from the spaces $C = \overline{HH}_0(\Pi), X_1, X_2, X_3, X_4, K_1, K_2$.

As a first step, we want to show that these spaces can actually be deduced from the knowledge of the Euler characteristic of $\overline{HC}_*(\Pi)$ together with C, X_1, X_3. Notice that:

- C lives in degrees $[1, h-3]$;
- X_1 lives in degrees $[2, h-2]$;
- X_2 lives in degrees $[3, h-1]$;
- $K_1[h]$ lives in degree h;
- X_3 lives in degrees $[h+1, 2h-3]$;
- X_4 lives in degrees $[h+2, 2h-2]$;
- $K_2[3h]$ lives in degree $3h$.

In particular, C, X_1 and X_2 are the only spaces among the $\overline{HC}_i(\Pi)$'s that live in degrees $[1, h-1]$. Therefore, if we know C, X_1 and $\chi_{\overline{HC}_*(\Pi)}(t)$, we get X_2 up to isomorphism. A similar argument applies to get X_4: the spaces X_3, X_4 and $X_3^*[3h]$ are the only ones among the $\overline{HC}_i(\Pi)$'s that live in degrees $[h+1, 2h-1]$.

Therefore, the knowledge of X_3 and $\chi_{\overline{HC}_*(\Pi)}(t)$ gives X_4 up to isomorphism. Furthermore, $K_1[h]$ and $K_2[3h]$ are the only spaces among the $\overline{HC}_i(\Pi)$'s that live in degree h and $3h$, respectively. Therefore, they can be deduced (up to isomorphism) directly from $\chi_{\overline{HC}_*(\Pi)}(t)$.

To sum up, we showed that the reduced cyclic homology $\overline{HC}_*(\Pi)$ can be deduced from the Euler characteristic $\chi_{\overline{HC}_*(\Pi)}(t)$ together with the spaces C, X_1 and X_3. In turn, these also give the Hochschild homology and cohomology of Π.

Therefore, the statement follows if we show that the computation of the spaces C, X_1 and X_3, together with $\chi_{\overline{HC}_*(\Pi)}(t)$, follows the one of $HH_0(\Pi)$, $HH_1(\Pi)$ and $HH_4(\Pi)$, together with $\chi_{\overline{HC}_*(\Pi)}(t)$. Indeed:

- $\overline{HH}_0(\Pi) \cong C$, so $HH_0(\Pi)$ gives C;
- $HH_1(\Pi) \cong X_1 \oplus C$, so $HH_1(\Pi)$ (together with C) gives X_1;
- $HH_4(\Pi) \cong K_1[h] \oplus X_3$. Since $K_1[h]$ is the only space between the $\overline{HC}_i(\Pi)$'s living in degree h, the knowledge of $HH_4(\Pi)$ together with the Euler characteristic of $\overline{HC}_*(\Pi)$ gives X_3.

To sum up, knowing $\chi_{\overline{HC}_*(\Pi)}(t)$, $HH_0(\Pi)$, $HH_1(\Pi)$ and $HH_4(\Pi)$ gives the reduced cyclic homology of Π and, in turn, the Hochschild homology and cohomology of Π. □

We now write explicitly how the spaces C, X_1, X_2, X_3, X_4, K_1 and K_2 from diagram (4.3.0.2) can be obtained from $\chi_{\overline{HC}_*(\Pi)}(t)$, $\overline{HH}_0(\Pi)$, $HH_1(\Pi)$ and $HH_4(\Pi)$.

Definition 4.3.0.6. For two natural numbers $0 \leq a \leq b$, define

$$\chi_{\overline{HC}_*(\Pi)}(t)|_a^b$$

to be the part of $\chi_{\overline{HC}_*(\Pi)}(t)$ in degrees $[a, b]$.

Corollary 4.3.0.7. The Hilbert series of the spaces in (4.3.0.2) are the following:

$$\begin{aligned}
h_C(t) =& h_{\overline{HH}_0(\Pi)}(t) \\
h_{X_1}(t) =& h_{HH_1(\Pi)}(t) - h_{\overline{HH}_0(\Pi)}(t) \\
h_{X_2}(t) =& \chi_{\overline{HC}_*(\Pi)}(t)|_2^{h-1} + h_{HH_1(\Pi)}(t) - h_{\overline{HH}_0(\Pi)}(t) \\
h_{K_1[h]}(t) =& -\chi_{\overline{HC}_*(\Pi)}(t)|_h^h \\
h_{X_3}(t) =& h_{HH_4(\Pi)}(t) + \chi_{\overline{HC}_*(\Pi)}(t)|_h^h \\
h_{X_4}(t) =& h_{HH_4(\Pi)}(t) + \chi_{\overline{HC}_*(\Pi)}(t)|_h^h - \chi_{\overline{HC}_*(\Pi)}(t)|_{h+1}^{2h-2} \\
h_{K_2[3h]}(t) =& -\chi_{\overline{HC}_*(\Pi)}(t)|_{3h}^{3h}
\end{aligned}$$

Proof. The first two formulas follow directly from the first two rows of diagram (4.3.0.2).

To get the third formula, notice that X_1 and X_2 are the only two spaces among the $\overline{HC}_i(\Pi)$'s that live in degrees $[2, h-1]$. Also, since X_1 (resp. X_2) is an odd (resp. even) reduced cyclic homology group, we have:

$$\chi_{\overline{HC}_*(\Pi)}(t)|_2^{h-1} = -h_{X_1}(t) + h_{X_2}(t). \tag{4.3.0.3}$$

Hence the third formula follows from (4.3.0.3) and the second formula.

We have that $K_1[h]$ is the only space among the $\overline{HC}_i(\Pi)$'s that lives in degree h. Therefore we get the fourth formula since $K_1[h] \cong \overline{HC}_3(\Pi)$ is an odd reduced cyclic homology group.

To get the fifth formula notice that, since $HH_4(\Pi) \cong X_3 \oplus K_1[h]$, then $h_{X_3}(t) = h_{HH_4(\Pi)}(t) - h_{K_1[h]}(t)$. Therefore the fifth formula follows from the fourth one.

To get the sixth formula, notice that X_3, X_4 are the only spaces among the $\overline{HC}_i(\Pi)$ that live in degrees $[h+1, 2h-2]$. Also, since X_3 (resp. X_4) is an even (resp. odd) reduced cyclic homology group, we have:

$$\chi_{\overline{HC}_*(\Pi)}(t)|_{h+1}^{2h-2} = h_{X_3}(t) - h_{X_4}(t). \tag{4.3.0.4}$$

Hence the sixth formula follows from (4.3.0.4) and the fifth formula.

Finally, since $K_2[3h] \cong \overline{HC}_{11}(\Pi)$ is the only space among the reduced cyclic homology groups that lives in degree $3h$, and since it is an odd reduced cyclic homology group, we get the last formula. $\square$

5

Computation of the Euler characteristic of $\overline{HC}_*(\Pi)$

In this chapter we present our attempts to compute $\chi_{\overline{HC}_*(\Pi)}(t)$, the Euler characteristic of the reduced cyclic homology of Π. In particular:

- In the first section we show that this can be deduced from the determination of $\det H_\Pi(t)$.
- In the second section we use the formula given in Proposition 2.3.3.11 to write $H_\Pi(t)$ as a quotient:

$$H_\Pi(t) = \frac{1 - Pt^h}{1 - Ct + Dt^2 - t^3},$$

  where P is the permutation matrix associated to the inverse Nakayama permutation ν^{-1}, C is the adjacency matrix of $\overline{Q}$ and D is the adjacency matrix of $\overline{Q}^*$, the opposite quiver of $\overline{Q}$.
- In the third section we compute $\det(1 - Pt^h)$.
- In the fourth section we give a conjectural formula for $\det(1-Ct+Dt^2-t^3)$, supporting it with examples and a GAP code.

5.1 Formula for $\chi_{\overline{HC}_*(\Pi)}(t)$

The following type of result first appeared in [22, Prop. 3.7.1] for preprojective algebras of non-Dynkin quivers.

Theorem 5.1.0.1. Let A be a preprojective algebra of a non-Dynkin quiver, and $\chi_{\overline{HC}_*(A)}(t) = \sum_{k=0}^{\infty} a_k t^k$ be the Euler characteristic of the reduced cyclic homology of A. Then:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s). \tag{5.1.0.1}$$

The formula (5.1.0.1) was later used in the proof of [21, Lem. 4.4.1] for preprojective algebras of Dynkin type, and in [27, §3.2] for other type of algebras.

It was outlined in private correspondence with Pavel Etingof that a formula like (5.1.0.1) actually holds for all graded algebras. However, this does not appear anywhere in print. Therefore, we state it as a conjecture, and in the following we will make sure to point out the results that rely on this.

Conjecture 5.1.0.2. Let A be a graded algebra, and and $\chi_{\overline{HC}_*(A)}(t) = \sum_{k=0}^{\infty} a_k t^k$ the Euler characteristic of the reduced cyclic homology of A. Then:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s). \tag{5.1.0.2}$$

Therefore, provided Conjecture 5.1.0.2 holds true, in order to compute the coefficients a_k that give the Euler characteristic of the reduced cyclic homology of Π, it is enough to compute the determinant of the Hilbert series of Π.

By Theorem 3.1.0.7(2) we know that the algebra Π is $(h-3,3)$-Koszul. Therefore, we can apply Proposition 2.3.3.11 to get a decomposition of $H_\Pi(t)$.

Theorem 5.1.0.3. Let P be the permutation matrix associated to the inverse of the Nakayama permutation ν^{-1}, and C, D the adjacency matrix of $\overline{Q}$ and of the opposite quiver $\overline{Q}^*$ of $\overline{Q}$, respectively. Then:

$$H_\Pi(t) = (1-Pt^h)(1-Ct+Dt^2-t^3)^{-1}. \tag{5.1.0.3}$$

Proof. By Proposition 4.1.0.2 we know that the fourth syzygy of the Π-bimodule Π is given by $\mathcal{N}[h] = \Pi_{\eta^{-1}}[h]$. Therefore, the fourth syzygy of the right Π-module $S \cong S \otimes_\Pi \Pi$ is given by $S_{\nu^{-1}}[h] \cong S \otimes_\Pi \mathcal{N}[h]$. Thus the numerator of (5.1.0.3) is given by $1-Pt^h$ by Proposition 2.3.3.11, once one notices that the Hilbert series of $S_{\nu^{-1}}$ is equal to P.

To get the denominator of (5.1.0.3) notice that, using the notation of Proposition 2.3.3.11, we have:

- $K_1 = V$, that has Hilbert series Ct;
- $K_2 = R$, that has Hilbert series Dt^2;
- $K_3 = VR \cap RV \cong S[3]$ by Lemma 4.1.0.1, that has Hilbert series the identity matrix multiplied by t^3.

Therefore the statement follows from Proposition 2.3.3.11. □

In view of Conjecture 5.1.0.2, we want to compute the determinant of $H_\Pi(t)$. By Theorem 5.1.0.3, it is enough to calculate $\det(1-Pt^h)$ and $\det(1-Ct+Dt^2-t^3)$ separately. This is the content of the next two sections.

5.2 Computation of $\det(1 - Pt^h)$

In this section we analyze the matrix $1 - Pt^h$, where P is the permutation matrix associated to the inverse Nakayama permutation ν^{-1}. In particular, we exhibit an indexing of the vertices of $\overline{Q}$ that makes the computation of $\det(1 - Pt^h)$ easy.

By the explicit construction of the Nakayama permutation ν of Π given in Lemma 3.2.0.8, we have:

$$\nu^{-1}(x_1, x_2, x_3) = (x_2, x_3, x_1)$$

for all $(x_1, x_2, x_3) \in \overline{Q}_0$. Therefore ν^{-1} has order 3, so it is possible to write it as a product of disjoint cycles of length three and cycles of length 1.

Now, $\nu^{-1}(x_1, x_2, x_3) = (x_1, x_2, x_3)$ if and only if $x_1 = x_2 = x_3$. This can happen if and only if $s \equiv 1 \pmod 3$ and $x_1 = x_2 = x_3 = \frac{s-1}{3}$.

Therefore, ν^{-1} can be written just as product of disjoint cycles of length 3 if $s \not\equiv 1 \pmod 3$, and as product of one cycle of length 1 and cycles of length 3 if $s \equiv 1 \pmod 3$. Now, the number of vertices of $\overline{Q}$ is given by:

$$|\overline{Q}_0| = |\{(i, j, k) \in \mathbb{Z}_{\geq 0}, i + j + k = s - 1\}| = \frac{s(s+1)}{2}.$$

Hence the permutation ν^{-1} can be written as follows:

$$\nu^{-1} = \begin{cases} C_1 \cdot C_2 \cdots C_{\frac{s(s+1)}{6}}, & \text{if } s \not\equiv 1 \pmod 3 \\ C_1 \cdot C_2 \cdots C_{\frac{s(s+1)-2}{6}} D_1, & \text{if } s \equiv 1 \pmod 3, \end{cases}$$

where the C_i's are disjoint cycles of length 3, and D_1 is the 1-cycle $id_{(\frac{s+1}{3}, \frac{s+1}{3}, \frac{s+1}{3})}$.

This means that we can choose a labeling for $\overline{Q}_0$ such that the permutation matrix P can be written as the following block diagonal matrix:

$$P = \begin{pmatrix} \boxed{A} & & & \\ & \boxed{A} & & \\ & & \ddots & \\ & & & \boxed{A} \end{pmatrix} \quad \text{if } s \not\equiv 1 \pmod 3$$

$$P = \begin{pmatrix} \boxed{A} & & & & \\ & \boxed{A} & & & \\ & & \ddots & & \\ & & & \boxed{A} & \\ & & & & 1 \end{pmatrix} \quad \text{if } s \equiv 1 \pmod 3.$$

The matrices A appearing in P are the 3×3 matrices that corresponds to the restriction of ν^{-1} to any 3-cycle. In particular, they are given by:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and they are in number equal to $\frac{s(s+1)}{6}$ if $s \not\equiv 1 \pmod 3$ and to $\frac{s(s+1)-2}{6}$ if $s \equiv 1 \pmod 3$.

Therefore, if we let $h = s+2$, $\det(1 - Pt^h)$ can be rewritten as follows:

$$\det(1 - Pt^h) = \begin{cases} \det(1 - At^h)^{\frac{(h-1)(h-2)}{6}}, & \text{if } s \not\equiv 1 \pmod 3 \\ \det(1 - At^h)^{\frac{(h-1)(h-2)-2}{6}}(1 - t^h), & \text{if } s \equiv 1 \pmod 3. \end{cases}$$

Now, we have:

$$1 - At^h = \begin{pmatrix} 1 & -t^h & 0 \\ 0 & 1 & -t^h \\ -t^h & 0 & 1 \end{pmatrix},$$

and hence $\det(1 - At^h) = 1 - t^{3h}$. Therefore, we proved the following.

Proposition 5.2.0.1. Let P be the permutation matrix associated to the inverse Nakayama automorphism ν^{-1}. Then:

$$\det(1 - Pt^h) = \begin{cases} (1 - t^{3h})^{\frac{(h-1)(h-2)}{6}}, & \text{if } s \not\equiv 1 \pmod 3 \\ (1 - t^{3h})^{\frac{(h-1)(h-2)-2}{6}}(1 - t^h), & \text{if } s \equiv 1 \pmod 3. \end{cases}$$

5.3 Computation of $\det(1 - Ct + Dt^2 - t^3)$

We now want to compute the determinant of the matrix $(1 - Ct + Dt^2 - t^3)$, where C is the adjacency matrix of $\overline{Q}$ and D is the adjacency matrix of $\overline{Q}^*$, the opposite quiver of $\overline{Q}$. More explicitly, C and D are $|\overline{Q}_0| \times |\overline{Q}_0|$ matrices whose (i,j)-th entry is given by:

$$C_{ij} = \begin{cases} 1, & \text{if } i \to j \in \overline{Q}_1 \\ 0, & \text{otherwise} \end{cases}, \qquad D_{ij} = \begin{cases} 1, & \text{if } j \to i \in \overline{Q}_1 \\ 0, & \text{otherwise.} \end{cases}$$

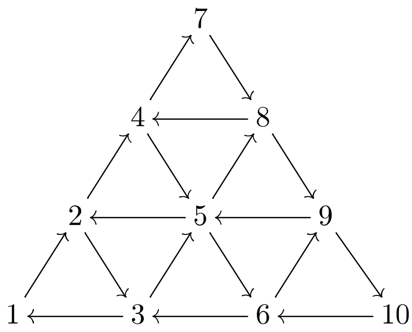
We consider the lexicographic order on triples of non-negative integers that sum up to $s-1$ as labeling of the vertices of the quiver Q.

Example 5.3.0.1. Let $s = 4$. Then we have:

$$\begin{aligned}(3,0,0) &< (2,1,0) < (2,0,1) < (1,2,0) < (1,1,1) \\ &< (1,0,2) < (0,0,3) < (0,2,1) < (0,1,2) < (0,0,3).\end{aligned}$$

With respect to this ordering of $\overline{Q}_0$, the quiver $\overline{Q}$ can be represented as follows.

$\overline{Q} =$



Therefore the matrix $1 - Ct + Dt^2 - t^3$ is the following 10×10 matrix.

$$
\left(\begin{array}{c|cc|ccc|cccc}
1-t^3 & -t & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
t^2 & 1-t^3 & -t & -t & t^2 & 0 & 0 & 0 & 0 & 0 \\
-t & t^2 & 1-t^3 & 0 & -t & t^2 & 0 & 0 & 0 & 0 \\ \hline
0 & t^2 & 0 & 1-t^3 & -t & 0 & -t & t^2 & 0 & 0 \\
0 & -t & t^2 & t^2 & 1-t^3 & -t & 0 & -t & t^2 & 0 \\
0 & 0 & -t & 0 & t^2 & 1-t^3 & 0 & 0 & -t & t^2 \\ \hline
0 & 0 & 0 & t^2 & 0 & 0 & 1-t^3 & -t & 0 & 0 \\
0 & 0 & 0 & -t & t^2 & 0 & t^2 & 1-t^3 & -t & 0 \\
0 & 0 & 0 & 0 & -t & t^2 & 0 & t^2 & 1-t^3 & -t \\
0 & 0 & 0 & 0 & 0 & -t & 0 & 0 & t^2 & 1-t^3
\end{array}\right)
$$

Notice that the 1×1 (resp. 3×3 , resp. 6×6) upper left submatrix is the matrix $1 - Ct + Dt^2 - t^3$ for $s = 1$ (resp. $s = 2$, resp. $s = 3$).

Using this ordering of the vertices, we see that the matrices $1 - Ct + Dt^2 - t^3$ for $k = 1, \ldots, s-1$ appear as submatrices of $1 - Ct + Dt^2 - t^3$, just like in Example 5.3.0.1. This could suggest the existence of a recursive formula that allows to compute $\det(1 - Ct + Dt^2 - t^3)$ for s in terms of $\det(1 - Ct + Dt^2 - t^3)$ for $k = 1, \ldots, s-1$.

This is possible for classical preprojective algebras of type ADE. In that setting one needs to compute the determinant of the matrix $1 - Ct + t^2$, where C is the adjacency matrix of the quiver. Choosing an appropriate ordering on the set of vertices, one gets that $1 - Ct + t^2$ is a tridiagonal matrix, and a recursive formula from [18] can be used to compute its determinant.

In our setting, the matrix $1 - Ct + Dt^2 - t^3$ is block tridiagonal. Molinari (see [54]) and Salkuyeh (see [59]) give recursive formulas for the determinant of a block-tridiagonal matrix A. However, in [54] the matrix A is required to have diagonal blocks all of the same size, while [59] requires the computation of the inverses of the diagonal blocks of A. Therefore, we have not been able to get such a formula.

Nonetheless, using the system for computational discrete algebra GAP [32], we have been able to give a formula for $\det(1 - Ct + Dt^2 - t^3)$ that has been verified for $s \leq 30$. This gives rise to the following conjecture.

Conjecture 5.3.0.2. Fix $s \geq 2$. Then the matrix $1 - Ct + Dt^2 - t^3$ has the following determinant.

$$\det(1 - Ct + Dt^2 - t^3) = \begin{cases} (1-t^3)(1-t^{3h})^{\frac{h-3}{2}}, & \text{if } s \equiv 1 \pmod 2 \\ \dfrac{(1-t^3)(1-t^{3h})^{\frac{h-2}{2}}}{(1-t^{\frac{3}{2}h})}, & \text{if } s \equiv 0 \pmod 2 \end{cases} \tag{5.3.0.1}$$

We include the GAP code with explanations.

```
IJKset := function(s)
    local T, i, j, k;
    T := [];
    for i in [0..s] do
        for j in [0..s] do
            for k in [0..s] do
                if i+j+k = s-1 then
                    Add(T, [i,j,k]);
                fi;
            od;
        od;
    od;
    return T;
end;

LexicoIJKorder := function(u, v)
    if u[1]>v[1] then return true; fi;
    if u[1]<v[1] then return false; fi;
    if u[2]>v[2] then return true; fi;
    if u[2]<v[2] then return false; fi;
    if u[3]>v[3] then return true; fi;
    if u[3]<v[3] then return false; fi;
    return false;
end;

det_denom_h_Pi := function(s)
local T, t, M, r, u, i, j, k, determ;
T := IJKset(s);
Sort(T, LexicoIJKorder);
t:=Indeterminate(Rationals, "t");
M:= [];
```

```
for r in [1..Length(T)] do
    M[r] := [];
    i := T[r][1];
    j := T[r][2];
    k := T[r][3];
    for u in [1..Length(T)] do
        if r=u then
            M[r][u]:=1-t^3;
        else
            M[r][u]:=0*t;
        fi;
    od;
    if i>0 then
        M[r][Position(T, [i-1,j,k+1])] := t^2;
        M[r][Position(T, [i-1,j+1,k])] := -t;
    fi;
    if j>0 then
        M[r][Position(T, [i+1,j-1,k])] := t^2;
        M[r][Position(T, [i,j-1,k+1])] := -t;
    fi;
    if k>0 then
        M[r][Position(T, [i,j+1,k-1])] := t^2;
        M[r][Position(T, [i+1,j,k-1])] := -t;
    fi;
od;
determ:=Determinant(M);
return determ;
end;

conj_form := function(s)
local Det1, conj, t;
t:=Indeterminate(Rationals, "t");
Det1:=det_denom_h_Pi(s);

if RemInt(s,2)=1 then
    conj:= (1-t^3)*(1-t^(3*(s+2)))^((s-1)/2);
else
    conj:= ((1-t^3)*(1-t^(3*(s+2)))^(s/2))/(1-t^(3*(s+2)/2));
fi;
return Det1=conj;
```

```
end;
```

The functions do the following.

- The function IJKset takes as an input an integer $s \geq 2$ and gives:

$$T = \{(i, j, k \in \mathbb{Z}_{\geq 0}) | i + j + k = s - 1\}.$$

- The function LexicoIJKorder takes as an input a pair of triples u, v, and gives as output `True` if $u > v$ with respect to the lexicographic order, and `False` otherwise.
- The function det_denom_h_Pi takes as input an integer $s \geq 2$, constructs the matrix $M = 1 - Ct + Dt^2 - t^3$ for that value of s, and gives as output its determinant.
- Finally, the function conj_form checks whether $\det(1 - Ct + Dt^2 - t^3)$ and the conjectural formula given in (5.3.0.1) agree for a fixed value of $s \geq 2$, returning `True` if they do.

5.4 Computation of $\chi_{\overline{HC}_*(\Pi)}(t)$

By Theorem 5.1.0.3, we have that:

$$\det H_\Pi(t) = \frac{\det(1 - Pt^h)}{\det(1 - Ct + Dt^2 - t^3)},$$

where P is the permutation matrix associated to the inverse Nakayama permutation ν^{-1}, and C and D are the adjacency matrices of $\overline{Q}$ and $\overline{Q}^*$, respectively. Therefore, we can use Proposition 5.2.0.1 and Conjecture 5.3.0.2 to get a conjectural formula for $\det H_\Pi(t)$.

Theorem 5.4.0.1. Fix $s \geq 2$, and let $h = s+2$. If Conjecture 5.3.0.2 holds true, then the determinant of $H_\Pi(t)$ is given by:

$$\det H_\Pi(t) = \begin{cases} \dfrac{(1 - t^{3h})^{\frac{(h-2)(h-4)}{6}}(1 - t^{\frac{3}{2}h})}{(1 - t^3)}, & \text{if } s \equiv 0, 2 \bmod 6 \\ \dfrac{(1 - t^{3h})^{\frac{(h-2)(h-4)+1}{6}}(1 - t^h)}{(1 - t^3)}, & \text{if } s \equiv 1 \bmod 6 \\ \dfrac{(1 - t^{3h})^{\frac{(h-2)(h-4)+3}{6}}}{(1 - t^3)}, & \text{if } s \equiv 3, 5 \bmod 6 \\ \dfrac{(1 - t^{3h})^{\frac{(h-2)(h-4)-2}{6}}(1 - t^{\frac{3}{2}h})(1 - t^h)}{(1 - t^3)}, & \text{if } s \equiv 4 \bmod 6 \end{cases}$$

Therefore, Conjecture 5.1.0.2 gives the formula for the Euler characteristic of $\overline{HC}_*(\Pi)$.

Theorem 5.4.0.2. Let $s \geq 2$ and $h = s + 2$. Provided conjectures 5.1.0.2 and 5.3.0.2 hold true, then the Euler characteristic of $\overline{HC}_*(\Pi)$ is given as follows.

$$\chi_{\overline{HC}_*(\Pi)}(t) = \begin{cases} \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)}{6}t^{3h}), & \text{if } s \equiv 0,2 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+1}{6}t^{3h}), & \text{if } s \equiv 1 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)-3}{6}t^{3h}), & \text{if } s \equiv 3,5 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+4}{6}t^{3h}), & \text{if } s \equiv 4 \bmod 6, \end{cases}$$

where

$$I = \{1,\dots,h-1\} \setminus \left(\{1,\dots,h-1\} \cap \left\{\frac{h}{2}, \frac{h}{3}, \frac{2h}{3}\right\}\right),$$

i.e., I is the set of numbers $1,\dots,h-1$ without $\frac{h}{2}, \frac{h}{3}, \frac{2h}{3}$ whenever they are integers.

Proof. Provided Conjecture 5.1.0.2 holds true, we know that:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{r=1}^{\infty} \det H_\Pi(t^r).$$

We can use Theorem 5.4.0.1 to rewrite the right hand side of this formula, and get the coefficients a_k. We divide the analysis in 4 steps, according to the residue class of s modulo 6.

- If $s \equiv 0, 2 \bmod 6$, then we have:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{r\geq 1}(1-t^{3r})^{-1}(1-t^{3rh})^{\frac{(h-2)(h-4)}{6}}(1-t^{\frac{3}{2}rh}).$$

Expanding the right hand side up to degree $3h$ we see that:

$$a_k = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod 3 \\ 1, & \text{if } k = 3r \text{ with } r \not\equiv 0, \frac{1}{2} \bmod h \\ 0, & \text{if } k = 3r \text{ with } r \equiv \frac{1}{2} \bmod h \\ -\frac{(h-2)(h-4)}{6}, & \text{if } k = 3r \text{ with } r \equiv 0 \bmod h. \end{cases}$$

Hence

$$\sum_{k=1}^{3h} a_k t^k = \sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)}{6}t^{3h},$$

and since the $a_k's$ are periodic of period $3h$, we get:

$$\sum_{k\geq 0} a_k t^k = \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)}{6}t^{3h}).$$

- If $s \equiv 1 \bmod 6$, then we have:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{r\geq 1}(1-t^{3r})^{-1}(1-t^{3rh})^{\frac{(h-2)(h-4)+1}{6}}(1-t^{rh}).$$

Expanding the right hand side up to degree $3h$ we see that:

$$a_k = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod 3 \\ 1, & \text{if } k = 3r \text{ with } r \not\equiv 0, \frac{1}{3} \bmod h \\ 0, & \text{if } k = 3r \text{ with } r \equiv \frac{1}{3} \bmod h \\ -\frac{(h-2)(h-4)+1}{6}, & \text{if } k = 3r \text{ with } r \equiv 0 \bmod h. \end{cases}$$

Hence:

$$\sum_{k=1}^{3h} a_k t^k = \sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+1}{6} t^{3h},$$

and since the $a_k{}'s$ are periodic of period $3h$, we get

$$\sum_{k\geq 0} a_k t^k = \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+1}{6} t^{3h}).$$

- If $s \equiv 3, 5 \bmod 6$, then we have:

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{r\geq 1}(1-t^{3r})^{-1}(1-t^{3rh})^{\frac{(h-2)(h-4)+3}{6}}.$$

Expanding the right hand side up to degree $3h$ we see that:

$$a_k = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod 3 \\ 1, & \text{if } k = 3r \text{ with } r \not\equiv 0 \bmod h \\ 1 - \frac{(h-2)(h-4)+3}{6} = -\frac{(h-2)(h-4)-3}{6}, & \text{if } k = 3r \text{ with } r \equiv 0 \bmod h. \end{cases}$$

Hence:

$$\sum_{k=1}^{3h} a_k t^k = \sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)-3}{6} t^{3h},$$

and since the $a_k's$ are periodic of period $3h$, we get:

$$\sum_{k\geq 0} a_k t^k = \frac{1}{(1-t^{3h})}(\sum_{k\in I_{3,5}} t^{3k} - \frac{(h-2)(h-4)-3}{6} t^{3h}).$$

- If $s \equiv 4 \bmod 6$, then we have

$$\prod_{k=1}^{\infty}(1-t^k)^{-a_k} = \prod_{r\geq 1}(1-t^{3r})^{-1}(1-t^{3rh})^{\frac{(h-2)(h-4)-2}{6}}(1-t^{\frac{3}{2}rh})(1-t^{rh}).$$

Expanding the right hand side up to degree $3h$ we see that:

$$a_k = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod 3 \\ 1, & \text{if } k = 3r \text{ with } r \not\equiv 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \bmod h \\ 0, & \text{if } k = 3r \text{ with } r \equiv \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \bmod h \\ 1 - (\frac{(h-2)(h-4)-2}{6} + 2), & \text{if } k = 3r \text{ with } r \equiv 0 \bmod h. \end{cases}$$

Now, we have $1 - (\frac{(h-2)(h-4)-2}{6} + 2) = -\frac{(h-2)(h-4)+4}{6}$, and hence:

$$\sum_{k=1}^{3h} a_k t^k = \sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+4}{6} t^{3h}.$$

Thus, since the $a_k's$ are periodic of period $3h$, we get:

$$\sum_{k\geq 0} a_k t^k = \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+4}{6} t^{3h}).$$

□

6

Computation of $HH_0(\Pi)$, $HH_1(\Pi)$, $HH_4(\Pi)$ and $HH^0(\Pi)$

Fix $s \geq 2$ and let $\Pi = \Pi^{(2,s)}$, $\overline{Q} = \overline{Q}^{(2,s)}$. For a set A, let $\langle A \rangle$ be the k-vector space spanned by A.

In this chapter we explicitly compute $HH_0(\Pi)$, $HH_1(\Pi)$, $HH_4(\Pi)$ and $HH^0(\Pi)$. Thanks to Proposition 4.3.0.5 and the (conjectural) formula for $\chi_{\overline{HC}_*(\Pi)}(t)$ given in Theorem 5.4.0.2 this will in turn give the computation of the whole Hochschild homology, cohomology and cyclic homology groups of Π.

We start with $HH_1(\Pi)$, since some results towards its computation will be useful to deduce $HH_0(\Pi)$ and $HH^0(\Pi)$.

6.1 Computation of $HH_1(\Pi)$

By Proposition 4.2.0.5, we know that the first Hochschild homology group $HH_1(\Pi)$ of Π can be obtained by computing the homology of the following complex:

$$(R \otimes \Pi)^S \xrightarrow{\mu_2'} (V \otimes \Pi)^S \xrightarrow{\mu_1'} \Pi^S \qquad (6.1.0.1)$$

where, if $p, q \in \{0, 1, 2\}$ and $p < q$ we have:

$$\begin{aligned} \mu_1'(v \otimes a) =& va - av \\ \mu_2'(e_x(\alpha_p\alpha_q - \alpha_q\alpha_p) \otimes a) =& e_x\alpha_p \otimes \alpha_q a + e_{t(e_x\alpha_p)}\alpha_q \otimes a\alpha_p \\ & - e_x\alpha_q \otimes \alpha_p a - e_{t(e_x\alpha_q)}\alpha_p \otimes a\alpha_q \end{aligned}$$

for all $x \in \overline{Q}_0$, $v \in V$ and $a \in \Pi$. Therefore:

$$HH_1(\Pi) \cong \frac{\ker \mu_1'}{\text{im } \mu_2'}.$$

In the rest of this section we use results from Section 3.2 to give a combinatorial description of the spaces Π^S, $(V \otimes \Pi)^S$ and $(R \otimes \Pi)^S$, and of the maps μ_1', μ_2'. Then we compute the component $HH_1(\Pi)(3)$ of weight degree 3 of $HH_1(\Pi)$, and show how to get the whole first Hochschild homology group

$HH_1(\Pi)$ from it.

6.1.1 A basis for Π^S

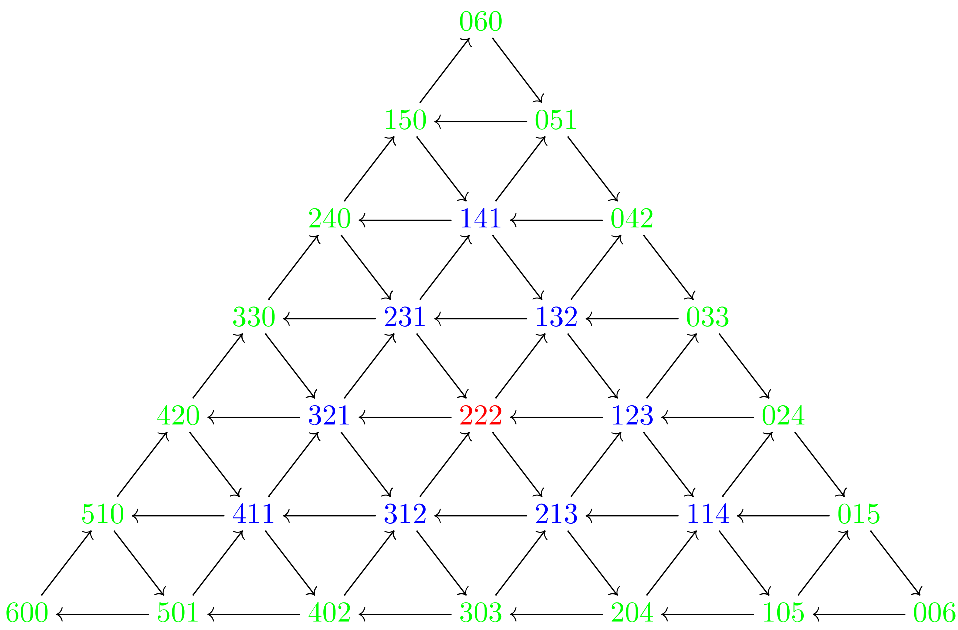
We define a partition on the set of vertices $\overline{Q}_0$ of $\overline{Q}$.

Definition 6.1.1.1. Define $(\overline{Q}_0)_\ell = \{(i,j,k) \in \overline{Q}_0 | \min\{i,j,k\} = \ell\}$.

For each vertex $(i,j,k) \in \overline{Q}_0$ we have $i+j+k = s-1$. This means that $\min\{i,j,k\} \leq \frac{s-1}{3}$, and therefore we get the following decomposition of $\overline{Q}_0$:

$$\overline{Q}_0 = \bigcup_{\ell=0}^{\lfloor \frac{s-1}{3} \rfloor} (\overline{Q}_0)_\ell. \tag{6.1.1.1}$$

Example 6.1.1.2. Let $s = 7$. Consider the quiver $\overline{Q}$ of $\Pi^{(2,7)}$.



The nonempty sets of vertices $(\overline{Q}_0)_\ell$'s are the following:

- $(\overline{Q}_0)_0 = \{\text{green vertices}\}$.
- $(\overline{Q}_0)_1 = \{\text{blue vertices}\}$.
- $(\overline{Q}_0)_2 = \{\text{red vertex}\} = \{(2,2,2)\}$.

For $(i,j,k) \in \overline{Q}_0$, let:

$$T_{(i,j,k)} := e_{(i,j,k)}\alpha_0\alpha_1\alpha_2.$$

Lemma 6.1.1.3. 1. The following decomposition holds:

$$\Pi^S = \bigoplus_{m=0}^{\lfloor \frac{s-1}{3} \rfloor} \Pi^S(3m). \tag{6.1.1.2}$$

2. For each $\ell \geq 0$, the map

$$\begin{array}{rccc} \theta_\ell : & \Pi^S(3\ell) & \to & \langle (\overline{Q}_0)_{\geq \ell} \rangle \\ & T^{\ell}_{(i,j,k)} & \mapsto & (i,j,k) \end{array} \tag{6.1.1.3}$$

is an isomorphism of vector spaces, where $(\overline{Q}_0)_{\geq \ell} = (\overline{Q}_0)_\ell \cup (\overline{Q}_0)_{\ell+1} \cup \dots$.

Equivalently, the set of vertices $(\overline{Q}_0)_{\geq \ell}$ can be identified with basis of $\Pi^S(3\ell)$.

Proof. Let $W = \{e_{(i,j,k)} | (i,j,k) \in \overline{Q}_0\}$. Then the S-bimodule generated by W is S, and we have:

$$(S \otimes \Pi) \cong \Pi.$$

Hence decomposition (6.1.1.2) follows from Proposition 3.2.0.5.

Now, the shortest path $(i,j,k) \to (i,j,k)$ is $e_{(i,j,k)}$ for all $(i,j,k) \in \overline{Q}_0$. Thus, by Proposition 3.2.0.5, the following elements generate $(e_{(i,j,k)}\Pi)^S$:

$$e_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^\ell = T^\ell_{(i,j,k)}, \qquad \ell \geq 0.$$

These elements, if nonzero, are clearly distinct. Hence, in order to show that (6.1.1.3) is an isomorphism, we just need to show that $T^\ell_{(i,j,k)} \neq 0$ if and only if $(i,j,k) \in (\overline{Q}_0)_{\geq \ell}$. This follows from a direct application of Lemma 3.2.0.2. Hence we get the statement. □

Example 6.1.1.4. Let $s = 7$ and consider Example 6.1.1.2.

- If $(i,j,k) \in (\overline{Q}_0)_0 = \{\text{green vertices}\}$, then:

$$T^0_{(i,j,k)} = e_{(i,j,k)} \in \Pi^S(0)$$

  is the only basis element of Π^S starting at (i,j,k).

- If $(i,j,k) \in (\overline{Q}_0)_1 = \{\text{blue vertices}\}$, then:

$$\begin{aligned} T^0_{(i,j,k)} =& e_{(i,j,k)} \in \Pi^S(0) \\ T^1_{(i,j,k)} =& e_{(i,j,k)}\alpha_0\alpha_1\alpha_2 \in \Pi^S(3) \end{aligned}$$

  are the basis elements of Π^S starting at (i,j,k).

- The basis elements of Π^S starting at $(2,2,2)$ are:

$$\begin{aligned}T^0_{(2,2,2)} =& e_{(2,2,2)} \in \Pi^S(0)\\ T^1_{(2,2,2)} =& e_{(2,2,2)}\alpha_0\alpha_1\alpha_2 \in \Pi^S(3)\\ T^2_{(2,2,2)} =& e_{(2,2,2)}(\alpha_0\alpha_1\alpha_2)^2 \in \Pi^S(6).\end{aligned}$$

Using the isomorphism (6.1.1.3) we can give a formula for $\dim \Pi^S(3)$.

Lemma 6.1.1.5.
$$\dim \Pi^S(3) = \frac{(s-2)(s-3)}{2}.$$

Proof. By Lemma 6.1.1.3, we know that $\{T_{(i,j,k)} | (i,j,k) \in \overline{Q}_0, \min\{i,j,k\} \geq 1\}$ is a basis of $\Pi^S(3)$. So $\dim \Pi^S(3) = |\{(i,j,k) \in \overline{Q}_0 | \min\{i,j,k\} \geq 1\}|$. We can compute the cardinality of the latter set in the following way:

- Choose i to be any number in $\{1,\ldots,s-3\}$;
- Choose j to be any number in $\{1,\ldots,s-i-2\}$;
- The value of k is determined by the choice of i,j.

Hence:

$$\begin{aligned}\dim \Pi^S(3) =& \sum_{i=1}^{s-3}(s-i-2) = (s-2)(s-3) - \sum_{i=1}^{s-2} i\\ =& (s-2)(s-3) - \frac{(s-2)(s-3)}{2} = \frac{(s-2)(s-3)}{2}.\end{aligned}$$

□

6.1.2 A basis for $(V \otimes \Pi)^S$

Similarly to what we did for Π^S, we introduce some sets of arrows that will later be useful to identify a basis for $(V \otimes \Pi)^S$.

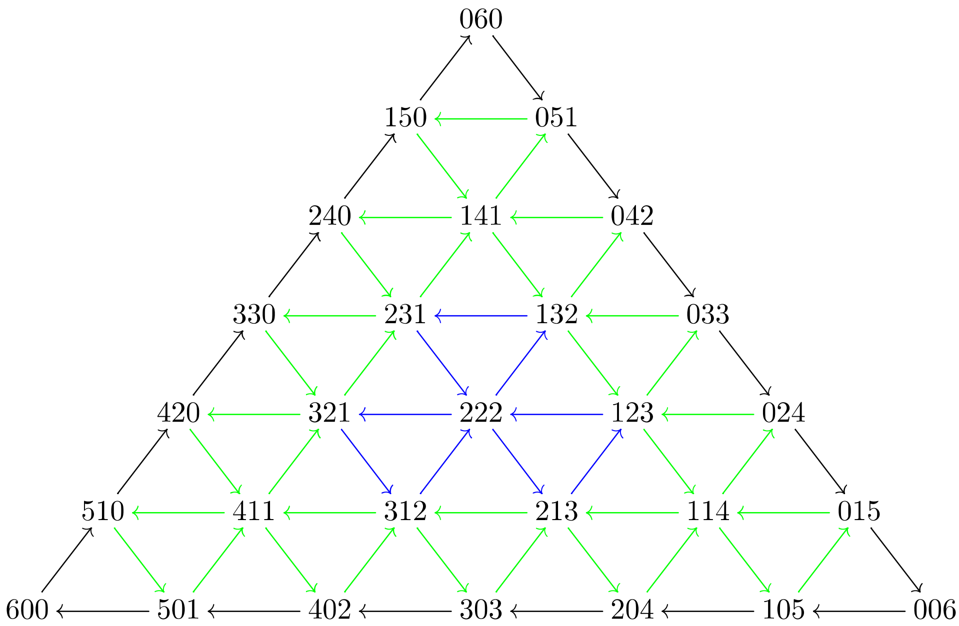
Definition 6.1.2.1. Fix $\ell \geq 1$ and define the following set of arrows:

$$D_\ell := \{\eta^r(e_{(i,j,k)}\alpha_0) | j,k \geq \ell \text{ and } i \geq \ell - 1 \text{ where } r = 0,1,2\}.$$

More explicitly, the set D_ℓ is the union of the following sets of arrows:

- $\{e_{(i,j,k)}\alpha_0 | j,k \geq \ell, i \geq \ell-1\}$;
- $\{e_{(i,j,k)}\alpha_1 | i,k \geq \ell, j \geq \ell-1\}$;
- $\{e_{(i,j,k)}\alpha_2 | i,j \geq \ell, k \geq \ell-1\}$.

Example 6.1.2.2. Let $s = 7$. Then the sets of arrows D_1, D_2 are the only nonempty D_i's, and can be represented on the quiver $\overline{Q}$ as follows.



The set D_1 is the union of the green and blue arrows, while the set D_2 consists of the blue arrows.

Let $\{p, q, r\} = \{0, 1, 2\}$. For an arrow $e_x\alpha_p \in \overline{Q}_1$ and $\ell \geq 1$, define:

$$d^\ell(e_x\alpha_p) = e_{t(e_x\alpha_p)}\alpha_p^{\ell-1}\alpha_q^\ell\alpha_r^\ell.$$

Lemma 6.1.2.3. 1. The following decomposition holds.

$$(V \otimes \Pi)^S = \bigoplus_{\ell \geq 1} (V \otimes \Pi)^S(3\ell) \tag{6.1.2.1}$$

2. The map

$$\begin{array}{llll} \varphi_\ell : & (V \otimes \Pi)^S(3\ell) & \to & \langle D_\ell \rangle \\ & e_x\alpha_p \otimes d^\ell(e_x\alpha_p) & \mapsto & e_x\alpha_p \end{array} \tag{6.1.2.2}$$

is an isomorphism of vector spaces.

Equivalently, the set of arrows D_ℓ can be identified with a basis of $(V \otimes \Pi)^S(3\ell)$.

Proof. The decomposition (6.1.2.1) follows directly from Proposition 3.2.0.5.

Now, fix $x \in \overline{Q}_0$ and an arrow $e_x\alpha_p \in \overline{Q}_1$. The shortest path $t(e_x\alpha_p) \to s(e_x\alpha_p)$ is given by $e_{t(e_x\alpha_p)}\alpha_q\alpha_r$. Hence, by Proposition 3.2.0.5 we know that the following elements generate $(e_x\alpha_p \otimes \Pi)^S \subset (V \otimes \Pi)^S$:

$$\begin{aligned} e_x\alpha_p \otimes \alpha_q\alpha_r(\alpha_0\alpha_1\alpha_2)^{\ell-1} &= e_x\alpha_p \otimes \alpha_p^{\ell-1}\alpha_q^\ell\alpha_r^\ell \\ &= e_x\alpha_p \otimes d^\ell(e_x\alpha_p), \qquad \ell \geq 1, \end{aligned}$$

where $\{p, q, r\} = \{0, 1, 2\}$.

Clearly the elements $d^\ell(e_x\alpha_p)$, if nonzero, are all distinct. Therefore, in order to prove the isomorphism (6.1.2.2), we just need to show that $d^\ell(e_x\alpha_p) \neq 0$ if and only if $e_x\alpha_p \in D_\ell$.

- Suppose that $e_x\alpha_p \in D_\ell$.

  If $p = 0$, then $e_x\alpha_p = e_x\alpha_0$ for some $x = (i, j, k) \in \overline{Q}_0$ such that $j, k \geq \ell$, $i \geq \ell - 1$. By definition:

  $$d^\ell(e_{(i,j,k)}\alpha_0) = e_{(i+1,j,k-1)}\alpha_0^{\ell-1}\alpha_1^\ell\alpha_2^\ell.$$

  Therefore $d^\ell(e_{(i,j,k)}\alpha_0) \neq 0$ by Lemma 3.2.0.2 since, by assumption, $i+1 \geq \ell$, $j \geq \ell$ and $k - 1 \geq \ell - 1$.

  We can use a similar argument to show that $d^\ell(e_x\alpha_p) \neq 0$ for $e_x\alpha_p \in D_\ell$, $p = 1, 2$.

- Suppose that $e_x\alpha_p \notin D_\ell$. We need to show that $d^\ell(e_x\alpha_p) = 0$.

  Assume w.l.o.g that $p = 0$, i.e., $e_x\alpha_p = e_x\alpha_0$ for some $x = (i, j, k) \in \overline{Q}_0$. Since $e_{(i,j,k)}\alpha_p \notin D_\ell$, then at least one of the following three conditions holds:

  - $i < \ell - 1$;
  - $j < \ell$;
  - $k < \ell$.

  Therefore, by Lemma 3.2.0.2:

  $$d^\ell(e_{(i,j,k)}\alpha_0) = e_{(i+1,j,k-1)}\alpha_0^{\ell-1}\alpha_1^\ell\alpha_2^\ell = 0.$$

  The same argument can be used for $p = 1, 2$.

□

Remark 6.1.2.4. Lemma 6.1.2.3 allows us to represent elements of $(V \otimes \Pi)^S$ as arrows in the quiver $\overline{Q}$. In particular, under this identification, all the arrows that are not on the outer rim of $\overline{Q}$ constitute a basis of $(V \otimes \Pi)^S(3)$ (see Example 6.1.2.2, where a basis of $(V \otimes \Pi)^S(3)$ is given by the green and blue arrows).

Using (6.1.2.2), we can get a formula for $\dim(V \otimes \Pi)^S(3)$.

Lemma 6.1.2.5.

$$\dim(V\otimes\Pi)^S(3)=\frac{3}{2}(s-1)(s-2)$$

Proof. By Lemma 6.1.2.3, a basis of $(V\otimes\Pi)^S(3)$ is given by arrows in D_1, that is, by the following arrows:

- $\{\eta^r(e_{(i,j,k)}\alpha_0)|j,k\geq 1, r=0,1,2\}$

Therefore we have:

$$\dim(V\otimes\Pi)^S(3)=3|\{(i,j,k)\in\overline{Q}_0|j,k\geq 1\}|.$$

We can count the vertices $(i,j,k)\in\overline{Q}_0$ such that $j,k\geq 1$ in the following way:

- We can choose $j\in\{1,\dots,s-2\}$;
- We can choose $k\in\{1,\dots s-1-j\}$;
- The value of i is determined by j,k since $i+j+k=s-1$.

Hence:

$$\begin{aligned}\dim(V\otimes\Pi)^S(3)=&3\sum_{j=1}^{s-2}(s-1-j)=3\left[(s-1)(s-2)-\sum_{j=1}^{s-2}j\right]\\=&3\left[(s-1)(s-2)-\frac{(s-1)(s-2)}{2}\right]=\frac{3}{2}(s-1)(s-2).\end{aligned}$$

□

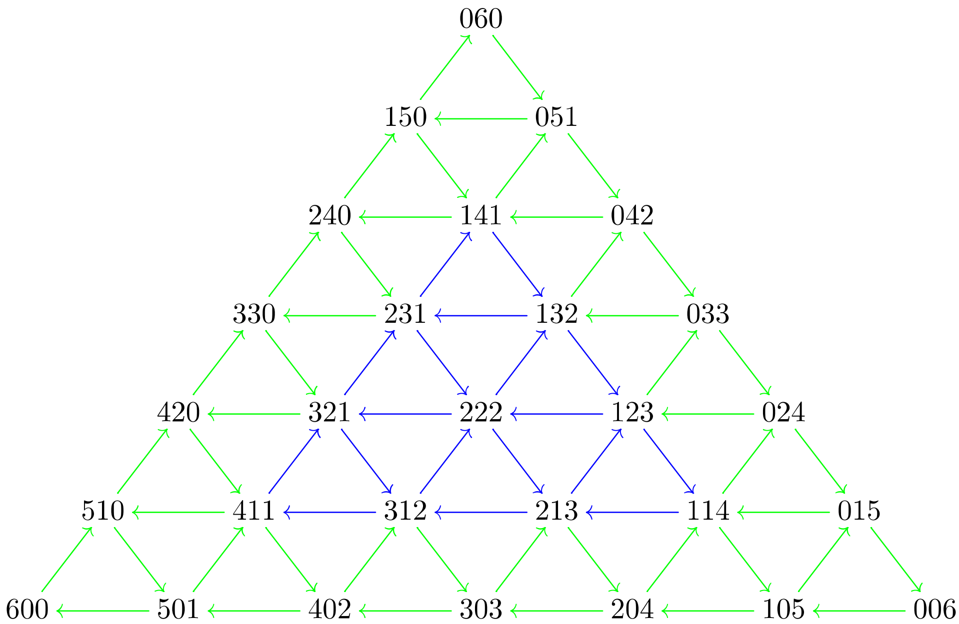
6.1.3 A basis for $(R\otimes\Pi)^S$

Similarly to $(V\otimes\Pi)^S$, we introduce some sets of arrows that allow us to identify a basis for $(R\otimes\Pi)^S$.

Definition 6.1.3.1. For $\ell\geq 1$ define the following set of arrows in $\overline{Q}$:

$$E_\ell:=\{v\in V|s(v),t(v)\in(\overline{Q}_0)_{\geq\ell-1}\}.$$

Example 6.1.3.2. Let $s=7$. The only nonzero E_i's are E_1,E_2, that can be represented on the quiver $\overline{Q}$ as follows.



The set E_1 consists of all arrows in $\overline{Q}$ (that is, the union of green and blue arrows), while E_2 consists of the blue arrows.

Remark 6.1.3.3. Let $\{p, q, r\} = \{0, 1, 2\}$. Any relation in R can be written as:

$$r' = e_x(\alpha_p\alpha_q - \alpha_q\alpha_p) \tag{6.1.3.1}$$

for some $x \in \overline{Q}_0$ and $p < q$, where one of the summands in (6.1.3.1) is zero if $x \in (\overline{Q}_0)_0$.

As already pointed out at the beginning of chapter 4, such relations can be identified with arrows in $\overline{Q}_1$. To be more precise, the following map is an S-bimodule isomorphism:

$$\begin{array}{ccc} R & \to & V^* \\ r' = e_x(\alpha_p\alpha_q - \alpha_q\alpha_p) & \mapsto & e_{t(r')}\alpha_r, \end{array}$$

where V^* is the S-bimodule spanned by the arrows in $\overline{Q}^*$, the opposite quiver of $\overline{Q}$.

We will use such identification in the following.

Let $\{p, q, r\} = \{0, 1, 2\}$, $p < q$. Using the identification given in Remark 6.1.3.3, let $e_x\alpha_r \in R$, and take $\ell \geq 1$. Define:

$$\varepsilon^\ell(e_x\alpha_r) = e_x\alpha_p^{\ell-1}\alpha_q^{\ell-1}\alpha_r^\ell. \tag{6.1.3.2}$$

Lemma 6.1.3.4. 1. The following decomposition holds.

$$(R \otimes \Pi)^S = \bigoplus_{\ell \geq 1} (R \otimes \Pi)^S(3\ell) \tag{6.1.3.3}$$

2. The map:

$$\begin{array}{rccc} \psi_\ell : & (R \otimes \Pi)^S(3\ell) & \to & \langle E_\ell \rangle \\ & e_x\alpha_r \otimes \varepsilon^\ell(e_x\alpha_r) & \mapsto & e_x\alpha_r \end{array}$$

is an isomorphism of vector spaces.

Equivalently, the set E_ℓ can be identified with a basis of $(R \otimes \Pi)^S(3\ell)$.

Proof. The decomposition (6.1.3.3) follows directly from Proposition 3.2.0.5.

Now, the fix $x \in \overline{Q}_0$ and, under the identification given in Remark 6.1.3.3, an element $e_x\alpha_r \in R$. The shortest path $s(e_x\alpha_r) \to t(e_x\alpha_r)$ is given by $e_x\alpha_r$. Hence, by Proposition 3.2.0.5, the following elements generate $(e_x\alpha_r \otimes \Pi)^S \subset (R \otimes \Pi)^S$:

$$\begin{aligned} e_x\alpha_r \otimes \alpha_r(\alpha_0\alpha_1\alpha_2)^{\ell-1} &= e_x\alpha_r \otimes \alpha_p^{\ell-1}\alpha_q^{\ell-1}\alpha_r^\ell \\ &= e_x\alpha_r \otimes \varepsilon^\ell(e_x\alpha_r), \quad \ell \geq 1, \end{aligned}$$

where $\{p, q, r\} = \{0, 1, 2\}$.

Clearly the elements $\varepsilon^\ell(e_x\alpha_r)$, if nonzero, are all distinct. Therefore, in order to prove the isomorphism (6.1.3.2), we just need to show that $\varepsilon^\ell(e_x\alpha_r) \neq 0$ if and only if $e_x\alpha_r \in E_\ell$.

- Suppose that $e_x\alpha_r \in E_\ell$ and, without loss of generality, that $r = 2$. Then $e_x\alpha_r = e_x\alpha_2$ for some $x \in \overline{Q}_0$ such that $s(e_x\alpha_2), t(e_x\alpha_2) \in (\overline{Q}_0)_{\geq \ell-1}$. By definition:

  $$\varepsilon^\ell(e_x\alpha_2) = e_x\alpha_0^{\ell-1}\alpha_1^{\ell-1}\alpha_2^\ell.$$

  A straightforward application of Lemma 3.2.0.2 therefore implies that $\varepsilon^\ell(e_x\alpha_2) \neq 0$.

- Suppose that $e_x\alpha_r \notin E_\ell$ for some $x = (i, j, k) \in \overline{Q}_0$ and, w.l.o.g., that $r = 2$. We need to show that $\varepsilon^\ell(e_x\alpha_2) = 0$. We have:

  $$\varepsilon^\ell(e_{(i,j,k)}\alpha_2) = e_{(i,j,k)}\alpha_0^{\ell-1}\alpha_1^{\ell-1}\alpha_2^\ell. \tag{6.1.3.4}$$

  Since $e_{(i,j,k)}\alpha_2 \notin E_\ell$, then we have two possibilities:

  - if $s(e_{(i,j,k)}\alpha_2) = (i, j, k) \in (\overline{Q}_0)_{<\ell-1}$, then $\min\{i, j, k\} < \ell - 1$. Hence Lemma 3.2.0.2 implies that (6.1.3.4) is zero.
  - if $t(e_{(i,j,k)}\alpha_2) = (i, j, k) \in (\overline{Q}_0)_{<\ell-1}$, then $\min\{i, j-1, k+1\} < \ell - 1$. Once again, we get that (6.1.3.4) is zero by Lemma 3.2.0.2.

  In any case, $\varepsilon^\ell(e_{(i,j,k)}\alpha_2) = 0$.

□

Remark 6.1.3.5. Lemma 6.1.3.4 allows us to represent basis elements of $(R\otimes\Pi)^S$ as arrows in the quiver $\overline{Q}$. In particular, under the identification ψ_1, a basis of $(R\otimes\Pi)^S(3)$ is given by all arrows of $\overline{Q}$ (see Example 6.1.3.2, where the arrows in E_1 are given by the green and blue arrows).

6.1.4 Rewriting of the complex computing $HH_1(\Pi)$

In this subsection we rewrite the complex (6.1.0.1) computing $HH_1(\Pi)$ in terms of the spaces $\langle(\overline{Q}_0)_{\geq\ell}\rangle$, $\langle D_\ell\rangle$ and $\langle E_\ell\rangle$.

Lemma 6.1.4.1. Fix $\ell \geq 1$, and define the maps:

$$\langle E_\ell\rangle \xrightarrow{g_\ell} \langle D_\ell\rangle \xrightarrow{f_\ell} \langle(\overline{Q}_0)_{\geq\ell}\rangle$$

by

$$\begin{aligned} f_\ell(v) &= s(v) - t(v) \\ g_\ell(\alpha_r) &= e_{t(\alpha_r)}\alpha_p + e_{t(\alpha_r\alpha_p)}\alpha_q - e_{t(\alpha_r)}\alpha_q - e_{t(\alpha_r\alpha_q)}\alpha_p, \end{aligned}$$

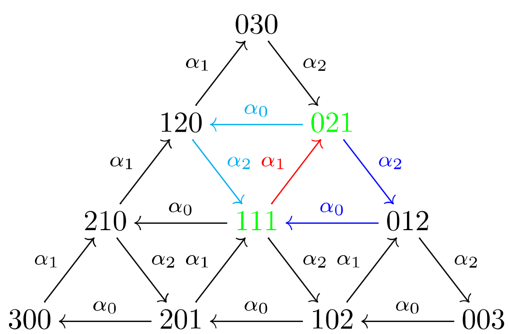
where $\{p,q,r\} = \{0,1,2\}$ and $p < q$. Then the following diagram commutes:

$$\begin{array}{ccccc} (R\otimes\Pi)^S(3\ell) & \xrightarrow{(\mu_2')_{|3\ell}} & (V\otimes\Pi)^S(3\ell) & \xrightarrow{(\mu_1')_{|3\ell}} & \Pi^S(3\ell) \\ \downarrow{\scriptstyle \psi_\ell} & & \downarrow{\scriptstyle \varphi_\ell} & & \downarrow{\scriptstyle \theta_\ell} \\ \langle E_\ell\rangle & \xrightarrow{g_\ell} & \langle D_\ell\rangle & \xrightarrow{f_\ell} & \langle(\overline{Q}_0)_{\geq\ell}\rangle \end{array} \qquad (6.1.4.1)$$

where the maps $\theta_\ell, \psi_\ell, \varphi_\ell$ are the isomorphisms introduced in Lemmas 6.1.1.3, 6.1.2.3 and 6.1.3.4 respectively, and $(\mu_1')_{|3\ell}, (\mu_2')_{|3\ell}$ are restrictions of the differentials defined in (6.1.0.1) to the graded component of degree 3ℓ of $(V\otimes\Pi)^S$ and $(R\otimes\Pi)^S$, respectively.

Before giving the proof of the Lemma, we show in an example how to interpret the maps f_ℓ and g_ℓ.

Example 6.1.4.2. Let $s = 4$. Then the quiver $\overline{Q}^{(2,4)}$ is given by:



Consider $\ell = 1$. The red arrow $e_{111}\alpha_1$ belongs both to D_1 and E_1, and therefore we can apply both f_1 and g_1 to it.

- The map f_1 acts on it as:

$$f_1(e_{111}\alpha_1) = (1,1,1) - (0,2,1),$$

  that is, it sends the arrow $e_{111}\alpha_1$ into the difference between its starting point and end point (coloured in green).

- The map g_1 acts on it as:

$$g_1(e_{111}\alpha_1) = e_{021}\alpha_0 + e_{120}\alpha_2 - e_{021}\alpha_2 - e_{012}\alpha_0,$$

  that is, it sends the arrow $e_{111}\alpha_1$ to the sum of the two light blue arrows minus the sum of the dark blue arrows.

Proof of Lemma 6.1.4.1. We need to prove that the maps f_ℓ, g_ℓ make the diagram (6.1.4.1) commutative. We have:

$$\begin{aligned}
&g_\ell\psi_\ell(e_x\alpha_r \otimes \varepsilon^\ell(e_x\alpha_r)) = g_\ell(e_x\alpha_r)\\
&= e_{t(\alpha_r)}\alpha_p + e_{t(\alpha_r\alpha_p)}\alpha_q - e_{t(\alpha_r)}\alpha_q - e_{t(\alpha_r\alpha_q)}\alpha_p\\
&= \varphi_\ell(e_{t(\alpha_r)}\alpha_p \otimes d^\ell(e_{t(\alpha_r)}\alpha_p)) + \varphi_\ell(e_{t(\alpha_r\alpha_p)}\alpha_q \otimes d^\ell(e_{t(\alpha_r\alpha_p)}\alpha_q))\\
&\quad - \varphi_\ell(e_{t(\alpha_r)}\alpha_q \otimes d^\ell(e_{t(\alpha_r)}\alpha_q)) - \varphi_\ell(e_{t(\alpha_r\alpha_q)}\alpha_p \otimes d^\ell(e_{t(\alpha_r\alpha_q)}\alpha_p))\\
&= \varphi_\ell(e_{t(\alpha_r)}\alpha_p \otimes \alpha_q\varepsilon^\ell(e_x\alpha_r)) + \varphi_\ell(e_{t(\alpha_r\alpha_p)}\alpha_q \otimes \varepsilon^\ell(e_x\alpha_r)\alpha_p)\\
&\quad - \varphi_\ell(e_{t(\alpha_r)}\alpha_q \otimes \alpha_p\varepsilon^\ell(e_x\alpha_r)) - \varphi_\ell(e_{t(\alpha_r\alpha_q)}\alpha_p \otimes \varepsilon^\ell(e_x\alpha_r)\alpha_q)\\
&= \varphi_\ell(\mu_2')_{|3\ell}(e_x\alpha_r \otimes \varepsilon^\ell(e_x\alpha_r)).
\end{aligned}$$

Therefore $g_\ell\psi_\ell = \varphi_\ell(\mu_2')_{|3\ell}$. As for the commutativity of the right square, we have:

$$\begin{aligned}
&f_\ell\varphi_\ell(e_x\alpha_p \otimes d^\ell(e_x\alpha_p)) = f_\ell(e_x\alpha_p) = s(e_x\alpha_p) - t(e_x\alpha_p)\\
&= \theta_\ell(T^\ell_{s(e_x\alpha_p)}) - \theta_\ell(T^\ell_{t(e_x\alpha_p)}) = \theta_\ell(e_{s(e_x\alpha_p)}\alpha_p d^\ell(e_x\alpha_p)) - \theta_\ell(e_{t(e_x\alpha_p)} d^\ell(e_x\alpha_p)\alpha_p)\\
&= \theta_\ell(\mu_1')_{|3\ell}(e_x\alpha_p \otimes d^\ell(e_x\alpha_p)).
\end{aligned}$$

Hence $f_\ell g_\ell = \theta_\ell(\mu_1')_{|3\ell}$, and thus (6.1.4.1) commutes. □

Remark 6.1.4.3. Thanks to Lemma 6.1.4.1, we can identify the maps f_ℓ and g_ℓ with $(\mu_1')_{|3\ell}$ and $(\mu_2')_{|3\ell}$, respectively. Therefore we have $\ker f_\ell \cong \ker(\mu_1')_{|3\ell}$, $\operatorname{im} g_\ell \cong \operatorname{im}\,(\mu_2')_{|3\ell}$ and, as a consequence:

$$HH_1(\Pi)(3\ell) \cong \frac{\ker f_\ell}{\operatorname{im}\, g_\ell}$$

for all $\ell \geq 1$.

6.1.5 Computation of $\ker f_1$

We want to study the kernel of the map

$$\begin{array}{rccc} f_1 : & \langle D_1 \rangle & \to & \langle (\overline{Q}_0)_{\geq 1} \rangle \\ & v & \mapsto & s(v) - t(v), \end{array} \tag{6.1.5.1}$$

where $s(v)$, $t(v)$ are interpreted as zero if they belong to $(\overline{Q}_0)_0$.

Lemma 6.1.5.1.

$$\dim \operatorname{im}(f_1) = \frac{(s-2)(s-3)}{2}.$$

Proof. By Lemma 6.1.1.5 we know that $\dim \Pi^S(3) = |(\overline{Q}_0)_{\geq 1}| = \frac{(s-2)(s-3)}{2}$; therefore we just need to show that $f_1 : \langle D_1 \rangle \to \langle (\overline{Q}_0)_{\geq 1} \rangle$ is surjective.

Let $(i, j, k) \in (\overline{Q}_0)_\ell$ for some $\ell \geq 1$. We prove the result by induction on ℓ.

- If $j = \ell = 1$, consider the arrow $\alpha_1 : (i+1, j-1, k) \to (i, j, k) \in D_1$. Since $(i+1, j-1, k) \in (\overline{Q}_0)_0$ we shall consider it as zero in $\langle (\overline{Q}_0)_{\geq 1} \rangle$. Therefore:

  $$f_1(e_{(i+1,j-1,k)}\alpha_1) = -(i, j, k),$$

  and so $(i, j, k) \in \operatorname{im}(f_1)$.
  If $i = \ell = 1$ or $k = \ell = 1$ we can apply a similar argument to conclude that $(i, j, k) \in \operatorname{im}(f_1)$.

- Let $(i, j, k) \in (\overline{Q}_0)_{\ell+1}$, $\ell > 0$. Suppose $j = \ell + 1$, and consider the arrow $\alpha_1 : (i+1, j-1, k) \to (i, j, k) \in D_1$. We have:

  $$f_1(e_{(i+1,j-1,k)}\alpha_1) = (i+1, j-1, k) - (i, j, k).$$

  But $(i+1, j-1, k) \in (\overline{Q}_0)_\ell$, and thus $(i+1, j-1, k) \in \operatorname{im}(f_1)$ by induction hypothesis. Hence also $(i, j, k) \in \operatorname{im}(f_1)$.
  We can apply an analogue argument to conclude that $(i, j, k) \in \operatorname{im}(f_1)$ also when $i = \ell + 1$ or $k = \ell + 1$.

Therefore all the elements $(i, j, k) \in (\overline{Q}_0)_{\geq \ell}$ are in im f_1; hence f_1 is surjective, and we get the statement. □

Corollary 6.1.5.2.

$$\dim \ker(f_1) = s(s-2).$$

Proof. Using Lemmas 6.1.2.5, 6.1.5.1 and the rank-nullity theorem, we have:

$$\begin{aligned}\dim\ker(f_1) &= \dim(V\otimes\Pi)^S(3) - \dim\operatorname{im}(f_1)\\ &= \frac{3}{2}(s-1)(s-2) - \frac{1}{2}(s-2)(s-3)\\ &= \frac{1}{2}(s-2)[3(s-1)-(s-3)] = s(s-2).\end{aligned}$$

□

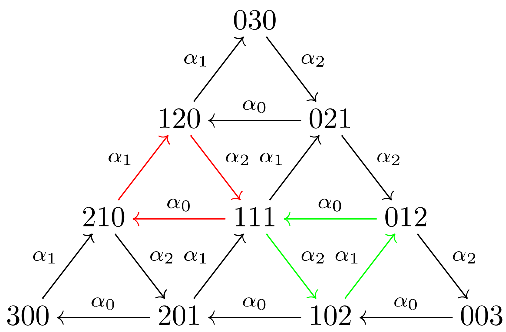
We now define some elements in $\langle D_1\rangle$ that have a relevant role in the computation of a convenient basis for $\ker(f_1)$.

Definition 6.1.5.3. Let $(i,j,k) \in \overline{Q}_0$.

- If $i \geq 1$, define $\Delta^U_{(i,j,k)} := e_{(i,j,k)}\alpha_1 + e_{(i-1,j+1,k)}\alpha_2 + e_{(i-1,j,k+1)}\alpha_0$.
- If $i, j \geq 1$, define $\Delta^D_{(i,j,k)} := e_{(i,j,k)}\alpha_2 + e_{(i,j-1,k+1)}\alpha_1 + e_{(i-1,j,k+1)}\alpha_0$.

Let $\Delta^U := \{\Delta^U_{(i,j,k)} | i \geq 1\}$, $\Delta^D := \{\Delta^D_{(i,j,k)} | i, j \geq 1\}$ and $\Delta := \Delta^U \cup \Delta^D$. We call the elements or Δ^U (resp. Δ^D, resp. Δ) **upward facing triangles** (resp. **downward facing triangles**, resp. **triangles**).

Example 6.1.5.4. Let $s = 4$. The upward and downward facing triangles can be visualized inside $\overline{Q}$ as follows.



- The three red arrows stand for the upward facing triangle

$$\Delta^U_{(2,1,0)} = e_{(2,1,0)}\alpha_1 + e_{(1,2,0)}\alpha_2 + e_{(1,1,1)}\alpha_0.$$

- The three green arrows stand for the downward facing triangle

$$\Delta^D_{(1,1,1)} = e_{(1,1,1)}\alpha_2 + e_{(1,0,2)}\alpha_1 + e_{(0,1,2)}\alpha_0.$$

The triangles given in Definition 6.1.5.3 belong to $\ker(f_1)$, as stated by the following Lemma.

Lemma 6.1.5.5. $\Delta \subseteq \ker(f_1)$.

Proof. Let $\Delta^U_{(i,j,k)}$ be an upward facing triangle. Then:

$$\begin{aligned} f_1(\Delta^U_{(i,j,k)}) =& f_1(e_{(i,j,k)}\alpha_2 + e_{(i,j-1,k+1)}\alpha_1 + e_{(i-1,j,k+1)}\alpha_0) \\ =& ((i,j,k) - (i-1,j+1,k)) + ((i-1,j+1,k) - (i-1,j,k+1)) \\ &+ ((i-1,j,k+1) - (i,j,k)) = 0. \end{aligned}$$

A direct computation also shows that all downward facing triangles belong to $\ker(f_1)$. □

Lemma 6.1.5.6.

$$|\Delta| = s(s-2) + 1.$$

Proof. The number of upward facing triangles is given by:

$$|\Delta^U| = |\{(i,j,k) \in \overline{Q}_0 | i \geq 1\}|.$$

To compute the cardinality of the latter we can:

- choose $i \in \{1, \ldots, s-1\}$;
- choose $j \in \{0, \ldots, s-i\}$;
- the value of k is determined by i, j.

Hence:

$$|\Delta^U| = \sum_{i=1}^{s-1}(s-i) = s(s-1) - \frac{1}{2}s(s-1) = \frac{1}{2}s(s-1).$$

A similar argument shows that $|\Delta^D| = \frac{1}{2}(s-1)(s-2)$. Therefore:

$$\begin{aligned} |\Delta| =& |\Delta^U| + |\Delta^D| = \frac{1}{2}s(s-1) + \frac{1}{2}(s-1)(s-2) \\ =& \frac{1}{2}(s-1)(s+s-2) = (s-1)^2 = s(s-2)+1. \end{aligned}$$

□

We now want to show that Δ spans $\ker(f_1)$ as a vector space. In order to do this, we need some preliminary results about triangles.

Remark 6.1.5.7. Let $v \in \overline{Q}_1$. Then:

- If v is on the outer rim of $\overline{Q}$, there is a unique upward facing triangle $\delta^u \in \Delta^U$ that has v as an arrow.
- If v is not on the outer rim of $\overline{Q}$, then there are exactly two triangles, an upward facing one $\delta^u \in \Delta^U$ and a downfacing one $\delta^d \in \Delta^D$, that have v as an arrow.

We now introduce a concept of distance between triangles in Δ.

Definition 6.1.5.8. Let $\delta, \delta' \in \Delta$. Define $h(\delta, \delta') \in \mathbb{N}$ recursively by:

- $h(\delta, \delta') = 0$ if $\delta = \delta'$;
- $h(\delta, \delta') = \ell$ if there exists $\delta'' \in \Delta$ such that $h(\delta, \delta'') = \ell - 1$ and δ'', δ' have exactly one arrow in common.

Lemma 6.1.5.9. Let $\delta, \delta' \in \Delta$, and $\ell = h(\delta, \delta')$.

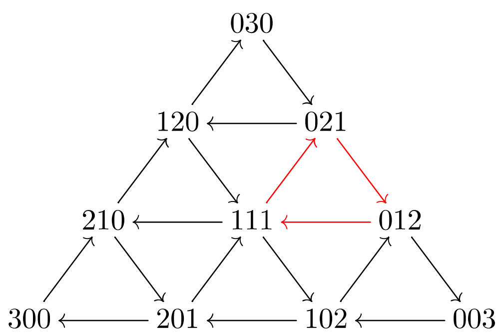
- If ℓ is even, then δ, δ' both belong to Δ^U or they both belong to Δ^D.
- If ℓ is odd, then $\delta \in \Delta^U$ and $\delta' \in \Delta^D$, or $\delta \in \Delta^D$ and $\delta' \in \Delta^U$.

Proof. We prove this by induction on ℓ.

- If $\ell = 0$, then $\delta = \delta'$, and the statement trivially holds true.
- Let $\ell > 0$. By definition of h, we can find a triangle $\delta'' \in \Delta$ such that $h(\delta, \delta'') = \ell - 1$ and δ'', δ' have one arrow in common. Then:
  - If ℓ is even, by induction hypothesis $\delta \in \Delta^U$ and $\delta'' \in \Delta^D$, or the other way around. In any case, since δ'', δ' have one arrow in common, we can use Remark 6.1.5.7 to conclude that δ, δ' both belong to Δ^U or they both belong to Δ^D.
  - If ℓ is odd, by induction hypothesis δ, δ'' both belong to Δ^U or they both belong to Δ^D. In any case, since δ'', δ' have one arrow in common, we can say that either $\delta \in \Delta^U$ and $\delta' \in \Delta^D$ or the other way around thanks to Remark 6.1.5.7, δ'', δ' have one arrow in common.

□

Example 6.1.5.10. Let $s = 4$. The quiver $\overline{Q}$ is as follows.



Consider the red upward facing triangle $\delta = \Delta^U_{(1,1,1)}$. We have:

- $1 = h(\delta, \Delta^D_{(1,2,0)}) = h(\delta, \Delta^D_{(1,1,1)})$;

- $2 = h(\delta, \Delta^U_{(2,1,0)}) = h(\delta, \Delta^U_{(1,2,0)}) = h(\delta, \Delta^U_{(2,0,1)}) = h(\delta, \Delta^U_{(1,0,2)})$;
- $3 = h(\delta, \Delta^D_{(2,1,0)})$;
- $4 = h(\delta, \Delta^U_{(3,0,0)})$.

Proposition 6.1.5.11.

$$\langle \Delta \rangle = \ker(f_1).$$

Furthermore, if we consider the map:

$$\langle \Delta^D \rangle \oplus \langle \Delta^U \rangle \to \langle \Delta \rangle$$

that sends each downward (resp. upward) facing triangle $\delta^d \in \Delta^D$ (resp. $\delta^u \in \Delta^U$) to itself, then its kernel is one dimensional, spanned by the following element:

$$\sum_{\delta^u \in \Delta^U} \delta^u - \sum_{\delta^d \in \Delta^D} \delta^d.$$

Proof. Consider the linear equation:

$$\sum_{\delta \in \Delta} \lambda_\delta \delta = 0, \qquad (6.1.5.2)$$

where $\lambda_\delta \in k$ for all $\delta \in \Delta$.

Suppose not all coefficients are zero. Take $\delta \in \Delta$ such that $\lambda_\delta \neq 0$. Suppose w.l.o.g. that $\delta \in \Delta^U$ is an upward facing triangle. By eventually rescaling, we can assume $\lambda_\delta = 1$.

We claim that:

$$\lambda_{\delta'} = \begin{cases} 1, & \text{if } \delta' \in \Delta^U \\ -1, & \text{if } \delta' \in \Delta^D. \end{cases}$$

We prove this by induction on $\ell = h(\delta, \delta')$.

- If $\ell = 0$, then $\delta = \delta'$, and therefore $\lambda_{\delta'} = 1$.
- Let $\ell > 0$.
  - If ℓ is even, we know that $\delta' \in \Delta^U$ by Lemma 6.1.5.9. By definition of h, we can find $\delta'' \in \Delta$ such that $h(\delta, \delta'') = \ell - 1$ and such that δ'', δ' have an arrow v in common. Now, since $\ell - 1$ is odd, then $\delta'' \in \Delta^D$ by Lemma 6.1.5.9, and therefore we have $\lambda_{\delta''} = -1$ by induction.

    Also, δ'' and δ' are the only triangles in Δ that have v as an arrow by Remark 6.1.5.7. Hence, using (6.1.5.2), we have that $\lambda_{\delta''} + \lambda_{\delta'} = 0$, that implies $\lambda_{\delta'} = -\lambda_{\delta''} = 1$.
  - If ℓ is odd, then $\delta' \in \Delta^D$ by Lemma 6.1.5.9. Using the same argument as above one can prove that $\lambda_{\delta'} = -1$.

Therefore, the only relation of linear dependence between the elements of Δ is given by:

$$\sum_{\delta^u \in \Delta^U} \delta^u - \sum_{\delta^d \in \Delta^D} \delta^d = 0.$$

This is equivalent to saying that

$$\ker(\langle \Delta^D \rangle \oplus \langle \Delta^U \rangle \to \langle \Delta \rangle) = \langle \sum_{\delta^u \in \Delta^U} \delta^u - \sum_{\delta^d \in \Delta^D} \delta^d \rangle.$$

Now, by Corollary 6.1.5.2 and Lemma 6.1.5.6, we have $|\Delta| = \dim \ker(f_1)+1$. Therefore, since $\Delta \subseteq \ker(f_1)$ by Lemma 6.1.5.5, we can conclude that:

$$\langle \Delta \rangle = \ker(f_1).$$

□

6.1.6 Description of the map g_1

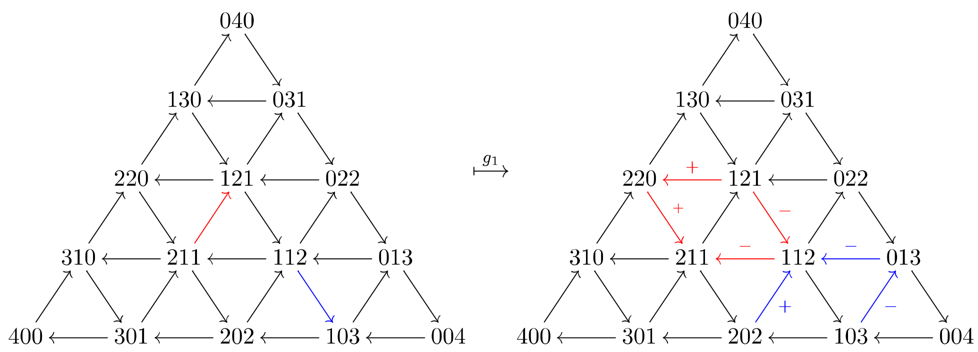
For $\ell = 1$, the set E_1 consists of all arrows in $\overline{Q}$, i.e., $\langle E_1 \rangle = V$. Therefore, we want to study the image of the map:

$$\begin{array}{rccc} g_1 : & V & \to & \langle D_1 \rangle \\ & \alpha_r & \mapsto & e_{t(\alpha_r)}\alpha_p + e_{t(\alpha_r\alpha_p)}\alpha_q \\ & & & -e_{t(\alpha_r)}\alpha_q - e_{t(\alpha_r\alpha_q)}\alpha_r, \end{array} \tag{6.1.6.1}$$

where $\{p, q, r\} = \{0, 1, 2\}$ and $p < q$.

Notice that, if an arrow appearing in the right hand side of (6.1.6.1) is on the external rim of $\overline{Q}$, then it is 0 by definition of D_1.

Example 6.1.6.1. Let $s = 5$. Then the map g_1 can be visualized as follows.



The picture means

$$e_{(2,1,1)}\alpha_1 \overset{g_1}{\mapsto} e_{(1,2,1)}\alpha_0 + e_{(2,2,0)}\alpha_2 - e_{(1,2,1)}\alpha_2 - e_{(1,1,2)}\alpha_0$$
$$e_{(1,1,2)}\alpha_2 \overset{g_1}{\mapsto} e_{(2,0,2)}\alpha_1 - e_{(1,0,3)}\alpha_1 - e_{(0,1,3)}\alpha_0$$

Notation 6.1.6.2. By Remark 6.1.5.7 we know that, for each arrow $v \in \overline{Q}_1$, there is either a unique upward facing triangle $\delta^u \in \Delta^U$ that has v as an arrow, or there is exactly a pair $(\delta^u, \delta^d) \in (\Delta^U, \Delta^D)$ that have v as an arrow.

In view of this, we can call Δ_v^U the unique upward facing triangle with v as an arrow, and Δ_v^D the unique downward facing triangle with v as an arrow, if it exists.

Lemma 6.1.6.3. Using the notation from above, the map g_1 can be rewritten in terms of the triangles in Δ as follows:

$$\begin{array}{rccc} g_1: & V & \to & \langle D_1 \rangle \\ & \alpha_0 & \mapsto & \Delta_{\alpha_0}^U - \Delta_{\alpha_0}^D \\ & \alpha_1 & \mapsto & \Delta_{\alpha_1}^D - \Delta_{\alpha_1}^U \\ & \alpha_2 & \mapsto & \Delta_{\alpha_2}^U - \Delta_{\alpha_2}^D. \end{array}$$

Proof. This is a straightforward computation. For example, if $v = e_{(i,j,k)}\alpha_0 \in \overline{Q}_0$ then, by definition of g_1 we have:

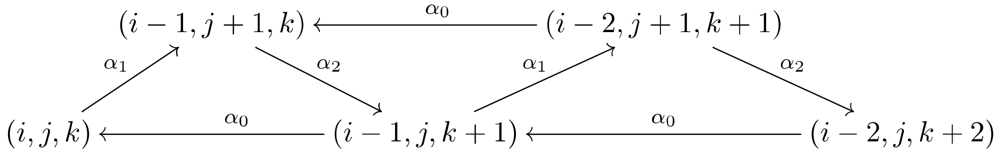
$$\begin{aligned} g_1(e_{(i,j,k)}\alpha_0) =& e_{(i+1,j,k-1)}\alpha_1 + e_{(i,j+1,k-1)}\alpha_2 - e_{(i+1,j,k-1)}\alpha_2 - e_{(i+1,j-1,k)}\alpha_1 \\ =& e_{(i+1,j,k-1)}\alpha_1 + e_{(i,j+1,k-1)}\alpha_2 + e_{(i,j,k)}\alpha_0 \\ & - e_{(i,j,k)}\alpha_0 - e_{(i+1,j,k-1)}\alpha_2 - e_{(i+1,j-1,k)}\alpha_1 \\ =& \Delta^U_{e_{(i,j,k)}\alpha_0} - \Delta^D_{e_{(i,j,k)}\alpha_0}. \end{aligned}$$

□

6.1.7 Computation of $HH_1(\Pi)$

In this subsection we use the results from the previous subsections to prove $HH_1(\Pi) = 0$.

Lemma 6.1.7.1. Let $(i, j, k) \in \overline{Q}_0$ be such that $i \geq 2$. Consider the following subquiver of $\overline{Q}$.



Then:

- $\Delta^U_{e_{(i-1,j+1,k)}\alpha_2} \in \mathrm{im}(g_1)$ implies $\Delta^D_{e_{(i-1,j+1,k)}\alpha_2} \in \mathrm{im}(g_1)$;
- $\Delta^D_{e_{(i-1,j,k+1)}\alpha_1} \in \mathrm{im}(g_1)$ implies $\Delta^U_{e_{(i-1,j,k+1)}\alpha_1} \in \mathrm{im}(g_1)$.

Proof. Suppose that $\Delta^U_{e_{(i-1,j+1,k)}\alpha_2} \in \text{im}(g_1)$. Then, using the description of g_1 given in Lemma 6.1.6.3, we have:

$$\begin{aligned}\Delta^D_{e_{(i-1,j+1,k)}\alpha_2} &= \Delta^U_{e_{(i-1,j+1,k)}\alpha_2} - (\Delta^U_{e_{(i-1,j+1,k)}\alpha_2} - \Delta^D_{e_{(i-1,j+1,k)}\alpha_2}) \\ &= \Delta^U_{e_{(i-1,j+1,k)}\alpha_2} - g_1(e_{(i-1,j+1,k)}\alpha_2) \in \text{im}(g_1)\end{aligned}$$

The other implication can be proved similarly. □

Using Lemma 6.1.7.1 and the description of g_1 given in Lemma 6.1.6.3, we can show that each triangle in Δ is in im g_1, that will imply $\ker(f_1) = \text{im}(g_1)$ by Proposition 6.1.5.11.

Proposition 6.1.7.2.

$$\text{im}(g_1) = \ker(f_1).$$

Equivalently,

$$HH_1(\Pi)(3) = 0.$$

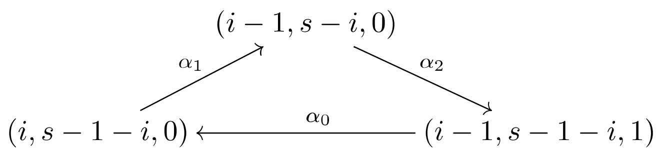
Proof. Recall that $HH_1(\Pi)(3) \cong \dfrac{\ker(f_1)}{\text{im }(g_1)}$ by Remark 6.1.4.3. Therefore, if we prove that $\text{im}(g_1) = \ker(f_1)$, we have also have $HH_1(\Pi)(3) = 0$.

We know that $\text{im}(g_1) \subseteq \ker(f_1)$ since:

$$\langle E_1\rangle \xrightarrow{g_1} \langle D_1\rangle \xrightarrow{f_1} \langle (Q_0)_{\geq 1}\rangle$$

is a complex.

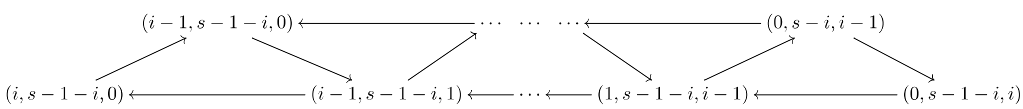
Therefore, in view of Proposition 6.1.5.11, it is enough to show that all triangles in Δ lie in $\text{im}(g_1)$. In order to do this, fix an upward facing triangle in the left rim of $\overline{Q}$ that is, for $i \geq 1$, consider the following subquiver of $\overline{Q}$:



By Lemma 6.1.6.3 we have:

$$g_1(e_{(i,s-1-i,0)}\alpha_1) = -\Delta^U_{e_{(i,s-1-i,0)}\alpha_1},$$

so $\Delta^U_{e_{(i,s-1-i,0)}\alpha_1} \in \text{im}(g_1)$. Repeatedly applying Lemma 6.1.7.1, we have that all triangles in the following subquiver of $\overline{Q}$ belong to $\text{im}(g_1)$.



Applying this procedure for all $i \geq 1$, we get that all triangles in Δ belong to $\mathrm{im}(g_1)$, and hence we get the statement. $\square$

We now want to use Proposition 6.1.7.2 to show that the whole first Hochschild homology group of Π vanishes.

Thanks to Lemma 6.1.2.3, we can say that:

$$HH_1(\Pi) = \bigoplus_{\ell \geq 1} HH_1(\Pi)(3\ell).$$

Thus, in view of Proposition 6.1.7.2, in order to show that $HH_1(\Pi) = 0$, it is enough to show that $HH_1(\Pi)(3\ell) = 0$ for all $\ell > 1$.

In order to do this we need to use different values of s, and therefore we will keep track of it using a superscript specifying what value of s we are considering.

We want to show that, for all $\ell \geq 1$, there is a commutative diagram:

$$\begin{array}{ccccc} \langle E_{\ell+1}\rangle^{(s+3)} & \overset{g_{\ell+1}^{(s+3)}}{\to} & \langle D_{\ell+1}\rangle^{(s+3)} & \overset{f_{\ell+1}^{(s+3)}}{\to} & \langle(\overline{Q}_0)_{\geq \ell+1}\rangle^{(s+3)} \\ \downarrow \Psi_2 & & \downarrow \Psi_1 & & \downarrow \Psi_0 \\ \langle E_\ell\rangle^{(s)} & \overset{g_\ell^{(s)}}{\to} & \langle D_\ell\rangle^{(s)} & \overset{f_\ell^{(s)}}{\to} & \langle(\overline{Q}_0)_{\geq \ell}\rangle^{(s)} \end{array} \tag{6.1.7.1}$$

where the vertical maps Ψ_0, Ψ_1 and Ψ_2 are isomorphisms.

We define Ψ_0, Ψ_1, Ψ_2 one by one, and then we carefully check that they are isomorphisms.

Proposition 6.1.7.3. The map:

$$\begin{array}{rccc} \Psi_0: & \langle(\overline{Q}_0)_{\geq \ell+1}\rangle^{(s+3)} & \to & \langle(\overline{Q}_0)_{\geq \ell}\rangle^{(s)} \\ & (i,j,k) & \mapsto & (i-1, j-1, k-1) \end{array}$$

is an isomorphism of vector spaces.

Proof. We need to show that

$$(i,j,k) \in (\overline{Q}_0)_{\geq \ell+1}^{(s+3)} \quad \text{if and only if} \quad (i-1,j-1,k-1) \in (\overline{Q}_0)_{\geq \ell}^{(s)}.$$

Let $(i,j,k) \in (\overline{Q}_0)_w^{(s+3)}$ for some $w \geq \ell+1$. This means that $i+j+k = s+2$ and $\min\{i,j,k\} = w$. Hence:

- $(i-1)+(j-1)+(k-1) = s-1$;
- $\min\{i-1, j-1, k-1\} = w-1$.

Thus $(i-1,j-1,k-1) \in (Q_0)_{w-1}^{(s)}$, that implies $(i-1,j-1,k-1) \in (\overline{Q}_0)_{\geq \ell}^{(s)}$. A similar argument can be used to prove the other implication. $\square$

Proposition 6.1.7.4. The map:

$$\begin{array}{rccc} \Psi_1: & \langle D_{\ell+1}\rangle^{(s+3)} & \to & \langle D_\ell\rangle^{(s)} \\ & e_{(i,j,k)}\alpha_p & \mapsto & e_{(i-1,j-1,k-1)}\alpha_p \end{array}$$

is an isomorphism of vector spaces.

Proof. By definition, bases of $\langle D_{\ell+1}\rangle^{(s+3)}$ and $\langle D_\ell\rangle^{(s)}$ are respectively given by:

- $\{\eta^r(e_{(i,j,k)}\alpha_0) | (i,j,k) \in \overline{Q}_0^{(s+3)}, j,k \geq \ell+1, i \geq \ell$ where $r = 0,1,2\}$;
- $\{\eta^r(e_{(i,j,k)}\alpha_0) | (i,j,k) \in \overline{Q}_0^{(s)}, j,k \geq \ell, i \geq \ell-1$ where $r = 0,1,2\}$.

One can easily check that these basis elements correspond under the map Ψ_1. For example, if we consider $e_{(i,j,k)}\alpha_0 \in \langle D_{\ell+1}\rangle^{(s+3)}$, we have $j,k \geq \ell+1$ and $i \geq \ell$. Hence the element $e_{(i-1,j-1,k-1)}\alpha_0$ is such that $(i-1,j-1,k-1) \in \overline{Q}_0^{(s)}$, $j,k \geq \ell$ and $i \geq \ell-1$, and thus belongs to $\langle D_\ell\rangle^{(s)}$.

The other checks can be done similarly. □

Proposition 6.1.7.5. The map:

$$\begin{array}{rccc} \Psi_2: & \langle E_{\ell+1}\rangle^{(s+3)} & \to & \langle E_\ell\rangle^{(s)} \\ & e_{(i,j,k)}\alpha_r & \mapsto & e_{(i-1,j-1,k-1)}\alpha_r \end{array}$$

is an isomorphism of vector spaces.

Proof. By definition, bases of $\langle E_{\ell+1}\rangle^{(s+3)}$ and $\langle E_\ell\rangle^{(s)}$ are respectively given by:

- $\{v \in \overline{Q}_1 | s(v), t(v) \in (\overline{Q}_0)^{(s+3)}_{\geq \ell}\}$;
- $\{v \in \overline{Q}_1 | s(v), t(v) \in (\overline{Q}_0)^{(s)}_{\geq \ell-1}\}$.

By definition of Ψ_2, it is immediate that $v \in \overline{Q}_1$ is such that $s(v), t(v) \in (\overline{Q}_0)^{(s+3)}_{\geq \ell}$ if and only if $\Psi_2(v)$ is such that $s(\Psi_2(v)), t(\Psi_2(v)) \in (\overline{Q}_0)^{(s)}_{\geq \ell-1}$. □

Proposition 6.1.7.6. The following diagram commutes.

$$\begin{array}{ccccc} \langle E_{\ell+1}\rangle^{(s+3)} & \overset{g_{\ell+1}^{(s+3)}}{\to} & \langle D_{\ell+1}\rangle^{(s+3)} & \overset{f_{\ell+1}^{(s+3)}}{\to} & \langle(\overline{Q}_0)_{\geq \ell+1}\rangle^{(s+3)} \\ \downarrow \Psi_2 & & \downarrow \Psi_1 & & \downarrow \Psi_0 \\ \langle E_\ell\rangle^{(s)} & \overset{g_\ell^{(s)}}{\to} & \langle D_\ell\rangle^{(s)} & \overset{f_\ell^{(s)}}{\to} & \langle(\overline{Q}_0)_{\geq \ell}\rangle^{(s)} \end{array}$$

Proof. Using the maps Ψ_0, Ψ_1, Ψ_2 defined above and Lemma 6.1.4.1, we have the following.

1. For all $v \in D_{\ell+1}^{(s+3)}$:

$$\begin{aligned} \Psi_0 f_{\ell+1}^{(s+3)}(v) =& \Psi_0(s(v) - t(v)) = (s(v) - (1,1,1)) - (t(v) - (1,1,1)) \\ =& f_\ell^{(s)}((e_{s(v)-(1,1,1)})v) = f_\ell^{(s)}\Psi_1(v). \end{aligned}$$

2. Let $\{p,q,r\}=\{0,1,2\}$, $p<q$. Then:

$$\begin{aligned}\Psi_1 g_{\ell+1}^{(s+3)}(\alpha_r) =& \Psi_1(e_{t(\alpha_r)}\alpha_p + e_{t(\alpha_r\alpha_p)}\alpha_q - e_{t(\alpha_r)}\alpha_q - e_{t(\alpha_r\alpha_q)}\alpha_p)\\ =& e_{t(\alpha_r)-(1,1,1)}\alpha_p + e_{t(\alpha_r\alpha_p)-(1,1,1)}\alpha_q\\ &- e_{t(\alpha_r)-(1,1,1)}\alpha_q - e_{t(\alpha_r\alpha_q)-(1,1,1)}\alpha_p\\ =& g_\ell^{(s)}(e_{s(\alpha_r)-(1,1,1)}\alpha_r) = g_\ell^{(s)}\Psi_2(\alpha_r).\end{aligned}$$

So the diagram commutes. □

As a consequence we get the following.

Corollary 6.1.7.7. Let $s \geq 2$. Then the first Hochschild homology group $HH_1(\Pi^{(2,s)})$ of $\Pi^{(2,s)}$ is zero.

Proof. Recall that:

$$HH_1(\Pi) = \bigoplus_{\ell\geq 1} HH_1(\Pi)(3\ell)$$

by Lemma 6.1.2.3. Therefore, it is enough to show that $HH_1(\Pi)(3\ell)=0$ for all $\ell \geq 1$.

First of all notice that, since $HH_1(\Pi)$ is a subquotient of $(V\otimes\Pi)^S$ and the maximal length of an element in $(V\otimes\Pi)^S$ is s by Proposition 3.2.0.6, we have that $HH_1(\Pi)(3\ell)=0$ if $3\ell > s$. Hence, we may assume $3\ell \leq s$.

Using the commutativity of the diagram (6.1.7.1) and the fact that the vertical maps are isomorphisms by Propositions 6.1.7.3, 6.1.7.4 and 6.1.7.5, we have:

$$\frac{\ker(f_\ell)^{(s)}}{\mathrm{im}(g_\ell)^{(s)}} \cong \frac{\ker(f_{\ell-1})^{(s-3)}}{\mathrm{im}(g_{\ell-1})^{(s-3)}}.$$

Hence, by Remark 6.1.4.3:

$$HH_1(\Pi^{(2,s)})(3\ell) \cong HH_1(\Pi^{(2,s-3)})(3(\ell-1)). \tag{6.1.7.2}$$

Repeatedly applying (6.1.7.2), we get:

$$HH_1(\Pi^{(2,s)})(3\ell) \cong HH_1(\Pi^{(2,s-3(\ell-1))})(3).$$

But the latter is 0 by Proposition 6.1.7.2, so the statement follows. □

6.2 Computation of $HH_0(\Pi)$

In this section we compute $HH_0(\Pi)$. In order to do this, we use Proposition 2.2.1.10, that states:

$$HH_0(\Pi) \cong \Pi/[\Pi,\Pi]. \tag{6.2.0.1}$$

The next theorem gives a description of $[\Pi,\Pi]$ and, in turn, of $HH_0(\Pi)$.

Theorem 6.2.0.1. Let $\Pi_+ = \bigoplus_{k\geq 1} \Pi(k)$. Then:

$$[\Pi, \Pi] = \Pi_+$$

and, as a consequence:

$$HH_0(\Pi) \cong S.$$

Proof. First of all, notice that:

$$\Pi = S \oplus \Pi_+, \tag{6.2.0.2}$$

since $S = \Pi(0)$.

Fix $x \in S$. If $x \in [\Pi, \Pi]$, then there must exist elements $y, z \in \Pi$ such that $x = [y, z]$. Now, if $y \in \Pi(i)$ and $z \in \Pi(j)$ for some $i, j \geq 0$, then $[y, z] \in \Pi(i + j)$. Therefore both y and z must lie in $S = \Pi(0)$.

However, we have:

$$[e_i, e_j] = 0$$

for all $i, j \in \overline{Q}_0$. Therefore, writing y, z as linear combination of e_i's, we see that $x = [y, z] = 0$. Therefore $[\Pi, \Pi] \subseteq \Pi_+$.

Now, to show the other inclusion, take a nonzero path $x \in \Pi_+$. If $x \notin \Pi_+^S$, then we have:

$$x = [e_{s(x)}, x] \in [\Pi, \Pi].$$

Therefore we can assume $x \in \Pi_+^S$, i.e., $s(x) = t(x)$. By Lemma 6.1.1.3 we have:

$$x = T^m_{(i,j,k)} = e_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m$$

for some $m \in \{1, \dots, \lfloor \frac{s-1}{3} \rfloor\}$ and $(i, j, k) \in (\overline{Q}_0)_\ell$ with $\ell \geq m$. We prove that $x \in [\Pi, \Pi]$ by induction on $\ell \geq m$.

- If $\ell = m$, then we have that $m = \ell = \min\{i, j, k\}$. Assume w.l.o.g. that $k = m$. Then we have:

$$[\Pi, \Pi] \ni [e_{(i,j,k)}\alpha_0^m, e_{(i+k,j,0)}(\alpha_1\alpha_2)^m] = T^m_{(i,j,k)} - T^m_{(i+k,j,0)}.$$

  But $(i + k, j, 0) \in (\overline{Q}_0)_0$, and therefore $T^m_{(i+k,j,0)} = 0$. Hence we have $x = T^m_{(i,j,k)} \in [\Pi, \Pi]$.

- If $\ell > m$, then we have $\ell = \min\{i, j, k\}$. Suppose w.l.o.g. that $k = \ell$. Then we have:

$$[\Pi, \Pi] \ni [e_{(i,j,k)}\alpha_0, e_{(i+1,j,k-1)}\alpha_0^{m-1}(\alpha_1\alpha_2)^m] = T^m_{(i,j,k)} - T^m_{(i+1,j,k-1)}.$$

  Now, since $(i + 1, j, k - 1) \in (\overline{Q}_0)^{\ell-1}$, we have $T^m_{(i+1,j,k-1)} \in [\Pi, \Pi]$ by induction hypothesis. Therefore also $x = T^m_{(i,j,k)} \in [\Pi, \Pi]$.

Hence $[\Pi,\Pi] = \Pi_+$ and, in view of (6.2.0.2) and (6.2.0.1), we can conclude that:

$$HH_0(\Pi) \cong S.$$

□

6.3 Computation of $HH_4(\Pi)$

By Proposition 4.2.0.5 we know that the fourth Hochschild homology group $HH_4(\Pi)$ can be deduced by computing the homology of the following complex:

$$(V\otimes\mathcal{N})^S[h] \overset{\mu_5'}{\to} \mathcal{N}^S[h] \overset{\mu_4'}{\to} \Pi^S[3], \tag{6.3.0.1}$$

where

$$\mu_4'(y) = \sum x_j^*\eta(y)x_j$$
$$\mu_5'(a\otimes x) = x\eta^{-1}(a) - ax$$

and η is the Nakayama automorphism of Π explicitly defined in Theorem 3.2.0.8, $\mathcal{N} = \Pi_{\eta^{-1}}$ and $\{x_j\}$ is a basis of Π consisting of all nonzero paths, with $\{x_j^*\}$ is its dual basis under the non-degenerate bilinear form $(.,.)$ given in Definition 3.2.0.7.

Therefore:

$$HH_4(\Pi) \cong \frac{\ker(\mu_4')}{\operatorname{im}(\mu_5')}.$$

In order to compute $HH_4(\Pi)$ we will use a similar strategy to the one we used for $HH_1(\Pi)$: we will give a combinatorial description of the spaces $\mathcal{N}^S$ and $(V\otimes\mathcal{N})^S$, and of the maps μ_4' and μ_5'.

6.3.1 A basis for $\mathcal{N}^S$

In this subsection we give a combinatorial description of the space:

$$\mathcal{N}^S = \bigoplus_{(i,j,k)\in\overline{Q}_0} e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \bigoplus_{(i,j,k)\in\overline{Q}_0} e_{(i,j,k)}\Pi e_{(j,k,i)}.$$

We thus want to find all nonzero paths $(i,j,k)\to(j,k,i)$.

Proposition 6.3.1.1. Let $(i,j,k)\in\overline{Q}_0$, and set $M = \max\{i,j,k\}$. Then $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} \neq 0$ if and only if one of the following holds:

- $M = i$ and $j \geq i-k$. In this case the shortest path $(i,j,k)\to(j,k,i)$ is given by $e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}$.
- $M = j$ and $k \geq j-i$. In this case the shortest path $(i,j,k)\to(j,k,i)$ is given by $e_{(i,j,k)}\alpha_0^{j-i}\alpha_2^{j-k}$.

- $M = k$ and $i \geq k - j$. In this case the shortest path $(i,j,k) \to (j,k,i)$ is given by $e_{(i,j,k)}\alpha_0^{k-i}\alpha_1^{k-j}$.

Proof. Suppose that $M = i$. Notice that the path $e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}$ ends at (j,k,i). Indeed:

$$t(e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}) = t(e_{(i-(i-j),j+(i-j),k)}\alpha_2^{i-k}) = t(e_{(j,i,k)}\alpha_2^{i-k})$$
$$= (j, i-(i-k), k+(i-k)) = (j,k,i).$$

By Lemma 3.2.0.3 this, if nonzero, is the shortest path $(i,j,k) \to (j,k,i)$. Hence $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} \neq 0$ if and only if $e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k} \neq 0$. But by Lemma 3.2.0.2 the latter holds if and only if the following two conditions hold:

- $i \geq i - j$;
- $j \geq i - k$.

Since the first condition trivially holds true, we can say that $e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k} \neq 0$ if and only if $j \geq i - k$, that gives the statement for the case $M = i$.

To prove the Proposition in the other cases, one can just apply the automorphisms η, $\eta^2 = \eta^{-1}$ of Π, noticing that

- $\eta(i,j,k) = (k,i,j)$ and $\eta(e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}) = e_{(k,i,j)}\alpha_2^{i-j}\alpha_0^{i-k}$
- $\eta^{-1}(i,j,k) = (j,k,i)$ and $\eta^{-1}(e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}) = e_{(j,k,i)}\alpha_0^{i-j}\alpha_1^{i-k}$.

□

Corollary 6.3.1.2. Let $(i,j,k) \in \overline{Q}_0$, and set $M = \max\{i,j,k\}$. Then $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} \neq 0$ if and only if

$$M \in \{\left\lceil \frac{s-1}{3} \right\rceil, \ldots, \left\lfloor \frac{s-1}{2} \right\rfloor\}.$$

Proof. First of all, notice that $M \geq \lceil \frac{s-1}{3} \rceil$ always holds true. Indeed:

$$3M \geq i+j+k = s-1 \quad \Rightarrow \quad M \geq \frac{s-1}{3} \quad \Rightarrow \quad M \geq \left\lceil \frac{s-1}{3} \right\rceil.$$

Now, suppose w.l.o.g. that $M = i$. Thanks to Proposition 6.3.1.1, it is enough to show that $i \leq \lfloor \frac{s-1}{2} \rfloor$ if and only if $j \geq i - k$.

If $i \leq \lfloor \frac{s-1}{2} \rfloor$, then:

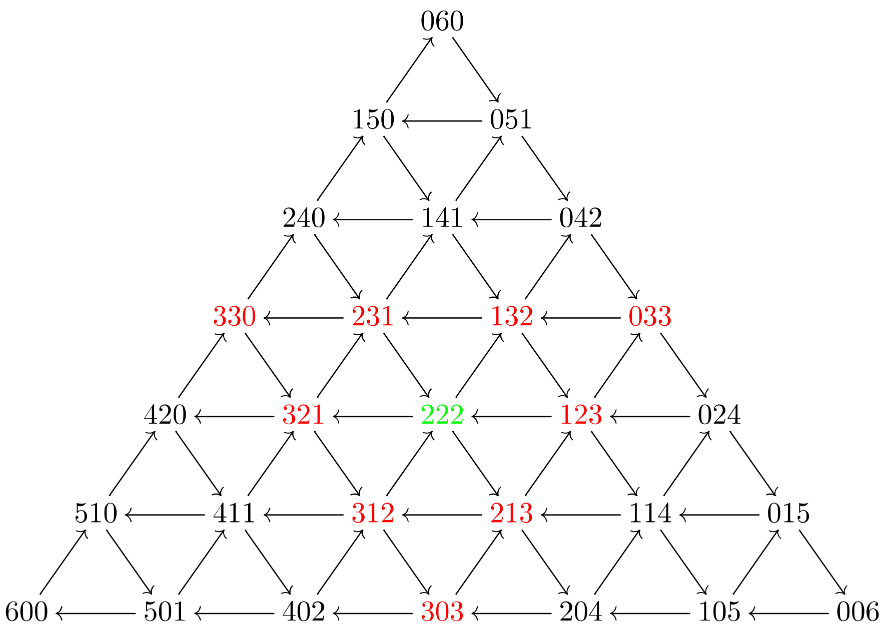
$$\begin{aligned} i - k &= i - (s - i - j - 1) \\ &= 2i + j - s + 1 \\ &\leq 2\left\lfloor \frac{s-1}{2} \right\rfloor + j - s + 1 \\ &\leq s - 1 + j - s + 1 = j. \end{aligned}$$

Conversely, if $j \geq i - k$, then:

$$i \leq k + j = s - 1 - i,$$

from which it follows $2i \leq s - 1$ or, equivalently, $i \leq \lfloor \frac{s-1}{2} \rfloor$. $\square$

Example 6.3.1.3. Let $s = 7$. The quiver $\overline{Q}$ is given as follows.



The coloured vertices (i, j, k) are the ones such that:

$$e_{(i,j,k)} \mathcal{N} e_{(i,j,k)} \neq 0.$$

In particular, the red vertices are the ones whose maximal component is equal to $\lfloor \frac{s-1}{2} \rfloor = 3$, while the green vertex $(2, 2, 2)$ has maximal component equal to $\lceil \frac{s-1}{3} \rceil = 2$.

Definition 6.3.1.4. Define

$$(\overline{Q}_0)^M = \{(i, j, k) \in \overline{Q}_0 | \max\{i, j, k\} = M\}.$$

This induces the following partition of the set of vertices of $\overline{Q}$:

$$\overline{Q}_0 = \bigcup_{M=\lceil \frac{s-1}{3} \rceil}^{s-1} (\overline{Q}_0)^M$$

Thanks to Corollary 6.3.1.2, we can therefore write:

$$\mathcal{N}^S = \bigoplus_{M=\lceil \frac{s-1}{3} \rceil}^{\lfloor \frac{s-1}{2} \rfloor} \bigoplus_{(i,j,k)\in(\overline{Q}_0)^M} e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}. \tag{6.3.1.1}$$

Definition 6.3.1.5. Let $(i,j,k) \in (\overline{Q}_0)^M$ for some $M \in \{\lceil \frac{s-1}{3} \rceil \dots, \lfloor \frac{s-1}{2} \rfloor\}$. Denote by $c_{(i,j,k)}$ the shortest path $(i,j,k) \to (j,k,i)$, given explicitly in Proposition 6.3.1.1.

We now study the summands of (6.3.1.1).

Theorem 6.3.1.6. Let $(i,j,k) \in (\overline{Q}_0)^M$ for some $M \in \{\lceil \frac{s-1}{3} \rceil \dots, \lfloor \frac{s-1}{2} \rfloor\}$. Then:

$$e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \langle c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m | m = 0, \dots, s-1-2M \rangle.$$

Proof. Suppose w.l.o.g. that $M = i$. By Proposition 6.3.1.1, we know that $c_{(i,j,k)} = e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}$. Furthermore, Lemma 3.2.0.3 implies that each path $(i,j,k) \to (j,k,i)$ is of the form:

$$c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m = e_{(i,j,k)}\alpha_0^m\alpha_1^{i-j+m}\alpha_2^{i-k+m} \tag{6.3.1.2}$$

for some $m \geq 0$. Notice that, for $m = s-1-2i$, the path (6.3.1.2) is nonzero by Lemma 3.2.0.2. Indeed, we have:

- $$\begin{aligned} i \geq i - j + (s-1-2i) &\Leftrightarrow i + (i+j) \geq s-1 \\ &\Leftrightarrow i + (s-1-k) \geq s-1 \\ &\Leftrightarrow i - k \geq 0, \end{aligned}$$

  that holds since $i = \max\{i,j,k\}$.

- $$j \geq i - k + (s-1-2i) \quad \Leftrightarrow \quad i+j+k \geq s-1,$$

  that holds since $i+j+k = s-1$.

- $$\begin{aligned} k \geq (s-1-2i) &\Leftrightarrow i + (i+k) \geq s-1 \\ &\Leftrightarrow i + (s-1-j) \geq s-1 \\ &\Leftrightarrow i - j \geq 0, \end{aligned}$$

  that holds since $i = \max\{i,j,k\}$.

Hence the path (6.3.1.2) is nonzero for $m = s-1-2i$, and so (6.3.1.2) is nonzero also for $m < s-1-2i$.
However the path (6.3.1.2) is zero for $m = s-2i$. This can be proved again using Lemma 3.2.0.2. Indeed:

$$i-k+s-2i = -i-k+s = j+1 > j.$$

□

Corollary 6.3.1.7. We have:

$$\mathcal{N}^S = \bigoplus_{M=\lceil \frac{s-1}{3} \rceil}^{\lfloor \frac{s-1}{2} \rfloor} \bigoplus_{(i,j,k)\in(\overline{Q}_0)^M} e_{(i,j,k)}\mathcal{N}e_{(i,j,k)},$$

where, for $(i,j,k) \in (\overline{Q}_0)^M$

$$e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \bigoplus_{m=0}^{s-1-2M} V^{(i,j,k)}_{3(M+m)-(s-1)}$$

with $V^{(i,j,k)}_{3(M+m)-(s-1)}$ a one dimensional vector space in weight degree $3(M+m)-(s-1)$ spanned by $c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m$, $m = 0, \dots, s-1-2M$.

Proof. The decomposition of $\mathcal{N}^S$ follows from (6.3.1.1). Hence, we just need to check that, for $M \in \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor$ and $(i,j,k) \in (\overline{Q}_0)^M$, the path $c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m$ has length $3(M+m)-(s-1)$ for all $m \in \{0, \dots, s-1-2M\}$. Assume w.l.o.g. that $M = i$. Then, by Proposition 6.3.1.1 we have:

$$c_{(i,j,k)} = e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k},$$

that has length

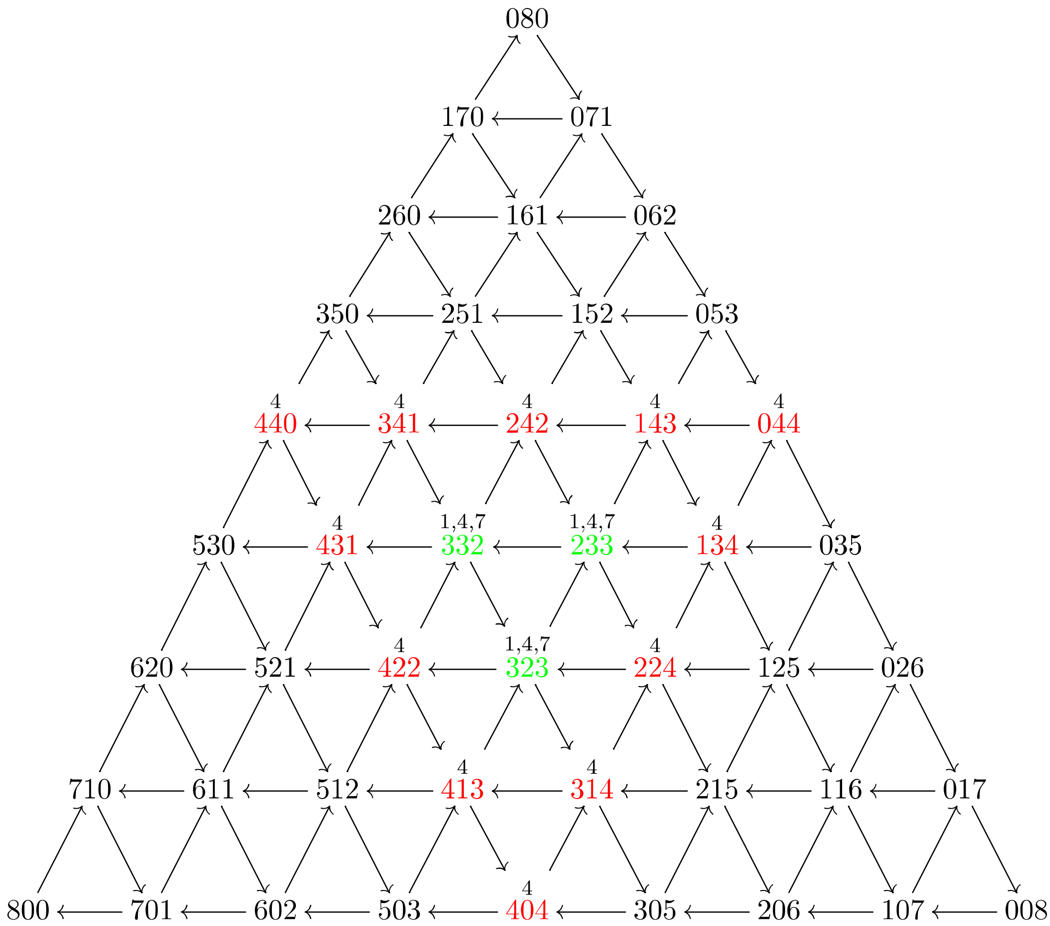
$$i-j+i-k = 2i-(j+k) = 2i-(s-1-i) = 3i-(s-1) = 3M-(s-1).$$

Therefore $c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m$ has length:

$$3i-(s-1)+3m = 3(i+m)-(s-1) = 3(M+m)-(s-1),$$

and this concludes the proof. □

Example 6.3.1.8. Let $s = 9$. Then the quiver $\overline{Q}$ is given by:



The coloured vertices are the start point of nonzero paths in $\mathcal{N}^S$. If the number ℓ is written above a coloured vertex (i,j,k), then $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ contains one path of length ℓ. In particular:

- the red vertices are the elements of $(\overline{Q}_0)^4$. By Corollary 6.3.1.7, each of them is the start point of a nonzero path in $\mathcal{N}^S$ of degree 4. More explicitly, if (i,j,k) is a red vertex, then:

$$e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \langle c_{(i,j,k)} \rangle$$

  where $c_{(i,j,k)}$ has length 4.

- The green vertices are the elements of $(\overline{Q}_0)^3$. By Corollary 6.3.1.7, each of them is the start point of three nonzero paths in $\mathcal{N}^S$, of degree 1,4,7. More explicitly, if (i,j,k) is a red vertex, then:

$$e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \langle c_{(i,j,k)}, c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2), c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^2 \rangle$$

  where $c_{(i,j,k)}$ has length 1.

6.3.2 A basis for $(V \otimes \mathcal{N})^S$

We now want to study the vector space:

$$(V \otimes \mathcal{N})^S = \bigoplus_{(i,j,k) \in \overline{Q}_0} e_{(i,j,k)}(V \otimes \mathcal{N})e_{(i,j,k)}.$$

In particular, we want to give a characterisation of the nonzero elements in $(V \otimes \mathcal{N})^S$ of the form $v \otimes a$, where $v \in \overline{Q}_1$ has start or end point that is a vertex in $(\overline{Q}_0)^M$, for some $M = \lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor$. Due to the description of a basis of $\mathcal{N}^S$ we gave in the previous subsection, this will turn out to be enough to give a complete description of im μ_5'.

We introduce some notation that will be useful throughout this section.

Definition 6.3.2.1. Let c be a nonzero path in Π, and $p \in \{0, 1, 2\}$.

- If α_p is an arrow in c, then define $d(c, \alpha_p)$ to be the path with start point $t(s(c)\alpha_p)$ obtained by removing α_p from c.
- If α_p is not an arrow in c and $t(s(c)\alpha_p) \in \overline{Q}_0$, then define $a(c, \alpha_p)$ to be the path with start point $t(s(c)\alpha_p)$ given by:

  $$a(c, \alpha_p) = e_{t(s(c)\alpha_p)} c \alpha_q \alpha_r,$$

  where $\{p, q, r\} = \{0, 1, 2\}$.

Remark 6.3.2.2. Let $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$ and $(i, j, k) \in (\overline{Q}_0)^M$. Consider the path $c_{(i,j,k)}$ given in Definition 6.3.1.5. It is straightforward to check that:

- $d(c_{(i,j,k)}, \alpha_\ell)$ has endpoint $(j, k, i) = \eta^{-1}(i, j, k)$ if α_ℓ is an arrow in $c_{(i,j,k)}$;
- $a(c_{(i,j,k)}, \alpha_\ell)$ has endpoint $(j, k, i) = \eta^{-1}(i, j, k)$ if α_ℓ is not an arrow in $c_{(i,j,k)}$.

Example 6.3.2.3. Consider the path $c = e_{(4,3,1)}\alpha_1\alpha_2^3 \in \mathcal{N}^S$ in Example 6.3.1.8. Then:

- $a(c, \alpha_0) = e_{(5,3,0)}\alpha_1^2\alpha_2^4 = 0$;
- $d(c, \alpha_1) = e_{(3,4,1)}\alpha_2^3$;
- $d(c, \alpha_2) = e_{(4,2,2)}\alpha_1\alpha_2^2$.

Notice that, if v is an arrow in $\overline{Q}$ and $s(v) \in (\overline{Q}_0)^M$ for some M, then there are 3 possibilities:

- $t(v) \in (\overline{Q}_0)^M$;
- $t(v) \in (\overline{Q}_0)^{M-1}$;

- $t(v) \in (\overline{Q}_0)^{M+1}$.

We will study the space $(v \otimes \mathcal{N})^S$ separately according to these cases.

Lemma 6.3.2.4. Let $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$, and $v \in \overline{Q}_1$ be such that $s(v), t(v) \in (\overline{Q}_0)^M$. Then:

$$(v \otimes \mathcal{N})^S = \langle v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m | m = 0, \ldots, s-1-2M \rangle.$$

Proof. Let $s(v) = (i,j,k)$, and suppose w.l.o.g. that $M = i$. By Proposition 6.3.1.1, the shortest path $(i,j,k) \to (j,k,i)$ is given by:

$$c_{(i,j,k)} = e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}.$$

We need to consider two distinct cases.

(a) If $i = k$, then necessarily $v = \alpha_1 : (i,j,k) \to (i-1, j+1, k)$. In this case we have that $d(c_{(i,j,i)}, v) = e_{(i-1,j+1,i)}\alpha_1^{i-j-1}$, and hence:

$$d(c_{(i,j,i)}, v)(\alpha_0\alpha_1\alpha_2)^m = e_{(i-1,j+1,i)}\alpha_0^m\alpha_1^{m+i-j-1}\alpha_2^m. \tag{6.3.2.1}$$

For $m = s-1-2i$, this path is nonzero by Lemma 3.2.0.2. Indeed:

- $m = s-1-2i = s-1-(s-1-j) = j \leq i$;
- $m+i-j-1 = s-1-2i+i-j-1 = i-1 \leq i-1$;
- $m = s-1-2i = j \leq j+1$.

As a consequence $v \otimes d(c_{(i,j,i)}, v)(\alpha_0\alpha_1\alpha_2)^m \neq 0$ for $m = 0, \ldots, s-1-2i$.

The second equation, together with Lemma 3.2.0.2, also tells us that (6.3.2.1) is zero for $m = s-2i$.

(b) If $i > k$, then necessarily $v = \alpha_2 : (i,j,k) \to (i, j-1, k+1)$. In this case we have $d(c_{(i,j,k)}, v) = e_{(i,j-1,k+1)}\alpha_1^{i-j}\alpha_2^{i-k-1}$, and hence:

$$d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m = e_{(i,j-1,k+1)}\alpha_0^m\alpha_1^{m+i-j}\alpha_2^{m+i-k-1}. \tag{6.3.2.2}$$

Once again, using Lemma 3.2.0.2, one can show that the path (6.3.2.2) is nonzero for $m = s-1-2i$. This implies that $v \otimes d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m$ is nonzero for all $m = 0, \ldots, s-1-2i$. Using the same argument, we have that (6.3.2.2) is zero for $m = s-2i$, that gives:

$$v \otimes d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^{s-2i} = 0.$$

□

Lemma 6.3.2.5. Let $M \in \{\lceil \frac{s-1}{3} \rceil + 1, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$, and $v \in \overline{Q}_1$ be such that $s(v) \in (\overline{Q}_0)^M$, $t(v) \in (\overline{Q}_0)^{M-1}$. Then:

$$(v \otimes \mathcal{N})^S = \langle v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m | m = 0, \ldots, s - 2M \rangle.$$

Proof. Suppose w.l.o.g. that $v = \alpha_1 : (i, j, k) \to (i-1, j+1, k)$. Then we necessarily have that $M = i$, $j + 1 \leq i - 1$ and $k \leq i - 1$. By Proposition 6.3.1.1, the shortest path $(i, j, k) \to (j, k, i)$ is given by:

$$c_{(i,j,k)} = e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}.$$

Hence we have $d(c_{(i,j,k)}, v) = e_{(i-1,j+1,k)}\alpha_1^{i-j-1}\alpha_2^{i-k}$, and thus:

$$d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m = e_{(i-1,j+1,k)}\alpha_0^m\alpha_1^{m+i-j-1}\alpha_2^{m+i-k}. \tag{6.3.2.3}$$

For $m = s - 2i$, we can check that (6.3.2.3) is nonzero using Lemma 3.2.0.2. Indeed:

- $m = s - 2i = j + k + 1 - i \leq k - 1 \leq k$, where the third inequality comes from the fact that $j + 1 \leq i - 1$;
- $m + i - j - 1 = s - i - j - 1 = k \leq i - 1$;
- $m + i - k = s - i - k = j + 1 \leq j + 1$.

This tells us that $v \otimes d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m \neq 0$ for $m = 0, \ldots, s - 2i$.

The third inequality also implies that (6.3.2.3) is zero for $m = s - 2i + 1$, and hence $v \otimes d(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^{s-2i+1} = 0$. □

Lemma 6.3.2.6. Let $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor - 1\}$, and $v \in \overline{Q}_1$ such that $s(v) \in (\overline{Q}_0)^M$, $t(v) \in (\overline{Q}_0)^{M+1}$. Then:

$$(v \otimes \mathcal{N})^S = \langle v \otimes a(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m | m = 0, \ldots, s - 2M - 2 \rangle.$$

Proof. Suppose w.l.o.g. that $v = \alpha_0 : (i, j, k) \to (i+1, j, k-1)$. Then we necessarily have $M = i$, and the shortest path $(i, j, k) \to (j, k, i)$ is thus given by:

$$c_{(i,j,k)} = e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}.$$

Hence $a(c_{(i,j,k)}, v) = e_{(i+1,j,k-1)}\alpha_1^{i-j+1}\alpha_2^{i-k+1}$, and so:

$$a(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m = e_{(i+1,j,k-1)}\alpha_0^m\alpha_1^{m+i-j+1}\alpha_2^{m+i-k+1}. \tag{6.3.2.4}$$

Once again, a straightforward use of Lemma 3.2.0.2 allows us to say that (6.3.2.4) is a nonzero path for $m = s - 2i - 2$. This implies $v \otimes a(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^m \neq 0$

for all $m = 0, \dots, s-2i-2$. However, Lemma 3.2.0.2 also implies that (6.3.2.4) is zero for $m = s-2i-1$, that gives $v \otimes a(c_{(i,j,k)}, v)(\alpha_0\alpha_1\alpha_2)^{s-2i-1} = 0$. □

We now just need to analyse the cases when the arrow v is such that:

- $s(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor}$ and $t(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor + 1}$;
- $s(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor + 1}$ and $t(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor}$

Lemma 6.3.2.7. Let $v \in \overline{Q}_1$ be such that $s(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor}$ and $t(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor + 1}$. Then:

$$(v \otimes \mathcal{N})^S = \begin{cases} 0, & \text{if } s \text{ is odd} \\ \langle v \otimes a(c_{s(v)}, v) \rangle, & \text{if } s \text{ is even.} \end{cases}$$

Proof. Suppose w.l.o.g. that $v = \alpha_0 : (i,j,k) \to (i+1, j, k-1)$. Then we necessarily have that $\lfloor \frac{s-1}{2} \rfloor = i$, and the shortest path $(i,j,k) \to (j,k,i)$ is therefore given by:

$$c_{(i,j,k)} = e_{(i,j,k)} \alpha_1^{i-j} \alpha_2^{i-k}.$$

Hence:

$$a(c_{(i,j,k)}, v) = e_{(\lfloor \frac{s-1}{2} \rfloor + 1, j, k-1)} \alpha_1^{\lfloor \frac{s-1}{2} \rfloor - j + 1} \alpha_2^{\lfloor \frac{s-1}{2} \rfloor - k + 1}. \tag{6.3.2.5}$$

We have:

- $\lfloor \frac{s-1}{2} \rfloor - j + 1 \le i = \lfloor \frac{s-1}{2} \rfloor$ if and only if $j \ge 1$;
- $\lfloor \frac{s-1}{2} \rfloor - k + 1 \le j$ if and only if $j + k \ge \lfloor \frac{s-1}{2} \rfloor + 1$. But $j + k = s - 1 - \lfloor \frac{s-1}{2} \rfloor$. So the inequality is true if and only if $s - 2\lfloor \frac{s-1}{2} \rfloor \ge 2$.

Now, the second inequality holds true only when s is even. In this case, the first inequality always holds true. Indeed, if by contradiction $j = 0$ then, since $i + j + k = s - 1$ and $i = \lfloor \frac{s-1}{2} \rfloor = \frac{s-2}{2}$, we should have $k = \frac{s}{2} > i$, against the assumption that $(i,j,k) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor}$. Hence, by Lemma 3.2.0.2, the path (6.3.2.5) is zero for s odd, and nonzero for s even. This implies that $v \otimes a(c_{(i,j,k)}, v)$ is zero for s odd and nonzero for s even.

The second inequality, together with Lemma 3.2.0.2, also implies that the path $a(c_{(i,j,k)}, v)\alpha_0\alpha_1\alpha_2$ is always zero, that gives $v \otimes a(c_{(i,j,k)}, v)\alpha_0\alpha_1\alpha_2 = 0$. □

Lemma 6.3.2.8. Let $v \in \overline{Q}_1$ be such that $s(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor + 1}$ and $t(v) \in (\overline{Q}_0)^{\lfloor \frac{s-1}{2} \rfloor}$. Then:

$$(v \otimes \mathcal{N})^S = \begin{cases} 0, & \text{if } s \text{ is odd} \\ \langle v \otimes d(c_{s(v)}, v) \rangle, & \text{if } s \text{ is even,} \end{cases}$$

Proof. Suppose w.l.o.g. that $v = \alpha_1 : (i,j,k) \to (i-1, j+1, k)$. Then we necessarily have that $\lfloor \frac{s-1}{2} \rfloor + 1 = i$, and the shortest path $(i,j,k) \to (j,k,i)$ is therefore given by:

$$c_{(i,j,k)} = e_{(i,j,k)} \alpha_1^{i-j} \alpha_2^{i-k}.$$

Hence

$$d(c_{(i,j,k)}, v) = e_{(\lfloor \frac{s-1}{2} \rfloor, j+1, k)} \alpha_1^{\lfloor \frac{s-1}{2} \rfloor - j} \alpha_2^{\lfloor \frac{s-1}{2} \rfloor - k+1}. \tag{6.3.2.6}$$

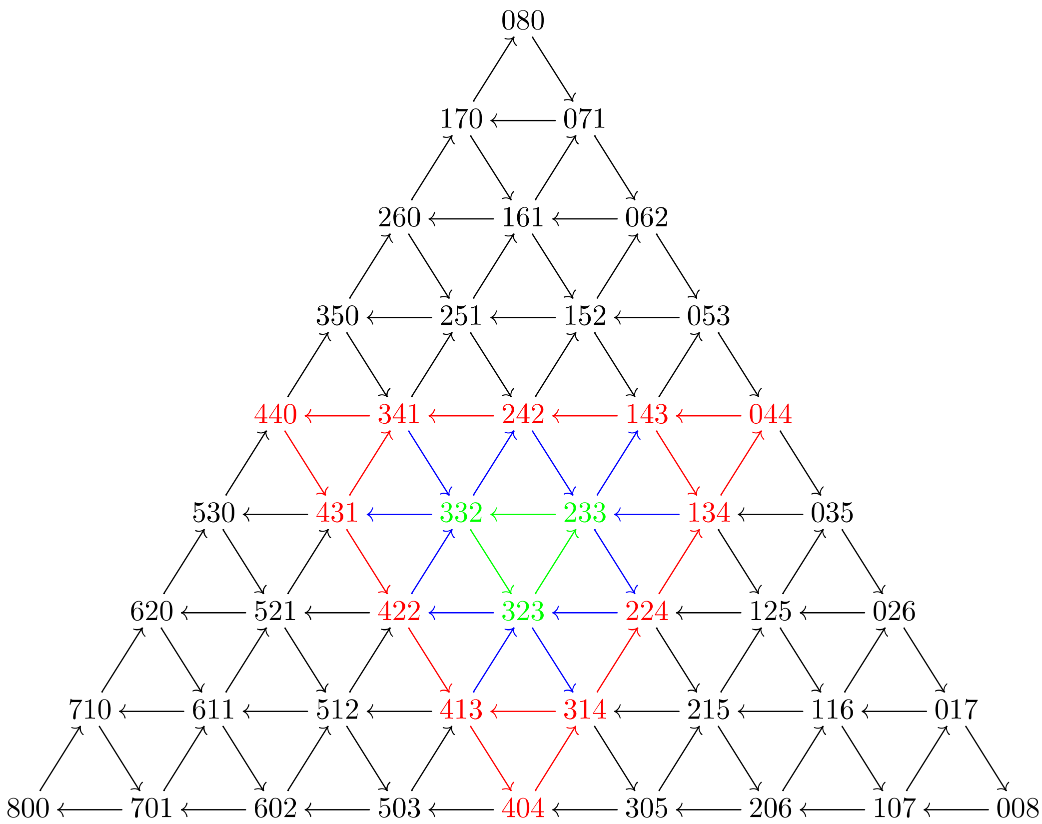
We have:

- $\lfloor \frac{s-1}{2} \rfloor - j \le i = \lfloor \frac{s-1}{2} \rfloor + 1$ if and only if $j \ge -1$;
- $\lfloor \frac{s-1}{2} \rfloor - k + 1 \le j+1$ if and only if $j+k \ge \lfloor \frac{s-1}{2} \rfloor$. But $j+k = s-1-\lfloor \frac{s-1}{2} \rfloor - 1 = s-2-\lfloor \frac{s-1}{2} \rfloor$. So the inequality is true if and only if $s-2 \ge 2\lfloor \frac{s-1}{2} \rfloor$.

The first inequality is always true, while the second one holds only if s is even. Therefore, by Lemma 3.2.0.2 the path (6.3.2.6) is zero for s odd, and nonzero for s even. This implies that $v \otimes d(c_{(i,j,k)}, v)$ is zero for s odd and nonzero for s even.

The second inequality, together with Lemma 3.2.0.2, also implies that the path $d(c_{(i,j,k)}, v)\alpha_0\alpha_1\alpha_2$ is always zero, that gives $v \otimes d(c_{(i,j,k)}, v)\alpha_0\alpha_1\alpha_2 = 0$. ☐

Example 6.3.2.9. Let $s = 9$. Thanks to the previous lemmas, we can represent all the nonzero paths of $(V \otimes \mathcal{N})^S$ directly on the quiver $\overline{Q}$.



The arrows v giving rise to nonzero elements of the form $v \otimes a \in (V \otimes \mathcal{N})^S$ are the coloured ones. In particular:

- if v is a red arrow (meaning it connects two red vertices in $(\overline{Q}_0)^4$), then by Lemma 6.3.2.4 the space $(v \otimes \mathcal{N})^S$ is one dimensional, generated by an element of degree 4;
- if v is a blue arrow (meaning it connects a red vertex in $(\overline{Q}_0)^4$ and a green vertex in $(\overline{Q}_0)^3$), then by Lemmas 6.3.2.5 and 6.3.2.6 the space $(v \otimes \mathcal{N})^S$ is two dimensional, generated by elements of degree 4 and 7;
- if v is a green arrow (meaning it connects two green vertices in $(\overline{Q}_0)^3$), then by Lemma 6.3.2.4 the space $(v \otimes \mathcal{N})^S$ is three dimensional, generated by elements of degree 1,4,7.

Notice that, since s is odd, Lemmas 6.3.2.7 and 6.3.2.8 imply there is no arrow v with either ($s(v) \in (\overline{Q}_0)^4$ and $t(v) \in (\overline{Q}_0)^5$) or ($s(v) \in (\overline{Q}_0)^5$ and $t(v) \in (\overline{Q}_0)^4$) such that $(v \otimes \mathcal{N})^S \neq 0$.

6.3.3 Computation of $\ker \mu_4'$

We want to study the kernel of the map

$$\begin{array}{rccc} \mu_4' : & \mathcal{N}^S[s+2] & \to & \Pi^S[3] \\ & y & \mapsto & \sum_j x_j^* \eta(y) x_j \end{array} \tag{6.3.3.1}$$

where $\{x_j\}$ is a homogeneous basis of Π and $\{x_j^*\}$ is its corresponding dual basis with respect to $(.,.)$. This map was defined in Proposition 4.2.0.5.

Theorem 6.3.3.1. The kernel of the map μ_4' is spanned by all nonzero paths of positive length in $\mathcal{N}^S$. In particular:

$$ker\mu_4' = \begin{cases} \mathcal{N}^S[s+2], & \text{if } s \not\equiv 1 \pmod 3 \\ \mathcal{N}^S[s+2]/\langle e_{(\frac{s-1}{3},\frac{s-1}{3},\frac{s-1}{3})}\rangle, & \text{if } s \equiv 1 \pmod 3). \end{cases}$$

Proof. Let $x \in \mathcal{N}^S[s+2](s+2+j)$ for some $j > 0$. Then:

$$\mu_4'(x) \in \Pi^S[3](s+2+j) = \Pi^S(s-1+j) = 0,$$

where the last equality follows from Proposition 3.2.0.6.

Now, if $x \in \mathcal{N}^S[s+2](s+2) = \mathcal{N}^S(0)$, then it needs to be of length 0, and its starting point needs to be of the form (i,i,i) for some i by Proposition 6.3.1.1. But $i+i+i = s-1$, and thus $i = \frac{s-1}{3}$. Hence such an element x exists only if

$s \equiv 1 \pmod 3$. In this case it is given by $e_{(\frac{s-1}{3},\frac{s-1}{3},\frac{s-1}{3})}$. Also:

$$\mu_4'(e_{(\frac{s-1}{3},\frac{s-1}{3},\frac{s-1}{4})}) = \sum_j x_j^* e_{(\frac{s-1}{3},\frac{s-1}{3},\frac{s-1}{3})} x_j \neq 0,$$

so $e_{(\frac{s-1}{3},\frac{s-1}{3},\frac{s-1}{3})} \notin \ker \mu_4'$. □

6.3.4 Description of im(μ_5')

We want to study the image of the map:

$$\begin{array}{rccc} \mu_5': & (V \otimes \mathcal{N})^S[s+2] & \to & \mathcal{N}^S[s+2] \\ & v \otimes x & \mapsto & x\eta^{-1}(v) - vx. \end{array}$$

Recall the decomposition of $\mathcal{N}^S$ from Corollary 6.3.1.7:

$$\mathcal{N}^S = \bigoplus_{M=\lceil\frac{s-1}{3}\rceil}^{\lfloor\frac{s-1}{2}\rfloor} \bigoplus_{(i,j,k)\in(\overline{Q}_0)^M} e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}.$$

For $(i,j,k) \in (\overline{Q}_0)^M$:

$$e_{(i,j,k)}\mathcal{N}e_{(i,j,k)} = \bigoplus_{m=0}^{s-1-2M} V^{(i,j,k)}_{3(M+m)-(s-1)},$$

with $V^{(i,j,k)}_{3(M+m)-(s-1)m}$ a one dimensional vector space in weight degree $3(M+m)-(s-1)$ generated by $c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m$, for all $m = 0,\dots,s-1-2M$.

Remark 6.3.4.1. The map μ_5' acts as zero on elements of the form $v \otimes a$ with $s(v) \in (\overline{Q}_0)^{M_1}$, $t(v) \in (\overline{Q}_0)^{M_2}$ and $M_1, M_2 > \lfloor\frac{s-1}{2}\rfloor$. Indeed, $\mu_5'(v \otimes a)$ is the difference of two paths in $\mathcal{N}^S$, one starting at $t(v)$ and the other at $s(v)$. But the decomposition of $\mathcal{N}^S$ recalled above implies that there are no such nonzero paths, and thus $\mu_5'(v \otimes a) = 0$.

Therefore, in order to describe im μ_5', we just need to check how μ_5' acts on elements of the form $v \otimes a$, where at least one between $s(v)$ and $t(v)$ lies in $(\overline{Q}_0)^M$ for some $M \in \{\lceil\frac{s-1}{3}\rceil,\dots,\lfloor\frac{s-1}{2}\rfloor\}$. Such elements are described in Lemmas 6.3.2.4, 6.3.2.5, 6.3.2.6, 6.3.2.7 and 6.3.2.8, and the next proposition gives their image under μ_5'.

Proposition 6.3.4.2. Let $v \in \overline{Q}_1$ be an arrow.

1. If $s(v), t(v) \in (\overline{Q}_0)^M$ for some $M \in \{\lceil\frac{s-1}{3}\rceil,\dots,\lfloor\frac{s-1}{2}\rfloor\}$, and $m \in \{0,\dots,s-2M-1\}$, then:

$$\mu_5'(v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m) = c_{t(v)}(\alpha_0\alpha_1\alpha_2)^m - c_{s(v)}(\alpha_0\alpha_1\alpha_2)^m.$$

2. If $s(v) \in (\overline{Q}_0)^M$ and $t(v) \in (\overline{Q}_0)^{M-1}$ for some $M \in \{\lceil \frac{s-1}{3} \rceil + 1, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$, and $m \in \{0, \ldots, s-2M-1\}$, then:

$$\mu_5'(v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m) = c_{t(v)}(\alpha_0\alpha_1\alpha_2)^m - c_{s(v)}(\alpha_0\alpha_1\alpha_2)^m$$
$$\mu_5'(v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^{s-2M}) = - c_{s(v)}(\alpha_0\alpha_1\alpha_2)^{s-2M}.$$

3. If $s(v) \in (\overline{Q}_0)^M$ and $t(v) \in (\overline{Q}_0)^{M+1}$ for some $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor - 1\}$, and $m \in \{0, \ldots, s-2M-3\}$, then:

$$\mu_5'(v \otimes a(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m) = c_{t(v)}(\alpha_0\alpha_1\alpha_2)^{m+1} - c_{s(v)}(\alpha_0\alpha_1\alpha_2)^{m+1}$$
$$\mu_5'(v \otimes a(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^{s-2M-2}) = c_{t(v)}(\alpha_0\alpha_1\alpha_2)^{s-2M-1}.$$

4. If s is even, $s(v) \in (\overline{Q}_0)^{\frac{s-2}{2}}$ and $t(v) \in (\overline{Q}_0)^{\frac{s}{2}}$, then:

$$\mu_5'(v \otimes a(c_{s(v)}, v)) = -c_{s(v)}(\alpha_0\alpha_1\alpha_2).$$

5. If s is even, $s(v) \in (\overline{Q}_0)^{\frac{s}{2}}$ and $t(v) \in (\overline{Q}_0)^{\frac{s-2}{2}}$, then:

$$\mu_5'(v \otimes d(c_{s(v)}, v)) = c_{t(v)}(\alpha_0\alpha_1\alpha_2).$$

Proof. The proof consists in applying the formula for μ_5' and using the description of $\mathcal{N}^S$ given above.

For example, if $M = i$ and $v = \alpha_2 : (i,j,k) \to (i, j-1, k+1)$ is such that $s(v), t(v) \in (\overline{Q}_0)^M$, then we have $d(c_{s(v)}, v) = e_{(i,j-1,k+1)}\alpha_1^{i-j}\alpha_2^{i-k-1}$. Hence, if we let $m \in \{0, \ldots, s-2M-1\}$, then:

$$\begin{aligned}
&\mu_5'(v \otimes d(c_{s(v)}, v)(\alpha_0\alpha_1\alpha_2)^m) = \mu_5'(e_{(i,j,k)}\alpha_2 \otimes \alpha_1^{i-j}\alpha_2^{i-k-1}(\alpha_0\alpha_1\alpha_2)^m) \\
&= e_{(i,j-1,k+1)}\alpha_1^{i-j}\alpha_2^{i-k-1}(\alpha_0\alpha_1\alpha_2)^m\eta^{-1}(e_{(i,j,k)}\alpha_2) - e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}(\alpha_0\alpha_1\alpha_2)^m \\
&= e_{(i,j-1,k+1)}\alpha_1^{i-(j-1)}\alpha_2^{i-(k+1)}(\alpha_0\alpha_1\alpha_2)^m - e_{(i,j,k)}\alpha_1^{i-j}\alpha_2^{i-k}(\alpha_0\alpha_1\alpha_2)^m \\
&= c_{t(v)}(\alpha_0\alpha_1\alpha_2)^m - c_{s(v)}(\alpha_0\alpha_1\alpha_2)^m.
\end{aligned}$$

The other computations are very similar, and give the statement. □

Lemma 6.3.4.3. Let $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$, and $(i,j,k), (i',j',k') \in (\overline{Q}_0)^M$. Then there is a path $v_1 \ldots v_n : (i,j,k) \to (i',j',k') \in k\overline{Q}$ such that $t(v_l) \in (\overline{Q}_0)^M$ for all $l = 1, \ldots, n$.

Proof. Consider the path

$$d = e_{(M,M,s-1-2M)}\alpha_1^M\alpha_2^M\alpha_0^M : (M, M, s-1-2M) \to (M, M, s-1-2M).$$

It is easy to check that the end point of all the arrows in d belong to $(\overline{Q}_0)^M$.

Furthermore, if (i,j,k) is an element of $(\overline{Q}_0)^M$, then it is the endpoint (resp. the starting point) of a subpath of d starting (resp. ending) at $(M,M,s-1-2M)$, that we will call $d_{(i,j,k)}$ (resp. $d^{(i,j,k)}$). For example, the path

$$d_{(i,j,k)} : (M,M,s-1-2M) \to (i,j,k)$$

is given by:

- if $i = M$, then $d_{(i,j,k)} = e_{(M,M,s-1-2M)}\alpha_1^{M-j}$;
- if $j = M$, then $d_{(i,j,k)} = e_{(M,M,s-1-2M)}\alpha_1^M\alpha_2^{M-i}$;
- if $k = M$, then $d_{(i,j,k)} = e_{(M,M,s-1-2M)}\alpha_1^M\alpha_2^M\alpha_0^{M-k}$.

Hence, if (i',j',k') is another element of $(\overline{Q}_0)^M$, then a path $(i,j,k) \to (i',j',k')$ as in the statement is given by:

$$v_1 \dots v_n = d^{(i,j,k)}d_{(i',j',k')}.$$

□

This lemma, together with Proposition 6.3.4.2, allows us to give a complete description of $\operatorname{im}(\mu_5')$. For the moment we consider the map

$$\mu_5'[-(s+2)] : (V \otimes \mathcal{N})^S \to \mathcal{N}^S,$$

and then we will shift by $s+2$ to get the description of the distinct equivalence classes of $HH_4(\Pi)$.

Theorem 6.3.4.4. We have:

$$\operatorname{im}(\mu_5') \cong P_1 \oplus P_2 \oplus P_3 \oplus P_4,$$

where the spaces P_i, $i = 1,2,3,4$ are described as follows.

1. P_1 is spanned by the elements:

$$c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m - c_{(i',j',k')}(\alpha_0\alpha_1\alpha_2)^m,$$

   with $(i,j,k),(i',j',k') \in (\overline{Q}_0)^M$, with $M = \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor$ and $m = 0,\dots,s-1-2M$.

2. P_2 is spanned by the elements:

$$c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m - c_{(i',j',k')}(\alpha_0\alpha_1\alpha_2)^m,$$

   with $(i,j,k) \in (\overline{Q}_0)^M, (i',j',k') \in (\overline{Q}_0)^{M+1}$, with $M = \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor - 1$ and $m = 1,\dots,s-1-2M$.

3. P_3 is spanned by the elements:

$$c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^{s-2M-1},$$

with $(i,j,k) \in (\overline{Q}_0)^M$ with $M = \lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor - 1$.

4. $P_4 = 0$ if s is odd. If s is even, P_4 is spanned by the elements:

$$c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2),$$

with $(i,j,k) \in (\overline{Q}_0)^{\frac{s-2}{2}}$.

Proof. First of all, we show that the sets P_1, P_2, P_3, P_4 all belong to im μ'_5.

1. Fix $(i,j,k), (i',j',k') \in (\overline{Q}_0)^M$ for some $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor\}$. By Lemma 6.3.4.3 we can find a path $v_1 \ldots v_n : (i,j,k) \to (i',j',k')$ such that $t(v_l) \in (\overline{Q}_0)^M$ for all $l = 1, \ldots, n$. Hence, by Proposition 6.3.4.2(1) we have the following:

$$\begin{aligned} &c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m - c_{(i',j',k')}(\alpha_0\alpha_1\alpha_2)^m \\ &= \sum_{l=1}^{n} \left(c_{t(e_{(i,j,k)}v_1\ldots v_l)}(\alpha_0\alpha_1\alpha_2)^m - c_{s(e_{(i,j,k)}v_1\ldots v_l)}(\alpha_0\alpha_1\alpha_2)^m \right) \in \mathrm{im}(\mu'_5) \end{aligned}$$

for all $m = 0, \ldots, s-1-2M$.

2. Fix $(i,j,k) \in (\overline{Q}_0)^M$, $(i',j',k') \in (\overline{Q}_0)^{M+1}$ for some $M \in \{\lceil \frac{s-1}{3} \rceil, \ldots, \lfloor \frac{s-1}{2} \rfloor - 1\}$. Let $(\tilde{i},\tilde{j},\tilde{k}) \in (\overline{Q}_0)^{M+1}$ be such that there is an arrow $v : (i,j,k) \to (\tilde{i},\tilde{j},\tilde{k})$. Then, thanks to the previous point and Proposition 6.3.4.2(2),(3), we have:

$$\begin{aligned} &c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m - c_{(i',j',k')}(\alpha_0\alpha_1\alpha_2)^m = (c_{(i,j,k)}(\alpha_0\alpha_1\alpha_2)^m - c_{(\tilde{i},\tilde{j},\tilde{k})}(\alpha_0\alpha_1\alpha_2)^m) \\ &- (c_{(\tilde{i},\tilde{j},\tilde{k})}(\alpha_0\alpha_1\alpha_2)^m - c_{(i',j',k')}(\alpha_0\alpha_1\alpha_2)^m) \in \mathrm{im}(\mu'_5) \end{aligned}$$

for all $m = 1, \ldots, s-1-2M$.

3. This follows directly from Proposition 6.3.4.2(2),(3).

4. This follows directly from Proposition 6.3.4.2(4),(5).

Proposition 6.3.4.2, together with Remark 6.3.4.1, implies that these are all the elements lying in im μ'_5. □

6.3.5 Computation of $HH_4(\Pi)$

We are almost ready to give a description of

$$HH_4(\Pi) \cong \frac{\ker \mu'_5}{\mathrm{im}\ \mu'_4}.$$

We introduce a diagram that contains all the information from Theorem 6.3.4.4.

Definition 6.3.5.1. Let $s \geq 2$. Define the diagram T_s in the following way.

- The rows of the diagram are labeled by the numbers $M = \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor$ from bottom to top. If s is even, add also the row $\lfloor \frac{s-1}{2} \rfloor + 1 = \frac{s}{2}$;
- The row labeled by M contains the numbers $3M-(s-1), 3(M+1)-(s-1), \dots, 3(M+(s-1-2M))-(s-1) = 2(s-1)-3M$ from left to right;
- Link the number ℓ in row M with the number ℓ in row $M+1$ for $M = \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor - 1$;
- Link the number $2(s-1)-3M$ in the row M with a blank space in the row $M+1$, for each M.

The next proposition gives an interpretation of the diagram T_s in terms of equivalence classes in $HH_4(\Pi)[-(s+2)]$.

Proposition 6.3.5.2. Let $h = s+2$. The diagram T_s may be interpreted as follows:

- the number ℓ in the row M of T_s is the equivalence class in $HH_4(\Pi)[-h]$ of a path in $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ of length ℓ, where (i,j,k) is any vertex in $(\overline{Q}_0)^M$;
- if two numbers in T_s are linked, then the corresponding equivalence classes in $HH_4(\Pi)[-h]$ are equal;
- if a number in T_s is linked to a blank space, then the corresponding equivalence class in $HH_4(\Pi)[-h]$ is zero.

Proof. By Corollary 6.3.1.7, if $(i,j,k) \in (\overline{Q}_0)^M$ for some $M \in \lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor$, then $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ is generated by $s-2M$ paths, of degrees $3M-(s-1), 3(M+1)-(s-1), \dots, 3(M+(s-1-2M))-(s-1) = 2(s-1)-3M$. Also, point 1. in Theorem 6.3.4.4 tells us that, if $(i',j',k') \in (\overline{Q}_0)^M$, then the paths of $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ and $e_{(i',j',k')}\mathcal{N}e_{(i',j',k')}$ of length $3(M+m)-(s-1)$ are equal in $HH_4(\Pi)[-h]$, for all $m = 0, \dots, s-1-2M$.

Thus, we may interpret the number ℓ in the row M of T_s as the equivalence class in $HH_4(\Pi)[-h]$ of any path of length ℓ in $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$, $(i,j,k) \in (\overline{Q}_0)^M$.

The link between numbers ℓ in rows $M, M+1$ in T_s reflects point 2. of Theorem 6.3.4.4, that is, the fact that the equivalence classes in $HH_4(\Pi)[-h]$ of paths of length ℓ in $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ and in $e_{(i',j',k')}\mathcal{N}e_{(i',j',k')}$ are equal, for $(i,j,k) \in (\overline{Q}_0)^M$ and $(i',j',k') \in (\overline{Q}_0)^{M+1}$.

The link between numbers of the form $2(s-1)+3M$ in row M with blank spaces in T_s expresses points 3. and 4. of Theorem 6.3.4.4 that is, the fact that the equivalence classes of paths of length $2(s-1)-3M$ in $e_{(i,j,k)}\mathcal{N}e_{(i,j,k)}$ are 0 in $HH_4(\Pi)[-h]$, for all $(i,j,k) \in (\overline{Q}_0)^M$. □

Example 6.3.5.3. Consider Example 6.3.2.9. Then the diagram T_9 defined above is given by:

The diagram tells us that $HH_4(\Pi)[-h]$ is a two dimensional vector space, with generators in degrees 1 and 4. Therefore $HH_4(\Pi)$ is also two dimensional, with generators in degrees $h+1=12$ and $h+4=15$.

Proposition 6.3.5.2 allows us to give an explicit description of $HH_4(\Pi)$.

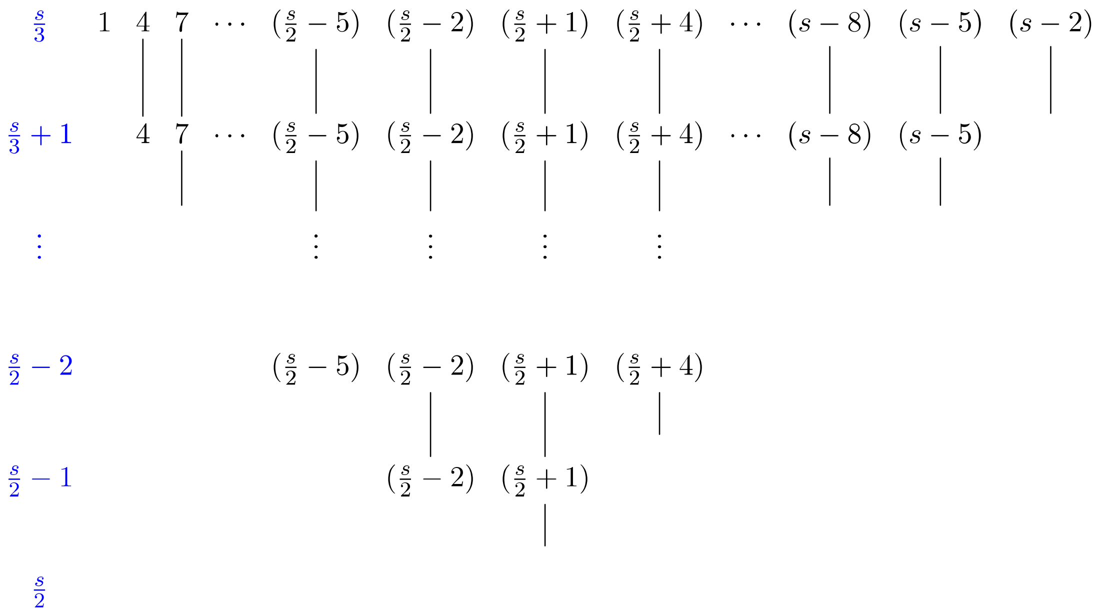
Corollary 6.3.5.4. Let $s \geq 2$. Then the Hilbert series of $HH_4(\Pi)$ is given by:

$$h_{HH_4(\Pi)}(t) = \sum_{k=\lfloor \frac{h}{3}+1 \rfloor}^{\lceil \frac{h}{2} \rceil - 1} t^{3k}.$$

Proof. By Theorem 6.3.3.1, we know that $\ker \mu_4'$ is generated by all nonzero paths of positive length in $\mathcal{N}^S$.

By Proposition 6.3.5.2, in order to compute $HH_4(\Pi)[-h]$ we can just look at the diagram T_s. Since the rows M of T_s run in $\lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor$ if s is odd (and in $\lceil \frac{s-1}{3} \rceil, \dots, \lfloor \frac{s-1}{2} \rfloor + 1$ if s is even), we need to consider 6 cases, according to the residue class of s modulo 6.

For example, for $s \equiv 0 \pmod 6$, the diagram T_s is the following:



The numbers in the right half of T_s, that is, the numbers $\frac{s}{2}+1, \frac{s}{2}+4, \dots, s-2$ are linked to a blank space, and hence the associated equivalence classes in $HH_4(\Pi)[-h]$ are zero. Furthermore, all the numbers on the left of the diagram T_s that lie in the same column are linked, meaning that they represent the same equivalence class in $HH_4(\Pi)[-h]$. Noticing that they can all be written in the form $1+3k$ for $k=0,\dots,\frac{s-6}{6}$, we get that:

$$h_{HH_4(\Pi)[-h]}(t)=\sum_{k=0}^{\frac{s-6}{6}} t^{3k+1}.$$

The same can be done for $s \not\equiv 0 \pmod 6$: one writes down the diagram T_s and looks at the numbers that are not linked and are not linked to a blank space. By Proposition 6.3.5.2 they give rise to all the distinct equivalence classes in $HH_4(\Pi)[-h]$. Doing this explicitly one gets that the Hilbert series of $HH_4(\Pi)[-h]$ is given as follows:

$$h_{HH_4(\Pi)[-h]}(t)=\begin{cases}\sum_{k=0}^{\frac{s-6}{6}} t^{3k+1}, & \text{if } s\equiv 0 \pmod 6\\ \sum_{k=0}^{\frac{s-7}{6}} t^{3k+3}, & \text{if } s\equiv 1 \pmod 6\\ \sum_{k=0}^{\frac{s-8}{6}} t^{3k+2}, & \text{if } s\equiv 2 \pmod 6\\ \sum_{k=0}^{\frac{s-3}{6}} t^{3k+1}, & \text{if } s\equiv 3 \pmod 6\\ \sum_{k=0}^{\frac{s-10}{6}} t^{3k+3}, & \text{if } s\equiv 4 \pmod 6\\ \sum_{k=0}^{\frac{s-5}{6}} t^{3k+2}, & \text{if } s\equiv 5 \pmod 6\end{cases} \tag{6.3.5.1}$$

Therefore the Hilbert series for $HH_4(\Pi)$ is obtained by multiplying (6.3.5.1) by t^h. This gives the statement. □

6.4 Computation of $HH^0(\Pi)$

In this section we compute the zeroth Hochschild cohomology group of Π. We use the description given in Proposition 2.2.1.10, that is:

$$HH^0(\Pi)\cong Z(\Pi)=\{a\in\Pi \mid ab=ba \text{ for all } b\in\Pi\}. \tag{6.4.0.1}$$

We compute an explicit basis for $Z(\Pi)$ using the results proved in subsection 6.1.1.

Recall that, for $\ell\in\{0,\dots,\lfloor\frac{s-1}{3}\rfloor\}$, we defined:

$$(\overline{Q}_0)_\ell=\{(i,j,k)\in\overline{Q}_0 \mid \min\{i,j,k\}=\ell\}.$$

For $(i,j,k)\in\overline{Q}_0$ we also defined:

$$T_{(i,j,k)}=e_{(i,j,k)}\alpha_0\alpha_1\alpha_2.$$

Proposition 6.4.0.1. The following decomposition holds:

$$Z(\Pi) = \bigoplus_{\ell=0}^{\lfloor \frac{s-1}{3} \rfloor} Z(\Pi)(3\ell). \tag{6.4.0.2}$$

Furthermore, for $\ell \in \{0, \ldots, \lfloor \frac{s-1}{3} \rfloor\}$, the space $Z(3\ell)$ is one dimensional, spanned by:

$$z_\ell = \sum_{x \in (\overline{Q}_0)_{\geq \ell}} T_x^\ell.$$

Proof. First of all, notice that $Z(\Pi) \subset \Pi^S$. Indeed, if $a \in Z(\Pi)$ is such that $a \notin \Pi^S$, we have:

$$e_{s(a)}a = a \neq 0 = ae_{s(a)},$$

so a is not central.

By Lemma 6.1.1.3 we have:

$$Z(\Pi) \subset \Pi^S = \bigoplus_{\ell=0}^{\lfloor \frac{s-1}{3} \rfloor} \Pi^S(3\ell).$$

Therefore $Z(\Pi)(k) = 0$ for all $k \not\equiv 0 \pmod 3$, and we get the decomposition (6.4.0.2).

We now want to prove that, for fixed $\ell \in \{0, \ldots, \frac{s-1}{3}\}$, then $Z(\Pi)(3\ell) \subset \langle z_\ell \rangle$. Using Lemma 6.1.1.3, we have that the following set is a basis of $\Pi^S(3\ell)$:

$$\{T_x^\ell | x \in (\overline{Q}_0)_{\geq \ell}\}.$$

Hence, a nonzero element $z \in Z(\Pi)(3\ell)$ can be written as:

$$z = \sum_{x \in (\overline{Q}_0)_{\geq \ell}} \lambda_x T_x^\ell \tag{6.4.0.3}$$

for some $\lambda_x \in k$ not all zero. Let $x \in (\overline{Q}_0)_{\geq \ell}$ be such that $\lambda_x \neq 0$ and suppose, by rescaling if necessary, that $\lambda_x = 1$.

Now, if $x' \in (\overline{Q}_0)_{\geq \ell}$ is such that there is an arrow $v : x \to x' \in \overline{Q}_1$ then, since $z \in Z(\Pi)$, we have $vz = zv$. Substituting this in the expression (6.4.0.3) for z, we get:

$$\lambda_{x'} v T_{x'}^\ell = \lambda_x T_x^\ell v = T_x^\ell v. \tag{6.4.0.4}$$

Notice that $vT_{x'}^\ell$ is a nonzero path. Indeed, if $x = (i, j, k)$ and, w.l.o.g. $v = e_x \alpha_0$, we have $x' = (i + 1, j, k - 1)$ and:

$$vT_{x'}^\ell = e_{(i,j,k)} \alpha_0^{\ell+1} \alpha_1^\ell \alpha_2^\ell.$$

We have:

- $i \geq \ell$;
- $j \geq \ell$;
- $k \geq \ell + 1$ since $x' = (i+1, j, k-1) \in (\overline{Q}_0)_{\geq \ell}$.

Therefore, $vT^{\ell}_{x'} \neq 0$ by Lemma 3.2.0.2. Hence, (6.4.0.4) implies that $\lambda_{x'} = \lambda_x = 1$.

Now, if $x' \in (\overline{Q}_0)_{\geq \ell}$ is arbitrary, consider a path $p = v_1 \dots v_n$ such that:

- $s(v_1) = x$;
- $t(v_i) \in (\overline{Q}_0)_{\geq \ell}$ for all $i = 1, \dots, n-1$;
- $t(v_n) = x'$.

Using what we just proved, we get:

$$1 = \lambda_x = \lambda_{t(v_1)} = \lambda_{t(v_2)} = \dots = \lambda_{t(v_n)} = \lambda_{x'}.$$

As a consequence, we have:

$$z = z_\ell = \sum_{x \in (\overline{Q}_0)_{\geq \ell}} T^{\ell}_{x},$$

and so $Z(\Pi)(3\ell) \subset \langle z_\ell \rangle$ is at most 1-dimensional.

Finally, we show that $z_\ell \in Z(\Pi)(3\ell)$. This will prove that $Z(\Pi)(3\ell) = \langle z_\ell \rangle$ is 1-dimensional. Let $c \in \Pi$ be an arbitrary nonzero path. Then:

$$z_\ell c = T^{\ell}_{s(c)} c, \qquad c z_\ell = c T^{\ell}_{t(c)}.$$

Using Lemma 3.2.0.1 to write the path c explicitly and Lemma 3.2.0.2, it is straightforward to show that $z_\ell c \neq 0$ if and only if $c z_\ell \neq 0$. Also, if they are both nonzero, they are equal since they have the same number of occurrences of each α_i, $i = 0, 1, 2$.. So we have:

$$z_\ell c = c z_\ell$$

for all $c \in \Pi$. Therefore $z_\ell \in Z(\Pi)$, and we get the statement. □

Corollary 6.4.0.2. The Hilbert series of $HH^0(\Pi)$ is given by:

$$h_{HH^0(\Pi)}(t) = \sum_{\ell=0}^{\lfloor \frac{h-3}{3} \rfloor} t^{3\ell}.$$

Proof. This follows directly from Proposition 6.4.0.1 and from the isomorphism (6.4.0.1) between $HH^0(\Pi)$ and $Z(\Pi)$. □

7

Hochschild homology, cohomology and cyclic homology of Π

Fix $s \geq 2$ and let $\Pi = \Pi^{(2,s)}$ be a 3-preprojective algebra of type A.

In this chapter we combine the results obtained in the last chapters to give a complete (conjectural) description of the Hochschild homology, cohomology and reduced cyclic homology groups of Π.

We start with the Hochschild homology and reduced cyclic homology.

Theorem 7.0.0.1. Let $s \geq 2$. If conjectures 5.1.0.2 and 5.3.0.2 hold true, then the Hochschild and reduced cyclic homology groups $\overline{HC}_i(\Pi)$ and $HH_i(\Pi)$ of $\Pi = \Pi^{(2,s)}$ are the following:

$$
\begin{aligned}
HH_0(\Pi) &\cong S & \overline{HC}_0(\Pi) &= 0\\
HH_1(\Pi) &= 0 & \overline{HC}_1(\Pi) &= 0\\
HH_2(\Pi) &\cong U & \overline{HC}_2(\Pi) &\cong U\\
HH_3(\Pi) &\cong U & \overline{HC}_3(\Pi) &= 0\\
HH_4(\Pi) &\cong W & \overline{HC}_4(\Pi) &\cong W\\
HH_5(\Pi) &\cong W & \overline{HC}_5(\Pi) &= 0\\
HH_6(\Pi) &\cong W^*[3h] & \overline{HC}_6(\Pi) &\cong W^*[3h]\\
HH_7[\Pi] &\cong W^*[3h] & \overline{HC}_7(\Pi) &= 0\\
HH_8(\Pi) &\cong U^*[3h] & \overline{HC}_8(\Pi) &\cong U^*[3h]\\
HH_9(\Pi) &\cong U^*[3h] & \overline{HC}_9(\Pi) &= 0\\
HH_{10}(\Pi) &= 0 & \overline{HC}_{10}(\Pi) &= 0\\
HH_{11}(\Pi) &\cong Z[3h] & \overline{HC}_{11}(\Pi) &\cong Z[3h]\\
HH_{12}(\Pi) &\cong Z^*[3h] & \overline{HC}_{12}(\Pi) &= 0\\
HH_{12+i}(\Pi) &\cong HH_i(\Pi)[3h], \quad i \geq 1 & \overline{HC}_{12+i}(\Pi) &\cong \overline{HC}_i(\Pi)[3h], \quad i \geq 1
\end{aligned}
$$

where $S = \Pi(0)$,

$$h_U(t) = \sum_{k=1}^{\lfloor \frac{h-1}{3} \rfloor} t^{3k},$$

$$h_W(t) = \sum_{k=\lfloor \frac{h}{2} \rfloor+1}^{\lceil \frac{2}{3}h \rceil -1} t^{3k}$$

and the Hilbert series of Z is a constant, with formula given by:

$$h_Z(t) = \begin{cases} \frac{(h-2)(h-4)}{6}, & \text{if } s \equiv 0,2 \bmod 6 \\ \frac{(h-2)(h-4)+1}{6}, & \text{if } s \equiv 1 \bmod 6 \\ \frac{(h-2)(h-4)-3}{6}, & \text{if } s \equiv 3,5 \bmod 6 \\ \frac{(h-2)(h-4)+4}{6}, & \text{if } s \equiv 4 \bmod 6. \end{cases}$$

Proof. The spaces C, X_1, X_2, $K_1[h]$, X_3, X_4 and $K_2[3h]$ in diagram 4.3.0.2, that allow to compute the Hochschild and reduced cyclic homology of Π, are given in terms of $\overline{HH}_0(\Pi)$, $HH_1(\Pi)$, $HH_4(\Pi)$ and $\chi_{\overline{HC}_*(\Pi)}(t)$ in Corollary 4.3.0.7.

By Theorem 5.4.0.2 we have that:

$$\chi_{\overline{HC}(\Pi)}(t) = \begin{cases} \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)}{6} t^{3h}), & \text{if } s \equiv 0,2 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+1}{6} t^{3h}), & \text{if } s \equiv 1 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)-3}{6} t^{3h}), & \text{if } s \equiv 3,5 \bmod 6 \\ \frac{1}{(1-t^{3h})}(\sum_{k\in I} t^{3k} - \frac{(h-2)(h-4)+4}{6} t^{3h}), & \text{if } s \equiv 4 \bmod 6, \end{cases}$$

where:

$$I = \{1,\dots,h-1\} \setminus \left(\{1,\dots,h-1\} \cap \left\{\frac{h}{2}, \frac{h}{3}, \frac{2h}{3}\right\}\right).$$

Furthermore, by Corollary 6.1.7.7 and Theorem 6.2.0.1 we have:

$$\overline{HH}_0(\Pi) = 0 = HH_1(\Pi).$$

Finally, Corollary 6.3.5.4 gives:

$$h_{HH_4(\Pi)}(t) = \sum_{k=\lfloor \frac{h}{3}+1 \rfloor}^{\lceil \frac{h}{2} \rceil -1} t^{3k}.$$

If we substitute these into the expressions for the Hilbert series of C, X_1, X_2, $K_1[h]$, X_3, X_4 and $K_2[3h]$ given in Corollary 4.3.0.7 we get the statement. □

Using Propositions 4.2.0.1 and 4.2.0.3, together with the description of $HH^0(\Pi)$ given in Corollary 6.4.0.2, we can get a description of the Hochschild cohomology groups of Π.

Theorem 7.0.0.2. Let $s \geq 2$. If conjectures 5.1.0.2 and 5.3.0.2 hold true, then the Hochschild cohomology groups $HH^i(\Pi)$ of $\Pi = \Pi^{(s)}$ are the following:

$$\begin{aligned}
HH^0(\Pi) &\cong L \\
HH^1(\Pi) &\cong U[-3] \\
HH^2(\Pi) &= 0 \\
HH^3(\Pi) &\cong Z^*[-3] \\
HH^4(\Pi) &\cong Z[-3] \\
HH^5(\Pi) &= 0 \\
HH^6(\Pi) &\cong U^*[-3] \\
HH^7(\Pi) &\cong U^*[-3] \\
HH^8(\Pi) &\cong W^*[-3] \\
HH^9(\Pi) &\cong W^*[-3] \\
HH^{10}(\Pi) &\cong W[-3h-3] \\
HH^{11}(\Pi) &\cong W[-3h-3] \\
HH^{12}(\Pi) &\cong U[-3h-3] \\
HH^{12+i}(\Pi) &\cong HH^i(\Pi)[-3h], \qquad i \geq 1
\end{aligned}$$

where the spaces U, W, Z are defined in Theorem 7.0.0.1 and the Hilbert series of L is given by:

$$h_L(t) = \sum_{k=0}^{\lfloor \frac{h-3}{3} \rfloor} t^{3k}.$$

8

Future work

There is one natural problem arising from this thesis.

Problem 1. Compute the (graded) Hochschild (co)homology and cyclic homology of $(d+1)$-preprojective algebras $\Pi = \Pi^{(d,s)}$ of type A for arbitrary $d \geq 2$.

The overall strategy described in this thesis should generalise in a straightforward way to arbitrary d. Indeed, the main tools we used to get the Hochschild (co)homology and cyclic homology groups are the following:

(i) a projective Π-bimodule resolution of Π;

(ii) a formula that gives $\chi_{\overline{HC}_*(\Pi)}(t)$, the Euler characteristic of the reduced cyclic homology of Π.

As for (i), by Theorem 3.1.0.7(2) we know that the algebra Π is $(s-1, d+1)$-Koszul. Therefore, the first $d+2$ terms in the Koszul bimodule complex are the start of a minimal projective Π-bimodule resolution of Π. Also, it should be possible to use an argument analogue to the one in the proof of Proposition 4.1.0.2 to get that the $(d+2)$-th syzygy $\Omega^{d+2}_{\Pi^e}(\Pi)$ is given by:

$$\Omega^{d+2}_{\Pi^e}(\Pi) \cong {}_1\Pi_{\eta^{-1}},$$

where η is the Nakayama automorphism of Π. Since η has order $d+1$ by Theorem 3.2.0.8, we would get that the algebra Π is periodic of period $(d+1)(d+2)$.

As for (ii), we are confident that the following formula (analogue to the one from Conjecture 5.1.0.2) holds:

$$\prod_{k=1}^{\infty} (1-t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_\Pi(t^s). \tag{8.0.0.1}$$

Also, Proposition 2.3.3.11 allows us to factor $H_\Pi(t)$ as:

$$H_\Pi(t) = (1 + (-1)^{d+1} P t^{s+d})(1 + \sum_{i=1}^{d+1} (-1)^i H_{K_i}(t))^{-1},$$

where P is the permutation matrix associated to η^{-1} and the K_i's are the $\Pi(0)$-bimodules appearing in the Koszul bicomplex.

However, there are two main issues:

(a) the explicit computation of $\det(1 + \sum_{i=1}^{d+1} (-1)^i H_{K_i}(t))$;

(b) the amount of Hochschild homology groups one has to explicitly compute.

At the moment (a) constitutes a major computational issue: as already underlined in this thesis we have no convenient way of computing this determinant also for $d = 2$. Therefore, considering bigger values of d will just make computations more complicated.

Point (b) is also a major computational obstacle. Indeed:

- For $d = 1$ the computation of $\chi_{\overline{HC}_*(\Pi)}(t)$ was enough to get all the homology groups required: indeed, in this case, different reduced cyclic homology groups live in different weight degrees.
- For $d = 2$ we saw in this thesis that the computation of $\chi_{\overline{HC}_*(\Pi)}(t)$ on its own is not enough in order to get all the required homology groups: we also need to compute explicitly $HH_1(\Pi)$ and $HH_4(\Pi)$ (together with $HH_0(\Pi)$ and $HH^0(\Pi)$).

Therefore, as d becomes bigger, along with $\chi_{\overline{HC}_*(\Pi)}(t)$, we will need to compute more and more Hochschild homology groups explicitly, and a more general strategy than the one used in this thesis is required.

To sum up, we think that at the actual state the computation of the Hochschild (co)homology and cyclic homology groups of all higher preprojective algebras of type A is very ambitious, and a few computational obstacles, together with the proof of Conjecture 5.1.0.2, need to be overcome. Nonetheless, the overall strategy to tackle the problem should still work, and if the aforementioned issues are overcome, we would be able to get the homology groups.

Another problem that could be interesting to investigate is the following.

Problem 2. Compute the Batalin-Vilkovisky structure of the Hochschild cohomology of Π.

Before explaining what the strategy to tackle this problem might be, we recall the appropriate definitions. We start by defining Gerstenhaber algebras.

Definition 3. A **Gerstenhaber algebra** $(\mathcal{V}^\bullet, \wedge, [,])$ over a field k is a $\mathbb{Z}$-graded supercommutative algebra $(\mathcal{V}, \wedge)$, together with a bracket $[,] : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ of degree -1, such that the induced bracket of degree zero on the shifted graded k-module $\mathcal{V}^{\bullet+1}$ is a Lie superbracket, satisfying the Leibniz identity:

$$[a \wedge b, c] = a \wedge [b, c] + (-1)^{mn} b \wedge [a, c],$$

for all $a \in \mathcal{V}^m$, $b \in \mathcal{V}^n$, $c \in \mathcal{V}$.

Definition 4. A **Batalin-Vilkovisky (BV) algebra** $(\mathcal{V}^\bullet, \wedge, \Delta)$ is a $\mathbb{Z}$-graded supercommutative algebra $(\mathcal{V}^\bullet, \wedge)$ equipped with an operator $\Delta : \mathcal{V} \to \mathcal{V}$ of degree -1 such that $\Delta^2 = 0$, and such that the bracket $[,]$ defined by:

$$(-1)^{m+1}[a, b] = \Delta(a\wedge b) - \Delta(a)\wedge b - (-1)^m a\wedge\Delta(b) + (-1)^{m+n} a\wedge b\wedge\Delta(1), a \in \mathcal{V}^m, b \in \mathcal{V}$$

endows $(\mathcal{V}^\bullet, \wedge, [,])$ with a Gerstenhaber algebra structure.

By Remark 4.1.0.4, the 3-preprojective algebras Π we studied in this thesis are (graded) periodic Calabi-Yau Frobenius. Therefore, [25, Thm. 2.3.64] implies that $(HH^*(\Pi), \cup, \Delta)$ is a BV algebra, where $\cup$ is the cup product, and Δ is obtained by duality from the Connes' differential. Explicitly, the operator Δ can be obtained by means of the following commutative diagram:

$$\begin{array}{ccc} HH_\bullet(\Pi) & \xrightarrow{B} & HH_{\bullet+1} \\ \mathbb{D}\big\downarrow\sim & & \sim\big\downarrow\mathbb{D} \\ HH^{12m+3-\bullet}(\Pi)[3mh+3] & \xrightarrow{\Delta} & HH^{12m+2-\bullet}(\Pi)[3mh+3] \end{array} \qquad (1)$$

where B is the Connes' differential, and $\mathbb{D}$ is the isomorphism between Hochschild homology and cohomology of Π that can be deduced from Propositions 4.2.0.3 and 4.2.0.1.

Therefore, in order to compute the BV structure on $HH^\bullet(\Pi)$, one just needs to calculate the cup product and the Connes' differential.

The BV structure for preprojective algebras A of non-Dynkin type was computed in [13], and for preprojective algebras of Dynkin type in [23].

In the latter paper the Connes' differential is computed in Proposition 6.1.3 by means of the Lie derivative $\mathcal{L}_{\theta_0}$ and the contraction map ι_{θ_0}, where $\theta_0 \in HH^1(A)$ is an element in weight degree 0. A formula for the Lie derivative $\mathcal{L}_{\theta_0}$ is deduced explicitly in Lemma 6.1.1, while the contraction map ι can be deduced from the cup product, since the following identity holds:

$$\mathbb{D}(\iota_\eta c) = \eta \cup \mathbb{D}(c), \qquad c \in HH_\bullet(A), \eta \in HH^\bullet(A), \qquad (2)$$

where $\mathbb{D}$ is the duality between Hochschild homology and cohomology given in [23, Sect. 6].

It should be possible to use the same strategy to compute the Connes' differential also for $HH_\bullet(\Pi)$. Indeed, equation (2) holds for arbitrary

Calabi-Yau Frobenius algebras (and so for Π) by [25, Thm. 2.3.27], and we are confident that the Lie derivative $\mathcal{L}_{\theta_0}$ can be deduced just as in [23]. This would give the Connes' differential on $HH_\bullet(\Pi)$ in terms of the cup product. Therefore, using diagram (1), this would in turn give the operator Δ on $HH^\bullet(\Pi)$ in terms of the cup product.

The main issue to use a strategy similar to [23] also for the cup product is the fact that it relies on the two computationally quite dense papers [20] and [24], where the cup product is computed for type A, and type D, E, respectively. In [25, Section 3.1.1] the authors use the fact that classical preprojective algebras of finite representation type are periodic Calabi-Yau Frobenius, with dimension 2 of shift 2, to show that the number of computations needed can actually be reduced.

Now, 3-preprojective algebras Π of type A are also periodic Calabi-Yau Frobenius, of dimension 3 and shift 3, since the isomorphism:

$$\Pi^\vee[3] \cong \Omega^4_{\Pi^e}(\Pi)$$

holds. In particular, by [25, Thm. 2.3.37] the stable Hochschild cohomology $\underline{HH}^\bullet(\Pi)$ is a graded Frobenius algebra, and one has that the following pairing is perfect:

$$\underline{HH}^i(\Pi) \otimes \underline{HH}^{7-i}(\Pi)[6] \xrightarrow{\cup} \underline{HH}^7(\Pi)[6] \xrightarrow{(-,id)} k.$$

A similar argument to [25, Sect. 3.1.1] implies that we need to consider unordered triples of Hochschild cohomology degrees $(|f|, |g|, |h|)$ (where $0 \leq |f|, |g|, |h| \leq 11$) such that $|f| + |g| + |h| = 7 \pmod{12}$. The graded commutativity of the cup product

$$(f \cup g, h) = (-1)^{|g||h|}(f \cup h, g) = (-1)^{|f|(|g|+|h|)}(g \cup h, f)$$

and the fact that the pairing $(,)$ is perfect implies that the cup product in any fixed two Hochschild cohomology degrees of a triple $(|f|, |g|, |h|)$ as above determines the other two pairs of cup products.

Such triples are the following:

$(0,0,7), (0,1,6), (0,2,5), (0,3,4), (0,8,11), (0,9,10), (1,1,5), (1,2,4), (1,3,3),$
$(1,7,11), (1,8,10), (1,9,9), (2,2,3), (2,6,11), (2,7,10), (2,8,9), (3,5,11), (3,6,10),$
$(3,7,9), (3,8,8), (4,4,11), (4,5,10), (4,6,9), (4,7,8), (5,5,9), (5,6,8), (5,7,7),$
$(6,6,7), (9,11,11), (10,10,11).$

Similarly to what the authors do in [25, Section 3.1.1] for preprojective algebras of type A, D, E, one can then use the explicit description of the Hochschild cohomology groups given in Theorem 7.0.0.2 to argue that some of the cup products between some of the Hochschild cohomology degrees written above need to be zero. This observation reduces further the amount of explicit computations needed to get a complete description of the cup product.

To sum up, we are confident that the above strategy could lead to a reasonable (computationally speaking) way to calculate the BV structure on $HH^*(\Pi)$.

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