The Universality Problem

A thesis submitted to the School of Mathematics at the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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June/2022

Abstract

The theme of this thesis is to explore the universality problem in set theory in connection to model theory, to present some methods for finding universality results, to analyse how these methods were applied, to mention some results and to emphasise some philosophical interrogations that these aspects entail.

A fundamental aspect of the universality problem is to find what determines the existence of universal objects. That means that we have to take into consideration and examine the methods that we use in proving their existence or nonexistence, the role of cardinal arithmetic, combinatorics etc. The proof methods used in the mathematical part will be mostly set-theoretic, but some methods from model theory and category theory will also be present.

A graph might be the simplest, but it is also one of the most useful notions in mathematics. We show that there is a faithful functor $\mathscr{F}$ from the category $\mathcal{L}$ of linear orders to the category $\mathcal{G}$ of graphs that preserves model theoretic-related universality results (classes of objects having universal models in exactly the same cardinals, and also having the same universality spectrum).

Trees constitute combinatorial objects and have a central role in set theory. The universality of trees is connected to the universality of linear orders, but it also seems to present more challenges, which we survey and present some results. We show that there is no embedding between an $\aleph_2$-Souslin tree and a non-special wide $\aleph_2$ tree T with no cofinal branches. Furthermore, using the notion of *ascent*

path, we prove that the class of non-special $\aleph_2$-Souslin tree with an ω-ascent path a has maximal complexity number, $2^{\aleph_2} = \aleph_3$.

Within the general framework of the universality problem in set theory and model theory, while emphasising their approaches and their connections with regard to this topic, we examine the possibility of drawing some philosophical conclusions connected to, among others, the notions of mathematical knowledge, mathematical object and proof.

Contents

Acknowledgements

There are many people that I want to thank and whose support has been essential in my being able to delve in this fascinating area of study, and in the conception and writing of this thesis.

I would like to express my deepest appreciation and sincere gratitude to my supervisor Prof. Mirna Džamonja for her consistent support and guidance throughout this research.

I would also like to thank prof. Pierre Wagner for his helpful advice and support throughout the project.

My sincere thanks go out also to Prof. Emilian Parau for his support and encouragement.

During these almost four years of cotutelle doctoral studies, the *School of Mathematics* from the *University of East Anglia* and the *Institut d'histoire et de philosophie des sciences et des techniques d'Université Paris 1 Panthéon-Sorbonne* offered me the opportunity to be part of a friendly research environment and benefit from many seminars and conferences.

Finally, I am grateful for the support and comprehension of extraordinary and dear friends and family who, fortunately, are too many to be mentioned in this restricted space.

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Introduction and Summary

The universality problem in set theory and model theory addresses the question of the existence of a universal element or a family of elements in certain classes of objects, such as linear orders, graphs or trees. To a certain extant, it represents a way of approaching the question of 'representatives' of these classes of mathematical objects.

This is of special interest in these areas of mathematics since both the existence or non-existence of such universal elements have significant mathematical consequences. Establishing such existential statements meets certain (mathematical) challenges, to be detailed throughout the text, so it happens that in certain case, we can hope to find universal elements (families) only for restricted classes of objects.

Historically, the configuration of the universality problem is connected to the development of the notion of a universal domain in the twentieth century, in such diverse fields as linear orders (Hausdorff in [82]), topology (Urysohn in [220]), algebraic geometry (Weil in [225]) and logic (Fraïssé in [64], Jónsson in [96] and [98], and, therefore, with the more general notion of extension. As we will show throughout the text, there is no unique methodology in addressing this issue, but its analysis invites further interrogations in both mathematics and philosophy.

It is not the purpose of this text to settle the nature of knowledge, not even of the presumably more restrictive term of mathematical knowledge. It is neither the purpose of this text to say what is the nature of philosophy, mathematics or the philosophy of mathematics. But no discussion can take place in a conceptual void, so I will start by making some distinctions that are necessary for an analysis of the universality problem in mathematics, specifically set theory and model theory. I have taken into account these two areas of mathematics because of certain similarities and common context in approaching this topic. There are some consequences following from these choices. Firstly, it uses their methods. Secondly, and after its description and analysis, it will hopefully become more clear how the universality problem is simultaneously determining the framework of the analysis.

The creation and development of set theory, starting in the last quarter of the 19th century, determined a precise configuration of the notion of structure and classes of structures (to the extent that Bourbaki got to proclaim that "[S]tructures are the weapons of the mathematician"). So it was only natural in this context to consider the concept of a universal structure for a class of structures and analysing its existence in different areas of mathematics.

Universality is inextricably linked to some mathematical notions, like that of a *model* or *structure*. As such, we will extend our analysis to them and to their meanings. It is connected with the idea of a universal domain in the theory of linear orders, model theory, topology or algebraic topology. It is also related to the part-whole relation, as the last is transposed into the context of set theory, within the mathematics of infinity. But the relation involved by universality is not determined in terms of size. Part - whole has new meanings here, so there is another consequence with regard to the way we understand mathematical objects.

The gradual configuration of the universality problem will underline some of its basic characteristics, first and foremost that of being a *mathematical object* of a specific kind, linked to the concept of method. Its nature expresses forms of generalisation and abstractisation, terms that will be examined in connection to notions such as re-conceptualisation and model extension.

But first, the universality problem is involved in the creation of mathematical knowledge, so I will start by delimiting a general conceptual framework to be used in establishing the nature of such knowledge. That survey will further lead us to epistemic distinctions related to the area of philosophy. The first chapter will describe the general framework of *mathematical knowledge*. This last notion will be determined in connection to the specific fields of set theory and model theory. That will require that we focus on specific aspects belonging to these theories and, in particular, the methods and types of poof employed in these contexts, with a focus on set theory, and their limits.

Universality problem is an open problem in both set theory and model theory. In model theory, classification theory in particular, it is used as a test problem. In set theory, is is fundamentally connected to combinatorics. And if we combine that with the fact that the concepts of set and the methods used in set theory are general and abstract enough that can be potentially applied to any field of mathematical inquiry, that the use of the axioms of set theory offers the possibility of showing the similarity, in the arguments, of various types of mathematical reasoning, this problem engenders new aspects to consider, including its role as a methodological tool.

There are philosophers of mathematics who minimise the value of mathematicians' judgements with regard to mathematical practice. I am aware that we could use Frege's distinction between judgement (Urteil) and (judgeable) content (beurteilbarer Inhalt) to leave them aside and avoid any form of *psychologism*[1]. But a focus on mathematical practice acknowledges the fact that mathematics is done by human beings, emphasising the necessity of taking into account the mathematicians' judgements, all the while being aware of the epistemological difficulties involved by such a positioning between subjectivity and objectivity. The objectivity of a mathematician's judgement related to their practice is sanctioned by a specific context. Consequently, I maintain that the informed opinion a mathematician has about their

[1] Content is true or false, leading to the logical notion of truth-value

mathematical practice constitutes a valuable aspect to consider when striving to achieve an objective conceptual analysis of what mathematics or mathematical practice represent. As such, I will take them into consideration whenever they are necessary and available.

This thesis is divided into four chapters (chapters number 3, 4, 5 and 6). The third and the sixth are mainly philosophical in substance, while the fourth and the fifth have a mathematical character. Such an operational division does not minimise the role of chapters 4 and 5 and their fundamental role for the comprehension of the thesis as a whole. The third chapter offers a general conceptual framework for the discussion, while the sixth one points to some philosophical distinctions determined by a specific mathematical practice and, in particular, the search for results regarding the universality problem.

Chapter 3 starts by fixing the idea of mathematical knowledge and making some conceptual distinctions, not with the purpose of offering a definitive answer, but rather to offer a general context in which mathematics can be seen as part of and in connection with general epistemic endeavours. I mention several ways in which mathematical knowledge could manifest itself. These forms are connected with the mathematical problem evaluated in this text: the universality problem.

Given that many interrogations concerning mathematics and mathematical knowledge are fundamentally philosophical, I was interested to see how exactly can philosophy be involved in analysing the constitution of mathematical knowledge through mathematical practice, and what role could it have for mathematics in general, and the universality problem in particular. So I took into consideration a general distinction among the following: the philosophical approach to mathematics; the philosophical beliefs of one mathematician or another; and mathematical practice. In order to emphasise some relevant aspects, I discussed several positions, coming both from mathematicians and philosophers. This last methodological point is one that I have used on several occasions throughout the text. I question some of the traditional approaches regarding the philosophy of mathematics and the roles of set theory in general, and I point several reasons which make set theory an interesting context for discussions, all the while mentioning various opinions in this regard. Its representational role in mathematics (different mathematical objects can be represented as sets) together with the methods it developed makes it particular interesting to interrogating the notion of *mathematical object*, with different topics in set theory offering further distinctions. And I will reprise this aspect in the sixth chapter.

Given that mathematical knowledge is not just juxtaposition of theorems, I took into account the role of proof and I analysed some approaches in this regard, including the limits of formal proof. Although proof does not constitute the only source of explanation (and therefore knowledge) in mathematics, it offers valuable insights when considered in a more global explanatory activity. For this reason, I also examined some aspects related to method, which were further developed in chapters 4 and 6. I maintain that there are different ways of approaching the notion of method and, in this chapter, I focus on discussing the axiomatisation method in the context of set theory and pointing to the fact that despite different limitations, it is epistemologically ampliative, particularly if we take into account the consistency results and independence results offered by forcing.

Through their objects (models, universes) and methods, set theory and model theory invite us to consider the idea of *extension*. Universality is also defined in terms of homogeneous saturation and, in model theory, saturation represents a generalisation of algebraic closure. I also mention some other examples, including the monster model or the concept of non-standard model, and I discuss some aspects related to asbtractisation in model theory. I analyse the notion of extension in relation to that of generalisation and idealisation in mathematics, and I also refer to extension in the sense of expansion, starting from Gödel's idea that no formal system can exhaustively describe the whole mathematical universe. Extension can also be considered as a process of reconceptualisation and, in this regard, I examine Manders' view.

I also analyse the idea of *universe* in set theory, a notion that involves certain distinctions, starting with the one between 'internal' and 'external' (or the metatheoretical universe of ZFC). It further involves one between semantic and syntactic. Universality will make its 'appearance' when trying to establish the order among the models of a class. I draw a connection between the role of different universes in set theory (and model theory) and some philosophical notions introduced by Plato. Finally, and in connection to the idea of extension as reconceptualisation and interpretability, I mention the possibility, proposed by some mathematicians, of regarding the universe as an infinite oriented graph (Krivine) or translating the Zermelo-Fraenkel axioms into graph-theoretic language (Nash-Williams). The extension of the set theoretic universe also involves the introduction of large cardinal axioms, used to measure the consistency strength of various set theoretic hypotheses and forming a linearly ordered hierarchy, with the inaccessible cardinals at the bottom. I examine some views regarding their role in set theory and the way they are used in the philosophy of mathematics (Steel, Feferman, Maddy, Shelah).

The fourth chapter contains an introduction to the problem of universality in set theory and in connection to model theory. We start by offering the general mathematical framework, in the form of basic definitions that will be further used throughout the text. Although universal objects can be found in several mathematical contexts (category theory, for example), we use set theory as a general framework because of its flexibility, axiomatic framework, and specific tools.

According to a general definition, a universal model of size κ of a theory T is a model in which any other model of T of cardinality κ embeds elementarily. We further extend and define the terms of the definition (to include the *universal family* and the *universality spectrum*), offer examples, and refer to different forms of universality results. For instance, a *weak* form of universality implies that there is a universal model of cardinality $\kappa > \lambda$, a model of a theory T, but not belonging to the collection of models of T of cardinality λ, into which every model of cardinality λ is elementary embedded. A proof for a first universality result goes back to Cantor: we say today that $\mathbb{Q}$ is the only countable linear order, up to isomorphism, which contains any other countable linear order, and every isomorphism between two finite subsets extends to an automorphism of $\mathbb{Q}$.

Universality (determining the universality spectrum for a theory) but also saturation (given that a universal model can also be defined model-theoretically as a saturated homogeneous model), and other mathematical phenomena are characterised by a rich interaction between set theory and model theory. Many approaches to universality (including non-universality theorems), which I mention, have taken into consideration the connection between various results in model theory and set theory. Consequently, the next section establishes the relation between set theory and model theory around the problem of universality. Universality in model theory is sensitive to results in set theory, specifically to extensions of its systems of axioms or combinatorial statements. Using model theoretic properties, the existence of a universal model is ensured in certain cases. In set theory, the results are determined and complicated by cardinal arithmetic and (infinite) combinatorics. I consider Baldwin's reference to a paradigmatic shift in two phases taking place in model theory (the first from a study of logics to the study of theories, the second brought forward by Shelah's introduction of classification theory), and I connect it to the universality problem, specifically in the third section of the chapter.

So we continue by offering an overview of the problem of universality in model theory, in particular classification theory, and ways to tackle it. The focus is on first order theories. We start by examining Fraïssé's and Jónsson's results, who approached the construction of universal domains from a purely semantic or algebraic point of view. Morley and Vaught determined a fundamental shift by replacing the *substructure* notion by *elementary submodel* and developing the central notion of *saturated model*; and then Shelah generalized even further. After introducing some definitions, we analyse in more detail some model-theoretic notions related to universality: structure, type, forking, definable sets, saturated models, homogeneity and stability. Saturation is a stronger property than universality. Universality represents an algebraic property describing a class of models, the embedding relation between models. Saturation refers to one model, describing the relation between M and the types over its subsets. Furthermore, when approaching saturation and universality, there is a difference between countable models and uncountable ones. Using the distinctions operated by the stability theory, theories have universal models in the cardinals where they are stable. But when a theory is unstable, one cannot say what happens when GCH does not hold. That being said, simple theories (unstable theories without the tree property, which is a weaker property than the strict order property) represent an interesting object for the study of universality, given that they are no more complicated than random graphs. We continue with an introduction to the use of classification theory in the approach to the universality problem. Very loosely speaking, classification theory involves organising classes of mathematical objects using a notion of equivalence by invariants. Classification theory generates general frameworks for comparing theories by providing dividing lines which depend on some test problems. Shelah first used the number of non-isomorphic models, and he is currently working on using the universality spectrum as a test problem. Baldwin emphasises the fact that Shelah's classification project also involves new problems and questions in the philosophy of mathematical practice.

The last section of this chapter represents an introduction to the universality problem in set theory and presents several methods used in this area to offers solutions. A theory can have universal models in some cardinals and not others, so the universality spectrum of a theory depends on cardinal arithmetic. Furthermore, there are differences between universes satisfying GCH and those without it. And in connection to the classification project, if some theories have universal models in the same cardinals, it may mean that they are somehow related. Although in some cases cardinal arithmetic can guarantee the existence of a universal model (CH, for instance, implies that every first order theory has a universal (saturated) model of cardinality $\aleph_1$), cardinal arithmetic is usually a fundamental aspect to consider. That being said, the first result which showed universal structures might exist despite the cardinal arithmetic was again a result by Shelah, who proved that it is consistent with $\neg$CH that there is a universal total order. Given the role of set theory in the configuration of the universality problem, we describe and analyse some of the instruments in dealing with the uncountable sets, concepts, techniques and methods that were used in set theory to find solutions to this problem. Some point to the existence of universal models in different contexts. Others show that a certain theory does not have a small universal family at a certain cardinal (club guessing methods). But they all refer to GCH-like assumptions: firstly, when we assume it; secondly, there are results where CH is violated, and then we analyse the possibility of the existence of universal models in forcing extensions. We discuss some notions and techniques related to the concept of limit and we start by acknowledging a distinction among different types of limit made by Tao. The ultraproduct construction offers the possibility of constructing new mathematical structures out of familiar ones. The Fraïssé limit represents a construction of a universal homogeneous countable relational structure from the class K of its finite substructures that satisfy certain properties. The Ehrenfeucht-Fraïssé games are used to gauge the similarities between structures and to prove when two models are equivalent.

The set theoretic combinatorics represents a complex and rich area of study and results, including extensions of Ramsey's theorem, especially partition calculus, transfinite trees and graphs, Martin's axiom, combinatorics of the continuum, singular cardinal combinatorics, PFA theory-related results, Todorčević's theory of minimal walks etc. The methods can be found in a large array of universality-related results. Generally speaking, infinitary combinatorics can be used to model and understand processes involving infinitely many steps (and their nature), while also being aware of the distinction among results connected to different types of cardinals (regular, singular and their successors). I continue by discussing guessing sequences, forcing and some techniques developed by Shelah in the PCF theory.

In the fifth chapter , we will define several terms, state some universality results that apply to all first order structures, give a context for the results of this thesis and give the said results. In the first section, we will give an account of the basic concepts and the definitions for the types of structures that will be used throughout the chapter. The notion of embedding captures the idea of relation between structures. The second chapter provides a general context for this notion together with definitions for the types of embeddings that will be used in this chapter for graphs and ordered structures. The third section contains some well-known universality results. There are more studies regarding the existence of universal models for first order theories. In the fourth section, we will focus on graphs. We start by establishing a context and mentioning some well-known results.

Furthermore, I will discuss the possibility of translating one type of structure into another (linear orders and graphs) in order to preserve the embedding related to the universality property. It also contains the following result, for whose proof we will use category theory notions (also presented).

Result. There is a faithful functor $\mathscr{F}$ from the category $\mathcal{L}$ of linear orders to the category $\mathcal{G}$ of graphs that preserves model theoretic-related universality results (classes of objects having universal models in exactly the same cardinals, and also having the same universality spectrum).

$\mathscr{F}$ can be considered as the functor from $\mathcal{L}$ to its essential image in $\mathcal{G}$, denoted by $\mathcal{E}$.

Section 5.5. is structured as a survey of universality results concerning trees.

Trees constitute combinatorial objects and have a central role in set theory. They can also be considered as a very natural generalisation of ordinals, given that they can be defined as partial orders on an ordinal. They express certain difficulties and complexities connected to set theory in general and the problem of universality in particular, and the solutions involving them (the existence or non-existence of a universal family of trees or a universal tree, in the context of this text) determined the creation and development of valuable proofs and techniques. The universality of trees is connected to the universality of linear orders, but it also seems to present more challenges. Furthermore, this problem is interesting in the case of those classes of trees that do not have an unbounded branch, given that an unbounded branch would automatically give a universal object. We structure this survey by taking into consideration several aspects that are involved in the search for and results concerning universal trees, all the while mentioning relevant known results and their proofs. We also offer some other proofs to already known results. We start by remarking the distinction between well-founded and non-well-founded when applied to trees, and then we describe the types of embeddings used when looking for this kind of universal objects. We continue by describing different types of trees (Aronszajn, Souslin, special, non-special, Canary), ways of constructing them, and existence results. We then show how different proofs and techniques are related to universality results, including the σ operation, forcing.

We give some proofs regarding the existence of some types of trees and their types of embedding ($\mathbb{Q}$-embeddable, $\mathbb{R}$-embeddable). We also prove some results connected to non-special trees: non-special Aronszajn trees, and the non-special trees constructed from a bi-stationary set. Put together, they show that there is no universal element in the class $\mathcal{T}(A)$ of bi-stationary trees, with A a bi-stationary subset of ω_1.

The σ operator offers the possibility of proving both existence and non-existence results. The tree $\sigma\mathbb{Q}$ is universal for all normal trees having strictly increasing embeddings into the reals. If we assume CH, the σ function forbids the existence of a universal family of size $\leq \omega_1$ for T_{ω_1}.

We then show that there is no embedding between an $\aleph_2$-Souslin tree and a non-special wide $\aleph_2$ tree T with no cofinal branches.

Using the notion of *ascent path*, we prove that the class of non-special $\aleph_2$-Souslin tree with an ω-ascent path has a maximal complexity number, $2^{\aleph_2} = \aleph_3$.

In the sixth chapter , we will show how the examination of various results regarding the universality problem in set theory and model theory offer new perspectives regarding method, mathematical object, the semantic/syntactic and the abstract/concrete distinctions.

Generally speaking, a mathematical theory can be described either internally or syntactically through axioms and theorems or externally or semantically, through its models. But results in model theory and classification theory or the method of forcing in set theory merge both a semantic and syntactic components. And it's the same case with the universality problem. Saturation refers to one model, whereas starting with the work of Fraíssé and then Jónsson in the 1950's, universal domains are constructed in a semantic way. In the mathematical practice there might be a preference for the semantic or the syntactic, but that is related to individual choices. That being said, combinatorics plays an important role in set theory and it represents a semantic approach. Logical equivalence is a syntactic notion, while equiconsistency (the use of large cardinals, so stronger axioms than ZFC) is a semantic one. The latter might be more informative from an epistemological point of view. But they all offer mathematical knowledge. It should be noted though that the notion of consistency was for Skolem not a syntactic but a semantic one, referring to the existence of a structure satisfying the axioms.

I continue by mentioning Cellucci's take on the mathematical object as hypothesis and present some further suggestions determined by the characteristics of set theory, model theory and results connected to the universality problem. Although not all mathematical object are sets, set theory offers the possibility of a homogeneous context to represent objects of the same type. As such, a mathematical object will be described as a set together with a certain structure on it. So I further maintain that the versatile nature of sets and the concepts and the methods used in their analysis can further offer the possibility of conceiving mathematical objects in general as heuristic devices. In the context of set theory and its methods of proof (particularly forcing), mathematical objects can also be analysed in relation to their *names*. But although it represents a philosophical question, the relation between names and objects has a specific meaning in set theory. And this last aspect questions again the configuration of a relation between mathematics and philosophy.

The analytic method in the philosophy of mathematics, which Cellucci is advancing, revolves around the idea of plausibility, not truth. Mathematical objects in such a view are considered to be hypotheses, which means that there will be no immediately justified premisses from which all knowledge is deduced. So there is no rejection of the infinite regress argument. I analyse this approach, while also including an account of Platos's view (from which it started). I think that the idea of mathematical object as hypothesis is an interesting one, but there are some limitations in Cellucci's view, first and foremost the fact that there are fundamental distinctions between mathematics and philosophy. Given that any *a priori* epistemological ideas represent a limitative approach in the process of understanding mathematical processes, I suggest we take into consideration different contexts, methods, and the mathematical practice in approaching a mathematical problem. The idea that I suggest is that the connection between mathematical objects and the world is not abstraction and idealisation, but interrogation and orientation.

I continue by making some remarks concerning the abstract-concrete distinction. I start by making a distinction between an external and an internal notion of abstractisation when dealing with mathematical objects. Such a distinction does not reflect the philosophical practice, but the mathematical one. I examine some views and give some examples connected to the universality problem.

I return to the idea of method, but in a way that could connect the meanings exposed in the previous chapters, encompassing the idea of an epistemic context for mathematical developments, a heuristic instrument, and as an object of study in itself. As such, I start by pointing the role of the infinite in establishing this general idea of method. First, it is not always possible to apply the same mathematical rules and intuition involved in the finite realm to the transfinite one. Secondly, the infinite as method is connected to set theory as mathematics of the infinite and its organising principles, expressed in the axioms. Regarding universality, it represents a test problem in model theory and classification theory and, as such, provides a context for establishing connections across different areas of mathematics, but also becoming a methodological instrument and a form of interrogation of method. Various results regarding universality show how it is inextricably connected to combinatorics in set theory. A common characteristic of the combinatorial principles involved is that they are independent of the usual system of axioms in set theory, and, as a result, they are particularly useful in proving non-existence. All in all, the analysis of universality determined the development of other central notions in model theory, like saturated models, homogeneous models and, eventually, non-forking.

Returning to the connection between mathematical knowledge or mathematics as a whole and philosophy, I entertain the possibility of taking them as different responses (maybe connected) to a more general human interrogation or interrogations. Consequently, I point to an idea of *order*. It is not a mathematical element, but it is an idea that mathematicians, as human beings, might take into consideration when referring to vague concepts such as 'intuition' or even the 'truth' expressed by a result. Some have pointed out that it constitutes a cultural phenomenon. Nonetheless, the idea of 'order' can be found 'behind' several mathematical endeavours. I discuss further aspects connected to the role of proof in mathematical knowledge, and in connection to that as well, I consider the idea of order as a unifying conceptual space.

2

Introduction et Sommaire

Le problème d'universalité dans la théorie des ensembles et la théorie des modèles aborde la question de l'existence d'un élément universel ou d'une famille d'éléments dans certaines classes d'objets, tels que les ordres linéaires, les graphes ou les arbres. Dans une certaine mesure, il représente une manière d'aborder la question des 'représentants' de ces classes d'objets mathématiques.

Ceci est d'un intérêt particulier dans ces domaines des mathématiques puisque l'existence ou la non-existence de tels éléments universels ont des conséquences mathématiques importantes. L'établissement de tels énoncés existentiels rencontre certains défis (mathématiques), à détailler tout au long du texte, si bien que dans certains cas, on ne peut espérer trouver des éléments ou familles universels que pour des classes restreintes d'objets.

Historiquement, la configuration du problème de l'universalité est liée au développement de la notion de domaine universel au vingtième siècle, dans des domaines aussi divers que les ordres linéaires (Hausdorff dans [82]), la topologie (Urysohn dans [220]), la géométrie algébrique (Weil dans [225]) et la logique (Fraïssé dans citeF54, Jónsson dans [96] et [98], et donc avec la notion plus générale d'extension. Comme nous le montrerons tout au long du texte, il n'y a pas de méthodologie unique pour aborder cette question, mais son analyse invite à de nouvelles interrogations en mathématiques et en philosophie.

Ce n'est pas le but de ce texte de fixer la nature de la connaissance, pas même des termes supposément plus restrictifs tel que savoir ou connaissance mathématique. Ce n'est pas non plus le but de ce texte de dire quelle est la nature de la philosophie, des mathématiques ou de la philosophie des mathématiques. Mais aucune discussion ne peut avoir lieu dans un vide conceptuel, donc je commencerai par faire quelques distinctions qui sont nécessaires pour une analyse du problème de l'universalité en mathématiques, en particulier dans la théorie des ensembles et la théorie des modèles. J'ai pris en compte ces deux domaines des mathématiques en raison de certaines similitudes et d'un contexte commun dans l'approche de ce sujet. Certaines conséquences découlent de ces choix. Tout d'abord, le problème de l'universalité utilise leurs méthodes. Dans un deuxième temps, et après sa description et son analyse, il deviendra, espérons-le, plus clair comment le problème de l'universalité détermine simultanément le cadre de l'analyse. La création et le développement de la théorie des ensembles, à partir du dernier quart du XIXe siècle, ont déterminé une configuration précise de la notion de structure et de classes de structures (au point que Bourbaki a proclamé que "les structures sont les armes du mathématicien"). Il était donc naturel dans ce contexte de considérer le concept de structure universelle pour une classe de structures et d'analyser son existence en différents domaines des mathématiques.

L'universalité est donc inextricablement liée à certaines notions mathématiques, comme celle de *modèle* ou de *structure*. À ce titre, nous allons étendre notre analyse pour les englober et dégager leur signification. Elle est liée à l'idée d'un domaine universel dans la théorie des ordres linéaires, la théorie des modèles, la topologie ou la topologie algébrique. Elle envoie aussi à la relation partie-tout, la façon dont ette dernière est transposé dans le cadre de la théorie des ensembles, au sein des mathématiques de l'infini. Mais le rapport qu'implique l'universalité n'est pas déterminé en termes de taille. La partie - le tout a ici de nouvelles significations, il y a donc une autre conséquence en ce qui concerne la façon dont nous comprenons les objets mathématiques.

La configuration progressive du problème d'universalité soulignera certaines de ses caractéristiques fondamentales, en premier lieu celle d'être un *objet mathématique* d'un genre spécifique, et lié aussi au concept de méthode. Sa nature exprime des formes de généralisation et d'abstraction, termes qui seront examinés en relation avec des notions telles que la re-conceptualisation et l'extension du modèle.

Mais d'abord, le problème de l'universalité est impliqué dans la création de la connaissance mathématique, donc je commencerai par délimiter un cadre conceptuel général à utiliser pour établir la nature de ce type de connaissance. Cette enquête nous conduira plus loin aux distinctions épistémiques liées au domaine de la philosophie. Le premier chapitre décrira le cadre général de *connaissance mathématique*. Cette dernière notion sera déterminée en lien avec les domaines spécifiques de la théorie des ensembles et la théorie des modèles. Cela nécessitera que nous nous concentrions sur des aspects spécifiques appartenant à ces théories et, en particulier, les méthodes et les types de démonstration employées dans ces contextes, avec un accent sur la théorie des ensembles, et leurs limites.

Le problème d'universalité est un problème ouvert à la fois dans la théorie des ensembles et la théorie des modèles. Dans la théorie des modèles, dans la théorie de la classification (classification theory) en particulier, il est utilisé comme problème test (test problem). Dans la théorie des ensembles, il est fondamentalement lié à la combinatoire. Et si nous combinons cela avec le fait que les concepts d'ensemble et les méthodes utilisées dans la théorie des ensembles sont suffisamment généraux et abstraits pour pouvoir être potentiellement appliqués à n'importe quel domaine de la recherche mathématique, que l'utilisation des axiomes de la théorie des ensembles offre la possibilité de montrer la similitude, dans les arguments, de divers types de raisonnement mathématique, ce problème engendre de nouveaux aspects à considérer, y compris son rôle d'outil méthodologique.

Il y a des philosophes des mathématiques qui minimisent la valeur des jugements des mathématiciens à l'égard de la pratique mathématique. Je suis consciente qu'on pourrait utiliser la distinction de Frege entre *jugement* (Urteil) et *contenu (jugeable)* (beurteilbarer Inhalt) pour les laisser de côté et éviter toute forme de *psychologisme*[1]. Mais si on prend la pratique mathématique comme centre d'attention, on reconnaît le fait que les mathématiques sont faites par des êtres humains, soulignant la nécessité de prendre en compte les jugements des mathématiciens, tout en étant conscient des difficultés épistémologiques qu'implique un tel positionnement entre subjectivité et objectivité. L'objectivité du jugement d'un mathématicien lié à sa pratique est sanctionnée par un contexte spécifique. Par conséquent, je maintiens que l'opinion informée qu'un mathématicien a sur sa pratique mathématique constitue un aspect précieux à considérer lorsqu'on s'efforce de parvenir à une analyse conceptuelle objective de ce que les mathématiques ou la pratique mathématique représentent. À ce titre, je les prendrai en considération chaque fois qu'elles seront nécessaires et disponibles.

Ce travail est divisé en quatre chapitres (chapitres numéros 3, 4, 5 et 6). Le troisième et le sixième sont principalement de nature philosophique, tandis que le quatrième et le cinquième ont un caractère mathématique. Un tel découpage opérationnel ne minimise pas le rôle des chapitres 4 et 5 et leur rôle fondamental pour la compréhension de la thèse dans son ensemble. Le troisième chapitre propose un cadre conceptuel général pour la discussion, tandis que le sixième indique certaines distinctions philosophiques déterminées par une pratique mathématique spécifique, en particulier la recherche de résultats concernant le problème de l'universalité.

Le troisième chapitre commence par fixer l'idée de connaissance mathématique et par faire quelques distinctions conceptuelles, non pas dans le but d'offrir une réponse définitive, mais plutôt de constituer un contexte général dans lequel les mathématiques peuvent être considérées comme faisant partie et en relation avec des démarches épistémiques générales. Je mentionne plusieurs

[1]Le contenu est vrai ou faux, conduisant à la notion logique de valeur de vérité

façons dont la connaissance mathématique pourrait se manifester. Ces formes sont liées au problème mathématique éxaminé dans ce texte : le problème de l'universalité.

Étant donné que de nombreuses interrogations concernant les mathématiques et le savoir mathématique sont fondamentalement philosophiques, j'étais intéressée de voir comment exactement la philosophie peut être impliquée dans l'analyse de la constitution du savoir mathématique à travers la pratique mathématique, et quel rôle pourrait-elle avoir pour les mathématiques en général, et le problème de l'universalité en particulier. J'ai donc pris en considération une distinction générale qu'on pourrait faire parmi les aspects suivants : une approche philosophique des mathématiques ; les croyances philosophiques d'un mathématicien ou d'un autre ; et la pratique mathématique. Afin de mettre l'accent sur certains aspects pertinents, j'ai discuté plusieurs positions, provenant à la fois de mathématiciens et de philosophes. Ce dernier point méthodologique est celui que j'ai utilisé à plusieurs reprises tout au long du texte.

Je remets en question certaines des approches traditionnelles concernant la philosophie des mathématiques et les rôles de la théorie des ensembles en général, et j'indique plusieurs raisons qui font de la théorie des ensembles un contexte intéressant pour les discussions, tout en mentionnant diverses opinions à cet égard. Son rôle représentationnel en mathématiques (différents objets mathématiques peuvent être représentés comme des ensembles) ainsi que les méthodes qu'il a développées le rendent particulièrement intéressant pour interroger la notion *d'objet mathématique*, avec différents sujets en théorie des ensembles offrant des distinctions supplémentaires. Et je reprendrai cet aspect dans le sixième chapitre.

Étant donné que les connaissances mathématiques ne sont pas seulement une juxtaposition de théorèmes, j'ai pris en compte le rôle de la preuve et j'analyse certaines approches à cet égard, notamment les limites de la preuve formelle.

Bien que la preuve ne constitue pas la seule source d'explication (et donc de connaissance) en mathématiques, elle offre des informations précieuses lorsqu'elle est considérée dans une activité explicative plus globale. Pour cette raison, j'ai également examiné certains aspects liés à la méthode, qui ont été détaillés dans les chapitres 4 et 6. Je soutiens qu'il y a différentes manières d'aborder la notion de méthode et, dans ce chapitre, je me concentre sur la discussion de la méthode d'axiomatisation dans le contexte de la théorie des ensembles et sur le fait que, malgré différentes limitations, elle est épistémologiquement ampliative, surtout si on tient compte des résultats de cohérence et les résultats d'indépendance offerts par la méthode de *forcing*.

Par leurs objets (modèles, univers) et leurs méthodes, la théorie des ensembles et la théorie des modèles nous invitent à considérer l'idée *d'extension*. L'universalité est également définie en termes de saturation homogène et, dans la théorie des modèles, la saturation représente une généralisation de la clôture algébrique. Je mentionne également quelques autres exemples, le *monster model*, par exemple, ou le concept de modèle non-standard, et j'aborde certains aspects liés à l'abstraction dans la théorie des modèles. J'analyse la notion d'extension par rapport à celle de généralisation et d'idéalisation en mathématiques, et je me réfère également à l'extension au sens d'expansion, en partant de l'idée de Gödel qu'aucun système formel ne peut décrire de manière exhaustive tout l'univers mathématique. L'extension peut aussi être considérée comme un processus de reconceptualisation et, à cet égard, j'examine le point de vue de Manders.

J'analyse également l'idée de *univers* dans la théorie des ensembles, une notion qui implique certaines distinctions, à commencer par celle entre 'interne' et 'externe' (ou l'univers métathéorique de ZFC). Une autre implique l'aspect sémantique et celui syntaxique. L'universalité fera son 'apparition' en essayant d'établir l'ordre entre les modèles d'une classe. J'établie une connexion entre le rôle des différents univers dans la théorie des ensembles (et la théorie des modèles) et certaines notions philosophiques introduites par Platon. Enfin, et en lien avec l'idée d'extension comme reconceptualisation et interprétabilité, je mentionne la possibilité, proposée par certains mathématiciens, de considérer l'univers comme un graphe orienté infini (Krivine), ou de traduire les axiomes de Zermelo-Fraenkel dans le langage de la théorie des graphes (Nash-Williams). L'extension de l'univers de la théorie des ensembles implique également l'introduction dex axiomes des grands cardinaux, utilisés pour mesurer la force de cohérence de diverses hypothèses de la théorie des ensembles, et tout en formant une hiérarchie ordonnée linéairement, avec les cardinaux inaccessibles en bas. J'examine quelques points de vue concernant leur rôle dans la théorie des ensembles et la manière dont ils sont utilisés dans la philosophie des mathématiques (Steel, Feferman, Maddy, Shelah).

Le quatrième chapitre contient une introduction au problème de l'universalité dans la théorie des ensembles et en relation avec la théorie des modèles. Nous commençons par proposer le cadre mathématique général, sous la forme de définitions de base qui seront ensuite utilisées tout au long du texte. Bien que les objets universels puissent être trouvés dans plusieurs contextes mathématiques (théorie des catégories, par exemple), nous utilisons la théorie des ensembles comme cadre général en raison de sa flexibilité, de son cadre axiomatique et de ses outils spécifiques.

Selon une définition générale, un modèle universel de cardinalité κ d'une théorie T est un modèle dans lequel tout autre modèle de T de cardinalité κ se plonge élémentairement. Nous étendons et développons les termes de la définition (pour inclure la famille universelle et le spectre d'universalité), proposons des exemples et nous nous référons à différentes formes de résultats d'universalité. Par exemple, une forme d'universalité *faible* implique qu'il existe un modèle universel de cardinalité $\kappa > \lambda$, un modèle d'une théorie T, mais n'appartenant pas à la classe des modèles de T de cardinalité λ, dans laquelle chaque modèle de cardinalité λ est élémentairement intégré. Une preuve d'un premier résultat d'universalité remonte à Cantor : on dit aujourd'hui que $\mathbb{Q}$ est le seul ordre linéaire dénombrable, à isomorphisme près, qui contient tout autre ordre linéaire dénombrable, et tout isomorphisme entre deux sous-ensembles finis se prolonge en un automorphisme de $\mathbb{Q}$.

L'universalité (détermination du spectre d'universalité d'une théorie) mais aussi la saturation (étant donné qu'un modèle universel peut aussi être défini modèle-théoriquement comme un modèle homogène saturé), et d'autres phénomènes mathématiques sont caractérisés par une interaction riche entre la théorie des ensembles et la théorie des modèles. De nombreuses approches de l'universalité (y compris les théorèmes de non-universalité), que je mentionne, ont pris en considération le lien entre divers résultats de la théorie des modèles et de la théorie des ensembles. Par conséquent, la section suivante vise à établir la relation entre la théorie des ensembles et la théorie des modèles autour du problème de l'universalité. L'universalité dans la théorie des modèles est sensible aux résultats de la théorie des ensembles, en particulier aux extensions de ses systèmes d'axiomes ou d'énoncés combinatoires. En utilisant des propriétés de la théorie des modèles, l'existence d'un modèle universel est assurée dans certains cas. Dans la théorie des ensembles, les résultats sont déterminés et compliqués par l'arithmétique cardinale et la combinatoire (infinie). Je considère la référence que Baldwin a fait à propos d'un changement de paradigme en deux phases ayant lieu dans la théorie des modèles (la première à partir d'une étude des logiques à l'étude des théories, la seconde apportée par l'introduction par Shelah de la théorie de la classification), et je la relie au problème de l'universalité, spécifiquement dans la troisième section du chapitre.

Nous continuons donc en offrant un aperçu du problème de l'universalité dans la théorie des modèles, en particulier la théorie de la classification, et les moyens de l'aborder. L'accent est mis sur les théories du premier ordre. Nous commençons par examiner les résultats de Fraïssé et Jónsson, qui ont abordé la construction de domaines universels d'un point de vue purement sémantique ou algébrique. Morley et Vaught ont déterminé un changement fondamental en remplaçant la notion de *sous-structure* par celui d'un *sous-modèle élémentaire* et en développant la notion centrale dans la théorie des modèles de *modèle saturé* ; et puis Shelah a généralisé encore plus loin. Après avoir introduit quelques définitions, nous analysons plus en détail certaines notions de la théorie des modèles liées à l'universalité : structure, type, *forking*, ensembles définissables, modèles saturés, homogénéité et stabilité. La saturation est une propriété plus forte que l'universalité. L'universalité (ou universalité homogène) représente une propriété algébrique décrivant une classe de modèles, la relation de plongement entre modèles. La saturation fait référence à un modèle, décrivant la relation entre M et les types sur ses sous-ensembles. De plus, quand on approche la saturation et l'universalité, on doit faire la différence entre les modèles dénombrables et les modèles indénombrables. En utilisant les distinctions opérées par la théorie de la stabilité, les théories ont des modèles universels dans les cardinaux où elles sont stables. Mais quand une théorie est instable, on ne peut pas dire ce qui se passe quand HGC ne tient pas. Cela dit, les théories simples (théories instables sans la propriété d'arbre, qui est une propriété plus faible que la propriété d'ordre strict) représentent un objet intéressant pour l'étude de l'universalité, étant donné qu'elles ne sont pas plus compliquées que les graphes aléatoires. Nous continuons avec une introduction à l'utilisation de la théorie de la classification dans l'approche du problème de l'universalité. Au sens large, la théorie de la classification consiste à organiser des classes d'objets mathématiques en utilisant une notion d'équivalence par invariants. La théorie de la classification génère des cadres généraux pour comparer les théories en fournissant des *lignes de division* (*dividing lines*) qui dépendent de certains *problèmes-test* (*test problems*). Shelah a d'abord utilisé le nombre de modèles non isomorphes, et il travaille actuellement sur l'utilisation du spectre d'universalité comme *problème-test*. Baldwin souligne le fait que le projet de classification de Shelah, à côté des résultats mathématiques, implique également de nouveaux problèmes et questions dans la philosophie de la pratique mathématique.

La dernière section de ce chapitre représente une introduction au problème d'universalité dans la théorie des ensembles et présente plusieurs méthodes utilisées dans ce domaine pour proposer des solutions. Une théorie peut avoir des modèles universels dans certains cardinaux et pas en d'autres, de sorte que le spectre d'universalité d'une théorie dépend de l'arithmétique cardinale. De plus, il existe des différences entre les univers satisfaisant HGC et ceux qui ne le satisfont pas. Et en ce qui concerne le projet de classification, si certaines théories ont des modèles universels dans les mêmes cardinaux, cela peut signifier qu'elles sont en quelque sorte connectées. Bien que dans certains cas l'arithmétique cardinale puisse garantir l'existence d'un modèle universel (HC, par exemple, implique que chaque théorie du premier ordre a un modèle universel (saturé) de cardinalité $\aleph_1$), l'arithmétique cardinale est généralement un aspect fondamental à considérer. Cela étant dit, le premier résultat qui a montré que des structures universelles pouvaient exister malgré l'arithmétique cardinale était à nouveau un résultat de Shelah, qui a prouvé qu'il est consistant avec $\neg$HC qu'il existe un ordre total universel. Étant donné le rôle de la théorie des ensembles dans la configuration du problème de l'universalité, nous décrivons et analysons certains de ses instruments pour traiter les ensembles non dénombrable, concepts, techniques et méthodes qui ont été utilisés dans la théorie des ensembles pour trouver des solutions à ce problème. Certains sont employés pour pouver l'existence de modèles universels dans différents contextes. D'autres montrent qu'une certaine théorie n'a pas de *famille universelle* restreinte à un certain cardinal (*club guessing guessing methods*). Mais ils se réfèrent tous à des hypothèses de type HGC : premièrement, lorsque nous l'assumons ; deuxièmement, il y a des résultats où HC est violée ; puis nous analysons la possibilité de l'existence de modèles universels dans les extensions de forcing. Nous abordons ensuite certaines notions et techniques liées au concept de limite et nous commençons par évoquer une distinction entre différents types de limite faite par Tao. La construction de l'ultraproduit offre la possibilité de construire de nouvelles structures mathématiques à partir de structures familières. La limite Fraïssé représente une construction d'une structure relationnelle dénombrable homogène universelle à partir de la classe K

de ses sous-structures finies qui satisfont certaines propriétés. Les jeux Ehrenfeucht-Fraïssé sont utilisés pour évaluer les similitudes entre les structures et pour prouver quand deux modèles sont équivalents.

La combinatoire de la théorie des ensembles représente un domaine d'étude et des résultats complexes et riches, comprenant des extensions du théorème de Ramsey, en particulier le calcul des partitions, les arbres et graphes transfinis, l'axiome de Martin, la combinatoire du continuum, la combinatoire des cardinaux singuliers, les résultats liés à la théorie PFA, la théorie de Todorčevič's de *minimal walks*, etc. Les méthodes peuvent être trouvées dans un large éventail de résultats liés à l'universalité. D'une manière générale, la combinatoire infinitaire peut être utilisée pour modéliser et comprendre des processus impliquant une infinité d'étapes, tout en tenant compte de la distinction entre les résultats liés à différents types de cardinaux (réguliers, singuliers et leurs successeurs. Je continue avec l'analyse en discutant *club guessing sequences*, la méthode de forcing et certaines techniques développées par Shelah dans la théorie PCF.

Dans le cinquième chapitre , nous définirons plusieurs termes, énoncerons quelques résultats d'universalité qui s'appliquent à toutes les structures du premier ordre, présentons un contexte aux résultats de cette thèse et donnerons lesdits résultats. Dans la première section, nous rendrons compte des concepts de base et des définitions des types de structures qui seront utilisées tout au long du chapitre. La notion du plongement capture l'idée de relation entre les structures. Le deuxième chapitre fournit un contexte général pour cette notion ainsi que des définitions pour les types de plongements qui seront utilisés dans ce chapitre pour les graphes et les structures ordonnées. La troisième section contient quelques résultats d'universalité bien connus. Il y a plus d'études concernant l'existence de modèles universels pour les théories du premier ordre. Dans la quatrième section, nous nous concentrerons sur les graphes. Nous commençons par établir un contexte et mentionner quelques résultats bien connus.

On continue par discuter la possibilité de traduire un type de structure dans un autre (ordres linéaires et graphes) afin de préserver le plongement (ou les plonngements) utilisé(s) pour obtenir des résultats d'universalité. Il contient également le résultat suivant, pour la preuve duquel nous utiliserons les notions de la théorie des catégories (également présentées).

Result. Il existe un foncteur fidèle $\mathscr{F}$ de la catégorie $\mathcal{L}$ des ordres linéaires à la catégorie $\mathcal{G}$ des graphes qui préserve les résultats d'universalité liés à la théorie des modèles (classes d'objets ayant modèles universels dans exactement les mêmes cardinaux, et ayant également le même spectre d'universalité).

$\mathscr{F}$ peut être considéré comme le foncteur de $\mathcal{L}$ vers son image essentielle dans $\mathcal{G}$, notée $\mathcal{E}$.

Section 5.5. est structurée comme une enquête sur des résultats d'universalité concernant les arbres. Les arbres constituent des objets combinatoires et ont un rôle central dans la théorie des ensembles. Ils peuvent également être considérés comme une généralisation très naturelle des ordinaux, étant donné qu'ils peuvent être définis comme des ordres partiels sur un ordinal. Ils expriment certaines difficultés et complexités liées à la théorie des ensembles en général et au problème de l'universalité en particulier, et les solutions les impliquant (l'existence ou l'inexistence d'un famille universelle d'arbres ou un arbre universel, dans le contexte de ce texte) a déterminé la création et le développement des méthodes de démonstration et des techniques de grande valeur. L'universalité des arbres est liée à l'universalité d'ordres linéaires, mais elle semble également présenter plus de défis. De plus, ce problème est intéressant dans le cas des classes d'arbres qui n'ont pas de branche cofinale, étant donné qu'une telle branche donnerait automatiquement un objet universel. Nous structurons cette enquête en prenant en considération plusieurs aspects qui interviennent dans la recherche et les résultats concernant les arbres universels, tout en mentionnant des résultats connus et leurs preuves. Nous commençons par remarquer la distinction entre *bien-fondé* et *non-fondé* lorsqu'elle est appliquée aux arbres, puis nous décrivons les types de plongements utilisés lors de la recherche de ce type d'objets universels. Nous continuons en décrivant différents types d'arbres (Aronszajn, Souslin, spéciaux, non spéciaux, Canary), les manières de les construire et les résultats d'existence. Nous montrons ensuite comment différentes preuves et techniques sont liées aux résultats d'universalité, y compris l'opération σ, le forcing.

On présente quelques preuves concernant l'existence de certains types d'arbres et leurs types du plongement (plongement dans $\mathbb{Q}$, plongement dans $\mathbb{R}$). Nous prouvons également des résultats liés aux arbres non spéciaux : les arbres d'Aronszajn non spéciaux, et les arbres non spéciaux construits à partir d'un ensemble bi-stationnaire. Ensembles, ils montrent qu'il n'y a pas d'élément universel dans la classe $\mathcal{T}(A)$ des arbres bi-stationnaires, avec A un sous-ensemble bi-stationnaire de ω_1.

L'opérateur σ offre la possibilité de prouver à la fois des résultats d'existence et de non-existence. L'arbre $\sigma\mathbb{Q}$ est universel pour tous les arbres normaux ayant des plongements strictement croissants dans les réels. Si on suppose CH, la fonction σ interdit l'existence d'une famille universelle de taille $\leq \omega_1$ pour T_{ω_1}.

Nous montrons qu'il n'y a pas de plongement entre un arbre $\aleph_2$-Souslin et un arbre $\aleph_2 - T$ large, non-spécial et sans branches cofinales.

En utilisant la notion de *ascent path*, nous prouvons que la classe des arbres Souslin non-speciaux, ayant cardinalit $\aleph_2$ et un *ascent path* de longueur ω a un nombre de complexité maximal : $2^{\aleph_2} = \aleph_3$.

Dans le sixième chapitre , je montre comment l'evaluation de divers résultats concernant le problème de l'universalité dans la theorie des ensembles et dans la théorie des modèles offre de nouvelles perspectives concernant la notion de méthode, d'objet mathématique, ou à propos des distinctions sémantique/syntaxique et abstrait/concret.

D'une manière générale, une théorie mathématique peut être décrite soit de manière interne ou syntaxique à travers des axiomes et des théorèmes, soit de manière externe ou sémantiquement, à travers ses modèles. Mais des résultats dans la théorie des modèles et dans la théorie de la classification, ou la méthode de *forcing* dans la théorie des ensembles combinent à la fois des composants sémantiques et syntaxiques. Et c'est le même cas avec le problème de l'universalité. La saturation renvoie à un modèle, alors qu'à partir des travaux de Fraíssé puis de Jónsson dans les années 1950, les domaines universels sont construits de manière sémantique. Dans la pratique mathématique, on peut y avoir une préférence pour la sémantique ou la syntaxe, mais cela est lié à des choix individuels. Cela dit, la combinatoire joue un rôle important dans la théorie des ensembles et représente une approche sémantique. L'équivalence logique est une notion syntaxique, tandis que l'équicohérence (l'utilisation de grands cardinaux, donc des axiomes plus forts que ZFC) est une notion sémantique. La dernière pourrait être plus informative d'un point de vue épistémologique. Mais ils offrent tous des connaissances mathématiques. Il faut cependant noter que la notion de cohérence n'était pas pour Skolem une notion syntaxique mais sémantique, renvoyant à l'existence d'une structure satisfaisant les axiomes.

Je continue en mentionnant l'opinion de Cellucci sur l'objet mathématique en tant qu'hypothèse et je présente quelques autres suggestions déterminées par les caractéristiques de la théorie des ensembles, de la théorie des modèles et des résultats liés au problème de l'universalité. Bien que tous les objets mathématiques ne soient pas des ensembles, la théorie des ensembles offre la possibilité d'un contexte homogène pour représenter des objets du même type. En tant que tel, un objet mathématique sera décrit comme un ensemble avec une certaine structure sur celui-ci. Je soutiens donc plus loin que la nature souple des ensembles et des concepts et méthodes utilisés dans leur analyse peuvent en outre offrir la possibilité de concevoir des objets mathématiques en général comme des dispositifs heuristiques. Dans le cadre de la théorie des ensembles et de ses méthodes de preuve (en particulier le *forcing*), les objets mathématiques peuvent également être analysés par rapport à leurs *noms*. Mais bien qu'il s'agisse d'une question philosophique, la relation entre les noms et les objets a un sens spécifique dans la théorie des ensembles. Et ce dernier aspect interroge à nouveau la configuration d'un rapport entre mathématiques et philosophie.

La méthode analytique dans la philosophie des mathématiques, que Cellucci avance, tourne autour de l'idée de plausibilité, pas de vérité. Les objets mathématiques dans une telle vue sont considérés comme des hypothèses, ce qui signifie qu'il n'y aura pas de prémisses immédiatement justifiées à partir desquelles toute connaissance est déduite. Il n'y a donc pas de rejet de l'argument de la régression infinie. J'analyse cette approche, tout en incluant un compte rendu du point de vue de Platon (qui est à l'origine de cette ideé). Je pense que la notion d'objet mathématique comme hypothèse est intéressante, mais il y a certaines limites dans la conception de Cellucci, en premier lieu le fait qu'il existe des distinctions fondamentales entre les mathématiques et la philosophie. Étant donné que toute idée épistémologique *a priori* représente une approche limitative dans le processus de compréhension des processus mathématiques, je suggère que nous prenions en considération différents contextes, méthodes et pratiques mathématiques pour aborder un problème mathématique. L'idée que je propose est que le lien entre les objest mathématiques et *le monde* n'est pas abstraction et idéalisation, mais interrogation et orientation.

Je continue en faisant quelques remarques concernant la distinction abstrait-concret. Je commence par faire un découpage entre une notion externe et une notion interne d'abstraction lorsqu'il s'agit d'objets mathématiques. Une telle séparation ne reflète pas la pratique philosophique, mais la pratique mathématique. J'examine quelques points de vue et donne quelques exemples liés au problème de l'universalité.

Je reviens à l'idée de méthode, mais d'une manière qui pourrait relier les significations exposées dans les chapitres précédents, englobant l'idée d'un contexte épistémique des développements mathématiques, d'instrument heuristique et d'objet d'étude en soi. Ainsi, je commence par souligner le rôle de l'infini dans l'établissement de cette idée générale de méthode. Premièrement, il n'est pas toujours possible d'appliquer les mêmes règles mathématiques et intuitions impliquées dans le domaine fini au domaine transfini. Deuxièmemen, l'infini comme méthode se rattache à la théorie des ensembles comme mathématiques de l'infini et ses principes organisateurs, exprimés dans les axiomes. En ce qui concerne l'universalité, elle représente un problème-test dans la théorie des modèles et dans la théorie de la classification et, en tant que telle, fournit un contexte pour établir des liens entre différents domaines des mathématiques, mais devient également un instrument méthodologique et une forme d'interrogation de la méthode. Divers résultats concernant l'universalité montrent comment elle est inextricablement liée à la combinatoire dans la théorie des ensembles. Une caractéristique commune des principes combinatoires impliqués est qu'ils sont indépendants du système habituel d'axiomes de la théorie des ensembles et, par conséquent, ils sont particulièrement utiles pour prouver l'inexistence. D'une manière générale, l'analyse de l'universalité a déterminé le développement d'autres notions centrales dans la théorie des modèles, comme les modèles saturés, les modèles homogènes et, éventuellement, le *non-forking.*

Revenant sur le lien entre la connaissance mathématique ou l'ensemble des mathématiques et la philosophie, j'entrevois la possibilité de les prendre comme des réponses différentes (peut-être liées) à une ou des interrogations humaines plus générales. Par conséquent, je propose l'idée *d'ordre.* Ce n'est pas un élément mathématique, mais c'est une idée que les mathématiciens, en tant qu'êtres humains, pourraient prendre en considération lorsqu'ils se réfèrent à des concepts vagues tels que 'intuition' ou même la 'vérité' exprimée par un résultat. Certains ont souligné qu'il s'agissait d'un phénomène culturel.

Néanmoins, l'idée *d'ordre* se trouve 'derrière' plusieurs démarches mathématiques. Je discute d'autres aspects liés au rôle de la preuve pour le savoir ou la connaissance mathématique et, en relation avec cela également, je considère l'idée d'ordre comme un espace conceptuel unificateur.

3

Mathematical knowledge

Synopsis

The chapter is divided into three parts. In the first section, we approach the idea of mathematical knowledge and focus on the field of set theory. In the second part, we analyse some aspects regarding the relation between proof and mathematical knowledge. The third section fixes the notion of mathematical knowledge by considering the idea of conceptual and mathematical extension mostly within the framework of set theory but also model theory.

> "Bohr's remark reminded me of Robert's comment, during our walk near Lake Starnberg, that atoms were not things. For although Bohr believed that he knew a great many details about the inner structure of atoms, he did not look upon electrons in the atomic shell as "things", in any case not as things in the sense of classical physics, which worked with such concepts as position, velocity, energy and extension. I therefore asked him: "If the inner structure of the atom is as closed to descriptive accounts as you say, if we really lack a language for dealing with it, how can we ever hope to understand atoms?" Bohr hesitated for a moment, and then he said: "I think we may yet be able to do so. But in the process we may have to learn what the word 'understanding' really means." (Heisenberg, [83], p. 41)

I mention some aspects related to mathematical knowledge and understanding, although I do not seek to find definitive answers in identifying the appropriate support or conditions for a claim to mathematical knowledge.

My approach in connecting mathematics and philosophy is based on the fact, also mentioned by Ewald, that "it is a common characteristic of the various attempts to integrate the whole of mathematics into a coherent whole - whether we think of Plato, of Descartes, or of Leibnitz, of arithmetization, or of the logistics of the nineteenth century - that they have all been made in connection with a philosophical system, more or less wide in scope; always starting from a priori views concerning the relations of mathematics with the twofold universe of the external world and the world of thought"([60], pp. 1266-7). My purpose in this chapter, to be continued in the fourth one, is to mention some aspects regarding these relation, and, given the subject, it cannot be other than limited in scope.

There are various elements that fall under the general term of *mathematical knowledge*, such as theorems, proofs, evaluation of correctness, and even hypotheses. I added this last factor because although hypotheses can be proved to be wrong, they are never arbitrary, there are always some contextual and structural constraints, mathematical ones, that determine their formulation.

Avigad maintains that given the central role of proof in mathematics - a theorem is to be considered true if is has a proof -, its clarification translates to the same status in the philosophy of mathematics. Mathematics, he continues, contains more than evaluations of correctness though: for instance, "we seem to feel that there is a difference between *knowing that* a mathematical claim is true, and *understanding why* such a claim is true. Similarly, we may be able to convince ourselves that a proof is correct by checking each inference carefully, and yet still feel as though we do not fully understand it". As such, "there is a gap between knowledge and understanding", made "pointedly clear by the fact that one often finds dozens of published proofs of a theorem in the literature, all of which are deemed important contributions, even after the first one has been accepted as correct. Later proofs do not add to our knowledge that the resulting theorem is correct, but they somehow augment our understanding. Our task is to make sense of this type of contribution" ([5], p. 319). For him, a theory of mathematical understanding should be a theory of mathematical abilities.

The fact that different proofs improve our mathematical understanding is clearly true, but it doesn't necessarily mean that the existence of different proofs point to a perceived insufficiency of the original one. The discovery of other proofs may involve a focus on certain aspects of the problem, they may be more 'elegant', they may establish relations to other areas of mathematics (or even introduce new sub-areas), conceive new methods etc. Overall, the effects could be identified in the creation of new relations within the mathematical landscape, (probably) revealing new patterns here.

But given the philosophical difficulties in definitively establishing the distinctions between *knowledge* and *understanding*, and the fact that it is not the purpose of this text to do that, I will use these two notions interchangeably, with their different connotations to be established in one context or another.

3.1 Conceptual framework - perspectives on the philosophy of mathematics

Do we need a theory of mathematical knowledge? What is then mathematical knowledge? And how is philosophy involved? Do we need to go outside mathematics to understand the universality problem?

Against the foundationalist view, I subscribe to the idea that knowledge "is not an edifice built up according to a plan fixed beforehand, but the plan develops as knowledge grows"([20], p. 37), that such a knowledge can indefinitely establish new 'floors', build new edifices and connected them. A *system* of knowledge is closed by definition, and although helpful through its own limitations, one could probably remember that "[P]hilosophers are despots who have no armies to command, so they subject the world to their tyranny by locking it up in a system of thought" ([157], p. 291).

It might be helpful to start with the traditional conceptual and procedural distinction summed up as *knowing that, knowing why* and *knowing how.* Of course philosophy does not offer a definitive view on such aspects, but the philosophical interrogations regarding such matters do determine conceptual distinctions, directions of interrogations and clarifications that make them fundamental for the human knowledge in general and when picked up in different sciences and mathematics, in particular.

It could therefore be helpful to acknowledge another distinction, more local this time, involving

- the philosophical approach to mathematics,
- the philosophical beliefs of one mathematician or another, and
- mathematical practice.

Mathematical practice is purely mathematical, the philosophy of mathematics strives to be an objective process, (at its best, it could be a self-interrogation), whereas the philosophical beliefs of one mathematician or another open the doors to other areas of knowledge. In 1979, R. Hersh wrote that "[B]y "philosophy of mathematics" I mean the working philosophy of the professional mathematician, the philosophical attitude toward his work that is assumed by the researcher, teacher, or user of mathematics" ([85], p. 31). That would reduce the philosophy of mathematics to psychology, sociology, or history and indeed, many mathematicians assume that the philosophy of mathematics is just the philosophy of mathematicians.

I think that up to a point Gowers is right when he states the following, although he is not always quite clear regarding the distinction between the philosophy of mathematics and the philosophical beliefs of one mathematician or another: "Suppose a paper were published tomorrow that gave a new and very compelling argument for some position in the philosophy of mathematics, and that, most unusually, the argument caused many philosophers to abandon their old beliefs and embrace a whole new -ism. What would be the effect on mathematics? I contend that there would be almost none, that the development would go virtually unnoticed. And basically, the reason is that the questions considered fundamental by philosophers are the strange, external ones that seem to make no difference to the real, internal business of doing mathematics. "([75], p. 198).

He considers his view as being *naturalist*, while also maintaining that there are "philosophical, or at least quasi-philosophical, considerations" (such as induction) that "do have an effect on the practice of mathematics" ([75], p. 199). Mathematicians' "outward behaviour" with regard to mathematics, he continues, is not influenced by their beliefs, whether they are Platonists or fictionalists.

Such a statement is not entirely correct if we take into account the history of mathematics. For instance, until the 19th century, the mathematicians would adhere to Aristotle's distinction between the actual and the potential infinite. It would be probably safer to assume that the last aspect involves psychological, sociological and philosophical aspects that, for the moment at least, are hard to unify into a definitive answer. He basically rests his view on a certain presupposition that philosophers have on the idea of truth in mathematics, particularly the ones identifying truth and provability. In that statement, he is ignoring the fact that such an approach represents one philosophical theory, not necessarily espoused by all philosophers. An alternative to the view that the aim of mathematics is truth is that the aim of mathematics is plausibility (see [20].

He does leave the door open to new approaches though when admitting that there might be an "important philosophical project": "the question of which informal arguments we find convincing and why" ([75], p. 196). On the other hand, he does favour a formalist approach: firstly because "[W]hen mathematicians discuss unsolved problems, what they are doing is not so much trying to uncover the truth as trying to find proofs"; and secondly because "the formalist way of looking at mathematics has beneficial pedagogical consequences" ([75], p. 199).

The role of philosophy in approaching mathematics is a sensitive one, given that a first encounter between the two has epistemological connotations. There is no definitive answer in this regard (how could it be possible?), but tackling the idea of knowledge and, in particular, mathematical knowledge, made possible the configuration of precious clarifications and delimitations, both for mathematics and philosophy.

Cellucci, for instance, is right when he considers that philosophy "may expose the inadequacy of some basic mathematical concepts"; that it "may provide an analysis of some basic mathematical concepts"(e.g. Turing's definition of effective calculability); or that it "may help to formulate new rules of discovery" (e.g. the principle of induction, attributed by Aristotle to Socrates [1]) ([20], p. 236).

What he is proposing is a reconfiguration of the philosophy of mathematics based on the idea of the analytic method, which will show that the aim of mathematics is plausibility and not truth, and that intuition has no role in mathematics. He maintains that a "genuine philosophy of mathematics" should offer answers to questions such as 'What is the nature of mathematical knowledge?', 'What is the role of mathematical knowledge in human life?', 'How is mathematical knowledge acquired?'. They all constitute typical philosophical questions, but the philosophy of mathematical could contribute to the "advancement of mathematics by further developing the analytic method. Even Frege acknowledges that "a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could easily be discovered than to discover particular truths, and all steps of scientific progress in recent times have had their origin in an improvement of method" (Frege 1967, 6)." ([20], p. 239).

For a recent account regarding plausibility, see Toffoli ([30]). She focuses her analysis on the individual mathematical justification, showing that "the main function of mathematical justification is to guarantee that the mathematical community can correct the errors that invariably arise from our fallible

[1] "For there are two things that may be fairly ascribed to Socrates - inductive arguments and universal definition, both of which are concerned with a starting-point of scientific knowledge" ([4] M 4, 1078 b 27-30)

practices" (p. 823). To that end, she offers a new definition of proof and introduces the related notion of *simil-proof*, allowing her to offer a non-factive account of justification. Although I agree with her that the existence of social norms for mathematical justification does not entail the idea of a 'socially-constructed' mathematics or any ontology of mathematics, there is always a certain peril when trying to propose some ideal norms regulating "regulating human mathematical practice".

I agree with Cellucci that philosophy could approach mathematics through the concept of method and I will expose further bellow the reasons in this regard, but first, I would like to point out some aspects to be considered.

3.1.1 Reluctance to fit mathematical knowledge into a theory of mathematical knowledge

There was a tendency to define the philosophy of mathematics from a foundationalist perspective, regardless of the 'school of thought' (logicism and neo-logicism, formalism and neo-formalism or intuitionism and neo-intuitionism).

To consider that the main directions of research is either the 'ontological question' - regarding the existence and nature of mathematical objects - or the 'epistemological question' - focusing on the nature and justification of the mathematical statements -, is reductionist and limiting, both for philosophy and mathematics. First, it involves a translation of philosophical topics into a new context: mathematics. But it also involved a certain interpretation the other way around (see [177]). Although it doesn't imply that each area of knowledge shouldn't be aware of the actual practice of the other. An example, concerning the *ontological questions* 'What is a number?' or 'What is a set?', they are not relevant to the mathematical practice, as so many mathematicians have emphasized. But even Frege, so opposed to any 'psychologist' approach, states that it is "a scandal that our science should be so unclear about the first and foremost among its objects" (Frege 1960, xiv).

For the other direction, when mathematical results are employed to support a philosophical tradition, we could mention Putnam's use of what would become known as the *model-theoretic argument against metaphysical realism* ([165]), but also Quine. Many of the most successful theories in various natural sciences are mathematically expressed. But Quine ([167]) might be too reductionist when maintaining that our best theories are our best scientific theories. A theory is connected to a larger context: the general human knowledge (whatever that might be taken to mean) or even a certain community of scholars. But it also has meanings, consequences, we can talk about its fruitfulness, all of these being aspects that are difficult to quantify, given that we would have to take into account such parameters as time (the entire human history) or space. Plato's work on different topics, for example, has had an impact on the general human knowledge for more than 2300 years and in different ways. In the end, sciences and mathematics are made by human beings, adhering to cultural norms and values.

So yes, the best theory or account regarding our physical reality will be scientific, or rather a pot pourri of scientific theories and facts pertaining to different sciences: physics, chemistry, biology, mathematics, etc. And they might represent our currently best account of what physically exists, a better understanding, but not necessarily of what we know or how we know it.

Given its foundational status, many of the problems belonging to the philosophy of mathematics are connected to set theory and the problems involved by the transfinite hierarchy. By applying Quine's naturalism to his philosophy of mathematics, Putnam ([166]) expressed his own ontological commitment: since mathematical theories are inherent to the most successful scientific theories, and experience globally confirms these scientific theories, it also empirically conforms the mathematical theories involved. Hence Putnam's mathematical ontology. But of course, given that the theories encompassing the infinite are less likely to be used in scientific theories, there is not as strong a Platonism as Gödel's Platonism.

Not all 'naturalistically'-inclined philosophers of mathematics adopt a realistic view regarding sets (see [132], for example). Maddy does not fully adhere to Quine's naturalistic stance. Mathematics could be considered a science in itself and the ontological commitments should be rather connected to the mathematical practice and its methods. Focusing on set theory ([133]), she is emphasising its unifying role of "bringing all mathematical structures together in a single arena and codifying the fundamental assumptions of mathematical proof" (p. 133). The axioms are justified as long as they facilitate this role and, as such, they need no further philosophical justification.

Feferman ([61]) imposes a predicativist framework and limitations in his approach to mathematics, which means that he might leave some areas outside (as not acceptable from a predicativist point of view), like the transfinite set theory. A more moderate view is expressed by Steel, for instance, who share "the Naturalist's reluctance to trim mathematics in order to make it fit some theory of mathematical knowledge. Nevertheless, - he continues - a solution to the Continuum Problem may need some accompanying analysis of what it is to be a solution to the Continuum Problem, and in this way, Philosophy may have a more active role to play at the foundations of mathematics than Maddy envisions." ([62], p. 433)

Generally speaking, the subject matter of mathematics consists of mathematical objects. Their descriptions are determined by different areas of mathematics. Their *nature* and their *justification* is nonetheless *philosophical* and only necessary to the degree that different members of the mathematical community consider it so for the development of a particular topic. On these lines, I also subscribe to the view that a proper philosophical account of mathematics should be grounded in the actual practice of mathematicians, that mathematics should not be made to fit already made philosophical theories, with results made to somehow adjust to already presumed perspectives, and less likely to determine and redefine the field of interrogations. But that might require a better reconfiguration of the role of philosophy in mathematics and vice-versa, an enormous task that is not the subject of this text.

But in emphasising the nature of mathematical knowledge, there is another aspect that I would like to mention regarding the connection between philosophy and mathematics, the fact that in the end, such a connection proves its value through the a posteriori insights and justification related to the mathematical practice. Mathematical results became subjects of human understanding. They are further connected, in one form or another, with human knowledge. At some point, in this very large context, it would be connected to some philosophical distinctions, which, at their turn, will determine new and various interrogations and investigations, including new mathematical inquiries.

As Hersh points out, "the philosophy of mathematics as practised in many articles and books is a thing unto itself, hardly connected either to living mathematics or to general philosophy. But how can it be claimed that the nature of mathematics is unrelated to the general question of human knowledge?" ([86], 68).

Thurston, an accomplished mathematician, emphasises as well the psychological and the social dimensions of the mathematical thinking, the fact that mathematics evolves by organically integrating these aspects: "as mathematics advances, we incorporate it into our thinking. As our thinking becomes more sophisticated, we generate new mathematical concepts and new mathematical structures: the subject matter of mathematics changes to reflect how we think" ([211], p. 162). So when mathematicians construct new and better ways of thinking, they should also be aware that people are not doing mathematics only for its own sake, that they are also driven by considerations related to a social setting, like status and economics.

That being said, Shelah points to another aspect related to the mathematical practice. He rejects the extreme formalist view reducing mathematics to a manipulation of symbols and offering an equal value to all set theories, and adopts a strong Platonism of sorts: "under axioms and logical rules, it seems that Mathematics is absolute. Plato's kind of argument according to which Mathematics is an idea which we discover will always be controversial, but to questions such as: is a given solution to a mathematical problem the right one, the answer is clear cut. True, we can have only evidence and not an absolute proof of the existence of Mathematics outside of man, because we only act in a human reality" ([107], p. 3).

3.1.2 How? and the relation to the universality problem

In this context, the thread I want to follow is the analysis of the continuous transformation and expansion of mathematical knowledge through methods of proof and inquiry. In particular, I would like to focus attention on the universality problem as a way of exemplifying the process of mathematical knowledge.

There is no plan a priori fixed determining the continuous expanding process characterising mathematics and mathematical knowledge. The existence of axioms, particularly set theoretic axioms, offers a strong framework constructed according to requirements determined by the mathematical practice [2]. But their hypothetical nature leaves open the possibility of new developments, new relations within the widening field of mathematics. Invoking Pascal, and changing 'numbers' with 'sets (which shouldn't be considered too far-fetched)'"[N]umbers imitate space, which is of such a different nature".

To that end, there are three main points I intend to consider:

1. The concept of mathematical object.

2. The notion of method.

3. Change and reconceptualisation in mathematics.

[2]For the axioms of set theory see the classic papers of Maddy, [131], [131].

Object This transformation or evolution is fundamentally connected to the specific practice of mathematics and the nature of its 'objects' or notions. They are distilled, abstract notions emphasising patterns and mediating analogies among them. The result are abstract theories allowing a large spectrum in the form and generality of proofs. That being said, there are two registers of abstraction or abstractisation that I will emphasise while approaching the universality problem.

Method As such, a theory becomes a method in itself. The concept of infinity can be conceived of as a method for a radical change of perspective in mathematics, it implies conceptual jumps (Gödel's term - the first infinite, the power set, I would add the diagonal argument) and the constitution of certain 'orders'. It determines a change with regard to different determinations, or levels of determination.

According to Dehornoy, one can "use infinity, and its specific tools, as a melting pot where previously hidden properties appear without this indicating any link between these properties and the framework that reveals them", such that set theory can be seen as "a catalyst", or even "a photographic film, which reveals a phenomenon but has no connection with it" ([32], p. 387). It is at this point that "set theory brings its main contribution at the very moment when it disappears as a proof tool" (p. 389).

I am not referring here to the axiomatic method. Although the axiomatic exposition is fundamental for mathematics to express its facts, given that the latter are not subject to experimental verification, and it represents "one of the great achievements of our culture", it is only a method. And "[W]hereas the facts of mathematics, once discovered, will never change, the method by which these facts are verified has changed many times in the past, and it would be foolhardy not to expect that it will change again at some future date" ([177], p. 166).

Change Change and reconceptualisation are determined and understood in relation to two elements: (i) different conceptual settings, such as set theory or model theory; and (ii) structural distinctions pertaining to these settings and related to the mathematical practice, the notion of model extension, for instance.

It should also be emphasised that change (in connection to mathematics) is not just an accumulation of facts, an increase in rigour or precision is another of the main ways mathematics changes. Change in this context means turning points: radical shifts but also gradual development, although, in this last case, we will need a certain period of time to appreciate the whole significance of a result. In the end, "a mathematician who solves a problem cannot avoid facing up to the historicity of the problem. Mathematics is nothing if not a historical subject par excellence" ([177], p. 174).

This multifaceted concept of mathematical change offers new insights into the meaning of mathematics, new questions regarding its internal changes. A problem becomes like a map that we are creating while moving among these turning points. How do we map change then? The most obvious way is that we get new results regarding a certain problem, sometimes not even a final solution. Sometimes, we can change the original context of a problem and translate the problem to another one. It also happens that mathematical statements receive different proofs and, sometimes, these different proofs involve methods and strategies that give way to new approaches, even new areas of study. Often, new mathematics appear from revisiting existing results and proofs, I would say in accordance with a very poetic principle:

"We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.
Through the unknown, remembered gate
When the last of earth left to discover
Is that which was the beginning;

At the source of the longest river
The voice of the hidden waterfall
And the children in the apple-tree
Not known, because not looked for
But heard, half-heard, in the stillness
Between two waves of the sea."
(T.S. Eliot, from "Little Gidding", *Four Quartets*).

Furthermore, the result becomes more general, an original problem becomes the special case of something more general. There are different ways of generalisation in mathematics, specific to different fields, but some of them involve weakening the hypotheses (so strengthening the conclusions), identifying specific properties, go to higher dimensions and use several variables, proving a more abstract result etc.

These developments are accompanied by a process of abstraction. As underlined by P. Mancosu, "Abstraction is of course "said in many ways" ([143], p. 1), but the meaning which is relevant to my investigation is not, as in his case, connected to the abstraction principles in neo-logicism. I am referring here to abstraction in the sense of degree of formalisation, to the discovery of a more general theorem or result such that all other particular and possibly interesting cases would follow, without the need to prove each one separately. Furthermore, such a situation would enable to acknowledge connections that otherwise might not be visible (even between different areas). "The processes of abstraction and generalization are therefore very important as a means of making sense of the huge mass of raw data (that is, proofs of individual theorems) and enabling at least some of it to be passed on" ([74]).

Reconceptualisation

As emphasised by Shelah, when explaining Cantor's discoveries in set theory, we meet an old solution, "very ancient and simple. We know that primitive peoples do not have large numbers, certainly not beyond 40. So how do they trade? Very simple: they trade one for one. Two sets of sheep are equivalent if there is a one to one correspondence between them ..." ([107], p. 6). This seemingly trivial aspect points to an essential trait of mathematical practice: whatever the impetus, one wants to understand a problem in depth. And the consequences of such inquiries and investigations might prove to be fundamental and surprising: the pursuit of Fermat's conjecture, for instance, determined developments in many areas of mathematics and, although a problem in number theory, the solution was found due to explorations in other mathematical fields. Along this text, I will follow the same movement with regard to results in set theory and model theory.

The aspect of reconceptualisation is fundamentally connected with the notion of a *context*, of a particular setting. Paraphrasing Feynman, when you ask why something happens, you have to be in a framework that allows something to be true. There is a great variety of set theoretic contexts one can choose from in dealing with a certain problem, and not in a trivial and artificial way. Different models or universes in set theory offer different frameworks of analysis. This evolving aspect represents a part of the method. And actually, "[P]assing from the study of a unique structure formed by the true [or pure] sets to that of abstract models of ZF is a Copernican revolution. From a practical point of view, this passage to a multiplicity of worlds enables the consideration of algebraic operations on the models: products of models, submodels, extension of models, surgery on the models, ..., which are all unthinkable without the notion of an abstract model" ([33], p. 353).

The choice and understanding of one particular setting represents an essential aspect of mathematical knowledge. One could even go as far as to maintain that "the recognition of mathematical settings, as features distinct from both mathematical structures and the systems which instantiate those structures, allows one to classify most of understandable misunderstandings in mathematics, and also to solve the identity problem" ([79], p. 1).

Reconceptualisation is closely connected to the notion of extension, and I will analyse the two in a subsection bellow. And if all these points of view are valid, it means that each of the different conceptual settings in analysing the universality problem (even when no solutions were found yet) could or will offer a different perspective on the same phenomenon/phenomena. They all catch different changes that are taking place in the way mathematics is done and mathematical knowledge is acquired.

Interpretability Connected to the notion of reconceptualisation is the process of interpretability. For any first order theory T, a model of T is a set equipped with functions and relations satisfying the appropriate axioms. A model of ZFC, for example, is a set M equipped with a binary relation satisfying etc. All mathematical objects can be represented as sets and, as such, every model of set theory (a structure satisfying the axioms of set theory) must include its own versions of *all* these objects, a complete rendition of the mathematical 'world', *the world according to that model.* Fundamental results (in model theory, for instance) show that given two signatures and provided that they are finite, every structure can be presented under a single binary relation. In particular, any structure in a countable language can be interpreted in a graph (see [90], [146]).

The possibility of interpreting any theory T in ZFC, in the sense that all the notions of T receive set-theoretic descriptions and, as such, make all the theorems of T provable in set theory points to a fundamental aspect: one doesn't need to 'understand' these notions, one just need to know how to 'operate' with sets. It doesn't even require a philosophical understanding of sets. Furthermore, through its interpretation, the theory T 'gets access' to the set theoretic proof apparatus, in particular to the independence proofs (consistency and forcing).

For example, we can represent the natural numbers $\mathbb{N}$ in an universe of set theory. That means that we will construct a number system *behaving mathematically* like the natural numbers, having the order and arithmetic. That being said, we are not saying that we are constructing the 'real' or the 'actual' natural numbers but objects having the same mathematical properties. We will take 0 as the empty set $\varnothing$, 1 for $\{0\}$, 2 for $\{0, 1\}$, $succ(x)$ for $x \cup \{x\}$, and $\mathbb{N}$ for $\{n : n \text{ is a natural number}\}$. As a result, $0, 1, 2, ..., n$ and $\mathbb{N}$ are symbols or names for some sets. There will also be theorems showing that $\in$ behaves on $\mathbb{N}$ exactly like the usual ordering $<$ on the natural numbers, and, in practice, people often use $<$ for $\in$ when writing about the natural numbers. The relation symbols $\leq, >$, and $\geq$ will also be used in their usual sense.

3.1.3 Set theory

Both Manders and Thurston (among others) downplay the role of set theory or the foundational programs, with the last going as far as maintaining that the foundations "[O]n the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do", with most mathematicians adhering "to foundational principles that are known to be polite fictions. For example, it is a theorem that there does not exist any way to ever actually construct or even define a well-ordering of the real numbers. There is considerable evidence (but no proof) that we can get away with these polite fictions without being caught out, but that doesn't make them right. Set theorists construct many alternate and mutually contradictory 'mathematical universes" such that if one is consistent, the others are too. This leaves very little confidence that one or the other is the right choice or the natural choice. Gödel's incompleteness theorem implies that there can be no formal system that is consistent, yet powerful enough to serve as a basis for all of the mathematics that we do ([211], pp. 170-171).

What he seems to ignore is that these constructions of universes are based on Gödel's work, it is mathematics. The completeness theorem shows that any consistent theory has a set-sized model (this includes ZFC). ZFC cannot prove that there is a set model of ZFC, since ZFC cannot prove Con(ZFC), but it is possible to form proper-class-sized models of ZFC from a given proper-class-sized model of ZFC, e.g., the inner model or thr constructible universe L. Gödel's completeness theorem tells us that if a first-order theory T is consistent, then there is a set model for it. Moreover, if T is a theory over a countable language, then the downward Löwenhein-Skolem theorem tells us that T has a countable model. If T is a set theory, then there is a distinction between the universe the model lives in and the universe inside the model.

Furthermore, these universes do not have an ontological existence, as a strong Platonist would argue. The language of set theory consists of the $\in$ symbol, and people are studying consequences of the axioms, not *real* universes, despite a certain naturalised language. Thurston is denying here value to a mathematical practice he didn't choose and he is also contradicting some of his remarks mentioned above, like the fact that the reliability of mathematics comes from mathematicians thinking carefully and critically about mathematical ideas, which is precisely what the mathematicians working in the foundational areas of mathematics are doing. A subjective assessment with regard to the value and quality of less known areas of mathematics does not fall under the valid ideas he otherwise espouses.

Manders maintains that the *universality* characterising the foundational role of set theory (or of other foundational approaches/schemes) "obscures" ([144], p. 202), since "[A]ll present and future mathematics is supposed to fit in", and even more, they "obscures the special role of fundamental structures (conceptual settings) and fundamental constructions for mathematical knowledge" ([144], p. 200). One reason is the universality of the language: "*if everything is expressible in a language, no special properties of things expressed follow from mere expressibility*" ([144], p. 200). Furthermore, unlike a more successful approach in model theory (where one uses the minimal language to formalise and display the "relevant underlying structure" of an object), set theoretic definability by itself does not demarcate the "relationships between (set theoretically definable) conceptual settings which constitute successful reconceptualisations, say p-adic completion, from the infinitely many completely uninteresting ones" ([144], p. 200). He does recognise though that "important special features can be displayed by recognisable syntactic features of set theoretic definitions rather than by set theoretic definability itself. For example, quantifier complexity of set theoretic definitions very effectively displays absoluteness features, as in the context of KPU and admissible languages, or of Schoenfeld absoluteness." Nonetheless, he maintains that "it is an entirely clear *empirical fact* that the special features of the fundamental structures and reconceptualisations of traditional mathematics, the features which would figure in an explanation of the leading role of these structures and reconceptualisations in enabling us to grasp and know, *cannot be discerned from commonly recognised syntactic features of set theoretic definitions*" ([144], p. 200)

His understanding of the framework offered by set theory or the role of model theory does not correspond to the way mathematicians engaged in both endeavours address the situation. Shelah, for instance, maintains that if one is interested in general results, then we have to use a set-theoretic framework: although "by now we know well how to generate 'generalized nonsense' which grinds water and tells us nothing new, many times a general framework shows you that isolated claims are parts of general phenomena (...) That is, if you want to know something about ALL structures of some kind (all groups, all manifolds, etc.), then you need to be able to deal with infinite unions or infinite products of sets, which are inherently set-theoretical concepts. Moreover, even if your main interest is in, say, finitely generated groups, you will be drawn into more general ones, e.g., taking some compactification or using infinite products" ([195], pp. 205-6).

Another criticism Manders brings to set theory and logical foundations is its assumed Platonism, the fact that the set theoretic universe is assumed to exists and to incorporate timelessly all relationships and settings, an aspect that would deny the possibility of reconceptualisations and new relationships ([144], p. 200). He refers to Plato, who maintained that since geometrical objects exist timelessly, geometric construction is impossible. At best, they are subjectively new, for human knowers?" ([144], p. 200)

I thinks that such an assessment is reductionist and uninformed. The "commonplace mathematical realism" is a very ambiguous notion, even in a pure philosophical context. Mathematicians do not adopt a philosophical framework according to which they conduct their mathematics. They may have personal preferences and philosophical values, but that represents another epistemic register. Conceptual innovation in mathematics characterises even the work of those mathematicians subscribing to a particular ontological stance with regard to the status of mathematical objects.

A traditional role for set theory was to provide a foundation by which, as mentioned above, every mathematical object could be represented as a set. Gödel's incompleteness theorems emphasised the limits of this foundational role, but that limitation characterises any axiomatic system. It made possible though the study of various extensions of the set theoretic axiomatisation. In light of more recent applications of set theory to other areas of mathematics (particularly in algebra and topology), P. Dehornoy points to a third role of set theory, i.e. "to crystallize some intuitions for which other frameworks would have been to rigid, and to elaborate them with the helps of its specific tools, including strong axioms." ([32], p. 379). The set theoretical hypothesis used will appear only as technical auxiliaries. He likens this use of set theory with that of theoretical physics, e.g. the Feynman integral ("a formalism that is not rationally founded, but turn out to be extremely fruitful" p. 388) "which gives heuristic intuitions that have to be rigorously justified afterwards" ([32], p. 380).

An example he used showed how a very strong axiom in set theory - the axiom of self-similar rank - had consequences for a purely algebraic property, before being eliminated by the construction of an alternative proof. In other words, one can "use infinity, and its specific tools", a reason for which he refers to set theory "as a melting pot where previously hidden properties appear without this indicating any link between these properties and the framework that reveals them" ([32], p. 387). It is already known that infinity can be used as a logical principle to establish certain properties of finite objects, which could otherwise have remained inaccessible (e.g. [160]). The heuristic role of set theory was emphasised by others as well, including Feferman. And, returning again to Dehornoy, "[T]he introduction of higher order infinities (...) can be likened to the introduction of an additional proof principle" ([32], p. 387) : "what is crucial now is no longer the truth of the axiom, but rather its potential richness and its proving power. And, from this viewpoint, the stronger the axiom is, and therefore the closer the contradiction, the more powerful it is likely to be in terms of applications" ([32], p. 388).

Furthermore, sets are useful in metamathematics as well, in proving statements, generally speaking, not about mathematical objects but about the process of mathematical reasoning itself. It is the consequence of using a very simple language with a small vocabulary and not too complex grammar, into which one can interpret all mathematical arguments.

3.2 Proof and method

3.2.1 Proof

If the purpose of mathematics is understanding, then the existence of different proofs constitute a consequence and a way in which such a process is manifesting itself within mathematics. In other words, one way of making sense of it is by looking at different directions in solving a problem, the different proofs for the same problems and the methods used, like forcing. In connection to the idea of mathematical knowledge, a proof represents a way of assuring and/or offering mathematical knowledge. As such, mathematical knowledge is not the result of a juxtaposition of theorems.

There is a distinction in the philosophical literature between explanatory proofs and non-explanatory proofs (see Steiner, [208], [207]). Introducing the concept of 'characterising property' to replace the more ambiguous ones of 'essence' or 'nature' of an entity, Steiner describes an explanatory proof as the one depending on this concept and being able to generalise it ([208], passim).

The distinction has its roots in another distinction, made by Aristotle in *Posterior Analytics* (A.13), although Steiner is not necessarily relying on that, between demonstrations "of the fact" (ὅτι, lit. "the that") and demonstrations "of the reasoned fact" (διότι, lit. "the why", "because of which"). The opposition between proofs "that convince but do not explain and proofs that in addition to providing the required conviction that the result is true also show *why* it is true" ([142]) could be place in a long philosophical tradition (see for instance [138], [140]). The 'explanatory proofs' would provide 'the why', while the 'non-explanatory' ones would provide 'the that'. Steiner's absolute distinction was questioned by others (Resnik and Kushner, Hafner and Mancosu).

In the same context, many philosophers of mathematics refer to *pure* and *impure* methods in mathematics (see for instance [34], [35]). Describing a mathematical problem as the set of mathematical resources including definitions, axioms and inferences such that any change in them would determined a change in the content of the problem, Detlefson and Arana call a solution to the problem *topically pure* if it uses only the resources that determine the topic of the problem ([35]). The *epistemic benefit* of a pure solution is that it is stable, unlike the *impure* one. From their point of view, an impure solution to a number theory problem (determining whether there are infinitely many primes) would be one using topological resources (due to Furstenberg). The relation between explanatoriness and purity has been emphasized and analysed by Skow ([203]) and Lange ([120]).

I subscribe to the importance of a conceptual analysis with regard to proof, although, and pointed out by Mancosu, proof would not be the only source of explanation (and therefore knowledge) in mathematics if one is considering a more global case of explanatory activity, extended to an entire disciple ([141]). Kitcher also points to rigorisation and systematisation as sources of explanation and mathematical understanding ([105]), and even unification as a model for explanation in mathematics and sciences ([106]). But we should not forget that these kind of distinctions are philosophical, and the conclusions based on their analysis and the mathematical practice to which they are connected belong to the field of philosophy. The question remains, of course, how to determine the relation between mathematics, in particular mathematical practice, and philosophy. I find it problematic then when the aim of the philosophy of mathematics, "broadly construed" is considered to be "to understand mathematics, and potentially engage with how mathematics should be done" ([80], p. 1114). A priori epistemological ideas represent a limitative approach in the process of understanding mathematical processes.

Using the aforementioned distinctions but also others in creating a context to analyse mathematical practice involved studies focused on pointing epistemological inadequacies in connection to some idealised and often artificial criteria. Of course there will always be constrains in introducing new mathematical concepts and principles, but they should be mathematical constrains, not philosophical ones. That does not negate the importance of individual abilities - reason, imagination, intuition - in doing mathematics: mathematics in done by human beings (as well).

Manders maintains that although proof theory does not lead to wrong or unjustified conclusions about knowledge, proofs are not sufficient for comprehension, for which reason we should "reject the traditional assumption of *deductive omnipotence* in epistemology of mathematics: *that the ideal cognitive agent should count as in intellectual possession of all propositions deducible from any proposition she possesses.* Deductive omnipotence would have us suppose results known even if only provable by "impenetrable and tedious computation" ([144], p. 199). Furthermore, he writes that "[C]onceptual unification, understandability, clarity, even length of proofs, fall outside the narrow justificational concern of foundational epistemology", and then he adds that the "process of establishing deductive relations is subsidiary to the larger goal of rendering understandable" ([145], p. 562)

Firstly, it is doubtful that foundational approaches are not interested in 'conceptual unification', 'understandability' or 'clarity'. Secondly, he does distinguish a proof aspect (the length of proof) that he considers overlooked, and which could be more conducive to knowledge than other aspects, so proof has methodological value. I agree that attributing conceptual omnipotence to the (ideal) cognitive agent imposes certain constrains regarding the possibility of mathematical knowledge, but I do not subscribe in his downplaying the importance of proof. That does not make cognitive agents "deductively omnipotent". On the contrary, it could offer new conditions for reconceptualisations. A proof could be a new argument and also innovative in the techniques that it might employ, it could open new directions of analysis and even research.

He further maintains that proof theoretic analysis of single theories, including their comparison, or even the "set theoretic description of relations between structures" cannot constitute premises for understanding the role played by reconceptualisations in approaching mathematical knowledge. But he does stress that such a role could be played by the model theoretic models ([144], p. 202). Such reductions are misleading at best, given that he also emphasized the importance of a "finer-grained individuation of theories ([144], p. 201).

3.2.2 Method

I will take *method* in its more general meaning, encompassing the idea of an epistemic context for mathematical developments, a heuristic instrument, and also as an object of study in itself. It involves mathematical definitions, theorems and complex developments, and it is expressed in various forms. An example to be discussed under the first meaning of this notion is the *axiomatic method*. Among others, Cellucci points to its limitations and argues for the *analytic method* as an alternative. *Forcing* represents another way to approach the concept of method in its heuristic form. This approach will be further developed in the fourth chapter. Furthermore, method could be conceived of as a kind of conceptual and epistemic unification, a feature that will be further developed in chapter six. But I claim that these various aspects of the notion of methods, as *context*, *instrument*, and as *object of study* are connected to the process of mathematical knowledge or understanding.

Given the role in supplying mathematical knowledge, and unlike Cellucci, I maintain that even deduction, characteristic of the axiomatic method, could be epistemologically ampliative. That last aspect comes from a certain advantage that set theory has over other fields of mathematics, considering that when a direct proof is not possible, it presents methods that could show that a proof (including a refutation) might be impossible. As such, we can distinguish between what is decidable and what is not. Shelah wrote though that even after forcing was developed, it seemed better to prove that something follows from GCH than just proving it is consistent", with statement of these kind being called "semi-axioms. Of course, the extent to which we consider a statement a semi-axiom is open to opinion and may change in time. I give statements in cardinal arithmetic a high score in this respect. Note that a semi-axiom may be (consistent with ZFC and) very atypical (= the family of universes satisfying it is "negligible") but still very interesting, since for some sets of problems it gives a nice coherent picture" ([195], p. 210).

The axiomatic method - limitations

That being said, there are some crucial limits to the axiomatic method. Skolem and Gödel had already shown limitations to Hilbert's idea that mathematical structures can be understood through their axiomatisation. Shelah also isolated instability phenomena in certain first-order theories, but he also introduced some techniques built on geometric and algebraic notions in order to approach the understanding of models of first-order theories. Universality represents a test problem in this regard. The stability of a theory is connected to its types, sets of formulas describing a generalised notions of elements. Intuitively, a theory is stable if the spaces of types in different cardinalities are not too big. But models containing an infinite linear order have too many types to be stable.

Furthermore, as emphasised by Rota, although the axiomatic exposition is fundamental for mathematics to express its facts, given that the latter are not subject to experimental verification, and it represents "one of the great achievements of our culture", it is only a method. And, what is more, "[W]hereas the facts of mathematics, once discovered, will never change, the method by which these facts are verified has changed many times in the past, and it would be foolhardy not to expect that it will change again at some future date" ([177], p. 166). As a result, one shouldn't confuse the content of mathematics with the way it is presented, with axiomatisation included in the presentation category. Confusing mathematics with the axiomatic method for its presentation is as preposterous as confusing the music of Johann Sebastian Bach with the techniques for counterpoint in the Baroque age" ([177], p. 171). Such a confusion can be found, he considers, among the "mathematising philosophers", the ones trying to imitate mathematics in dealing with philosophical topics, and who mistakenly believe that mathematicians use the axiomatic method to solve and prove theorems.

Furthermore, Baldwin points to the questions raised by classification theory regarding the nature of axiomatisation: "the study of arbitrary theories in model theory reflects the view of axioms not as 'self-evident' or even 'well-established' fundamental principles but as tools for organising mathematics" ([7], p. 365). Shelah's classification project is taking this aspect to a higher level of generalisation and abstraction given that it is providing general schemes for comparing theories. Consequently, one could ask question criteria used for evaluation the axiom system, for instance. But in the same time, "[W]hat are the connections among the justificatory and explanatory functions of axioms? For example, are there criteria for choosing among first-order, second-order, or infinitary logic? In what sense is second-order logic simply a natural avatar for set theory (Väänänen, 2012)? What principles underlie the development of a taxonomy of mathematics (or at least formal theories)?" (Ibid., p. 365).

Cellucci emphasises the role of the analytic method of demonstration. Demonstration has a heuristic role and the analytic method lead to the analytic notion of demonstration, which "consists, first, in a non-deductive derivation of a hypothesis from a problem, and possibly other data, where the hypothesis is a sufficient condition for the solution of the problem and is probable; then, in a non-deductive derivation of another hypothesis from this hypothesis, and possibly other data, where this other hypothesis is a sufficient condition for the solution of the problem posed by the former hypothesis and is plausible; and so on, *ad infinitum*. The goal of demonstration is to discover hypotheses that are sufficient conditions for the solution of a problem and are plausible" ([18], pp. 34-5). A hypothesis will always guide the observation, experiment and data-finding, as a conduit to interpretation and understanding. Unlike the axiomatic notion of proof, which is "only a sort of super spell checker that is intended to validate a mathematical statement relative to basic axioms - and, by Gödel's incompleteness theorems, fail even to do that" ([18], pp. 35), the analytic option represents a basic tool for acquiring mathematical knowledge.

In this text, I will also maintain the role of the universality problem as a type (that will be gradually configured along the development of the text) of hypothesis that guides. Such an approach is connected to the fact that universality is still an open problem in both set theory and model theory.

Proof and independence results

A further aspect related to the limits of the axiomatisation method involves independence phenomena in set theory. There is a difference between consistency results and independence results offered by forcing. The consistency (strength) proofs offers more than independence, given that some of them require large cardinals. Forcing, on the other hand, is used to show that one cannot prove a theorem. It is more 'fluid' in its uses, even when large cardinals are used. Of course, one can also use inner models to show that large cardinals are necessary and to get equiconsisteny results.

Gödel's results from 1938 ([78]) and 1939 constitute the first part of the (independency) solution of the continuum hypothesis (CH). By using what is now called an inner model of set theory (in relation to the typical set theory model V), he showed that both the axiom of choice and the CH hold in this model. After this relative consistency proof, Cohen offered in 1963 ([25]) the second half of the solution, introducing the forcing method and what is now considered to be an outer model of V, the forcing extension $V[G]$, obtained by adjoining a V-generic filter G over a partial order $\mathbb{P} \in V$.

Writing about Cohen's result, Church made the following remarks: "[T]he feeling that there is an absolute realm of sets, somehow determined in spite of the non-existence of a complete axiomatic characterization, receives more of a blow from the solution (better, the unsolving) of the continuum problem than from the famous Gödel incompleteness theorems. It is not a question of realism (miscalled "Platonism") versus either conceptualism or nominalism, but if one chooses realism, whether there can be a "genetic" realism without axiomatic specification. The Gödel-Cohen results and subsequent extensions of them have the consequence that there is not one set theory but many, with the difference arising in connection with a problem which intuition still seems to tell us must "really" have only one true solution" ([24], p. 18).

Although he mentioned that he prefers proofs in ZFC to independence results, Shelah was for a long time keen to find additional methods of independence in addition to forcing and large cardinals and consistency strengths methods or prove the uniqueness of such methods, according to one of his "dreams" exposed in [195]. To that end, he developed *dividing lines* in model theory. His aim though is to find those proofs and techniques that offer a (or the most) general method. Hence his interest, with all complex and fundamental developments that come from that, for forcing. The forcing framework "in its strong form tell us, in essence, that all universes are equally valid, and hence we should, in fact, be interested in extreme universes. ... But in the moderate sense, this position is quite complementary to the ZFC position: One approach gives the negative results for the other, so being really interested in one forces you to have some interest in the other" ([196], p. 3). Furthermore, it also "works for us like a sieve: when we have a myriad of problems in some directions and we have tried to prove independence (and many times this results in discarding most of them), we are left with strong candidates for theorems of ZFC ([195], p. 219).

So forcing involves the notion of extension, in the form of transitive models. But generally speaking, the major obstacle to forcing for various mathematicians is probably the Platonic character attached to the set-theoretic framework. Nonetheless, one should not forget that the language of set theory is made up of a single relation, $\in$, and that different forcing techniques evolve around a complex formal apparatus and not in describing some *real* universes of sets.

Limits of formal proof In the same context, there are also discussions regarding the limits of formal proofs as embodying the general idea of a mathematical proof.

As pointed out by Wilf, "[I]n many branches of pure mathematics it can be surprisingly hard to recognize when a question has, in fact, been answered. A clearcut proof of a theorem or the discovery of a counterexample leaves no doubt in the reader's mind that a solution has been found. But when an "explicit solution" to a problem is given, it may happen that more work is needed to evaluate that "solution," in a particular case, than exhaustively to examine all of the possibilities directly from the original formulation of the problem. In such a situation, other things being equal, we may justifiably question whether the problem has in fact been solved" ([228], p. 289). He uses the case of combinatorics.

As with many other aspects then in the mathematical practice, one must be aware of the context. Otherwise, one might be in danger of creating sterile generalisation or even, when aspects external to the actual mathematical practice are involved, wrong ones.

Rav emphasises the distinction between proof as "a conceptual proof of customary mathematical discourse, having an irreducible semantic content" and "derivation, which is a syntactic object of some formal system" ([171], p. 11), in other words, between a formal and informal model of proof. He maintains that the informal (conceptual) proofs present some features that are not captured by the model of the formal, derivative ones. The first is the semantic content defining the conceptual, informal proof. Although ordinary mathematical proofs (the informal ones) employ deductive reasoning, they cannot "be rendered intelligible by attending only to their deductive components" ([171], p. 12). So they can be used "to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (Ibid., p. 13). They determine the development of new mathematical tools, methods, other concepts. As such, they have greater epistemological significance, they are "the bearers of mathematical knowledge", while theorems being "in a sense just tags, labels for proofs, summaries of information, headlines of news, editorial devices": "[T]he whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systematisation of results - the entire mathematical know-how is embedded in proofs" (Ibid., p. 20).

3.3 Extension

> ”Tant qu’on a de nouveaux elements a introduire, on doit craindre d’avoir a recommencer tout son travail; or il n’arrivera jamais qu’on n’ait plus de nouveaux elements a introduire (...) Imaginons au contraire qu’on veuille classer les points de l’espace et que l’on distingue ceux qui peuvent être définis en un nombre fini de mots et ceux qui ne le peuvent pas. Parmi les phrases possibles, il y en aura qui feront allusion à la collection tout entière, c’est-à-dire à l’espace, ou à des parties de l’espace. Quand nous introduirons de nouveaux points dans l’espace, ces phrases changeront de sens, elles ne définiront plus le même point; ou bien elles perdront toute espèce de sens; ou encore elles acquerront un sens alors qu’elles n’en avaient pas auparavant. Et alors des points qui n’étaient pas définissables deviendront susceptibles d’être définis; d’autres qui l’étaient cesseront de l’ètre. Ils devront passer d’une catégorie dans une autre” ”. (Poincaré, when dealing with infinite sets, [162], pp. 463-464))

The idea of *extension* is present when we point out the fact that first-order logic was gradually developed as a powerful language for expressing formal mathematics. In the context of this text, determined by the notion of universality, *extension* is closely related to the part-whole distinction transposed in the context of the mathematics of infinity (in particular set theory), i.e. set-subset, model-submodel, but we can also refer to the extension of the different (formal) systems, for instance. By the Löwenhein-Skolem theorems, if a first-order theory has an infinite model, then it has a model of every infinite cardinality, so the theory cannot determine the size of its models. We can therefore distinguish among different types of contextual change and the epistemological consequences of these movements/shifts. Examples of methods for producing such extensions include forcing, large cardinal axioms, the concept of limit, forking.

3.3.1 Generalisation

Generally speaking, the extensions of different formal systems (like the number systems) respond to the necessity of accounting for a greater variety of problems and phenomena. So if we want to find solutions to equations in higher degrees, to do trigonometry or take algorithms, for instance, we need irrational numbers.

An extension of the rational number system to a larger field will be represented by the real numbers. These elementary examples represent a certain idealisation. In Gödel's view, "[W]ithout idealizations nothing remains: there would be no mathematics at all, except the part about small numbers. It is arbitrary to stop anywhere along the path of more and more idealizations. We move from intuitionistic to classical mathematics and then to set theory, with decreasing certainty. The increasing degree of uncertainty begins [at the region] between classical mathematics and set theory. Only as mathematics is developed more and more, the overall certainty goes up. The relative degrees remain the same" ([224] 7.1.11). A cessation to this process of idealisation represents an act of arbitrariness: "We idealize the integers (a) to the possibility of an infinite totality, and (b) with omissions. In this way we get a new concretely intuitive idea, and then one goes on. There is no doubt in the mind that this idealization - to any extent whatsoever - is at the bottom of classical mathematics. This is even true of Brouwer. Frege and Russell tried to replace this idealization by simpler (logical) idealizations, which, however, are destroyed by the paradoxes. What this idealization - realization of a possibility - means is that we conceive and realize the possibility of a mind which can do it. We recognize possibilities in our minds in the same way as we see objects with our senses" ([224], 7.1.19, p. 220).

This idealisation implies that there is a change in the nature and the role of the mathematical objects. The real numbers, for instance, will not be used directly, or not only, for the purpose of measurement; we need them if we want to mathematically describe *the world* and theoretically reason about it. Although rejecting set theory, Thurston is therefore right when he writes that "as mathematics advances, we incorporate it into our thinking. As our thinking becomes more sophisticated, we generate new mathematical concepts and new mathematical structures: the subject matter of mathematics changes to reflect how we think" ([211], p. 162). But we can go deeper or, as it is, generalise our epistemic stance, given that "for a general theory to give interesting results when specialized to older contexts is strong evidence of its being deep (though certainly not a necessary condition)" ([195], p. 205). According to Shelah's *Generality Thesis*, "If you would like to have general results, you have to use a set-theoretic framework" ([195], p. 205). So if one wants to find results about "ALL structures of some kind (all groups, all manifolds, etc.), then you need to be able to deal with infinite unions or infinite products of sets, which are inherently set-theoretical concepts. Moreover, even if your main interest is in, say, finitely generated groups, you will be drawn into more general ones, e.g., taking some compactification or using infinite products" (Ibid., pp.205-206).

3.3.2 Extension - as expansion

By Gödel's second incompleteness theorem, no formal system can exhaustively describe the whole mathematical universe. In other words, it legitimises the use of different extensions of the ZF axiomatisation. "In this perspective, the point is no longer to actually prove the properties, but rather to calibrate them in a scale of increasingly strong logical axioms "([32], p. 379).

Given two signatures σ_1 and σ_2, with $\sigma_1 \subseteq \sigma_2$, and a σ_2-structure S_2, if we delete the symbols not in σ_1 without deleting any element of S_2, we get the reduct of S_2 to σ_1, i.e., S_1. S_2 is the *expansion* of S_1 to σ_2. If an expansion adds new symbols but no new elements, an *extension* adds new elements without adding new symbols.

To the question of 'How far should one go?', Shelah offers the following suggestion: "The best framework, the best foundation, is the one that governs you least; that is you do not notice its restrictions (except when really necessary, like arriving at a contradiction). ... Would you object to such a proof? Or would you stop at the power set $\mathcal{P}(\mathbb{N})$ of N? First, it seems to me unnatural not to have $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, and second you will not gain much; e.g., the problems of cardinal arithmetic are replaced by relatives even if you consider only cardinalities of sets of reals (on pcf theory ...; also definable sets may give such phenomena).... There is a natural scale of theories, some stronger than ZFC (large cardinals), some weaker (e.g., PA = Peano Arithmetic...). PA already tells us that the universe is infinite, but PA 'stops' after we have all the natural numbers. ZFC goes beyond the natural numbers; in ZFC we can distinguish different infinite cardinalities such as 'countable' and 'uncountable', and we can show that there are infinitely many cardinalities, uncountably many, etc. But there are also set theories stronger than ZFC, which are as high above ZFC as ZFC is above PA, and even higher. ... Even if you feel ZFC assumes too much or too little (and you do not work artificially), you will end up somewhere along this scale, going from PA to the large cardinals" ([195], p. 206).

So if we don't have a proof of contradiction of one theory or a statement is undecidable there, we can move along the scale of theories. Oftentimes, important information implicit or connected to one theory, but also structure, becomes visible when we pass to an extension. In a general and less restrictive (in the sense of axioms and rules) theory, many aspects become more transparent. For example, one could get a deeper understanding of the class of countable models of a first order theory by 'passing' or looking to κ-saturated models of T of cardinality $> \kappa$. One could also look to prove that an 'external' property $P(T)$ (λ) of a first order theory T does not depend on λ by proving it equivalent to an 'internal' syntactical property (the syntactic/semantic aspects will be analysed below). Other logics will bring some light on the elementary case but also on some uncountable models. From models with sufficient large cardinals we can construct models with determinacy or inner-model-theoretic fine structure and vice versa. And while we have relative consistency results and equiconsistencies and even mutual interpretations, we will have no nontrivial bi-interpretations.

3.3.3 Domain extension as reconceptualisation (Manders)

As emphasised by Manders, theoretical understanding represents the primary intellectual goal of mathematical activity. He considers that the epistemological tradition concerning the philosophy of mathematics is centred on problems of reliability, on the question of legitimacy of the non-logical axioms in foundational studies, or in equating mathematics with a juxtaposition of theorems. Understandability is a global feature associated primarily with entire conceptual settings and with what he calls *reconceptualisation* relations between conceptual settings. He takes a conceptual setting to be a structure or a theory. The purpose of this last aspect is the comprehensibility of the original setting, but, incidentaly, it also facilitates proofs of theorems, thus pointing out to its success ([144], p. 195). Given that he refers to "a broad pre-theoretical notion such as mathematical understanding", which could be treated philosophically, he considers adequate to present some important paradigm cases, all the while being aware that the reconceptualisation methods have their limitations and there is not a definitive "good way to decide how many cases make a method" ([144], p. 203).

Overall, he is quite vague with regard to the notion of understanding, and the examples he provides are not enough to lessen the ambiguity he is aware of, but try to dismiss, by mentioning that "understandability is not something essentially non-propositional, the relationships involved are essentially 'logical', in a broad sense"([144], p. 197). Discussing the relationship between conceptual settings, he introduces the notion of *accessibility properties*, endowed with an explanatory role and able to offer metamathematical evidence "having finite or surveyable sets of axioms, effective decidability (now refined in terms of computational complexity) of theories; model completeness and elimination of quantifiers."([144], p. 204). He also makes a distinction between *context internal shifts* and *context changing*, with model completion being an exemplification of the latter, and he maintains that domain extensions "unify concepts, in a technical sense which covers the widely cited advantages of simplification and clarity" ([145], p. 554)

He gives different examples of interaction. The simplification constructions include interpretation of one theory into another and model completion. The so-called *local-global method* for analysing a structure "uses a homogeneous family of simplification constructions to decompose a problem" ([144], p. 207) with each simplification determining a structure or a theory that contains information about one aspect of the original problem. There are local-global methods in commutative algebra ([144], pp. 207-9), valuation-theoretic criteria in algebraic geometry, local-global methods with respect to completions or others in formally real fields ([144], p. 209).

This is a valid approach, but I think that a weakness of this view is that it is ignoring some historical aspects. As Gowers rightly emphasizes, "mathematics carries its history with it" ([73], p. ix). A more appropriate stance, one that would incorporate the historical aspect, could be Thurston's idea that the question is not "How do mathematicians prove theorems?", "not even 'How do mathematicians make progress in mathematics?'", but rather "as a more explicit (and leading) form of the question, I prefer 'How do mathematicians advance human understanding of mathematics?'", given that "what they really want is usually not some collection of "answers" - what they want is understanding" ([211], pp. 161-2). But to go further, Manders seems to ignore, and again in connection to some objective measure related to historical reasons, that "having a good test problem is usually crucial to the advance of mathematics" ([196], p. 8). It is true that reliability "is a necessary feature of knowledge" (194), but if it is the main issue in the theory of knowledge, it leads to "a badly distorted theory of knowledge ([144], p 194). And Thurston is on the same line: "The reliability of mathematics "does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas" ([211], p. 170).

Manders' model extension does not represent context-internal inferences, it has a syntactic meaning: it involves the passage from one formalisation to another, from one theory to another, and it is achieved in the model-completed setting. Set theory is insufficient from his point of view. Domain extensions, he continues, do "not start a completely new intellectual enterprise, abandoning one subject in favor of a new one ... even moving from Euclidean to hyperbolic geometry. Rather, a domain-extension move brings an existing intellectual enterprise forward, realizing general formal conditions, which allow a more systematic understanding of the previous theoretical setting". ([145], p. 561) Furthermore, an epistemological consequence of domain extensions is that they "are truth-destroying, not deductive, truth-preserving inferences" ([145], p. 561). Although he gives some examples, such a statement cannot be maintained, as examples from model theory itself could testify.

3.3.4 Examples - Conceptual extensions in set theory and model theory

Universality is also defined in terms of homogeneous saturation and, in model theory, saturation represents a generalisation of algebraic closure.

Saturation represents a central notion in model theory. Information that is only implicit in a structure can be made manifest by taking a saturated extension of the theory. Countable saturation is a completeness property, analogous to metric completeness. Loosely speaking, a saturated model realises as many complete types (i.e. a consistent sets of equations) as possible. A structure is $\aleph_1$-saturated, for example, if every countable type has a solution. The idea of types over sets can be found to have some roots in the notion of generic point, that is, the notion of a generic point a over a variety X defined over a field K. From a model theoretic point of view, a generic point a of X is a point in the extension field of K: if K' represents the algebraic closure of K, a is a realisation in an elementary extension of K' of a non-forking extension of the types that contains the formula X.

Saturation can be generalised to any cardinal κ: a structure M is κ-saturated if any set of formulas (a type) using $< \kappa$ parameters that is finitely realisable in M is also realised in M. A complete theory T has a saturated model of cardinality κ in two cases: either when $\kappa = \kappa^{<\kappa} \geq |D(T)|$, where D(T) is the set of all complete types in finitely many variables over the empty set, or T is stable in κ.

In conformity with results in model theory, the countable saturated models are universal, but the converse does not hold: a universal model is saturated if it is homogeneous. In homogeneous models, partial elementary maps are just restrictions of automorphisms. Saturation is therefore a stronger property than universality. A countable model is saturated if and only if is universal and homogeneous.

Universal domains - Monster model Another way to approach abstractisation in model theory is the monster model, $\mathfrak{C} = \mathfrak{C}_T$, a saturated enough model of T, or κ-saturated for a κ larger than all parameters or cardinal connected to T. All models of T are $\prec \mathfrak{C}$ and all its submodels have sizes less than the saturation number of $\mathfrak{C}$. Furthermore, given that this monster model is closed under all the normal theoretic operations means there is a context specific aspect of its universality.

The notion of a *universal domain* originated in A. Weil's *Foundations of Algebraic Geometry* (first published in 1946), but was introduced in its modern form in model theory as the *monster model* by S. Shelah in 1978 ([197]). Weil's notion of a universal domain is, in model theoretic terms, an $\aleph_1$-saturated model of the theory of algebraically closed fields. As pointed out by Baldwin, the notion of a universal domain/monster model contains two paradigmatic shifts expressed in the work of Shelah: "[F]or a fixed characteristic p he moves from an arbitrary field to an *algebraically closed* field (context changing). Then he requires the algebraically closed field to have *infinite dimension* (context internal) so it will contain any possible solution to a system of polynomial equations over (a finite extension of) the prime field" ([6], p. 124).

Linear orders There are also a number of interesting extensions regarding the theory of linear orders. For example, we could add the sentence $\forall x \forall y (x < y \rightarrow$

$\exists z(x < z \wedge z < y))$ to get the theory of dense linear orders, or we could instead add the sentence $\forall x \exists y(x < y \wedge \forall z(x < z \rightarrow (z = y \vee y < z)))$ to get the theory of linear orders where every element has a unique successor. We could also add sentences that either assert or deny the existence of a minimum or maximum element.

Non-standard model Non-standard models of mathematics, such as the non-standard model of the natural numbers, $N^\star$, enjoy the property of countable saturation. *Nonstandard elements* are a certain kind of *numbers* that extend the scope of the standard model to models that are larger, but still satisfy the axioms of the standard structure. A non-standard model may include the infinite integers. As emphasised by Tao, "while we identify standard objects x with their nonstandard counterparts $\lim_{n\to\alpha} x$, we do *not* identify standard spaces with their nonstandard counterparts ${}^*X = \prod_{n\to\alpha} X$; in particular, we do not view standard groups as examples of nonstandard groups (except in the finite case), nor do we view standard fields as examples of nonstandard fields, etc. Instead, one can view the nonstandard space ${}^*X$ as a 'completion' of the standard space X, analogous to metric completion" ([210]).

Non-standard models were introduced by Skolem, in a series of papers from 1922 to 1934, for set theory and Peano arithmetic, although today they are less used in the first case. When it is used in set theory, it is usually applied not to countable models, but to models whose ordinals are not well-ordered ([69], p. 4). In the first case, he considered he found an anti-realist approach to uncountable sets, also due to what is now known as *Skolem paradox* (and which is not, actually, a paradox); uncountability is a property that a member of the model might have 'inside' some countable structure, not 'absolutely', in the 'real universe'. Uncountability is a model-dependent, 'non-absolute' property. He then introduced them for Peano arithmetic, but in this case, as Gaifman puts it, "he saw evidence for the impossibility of capturing the intended interpretation by purely deductive methods.... The standard model of natural numbers is the best candidate for an intended interpretation that cannot be captured by a deductive system". ([69], p. 1).

There are various fascinating results regarding the non-standard models of the Peano arithmetic. H. Friedman, for instance ([67]), showed that every countable nonstandard model of arithmetic is isomorphic to a proper initial segment of itself. The elements of every countable saturated model are linearly ordered. The model has an initial segment isomorphic to the standard natural numbers, which is followed by additional element. Its order type is $\omega + (\omega^*) \cdot \eta$, i.e., the natural numbers followed by densely many copies of the integers. But the operations for these models are much more complicated: the arithmetical structure of these models is much more complex than the structure of the rationals. Kirby and Paris use the structure of nonstandard models of arithmetic and combinatorial techniques to prove and improve purely number-theoretic statements(e.g., [104]).

By the compactenss theorem, there are countable nonstandard models of the Peano arithemtic (or indeed of any set of arithmetic statements). The models constructed by the ultraproduct method are uncountable. We can *approach* both the standard and the non-standard models *externally* by placing them in a larger universe, a model of ZFC. Strictly speaking, this other universe belongs to the metatheory. But that does not impede us in adding large cardinal axioms and/or other constructions on these structures.

Abstractisation in model theory

Given a set of first order *laws* (that is, they can be written as first order formulas), we can pick a class K of structures satisfying these laws. Two structures M and N are elementarily equivalent if M and N have the same signature and satisfy the same *laws*. This is an equivalence relation, and all its equivalence classes are the first-order complete classes. But such a division into first-order complete classes "need not correspond to any natural mathematical division between the structures"([89], p. 212), although it does so in a few cases, such as the class of algebraically closed fields, whose complete components are the classes of algebraically closed fields with fixed characteristic.

The division of every first-order class K into a disjoint union of first-order classes represents the first step in the model-theoretic classification of the class K. Tarski studied classification up to elementary equivalence. In his classification program, Shelah took structures that are already elementarily equivalent and classified them up to isomorphism. Due to its high degree of abstraction and levels of formalisation, classification theory represents an extraordinary instrument to investigate specific areas of mathematics and a systematic way of establish and examine relations between fields. But Shelah continued to obtain further generalisations in [186] and [185], and created a more abstract framework for classification theory, a context to study non elementary classes. As such, he introduced the notion of *Abstract Elementary Classes* (AECs) in his overall program of clarification and unification, a class of structures with a strong substructure relation satisfying variants of the axioms Jónsson's introduced in the 1950's. Jónsson's work ([97], [98]) had already influenced Morley and Vaught and their introduction of saturated models in their fundamental paper from 1962 ([155]). Among other things, this last paper has extended the syntactic aspect, from realising formulas to realising types, but in connection with the semantic aspect (see [6]).

But as emphasised by Grossberg, the context of AECs is much more general than that of homogeneous model theory, model theory for $\mathbb{L}_{\omega_0,\omega}$ or even the framework of submodels of a given structure. For stable AEC with amalgamation, the analogue of saturation are limit models ([182], [202], [222]).

3.3.5 Universe, universes in set theory

The standard universe embeds into the non-standard universe and they both exist inside a 'larger', 'external' universe, i.e., the metatheoretical universe of ZFC. This aspect allows us to study mathematical objects/structures on different levels: *internal* and *external*, but in this last case, from two points of view. The *internal* level corresponds to the syntactical aspect. The *external* one implies a distinction between the relation to other models (like the embedding between the standard models into the non-standard one) and the connection (and the possible determinations) to the ZFC universe. Universality will make its 'appearance' when trying to establish the order among the models of a class.

Roughly speaking, a universe ($\mathfrak{U}$) of set theory is a structure provided with a binary relation $\in$ - the membership relation - satisfying the axioms of set theory. Strictly speaking, the membership relation is established between the elements of the collection $\mathfrak{U}$ (sets) and not between the sets and $\mathfrak{U}$[3]. The set theoretic axioms express the properties of the binary relation.

If M is a countable elementary model, it cannot refer to things outside it. But if we *situate* ourselves in the model M, and $A \in M$, the model 'knows' about A, i.e. we get certain information, facts about the objects in M of different kinds:

- Any finite subset of M is a member of M.
- Any countable member of M is a subset of M.
- If $A \in M$, $A \subseteq \omega_1$ and $\delta \in A$ is a limit ordinal, then A is a stationary set in ω_1, and as such, is uncountable. This is a fact used in many fundamental results in set theory.
- Not every set of M must necessarily be a subset of M. This is another fact of crucial importance in set theory.

The existence of an ambient universe, the metauniverse mentioned above, does not pose any conceptual problems. In fact, we can even talk formally and informally about universes or set theories. As pointed out by Shelah, "Cantor understood set theory quite well and understood the Continuum Problem without sticking to a formalization. Alternatively, we may work within a "bare bones set theory", just enough to formalize first order theories, proofs, and, say, having the completeness theorem. We may well agree we are in a universe which is set theory and discuss it without a priori having a common agreement on all its properties" ([195]), p. (214).

[3]Although we do intuitively say that a set X belongs $\mathfrak{U}$.

Among other things, the existence of different set theoretic universes, in addition to certain strong methods (such as forcing), offers an advantage of set theory over many other fields of mathematics due to the fact that one can show that a proof or its refutation may be impossible. This aspect offers the basis for new and valuable conceptualisations, results or new methods. One can thus find what is decidable, for instance, and maybe find new patterns in detecting new problems and questions.

The 'space' of the metatheoretic universe, the level of discourse, represents a condition of possibility, an ambient framework where every *structure* or *order* one imposes represents an *in between, an interval space.*

Philebus The process of determining regions of structure might find a possible analogy in the way Plato refers to the circumscribing of domains in *Philebus.* Plato considers that there are couples of relatives, like 'more or less', which describe the multiple aspects of the *apeiron* (24a3). As such, these couples have at least one purpose, that of circumscribing a domain. The *apeiron* is not reduced to extension, quantity or a material cause (as Aristotle wrongfully maintains). For Plato in *Philebus* (but also in *Republic* 438b, *Statesman* 28434-5, and the *Sophist* 255c12-13), the relatives or the relative contraries ('more and less') which characterizes the *apeiron* are opposed to the things themselves, to the limit (*peras*).

In this capacity, they imply a logical transformation visible in a discursive one: in one of the example, the change in modality from a "higher" or "lower" pitch to a "high" or "low" pitch. But, as A.G. Wersinger emphasizes, the couple high - low cannot be substituted for the large - small one or the cold - warm one: "there is 'something' which allows the distinction in the *apeiron* of various domains. This 'something', translated in the passage from the comparative and relative form to the absolute form of opposites, must be understood. Or, the musical example shows that this passage is tightly correlated to a particular device, the interval (*diastema*)" ([227], p. 351). The essence of the mechanism of the limit is to suspend the conflicts between opposites and to operate substitutions.

Wersinger artfully emphasizes the fact that the act of substituting the *apeiron* by the limit must be conceived of as if it would be accomplished in the same place, hence the reference "their own territory" (*tès hauton chôras*). So the idea is to describe the element which has the role of 'residence' for both the *apeiron* and the quantitative limit, that is, the interval. The answer is determined by the consideration of the progressions offered in *Timaeus*: the geometric progression (*analogia*), corresponding to *summetria* in *Philebus*, and the arithmetic and harmonic progressions, corresponding in *Philebus* to the notion of *sumphônia*. In other words, 'filling' the interval means understanding the fact that the mix is an operation of division: "to fill (*pléróun*) does not mean to fill a void, but to divide an interval" ([227], p. 352). In the circumstances, the knowledgeable one is the one who is able to learn the number of the intervals, their determination and character, in other words, distinguish the modes of harmony (*Philebus*,17c-d). To fill the interval means to produce a measure (*meson*), therefore only the mode of division is different. One can think of measure here as a transfinite number with certain properties.

The mixture (*mikton*) assures something common for the things to be mixed, the lien of the division. What is different is the way of division, of producing limits or units, difference determined by the different kinds of progressions (arithmetic, geometric, harmonic). Passing from one mode to another emphasizes the insufficiency of the previous one: at a certain moment, the mode of unity reveals itself insufficient, leaving an epistemological hole which is not void in an absolute way, but full of the medium which proved to be insufficient ([226], p. 261).

Inside the interval, the *apeiron* becomes a relative limit, that is, a domain, or a difference which delimits a frontier, offering the solution to the problem of assimilating the relative opposites to the absolute ones. Just like the *chôra*, it is not a space in the sense of geometrical extension, nor undetermined matter, nor the great-small dyad in the reading of Aristotle (*Physics* 209b35-210a2), but the liminal interval of the mixture, the locus in which the relative opposites make place to the absolute ones to the end of delimiting a domain in which the mixture can operate its successive modes" ([226], p. 262). That is why Aristotle wrongfully identifies the *apeiron* with *chôra*.

These ideas could be connected to arrangements of ordinal numbers. Cantor himself was aware of and pointed out Plato as one of his sources for his conceptual discoveries (particularly in the *Grundlagen*, see [16]). Or to put everything in another, related, mathematical framework, "my typical universe is quite interesting (even pluralistic): It has long intervals where GCH holds, but others in which it is violated badly, many λ's such that λ^+-Souslin trees exist and many λ's for which every λ^+-Aronszajn is special, and it may have lots of measurables, with a huge cardinal being a marginal case..."([196], p. 5).

3.3.6 The universe as a graph

I mentioned above the notion of interpretability, but there are some results that, although connected, emphasise further aspects. One of them is Krivine's observation that the universe $\mathfrak{U}$ can be regarded as an infinite oriented graph ([113], p.8).

Since the elements of the collection $\mathfrak{U}$ are sets, they could also constitute the base of a model M, with M in $\mathfrak{U}$. Both relations $\in_{\mathfrak{U}}$ and $\in_{\mathrm{M}}$ are interpretations of the binary relation $\in$ of the language of set theory (ZF, usually ZFC), but the interpretation of the binary relation $\in_{\mathrm{M}}$ ($\in_{\mathrm{M}} \subset_{\mathfrak{U}} M \times_{\mathfrak{U}} M$) need not be the same as the interpretation of $\in_{\mathfrak{U}}$ [4]. In this last case, both the model M and its binary relation $\in_{\mathrm{M}} \subset_{\mathfrak{U}} M \times_{\mathfrak{U}} M$ represent a point in the infinite graph $\mathfrak{U}$. A submodel of a model of set theory, viewed as a directed graph, is just an induced subgraph.

[4] Since $(\mathfrak{U}, \in_{\mathfrak{U}})$ is a model of set theory, we can define operations on sets, such as inclusion, product, union, intersection etc.

In [158], Nash-Williams proposes a translation of the Zermelo-Fraenkel axioms (for set theory) into graph-theoretic language. The notion of *set* is replaced with that of *vertex* and the membership relation $\in$ with a relation $\rightarrow$ between pairs of vertices, expressed as *preneighbourhood*. Sets could be thus considered as the vertices of a digraph in which there is a a directed edge from a vertex/set x to a vertex/set y if and only if $x \in y$. The set theoretical terminology is not used any longer to describe the new axiomatically defined system, but it can still provide less precise "words and symbols" and serving as a means of communication. In this "framework of intuitive set theory", a digraph D is "an ordered pair $(V(D), E(D))$, where $V(D)$ is a class and $E(D)$ is a subclass of $V(D) \prod V(D)$, i.e., each element of $E(D)$ is an ordered pair of elements of $V(D)$" ([158], p.748). After translating the axioms in the new language, any digraph Δ which is "extensional, infinitive, regulated, powerful and replacing" (p. 751) is called *congregational* and is a model of ZF set theory. One could add the Axiom of Choice, but Nash-William uses that fact that AC is independent of ZF to show that even independence can be formulated as a theorem of graph theory (asserting the existence of certain kind of digraphs), therefore in a way that avoids the concepts of symbolic logic, and, as such, it could even increase our understanding and provide insight.

And "[W]hilst it might be arguable that an approach based on symbolic logic is more rigorous for the foregoing reason, the aim of a graph-theoretic approach to the proofs of independence theorems etc. would be understandability rather than rigour: when the arguments have been well understood in graph-theoretic language, it should be a fairly mechanical operation to translate them into a form based on symbolic language if that is thought more rigorous" (p. 756). That being said, this kind of translations (of theorems from one language into another), "would presumably have to justify their existence by achieving some substantial progress in either set theory or graph theory" (757).

In [81], Hamkins adopts a graph theoretic view on the models of set theory (described as special acyclic digraphs) and uses universal digraph combinatorics to prove some embedding results.

3.3.7 New axioms

Large cardinal axioms assert the existence of certain *large* cardinals that have strong reflection properties and they provide a strengthening of ZFC. They are not known to be consistent with the ZFC though and, actually, they cannot be consistent with the ZFC. An alternative way to define these large cardinals is via elementary embeddings of the set theoretic universe. These axioms form a linearly ordered hierarchy, with the inaccessible cardinals at the bottom, and they are used to measure the consistency strength of various set theoretic hypotheses.

Definition 3.3.1. A cardinal κ is a strong limit cardinal if $\lambda < \kappa$ implies $2^{\lambda} < \kappa$. A regular strong limit cardinal is called inaccessible.

An important theorem about inaccessible cardinals is the following

Theorem 3.3.2. *Let κ be an inaccessible cardinal. Then V_{κ}, the universe of sets constructed up to level κ, is a model of ZFC.*

In other words, if ZFC could prove that the existence of inaccessibles is consistent, then ZFC could prove its own consistency, which would be contrary to Gödel's Second Incompleteness Theorem.

We have set theoretic principles and they are axiomatised or are embodied in different systems of axioms, like ZFC. There are others (ZF, NBG, MK, NF), and there may appear others (which could settles CH inside). The question as to whether the ZFC axioms embody all the *ordinary set theoretic principles* is still open. One can derive from them all the current mathematics. But, as emphasised by Kunen, "future generations of mathematicians may come to realize some "obviously true" set-theoretic principles which do not follow from ZFC. Conceivably, CH could be then settled using those principles" ([114], p. 1). So extension with regard to set theory also involves discussions regarding the "need" for new axioms, specifically, large cardinals. Many (well known) problems in the more theoretical parts of pure mathematics, such as the Continuum Problem or Suslin's Problem, require new axioms for their solution. But there is already a certain attitude of mistrust towards set theory coming from mathematicians from other fields.

Steel considers that "[T]he old self-evidence requirement on axioms is too subjective, an more importantly, too limiting. (...) The self evidence requirements would block this kind of progress toward a stronger foundation" ([62], p. 422). "In extending ZFC, we are attempting to strengthen this foundation", i.e., the interpretative power of set theory (p. 423). Maddy uses the word 'maximize' in her book *Naturalism in Mathematics*. So to believe that there are measurable cardinals is to seek to naturally interpret all mathematical theories of sets, to the extent that they have natural interpretations, in extensions of ZFC + 'there is a measurable cardinal'. Many set theorists consider that $V = L$ is too restrictive with respect to the interpretative power of the language of set theory and can be translated into any other language of set theory. It actually "prevents us from asking as many questions, since we are then forbidden to ask about the world outside L" ([62], p.423). If we adopt a measurable cardinal, we have $0^{\#}$ "and with that, a much clearer view of L than we get if we are only allowed to look at it "from inside". Furthermore, the proponents of large cardinal axioms emphasise the fact that they seem to decide all 'natural' questions in the language of second order arithmetic. And some metamathematical evidence of this completeness can be found in the fact that using forcing, no sentence in the language of second order arithmetic can be shown independent of existence of arbitrarily large Woodin cardinals.

Feferman, adopting an instrumentalist view, maintains that in "the case of set theory, it is at the next level (over $\mathbb{N}$) that issues of evidence, vagueness, and truth arise" ([62], p.410). The axioms of second-order arithmetic would constitute the right choice for the conception of the structure of 'arbitrary sets of naturals number', but the problem is, in his view, that the meaning of 'arbitrary subset of ℕ' is vague, resisting distinctions such as the 'truth' or 'falsity' of the analytic statements involved. And the same vagueness characterises the CH as well, since, he maintains, despite being able to ascribe some evident properties (impredicative comprehension axiom scheme, for instance) to the notion of 'arbitrary subsets of the set of natural numbers', we cannot fix the object. So "it follows that the conception of the whole of the cumulative hierarchy, i.e., the transfinitely cumulatively iterated power set operation, is even more so inherently vague, and that one cannot in general speak of what is a fact of the matter under that conception ([62], p.405). In a more 'pragmatic instrumentalist' approach, he would contend that the necessity of large axioms represents a philosophical question (hence a large variety of answer), and from this point of view the answer is definitely affirmative, and as such, the question has not much to do with the mathematical practice. But I think that he is right to state that even "if mathematics doesn't convincingly need new axioms, it may need for instrumental and heuristic reasons the work that has been done and continues to be done in higher set theory ([62], p.408).

On the other hand, Maddy, adhering to a naturalistic point of view, maintains that mathematical practice, and set-theoretical practice in particular, is not in need of philosophical justification. "Justification (...) comes from within (...) in (...) terms of what means are most effective for meeting the relevant mathematical ends. Philosophy follows afterwards, as an attempt to understand the practice, not to justify or to criticize it", rather, "it would be more appropriate to ask whether or not some particular axiom (...) would or would not help this particular practice (...) meet one or more of its particular goals" ([62], p.408). The example given, from contemporary set theory, is the assumption of many Woodin cardinals (Ibid., p. 409). There are 'intrinsic' and 'extrinsic' mathematical reasons for adding an axiom to ZFC, she continues. The first involve terms like 'self-evident', 'intuitive', and 'part of the very concept of set'. The 'extrinsic' reasons refer to the consequences of such an axiom, and the justification for its adoption resides with these kind of reasons. But in this context, set theoretic foundations do not meet two of the essential expectations of earlier thinkers: provide ontological explanations and epistemic foundation. If one still adheres to these goals though, basically ignored by the development of mathematics, the foundational axioms would be considered absolutely certain, self-evident truths. Furthermore, an epistemic foundationalist stance could not accept any extrinsic justification. The 'relativism' of naturalism is expressed in the justification for a given mathematical method in terms of the goals imposed by the practice involved, but also in taking no stand in maintaining that certain questions in set theory have a determinate answer, despite their independence (Ibid., p. 418).

Shelah maintains, against a pure Platonic view, that there are "many possible set theories all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality" ([196], p. 211). The large cardinal axioms represent 'natural statements', "as their role in finding a quite linear scale of consistency strength on statements (arising independently of them) shows" ([195], p. 211). But he rejects the stronger beliefs of the 'Californian school of set theory' in this respect. $V = L$ is very helpful for building examples and it offers a coherent framework for an important group of problems in Abelian group theory. The analogy of arriving at large cardinals with ZFC is problematic, he argues: "we arrive at ZFC by considering natural formations of sets (the set of natural numbers, taking Cartesian products and power sets); even the first strongly inaccessible cardinal has no parallel justification. If you go higher up in the large cardinal hierarchy, the justification for their existence decreases, so large cardinal axioms are great semi-axioms but not to be accepted as true" ([195], p. 214).

A semi-axiom, from his point of view, must fulfil some conditions, which are, in fact, contradictory: it must have many consequences, "preferable that it is reasonable and "has positive measure", and "it is preferred that it leads to no contradiction (so lower consistency strength is better)". For instance, $V = L$ is preferable to GCH, having more consequences, but the latter is "more reasonable" and still many consequences ([195], p. 214). The large cardinals are "axioms" only for extremists, their existence is only a 'semi-axiom' ([196], p.4). They do represent natural notions and they are indispensable for our knowledge, more specifically, for the linear order phenomena the imply and their roles in equiconsistency. But there is a 'hole' in this program to assure the linearity of consistency strength for large cardinals, i.e., the supercompact cardinal. Although widely used in consistency proofs, we do not know how to prove the consistency of ZFC + 'there is a supercompact cardinal' from 'low level statements'([195], p. 215).

4

Universality

Synopsis

This chapter contains an introduction to the the problem of universality in set theory and in connection to model theory. It starts by offering the general mathematical framework, in the form of basic definitions that will be further used throughout the text. The next section establishes the relation between set theory and model theory around the problem of universality. We continue by offering an overview of the problem of universality in model theory and ways to tackle it. The last section of this chapter represents an introduction to the universality problem in set theory and presents several methods used in this area to offer solutions.

4.1 Introduction

Universal objects can be found in several mathematical contexts, and there are different frameworks in which to develop a theory of universality (category theory, model theory, set theory), but we use set theory as a framework because it is able "to crystallize some intuitions for which other frameworks would have been too rigid, and to elaborate them with the help of its specific tools, including strong axioms." ([32], p. 379)

In category theory, the definition of a universal object also defines a universal property. And universal properties are central to the way category theory describes structures considered 'canonical'. As such, a universal object is an initial object or a terminal object depending on the context. Given that universal properties define objects uniquely up to a unique isomorphism, one can prove that two objects are isomorphic by showing that they satisfy the same universal property. A fundamental lemma (the Yoneda Lemma) implies that any mathematical object can be characterised by its universal property ([173], p. 62). And this property will express one of the roles played by that object. For example, the category theoretical proof that the tensor product commutes with the direct sum of vector spaces uses only the universal properties of the tensor product and the direct sum constructions in appropriate categories (or mathematical contexts).

Universality is an open problem in both set theory and model theory. The non-universality theorems have both set theoretic and model theoretic assumptions, the first in the form of certain combinatorial statements, to be discussed bellow.

4.2 Preliminaries

$|X|$ is the cardinality of the set X. The relation $X \subseteq Y$ means that X is either a proper subset of Y or equal to Y. For a function f with domain D_f and $A \subset D$ $f \restriction A$ denotes the restriction of f to the set $A \cap D_f$. $[X]^\kappa$ is the set of all subsets of X of cardinality κ. The set $[X]^{\leq\kappa}$ is defined analogously as the set of all subsets of X of cardinality $\leq \kappa$.

Definition 4.2.1.

If X is a set of ordinals, a limit point of X is an ordinal α such that $\alpha = sup(X \cap \alpha)$ or, equivalently, $\alpha = sup(Y)$ for some $Y \subset X$. $\mathrm{Lim}(X)$ is the set of limit points of X.

Definition 4.2.2. Suppose α is a limit ordinal. A subset $X \subset \alpha$ is *bounded* if $\sup(X) < \alpha$, and *unbounded* if $\sup(X) = \alpha$.

A subset $C \subset \alpha$ is *closed* if and only if for all limit $\delta < \alpha$, if $C \cap \delta$ is unbounded in δ, then $\delta \in C$.

C is *club* (or a *club subset*) if and only if C is *closed and unbounded* in α.

Definition 4.2.3. If κ is a regular uncountable cardinal, a set $C \subset \kappa$ is a *club subset* of κ if C is unbounded in κ and if it contains all its limit points $< \kappa$.

A subset $S \subset \kappa$ is *stationary* if for every club C of κ, $S \cap C \neq \varnothing$.

Definition 4.2.4. Let α and β be ordinals and a function $f : \alpha \to \beta$. The function f maps α *cofinally* if and only if $ran(f)$ is unbounded in β.

The *cofinality* of β is the least α such that there is a cofinal map from α into β.

Lemma 4.2.5. *For every limit ordinal α, $cf(\alpha)$ is a regular cardinal.*

An infinite cardinal $\aleph_\alpha$ is regular if $cf(\omega_\alpha) = \omega_\alpha$, and it is singular if $cf(\omega_\alpha) < \omega_\alpha$.

4.3 What is an universal object

A classic model theoretic definition stipulates that a model $\mathfrak{A}$ is said to be countably universal if and only if $\mathfrak{A}$ is countable and every countable model $\mathfrak{B} \equiv \mathfrak{A}$ is elementarily embedded in $\mathfrak{A}$. We can generalise and say that a universal model of size κ of a theory T is a model in which any other model of T of cardinality κ embeds elementarily. This definition differs from that in [21] (p. 297), which states that a model M is κ-universal if it elementarily embeds models of a theory T of cardinality $< \kappa$. In classical model theory, a model M of a theory T is κ-universal if and only if every model N of the theory of cardinality $\leq \kappa$ can embed into M. According to this definition, the size of M is not required to be κ, but it will be all along this text. We will say that M is a universal model if and only if M is $|M|$-universal. From the definition definition of a κ-universal model, it follows that every model of cardinality $\leq \kappa$ embeds into the universal model of size κ.

We say, for example, that $\mathbb{Q}$ is the only countable linear order, up to isomorphism, which contains any other countable linear order and every isomorphism between two finite subsets extends to an automorphism of $\mathbb{Q}$. The proof goes back to Cantor, for whom the extension property is equivalent to saying that the rational numbers represent the unique countable dense linear order with no endpoints, and who constructed the automorphism inductively, by interchanging the domain and the codomain at each step, in what is now called the *back-and-forth method*. Urysohn found an analogue of the method for metric spaces (any countable dense subset of the Urysohn space is countably universal in the class of metric spaces with almost isometric embedding). And in the 1950's, Fraïssé approached the argument in a model theoretic framework.

These aspects could lead to a further distinction, between a *weak* and a *strong* form of universality. As such, a *weak* form of universality implies that given a theory T, a collection $\mathcal{K}$ of models of cardinality λ of the theory, there is a universal model (of the theory T) of cardinality κ, $\kappa > \lambda$, not belonging to the collection $\mathcal{K}$, into which every model of cardinality λ is elementary embedded. The strong form corresponds to the definition mentioned above.

Cofinality offers the possibility of more nuanced generalisations. So universality can also be rephrased in terms of cofinality: it represents the minimal cardinality of a subcollection of elements (or the dominating family) such that every element in the given class is smaller or equal to one of the elements in the subcollection. But although cofinal is the same as unbounded for linearly ordered sets, it is not the same for partially ordered sets.

Given a class of structures, $\mathcal{K}_\lambda$, each of size λ, we define its *universal family* (as the smallest size of a family of structures of $\mathcal{K}_\lambda$ which embeds the rest. The minimal cardinality of the universal family is called the universality number of the class. When the universality number of $\mathcal{K}$ is 1, the unique element of the universal family is called a *a universal model*. The *universal spectrum* for a class of structures $\mathcal{K}$ is the family of cardinals for which $\mathcal{K}$ has a universal model.

The universality spectrum of a theory depends on cardinal arithmetic, so the problem of determining the universal spectrum of a theory could be phrased as: "under which cardinal arithmetic assumptions *can* a given theory (class) possess a universal model in a given cardinality λ?" ([109], p.875). As emphasized by M. Kojman and S. Shelah in [110], the universal model problem can also be stated as a question about a partial order: the class $\mathcal{K}$ of structures together with a class of embeddings, such that for two models $A, B \in \mathcal{K}, A \leq B$ if there is an allowed mapping of A into B (p. 57).

Universal objects appear in several mathematical contexts. And there are also different frameworks in which to develop a theory of universality: category theory, set theory, model theory. There are many examples in almost all areas of mathematics. They include: $\mathbb{Q}$, the rational numbers (mentioned above) considered as a linear order embedding every countable linear order; the random graph embedding every countable graph; $[0,1]^\kappa$ containing a closed copy of every compact space of weight κ. The Cantor set is a universal compact metrizable space (Alexandroff and Hausdorff). A universal Turing machine constitutes a universal object, given that it could theoretically and conveniently simulate any other Turin machine. The Rado graph is a universal countable graph; the Urysohn universal space is a universal separable metric space; the Hilbert Cube $[0,1]^\omega$ is a universal Polish space. In the case of an injective homomorphism, we would refer to a subgraph, and every Rado graph contains every countable graph as a subgraph. But when studying universality in graph theory, a more interesting result would be to refer to a universal countable graph containing every countable graph as an *induced* subgraph.

4.4 A context - the relation between model theory and set theory

Universality (determining the universality spectrum for a theory) but also saturation and other mathematical phenomena are characterised by a rich interaction between set theory and model theory.

Universality in model theory is sensitive to set theory, extensions of its systems of axioms, and other set theoretic assumptions taking the form of some combinatorial statements. According to a classical result from model theory ([21]), if GCH holds, then every countable first order theory admits a universal model in every uncountable cardinal. Given the definition of the universality number/spectrum of a theory T mentioned above, it follows that under GCH, this number for a countable T and an uncountable λ is 1.

Many approaches to universality (including non-universality theorems) have taken into consideration the connection between various results in model theory and set theory. The results appearing in the works of Kojman and Shelah ([109]), Kojman ([108]), Shelah ([188], [191], [199]), Džamonja and Shelah ([49], [50]) involve the thesis of maintaining a relation between the complexity of a theory and its amenability to the existence of universal models.

Using model theoretic properties, the existence of a universal model is ensured in the following cases: if it is ω-stable, any theory (complete first order) with an infinite model will have a universal model in every infinite cardinal, starting with $\aleph_0$; a superstable theory will have universal models in every cardinality greater or equal to the continuum; every stable theory has universal models in every cardinal κ satisfying $\kappa^{\aleph_0} = \kappa$.

In set theory, the results are determined and complicated by cardinal arithmetic and (infinite) combinatorics. But it also represents a fundamental context of analysis. As pointed out by Blass, taking a theory T, "[T]he simplicity and efficiency of ZFC and the fact that T can be interpreted in it (i.e., that all the concepts of T have set-theoretic definitions which make all the axioms of T set-theoretically provable) have, as far as I can see, two main uses. One is philosophical (...) The other is in proofs of consistency and independence. To show that some problem, say in topology, can't be decided in current mathematics means to show it's independent of T. So you'd want to construct lots of models of T to get lots of independence results. But models of T are terribly complicated objects. So instead we construct models of ZFC, which are not so bad, and we rely on the interpretation to convert them into models of T. And usually we don't mention T at all and just identify ZFC with "current mathematics" via the interpretation" ([13]).

In [6] Baldwin analyses what he calls a paradigm shift in model theory that, he maintains, had taken place in two phases and which determines the general framework for mathematics itself: the first involved a shift from identifying higher logic with this general framework (or the Russell-Hilbert-Gödel conception) to a focus on first-order theories with the purpose of studying distinct areas of mathematics (describing the Robinson-Tarski approach), from a study of logics to the study of theories, from a focus on one structure to a collection of distinct or non-isomorphic structures modelling a set of axioms. As a result, one starts by fixing a vocabulary with a fixed set of symbols relevant to a certain area of study (a theory) and quantifies over individuals instead of predicate variables of all *arities* and orders belonging to a logic. An example is Henkin's proof of the completeness theorem, allowing a focus on a particular area of mathematics and which is formalised in a specific vocabulary.

In 1949, Henkin ([84], containing results from his 1947 dissertation) proves the completeness theorem (the strong version) by expanding a given vocabulary only by constants. Gödel had used an extension of the vocabulary by additional relations, thus moving outside the original context; although quantification is restricted to individuals, there are still predicate symbols of all orders, so it was meant as a framework for all mathematics. It is in this context that Robinson introduced the concept of model completion [174] and M. Morley [154] showed that a countable theory is $\aleph_1$-categorical if and only if it is κ-categorical for every uncountable cardinal κ. Another example is to take the natural numbers, which represent a structure. But the focus of analysis is now the collection of non-isomorphic structures satisfying a set of axioms, of algebraically closed fields for instance. And there can be models of every infinite cardinality.

The second phase mentioned by Baldwin is determined by Shelah's introduction of the classification theory with the search for *dividing lines* among general families of structures and a stability hierarchy for theories. Many areas of mathematics could be formalised by first order theories that behave well in the stability classification. A dividing line is a method or a mathematical technique, constitutes a choice of classification, and it depends on a test problem. The stability hierarchy is one test problem, but other candidates include saturation of ultrapowers and the Keisler order and universality. They all represent ways of organising the first order theories. A consequence of this is that they could also provide contexts for establishing connections across different areas of mathematics (due to the formalisation of different topics). The Keisler order and the universality spectrum determine a refining of the stability hierarchy (which contains only a finite number of classes) to infinitely many of them. But Malliaris and Shelah had showed that there is a maximal number of classes ($2^{\aleph_0}$) for the Keisler order ([137]).

Shelah also extended (1974) Morley's categoricity theorem to uncountable languages by showing that if the language has cardinality κ and a theory is categorical in some uncountable cardinal $\geq \kappa$, then it is categorical in all cardinalities greater than κ.

Set theory is fundamental in the development of infinitary model theory, and the use of dividing lines in this context materialised in the study of abstract elementary classes, involving a semantic approach to infinitary logic. Shelah's research showed a deep connection between model theory and set theory regarding certain properties of first-order theories, including the categoricty in power for infinitary logic. Furthermore, many model theoretic properties and constructions can be carried out in ZFC by a restriction to theories that behave well in the stability classification.

According to Pillay, certain developments in model theory - stability theory and what one might call a "generalised stability theory" (see Shelah's generalisation of stability theory techniques to simple theories) with the "machinery" of independence", dimension theory, orthogonality - offered a certain "geometric sensibility" that complements the "set-theoretic sensibility" and its role in the foundations of mathematics ([15], p. 184). That had consequences for the set theoretically more complicated objects of mathematics (ex: locally compact fields), and "once their set-theoretic genesis is forgotten, we have access to, via quantifier-elimination and decidability theorems" (Ibid., p. 186). When working with a κ-stable theory T, we can eliminate any cardinal exponentiation assumption.

There are actually two aspects of research in set theory, according to A. Kechris, and they are often quite interrelated : there is an internal or foundational aspect, and then there is an external or interactive one. The first "aims at providing a foundation for a comprehensive and satisfactory theory of sets" ([15], p. 190), while the second uses set theoretic concepts, methods and techniques in connection with other areas of mathematics. Mutatis mutandis (and in the same article), A. Pillay makes the case for the same distinction with regard to model theory. He mentions homogeneous-universal or saturated models (initially developed by Morley and Vaught) as a focus point for an inward-looking approach (Ibid., p. 183), but since the universal objects were present in set theory from its beginnings, the universality problem represents an example of the interaction between internal and external perspectives with significant consequences for all parties involved.

In fact, in its early days, many of the model theory main problems were quite set-theoretically, since different questions were dependent on specific models of set theory (e.g. two-cardinal theorems, generalised omitting types). The interplay between the two areas can be seen in Morley's categoricity theorem, the Ehrenfeucht-Mostowski models, Vaught's conjecture, Chang's conjecture, etc. Nowadays, the connection can be spotted in the study of abstract elementary classes, for example, and new uses of large cardinals and forcing. Shelah's research in model theory, particularly classification theory, is characterised by deep set-theoretical preoccupations and vice-versa. Concurrently, model theoretic methods are used to extend models of set theory while leaving specified sets fixed. Specifically, every countable model μ of ZF has: (i) an extension leaving every set in μ fixed, and (ii) for each regular cardinal κ in M, an extension enlarging κ but leaving each cardinal less than a fixed ([102]).

That being said, there is a certain kind of limitation regarding this connection between model theory and set theory. Forcing represents an example. Since the introduction of forcing by Cohen in 1963, there were some attempts to modify Cohen forcing into a model theoretic framework or construction. Reyes, for instance, connected forcing with homogeneous universal models using Baire category approach. He would obtain the notion of infinite forcing, further developed by Robinson (in 1969-1979). The last also developed finite forcing, known as model-theoretic forcing (see [129], p. 159). Another limitation involves a certain reserve on the part of model theorists towards cardinal arithmetic and the higher levels in the infinity hierarchy.

4.5 Universality and model theory

One can approach the problem from the point of view of model theory, and more specifically, classification theory, with a focus on first order theories.

Fraïssé ([65]) and Jónsson ([98] and [97]) approached the construction of universal domains from a purely semantic or algebraic point of view. A class K of countably many finite models closed under the natural relations of amalgamation, joint embedding, and substructure have a countable structure M which is both ultrahomogeneous and universal. In an ultrahomogeneaous structure, any two isomorphic finite structures are automorphic in M. A universal structure is one in which any finite member of K can be embedded in M. But Jónsson goes beyond the countable realm and analyses κ-universal structures for arbitrary κ. Furthermore, he described a collection of axioms to be satisfied by a class of models that guaranteed the existence of a homogeneous-universal model. The substructure relation constituted an essential part in this approach. Morley and Vaught ([155] replaced the substructure notion by *elementary submodel* and they developed the central notion of saturated model. Then Shelah ([186], [185]) generalized this approach in two ways: firstly, by considering the amalgamation property a constraint rather than a basic theorem (similar to a practice in universal algebra). Secondly, after making the substructure notion a *free variable*, he introduced the notion of *abstract elementary class*, i.e., a class of structures with a strong substructre relation satisfying variations of Jónsson's axioms.

4.5.1 Definitions and proof notions

We will start by introducing some definitions. A *signature* is a set of constant, function and relation symbols. We also choose a *language* L, in which we can talk about the distinguished functions, relations, and elements, and usually that is the language of first order logic. The symbol for equality is taken as part of the language by default and it is interpreted as identity. The cardinality of L, $\|L\|$, is the cardinal $\geq$ the number of symbols in the signature of L and is the same as the size of the set of essentially different well-formed formulas in the signature. Two formulas are not (essentially) different if and only if one comes from the other by a one-to-one change of variables.

A structure $\mathcal{M}$ in model theory is a triple $(\mathcal{L}, M, I)$, where $\mathcal{L}$ is the signature, M is a non-empty set representing the domain of the structure, also written as $\|M\|$, and I is the interpretation of the signature on the domain M. For example, the language of graphs is $\mathcal{L} = R$ where R is a binary relation symbol. We also say that given a signature, a model for a collection of axioms stated in the first-order language is a mathematical structure for which the axioms, properly interpreted, are true: $\mathcal{M}$ is a model of T, $\mathcal{M} \models T$, if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. In other words, the semantics of the first order logic is defined in a structure.

The axioms could be understood as properties of the relations. A $\mathcal{L}$-theory T is a set of $\mathcal{L}$-sentences. By T we are referring to a first order theory with infinite models, usually countable when we situate ourselves in a model-theoretic framework. Although we focus on first order theories, the properties of models that we consider - saturated, homogeneous, universal - will be second order.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an *elementary class* if there is a $\mathcal{L}$-theory T such that $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$. Let $\mathcal{L}_1 \supset \mathcal{L}$. If $\mathcal{M}_1$ is an $\mathcal{L}_1$-structure, then by ignoring the interpretations of the symbols in $\mathcal{L}_1 \backslash \mathcal{L}$ we get an $\mathcal{L}$-structure $\mathcal{M}$. We call $\mathcal{M}$ a *reduct* of $\mathcal{M}_1$ and $\mathcal{M}_1$ an *expansion* of $\mathcal{M}$ ([146], ex. 1.4.15). These last definitions permit the interpretation of (very) complicated structures in (seemingly) simpler ones. For instance, any structure in a countable language can be interpreted in a graph (see 5.4.2).

Usually, in model theory, *model* and *structure* are used interchangeably, but there is a distinction. Given the structures of a certain language $\mathcal{L}$, the models are the ones satisfying a particular theory. In this case, structure is a more general term, used when the properties/axioms are not completely known or specified. A concrete graph, for instance, is a model for the axioms of graph theory. Models for set theory, although essential from many points of view, are another example. We can take different classes of models, such as: the class of models of a certain cardinality for a first order theory ordered by elementary embeddings, the class of models of a given cardinality of an abstract elementary class quasi-ordered by the inherited order, the class of models for which we consider another cardinal invariant, like topological weight. A substructure is said to be proper if its domain is a proper subset of the domain of the extension, in which case we write $M < N$. The relation $M \leq N \leftrightarrow M < N \bigvee M = N$ defines a partial order.

Type Theories describe structures, but types describe the elements in a structure. The *type* is a central model-theoretic notion allowing a deeper analysis of the models of a theory. The number of types is related to the number of models of a certain theory T. If T has 'few' types, for instance, then the class of models of T contains a smallest model (uniquely defined) that can be elementarily embedded into any other model in the class. It extends the notion of a theory to include formulas, in addition to sentences. A type is a consistent set of sentences, while a complete type is a maximal consistent set of sentences.

Definition 4.5.1. Let M be an L-structure and $A \subseteq M$. L_A represents the language obtained by adding to L all $a \in A$ as constant symbols. By interpreting the new symbols, M can be seen as a L_A-structure. Let $Th_A(M)$ be the set of all L_A-sentences true in M.

Let p be the set of L_A-formulas in free variables $x_1, ...x_n$. p is an *n-type* if $p \cup Th_A(M)$ is satisfiable. p is a *complete n-type* if for all L_A-formulas φ with free variables either $\varphi \in p$ or $\neg\varphi \in p$. $D_n(A)$ is the set of all complete types.

Let p be an n-type over A, and $\bar{a} \in M$ a sequence $\bar{a} = \langle a_1, ..., a_n \rangle$. We say that $\bar{a}$ realises p if $M \models \varphi(\bar{a})$ for all $\varphi \in p$. If p is not realised in M, we say that M omits p.

The type of $\bar{a}$ over A $(tp(\bar{a})/A)$ represents a complete description of how $\bar{a}$ relates to the subset A. The complete types represent the building blocks of a model, and the more types there are for a theory, the more options there are to construct models for that theory. Not all types we can construct represent actual elements of a model M, they can be realised in some elementary extension of M. Consequently, trying to find the types from M determines questions such as *saturation* of models.

Forking - Non-forking

Various proofs in model theory, classification theory specifically, involve the notions of *forking* and *non-forking.* In stability theory and its generalizations, they are used to make precise the idea of *generic extensions* involving types and of independence. Let A be a set o parameters. A formula which forks over a set of parameters A is considered to be 'non-generic'. If p is a complete type over a set A, the non-forking extensions of p are those extensions containing the formula forking over A, and they are the 'generic extensions' of the type p. In other words, these extensions of p involve a larger set of parameters that doesn't add new constraints to the set of its solutions.

Definition 4.5.2. If $tp(\bar{a}; B)$ is not free over A, then $\bar{a}$ must satisfy more relations over B than it does over A.

A type p forks over A if there is a finite conjunction of formulas from p which forks over A. It follows that if a partial type p does not fork over A then there is a global type $p' \in S(\mathbb{M})$ extending p that does not fork over A. Intuitively speaking, the ternary relation involved by non-forking means that A is free from B relative to C.

With regard to the independence aspect involved by the notion of forking, one says that the tuples a and b are independent over a set A if tp(a/bA) does not fork over A. More generally, we say that

Definition 4.5.3. For each type $p \in S(M)$ and each $N \succ M$, p has a unique extension p' to $S(N)$ if every realisation of p' is independent from N over M. This distinguished extension is called *non-forking* (or generic).

The notion of independence offered by non-forking (in the context of first-order theories) generalizes to a great extent the notion of linear independence (in the special case of vector spaces) and algebraic independence (in the case of algebraically closed fields) (see Van der Waerden ([223]) by allowing one to study structures with a family of dimensions. It is not a strict generalisation, given that it is stronger in some ways and weaker in others, but it retains a crucial characteristic, i.e., the ability to assign a dimension to each member of certain classes of models. The notion of a *non-forking* extension of a type was introduced by Shelah in his classification program ([197]). An essential property of forking for simple theory is the *Kim-Pillay independence theorem*), an example of amalgamation property (3-amalgamation). Non-forking itself is referred to by Shelah as 'free amalgamation', with amalgamation playing an essential role in proofs of consistent existence of universal objects.

Another central question that arises in this context is: how many non-forking extensions can there be? To make the question more precise, given a complete first-order theory T, one can associate to it its *non-forking spectrum*, a function $f_{\mathrm{T}}(\kappa, \lambda)$ of cardinals giving the supremum of the possible numbers of types over a model of cardinality λ that do not fork over a submodel of size κ. The function is defined as follows: $f_{\mathrm{T}}(\kappa, \lambda) = \sup\{S^{nf}(M, N) | M \preceq N \vDash T, |M| \leq \kappa, |N| \leq \lambda\}$, where $S^{nf}(A, B) = \{p \in S_1(A) |\ p$ does not fork over $B\}$. The choice of 1-types over n-types is essential since the value may depend on the arity [23].

Given that the stability function of a theory is defined as $f_T(\kappa) = \sup\{S(M) | M \vDash T, |M| = \kappa\}$, the question above can be considered as a generalisation of the classical question (intensively studied by Keisler and Shelah) of 'how many types can a theory have over a model?': in other words: $f_{\mathrm{T}}(\kappa, \kappa) = f_{\mathrm{T}}(\kappa)$.

Structure

As emphasised by Hodges, in "a well-behaved class, the larger structures are built up from the smaller ones by adding pieces of certain fixed types; the structures are determined up to isomorphism by how many pieces of each type were added as they were built up from the smallest structures (...) then one ought not to keep finding essentially new phenomena as one moves from the small structures in K to the large ones" ([89], p. 216). In such a class, the constructed structure is similar to the smaller ones but larger in certain directions.

The classification of finite simple groups is an example. Another classic one is represented by Hausdorff's results regarding the countable scattered linear orderings (in [63]). A linear order is *scattered* if it does not embed the order type of the rationals. He shows that such an order can be built from simpler ones through ordered sums indexed by ordinals. It should be emphasised that the order in the sum is essential. A corollary is that in a well-structured class K, we can define an equivalence relation that splits the class into families that might be distinguished by parameters, such as "structure B can be reached from structure A through a chain of intermediate structures in K, by adding or removing pieces" (Ibid.). This represents an ideal case though, rather an heuristic idea, since we can always find structure theories that constitute counterexamples to such a process. Determining each structure from the family and up to isomorphism by a set of parameters that measure the sizes of certain parts of the structure. And this is a pattern to be found in several model theories approaches.

Within the framework of model theory, set theory has a particular 'place' since, from a set-theoretic point of view, the model of any given theory T is a set-theoretic structure, i.e., they are the members of a background set theoretic universe. It is particularly obvious when we consider the theory T to be ZFC itself. In such a case, we should be aware of the distinction between a model of ZFC and the background universe in which this model 'resides', as any other model of any other formal theory. One usually refers to the 'monster model' in the practice of model theory. Working in a monster model assures that all types over smaller parameter sets are realised. To put it another way, models of a formal theory are members of a universe of sets which, in turn, can be seen as being itself a model, of course not of a formal theory, but rather of the informal set theory that one presupposes when doing mathematics. All mathematical objects can be represented as sets and, therefore, every model of ZF should include its own version of all these objects. Now strictly speaking, no model of ZF is ever going to be described or constructed within the framework of ZF, due to essential obstructions. So we will have to maintain a certain degree of awareness when migrating between a mathematical level, the level of objects and a metamathematical level making possible the discourse about these objects.

4.5.2 Definable sets

Mathematical logic contains some threads that can be found in different forms in all its developments: definability, categoricity (starting with the works of Łos and Morley), stability.

Definition 4.5.4. A theory T is $\aleph_0$-categorical if and only if there are only finitely many formulas in n free variable up to T-equivalence.

In other words, the countably categorical theories are those theories that are complete and have few formulas (finitely many in n free variables for each n). This is equivalent to

Lemma 4.5.5. *A σ-structure M having universe $|M|$ is $\aleph_0$-categorical iff for each $n \in \mathbb{N}$, only finitely many subsets of $|M|^n$ are σ-definable.*

Categoricity is central in Shelah's development of stability theory and, given the nature of Shelah's work, it also interacts with set theory. Zilber analyses categoricty in connection with the pseudo-analytic structures. Stability, albeit a central concept in model theory, has some related preceding elements in Banach space theory, particularly in the work of Grothendieck, Krivine and others.

Väänänen described definability as a "wormhole that somehow travels through all of mathematical logic. (...) (its appearances in disguised form sometimes, linking first order, second and higher order logic, linking model theory, proof theory, set theory and computability theory, appearing in theorems both classic and cutting edge) is an important reminder of what unites logic, beyond and together with the shifting dynamics of its internal movements and external interactions" ([88], pp. XIII-XIV).

According to Pillay, understanding a structure means understanding the category of definable sets: "[T]here is now a reasonably coherent sense of what it means to understand a structure: it means understanding the category of definable sets (including quotients by definable equivalence relations). Generalized stability theory gives a host of concepts and tools which inform this analysis: dimension theory (the assignment of meaningful ordinal-valued dimensions to definable sets, invariant under definable bijection), orthogonality, geometries, definable groups and homogeneous spaces. As mentioned earlier, the contexts in which such tools are applicable tend to rule out Gödel undecidability phenomena. Interpretability is a key (even characteristic) notion, and in a tautological sense the business of "pure" model theory becomes the classification of first order theories up to bi-interpretability ([15]).

Descriptive set theory, or definability theory of the continuum, is the study of definable (Borel, projective, etc.) sets and functions on Polish spaces. Given that a structure can be defined as the set with special subsets having names, a first-order structure is a structure having names for those sets that are definable by a first-order formula. In general, most subsets belonging to a structure's universe are not definable.

Furthermore, in first-order logic we cannot say that two definable subsets have the same size. We can do that in second-order logic, but in this logic we do not have compactness, nor the Löwenstein-Skolem theorem, and we cannot provide a set of rules from which to deduce all truths of second-order logic. Model completeness reduces the definable relations of a theory to those defined using $\exists$ formulas, which are more comprehensible and of lesser complexity. Each definable relation of M is defined by a formula, and the formula shows how the relation is built up from simpler relations. Two first order formulas are equivalent if they define the same relation. As a consequence, the family of definable relations might not be as rich as the language used to describe them. Every definable subset is a finite union of intervals. For example, Tarski's theorem states that every definable subset of the reals is a finite union of intervals.

But there is no uniform way of determining the various first order definable relations on a model M. A few of them point to minimality, while arithmetic is connected to the existence of many definable subsets. In the case of stable theories, all complete types are definable. If T is a ω-stable theory, there are saturated models of T of cardinality κ for each regular cardinal κ.

A theory T is called *strongly minimal* if every definable set in the model with parameters is either finite or cofinite. Intuitively speaking, the Morley rank determines how many infinite collections of pairwise incomparable definable sets are in a model.

The model M itself can be identified with the subset of its realised types, i.e. $\{tp(a/M) : a \in M\} \subset S_1(M)$. It follows that $S_1(M)$ is the topological closure of this set. So one can think of the space of types as as 'compactification of the model'.

The countably categorical theories are those theories that are complete and have few formulas (finitely many in n free variables for each n). So a definition of an $\aleph_0$-categorical theory can also be given in terms of definable sets: for each $n \in \mathbb{N}$, only finitely many submodels of M^n are definable. This definition is equivalent to another one: there are finitely many formulas in n free variables up to T-equivalence for each n.

A subset $X \subset M$ is *type-definable* if it is a possibly infinite conjunction of definable sets. Given A the set of parameters, definability of n-types is related to the cardinality of $S^n(A)$. If a type $p \in S(M)$ is not definable, it has 2 heirs.

Many types imply many models, few types imply few models. Few models means that the theory admits a structure theory, i.e., it admits a freeness relation satisfying the independence axioms and showing this through the notion of 'definability of types'. If a theory has few types, then every type is definable. Roughly speaking, if there are few possibilities for a given phenomenon, then each possibility is definable (Beth theorem). If T is stable, the number of complete φ-types over A is bounded by the number of definitions over A. So, for each A, $|S_\varphi(A)| \leq |A|$.

4.5.3 Models and extensions

An end-extension of a model of set theory $\langle M, \in^M \rangle$ is another model $\langle N, \in^N \rangle$, such that the first is an $\in$-initial substructure of the second, i.e., $M \subseteq N$ and $\in^M = \in^N \restriction M$. The extension model does not add new elements to the original set: if $a \in^N b \in M$, then $a \in M$ and hence $a \in^M b$.

Theorem 4.5.6 (Keisler-Morley)**.** *Every countable model of ZF has an elementary end-extension.*

The result cannot be generalised to all uncountable models. If κ is the least inaccessible cardinal, then $\langle V_\kappa, \in \rangle$ has no elementary end-extension.

Theorem 4.5.7 (Barwise extension theorem)**.** *Every countable model of ZF has an end-extension to a model of* $ZFC + V = L$.

4.5.4 Saturation

Saturation is a model-theoretic generalisation of algebraic closure and it is defined in terms of realising types.

Given a theory T, a model $\mathcal{M}$ of cardinality κ is κ-saturated if and only if for very subset A of the universe of M ($A \subseteq \mathcal{M}$) of cardinality less than κ ($|A| < \kappa$) and every $n \in \mathbb{N}$, every complete n-type p over A is realised in $\mathcal{M}$.

A model M with its universe of cardinality κ is saturated if and only if it is κ-saturated.

We could say that the types over a subset $A \subset M$ can be realised as orbits of automorphism in M having fixed A.

Saturated models represent a fundamental model-theoretic notion and structure theorems almost always start from saturated models and develop *outwards*. An example is Morley's theorem that every uncountable structure in a class $\mathcal{K}$ of models must be saturated. Complete first-order classes also present a (first-order) property known as *elementary amalgamation*.

Theorem 4.5.8. *Let M and N be L-structures. Let A be a subset of the universe of M such that there is a partial elementary embedding $f : A \to N$. Then there is an elementary extension B of M and an elementary embedding $g : N \to B$ such that $g(f(a)) = a$ for all $a \in A$.*

As a result, first-order classes present a strong homogeneity and contain many structures presenting a high degree of saturation.

Lemma 4.5.9. *A theory T has a countable saturated model if and only if $(\forall n)\,|D_{\mathrm{n}}(T)| \leq \aleph_0$.*

Regarding the existence of saturated models, we mention here the following known theorem, given without proof:

Theorem 4.5.10. *A saturated model of cardinality κ exists for a theory of cardinality $< \kappa$ when $\kappa = \kappa^{<\kappa}$.*

One can also create saturated models "by amalgamating together all possible models, rather in the spirit of Fraïssé's construction" ([90]).

Mekler pointed out that the question of determining the spectrum of saturated models in cardinality κ for a complete first order theory T is completely understood and depends on the stability class of the class $\mathcal{K}$ of models and the cardinal arithmetic in the set theoretic universe. There are two cases for that to happen:

- If $\kappa = \kappa^{<\kappa} \geq |D(T)|$, where $D(T)$ is the set of all complete types in finitely many variables over the empty set.
- If T is stable in κ.

In connection to our discussion from chapter 3, which will be further reprised in the sixth chapter, Baldwin emphasises that model completion is a *context-changing*, whereas a saturated extension represents a *context-internal* operation. Saturation is a property of one structure and its complete theory. "Having fixed a theory which admits quantifier elimination, perhaps by fiat (...), can we extend the domain of an arbitrary model M to one that solves problems as fundamental but more complicated than those specified by single formulas?"([6]). In algebraic geometry, if we search for a 'generic' solution and not just a solution to a polynomial equation, we must realise a type over the parameters of the equation. So if we want to find a model of the same theory - hence context internal - that meets the aforementioned requirements, we will have to take into account the notions of universal, homogeneous and saturated models.

A necessary and sufficient condition for countable saturation is the following

Lemma 4.5.11 ([146], (ex 2.3.12))**.** *A countable model M is countably saturated if and only if for every finite subset X of M, M_X is countably universal.*

In a way, saturation imposes a strong constrain on a structure. If two structures are elementarily equivalent, have the same cardinality and are saturated, then they are necessarily isomorphic.

The **ultrapower** construction represents another way of constructing a saturated model, in addition to amalgamating together all possible models. Malliaris and Shelah have showed that regular ultrapowers are universal for models of cardinality not larger than the size of the index set, and thus failures of saturation come from failures of homogeneity ([136]). Consequently, the saturation of regular ultrapowers can be analysed using the existence or the absence of the internal structure-preserving maps between small subsets of the ultrapower.

The construction of saturated models for regular cardinals is solved for stable theories. In the case of limit cardinals, we can use special models. A special model represents a weaker notion than saturation. The notion of *special model* was developed by Morley and Vaught (1962), but also by Chang and Kiesler (1973) or Hodges (1993) with the aim of using GCH to find saturated models. A model M of size κ is special if it is the union of an elementary chain $\langle M_\lambda : \lambda < \kappa \rangle$ such that each M_λ is λ^+-saturated. Every saturated model is special and every special model is universal (see [21]). Special models are useful in providing a substitute for saturation in the case of limit cardinals.

Homogeneity Another fundamental model-theoretic property is *homogeneity.*

Definition 4.5.12. Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is κ-homogeneous if whenever A $\subset$ M with $|A| < \kappa$, $f : A \to M$ is a partial elementary map, and $a \in M$, there is a function g with $f \subseteq g$ such that $g : A \cup \{a\} \to M$ is partial elementary. We say that $\mathcal{M}$ is homogeneous if it is $|M|$-homogeneous.

In other words, in homogeneous models, partial elementary maps are just restrictions of automorphisms. We can also see homogeneity as a weak form of saturation. The best known examples of strongly universal graphs are also homogeneous.

Now given all of the above-mentioned notions, we have the following results:

Definition 4.5.13. A model $M \models T$ is κ-universal if for all $N \models T$ with $|N| < \kappa$ there is an elementary embedding of N into M. We say that M is universal if it is $|M|^+$-universal.

Lemma 4.5.14. *Let $\kappa \geq \aleph_0$. If M is κ-saturated, then M is κ^+-universal.*

Theorem 4.5.15 ([21])**.** *Let $\kappa \geq \aleph_0$. The following are equivalent:*

i) M is κ-saturated.

ii) M is κ-homogeneous and κ^+-universal. If $\kappa \geq \aleph_1$ i) and ii) are also equivalent to:

iii) M is κ-homogeneous and κ-universal.

In other words, given a theory T, a model/structure M is saturated if and only if it universal and homogeneous for the class of models of T with cardinality $< |M|$ and under the elementary substructure relation. The countable saturated models are universal, but the converse does not hold: saturation is not a condition for universality, since one can find a universal model which is not saturated. An example from [45] is $\omega + \mathbb{Q}$, a universal countable linear order which is not a saturated model). Another example can be found in [21], p. 104.

Saturation is therefore a stronger property than universality. Furthermore, when approaching saturation and universality, there is a difference between countable models and uncountable ones. Universality (or homogeneous saturation) represents an algebraic property describing a class of models, the embedding relation between models. Saturation refers to one model, describing the relation between M and the types over its subsets (see [6], pp. 126-7).

Assuming CH, there is a general theorem of model theory asserting that for any first-order theory which has infinite models, all countably saturated models of cardinality $\mathfrak{c}$ are isomorphic. It is a result by Hausdorff. In fact, it represents a particular case of a more general result, namely, given any first order theory, all κ-saturated models of cardinality κ are pairwise isomorphic. Haudorff proved the uniqueness of a prime countably saturated order under CH.

Theorem 4.5.16. *Assume CH. Then every two countably saturated linear orders of cardinality $\mathfrak{c}$ are isomorphic.*

This result is not true without CH: if $2^\omega \geq \omega_2$, then both of them are countably saturated linear orders of cardinality continuum, but are not isomorphic, since the latter contains an isomorphic copy of the ordinal ω_2.

Assuming GCH, there are universal models in every uncountable cardinals for a complete first-order theory of countable size; when considering regular cardinals, the model is saturated, while in the case of singular cardinals, we refer to a special model, as I mentioned above. Given a cardinal λ, if a first order theory T has infinite models and $|T| \leq \lambda$, then there is a saturated model in every inaccessible cardinal $\mu > \lambda$. Universality still raises questions when GCH fails or we consider non first-order theories. Regarding the countable models of countable theories, the essential results were offered by G.L. Cherlin in [22] and Schmerl in [179].

4.5.5 Stability

There is a connection between stable theories and saturated models, with results proved for regular and singular cardinals κ. That could mean that κ-stable theories realise few types because they "have trouble making distinctions among elements of their models" ([96], p. 18). Stable theories, in a sense, have no partial order. The theory T is stable if for every $A \subset M$ and every $p \in S(A)$, p is definable over A. A definable extension of a type is a distinguished extension of a type: picking it is a freeness relation. Definable extensions of a type are free. For stable theories, all complete types are definable. Stable theories permit the minimum number of types in their models. If T is stable, it is characterised by the definability of types over all models of T. But there are some unstable theories which have certain special models over which all types are definable.

In any sufficiently large subset of a model for any stable theory, we can find large indiscernible sequences, which means that stable theories, in a way, do not posses "a certain kind of descriptive power" ([96], p. 2). Indiscernible sequences are sequences of elements in a model over a set of parameters such that any two n-tuples satisfy the same formulas. Infinite indiscernible sets are sets of indiscernibles and they are used to realise types and to 'blow-up' models. If I is a set of indiscernibles over a subset A of M, the elements of I cannot be distinguished by formulas over A: all the elements of I realise the same strong types over A. Furthermore, when not realising new types, the cardinalities of maximal sets of indiscernibles can be used as invariants in classifying models as bases for models in stable theories. In the case of vector spaces, a structure is determined by one magnitude or dimension, but models of other theories will depend on more than one dimension. Indiscernibles are identified with independent elements in vector spaces, ACF and free algebra. But independence does not quite guarantee indiscernability. A sequence may be indiscernible without being independent. Furthermore, there is an independent set without being a set of indiscernibles. Non-forking preserves indiscernability.

Using the distinctions operated by the stability theory, theories have universal models in the cardinals where they are stable. But when a theory is unstable, one cannot say what happens when GCH does not hold. The question of establishing a connection among different theories regarding the universality spectrum and the non-existence results is still being developed.

Simple theories

Every unstable theory possesses the strict order property or the independence property or both ([109]) and any theory with the tree property is unstable. But simple theories are unstable theories without the tree property, which is a weaker property than the strict order property. A well-known example of a simple unstable property is the theory of the random graph. The simple theories constitute a subclass of the theories without the strict order property and they include stable theories. Due to a result by Dzamonja and Shelah, no theory with the oak property is simple ([50]). Moreover, simple theories are those theories that can code certain set theoretic information, like stationary sets (rather than generalised dimension). So the models of a simple theory are from a certain point of view no more complicated than random graphs ([200]). As shown by Shelah in [191], it is easier for a simple first order theory rather than for the theory of linear order to have a universal model in some cardinalities. He proved that a first order theory with the 4-strong order property behaves like linear orders concerning existence of universal modes.

4.5.6 Universality and Classification theory

In establishing and developing classification theory, Shelah brought forward a new set of questions about mathematics, and their implications expanded beyond model theory. Loosely speaking, classification involves organising classes of mathematical objects using a notion of equivalence by invariants. The theory developed from a "natural problem", that is, from the study of categoricity in relation to cardinality. Morley conjectured that except for the obvious exception (there are $\aleph_1$-categorical theories with infinitely many countable models), a countable, first order theory should have at least as many models in κ^+ as in κ, for uncountable κ.

Theorem 4.5.17 (Morley's Categoricity Theorem)**.** *Let T be a countable first order theory. T is categorical in one uncountable power if and only if T is categorical in all uncountable powers.*

Shelah proved Morley's conjecture by splitting theories into a finite number of classes and providing for each class a schema of explicitly defined (increasing) functions. But he extended Morley's Theorem to uncountable languages, and the uncountable spectrum of a theory T becomes fundamental in the further developments. These results proved that there is regularity in the universe: the property of having the same uncountable spectrum could determine a partition among complete theories that is natural and that can have other applications as well. Another aspect emphasized by Shelah's work was that certain fundamental properties of first order theories (such as categoricity) are closely connected to set theory, which was therefore required to provide new techniques in tackling model theory problems.

Classification theory generates general frameworks for comparing theories by providing dividing lines which depend on some test problems: one applies the method of diving lines to a specific test question, the latter having a *wild* or a *tame* answer for each theory. In Shelah's words, "[A]s we view model theory also as an abstract algebra (i.e., dealing with any T, not just a specific one), we want to find a general structure theorem for the class of models of T like those of Steinitz (for algebraically closed fields) and Ulm (for countable torsion abelian groups). So, ideally, for every model M of T we should be able to find a set of invariants which is complete, i.e., determines M up to isomorphism. Such an invariant is the isomorphism type, so we should better restrict ourselves to more reasonable ones, and the natural candidates are cardinal invariants or reasonable generalizations of them. For a vector space over $\mathbb{Q}$ we need one cardinal (the dimension); for a vector space over an algebraically closed field, two cardinals; for a divisible abelian group G, countably many cardinals (...); and for a structure with countably many one-place relations (i.e. = distinguished subsets), we need $2^{\aleph_0}$ cardinals (the cardinality of each Boolean combination)" ([184], p. 227).

Basically, there are two different levels of classification. Firstly, there is a structure theory for a class K and it involves classifying its structures. Secondly, he classifies classes K according to whether they have or not structure theories. One can usually have a rough intuition about what constitutes a good structure theory, for instance, the structure theory of finitely abelian groups. The *Main Gap Theorem* involves this second level of classification and represents a dichotomy (the *dichotomy theorem* is another name for it) between the theories whose models are classifiable (or controlled) and those which are *wild*. In the first case, each model is determined by a system of complex invariants, i.e., a well-founded tree of countable height and width λ of cardinal invariants, each at most λ. So any defined class K either has a structure theory of a certain form or it is too complicated to have one.

A dividing line represents a property such that both it and its negation are virtuous, with a property of a theory being virtuous if it has significant consequences for any theory satisfying it, if it impacts the understanding of the models of the theory. Finding *dividing lines* is one of Shelah's "dreams" (see [195]), particularly important, given the connection with set theory: "Can we find important dividing lines and develop a theory for combinatorial set theory? Now Jensen has a different dream (I do not believe that it will materialize). (...) Considering inner models, we find there are good dividing lines for descriptive set theory" (p. 218). Examples of test problems include Morley's conjecture and the numbers of models, with few models representing the strongest form of *non-chaos*. Two meaningful dividing lines for the classification of unstable theories are the strict order property and the independence property and their disjunction is equivalent to unstability ([191]). The strict order property is stronger than the strong order property. There are test problems that can be used to study theories with many non-isomorphic models, in particular unstable theories without the SOP (simple theories, for instance): the existence of saturated models [188], the Keisler order [181], or the existence of universal models [109].

As emphasised by Shelah ([200]), good test problems help finding the right dividing lines. Universality constitutes one of the ways of organising the collection of first order theories, with the measure of complexity involving a different ordering for first order theories: complexity for a theory could signify now fewer universal models. And he offers the following definition:

Definition 4.5.18 (2.6. in [200])**.** $T_1 \leq_{\text{univ}} T_2$ if "$\lambda \in \text{univ}(T_2) \Rightarrow \lambda \in (T_1)$" holds also in every larger "universe of sets" (i.e. in every forcing extension where $\lambda \geq 2^{\aleph_0}$, or even $\lambda = \lambda^{\aleph_0}$ holds).

So the universality spectrum problem involves the understanding of the quasi-order determined by $\leq_{\text{univ}}$. The *small* or *low* theories will have a large universality spectrum, while the *large* or *high* theories will have small universality spectrum. 'Nicer' theories have more universals. In the *high* case, there is also the following *external* definition:

Definition 4.5.19 (2.7. in [200]). K is almost $\leq_{\text{univ}}$-maximal if in every forcing extension, K has no universal model in λ for any $\lambda = \lambda^{\aleph_0}$ with $\mu^{++} = \lambda < 2^\mu$ for some μ.

If T has the strict order property, i.e., some formula $\phi(\bar{x}_n, \bar{y}_n)$ defines in some $M \in \mathcal{K}_T$ a partial order with infinite chains, then it is almost $\leq_{\text{univ}}$-maximal (by [109]).

Theorem 4.5.20 (9.4. in [200]). *1) The class of linear orders (and thus many other T's) is almost $\leq_{\text{univ}}$-maximal, (even $\lambda = \mu^{++} < 2^\mu$ suffices and less), that is, it is $\leq_{univ,\{\lambda:(\exists\mu)(\lambda=\mu^{++}=\lambda^{\aleph_0})\}}$-maximal.*

2) Moreover, the StOP (= the strict order property) suffices (StOP means that some formula $\varphi(\bar{x_n}, \bar{y_n})$ defines a partial order in every model of T, and it has an infinite chain in some model of T).

SOP_n's represent approximations to the strict order property, strengthen instability (the order property) and, despite the sparse evidence, might be considered possible candidates for dividing lines ([200], p. 287).

Definition 4.5.21 (9.7 in [200]). 1) For $n \geq 3$, T has SOP_n if there is a formula $\varphi(\bar{x_n}, \bar{y_n})$ that defines on some model of T a directed graph with an infinite chain and no cycles of length $\leq n$.

2) We write $NSOP_n$ for the negation of SOP_n. A prototypical class for SOP_n (for $n \geq 3$) is the class of directed graphs with no $(\leq n)$-cycle.

$NSOP_4$ was considered for a long time a good dividing line for universality ([200], p.317), but, in accordance with [191], SOP_4 would be enough for almost $\leq_{\text{univ}}$-maximality in the theorem above. Furthermore, the results of the theorem hold for theories having the *olive property*. The olive property, introduced by Shelah in [198], represents a sufficient condition for a class to have a universal member in λ, but only if λ is 'close to satisfying GCH', similar to the case of linear orders. The class of all groups were known to have $NSOP_4$ and SOP_3 and more universal elements, but the olive property is weaker than SOP_4, implies SOP_3 and very few universal groups.

So a central idea in [200], and in other work in progress, is that a good dividing line revolves around a combination of being a tree and the olive property. But such an approach has to take into consideration several other aspects: the olive property "seems *ad hoc*, and it is doubtful that it is a good candidate for a successful dividing line" [200]. Then any theory with the tree property is unstable and every unstable theory possesses the strict order property or the independence propery or both ([109]). By [109], the theory of linear order and, more generally, theories with the strict order property have universal models in 'few' cardinals. The tree property is weaker than the strict order property. But simple theories are unstable theories without the tree property, so not having this property can be considered as a weakening of stable ([191]).

As emphasised by Baldwin, classification theory "raises questions about the nature of axiomatisation": "the study of arbitrary theories in model theory reflects the view of axioms not as 'self-evident' or even 'well-established' fundamental principles but as tools for organising mathematics". Nonetheless, "Shelah's classification project takes this to a higher level of abstraction by providing general schemes for comparing theories. This raises new problems in the philosophy of mathematical practice.

- What are criteria for evaluating axiom systems?
- What are the connections among the justificatory and explanatory functions of axioms? E.g., are there criteria for choosing among first order, second-order, or infinitary logic? In what sense is second order logic simply a natural avatar for set theory?
- What principles underlie the development of a taxonomy of mathematics

(or at least formal theories) such as the ones described here?" ([7], p. 365).

4.6 Universality and set theory

A theory can have universal models in some cardinals and not others. The complexity of a set of structures is dependent on the set theory and the cardinal arithmetic one is using. It differs in set-theoretic universes satisfying GCH and in those without it. And even when one is classifying theories, the criterion is the family of cardinals where they have universal objects, given a specific set-theoretic framework and cardinal arithmetic assumptions. So if some theories have universal models in the same cardinals, it may mean that they are somehow related. In the circumstances, set theory represent a/the fundamental condition in approaching the universality problem.

In some cases, cardinal arithmetic can guarantee the existence of a universal model: CH, for instance, implies that every first order theory has a universal (saturated) model of cardinality $\aleph_1$. But Shelah's work shows that the properties of the models of a theory can vary in an essential way depending on the cardinality of the model and on the cardinal arithmetic. The largest universality number, 2^λ could be obtained by forcing, as seen in [109]. The large part of mathematical areas use models of cardinality at most the continuum and involve statements whose truth is independent of the structure's cardinality. The first result which showed universal structures might exist despite the cardinal arithmetic was again a result by Shelah in [180]: it is consistent with $\neg CH$ that there is a universal total order.

But even in results pointing to the connection between the complexity of a theory and the existence of universal models, set theoretic elements play a fundamental role. Džamonja, for instance, shows that for certain theories, such as the theory of graphs, it is possible to violate the GCH as much as one wishes and still have a low universality numbers. This aspect implies "that the ability of having a small universal number in 'reasonable' forcing extensions in which the relevant instances of GCH are violated is a property of the theory itself, which is not possessed by all theories" ([45], p. 5). The connection between the complexity of a theory and amenability to the existence of universal models was explored in other works as well ([109], [110], [108], [188], [49]).

The role of set theory is particularly fundamental in the configuration of the universality problem due to a more sophisticated apparatus in dealing with the uncountable sets, as the complexity of some of the methods, mentioned below, can testify.

Interpretability In any model of one system of set theoretic axioms one can define models of other systems of set theoretic axioms and vice versa. We say that

Definition 4.6.1 (1.3.9 in [146])**.** An $\mathcal{L}_\infty$-structure M is interpretable in a $\mathcal{L}_\in$-structure N if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X and for each symbol of $\mathcal{L}_\infty$ we can find definable E-invariant sets on X such that X/E with the induced structure is isomorphic to M.

Theorem 4.6.2 (5.5.1 in [90])**.** *Let K be a first order language whose signature consists of the binary relation R, and let L be a first-order language with finite signature. Then there is a sentence χ of K such that*

(a) every model of χ is a graph;

(b) the class of models of χ is bi-interpretable with the class of all L-structures which have more than one element.

Moreover, both the interpretations in (b) preserve embeddings.

Furthermore, the theory of linear orders $\leq$ is bi-interpretable with the theory of strict linear order $<$, since from any linear order $\leq$ we can define the corresponding strict linear order $<$ on the same domain and vice versa. When two theories are mutually interpretable they are equiconsistent, given that from any model of one we can produce a model of the other. The most obvious example is the fact that any model of ZFC can define models of $ZFC + CH$ and also $ZFC + \neg CH$. But there is no bi-interpretation in set theory. In [58], Eli Enayat proves that distinct theories extending ZF are not bi-interpretable, and that models of ZF are bi-interpretable only when they are isomorphic.

Universality and proof methods The current results about universality involve various techniques and methods. Some point to the existence of universal models in different contexts. Others show that a certain theory does not have a small universal family at a certain cardinal (club guessing methods).

But they all refer to GCH-like assumptions: firstly, when we assume it; secondly, there are results where CH is violated, and then we analyse the possibility of the existence of universal models in forcing extensions (where CH could fail), and particularly at $\aleph_1$. Jónsson had showed that the classes of structures that satisfy GCH and six other axioms have universal structures in all uncountable cardinals ([97]). GCH implies that all first order theories T with $|T| = \kappa$ have universal models in all uncountable cardinals $\lambda > \kappa$. But things get complicated with the negation of GCH, with Shelah obtaining many independence results regarding the existence of universal structures in uncountable cardinalities ([180], [183], [187]).

4.6.1 Limits

"Roughly speaking, - writes T. Tao - one can divide limits into three categories" ([210]: topological and metric limits, categorical limits and logical limits. Regarding the third kind of limit, "one starts with a sequence of objects $x_{\mathbf{n}}$ or of spaces $X_{\mathbf{n}}$, each of which is (a component of) a model for given (first-order) mathematical language (e.g. if one is working in the language of groups, $X_{\mathbf{n}}$ might be groups and $x_{\mathbf{n}}$ might be elements of these groups). There are several devices one can use in this regard. One is the ultraproduct construction, which

Tao analyses as a 'bridge between continuous and discrete analysis'. Another is the compactness theorem in logic. Still another is the Fraïssé construction. We start with the first.

The new object $\lim_{\mathbf{n}\to\alpha} x_{\mathbf{n}}$ or the new space $\prod_{\mathbf{n}\to\alpha} X_{\mathbf{n}}$ that we create is still a model of the same language. The limiting object (or space) is 'close' to the objects (or spaces) we started of with, in the sense that any assertion that is true of the new objects is also true for many of the original objects and vice versa. For instance, if $\prod_{\mathbf{n}\to\alpha} X_{\mathbf{n}}$ is an abelian group, then the $X_{\mathbf{n}}$ will also be abelian groups for many $\mathbf{n}$. The importance of this type of limit (the logical one) is that it represents "a 'universal' limiting procedure that can be used to replace most of "other types of limits".

Definition 4.6.3 (Ultraproduct)**.** Let L be a language, I and infinite set, and U an ultrafilter on I. Suppose that M_i is an L-structure for every $i \in I$. We define a new structure, $M = \prod M_i/U$ the ultraproduct of the M_i using the filter U. Given

$X = \prod M_i = \{f : I \to \bigcup_i M_i \mid \forall i \in I, f(i) \in M_i\}$

we define the equivalence relation $\sim$ on X by $f \sim g$ if and only if $\{i \in I : f(i) = g(i)\} \in U$.

When we take the product of the same structure, we get the ultrapower.

Definition 4.6.4. Let M be a fixed L-structure and let $M_i = M$ for every $i \in I$. Let U be a nonprincipal ultrafilter on ω. Let $M^* = \prod M_i/U$. We call M^* an ultrapower of M.

The ultraproduct construction in particular, writes Tao, has "two very useful properties which make it particularity useful for the purpose of extracting good continuous limit objects out of a sequence of discrete objects" (Ibid.): Łoś' theorem and the countable saturation property that ultraproducts automatically enjoy.

Theorem 4.6.5 (Łoś' theorem)**.** *Let L be a language, I and infinite set, U an ultrafilter on I and $\sim$ an equivalence relation defined as above. Let $\varphi(x_1, ..., x_n)$ be an L-formula. Then*

$$M \models \varphi(g_1/\sim, ...g_n/\sim) \text{ if and only if } \{i \in I : M_i \models \varphi(g_1(i), ..., g_n(i))\} \in U.$$

Roughly speaking, the theorem asserts that any first-order statement satisfied by a sequence of objects will be satisfied by their ultralimit and conversely. Or, we could say that this theorem is continuous with respect to ultralimits. But the general idea is that the ultraproduct construction offers the possibility of constructing new mathematical structures out of familiar ones, and Łoś' theorem assures that the new model is elementarily equivalent to the original one. It is also a technique used to construct non-standard models.

Compactness I mentioned above the notion of compactness as one way of creating generalisation.

Theorem 4.6.6 (Compactness Theorem)**.** *A theory T is satisfiable if and only if every finite subset of T is satisfiable.*

It is a very simple (and arguable fundamental) consequence of a corollary to Gödel's Completeness Theorem:

Corollary 4.6.7. *Let T be an L-theory. T is consistent if and only if T is satisfiable.*

For instance, a linear order is compact if it is compact in the order topology, i.e. every set has both supremum and infimum, in particular both endpoints exist. We often call such orders *compact lines.* A linear order is a linearly ordered continuum if it is compact and connected in the order topology, i.e., it is compact and dense as a linear order. Compactness represents a consequence of Łoś' theorem. And countable saturation (a property analogous to compactness in topological spaces) can be used to ensure that the (continuous) objects obtained by the ultraproduct construction are 'complete' or 'compact'.

Theorem 4.6.8. *For any countable language L and L-structures M_i, and a free ultrafilter U on $\mathbb{N}$,*

$$\prod_{i\in\mathbb{N}} M_i/U \text{ is countably saturated.}$$

The following theorem shows how ultraproducts can give a different proof of the Compactness theorem. If T is a finite theory, then it is satisfiable.

Theorem 4.6.9 (Ultraproducts and the Compactness Theorem)**.** *Suppose that T is a infinite theory, finitely satisfiable. Let $I = \{\triangle \subseteq T : \triangle \text{ is finite}\}$. Then*

(i) For $\varphi \in T$, let $X_\varphi = \{\triangle \in I : \varphi \in \triangle\}$. Let $D = \{Y \subseteq I : (\exists\varphi \in T) \quad X_\varphi \subset Y\}$. Then D is a filter on I.

(ii) For $\triangle \in I$, let $M_\triangle \models \triangle$. Let U be an ultrafilter on I with $D \subseteq U$. Then $\prod_{\triangle\in I} M_\triangle/U \models T$.

In other words, any class of models that is closed under ultraproducts is also compact. Ultraproducts are also connected to first-order definability, in the sense that if a class of L-structures is first-order definable, the it is closed under ultraproducts. Given that the class of finite L-structures is not closed under ultraproducts, 'finiteness' is not first-order definable. Taking ultrapowers of models determines the Keisler order, which constitutes another test problem in model theory.

Fraïssé limit

So many times, the limit models offer a new perspective on the original one. A general way to construct countable limit structures is the Fraïssé construction. The Fraïssé limits are now studied in different mathematical contexts, like combinatorics, descriptive set theory, permutation group theory, topological dynamics (see [130] for further references). The substructure relation is an integral part in its definition, but since the construction was introduced by

Fraïssé in 1948 ([65]), it received certain generalisations (e.g., the Hrushovski construction).

The Fraïssé limit represents a construction of a universal homogeneous countable relational structure from the class K of its finite substructures that satisfy certain properties:

- Amalgamation: For every $M, N \in K$ that agree on $M \cap N$, there is a structure $M^* \in K$ such that M is a substructure of M^* and N embeds into M^* preserving $M \cap N$.
- Joint embedding property: For every $M, N \in K$ there is $N^* \in K$ such that M, N embed into N^*.
- The class K is closed under isomorphisms;
- The class K is hereditary: if M is an L-structure, $M \in K$ and N is a substructure of M, then $N \in K$.

Theorem 4.6.10 ([65]). *Let Let L be a relational language. If K is a hereditary class of finite relational structures closed under isomorphism and also satisfying the amalgamation and joint embedding properties, then there is a unique countable structure $F(K)$ whose all finite substructures are exactly the structures in K. Moreover, $F(K)$ is ultrahomogeneous, universal and $Th(K)$ has a unique countably infinite model, up to isomorphism.*

The last aspect means that the generic model obtained, $F(K)$, is $\aleph_0$-categorical (the language is finite). This type of model construction appears in a variety of contexts, including linear orders and graphs. The ordered set of rationals ($\mathbb{Q}$) represents the Fraïssé limit of the class of all finite linear orders, while he random graph is the Fraïssé limit of the class of all finite graphs.

The Fraïssé limit was reconfigured by Hrushovski to obtain relational structures and to yield simple unstable structures: the *Hrushovski construction* generalizes the Fraïssé limit by working with a notion of strong substructure, using the $\leq$ relation rather than $\subseteq$ one.

4.6.2 Ehrenfeucht-Fraïssé games

The Ehrenfeucht-Fraïssé games are used to gauge the similarities between structures. They were introduced by Ehrenfeucht ([56]), building on work by Fraïssé ([64]) as a method for proving that two models are equivalent. They are similar to Scott's result that countable structures are determined up to isomorphism by a single infinitary sequence. In its general form, Scott's Isomorphism Theorem states that every countable L-structure is described up to isomorphism by a single $L_{\omega_1,\omega}$-sentence.

Theorem 4.6.11 (Scott's Isomorphism Theorem)**.** *Let L be a countable language and A a countable L-structure. Then there is Scott sentence whose countable models are just the isomorphic copies of A.*

It is an approach particularity characteristic to the Helsinki school (see [91], [221]). Given two models, A and B, there are two players, a non-isomorphism player ($\exists$) and an isomorphism player ($\forall$), who alternately choose elements from A and B. $G_\kappa(A, B)$ denotes the Ehrenfeucht-Fraïssé game of length κ (or κ rounds). At every round i, let a_i be the element chosen from the structure A, and b_i be the element chosen from the structure B. The analogue in this context of Scott isomorphism theorem is the result that two structures of a given cardinality are isomorphic if and only if the isomorphism player has a winning strategy.

Proposition 4.6.12. *If A and B are countable models, then the player $\forall$ has a winning strategy in the game $G_\omega(A, B)$ if and only if $A \cong B$.*

If we are 'playing' the game on ω_1, it will have ω_1 moves and the isomorphism player wins if an isomorphism between the chosen substructures has been constructed. The non-isomorphism player will win the game if the resulting mapping $a_i \mapsto b_i$ is a partial isomorphism, and otherwise, $\forall$ wins. A player *wins* if it has a winning strategy. If $\kappa = \omega$, we have a *ranked* game.

Definition 4.6.13. Let τ be a winning strategy for $\forall$ in $G_\omega(A, B)$. The *Scott rank* of A is the smallest α such that if $A \ncong B$, then for some winning strategy τ of $\forall$ in the game, the rank of (A, B, τ) is at most α.

If $\kappa > \alpha$, we cannot use ordinals: if at every round of the game $\forall$ plays its winning strategy τ, the rank will go down on each move and after a finite number of moves it will reach its 'victory'. Instead, the winning strategy was introduced in the form of a tree of all possible sequences of successor length of moves by the non-isomorphic player $\exists$ against τ, such that $\exists$ has not lost the game yet.

As such, the analogue of the Scott height are trees with no uncountable branches, in the case of tress of cardinality $\aleph_1$, and with no branches of size κ for trees of height κ (games of lenght κ). If T is a tree, the game $G_T(A, B)$ is defined as follows: at any stage, the non-isomorphism player chooses an element from either A or B and a node of the tree lying above the nodes already chosen by this player. The isomorphism player responds with an element of B if the other player had chosen an element of A, and an element of B in the other case. The resulting sequences of moves from A and B form a partial isomorphism. The first player who is unable to move loses.

Similar to Scott height, if A and B are non-isomorphic structures of cardinality $\aleph_1$, then there is a tree of cardinality at most $2^{\aleph_0}$ with no uncountable branches and such that the non-isomorphism player has a winning strategy in the game (the tree T can be chosen to be minimal). The difference as to the Scott height is that the choice of the trees depends on the pair of structures (A, B): if we start by taking a specific structure A, it doesn't work for all structures B.

Definition 4.6.14. Let A and B be structures of cardinality $\aleph_1$. A tree T is called a universal non-equivalence tree for A if T has an uncountable branch and for every non-isomorphic structure B, the non-isomorphic player has a winning strategy in $G_T(A, B)$.

The difficulty in using trees to analyse similarities of structures is that, unlike the ordinals, the structure of the trees is not well understood. Furthermore, another aspect to be considered is that for models of cardinality $> \kappa$, the game $G_\kappa(A, B)$ need not be determined:

Theorem 4.6.15 ([150]). *There are models A and B of cardinality ω_3 such that the game $G_{\omega_1}(A, B)$ is non-determined. It is consistent relative to the consistency of a measurable cardinal that $G_{\omega_1}(A, B)$ is determined for all models of cardinality $\leq \omega_2$. It is consistent relative to the consistency of ZFC that $G_{\omega_1}(A, B)$ is not determined for some models of cardinality $< \omega_2$.*

As emphasised by Hodges ([90], p. 335), the existence of a winning strategy represents an enforceable property, and this kind of properties have been studied also under the name of omitting types in model theory or forcing in set theory. The essence of this method is to break down the overall task into infinitely many smaller tasks, which can be carried out independently without interfering with each other.

4.6.3 Amalgamations

As Hodges remarked, "[A]malgamation theorems (...) tend to spawn offspring of the following kinds: (i) criteria for a structure to be expandable or extendable in certain ways, (ii) syntactic criteria for a formula or set of formulas to be preserved under certain model-theoretic operations (results of this kind are called preservation theorems), (iii) interpolation theorems" ([90], p. 295).

The amalgamation process is a method to obtain simple structures in a direct construction from finite (or finitely generated) substructures. It initially appeared in the works of Ehrenfeught, Fraïssé, and Jónsson. In connection to syntactic criteria, amalgamation represents a technique for realising many types simultaneously inside one and the same structure. Furthermore, non-forking, central element in classification theory, is referred to by Shelah as *free* *amalgamation*. As a semantic notion, it proved to be of crucial importance in proofs establishing the existence of universal objects.

4.6.4 Combinatorics

The *set theoretic combinatorics* represents a complex and rich area of study and results, including extensions of Ramsey's theorem, especially partition calculus, transfinite trees and graphs, Martin's axiom, combinatorics of the continuum, singular cardinal combinatorics, PFA theory-related results, Todorčevič's theory of minimal walks etc. An informative account of its development and connection to other areas can be found in [121], a decade by decade account of different threads, starting with Cantor at a look "at *freedom* as shown in the generalization of notions of largeness, first (...) those used in successive versions of the Regressive Function Theorem, then in generalizations of the Pigeonhole Principle, and in applications to partition relations and trees", then examining "the *winnowing* process mentioned by Cantor that shaped the notion of uncountable tree from the ramified sets, tables, and suites of Kurepa to the family of ω_1-trees classified as to whether they are special Aronszajn trees, non-special Aronszajn trees, Suslin trees, or Kurepa trees" (p. 324).

Combinatorics simultaneously points to few restraints in its practice, but it also determines patterns that complicate structures.

Finer is more

As emphasized by the Soukups, "[S]olutions to combinatorial problems often follow the same head-on approach: enumerate certain objectives and then inductively meet these goals" ([205], p. 1247). And although the techniques vary from problem to problem, the idea is the same: finding the right enumeration of infinitely (including uncountably) many objects involves a recurring feature, that is "to write our set of objectives χ as a union of smaller pieces $\langle \chi_\alpha : \alpha < \kappa \rangle$ so that each χ_α resembles the original structure χ" (Ibid.). In other words, we use a filtration.

Definition 4.6.16. Let κ be an uncountable cardinal and X a set such that $|X| = \kappa$. A *filtration* of X is an increasing and continuous sequence $\langle X_i : i < \kappa \rangle$ such that $X_i \subseteq X$, $X = \cup_i X_i$ and $|X_i| < \kappa$.

A key property is that given filtrations X_i and X'_i, we have $X_j = X'_j$ for a club set of j. Although there are certain limitations, the Soukoups rightfully emphasise the fact that the "introduction of elementary submodels to solving combinatorial problems was truly revolutionary. It provided deeper insight and simplified proofs to otherwise technical results" ([205], p. 1247). Filtrations are used in the proof of many universality results. They involve a fine structure analysis of the objects involved.

There are other examples of fine structure analyses and they all determine fascinating results. I used 'fine' here in a generic sense.

But there is a specific, technical way in which one uses this notion, i.e., when referring to the *fine structure theory* developed by Jensen. He investigated the growth of constructible hierarchy by examining its behaviour at arbitrary levels and he introduced the *Jensen hierarchy*, switching from the hierarchy of L_α's to a hierarchy of J_α's, where $J_{\alpha+1}$ is the closure of $J_\alpha \cup \{J_\alpha\}$ under the "rudimentary" functions. That was his primary goal, but the 'offshoot' was the creation of a complex machinery for this investigation. As Kanamori puts it, a "pivotal question became: when does an ordinal α first get "singularized", i.e. what is the least β such that there is in $L_{\beta+1}$ an unbounded subset of α of smaller order-type, and what definitional complexity does this set have? One is struck by the contrast between Jensen's attention to such local questions as this one, at the heart of his proof of $\square_\kappa$, and how his analysis could lead to major large-scale results of manifest significance" ([99], p. 39).

So generally speaking, combinatorial principles, small 'fragments' of the constructible universe L expressed as assertions like *diamond* ($\diamondsuit$), *square* ($\Box$) etc., have applications in many areas of mathematics, but mostly set theory.

Singular cardinals pose specific problems in set theory, but, as Džamonja writes: "[I]f the infinite is the limit of the finite, a singular cardinal is a limit of the successors of regulars, and maybe it is at such limits that the unruly universe of set theory wishes to express its more tame behaviour. It seems possible that by investigating finer combinatorics than that expressed by the power set function we may find combinatorial versions of SCH which are just outright true" ([46], p. 145).

Combinatorics, both finite and infinite, plays a fundamental role in the configuration of the universality theory. The finite case include the Szemerédi Regularity Lemma, Ramsey theory, probabilistic methods. etc. But of essential importance is infinite combinatorics. Infinite combinatorics involves an extension of finite combinatorics ideas into the infinite realm. Generally speaking, infinitary combinatorics can be used to model and therefore to (try to) understand processes involving infinitely many steps and their nature. And actually, as emphasised by Kanamori, "the direct investigation of the transfinite as extension of number was advanced, gingerly at first, by the emergence of infinite combinatorics" ([99], p. 17). A common characteristic of the combinatorial principles involved is that they are independent of the usual system of axioms in set theory. As a result, they are particularly useful in proving non-existence results regarding universality.

There are two types of cardinals, regular and singular, but the combinatorics of regular cardinals largely overpasses the combinatorics of the singular ones. The infinite combinatorics is related to a cardinal κ and implies an interplay of various appropriate properties $\varphi(\kappa)$ that it has. As emphasised by Dzamonja, "[O]ne may think of κ as a parameter here" ([47], p. 164). An example is represented by the existence of certain objects of size κ, such as a graph or a tree on κ having certain properties, with the combinatorics of non-special trees (discussed later) generalising the usual combinatorics of the uncountable cardinals. We will use a strengthening to trees of the classical Pressing Down Lemma of ω_1: every regressive mapping on a non-special tree is constant on a non-special subtree.

We can also talk about the combinatorics associated with the cardinal invariants of the Cichón diagram. Club guessing sequences and their associated filters play a vital role in singular cardinal combinatorics and other subjects. Other such properties include the existence of a topological measure, of a measure-theoretic object of size κ, extensions of Ramsey's theorem, Martin's axiom etc.

The most notable combinatorial principal is the *General Continuum Hypothesis*, by which every infinite set has the least possible number of subsets. In 1908, Hausdorff proved, using GCH, that there is a universal linearly ordered set in every infinite power. A universal linearly ordered set of cardinality $\aleph_\alpha$ is a linearly ordered set of cardinality $\aleph_\alpha$ with the property that every linearly ordered set of cardinality $\aleph_\alpha$ embeds into it. Hausdorff uses what is now called a *saturation argument* (introduced by Vaught and Morley in the 1950's), i.e., he counts types and realises them to obtain saturated order-types. If CH holds, there is a universal linearly ordered set of cardinality $\aleph_1$. But this fact does not necessarily hold without the CH. He also showed that there is a universal linearly ordered set of cardinality κ for every strong limit cardinal κ. A partial converse was proved by Kojman and Shelah in [109] for an initial segment of the singular cardinals:

Theorem 4.6.17. *For every singular cardinal μ below the least fixed point of second order, if μ is not a strong limit, then a universal linear ordered set of size μ does not exist.*

A fixed point is a cardinal $\mu = \aleph_\mu$, and a fixed point of the second order is a cardinal μ which is the μth point. One is still to find out if this converse holds for all cardinals, if it fails at the least point of second order or at a higher point.

Laver uses Nash-Williams's combinatorial results and show that embeddability regarding the countable order types is well-quasi-ordered ([123]).

4.6.5 Guessing sequences

The guessing principles, diamond ($\diamondsuit$), square, $\square$ or club ($\clubsuit$), can *capture* in a non-trivial way at least λ^+ many objects from $[\lambda]^\lambda$, in a sequence of length λ. They can be used to construct various other combinatorial objects. For instance, we have that $\diamondsuit \to \neg SH$.

Definition 4.6.18. Let S be a stationary set of λ. The sequence $\bar{C} = \langle A_\delta : \delta \in S\rangle$ is called a *diamond sequence*, $\diamondsuit(S)$ if for every $X \in [\lambda]^\lambda$, the set $\{\delta \in S : X \cap \delta = A_\delta\}$ is stationary.

The $\diamondsuit$ and $\square$ principles (with variations) were introduced by Jensen in his paper on the fine-structure theory ([95]), showing that they follow from the assumption that $V = L$. The sequence $\langle A_\delta\rangle_{\delta<\omega_1}$ offers types of universal approximations for the subsets of ω_1.

$\diamondsuit$ expresses combinatorial constrains on the subsets of ω_1: the minimal character of the constructible universe L and the fact that it contains only a 'strict number' of sets. It implies the CH. It can be generalised to a cardinal κ, where it implies the GCH. Shelah proves that for every $\kappa > \aleph_0$, the converse hold.

We can obtain a weaker principle ($\clubsuit$, introduced by Ostaszewski in 1975) by replacing the equality in the definition with $\subseteq$, and removing the cardinals arithmetical assumptions present in $\diamondsuit$.

Definition 4.6.19. Let S be a stationary set of λ. The sequence $\langle A_\delta : \delta \in S\rangle$ is a $\clubsuit(S)$-sequence if $A_\delta \subseteq \delta$ is unbounded, and for every $X \in [\lambda]^\lambda$, the set $\{\delta \in S : A_\delta \subseteq X\}$ is stationary.

$\diamondsuit$ implies the CH, but $\clubsuit(\omega_1)$ is consistent with a large continuum. The principle was used by Macintyre to prove that no Abelian locally finite group of size $\aleph_0$ is embeddable in all universal locally finite groups of size $\aleph_0$ ([128]). Komjáth and Pach use the $\diamondsuit$ principle to prove that there is no universal graph in cardinality $\aleph_0$ among the graphs that K_{ω_0,ω_1}.

The square principles - $\Box$ There are several *square principles*, with the first being discovered by Jensen. In the context of the fine structure theory of the constructible universe L, he proved that $\Box_\kappa$ holds in L for every uncountable cardinal κ. He used this principle to prove the existence of an ω_2-Souslin tree in the constructible universe (see [95]). Given that in $V = L$ they hold for all uncountable cardinals, these principles were used since then as a way to express various properties for the cardinals in L. The related principle $\Diamond$ was used to construct an ω_1-Souslin tree in the same universe (L).

Definition 4.6.20. Let κ be a regular cardinal. A *square sequence* on κ is a sequence $\bar{C} = \langle C_\alpha : \alpha \text{ is a limit ordinal in } \kappa^+ \rangle$ such that

1. C_α is a club subset of α.
2. If $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.
3. If $cf(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

$\Box_\kappa$ represents the statement that there is a square sequence on κ.

The second condition assures the *coherence* property. So the square principle on a cardinal κ refers to the existence of a sequence $\langle C_\alpha \rangle$ indexed by limit ordinal in $[\kappa, \kappa^+)$ such that each C_α is a club of α with order type $\leq \kappa$, and which is also coherent: if β is a limit point of α, then $C_\beta = C_\alpha \cap \beta$.

A generalisation and hierarchy of square principles was provided by Schimmerling in [59]. There are different forms of square principles and they are relatively easy to be obtained by forcing.

4.6.6 The PCF theory

Other options involving combinatorial principles could also be found in Shelah's work, particularly in his PCF theory ([189]), an acronym for the theory of reduced products of small sets of regular cardinals. Shelah's approach was determined by problems related to singular cardinal arithmetic. It is a powerful theory that changed the view on cardinal arithemtic, with Jech maintaining that is more fundamental than cardinal arithmetic ([93], p. 417).

A set $pcf(A)$ of possible cofinalities defined for every set A of regular cardinals is the collection of all cofinalities of ultraproducts $\Pi A/D$ with ultrafilters D over A. A surprising result is: if $2^{\aleph_n} < \aleph_\omega$ for $\forall n = 0, 1, 2, ...$, ($\aleph_\omega$ is a strong limit cardinal), then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Shelah obtained a plethora of combinatorial results and his ZFC combinatorics on uncountable ordinals was used in many studies regarding embeddability. In [109], for instance, it was shown that if $\lambda > \aleph_1$ is regular and $\lambda < 2^{\aleph_0}$, then there is no universal linear ordering in λ. Analogous results were proved for models of first order theories ([110]) and infinite Abelian groups ([111]), but his independence results prove them impossible for the class of all graphs ([187]).

On how the problem of the existence of universal objects is connected to the PCF theory, see [109], [190], [194]. In [108], Kojman uses some of Shelah's combinatorics to establish a representation theorem regarding the connection between the structure of embeddability over a monotone class of infinite graphs and the relation of set inclusion over subsets of reals of bounded cardinality. As such, the result shows that the relation of embeddability among the class of graphs is at least as complicated as the relation of inclusion among the subsets of reals of cardinality at most λ, and, moreover, that the structure of embeddability over the class described is not independent with regard the negations of the GCH. Specifically, the theorem assert that there is a surjective homomorphism from the former relation onto the latter.

Theorem 4.6.21 (1.8. in [108])**.** *If $\lambda > \aleph_1$ is regular, then there is a surjective homomorphism*

$\varphi : \langle \mathcal{G}_\lambda, \leq_w \rangle \longrightarrow \langle [\mathbb{R}]^\lambda, \subseteq \rangle.$

A corollary asserts that the structure of embeddability over the class studied is dependent on negations of GCH.

The proofs in PCF theory do not depend much on the complex developments of set theory in the 1960s (forcing, in particular). As such, and as emphasised by Shelah in the *Introduction* to [189], Cantor, arising from his grave, would be able to understand them, or at least the theorems.

Club guessing Club guessing is a weakening of $\diamondsuit$: in this case, only club sets are guessed. For a comparison of various club guessing principles, see [92].

Definition 4.6.22. Let $\kappa < \lambda$ be regular cardinals with $\kappa \leq \mu < \lambda$ and S a stationary subset of λ consisting of points of cofinality κ. A sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a club guessing sequence iff

1. For every $\delta \in S$, the set C_δ is a subset of δ with $\text{otp}(C_\delta) = \mu$.
2. For every club E of λ there is $\delta \in S$ such that $C_\delta \subseteq E$.
3. For every $\alpha \in \lambda$, $|\{C_\delta \cap \alpha : \delta \in S \& \alpha \in (C_\delta \setminus \lim(C_\delta))\}| < \lambda$.
4. $\sup(C_\delta) = \lambda$.

Unlike Jensen's *diamond*, non-trivial club guessing sequences exist in ZFC, are provable in the ZFC. Shelah started his work involving club guessing to prove non-structure theorems. Gradually, the method was used in different applications in model theory ([110]). But the variety and the number of results in set theory using this construction is extraordinary.

A fundamental result in combinatorial set theory is Shelah's proof of club guessing for regular cardinals of cardinality $> \aleph_1$. Club guessing sequences $\bar{C} = \langle C_\delta : \delta \in S \subset \lambda \rangle$ are used in the proof of certain universality results, starting with [109]. The proof uses invariants of δ along the places in the filtration of a linear order of cardinality λ that are determined by the club guessing sequence. The construction is made such that a club E will witness the embedding between the linear orders (their filtrations): at any $\delta \in E$, the required invariant (defined for elements L_j in the filtration $\bar{L} = \langle L_j : j < \lambda$ of a linear order L of cardinality λ) is achieved only if $D_\delta \subseteq E$. In other words, for any two filtrations L and L' of any two linear orders, the dependence of the invariant on the filtration is only up to a club E.

The construction is clearly synthesised in [42] or [50]: "This simple proof has three important elements: invariants, construction, and preservation. Specifically, to each well order we have associated an invariant, namely, its order type, then we observed that the invariant is preserved in the sense that it can only increase under embedding, and finally, we have constructed a family of well orders of size λ where many different values of the invariant are present (namely, the ordinals in $[\lambda, \lambda^+]$), so showing that no single well order of size λ can embed them all" ([42], p. 284).

Another aspect that these studies emphasise is that the use of the club guessing method establishes connections between a certain PCF statement and the 'desired' negative universality result (one showing that the universality numbers has at least a given value), such that the negative universality results hold in more that just one (specifically) constructed universe.

4.6.7 Forcing

"To prove or to force, this is the question" ([193], p. 1)

Forcing offers another very abstract description of the mathematical process of creation and, moreover, it "supplies us with a general method" ([196], p. 3). Before Cohen's introduction of this technique, Gödel's second incompleteness theorem had already 'legitimised' the use of different extensions of the ZF axiomatisation: no formal system can exhaustively describe the whole mathematical universe. In this context, the point is probably no longer to just prove some specific properties, but rather to connect them in an increasingly expanding *Universe*.

Independence phenomena in various contexts — that is of mathematical statements which cannot be settled in a particular mathematical theory — remain of central importance in set theory. There are two major forms of independence proofs (or unprovability results): forcing and consistency strength. In forcing, one starts from models of ZFC and, given a specific statement φ, proves that both $ZFC + \varphi$ and $ZFC + \neg\varphi$ are consistent. So the consistency strength of 'ZFC', '$ZFC + \varphi$' and '$ZFC + \neg\varphi$' are all equal, unlike the case using the other context. Furthermore, independence results involving consistency strength can be obtained over weaker theories than ZFC, like Peano Arithmetic. Another difference between the two methods is that although a theorem can be proved in both contexts, forcing gives a definitive answer, whereas the result obtained by comparing consistency strength may be open for debate. The latter is due to the different approaches regarding the large cardinals. That being said, forcing can also start with large cardinals.

Regarding the large numbers of independence result, Shelah remarked that they help to "sort out possible theorems – after throwing away all relations which do not hold, you no longer have a heap of questions which clearly are all independent, the trash is thrown away and in what remains you find some grains of gold. This is in general a good justification for independence results; a good place where this had worked is cardinal arithmetic – before Cohen and Easton, who would have looked at $2^{\aleph_{\omega_1}}$?" ([196], p. 7).

A forcing notion $\mathbb{P}$ has certain properties that makes it similar to a construction by recursion. One can construct a generic object by partially ordered approximations that get bigger and bigger and which are, in the end and due to the way they were defined, glued together (usually through union) to obtain the generic set ($\cup G$), the *forced* object (a linear order, for instance). Such a generic set must have certain properties, which are assured by the intersection with the dense sets of the forcing notion. The approximations will fit together in this new object because we won't take the union of the whole forcing notion $\mathbb{P}$ but of some filter G on $\mathbb{P}$. So the approximations will fit together, and the generic object will be consistent. By requiring G to intersect every dense set in the original model M prevents us from predicting from within M the construction of, i.e., the result of gluing together the consistent approximations.

The apparent lack of any control on the construction deepens if we take into consideration the fact that the $\mathbb{P}$-names, necessary in the construction of the objects in the extension model are themselves an expansion of the universe, given that a $\mathbb{P}$-name is a set. The enumeration of the dense sets also takes place outside M. And yet, as Kunen plastically points out, the people "living in M cannot construct a G which is $\mathbb{P}$-generic over M, but they can "figure out certain properties of G" and $\cup G$, and, "[M]ore generally, they can construct a *forcing language*, where a sentence ψ of the forcing language uses the names in $M^{\mathbb{P}}$ to assert something about $M[G]$" ([114], p. 193).

Forcing is a technique discovered by Cohen ([25]) and subsequently developed by Solovay and many others for producing 'generic' sets. It enables the extension of a universe V of set theory (i.e. a model of ZFC) to another one, $V[G]$, such that $V[G]$ has the same ordinals as the original universe, most often has the same cardinals, and satisfies a desired formula φ. The ccc forcing, for instance, mentioned below, represents a property of forcing that guarantees that the forcing preserves all cardinals. The weak forcing relation, $\Vdash^*$, refers to a sentence $\varphi(\tau_1, ..., \tau_n)$ formally defined in the metatheory, but intuitively placed in a *forcing language*.

Definition 4.6.23. Let M be a countable transitive model of ZFC, $\varphi(x_1, ..., x_n)$ a formula in the language of set theory, and P a partial order in M. Then for $p \in P$ and names $\tau_1, ..., \tau_n \in M^P$, $p \Vdash^*_{P,M} \varphi(\tau_1, ..., \tau_n)$ iff $\forall G$ [(G is P generic over $M \wedge p \in G) \rightarrow M[G] \Vdash val(\tau_1, G), ...val(\tau_n, G)$].

This definition involves all possible filters G so it is not expressible in M. Hence the semantic nature: it depends on the interpretation of the forcing relation. The purpose of the $\Vdash^*$ is to show, in conjunction with the Main Theorem of Forcing, that $\Vdash$ is expressible as a relation in M, and, as such, pointing to the syntactic nature of the forcing definition, i.e., it depends on the syntactic properties of the formulas. The Main Theorem of Forcing shows that the two independently defined forcing relations, $\Vdash^*$ and $\Vdash$, are the same:

Theorem 4.6.24 (Main Theorem of Forcing)**.** *Let M be a countable transitive model of ZFC, $\varphi(x_1, ..., x_n)$ a formula in the language of set theory, and P a partial order in M. Then for any $p \in P$ and names $\tau_1, ..., \tau_n \in M^P$, we have $p \Vdash^* \varphi(\tau_1, ..., \tau_n) \longleftrightarrow (p \Vdash \varphi(\tau_1, ..., \tau_n))^M$.*

In iterating forcing, the semantic point of view means that we start with $M = M_0$, extend it by the filter G_0, which is P_0-generic over M_0 and we obtain $\mu_1 = M[G_0]$. We then pick a poset $P_1 \in M_1$ and extend by the filer G_1, which is P_1-generic over M_1, obtaining $M_2 = M_1[G_1]$, and so on. But given the difficulties related to a limit stage, like ω, we need to make some further adjustments, which were found by Solovay and Tennenbaum ([204]): the entire process of constructing models is controlled in the original model, where, with the help of *names*, one defines a forcing notion which encodes all future extensions.

The *countable chain condition (ccc)* represents a topological property that became a central concept in forcing.

Definition 4.6.25. A forcing notion P satisfies the countable chain condition (ccc) if every antichain in P is at most countable.

An iteration of ccc forcing with finite supports is ccc, which means that one can keep extending the universes (the models of ZFC) and, in the end, we will still have a model of ZFC in which all ordinals are preserved and cardinals are preserved. But we formulate in the original model that which is true in that final universe. Given that we controlled the construction of the models in the original universe, we will have some knowledge about that model by ensuring that we have iterated the ccc forcings for as much as possible for them to be formulated as MA (Martin's Axiom):

Definition 4.6.26 ($MA(\aleph_1)$)**.** Whenever $\mathbb{P}$ is a non-empty ccc forcing and for every family $\mathcal{D}$ of $< 2^\omega$ many dense sets in $\mathbb{P}$ there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathcal{D}(G \cap D \neq 0)$.

MA represents a very interesting and useful instrument: it is entirely combinatorial, and we don't even need first-order logic in order to use it.

By replacing the countable chain condition with *properness*, a much weaker condition, in the statement of $MA(\aleph_1)$, we obtain the *proper forcing axiom* (PFA).

Definition 4.6.27 (PFA)**.** If $\mathbb{P}$ is a proper forcing poset and $\mathcal{F}$ is a family of $\aleph_1$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ which intersects all sets in $\mathcal{F}$.

PFA is a much stronger axiom than $MA(\aleph_1)$ and decides many problems that were left open by $MA(\aleph_1)$. As emphasised by Kanamori, Shelah's variants and augmentations "revamped forcing for combinatorics and the continuum with systemic proofs of new and old results" ([99], p. 55).

In forcing, the different universes have different opinions about different statement in set theory. But there are some statements that cannot be changes by forcing: $(\forall n)2^{\aleph_n} < \aleph_\omega \Rightarrow 2^{\aleph_\omega} < \aleph_{\omega_4}$ ([189]). And another important aspect is that although we have a certain proof using forcing, it is not clear which theories cannot undergo that forcing.

Cohen's first use of forcing was to generate new *reals* (i.e. an element of $2^{\aleph_0}$), resulting in the consistency of the negation of the Continuum Hypothesis (CH) from the consistency of ZFC. Because Gödel had previously established the consistency of the CH with ZFC, the two results combined showed that ZFC does not settle the CH. And it was discovered quite early that in contrast to the combinatorics of singular cardinals, the one of the regular cardinals can be 'easily' manipulated by forcing (the classic examples is Easton's theorem).

If GCH is dropped, it becomes harder to construct universal objects, but it is easier to obtain negative consistency result by adding Cohen subsets to the universe: if $\aleph_2$ Cohen reals are added, any non-$\aleph_0$-stable countable theory T has no universal model in $\aleph_1$ ([187]). If $\kappa = \kappa^{<\kappa}$ and μ Cohen subsets of κ are added, no unstable theory T has a universal model in any $\chi \in (\kappa, \mu)$ ([148]). And also see [109].

Much of what is known concerns the successors of regular cardinals, since in the case of a singular there are fewer forcing constructions available. A solution, offered by Dzamonja and Shelah in [49] is to prepare a large cardinal κ by iterated forcing, which preserves its large cardinal character, and only at the end of the construction force κ to become a singular cardinal. The forcing used in the final step is the Prikry forcing, such that κ in the final model is still large (it is still a cardinal fixed point). But other papers make use different methods. In [27], the authors use a forcing poset defined by Foreman and Woodin (which is a variation of the Prikry forcing that adds a Prikry sequence κ_i of inaccessible cardinals cofinal in κ and, in addition, collapses all but finitely many cardinals between successive points on the Prikry sequence, such that κ become $\aleph_\omega$). In another paper ([28]), the final step is characterized by the use of Radin forcing, which changes the cofinality of κ to uncountable values, like ω_1, SCH fails at κ and $u_{\kappa^+} < 2^\kappa$.

Miller's variant to Laver's forcing ([151]) has a particular role in understanding forcing and the continuum. The forcing notion was defined originally as the partial order of perfect subsets of the real line in which the rationals are dense, and it was described as an "intermediate between Sacks perfect set forcing and Laver forcing". It is actually equivalent to forcing with infinitely branching trees. The conditions do not have to branch at every node, but only cofinally many times (unlike the case of Laver partial order). The new real added in the extension is of minimal degree.

Forcing can also have an essential role for the classification project in model theory when using the universality test problem since, as Shelah remarks, it enables us to "prove consistency results that can show that the obtained results are best possible (and thus really constitute a dividing line among theories)" ([200], p. 279).

5

Universality results

Synopsis

This chapter contains results regarding the universality problem. We will start by introducing further notations and definitions focusing on graphs and orders. We will state some universality results that apply to first order structures and offer a context in which to state the mathematical result of this thesis. Another section will analyse the role of embeddings.

The next section will give a brief overview of some universality results on graph theory and then we will state our own result using concepts of category theory.

The last section of this chapter represents an overview of results regarding tree, with a focus on Aronszajn trees.

It is relatively easy to obtain universal objects for a finite class of elements or models. But the situation changes drastically when we enter the domain of the infinite. A fundamental aspect of the universality problem is to find what determines the existence of universal objects. That means that we have to take into consideration the methods that we use in proving their existence or nonexistence, the role of the cardinal arithmetic. Or is there a way in which the nature of a theory determines its universality spectrum (see the classification theory)?

5.1 Preliminaries

5.1.1 Graphs

A graph is a pair (X, R) with X a set and R a symmetric and reflexive binary relation on X.

Definition 5.1.1. A *graph* is a pair (G, R), with G a set and R a symmetric and reflexive binary relation on G. The members of the set are called *vertices* or *nodes*, and R will be the *edge relation*. An edge is a pair of vertices (v_1, v_2) such that $v_1 R v_2$.

A *subgraph* of a graph G is a substructure of G.

A *path* of length n is a sequence of edges $\{v_0, v_1\}, \{v_1, v_2\}...\{v_{n-2}, v_{n-1}\}$. The path is a cycle if $v_n = v_0$.

A *directed graph* or a *digraph* is a set together with a symmetric and reflexive binary relation, called the *directed edge relation*.

An *oriented graph* is a directed graph that excludes multi-edges (i.e. there is at most one directed edge between any two vertices) and cycles.

The graphs used in this text will be irreflexive. When there is no confusion, we will refer to a graph as G.

5.1.2 Orders

Definition 5.1.2. A *partial order* is a structure $(P, \leq)$, where P is the universe (or the underlying set), and $\leq_P$ is a relation which is reflexive $(\forall x(x \leq_P x))$, transitive $(\forall x, y, z(x \leq_P y \wedge y \leq_P z \rightarrow x \leq_P z))$.

$(P, \leq)$ is a partial order in the strict sense if and only if it is irreflexive, transitive, and it also satisfies antisymmetry: $\forall x, y[\neg(x \leq_P y \wedge y \leq_P x)]$. This will be the type of partial order to be considered throughout this text. We will abuse notation by referring to 'the partial order P' or '$<$' instead of $(P, \leq)$ if no confusion should arise in the context. We will often call a partial order a *poset*.

Definition 5.1.3. A sub-poset $(S, <)$ of a poset $(P, \leq)$ is a subset $S \subseteq P$ such that $\leq_S = \leq_P \restriction S$. That is, $\forall s_1, s_2 \in S(s_1 \leq_S s_2 \text{ iff } s_1 \leq_P s_2)$.

We say that two elements x, y in P are *incomparable* if and only if $x \not< y$ and $y \not< x$. Two elements x, y in P are *incompatible*, $x \perp y$, if and only if there is no $r \in P$ such that $r > x, y$.

Definition 5.1.4. A *chain* in $(P, \leq)$ is a subset of P in which all elements are pairwise comparable.

Definition 5.1.5. An weak (strong) antichain in $(P, \leq)$ is a subset of P in which all elements are pairwise incomparable (incompatible).

Definition 5.1.6. A *linear order* or a *total order* is a partial order such that all elements are pairwise comparable.

A *well-founded partial order* is a partial order that has no infinitely decreasing sequences.

A *well-order* is a totally ordered well-founded partial order.

According to these definitions, any chain in a well-founded partial order is a well-order.

Trees

A *tree* is a special kind of partial order. And although there are many definitions of trees, the one used in this text fixes a tree as a special type of well-founded partial order.

Definition 5.1.7. In set theory, a *tree* $(T, <_T)$ is a set T together with a relation $<_T \subseteq T \times T$ such that:

- $<_T$ is a partial ordering of T
- and for any $t \in T$, $pred_T(t) = \{s \in T \mid s <_T t\}$ is well-ordered.

For $t \in T$, $ht_T(t) = otp(pred_T(t), <_T)$.

For an ordinal α, T_α is the set of $t \in T$ such that $ht_T(t) = \alpha$, and the height of the tree, $ht(T)$, is the least α such that $T_\alpha = \varnothing$.

Definition 5.1.8. A subtree $(X, <_T \cap (X \times X))$ of $(T, <_T)$ under the induced partial order is a subset $X \subset T$. The height, the width and the level structure of the subtree might not be retained.

Definition 5.1.9. A subset b of a tree T is a *chain* or a *branch* in T if b is linearly ordered by $<_T$.

A *path* or a *cofinal branch* though T is a chain $c \subseteq T$ such that $c \cap lv_\alpha(T) \neq \varnothing$ for every $\alpha < ht(T)$. Or we could say that a branch b is *cofinal* if the set $\{ht_T(t) : t \in b\}$ is cofinal in $ht(T)$.

An antichain is a subset $A \subseteq T$ whose elements are pairwise incomparable (incompatible).

Incompatibility is a stronger condition than *incomparability* and it is usually used in this context in relation to forcing. In a tree, two elements are incomparable if and only if they are incompatible, i.e. there is no common majorant. We will often abuse notation and write T instead of $(T, <_T)$.

5.1.3 Model theoretic definitions

In this text, *embedding* will mean *elementarily embedding* when we are referring to a first-order complete theory. Otherwise, the term embedding will be as specified. If M is embedded into N we write $M \prec N$. If M satisfies exactly the same sentences as N, i.e. they are elementarily equivalent, we write $M \equiv N$. A structure M is homogeneous if any isomorphism between finite substructures of M can be extended to an automorphism of M.

Definition 5.1.10. ([21]) A theory T has the *amalgamation property* if for any three models $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ of T and isomorphic embeddings $f : \mathfrak{C} \to \mathfrak{A}$, $g : \mathfrak{C} \to \mathfrak{B}$ there is a model $\mathfrak{C}'$ of T and isomorphic embeddings $f' : \mathfrak{A} \to \mathfrak{C}'$, $g' : \mathfrak{B} \to \mathfrak{C}'$ such that the diagram commutes. The model $\mathfrak{C}'$ is said to *amalgamate* $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathfrak{C}$

As pointed out in [112], a class of structures which weakly omit some collection of finite structures possesses a countable strongly universal homogeneous member if and only if its finite elements form an amalgamation class. That means that any two finite elements can be isomorphically embedded in a common finite extension in the class, and if any two finite elements in the class have a common substructure, then there is a finite extension of both in the class. The countable strongly universal homogeneous element is also unique up to isomorphism. When the finite graphs in the class do not form an amalgamation class there are several examples to show that universal graphs may not exist.

Definition 5.1.11. (([21], p. 195) A theory T has the *joint embedding property* if for any two models $\mathfrak{A}, \mathfrak{B}$ of T there is a model $\mathfrak{C}$ of T such that both $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphically embeddable in $\mathfrak{C}$.

Every complete theory has the joint embedding property. And, as a special case, it follows that every model complete theory has the amalgamation property ([21]).

Assuming C satisfies the property, there are various complete first order theories T such that T has a model which embeds every countable member of C. If such a theory T has a universal model in power κ, then C has a universal element in power κ.

The notions of homogeneity and universality are algebraic properties related to the models of a class, whereas saturation is a concept characterizing a single structure, the relations of the model to its types over the subsets of the model.

I mentioned above (subsection 4.5.4) the notion of *special* model. For completeness, I will mention the following results connected to special models:

Proposition 5.1.12.
- *Every saturated model is special.*
- *A model of power κ^+ is saturated if and only if it is special.*
- *If κ is a regular limit cardinal, then a model of power κ is saturated if and only if it is special.*
- *If M is special, then every reduct of M is special.*
- *Assuming GCH, if a theory T has an infinite model, it has a special model in each power $\kappa > ||\mathcal{L}||$.*

Every saturated model is special, but not all special models are saturated. What is more, the existence of special models does not require the GCH or inaccessible models.

5.1.4 Universality

Even if we won't always refer to first order structures, so we won't be able to use first-order model definitions, *model* and *theory* will have the same meaning. So we will say that given a theory T and a class of models K of T, a universal model in K exists if and only if there exist a structure U such that for all $M \in K$, there is an embedding $f : M \to U$.

There are positive universality results and negative universality results. The first state the existence of a universal model or a universal family, while the latter show under what conditions a universal model cannot exist. Furthermore, there may not be *a* universal model for a class of models of a certain cardinality. In this case, we are referring to the smallest family of sets in the class (the *universal family*) able to embed all the other models. This *covering number* is called the *complexity* of the class or the *universality number*.

Definition 5.1.13. The *complexity* C or the *complexity number* of a class $\mathcal{K}$ of models is the minimal cardinality of a *universal family* $\mathcal{F}$, such that $\forall M \in \mathcal{K}$, there exists $X \in \mathcal{F}$ and an embedding $f : M \to X$.

If the complexity of $\mathcal{K}$ is 1, we say that it has a universal model.

Given a set theoretic universe, how do we find out the complexity of a class of models? Given a cardinal κ, let $\mathcal{K}_\kappa$ denote the set of structures with the underlying set of cardinality κ. We can identify isomorphic structures with their isomorphic image having a universe of size κ. If the complexity of $\mathcal{K}_\kappa$ is $\leq \kappa$, then the models in $\mathcal{F}$ can be let to form a universal model of size κ: for instance, we could take a disjoint union of all these models of $\mathcal{F}$ of cardinality κ. The maximal complexity is 2^κ, i.e., the maximal cardinality of $\mathcal{K}_\kappa$. In other words, if we assume the GCH and $\mathcal{K}_\kappa$ is closed under taking disjoint unions of size κ, then the complexity number is either 1 or $2^\kappa = \kappa^+$.

Without the GCH, the complexity number ranges between $\kappa^+ \leq C < 2^\kappa$. If the cardinality of a universal family belongs to this range, it will have a *small universal family.* In connection to the classification theory, mentioned above, and the use of universality as a test problem to find dividing lines, we get the notion of the *universality spectrum.*

Definition 5.1.14. Let V be a set theoretic universe and $\mathcal{K}$ a class of structures, the *universality spectrum* for $\mathcal{K}$ is the family of cardinals κ for which $\mathcal{K}_\kappa$ has a universal model in κ.

In this framework, if two theories have universal models in the same cardinals, then the these theories are somehow connected (it is the project of current work in classification theory to explore how). Notice that whereas the complexity of a class of models is dependent on the set theory and the cardinal arithmetic, assuming or not GCH, for instance, the universality spectrum also depends on the embedding type. Many results mentioned will be model-theoretic in statement and proof.

5.2 Embeddings

I mentioned the notion of *structure* in the sections above. Historically, the abstract structures, found all over mathematics now, emerged as generalizations from concrete instances. But it is also a common practice in mathematics to study how these structures are related to each other. They are usually connected through functions from the domain - one structure - to the co-domain - another structure. It is said that these functions preserve some 'structure', with 'structure' referring here to the characteristics determined by the additional features (relation, topology, etc.) attached to the underlying set. So structure present in the domain can be found in the co-domain. An example is the notion of *embedding*, open to various definitions according of the domain of study. An endomorphism implies a part having similar structure as the whole, but some details of the structure may be lost. A self-embedding means that the part and whole have the same structure. In the case of an automorphism, although no part is involved, the structure of the whole is 'expressed' in various ways on the same whole. Given that the whole is trivially a part of itself, any automorphism is an embedding and vice-versa. In the case of finite structures, the notions of self-embedding and automorphism coincide. Furthermore, when a lot of structure needs to be preserved, the notion of self-embedding and endomorphism coincide.

Embeddings constitute a natural choice in set theory. An embedding will display the domain as a substructure of the co-domain, but we will also be able to ask *how* the domain embeds into the co-domain. In the case of first-order structures, we can check if the embedding is elementary, i.e. whether the truth of every first-order sentence having parameters in the domain is preserved by the embedding. Every elementary embedding is a strong homomorphism, and its image is an elementary substructure. In homogeneous models, partial elementary maps are restrictions of automorphisms. In model theory, elementary embeddings are of a crucial importance. But they also play an important role in set theory. Elementary embeddings whose domain is V (the universe of set theory) are central in the theory of large cardinals.

Definition 5.2.1. An elementary embedding of a structure M into a structure N of the same signature σ is a map $f : M \to N$ such that for every first-order σ-formula $\varphi(x_1, \cdots, x_n)$ and all elements $a_1, \cdots, a_n$ of M, $M \models \phi(a_1, \cdots, a_n)$ if and only if $N \models \varphi(f(a_1), \cdots, f(a_n))$.

Initial embedding We could also ask whether the domain is embedded *initially* in the co-domain. This kind of embedding receives various formulations in set theory, corresponding to different strengths. In the weakest form, it means that the image of the embedding is downwards closed under $\in$, that is, if an element x is an element in the image and the co-domain satisfies that $y \in x$, then y is also in the image. This kind of embeddings is trivial for well-founded structures. It is a fact following from the Mostowski Collapse Lemma. Specifically, if $f : M \to N$ is an initial embedding between well-founded extensional structures, then M and N are isomorphic to transitive sets M' and N', respectively, with the inherited $\in$-structure, and f is induced by the inclusion function of M' into N'. As a result, the interesting cases are offered by the non-standard models, but this subject is not part of the purpose of this text.

Consequently, the nature of embedding represents an essential aspect when studying the structure of a class of models. For ordered structures (linear orders and partial orders), the embeddings could be constructed as injective maps that preserve order. But there are alternative definitions, to be used later, in which *embedding* will refer to non-injective functions.

As pointed out by M. Kojman and S. Shelah ([109, p. 2]), if we are dealing with the class of models of a first-order theory T, then 'embedding' should be understood as 'elementary embedding' when T is complete, and 'universal' is considered with respect to elementary embeddings; whereas when T is not complete, as in the case of the theory of linear orders, the theory of graphs or the theory of Boolean algebras, 'embedding' is an ordinary embedding, a one-to-one function that preserves all relations and operations, with 'universal' considered with respect to ordinary embeddings. The distinction is necessary since there are theories for which universal models in the sense of an ordinary embedding can exist, but universal models in the sense of an elementary embedding do not. The elementary embeddings involve a restrictive class of graphs, which preserve all first-order properties; the isomorphisms are elementary maps, bijective 'ordinary' embeddings.

It should be emphasized that embedding is not the same as inclusion. The inclusion map is a topological embedding. The embedding is given by some injective and structure preserving map $f : X \to Y$. The structure preserving depends on the kind of mathematical structure of which X and Y are instances. Several different embeddings are possible between X and Y. In many cases, there is a standard/canonical one and in such cases, it is common to identify the domain X with its image, $f(X)$, contained in Y such that $f(X) \subseteq Y$.

Embeddings for graphs

Definition 5.2.2 (Embeddings for graphs)**.** Given two graphs G_1 and G_2, $f : G_1 \to G_2$ is a *weak embedding* (or just *embedding*) for graphs if it is an injective function which preserves edges, i.e., if (g_1, g_2) is an edge in G_1, then $(f(g_1), f(g_2))$ is an edge in G_2. A *strong embedding* for graphs is one which also preserves non-edges (such that there is no edge relation between two vertices), that is, G_1 is embedded in G_2 as an induced graph.

There are some results that show that if the continuum has a regular value, embeddability among graphs at a regular uncountable cardinal κ is indifferent to the size of the continuum. Such a behaviour is in contrast to the embeddability of linear orders, which cannot have a universal at a regular cardinal $\kappa > \aleph_1$ below the continuum ([109]). A singular $2^{\aleph_0}$ affects the structure of embeddability in a broad spectrum of classes of infinite graphs below the continuum. If cf $2^{\aleph_0} = \aleph_1$, meaning that the continuum is singular if CH does not hold, there is no universal graph (in the class of all graphs) in all uncountable $\kappa < 2^{\aleph_0}$.

In [187], Shelah proves the consistency of the existence of a universal graph of cardinality λ, with arbitrary κ and λ such that $\kappa = \kappa^{<\kappa} < \lambda = cf(\lambda) < 2^{\kappa}$. Mekler [148] generalised Shelah's result to a broader collection of classes of relational structures to show the consistency of non-existence of universal elements in uncountable κ below a regular continuum is easy (see [109], Appendix).

In [81] J.D. Hamkins proves that every countable model of set theory $\langle M, \in^M \rangle$, including every well-founded model, is isomorphic to a submodel of its own constructible universe $\langle L^M, \in^M \rangle$ by means of an embedding $j : M \to L^M$. It follows that the countable models (of set theory) are linearly pre-ordered by embeddability: for any two such models, one of them is isomorphic to an induced subgraph of the other (= then either M is isomorphic to a submodel of N or conversely). The countable well-founded models are ordered by embeddability in accordance to the heights of their ordinals (the shorter model embeds into every taller model), so this pre-well-ordered embeddability has order type $\omega_1 + 1$ (or there are $\omega_1 + 1$ many bi-embeddability classes). Two models with the same ordinals are said to be bi-embeddable.

Furthermore, the proof shows that $\langle L^M, \in^M \rangle$ contains a submodel that is a universal acyclic digraph of rank $\mathrm{Ord}^{\mathrm{M}}$, so every model M of set theory is universal for all countable well-founded (acyclic) binary relations of rank at most $\mathrm{Ord}^{\mathrm{M}}$; and every ill-founded model of set theory is universal for all countable acyclic binary relations. So actually, every nonstandard model of set theory is universal for all countable acyclic digraphs.

Embeddings for ordered sets

Definition 5.2.3 (Embeddings for ordered sets)**.** A *weak embedding* (or just *embedding*) for ordered sets is an injective function that preserves order, i.e., if $(M, <_M)$ and $(N, <_N)$ are ordered sets and $f : M \to N$ is an embedding, then $\forall m_1, m_2 \in M(m_1 <_M m_2)$, we have $f(m_1) <_N f(m_2)$. A *strong embedding* is one which also preserves incomparability, that is, for M and N as above, if m_1 and m_2 are incomparable, then $f(m_1)$ and $f(m_2)$ are incomparable.

If P is a partially ordered set, we say that S is P-embeddable if there is a strictly increasing mapping $f : S \to P$. The function f need not be 1-1.

Proposition 5.2.4. *(3.1. Tod) Let* $(L, \leq_L)$ *be a linearly ordered set. Then* $d(L, \leq_L) = min\{\kappa | (L, \leq_L)$ *is embeddable into* $(\mathcal{P}(\kappa), \subseteq)\}$

Proof. $d(L, \leq_L) \leq d(D \cup L, \subseteq) \leq \kappa$. □

In the case of partial orders, such as trees, we may also require that incomparability is preserved, dealing therefore with the notion of strong embedding.

Other options involving combinatorial principles could also be found in Shelah's work, particularly in his PCF theory ([189]). A stronger type of embedding, determining a specific type of universality results with regards to a certain kind of well-founded partial orders, involves a club guessing method used by Shelah and Kojman ([109]), and later generalised by Kojman ([108]). Kojman, for instance, finds a surjective homomorphism between certain subsets of $\mathcal{P}(\omega_0)$ (subsets of reals of bounded cardinality) ordered by the subset relation, on one hand, and the structure of enbeddability over well-founded partial orders (the monotone class of infinite graphs), on the other. These embeddings preserve both rank and order and is particularly useful in studying trees, as Väänäen and Todorčević show in [219]).

So embeddability also employs combinatorics, but a common characteristic of the combinatorial principles involved is that they are independent of the usual system of axioms in set theory, with infinite combinatorics being used in connection to non-existence results regarding universality.

5.3 General universality results - First order and non-first-order theories

There are more studies regarding the existence of universal models for first order theories (i.e. elementary classes of all models of such theory), and they use model-theoretic properties that guarantee the existence of universal models. Without these model-theoretic assumptions, it is hard to show that a first order theory has a universal model (for discussions see [188]).

If GCH holds, there is a universal model of a theory T of cardinality λ for every $\kappa > |T|$. For a model of T of cardinality κ, if $2^{<\kappa} = \kappa > |T|$, then there is a universal model of T of cardinality κ. CH implies that any complete first order theory has a saturated and, therefore universal model of cardinality $\aleph_1$.

Stability Another property implying the existence of universal models is *stability.*

Definition 5.3.1. A complete theory T is ω*-stable* if for every set A of cardinality $\aleph_0$, the set of complete types over A has cardinality $\aleph_0$.

Let κ an infinite cardinal. A complete theory T in a countable language is κ-stable if whenever $M \models T, A \subseteq M$, and $|A| = \kappa$, then $|S_n^M(A)| = \kappa$ ([146], p. 135).

M is κ-stable if $Th(M)$ is κ-stable.

What is more,

Theorem 5.3.2 (see [197] and [146]). *If a theory is ω-stable, then it is κ-stable for all infinite cardinals κ.*

Since the stability of a theory shows that it does not have too many types, it is used in classification theory as a condition in dividing the complete theories into those whose models can be classified and those whose models are too complicated to classify.

According to Morley and Vaught ([155]), under GCH ($\kappa = \kappa^{<\kappa}$), every theory has a universal model in every uncountable cardinal. But we can obtain arbitrarily large saturated and therefore universal models without assuming the GCH.

We start with the following theorem by Shelah:

Theorem 5.3.3 ([197], Th. 3.12.). *If T is λ-stable then T has a λ-saturated model of power λ.*

According to the model-theoretic definitions of saturation and universality, every saturated model is universal. So it follows that a κ-stable theory has a universal model of power κ.

Then taking into account the definition of stability in term of types, we can construct a saturated elementary extension for every model of a stable theory. These results can be found in [197], ch.III. In the following, we show how to construct a saturated elementary extension for a model M_0 of cardinality κ. The idea is to realise all types over it in a bigger model, and repeat the process κ^+ times.

Proposition 5.3.4. *If T is a κ-stable theory and $M_0 \models T$ with $|M_0| = \kappa$, then there is a saturated elementary extension $M \models T$ of M_0 with $|M| = \kappa$.*

Proof. We take an elementary chain of models $(M_\alpha : \alpha < \kappa)$, with $|M_\alpha| = \kappa$, $M_\alpha \prec M_{\alpha+1}$, and such that:

- $M_0 \models T$ and $|M_0| = \kappa$.

- If α is a limit ordinal, we have $M_\alpha = \cup_{\beta<\alpha} M_\beta$.
- If $p \in S_1^{M_\alpha}(M_\alpha)$, then p is realised in $M_{\alpha+1}$.

Given that T is stable, if $|M_\alpha| = \kappa$, then $|S_1^{M_\alpha}| = \kappa$. So we can find models $M_\alpha \prec M_{\alpha+1}$ such that $|S_1^{M_\alpha}| = \kappa$ and $M_{\alpha+1}$ realises all the types from $S_1^{M_\alpha}$. Now let $M = \cup M_\alpha$. Since M represents the union of κ models of cardinality κ, it also has cardinality κ.

We need to show that is saturated. Let $A \subset M$ with $|A| < \kappa$. Given that κ is regular, there is $\alpha < \kappa$ such that $A \subseteq M_\alpha$. If there is a type $p \in S_1^M(A)$, then there is a type $q \in S_1^M(M_\alpha)$ and $S_1^M(M_\alpha) = S_1^{M_\alpha}(M_\alpha)$ with $p \subseteq q$. But the type p is realised in $M_{\alpha+1}$, therefore p is realised in M. It follows that M is saturated. □

For the proof regarding saturated models of singular cardinality for ω-stable theories see [197], ch.III.

Given the results mentioned above, we can also obtain the following results:

Proposition 5.3.5. *If a complete first-order theory T is ω-stable and has an infinite model, it has an universal model in every cardinal, including $\aleph_0$.*

Proof. It follows from the Th.5.3.3. and the Pr. 5.3.4. above and the definitions of saturation and universality. □

Other corollaries that we state without proof (see [200]), are the following

Proposition 5.3.6. *If a theory T is superstable and has an infinite model, it will have universal models in every cardinality $\geq 2^{\aleph_0}$.*

Proposition 5.3.7. *If a theory T is stable, it has universal models in every cardinal κ satisfying $\kappa^{\aleph_0} = \kappa$.*

Mekler characterized the class of theories for which it is consistent to have a universal model at $\aleph_1 < 2^{\aleph_0}$. He showed in [148], continuing [183], that it is consistent with ¬CH that every universal theory of relational structures with the joint embedding property and amalgamation for $P^{-}(3)$-diagrams and only finitely many isomorphism types at every finite power, has a universal model at $\aleph_1$. In [109], M. Kojman and S. Shelah show that there can be a universal linear order at a regular cardinal κ only if $\kappa = \kappa^{<\kappa}$ or if $\kappa = \mu^{+}$ and $2^{<\mu} \leqslant \kappa$, and they prove in ZFC several non-existence theorems for universal linear orders in regular cardinals. They find a covering lemma which shows, as one corollary, that if $2^{\aleph_0} = \aleph_{\omega_1}$, then there are no universal models for non-ω-stable theories in every regular κ below the continuum. The problem becomes harder when κ is a singular cardinal. But they prove non-existence theorems for universal linear orders in singular cardinals. For example, if κ is not a strong limit and is not a fix point of the $\aleph$ function, then there is no universal linear order in κ. If GCH is removed, Kojman and Shelah [109] have an example of a countable theory which has a universal model of size $\aleph_1$ exactly when CH holds. So every theory that doesn't have an uncountable $D(T)$ [1] has a universal model at $\aleph_1 < 2^{\aleph_0}$ (cf. [109]). Kojman and Shelah show that a theory T with $|D(T)| = \aleph_0$ (which is even $\aleph_0$-categorical) has a universal model in $\aleph_1$ if and only if CH holds.

As pointed out in [49], there are "many natural theories which are not first order", approached from "the point of view of abstract elementary classes" (introduced by Shelah in), "and in a more specialized form earlier by Bjarni Jónsson" (see [21]). Given such a class $\mathcal{K}$ and λ a cardinal, the family of elements of $\mathcal{K}$ with size λ will be denoted by $\mathcal{K}_\lambda$.

In [188] S. Shelah introduced the notion of an *approximation family* and studied elementary classes with a "simple"/"workable" (in [49]) λ-approximation family. If λ is an uncountable cardinal satisfying $\lambda = \lambda^{<\lambda}$, it is consistent that every abstract elementary class $\mathcal{K}$ with a "workable" λ-approximation family has an element of size λ^{++}, i.e. $\mathcal{K}_{\lambda^+}$. K_{ap} can be seen a *forcing notion* whose generic gives an element of $\mathcal{K}_{\lambda^+}$.

[1] D(T) is the set of all complete n-types over the empty set, $n < \omega$. If it is uncountable, it has size $2^{\aleph_0}$. Every type in D(T) must be realized in a universal model

5.4 Universal graph

A graph might be the simplest but simultaneously it is one of the most useful notions in mathematics. In a general sense, the question is whether or not a class of graphs has a universal element in an infinite cardinal, that is, one that contains any other element of that cardinality as a subgraph. The existence of a universal graph on a fixed cardinal κ in various contexts represents another example of infinite combinatorics, another combinatorial question. As emphasised by Džamonja, when studied in connection with graphs, it seems that (a) "singular cardinals are more manageable than the regular ones and (b) in some models obtained from large cardinals the successors of singular cardinals actually behave quite close to how they do in L" ([46], p. 139), even when the cardinal arithmetic is changed dramatically.

5.4.1 Results

So given a cardinal κ, what is the smallest size of a family of graphs of size κ which embeds every graph of size κ as an induced subgraph?

With GCH In ZFC, there is a unique (up to isomorphism) graph of size $\aleph_0$, known as the *Rado graph* (and also the *random graph* or the *Erdös-Rényi graph*). Although this graph was discovered independently by several mathematicians, starting from Ackerman, the universality properties were proved by Rado in [168] and [169]. This graph has the property that for every finite graph G_n and every vertex v of G_n, every strong embedding of $G \setminus v$ into G can be extended to a strong embedding of G_n into G. As a result, G strongly embeds all countable graphs. When $\kappa = \kappa^{<\kappa}$, by a similar proof as in the case of a saturated model, one can obtain a *special model*, which is also universal. By fixing $\langle \kappa_i : i < cf(\kappa) \rangle$ a sequence of regular cardinals which is cofinal in κ, one can build a graph $\mathcal{G}$ which is the union of an increasing sequence of induced subgraphs $\mathcal{G}_i$, where $\mathcal{G}_i$ is a saturated graph on κ_i, and argue by repeated application of saturation that

$\mathcal{G}$ is universal ([27, p. 541]).

From the existence of saturated and special models in first-order theory, mentioned above, a universal graph exists at every infinite cardinal κ: $\lambda < \kappa \Rightarrow \kappa^\lambda = \kappa$ means that there is a saturated and therefore universal graph of size κ. This holds even if 2^{κ^+} is large. It is also known that if κ is regular, then it is consistent to have a jointly universal family on κ^+ of size κ^{++}, while 2^{κ^+} is arbitrarily large (cf. [49]). So it follows that if κ is singular and GCH holds, then $u_\kappa = 1$.

In [112], the authors show that every countable graph G is strongly universal in the class of graphs which weakly omit the same finite graphs as G, with the best known examples of strongly universal graphs being also homogeneous (i.e. saturated). But there are several classes of graphs without saturated graphs but universal graphs:

Theorem 5.4.1. *Let $s \in \mathbb{N}$. Then*

1. *The class of graphs omitting all odd circuits of length at most $2s+1$ contains a countable strongly universal element [Th. 1.3.].*
2. *There is a strongly universal members of any infinite cardinality in*
    - 2.1. *in the class of graphs omitting paths of length s [Th. 1.12.]*
    - 2.2. *in the class of graphs omitting all circuits [a cycle with unrepeated vertices] of length at least s [Th. 1.14].*

Their technique in constructing universal graphs was to show that the class of graphs intended to be constructed are *reducts* to the language of graphs of a class of structures in a larger language (one defined by additional relations), where universal objects are proved to exist. Referring to uncountable cardinals κ, they observe that the question of whether a strongly universal graph of cardinality κ exists or not is more a model-theoretical question than a graph-theoretical one ([112], p. 160). As such, if $\mathcal{K}$ is a class of graphs (so models of a first-order theory), a necessary condition for $\mathcal{K}$ to have universal elements is to have the joint embedding property, i.e., that any two elements of $\mathcal{K}$ be isomorphically embedded into a common element. In the classes of graphs omitting paths of length s and classes of graphs omitting all circuits of at least s, every graph has an ω-stable theory, and therefore the classes allow universal graphs for cardinals $\geqslant$ to the continuum.

Without GCH But the removal of the GCH condition involves a more difficult context to find universal graphs and gives way to various results. Different classes behave differently. When GCH fails sufficiently, for some classes of graphs there are no universal objects ([108], [191]), while for others there can exist consistently a small family of the class that acts jointly as a universal object for the class at the given cardinality ([188], [49]).

In [183], Shelah proves that by adding $\aleph_2$ Cohen reals to a model of CH, there is no universal graph of size $\aleph_1$. In such a model, there can be $2^{\aleph_1}$ subsets of $\aleph_1$ each of power $\aleph_1$, with finite intersection of any two of them. The universality number for a family $\mathcal{F}$ in this model has the greatest value, namely, $2^{\aleph_1}$. He proves that there is a universal graph of power $\aleph_1 < 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Furthermore, for any $\kappa = \kappa^{<\kappa}$, 2^κ can be arbitrary large, and there is a universal graph in every $\lambda \in [\kappa, 2^\kappa)$.

If GCH fails ($\kappa < \kappa^{<\kappa}$), one can make a distinction between theories that for many cardinals have the largest possible universality number in a cardinal κ whenever GCH fails, and those for which it is possible to construct a model in which GCH fails, but the theory has a small universality number at the desired cardinality ([50]).

All in all, if GCH is dropped, it becomes harder to construct universal objects, but it is easy to obtain negative consistency result by adding Cohen subsets to the universe: if $\aleph_2$ Cohen reals are added, any non-$\aleph_0$-stable countable theory T has no universal model in $\aleph_1$ ([187]). If $\kappa = \kappa^{<\kappa}$ and μ Cohen subsets of κ are added, no unstable theory T has a universal model in any $\chi \in (\kappa, \mu)$ ([148]). Also see [109].

Singular cardinals When dealing with the more difficult case of a singular cardinal, Džamonja and Shelah introduced in [48] a new approach: they started with a supercompact cardinal κ, added functions through a preparatory iteration that would become embeddings into a family of jointly universal graphs after the Prikry forcing - while preserving some of the character of κ -, and only at the end apply Prikry forcing to change the cofinality of κ and make it singular. This way, one can produce models where κ is a singular strong limit of cofinality ω, 2^κ is arbitrarily large and $u_{\kappa^+} \leqslant \kappa^{++}$. The positive consistency result regarding the existence of a small family of such graphs that act jointly as universal for the graphs of the same size shows that there are κ^{++} universal graphs of size κ^+ for the successor of a strong limit singular of cofinality $\aleph_0$ with $2^{\kappa^+} > \kappa^{++}$ (assuming the consistency of the existence of a supercompact cardinal).

In [27] the authors are considering a successor cardinal κ and $2^\kappa > \kappa^+$, and they prove that

Theorem 5.4.2 (Th. 6.1.). *It is consistent from large cardinals that $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = 2^{\aleph_{\omega+1}} = \aleph_{\omega+3}$, and there is a family of size $\aleph_{\omega+2}$ of graphs on $\aleph_{\omega+1}$ which is jointly universal for all such graphs*

They use a forcing poset defined by Foreman and Woodin, which is a variation of the Prikry forcing that adds a Prikry sequence κ_i of inaccessible cardinals cofinal in κ and, in addition, collapses all but finitely many cardinals between successive points on the Prikry sequence, s.t. κ become $\aleph_\omega$) by which κ becomes $\aleph_\omega$.

In another paper ([28]), the final step is characterized by the use of Radin forcing which changes the cofinality of κ to uncountable values, like ω_1, SCH fails at κ and $u_{\kappa^+} < 2^\kappa$.

Shelah ([187]) proved that there is a positive consistency result concerning the existence of a universal graph at the successor of singular cardinal that is not a strong limit: a universal graph at λ exists if there is a κ s.t. $\kappa = \kappa^{<\kappa} < \lambda = cf\lambda < 2^\kappa = cf(2^\kappa)$.

Other graphs But there are independence results for other classes of graphs as well. In [187], Shelah shows that in the same model in which there is no universal graph of cardinality $\aleph_1$, if $\aleph_2$ Cohen reals are added to the universe, there is no universal triangle-free graph of cardinality $\aleph_1$. Mekler ([148]) has extended this result to other classes of structures. In [49], the authors give a precise criterion for a class to be amenable to the theorem about consistency of the existence of a small family of models in $\mathcal{K}_{\lambda^+}$ that are universal for $\mathcal{K}_{\lambda^+}$. Among the classes satisfying this criterion are the classes of triangle-free graphs under embeddings (as shown in [188]) or the elementary class of models of any simple theory, as shown in [191].

5.4.2 A universality result involving linear orders and graphs

I will discuss here the possibility of translating one type of structure into another in order to preserve the embedding related to the universality property. The structures considered to this end are the linear orders and the graphs. And since neither the theory of linear orders nor the theory of graphs are complete theories, 'embedding' should first of all be understood as ordinary embedding (not elementary embedding), namely an injective (one to one) function that preserves all relations and operations. To this end, and considering the assumption that "the assignment of a mathematical object of one type to mathematical objects of another type is functorial" ([173, p. 15]), we will use category theory notions: expressing and building the problem in this context will offer new insights into it, allowing the conclusion to follow smoothly.

It should be emphasized that the model theoretic notion of universality used in proving that classes of objects have universal models in exactly the same cardinals or having the same universality spectrum differs from the notion of universality used in category theory. In category theory, the universality property from 'universal object' characterizes the object up to isomorphism.

Universality, *pcf* theory and linear orders M. Kojman and S. Shelah [109] reduced the existence of a universal linear order in cardinality κ to the existence of a universal model for any theory possessing the strict order property. There can be a universal linear order at a regular cardinal λ only if $\lambda = \lambda^{<\lambda}$ or if $\lambda = \mu^+$ and $2^{<\mu} \leqslant \lambda$. They show that the non-existence theorems they prove for linear orders also hold for a large collections of theories: the theories of linear orders, partial orders, Boolean algebras, lattices, ordered fields, ordered groups, number theory, p-adic rings. Consequently, an important element that should be taken into consideration is that the universal spectrum of the theory of linear orders is dependent on the axiomatisation of set theory that we choose to adopt and represents an example of the set theoretic independence phenomena. Under GCH, every first order theory (including the class of linear orders) in a countable language has a universal model in all uncountable cardinals.

The relation between the structures

A fundamental question that we need to approach is if we can translate one type of structure into another in order to preserve certain embedding-related properties, specifically how universality results for linear orders are related to universality results for graphs. We start by emphasizing the fact that any linear ordering can be coded up into a graph, "[A]ctually this method allows us to code up any kind of structure of finite or countable signature as a graph. (We have two layers of coding: structure into graph, then graph into bipartite graph.) So our bipartite graphs in K have the maximum possible amount of complication in them ([89], 215).

Interpretation as graph As I mentioned above, the definition of a structure permits the interpretation of 'complicated' structures in simpler ones. For instance, any structure in a countable language can be interpreted in a graph. Let $\mathcal{L} = \{R\}$, where R is a binary relation. It is possible to describe a $\mathcal{L}$-theory T of graphs such that every model of T is G_A for a unique linear order A ([146], pp. 25-27 Passim). Since any structure in a countable language can be interpreted in a graph, one can use model theory to show how to describe a $\mathcal{L}$-theory T of graphs such that every model of T corresponds to the graph G_A for a unique linear order A described by $\mathcal{L} = \{R\}$ ([146], pp. 25-27): if $(A, <)$ is a linear order, then $G_A \models T$. If we can interpret linear orders into every model of T means that, in fact, $(A, <) \mapsto G_A$ is a bijection between linear orders and models of T.

We will start by defining a structure as a category with the appropriate embeddings as its morphisms. Then we will show how an interpretation between theories corresponds to a functor. A functor consists of a mapping on objects and a mapping on morphisms that preserves all of the structure of a category, namely composition, identities, domains and co-domains. But since not every functor corresponds to an interpretation between theories, some conditions are to be specified on the functor, such that it preserve the relevant structure. It must be shown how it works on the objects of a category and how it works on the morphisms.

Difference between the theory of order and the theory of graphs regarding universality This functorial approach to the connection between the linear orders and graphs must take into consideration the differences between the behaviour of these two structures. For example, as emphasized in [109], "While the proof of the consistency of having a universal graph in $\aleph_2 < 2^{\aleph_0}$ generalizes the proof for the case $\aleph_1 < 2^{\aleph_0}$, the consistency of universal linear order is true for the former case and is *false* for the latter" ([109], p. 3).

Definitions

Category:

Definition 5.4.3. A *category* $\mathcal{C}$ consists of: 1. A collection of objects $Ob(\mathcal{C})$ denoted by A, B, C,...

2. A collection of arrows (also called morphisms) $Mor(\mathcal{C})$ denoted by f, g, h,...

3. A rule assigning to each $f \in Mor(\mathcal{C})$ two objects $domf$ and $codf$, its domain and codomain.

4. For each pair (f, g) of morphisms with $codf = domg$ we have a composite morphism $gf : domf \to codg$ subject to the axiom $h(gf) = (hg)f$ whenever gf and hg are defined.

5. For each object A we have an identity morphism $1_A : A \to A$, subject to the axioms $1_B f = f = f 1_A$ for all $f : A \to B$.

Functor

Definition 5.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ consists of:

1. A mapping $A \mapsto \mathscr{F}A : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

2. A mapping $f \mapsto \mathscr{F}f : Mor(\mathcal{C}) \to Mor(\mathcal{D})$

such that $dom\mathscr{F}f = \mathscr{F}(domf), cod\mathscr{F}f = \mathscr{F}(codf), \mathscr{F}(1_A) = 1_{\mathscr{F}A}$, and $\mathscr{F}(gf) = (\mathscr{F}g)(\mathscr{F}f)$ whenever gf is defined in $\mathcal{C}$.

Essential image

Definition 5.4.5. The *full subcategory* of a category $\mathcal{C}'$ is the category consisting of some of the objects of $\mathcal{C}'$, but all of the morphisms the pairs of these objects.

The *essential image* of a functor $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$ is the full subcategory of $\mathcal{C}'$ consisting of objects $c' \in \mathcal{C}'$ that are isomorphic to $\mathscr{F}(c)$ for some $c \in \mathcal{C}'$.

The category of linear orders:

Definition 5.4.6. For a signature $\sigma = <$, with '$<$' a binary relation, let $\mathcal{L}$ be the category whose objects are σ-structures that model the theory of linear orders, having as morphisms linear order embeddings, that is injective order-preserving maps between structures.

The category of graphs

Definition 5.4.7. Let $\mathcal{G}$ be the category of graphs. The objects are simple graphs (undirected graphs containing no loops or multiple edges), each one defined by a set of vertices V and a binary, symmetric relation R, such that for every vertices $v, v' \in V$, there is and edge $\{v, v'\}$ between v and v' iff $(v, v') \in R$ and $v \neq v'$.

The morphisms will be embeddings: a morphism $f : (V, R) \to (V', R')$ is an injection $f : V \hookrightarrow V'$ such that for each $(v, v') \in V \times V$ (or $\{v, v'\} \in \binom{V}{2}$), we have $\{v, v'\} \in R$ if and only if $\{f(v), f(v')\} \in R'$.

Or we could say that morphisms will be embeddings $f : (V, R) \to (V', R')$ with the property that $f : (V, R) \subseteq f : (V', R')$. In other words, a morphism in the category of graphs is an isomorphism from (V, R) onto an induced subgraph of (V', R').

Concrete categories: Both $\mathcal{L}$ and $\mathcal{G}$ represent categories since embeddings are closed under composition. Furthermore, we are dealing here with *concrete categories.* In a concrete category the objects have underlying sets whose morphisms are functions between these underlying sets, structure-preserving morphisms. The embeddings, in each case, represent a subset of the set of all functions between the underlying sets used in defining the objects. We will be dealing with weak embeddings.

Embeddings and functors: There is no single meaning of *embedding* in category theory, but the notion can be described in terms of properties of morphisms.

In a category: The notion of *monomorphism* is the generalization to arbitrary categories of the notion of injective map of sets. But many monomorphisms are not embeddings. They are in the category *Set*, where being a monomorphism is sufficient for being an injective function, but not in *Ord*, for example, where monomorphisms are injective monotone functions, but not embeddings. The embedding required by the concept of universality spectrum does not translate into requiring that the function on the objects O of a category C to U - a universal object in O - be monic, i.e a monomorphism. For example, a monomorphism of a graph is a subgraph, while a universal graph should contain other graphs as induced subgraphs. Consequently, one of the requirements in describing the embeddings in our case is to require that there be an embedding $A \to U$ for every object of A.

Furthermore, we will ask it to be an initial morphism: if g is a function from the underlying set of an object C to the underlying set of object A, and if its composition with f is a morphism $f \circ g : C \to B$, then g itself is a morphism. Therefore, an embedding between two objects A and B in a concrete category is a morphism $f : A \to B$, which is an injective function from the underlying set of A to the underlying set of B and it is also an initial morphism. The composition of two embeddings is always an embedding itself.

Between categories:

Definition 5.4.8. A functor $\mathscr{F}$ is faithful (respectively full) if for any two objects $c_1, c_2 \in \mathcal{C}$, the map $Hom(c_1, c_2) \to Hom(\mathscr{F}(c_1), \mathscr{F}(c_2))$ is injective (respectively, surjective).

An embedding can be defined as a functor that is injective with respect to morphisms. In other words, a functor is an embedding if and only if it is faithful and injective on objects. As emphasised by Riehl, "Fullness and faithfulness are *local* conditions; a *global* condition, by contrast, applies "everywhere". A faithful functor need not be injective on morphisms; neither must a full functor be surjective on morphisms. A faithful functor that is injective on objects is called an *embedding* and identifies the domain category as a subcategory of the codomain; in this case, faithfulness implies that the functor is (globally) injective on arrows. A full and faithful functor, called *fully faithfull* for short, that is injective-on-objects defines a *full embedding* of the domain category into the codomain category. The domain then defines a *full subcategory* of the codomain" ([173], p. 31). So given two categories $\mathcal{C}$ and $\mathcal{C}'$, a faithful functor doesn't need to be injective on objects or morphisms, for which reason the range of a full and faithful functor is not necessarily isomorphic to $\mathcal{C}'$, and two morphisms from $\mathcal{C}$ with different domains and codomains may map to the same morphism in $\mathcal{C}'$. The same applies to a full functor. But a fully faithful functor is necessarily injective on objects up to isomorphism.

In other words, a faithful functor (respectively a full functor) is a functor that is injective (respectively surjective) when *restricted* to each set of morphisms that have a given source and target. And then the induced maps $\mathscr{F}_{c_1,c_2}$ are injective for all objects c_1, c_2 in C. But when we require the faithful functor to be injective on the hom-sets between every pair of objects in the target category, we end up with a functor injective on morphisms. A functor that is injective on morphisms is automatically faithful and is also injective on objects, but the contrary need not hold.

Further requirement: Furthermore, in order to insure that the translation of one type of structure into another preserves the embedding related to the universality property, we need a stronger version of a functor to act as arrows, one that involves a universal morphism. Even if f is an embedding in the concrete category $\mathcal{L}$, it doesn't follow that $\mathscr{F}(f)$ is an embedding in the concrete category

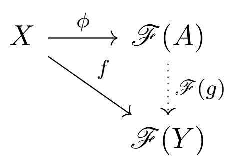
$\mathcal{G}$, not even if they were equivalent under the functor $\mathscr{F}$. What we need is a functor that has an inverse, even though that inverse does not have to be defined on the whole codomain. In other words, we need the notion of *initial morphism.*

Initial morphism:

Definition 5.4.9. Suppose $\mathscr{F}$ is a functor $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$.

Let X be an object of $\mathcal{C}'$. An *initial morphism* is an initial object in the category $(X \downarrow \mathscr{F})$ of morphisms from X to $\mathscr{F}$. In other words, it consists of a pair (A, ϕ), where A is an object of $\mathcal{C}$ and $\phi : X \to \mathscr{F}(A)$ is a morphism in $\mathcal{C}'$, such that the following universal property is satisfied:

whenever Y is an object of $\mathcal{C}$ and $f : X \to \mathscr{F}(Y)$ is a morphism in $\mathcal{C}'$, then there exists a unique morphism $g : A \to Y$, such that the following diagram commutes:



Result:

Theorem 5.4.10. *There is a faithful functor $\mathscr{F}$ from the category $\mathcal{L}$ of linear orders to the category $\mathcal{G}$ of graphs that preserves model theoretic-related universality results (classes of objects having universal models in exactly the same cardinals, and also having the same universality spectrum).*

$\mathscr{F}$ can be considered as the functor from $\mathcal{L}$ to its essential image in $\mathcal{G}$, denoted by $\mathcal{E}$.

Proof. We will define the functor $\mathscr{F} : \mathcal{L} \to \mathcal{G}$.

The functor between categories must preserve the size of the objects and the morphisms (embeddings here), in other words, it has to map objects to objects and morphisms to morphisms preserving identity morphism and composition of morphisms.

Firstly, and in accordance with results in model theory mentioned above (see [146, pp. 25-27]), linear orders can be interpreted into graphs.

The elements of a linear order $(A, <_A)$ are the vertices in a graph G_A. Given a linear order $(A, <_A)$, the graph G_A will have vertices A and an edge $\{\mathscr{F}(a), \mathscr{F}(a')\} \in R$ whenever $a <_A a'$.

We have then a functor $\mathscr{F} : Ob(\mathcal{L}) \to Ob(\mathcal{G})$ preserving the size of the objects.

We have to show now that $\mathscr{F}$ preserves identity and composition. Although $\mathcal{L}$ and $\mathcal{G}$ have different collections of objects, we are dealing here with concrete categories, therefore the morphisms will constitute a subset of the class of all functions between the underlying sets of the objects of the two categories.

Each linear order $(A, <_A)$ is a relation $R_A \subseteq A \times A$. A morphism in $\mathcal{L}$ is a function $f : (A, <_A) \to (B, <_B)$, such that for any $(a, a') \in A \times A$, if $a <_A a'$, then $f(a) < f(a')$.

In other words, an embedding $(A, <_A) \hookrightarrow (B, <_B)$ is an isomorphism from $(A, <_A$ $)$ to an elementary substructure of $B, <_B)$. Or, we can say that there is a dotted arrow making the following diagram of sets commute:

$$\begin{array}{ccc} R_A & \longrightarrow & A \times A \\ \downarrow & & \downarrow{\scriptstyle f \times f} \\ R_B & \longrightarrow & B \times B \end{array}$$

Since $R_B \to B \times B$ is a monomorphism, there cannot be two different dotted arrows making the above diagram commute. An embedding in $\mathcal{G}$ is a graph isomorphism from an object G to an induced subgraph of object G'.

The following diagram will show that the image of the function f will act on the graphs the same way it acts on linear orders. Let $f : (A, <_A) \to (B, <_B)$ be an embedding in $\mathcal{L}$. We have the following commutative diagram:

$$\begin{array}{ccc} (A, <_A) & \xrightarrow{\quad f \quad} & (B, <_B) \\ \downarrow \mathscr{F} & & \downarrow \mathscr{F} \\ (\mathscr{F}(A), \mathscr{F}(<_A)) & \xrightarrow{f} & (\mathscr{F}(B), \mathscr{F}(<_B)) \end{array}$$

If (v, v') is an edge in $\mathscr{F}(A, <_A)$ (or G_A), then $(f(v), f(v'))$ is an edge in $\mathscr{F}(B, <_B)$ (or G_B).

But f is an embedding, so $f(v) <_B f(v')$, which further corresponds to and edge $(f(v), f(v')$ for $\mathscr{F}(B, <_B)$.

$\mathscr{F}$ was thus defined on objects and morphisms and showed to preserve the size of the underlying sets (not the size of the class of objects - we would have then isomorphism), identity and composition.

It follows that every graph morphism $(A, <_A) \hookrightarrow (B, <_B)$ is in the image of $Mor_{\mathscr{F}} : Mor_{\mathcal{L}(A,<_A),(B,<_B)} \to Mor_{\mathcal{G}}(G_A, G_B)$

In other words, every graph morphism between G_A and G_B comes from a linear morphism between $(A, <_A)$ and $(B, <_B)$. It is also clear that it is not a surjective function on the objects: not every graph is in the image of $\mathscr{F}$.

The image of $\mathscr{F}$

Given a linear order $(A, <_A)$, we have the graph $G_A = \mathscr{F}(A, <_A)$, with vertices A and an edge (a, a') whenever $a <_A a'$.

Given that every function $f : A \to B$ has an image $I(A) \subseteq B$, we could also construct an image functor $I : \mathcal{G} \to \mathcal{L}$. To be more specific, I is a functor from $\mathcal{E}$ to $\mathcal{G}$.

The image of an object G of $\mathcal{G}$ is a binary relation $R \subseteq V \times V$, and not necessarily a linear order, although we can generate one for each G under I: its elements are

V, the vertices of G. Since graphs morphisms send paths to paths, I preserves the morphisms, and identity is obvious. I sends G_A back to $(A, <_A)$. $(A, <_A) = I(\mathscr{F}(A, <_A))$ is therefore in the image of I.

Furthermore,

$$I \circ \mathscr{F} = I(\mathscr{F}(A, <_A)) = I(G_A) = (A, <_A) = 1_{\mathcal{L}}.$$

The functor I is faithful, that is it shows how to recover a morphism $f : (V, R) \to (V', R')$ in the domain of I given $(V, R), (V', R')$ and $I(f)$. $\mathscr{F} \circ I$ is also the identity functor: a graph is turned into a linear order and then back into a graph. In other words, when a linear order is turned into a graph and then back into a linear order, it returns unchanged. And when a linear order morphism is turned into a graph morphism and then back into a linear order morphism, it remains unchanged.

$\mathscr{F}$ and I preserve the size of the objects and the embedding characteristic of the two categories.

Now we have to show how the objects of the two categories embed into their images under functor composition.

Universal model We have to show that $\mathcal{L} \to I(\mathscr{F}(L))$ is faithful. Suppose that there is a universal model U in $\mathcal{L}$. We have to show that $\mathscr{F}(U)$ is universal in $\mathcal{G}$. For any object G in $\mathcal{G}$, $I(G)$ is an object in $\mathcal{L}$, so $I(G) \hookrightarrow U$. Since $\mathscr{F}$ preserves embeddings, we have $G \hookrightarrow \mathscr{F}(I(G)) \hookrightarrow \mathscr{F}(U)$.

Conclusion It was showed that $\mathscr{F}$ and I are faithful, and we also have $I \circ \mathscr{F} = 1_{\mathcal{L}}$. It follows that $\mathscr{F}$ is fully faithful, or an embedding. Consequently, if there is a universal order, then there is a universal graph. However, the existence of a universal graph does not guarantee the existence of a universal order.

$\square$

The conclusion shows that it is easier to have a universal graph than a universal order. As shown by M. Kojman and S. Shelah in [109], if there is no universal linear order in a cardinal λ, then there is no universal model in λ for any countable theory possessing the strict order property. Furthermore, the proof of the consistency of having a universal graph in $\aleph_2 < 2^{\aleph_0}$ generalizes the proof for the case $\aleph_1 < 2^{\aleph_0}$, but the consistency of universal linear order is true for the former case and is *false* for the latter.

5.5 Trees

5.5.1 Introduction

Trees occupy a central place in set theory, given that "[M]any problems in combinatorial set theory can be formulated as problems about trees" ([94], p. 113). Or, according to Devlin,

Lemma 5.5.1 (L. 7.2.2. in [39])**.** *Every set can be pictured by a tree.*

This result uses graph-theoretic terminology (in which a tree is defined as a graph such that for every node N there is a unique path starting from the top node and terminating at N), and it is also the case that such representations are not unique, presenting different or nonisomorphic trees, but it still points to interesting connections.

Like many combinatorial objects or properties, trees present a 'hidden depth'. As such, we could say that they represent the expression of certain difficulties and complexities connected to set theory in general and the problem of universality in particular. Consequently, the existence or non-existence of a universal family of trees or a universal tree emphasises the importance of the proofs and the techniques such results involve. Trees are partial orders where the set of predecessors to every element is well-ordered. A partial order satisfies all the requirements of a total order, with the exception of the trichotomy requirement. They are used in partition calculus and large cardinals. Trees can be considered as a very natural generalisation of ordinals, given that they can be defined as partial orders on an ordinal.

A trivial example is any ordinal δ with the usual order, with $ht(\delta) = \delta$ and $ht_\delta(\alpha) = \alpha$. Another example is the complete I-ary tree of height δ: $T =^{\alpha} I = \cup\{^{\alpha}I : \alpha < \delta\}$. A tree of size $\aleph_1$ with no uncountable branches could be understood as a way of coding all ω_1-sequences of infinite subsets of ω. By CH, $2^{<\omega_1}$ has cardinality $\aleph_1$. Many κ-trees are isomorphic to a subtree of the full tree $(^{<\kappa}\kappa, \subset)$.

The investigation of the structure of $\mathcal{T}$, besides trying to find the kind of chains and antichains that $\mathcal{T}$ has, the kind of hierarchies that can be isolated or the particular classes that it encompasses, it also involves the issue of the existence of a universal object. The universality of trees is connected to the universality of linear orders, but it also seems to present more challenges. The problem is interesting in the case of those classes of trees that do not have an unbounded branch, given that an unbounded branch would automatically give a universal object. But the structure of these classes is not completely understood and it became clear that ZFC alone is not sufficient for deciding statements about these trees. This chapter will focus on the class $\mathcal{T}$ of trees of size and height ω_1 with no uncountable branches. The framework for analysing them is determined by cardinal arithmetic and the nature of embeddability.

Well-founded - non-well-founded

We cannot formulate well-foundedness in first-order logic, but every well-founded partial order can also be extended to a well-order with the same universe ([65]), which implies that every universal well-order is a universal well-founded partial order.

M. Džamonja and K. Thompson show in [52] that well-founded partial orders have universal models in exactly the same cardinalities as the ordinals: for every $\lambda \geq \aleph_1$, there are λ^+ many well orders jointly universal for the well-founded partial orders of size λ, meaning that the universality spectrum for both the well-founded partial orders and the ordinals is the set of finite cardinals.

So when approaching trees, we could also make a distinction between well-founded models - when we are dealing with countable ones - and the non-well-founded case, when the models are uncountable.

Any two well-founded trees are comparable by order-preserving embeddability, and there is a family of ω_1 countable well-founded trees such that each one of them is order-preservingly mappable to another member of the family. According to a result due to Mekler and Väänänen ([147]), the same problem in the case of non-well-founded case is undecidable in ZFC+CH. There are non-comparable trees in the non well-founded case and the order structure in such class of such trees is complex.

The tree order itself though, $<_T$, is well-founded, and this aspect is used when doing induction and recursion on T.

Further definitions

Before going further, we will introduce some further definitions.

Definition 5.5.2. A *wide Aronszajn tree* is a tree of height and size κ with no restrictions on the cardinality of the levels.

We will use $\mathcal{T}$ for the class of wide Aronszajn trees, and $\mathcal{A}$ for the class of κ-Aronszajn trees.

Definition 5.5.3. A subset X of a tree T is *unbounded* in T if and only if the set of levels to which the elements of X belong $\{h(x) : x \in X\}$ is an unbounded set in the height of the tree T, $h(T)$.

Definition 5.5.4. A subset X of an ω_1-tree is called *stationary* (respectively a *club*) if and only if the set of levels to which the elements of X belong, $\{h(x) : x \in X\}$, is stationary (club, respectively) in ω_1.

Definition 5.5.5. Let $S \subseteq \omega_1$ be a stationary subset. A *regressive function* (or a *pressing-down function*) is a function $f : S \to \omega_1$ such that $(\forall \alpha \in S) f(\alpha) < \alpha$.

Theorem 5.5.6. *(Pressing Down Lemma - Neumar's Theorem) Let S be a stationary subset of an uncountable, regular cardinal κ and $f : S \to \kappa$ a regressive function. Then there is a $\beta \in \kappa$ and a $S' \in [S]^\kappa$ such that $(\forall \alpha \in S') f(\alpha) < \beta$ ($[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$). = For every regressive function $f : X \to \kappa$, there is some $\alpha < \kappa$ such that $f^{-1}(\alpha)$ is unbounded below κ.*

In a stronger form, due to Fodor, S' can be chosen to be a stationary set.

Theorem 5.5.7 (Pressing Down Lemma (Fodor))**.** *Let $S \subseteq \kappa$ be stationary and $f : S \to \kappa$ a regressive function. Then there is a stationary set $S' \subseteq S$ on which the function f is constant, i.e., there is some $\alpha \in \kappa$ such that $f^{-1}(\alpha)$ is stationary.*

Among the subsets of a tree, we can find a subtree or a branch.

5.5.2 Types of embeddings

For ordered structures, the embeddings constitute injective maps that preserve order. Both the linear order embeddings and the partial order embeddings are one-to-one functions that preserve order. In the case of partial orders, such as trees, we may also require that incomparability is preserved, dealing therefore with a notion of *strong embedding*. A *weak embedding* doesn't have to be one-to-one.

There are various ways of approaching the ordering of trees. Let T and T' be trees in a family of trees. Firstly, $T \leq T'$ if there is a an order-preserving function $f : T \to T'$ satisfying $x <_T y$ implies $f(x) <_{T'} f(y)$). The strict ordering $T < T'$ holds if $T \leq T'$ and $T' \nleq T$; $T \equiv T'$ if $T \leq T'$ and $T' \leq T$. Such a function, also called *reduction*, need not be injective or surjective, but any reduction of T to T' has the property that $ht_{T'}(x) \geq ht_T(x)$. A *weak embedding* from one tree to another is one which preserves levels, $ht_{T_1}(x) = ht_{T_2}(f(x))$ for all $x \in T_1$: the functions f map branches of T into branches of T. This kind of strict order-preserving function also measures how non-isomorphic the trees/models are. And we could also use the weak embeddings to quasi-order the classes $\mathcal{T}$ and $\mathcal{A}$.

The ordering (quasi-ordering) of trees arises naturally in infinitary model theory, particularly in the investigation of Ehrenfeucht-Fraïssé games (see [91]) and was also used to study non-well-founded inductive definitions ([221], for example). Trees of size $\aleph_0$ with no uncountable branches can be used in infinitary model theory as clocks of generalised Ehrenfeucht-Fraïssé games. In such cases, the ordering of trees can be defined in terms of a comparison game $G(T, T')$ with two players $\exists$ and $\forall$. The first player $\forall$ starts and moves an element of T', while the second one, $\exists$, responds with an element of T. The game goes on with $\forall$ playing elements of T' and $\exists$ playing elements of T, both in a strictly ascending order. The first player unable to move loses. In other words, $T \leq T'$ means that a game clocked by T is less complicated than a game clocked by T'. Hyttinen and Väänänen ([91]) used trees with no uncountable branches as invariants of uncountable models in the same way as ordinals (which, as I mentioned before, can be considered trees with no infinite branches) are used as invariants of countable models and they established the following lemma:

Lemma 5.5.8 (Hyttinen and Väänänen). *1. $T' \leqslant T$ iff $\exists$ wins $G(T, T')$.*

2. $T \ll T'$ iff $\forall$ wins $G(T, T')$.

Other forms of reduction include the injective reduction used by Mekler and Väänänen in [147], or the homomorphic reduction $\preceq$, which is an injective reduction function satisfying $x \leq_T y \iff f(x) \leq_{T'} f(y)$.

There are then reductions which have the property of preserving the 'distance' between nodes, which is another way of conceiving trees in $\mathcal{T}$ as subtrees of

${}^{\omega_1>}\omega_1$, and denoting by $\triangle(x, y)$ the first ordinal α where $x(\alpha) \neq y(\alpha)$. In [43], Džamonja and Väänänen study the reductions that preserve the value of $\triangle(x, y)$, with 'preserving $\triangle$' meaning that it corresponds to isometries of ω_1-metric spaces (Th.04).

A stronger type of embedding, determining a specific type of universality results with regards to a certain kind of well-founded partial orders, involves a club guessing method used by Shelah and Kojman ([109]), and later generalised by Kojman ([108]). By this method, Kojman finds a surjective homomorphism between certain subsets of $\mathcal{P}(\omega_0)$ (subsets of reals of bounded cardinality) ordered by the subset relation, on one hand, and the structure of enbeddability over well-founded partial orders (the monotone class of infinite graphs), on the other. These embeddings preserve both rank and order and is particularly useful in studying trees, as Väänäen and Todorčević show in [219]).

5.5.3 Types of trees

Aronszajn trees

It cannot be proved that an uncountable tree T ($|T| \geq \omega_1$) has an uncountable branch. But it is possible to construct an uncountable tree with countable levels and with no uncountable branch, i.e., an *Aronszajn tree.*

Definition 5.5.9. A κ*-tree* is a tree of cardinality κ in which each level has cardinality $< \kappa$. If, additionally, such a tree has no chains (branches) of cardinality κ, then we refer to a κ*-Aronzajn tree.* For $\kappa = \omega_1$, we simply refer to an *Aronszajn tree.*

Such a tree also presents *normality properties*: it is a rooted ω_1-tree such that all nodes on the same limit level have different sets of predecessors, every element of the tree has successors on all higher levels, and every non-maximal element has infinitely many immediate successors. We also say that all these properties describe a full well pruned Hausdorff ω_1-tree. Throughout this text, by *tree* we will generally mean *normal tree*; that is, each node of limit rank is determined by its set of predecessors, and every node has successors of arbitrarily high countable rank. It should also be pointed out that an ω_1-tree T whose levels are only finite does have an ω_1-branch, a consequence of König's lemma,

Theorem 5.5.10 (König's lemma)**.** *A finitely branching tree is infinite if and only if it has an infinite path.*

The generalisation of König's theorem for $\kappa = \aleph_1$ is false, as the existence of an Aronszajn tree testifies: there is an $\aleph_1$-tree without cofinal branches. But a generalisation of the König's theorem, due to Kurepa, is still possible. A *κ-Kurepa tree* is a κ-tree with at least κ^+ maximal branches or paths.

A cardinal κ for which no κ-Aronszajn tree exists is said to have the *tree property* (TP).

Definition 5.5.11. Let κ be a regular uncountable cardinal. κ has the *tree property* if every tree of height κ with levels of cardinality $< \kappa$ has a branch of cardinality κ.

The tree property is a combinatorial assertion having certain characteristics: it is a compactness principle, it is related to the behaviour of the continuous function, it it sensitive to forcing, and it is difficult to prove that it belongs to two adjacent cardinals. We can find the tree property at $\kappa = \omega$. No singular cardinal has the tree property, but it should also be emphasized that this property doesn't hold for all regular cardinals in general.

That being said, there are κ-trees that are not Aronszajn. An example is the ordinal κ itself, having exactly one branch of cardinality κ (or maximal chain of cardinality κ or a path). Another example is the tree of binary sequences of κ, $T = \{s \in^{<\kappa} 2 : |\alpha \in dom(s) : s(\alpha) = 1| < \omega\}$, having κ maximal branches.

Strengthenings of the notion of Aronszajn tree include the *Souslin tree* and the *special Aronszajn tree.* A Souslin tree is an ω_1-Aronszajn tree with no uncountable antichain. We can generalise these notions to arbitrary cardinals κ.

Definition 5.5.12. For any infinite cardinal κ, a κ-Souslin tree is a tree T such that $|T| = \kappa$ and every chain and every antichain of T has cardinality $< \kappa$.

By König's Lemma, there are no $\aleph_0$-Aronszajn trees. The existence of an Aronszajn tree is an essential property of the first uncountable cardinal: there is always an Aronszajn tree. In ZFC, there is always an Aronszajn tree (i.e. an ω_1-Aronszajn tree). But the existence of a Suslin ω_1-tree is independent of ZFC. Every κ-Suslin tree is a κ-Aronszajn tree. Without ZFC, there need not be a ω_2-Aronszajn tree. The existence of such a tree is consistent but independent of $ZFC + 2^\omega = \omega_2$ (Mitchell 1972).

We can also build special Aronszajn trees using Todorčević's method of walk on the ordinals ([217]), but we will not focus on this method here.

Construction of an Aronszajn tree. Special trees

CH implies that there are many different Aronszajn trees.

The definition of special Aronszajn tree has several equivalent variants and the literature also contains generalisations of the definition of a special Aronszajn tree.

The type of embedding is a way of approaching this aspect. We will start with the distinction $\mathbb{Q}$-embeddable - $\mathbb{R}$-embeddable.

Definition 5.5.13. If P is a partially ordered set, we say that T is *P-embeddable* if there is a strictly increasing mapping $f : T \to P$. A tree T is called *$\mathbb{R}$-embeddable* if and only if there is $f : T \to \mathbb{R}$ such that $x <_T y \to f(x) < f(y)$. $\mathbb{Q}$-embeddability is defined similarly.

As emphasised by Abraham, the construction of an Aronszajn tree involves a tension between compactness at ω_0 and incompactness at ω_1, the fact that the König's Lemma (any tree with infinitely many finite levels has an infinite branch), essentially about ω, cannot be applied to ω_1.

The existence of ω_1 trees with countable levels and without ω_1 branches represents a classical result due to Aronszajn (but published by Kurepa ([117]. Such a tree is constructed together with an order embedding of the tree into the rationals (the reason for being a special tree). So the construction of an Aronszajn tree is connected to the notion of *special* tree.

If we construct an $\aleph_1$-tree using sequences of rationals (as it was done originally), the tree T will be $\mathbb{Q}$-embeddable. When we say that a special Aronszajn trees is $\mathbb{Q} - \textit{embeddable}$, we do not require f to be injective, so it is not an embedding in the usual sense.

The members of the initial Aronszajn tree are strictly increasing finite and countable bounded sequences of rational numbers, the ordering is sequence extension, and the tree is built by recursion on the levels. In the construction, a special attention is paid at the limit level, conceived as the set theoretic union. An uncountable branch, i.e., a linearly ordered subset in an Aronszajn tree would meet level T_α for every countable ordinal α, so we would get an uncountable strictly increasing sequence of rationals, contrary to the fact that rationals are a countable set. Consequently, the Aronszajn tree constructed from ascending sequences of rationals does not consist of all sequences of rational numbers, since in that case, the tree would already be uncountable at level ω, so we have to make a selection among the sequences. The tree is constructed by recursion: we construct the αth level T_α (consisting of all α-sequences of rationals) after having constructed the subtree $T \restriction \alpha$ and in accordance to certain rules. At each limit stage, and for each $q \in \mathbb{Q}$, we pick inductively one sequence bounded by q to extend, by adding q as the maximal element. The function sending each such sequence to its supremum is a specializing function. The $\mathbb{Q}$-embeddable Aronszajn tree we obtain is a special

Aronszajn tree of cardinality $< 2^{\aleph_0}$, with no uncountable branch.

The process of *specialising* can thus be seen as a way to use the countability of $\mathbb{Q}$ to make the tree more *thin* given the levels of size continuum.

Baumgartner showed that if ZFC is consistent, then so is *ZFC + 'Every Aronszajn tree is* $\mathbb{Q}$*-embeddable'* ([11]). But whether or not every Aronszajn tree is $\mathbb{Q}$-embeddable is a question which cannot be decided under ZFC.

One can make trees special by forcing. For instance, if T is an $\aleph_1$-Aronszajn tree, there exists a ccc forcing $\mathbb{P}_T$ that adds a specialising function for the tree T. Baumgartner, Malitz and Reinhardt showed that $MA + \neg CH$ implies that all tree are special ([8]). Laver and Shelah ([124]) extended their result to κ^+, for κ regular, relative to/starting with a weakly compact cardinal bigger than κ. Golshani and Hayut ([72]) managed to force 'all κ-Aronsazajn trees are special' at many successors of regular cardinals.

From the definitions above, we could state the following facts:

Fact 1. A special κ^+-Aronszajn tree is a κ^+ Aronszajn tree.

Fact 2. A κ^+ Souslin tree is a non-special κ^+ Aronszajn tree.

Fact 3. A special κ^+-Aronszajn tree remains special after cardinal preserving forcing.

That being said, having no uncountable chains is necessary but not sufficient for being $\mathbb{Q}$-embeddable. There are trees with uncountable chains that are not $\mathbb{Q}$-embeddable. $\sigma\mathbb{Q}$ is an example and such trees have cardinality $2^{\aleph_0}$. So $\sigma\mathbb{Q}$ is not the union of countable many antichains (Kurepa). We could also construct κ-trees without cofinal branches which are not $\mathbb{Q}$-embeddable and which have cardinality $2^{\aleph_0}$ by using bi-stationary sets (more on that below).

Baumgartner ([10]) extends the usual definition of *special* by introducing a function $f : T \to \omega$, such that $\forall s, t, u \in T$, if $s < t, u$ and $f(s) = f(t) = f(u)$, then t and u are comparable (p. 41). It follows that

Definition 5.5.14. A tree T is special (in the usual sense) if the condition above holds and it has no uncountable branch.

In this context, he proves the following theorem:

Theorem 5.5.15 (8.1.). *If T is special then there are at most $\aleph_1$ uncountable branches through T.*

Antichains

We can also approach the notion of a *special* Aronszajn tree using the notion of an *antichain*. An antichain of T is a pairwise incomparable (or incompatible in a stronger sense, usually used in forcing) subset of T. For any partially ordered set P (not necessarily a tree) there is a strictly increasing function $f : P \to Q$ if and only if P is the union of countably many antichains.

An Aronszajn tree is special if it is the union of a countable collection of antichains. It is clear that a special Aronszajn tree cannot be Souslin, as it is impossible for an uncountable tree to be a union of countably many antichains all of which are countable. But as we will show bellow, an ω_1-tree could be *an almost-Souslin tree* if and only if it has no stationary antichain.

Before going further, it should be pointed out that the existence of an uncountable chain in T is related to the existence of an uncountable antichain, due to the fact that every node has two incompatible extensions at the next level: if b_α is an uncountable chain, i.e. a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$, and the height of each element in the chain $h(a_\alpha) = \zeta_\alpha$ for an increasing sequence of ordinals $\zeta_\alpha : \alpha < \omega_1$, we could choose an element $x_\alpha \neq a_{\alpha+1}$ of height $\zeta_{\alpha+1}$ such that $a_\alpha < x_\alpha$. All the elements x_α, $\langle x_\alpha : \alpha < \omega_1 \rangle$, would constitute an uncountable antichain.

Theorem 5.5.16. *Let $(P, <_P)$ be a partial order. There is an equivalence between the following two statements:*

(i) There is an order homomorphism between P and $\mathbb{Q}$, $f : P \to \mathbb{Q}$, such that

$$s <_P t \Longrightarrow f(s) < f(t).$$

(ii) If A_n, $n < \omega$, is an antichain, P is a union of countably many antichains, $P = \bigcup_{n<\omega} A_n$.

Kurepa had already proved ([118], Th.1, p. 172) the equivalence for all partial orders $(P, <_P)$, although the distinction between *special* and *non-special* trees was considered for a while only in relation to Aronszajn trees.

Consequently, we could say that a tree T is special if any of the following equivalent conditions hold: (i) T is the union of countably many antichains; (ii) there is a function $f : T \to \omega$ which is injective on chains; (iii) there is an increasing monotone function $f : T \to \mathbb{Q}$ on the chains of T.

We can generalise to any κ regular.

Lemma 5.5.17. *Let X be a partially ordered set $(X, <)$. X is the union of at most κ many antichains if and only if there is an embedding $f : X \to \kappa$ such that if $x < y$, then $f(x) \neq f(y)$, for every $x, y \in X$.*

Again, the function f is not required to be one-to-one.

Lemma 5.5.18. *Let κ be a regular cardinal and assume $\kappa^{<\kappa} = \kappa$. We say that a κ^+ Aronszajn tree is special if and only if any of the following equivalent conditions hold: (i) there is an embedding $f : T \to \kappa$ such that if $x < y$, then $f(x) \neq f(y)$, for every $x, y \in X$. (ii) T is the union of κ-many antichains.*

Equivalently, a tree $T \subseteq^{<\kappa} 2$ for $\kappa = \lambda^+$ is special if and only if there is a function $f : T \to \lambda$ such that for any path $c \subseteq T$, $f \restriction c$ is injective.

Non-stationary

A generalisation of the definition of special trees of successor height κ^+ defined as the union of κ many antichains was offered by Todorčević. This approach uses the theory of stationary sets and its distinctions. A starting point is Neumar's theorem: for every regressive function $f : X \to \kappa$, there is some $\alpha < \kappa$ such that $f^{-1}(\alpha)$ is unbounded below κ.

Definition 5.5.19 ([213]). Let κ be an uncountable, regular cardinal, and T a tree of height κ and $X \subseteq \kappa$.

1. X is *non-stationary* in T if there exists a regressive mapping $f : T \restriction X \to T$ such that for every $t \in T$, $f^{-1}(t)$ (that is the complete f-preimage of t), is the union of $(< \kappa)$-many antichains in T.
2. $T(X)$ is special if and only if X is non-stationary in T i.e. $f^{-1}(t)$ is a special subtree of T, for every $t \in T$.

In other words, if X is nonstationary, then $T \restriction X$ is a union of $|X|$ antichains. So if a tree is special, then all of its subtrees are special and therefore nonstationary. Consequently, if T is a tree of cardinality κ^+, then every κ-special subtree of T is a nonstationary subtree.

Proposition 5.5.20. *Let κ be a regular uncountable cardinal and S a stationary set. If S is non-stationary in T, then T is special and therefore it has no cofinal branches.*

Existence

Wide κ-Aronszajn trees exist for any κ. To obtain one, we could take a disjoint union of ordinals $< \kappa$.

If GCH holds, then for every regular cardinal κ there is a κ^+ Aronszajn tree (a result usually ascribed to Specker, [206], but he attributes it to Sikorski): we just imitate the construction of Aronszajn trees on $\aleph_1$ and find a κ^+ Aronszajn tree.

Proof. $\kappa = \kappa^{<\kappa}$ implies that κ is regular. But it is also equivalent to the GCH. Let κ be $\aleph_1$. If CH holds, then $\aleph_1^{\aleph_0} = \aleph_1$. So it follows that there is an $\aleph_2$

Aronszajn tree. □

Another proof for the existence of an $\aleph_2$ Aronszajn tree, for which we take into consideration hints from [115], would go as follows:

Proposition 5.5.21. *CH implies that there is an ω_2-Aronszajn tree.*

Proof. We will also imitate the construction of an (ω_1) Aronszajn tree, but we will replace ω_1 by ω_2 and, in accordance with [214], $\mathbb{Q}$ with a densely ordered set, $\mathbb{Q}^*$, such that every non-trivial interval of $\mathbb{Q}^*$ contains an ordinal $< \omega_2$: $\mathbb{Q}^* = \{g \in {}^{\omega}\omega_1 : \{n < \omega : g(n) \neq 0\}$ is finite$\}$. As a set, the tree T (Hausdorff or normal) is the ordinal ω_2. Each node has $\aleph_1$ immediate successors. The levels are constructed as follows: $\mathcal{L}_0 = \{0\}$, and for $0 < \alpha < \omega_2$, $\mathcal{L}_\alpha$ is uncountable. $T_\gamma = \cup_{\alpha<\gamma}\alpha$.

We will build the tree T as a sequence of functions $\langle f_\alpha : \alpha < \omega_2 \rangle$, such that, in the end, $T = \cup_{\alpha<\alpha_2}\{t \in \mathcal{L}_\alpha(T) : t = f_\alpha\}$.

By recursion on $\alpha < \omega_2$, we choose the functions $f_\alpha \in \mathcal{L}_\alpha(T)$ such that:

1. $f_\alpha : \alpha \to \aleph_1$ is one-to-one.
2. $\omega_1 \setminus ran(f_\alpha)$ is uncountable.
3. For $\beta < \alpha < \omega_2$, $f_\alpha \restriction \beta = f_\beta$. f_β and $f_\alpha \restriction \beta$ differ only on countably many places.

f_{ω_1} is the identity function. The induction step from α to $\alpha + 1$ does not pose any problems. Given $f_\alpha, f_{\alpha+1} = f_\alpha \cup \{(\alpha, i)\}$ for some $i \in \omega_1 \setminus \ ran(f_\alpha)$.

For λ a limit ordinal, we need to make a distinction between limit ordinals with cofinality ω and the ones having cofinality ω_1.

Assume λ is a limit ordinal and we have f_α for $\alpha < \lambda$. We choose $\alpha_0 < \alpha_1 < ...$, such that $\sup\{\alpha_i : i < \omega_1\} = \lambda$. Then we choose $t_i \in \mathcal{L}_{\alpha_i}(T)$ such that $t_0 =$

$f_{\alpha_0}, t_{i+1} = f_{\alpha_i+1}$ and $t_{i+1} \upharpoonright \alpha_i = t_i$. Let $t = \cup_i t_i$. Then $t : \lambda \to \omega_1$ is a one-to-one function, so $t \in \mathcal{L}_\lambda(T)$.

But when we want to construct an ω_2-Aronszajn tree, the cardinality of the nodes on level $\mathcal{L}_\alpha(T), \{t \in \mathcal{L}_\alpha(T) : t = f_\alpha\}$ is $2^{\aleph_0}$, not $\aleph_1$. As such, the construction works only if we assume CH.

The problem appear at the limit ordinals with countable cofinality, in the case where $\omega_1 \setminus \cup_i(ran(f_\alpha))$ is countable. So we will take $A = \omega_1 \times \omega_1$, $g_\alpha : \omega_1 \to \omega_1$, and finally $ran(f_\alpha) \subseteq \{(\delta, \gamma) : \gamma < g_{\alpha(\delta)}\}$ for some $g_\alpha : \omega_1 \to \omega_1$.

So it will follow that under CH, $f_\alpha \upharpoonright \beta : \beta \leq \alpha < \omega_2$ (ordered by inclusion) is an ω_2-Aronszajn tree. □

GCH represents a sufficient condition for the existence of κ^+ Aronszajn trees for regular κ. GCH is needed to exclude singular cardinals κ in the construction at limit stages λ: the number of all λ branches in $T(\lambda)$ is $\kappa^{cf(\lambda)}$, so we must assume $\kappa^{<\kappa} = \kappa$. As a consequence, if we look for a model of the universe without ω_2 Aronszajn tree, we would need to assume $\neg CH$.

Although GCH implies that there are κ-Aronszajn trees whenever κ is the successor of a regular cardinal, there are some (set-theoretic) difficulties involved when we are considering the successor of a singular cardinal. Jensen (mentioned in [95]) has shown that all such cardinals have κ-Aronszajn trees and even κ-Souslin trees if $V = L$. Magidor and Shelah ([135]) proved that there are no κ^+-Aronszajn trees if κ is singular and the strongly compact cardinals are unbounded below κ. A cardinal κ is strongly compact if and only if every κ-complete filter can be extended to a κ-complete ultrafilter; is a lot larger than a weakly compact cardinal, and its existence is (relatively) consistent with the GCH. But there is always a κ-Aronszajn tree when κ is the least strongly inaccessible cardinal ([115], ex. III.6.33).

Under CH, there are also many isomorphism types of Aronszajn trees. According to a result by Gaifman and Specker ([70]), there are $2^{\aleph_1}$ non-isomorphic Aronszajn trees. Also assuming $\kappa^\lambda = \kappa$, there are 2^{κ^+} non-isomorphic normal Aronszajn κ^+-trees ([70]). The number of isomorphism types of normal Aronszajn trees is $2^{\aleph_1}$.

The Souslin Hypothesis (SH) asserts that there are no Souslin trees. Jech and Tennenbaum proved the consistency of ZFC with the negation of SH.

Theorem 5.5.22. *The existence of $\mathbb{Q}$-embeddability of Aronszajn trees implies SH.*

Proof. Let T be a $\mathbb{Q}$-embeddable Souslin tree. Then T can be written as the reunion of countable many antichains $T = \cup\{T_q : q \in \mathbb{Q}\}$ with $T_q = \{t \in T : f(t) = q\}$. Since $|T| = \aleph_1$, it follows that some T_q is uncountable and since a Souslin tree has no uncountable antichains, we get a contradiction. □

Since ¬SH is consistent, so is the negation of the existence of a special Aronszajn tree and, therefore of a $\mathbb{Q}$-embeddable/special ω_1 tree.

The construction can be generalised to any κ. And every branch can be coded as a subset of κ.

In other words, the statement that all Aronszajn trees are $\mathbb{Q}$-embeddable, and therefore $\mathbb{R}$-embeddable, represents a strong form of SH ('there are no Souslin trees').

There are a few more aspects to consider in relation to the embeddability of trees. Due to a result by Baumgartner, we know that

Theorem 5.5.23. *If ZFC is consistent, then so is 'ZFC + Every Aronszajn tree is $\mathbb{Q}$-embeddable.*

If T is an ω_1 tree, we can define a $\mathbb{R}$-embeddable Aronszajn trees in relation to $\mathbb{Q}$-embeddable Aronszajn trees in accordance to [11]:

Proposition 5.5.24. *Let T be an ω_1 tree. T is $\mathbb{R}$-embeddable if and only if $T = \bigcup_{\alpha<\omega_1} T_{\alpha+1}$ is $\mathbb{Q}$-embeddable.*

Proof. Since $T = \bigcup_{\alpha<\omega_1} T_{\alpha+1}$ is $\mathbb{Q}$-embeddable it follows that $T = \bigcup_{n<\omega} T_{A_n}$, where A_n is an antichain. But $\{\alpha+1 : \alpha < \omega_1\}$ is a stationary set in ω_1, so some A_n is stationary. It follows that $T \restriction S$ is $\mathbb{Q}$-embeddable. □

We can generalise this result to any κ regular to describe $\mathbb{R}_\kappa$-embeddable Aronszajn trees using the notion of $\mathbb{Q}_\kappa$-embeddable Aronszajn trees:

Theorem 5.5.25. *Assume $\kappa^{<\kappa} = \kappa$. Let T an κ^+ tree. T is $\mathbb{R}_\kappa$-embeddable if and only if $T^* = \bigcup_{\alpha<\kappa^+} T_{\alpha+1}$ is $\mathbb{Q}_\kappa$-embeddable.*

We could also describe the specialising function as a function from a subtree $T(\alpha)$ (of height $T_{\alpha+1}$) of T , for every ordinal α, into ω, such that the pre-image of $\{q\}$ (for every $q \in \mathbb{Q}$) is an antichain in $T(\alpha)$. $T(\alpha)$ is both the set $\cup_{\beta<\alpha} T(\beta)$ and the tree on this set determined by the inclusion order (which is the ordering of T).

Since the ordering is inclusion, we are only concerned with which sequences each T_α will contain. The definition is by recursion on the levels. That is, we define T_α from $\bigcup_{\beta<\alpha} T_\beta$.

According to another result of Baumgartner from [11],

Proposition 5.5.26. *If T is $\mathbb{R}$-embeddable, then T is Aronszajn and every uncountable subset of T contains an uncountable antichain.*

It follows that no Souslin tree is $\mathbb{R}$-embeddable.

So given the definition of $\mathbb{R}$ or $\mathbb{Q}$-embeddability, an Aronszajn tree T is called non-Souslin (or non-Souslin in a strong sense) if every uncountable subset of T contains an uncountable antichain. But the converse is not true, in accordance to a result by Baumgartner :

Theorem 5.5.27 ([11])**.** *Assume $V = L$. Then there is an Aronszajn tree which is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.*

It means that $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable Aronszajn trees cannot be constructed in ZFC. It also means that if the axiom of constructibility is assumed, there is a Souslin tree. In other words, there are non-Souslin-trees which are not $\mathbb{R}$-embeddable. The statement is equivalent with the fact that if we assume $\diamondsuit^*$, then there is a $\mathbb{R}$-embeddable Aronszajn tree with no stationary antichains, a result due to Todorčević. Furthermore, by the Pressing Down Lemma, if the levels at which the nodes of a special subtree of T form a stationary set, then there is a stationary antichain (by the Pressing Down Lemma). An ω_1-Suslin tree has no uncountable antichain, and hence no stationary antichain, while every special ω_1-Aronszajn tree has a stationary antichain.

Devlin ([36]) uses a weaker assumption than $V = L$, i.e. $\diamondsuit$, which follows from $V = L$ (see Jensen), and proves

Theorem 5.5.28. *Assume $\diamondsuit$. Then there are $2^{\aleph_1}$ non-isomorphic Aronszajn trees $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.*

Forcings that add $\diamondsuit$ show how to generalise to any cardinal κ regular. According to a theorem of Beaudoin ([12]), we can generalise to any cardinal κ regular the fact that there are $\mathbb{R}$-embeddable κ Aronszajn trees that are not special and do not contain a Souslin substree of cardinality κ^+.

Theorem 5.5.29 (Th. 1.10.)**.** *Assume κ is a regular cardinal $\geq \omega_1$, $\kappa^{<\kappa} = \kappa$ and $\diamondsuit(E_\kappa^{\kappa^+})$. Then there is a normal κ^+-Aronszajn tree T which has no special κ-Aronszajn nor κ-Suslin subtrees of size κ^+.*

Solovay and Tennenbaum have proved that it is consistent to assume there is no Souslin tree. Shelah found ways to specialise trees that are still incompatible with the property of Souslin trees, and thus he showed that Souslin's Hypothesis does not imply every Aronszajn tree is special. He found forcings that *specialise* Aronszajn trees in a weak sense, with iteration preserving at least one non-special Aronszajn tree (see chapter IX of [192]). So we could take the following tree

Definition 5.5.30. An ω_1 tree T is an *almost-Souslin tree* if and only if it has no stationary antichain.

Every Souslin tree is an almost-Souslin tree.

Square principles

Both the square principles and Aronszajn trees represent combinatorial principles manifesting incompactness. A κ-Aronszajn tree is an incompleteness phenomenon because it does not have an unbounded branch. $\square$ represents a combinatorial principle exemplifying incompactness by asserting the existence of a cohering sequence of 'short' club sets, but such that no ('longer') club set threads or coheres with all of them. A generalization of this principle is due to Schimmerling ([59]). Jensen showed ([95]) that in $V = L$, the square principles hold for all uncountable cardinals. And in fact, there are also results establishing connections between square principles and Aronszajn trees. We will start with some definitions.

Definition 5.5.31 ($\square_\kappa$)**.** Let κ be a regular cardinal. A *square sequence*, $\square_\kappa$ on κ is a sequence $\langle C_\alpha : \alpha \text{ is a limit ordinal in } \kappa^+ \rangle$ such that

1. C_α is a club subset of α.
2. If $\beta \in \text{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.
3. If $cf(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

Definition 5.5.32 ($\square^*_\kappa$)**.** Let κ be a regular cardinal. A *weak square sequence*, $\square^*_\kappa$ on κ is a sequence $\langle \bar{C}_\alpha : \alpha \text{ is a limit ordinal in } \kappa^+ \rangle$, such that

1. For all $\alpha < \kappa^+$, C_α is a set of club subsets in α with $0 < |C_\alpha| < \kappa$).
2. For all $\alpha < \beta < \kappa^+$, α and β limit ordinals, and all $C \in C_\beta$, if $\alpha \in acc(C)$, the $C \cap \alpha \in C_\alpha$.
3. For all $\alpha < \kappa^+$ and all $C \in C_\alpha$, $otp(C) \leq \kappa$.

Todorčević introduces a weakening of $\Box_\kappa$ by replacing the order type restriction with the anti-thread aspect. So we get a sequence $\Box(\kappa)$ of clubs of α, endowed with the coherence property of a $\Box$-sequence, but which is not trivialised by a club of κ:

Definition 5.5.33 (general form)**.** Let κ be a regular, uncountable cardinal and $\lambda > 1$ another cardinal. A sequence $\Box(\kappa, < \lambda)$ is a sequence $\langle \bar{C}_\alpha : \alpha \text{ is a limit ordinal in } \kappa \rangle$ such that

1. For all $\alpha < \kappa$, C_α is a set of club subsets in α with $0 < |C_\alpha| < \lambda$).
2. For all $\alpha < \beta < \kappa$, α and β limit ordinals, and all $C \in C_\beta$, if $\alpha \in acc(C)$, then $C \cap \alpha \in C_\alpha$.
3. There is no club $D \subseteq \kappa$ such that for all $\alpha \in acc(D)$, $D \cap \alpha \in C_\alpha$.

The first result connecting square principles to trees belongs to Jensen.

Theorem 5.5.34 ([95])**.** *There is a special κ^+-Aronszajn tree if and only if $\Box^*_\kappa$ holds.*

Further results by Solovay, Gregory and Shelah show that if κ is uncountable, then $GCH + \Box_\kappa$ imply the existence of a κ^+ Souslin tree. In [217], Todorčević proved that if κ is a regular, uncountable cardinal, then $\Box_{\kappa,<\kappa}$ implies the existence of a κ-Aronszajn tree. Consequently, these results show that we don't need the GCH assumption when constructing a κ^+-Aronszajn tree, but we need it if we want to construct a Souslin tree (for $\kappa \geq \aleph_1$). For regular cardinal κ, $\Box^*_\kappa$ follows from GCH and the consistency of $ZFC + \neg\Box_\kappa$ is equivalent to that of 'ZFC + There is a Mahlo cardinal' ([95]).

Non-special trees

Laver had shown in ZFC that there is a tree of power $2^{\aleph_0}$ embeddable in the reals but not in the rationals, and such that each uncountable subset contains an uncountable antichain (a proof can be found in [11], mentioned in [8]). According to [8], Galvin found a tree of power $2^{\aleph_0}$ that cannot be embedded into the reals. It follows that there are non-special Aronszajn trees. So non-special Aronszajn trees have cardinality $\geq 2^{\aleph_0}$.

While working on trees in his approach to the Souslin Hypothesis, Kurepa considered their width, the cardinality of its levels ([117]), with a special focus on Aronszajn trees. So the distinction between special and non-special was not the defining characterisation of Aronszajn trees, although it was generally considered since then mainly in connection to Aronszajn trees. Being an Aronszajn tree involved a condition on the width of the trees, the cardinality of the levels. In 1935, Kurepa credits N. Aronszajn with showing that Aronszajn trees exist in ZFC (Theorem 6 in [117]). But he also proves the existence of a non-special tree with no uncountable branch, namely $w\mathbb{Q}$ (with its variant σ) ([116]), i.e., the collection of all well-ordered subsets of $\mathbb{Q}$ ordered by end-extension and so $w\mathbb{Q} = \mathfrak{c}$. So in trying to understand the class of trees of height ω_1 without uncountable branches, we cannot ignore the methodological distinction between special and non-special trees, in addition to aspects related to their width or the size of their levels. The distinction is related to the notion of forcing as well since the countability of the levels is essential in some forcing posets.

Given that for any infinite cardinal κ, a tree T is a κ-special tree if it can be written as a union of $\leq \kappa$ many antichains, otherwise T is a non-special κ tree, it follows that a non-special tree cannot embed weakly into a special one. But then there is also the option of specialising a tree in a *weak* sense and follow the consequences regarding universality.

In [192], Shelah proves that Souslin's Hypothesis (SH) does not imply that every Aronszajn tree is special by investigating weak notions of 'special Aronszajn tree' that are still incompatible with a tree being Souslin: *S-special* or special on a stationary set S, but also *S-r-special* and *S-st-special*. All types mean that every uncountable subset contains an uncountable antichain. These forcing notions "specialize" Aronszajn trees in a weak sense, and they can be iterated while still preserving at least one non-special Aronszajn tree. The concept of (T^*, S)-preserving, for instance (Definition 4.5), represents the property of forcing notions that ensures the tree T^* never gets fully specialized.

The work of Todorčević represents another important contribution to the study of trees and, in particular, non-special trees, as many results mentioned here testify.

We can show now that a non-special Aronszajn tree cannot be embedded into a special one:

Theorem 5.5.35. *A non-special Aronszajn tree cannot be embedded into a special one.*

Proof. Assume $\kappa^{<\kappa} = \kappa$.

Since T is special, there is an embedding $s : T \to \mathbb{Q}$ preserving the strict order: whenever $x < y$, then $f(x) < f(y)$.

Let T' be a non-special tree and T a special one. Suppose there is a weak embedding $f : T' \to T$ such that for all $x, y \in T'$, $x <_{T'} y \Rightarrow f(x) <_T f(y)$. A weak embedding is one that preserves levels, namely $ht_{T'}(x) < ht_T(f(x))$, for all $x \in T'$.

So there would be a function $f : \kappa^+ \to \kappa$ s.t. $f \circ s$ embeds T' in $\mathbb{Q}_\kappa$. □

Trees constructed from bi-stationary sets

The various definitions of a *special tree* mentioned above show that the non-existence of a cofinal branch for a special tree of regular cardinality cannot be destroyed by forcing, in the outer transitive models of a universe V we start with. But there are also tree without cofinal branches, non-special ones, that could also maintain this last property in an absolute way.

A bi-stationary set can be used to construct a tree $A(T)$ of height ω_1 of ascending and closed (bounded by a least element in ω_1) sequences of elements of A, partially ordered by inclusion: $s \leq t$ if s is an initial part of t. $T(A)$ trees were first studied by Todorčević in [213]. They represent a class of non-special trees that, when CH is assumed and they are part of $\mathcal{T}$, they are incomparable with the Aronszajn trees (see [219]). We will start with some facts about stationary and bi-stationary sets.

Fact 4. According to a result due to Ulam for successor ordinals and Solovay for the general case, if κ is a regular uncountable cardinal and S a stationary subset of κ, there is a family $\mathcal{F}$ of pairwise disjoint stationary subsets of κ, each contained in S, such that $|\mathcal{F}| = \kappa$. In other words, every stationary subset of κ is the disjoint union of κ stationary subsets.

It follows that there are 2^κ many distinct stationary sets in κ.

Given a stationary set S, its complement is co-stationary. A stationary and co-stationary set is called bi-stationary. They are non-Borel subsets of ω_1 and their existence, under the AC, was proved by Rudin in [178].

Theorem 5.5.36. *There is a subset $S \subseteq \omega_1$ such that neither S nor $\omega_1 \setminus S$ contains a club set.*

In other words,

Fact 5. If A is a bi-stationary set in κ, then there is no continuous map from a club of κ onto a cofinal subset of A.

There are ω_1 disjoint stationary subsets of ω_1 and therefore 2^{ω_1} bi-stationary subsets: for every $\alpha \neq \beta$ and bi-stationary subsets A_α and A_β, $A_\alpha \setminus A_\beta$ is bi-stationary.

Fact 6. Although S and its complement do not contain a club, each one of them intersects every club. If S doesn't, then $\omega_1 \setminus A$ would contain a club. It goes the same for the complement.

We generalise now to a cardinal κ.

Lemma 5.5.37. *Let S and T be bi-stationary subsets of κ, with $S \neq T$. Then there is no continuous map from S onto a cofinal subset of T and the class of all such sets has cardinality 2^κ.*

Proof. Suppose there is a continuous map from S onto a cofinal subset of T. Then T is stationary, and being a cofinal subset, $|f(S)| = \kappa$. It follows that $S \setminus T$ is not stationary. But that would contradict the bi-stationarity of T.

In accordance with Solovay's theorem, every stationary subset of κ is the disjoint union of κ stationary subsets. For every $\alpha \neq \beta$, $\alpha, \beta < \kappa$, we have $|\{S_\alpha : \alpha < \kappa\}| = |\{T_\alpha : \alpha < \kappa\}|$, where $S_\alpha \neq S_\beta$, $T_\alpha \neq T_\beta$, and $S_\alpha \neq T_\alpha$.

Each bi-stationary subset $X \subset \kappa$ contains two disjoint members of the collection of κ stationary sets, such that $U(X) = \bigcup\{S_\alpha : \alpha \in X\} \cup \bigcup\{S'_\alpha : \alpha \in \kappa \setminus X\}$.

If X and Y are distinct subsets of κ, $U(X) \setminus U(Y)$ contains a stationary set of κ.

So we have 2^κ bi-stationary sets X_α of κ, such that $X_\alpha \setminus X_\beta$ is bi-stationary whenever $\alpha \neq \beta$. □

The trees $T(A)$ constructed from a bi-stationary set A have the following characteristics. Every element $t \in T(A)$ is a closed, bounded, countable set of ordinals, representing initial segments of the stationary set A. It follows that the width of the tree is $2^{\aleph_0}$. Firstly, we show that

Lemma 5.5.38. *$T(A)$ is a tree of height ω_1 with no uncountable branches.*

Proof. A stationary set of ω_1 has closed subsets of all order types $< \omega_1$. It follows that $T(A)$ is a tree of height ω_1.

If T were the tree constructed starting from a stationary set, it would contain an uncountable branch. The tree $T(A)$ is bi-stationary, so the existence of a club in A, implied by an uncountable branch in $T(A)$, would lead to a contradiction with the co-stationarity of A. It follows that $T(A)$ is a tree of height $2^{\aleph_0}$ ($= \omega_1$ under CH) with no uncountable branches. □

Secondly, we prove that $T(A)$ is non-special using some elements from Todorčević ([214]):

Lemma 5.5.39. *$T(A)$ is not special.*

Proof. Every countable increasing sequence in ω_1 is bounded and has a supremum belonging to ω_1: if $\langle \alpha_n : n \in \mathbb{N} \rangle$ is any sequence of ordinals $< \omega_1$, $\alpha_\omega := \bigcup\limits_{n \in \mathbb{N}} \alpha_n$ is a countable ordinal and therefore $< \omega_1$; so $\alpha_n < \alpha_\omega + 1 < \omega_1$, $\forall n \in \mathbb{N}$.

We also know that every stationary set of ω_1 has closed subsets of all order types $< \omega_1$. So $T(A)$ will be the set of all countable subsets, closed in ω_1, ordered by end extension, and having no uncountable chains. $T(A)$ has cardinality $2^{\aleph_0}$, so given an uncountable subset S of $T(A)$, we could take a linear ordering on it, $\langle S, \prec \rangle$, as order isomorphic with a set of reals. So $\langle S, \prec \rangle$ is separable, that is, it has no end points and every infinite interval is dense.

Its nodes are sequences of countable ordinals and the $\prec$ also induces a linear ordering of each node.

Let D be the set of all maximal chains or branches of $T(A)$, ordered as follows. Let $s = min(b_1 - b_2)$ and $t = min(b_2 - b_1)$. Using the $\prec$ ordering, we take $b_1 \lhd b_2$ iff $max(s) \prec max(t)$.

The set D is then a linearly ordered continuum, but it is not separable. Given that the tree $T(A)$ has no uncountable chains, D has no uncountable well-ordered subsets. Furthermore, for every $\alpha < \omega_1$ (the tree $T(A)$ has length ω_1), let b_α be the closure of $\{b \in D : ot(b) \leq \alpha\}$. Given that each chain is countable, the closure is a nowhere dense set. D is then the reunion of ω_1 nowhere dense sets. □

Proposition 5.5.40. *There is a class of 2^κ pairwise incomparable non-special*

Aronszajn trees. There is a class of 2^{ω_1} pairwise incomparable non-special Aronszajn trees.

Proof. We simply apply the result above to the trees T(A) constructed from a bi-stationary set A.

Let A be a bi-stationary set of κ and $\mathcal{T}(\mathcal{A})$ the class of trees constructed from the bi-stationary sets of κ: $\{T(S)_\alpha : \alpha < 2^\kappa\}$.

There are κ disjoint stationary subsets of ω_1.

For the case of ω_1 and a subset S a bi-stationary set of ω_1, T(A) represents the tree constructed from S, and $\mathcal{T}(\mathcal{S})$ the class of these trees, $\{T(A)_\alpha : \alpha < 2^{\omega_1}\}$.

It follows that there are 2^{ω_1} bi-stationary subsets S_α of ω_1 such that $S_\alpha\ S_\beta$ is bi-stationary whenever $\alpha \neq \beta$. □

Theorem 5.5.41. *There is no universal element in the class of $\mathcal{T}(\mathcal{A})$ trees.*

Proof. It follows immediately from the results above. □

Remark. Since it doesn't have an uncountable branch, the tree $T(A)$ is embeddable in the reals: an $\mathbb{R}$-embedded uncountable branch would determine an uncountable well-ordered subset of the reals. It follows that it does not contain a Souslin tree. But it is not special either.

Forcing with $T(A)$ Forcing with $T(A)$ is equivalent to the forcing to shooting a club through the stationary A. This forcing does not collapse ω_1 and $T(A)$ is not special. The elements of $T(A)$ as a forcing notion constitute the set of all countable closed in ω_1 subsets of A ordered by end extension. Since it is a σ-closed forcing notion, it doesn't add reals, so the tree preserves its height, ω_1. Furthermore, since $\mathbb{P}$ doesn't add reals, $T(A)$ will be evaluated as $T(A)$ in $V^{\mathbb{P}}$, therefore still σ-closed and non-special. The tree will remains non-special in every forcing extension obtained by non adding reals. Such an aspect has an important consequence. We know that it is consistent with CH that the wide Aronszajn trees which can be embedded into an ω_1 tree are special. But the $T(A)$ trees are wide Aronszajn trees which *cannot* be embedded into ω_1 trees under ZFC.

Theorem 5.5.42. *Let $T(A)$ be a bi-stationary tree and $T \in \mathcal{T}$. There is no embedding between $T(A)$ and T.*

Proof. We point again the fact that the $T(A)$ trees are in the class $\mathcal{T}$ only if CH is assumed. We also know that if a tree is special, then all its subtrees are special. Furthermore, the forcing to specialise the T tree adds many $|T|$-many reals. Suppose that there is an embedding between $T(A)$ and T. Forcing to specialise the tree T will also specialise the tree $T(A)$, a contradiction. □

Other results include Todorčević (81, 84), Hyttinen and Väänänen (using Ehrenfeught Fraisee games) and Todorčević and Väänänen. For instance:

Proposition 5.5.43 (Pr. 39. in [219])**.** *Let $\leqslant$ denote a week embedding between two trees. If $A \subseteq \omega_1$ is bi-stationary and T is Aronszajn, then $T(A) \nleqslant T$ and $T \nleqslant T(A)$.*

Mekler and Shelah ([149]) have proved that the existence of a universal tree for the class of $T(A)$ - *the canary tree* (see below) - is independent of $ZFC + GCH$ and assuming $\diamondsuit$. Todorčević and Väänänen improve this result using a stronger form of $\diamondsuit$, $\diamondsuit^+(\prod_1^1)$ (a $\prod_1^1$-reflecting sequence).

Canary trees In [149], Mekler and Shelah introduce the notion of a *canary tree*, a tree of cardinality continuum with no uncountable branch, which gains an uncountable branch in any extension of the universe in which no reals are added, but whenever a stationary set is destroyed. They prove the following

Theorem 5.5.44. *The existence of a canary tree is independent of $ZFC{+}GCH$.*

The way to destroy a stationary co-stationary subset A of ω_1 is to force a club through its complement using as conditions in the partial order $\mathbb{P}$ closed subsets of the complement, ordered by end-extension ([9]). The forcing $\mathbb{P}$ will add an uncountable branch, which will be a club contained in A, but it will add no reals. And in any forcing extension in which no new reals are added and $\aleph_1 \setminus A$ is non-stationary, A will contain an uncountable branch.

An immediate consequence is that under CH, specifically, $2^{\aleph_0} = 2^{\aleph_1}$, there is a Canary tree. And if we take the tree of all closed bounded subsets of a bi-stationary set A, T(A), we can construct a Canary tree which almost contain the union of all trees T(A) (Theorem 1 in [149]).

In [147], Mekler and Väänänen improve on Theorem 1 in [149] and show, using Ehrenfeucht-Fraïssé games, that if T is a Canary tree, then $T(A) \leq T$ for all bi-stationary subset $A \subseteq \omega_1$.

Theorem 5.5.45 (Th. 23.)**.** *Suppose T is a Canary tree and $T(A)$ the tree constructed from any bi-stationary set $A \subseteq \omega_1$. Then $T(A) \leq T$.*

In other words, T is a Canary tree if and only if for every bi-stationary set A there is an order preserving function from T(A) to T. If $MA+\neg$CH hold, then all Aronszajn trees are special, but T(A) for stationary A is never special (according to Theorem), so no Canary tree can exist under these conditions.

Forcing and special trees

Many significant results mentioned above imply a connection between embedding, finding universal trees and specialisation. Any Aronszajn tree can be forced to be special and the specialising forcing is ccc and absolute (consisting of finite partial specialising functions). It follows that this forcing remains ccc in any extension in which T remains Aronszajn. The connection between forcing and special trees was first investigated in [8]. The ω_1-closed forcing poset used to add a specialising function with finite approximations is ccc and therefore $MA+\neg CH$ implies that all trees of height $\aleph_1$ with no uncountable branches and cardinality $< 2^{\aleph_0}$ are special. This poset adds reals. So under $MA(\aleph_1)$, there is no Souslin trees and, still more, every Aronszajn tree is special.

A Souslin tree is ccc and can be used as a forcing poset, $\mathbb{P}_T$. For any ω_1 tree T, $\mathbb{P}_T$ is the forcing whose conditions are finite partial functions $f : T \to Q$ s.t. if $t, s \in dom(f)$ and $t <_T s$, then $f(t) < f(s)$ (in $\mathbb{Q}$, the rational ordering). A Souslin poset is ccc, but the product of two Souslin posets is not ccc since the set of pairs of immediate successors of each node in the tree gives an uncountable antichain.

Forcings of the form $\mathbb{P}_T$ can collapse cardinals. If T is special, forcing with T collapses ω_1: given any cofinal branch b in the extension and any specialising function f, the restriction $f \restriction b : b \to \omega$ would be an injection of a set of size $(\omega_1)^V$ into ω_0: the branch has size ω_1^V and will contain at most one node from each of the countable number of antichains witnessing that T is special. In other words, a special Aronszajn tree must remain Aronszajn in any ω_1-preserving forcing extensions, including ccc extension. Furthermore, this kind of forcing adds $|T|$-many reals. The generic specialising function is a Cohen real. So one could ask if there is a similar poset for specialising trees that does not add reals so that these results could be studied in the CH context.

Of course, ccc is a desirable property for a forcing notion to have, given that the forcing will preserve cardinals and cofinalities. But forcing may also require iteration and, in that case, one needs to use finite supports in order to preserve the property. Properness (and the Proper Forcing Axiom, PFA) introduced by Shelah, represents a more general property and it does not collapse $\aleph_1$. In addition to that, if the forcing notion also has the $\aleph_2$-chain condition, all cardinals are preserved. In [192], Shelah uses stationary sets to define his forcing notion: a partial order $\mathbb{P}$ is proper if it preserves stationary subsets of λ^{ω_0} for all uncountable cardinals λ. Another equivalent definition uses elementary substructures of H_λ ([192], p. 102).

In [192], Shelah introduced new proper forcing techniques for specialising an Aronszajn tree, including one that did not add reals, and others that would specialise an Aronszajn tree on a stationary set, but not on the rest of the tree. He was thus able to build a model with no Souslin trees and without making all Aronszajn trees specials. Among other things, PFA implies that any two Aronszajn trees are club-isomorphic. MA is consistent with arbitrarily large values for the continuum, but according to a result by Todorčević and Velickovic, PFA implies that the continuum must be $\aleph_2$.

Another generalisation of ccc forcings but also of the Proper Forcing Axiom (PFA) is MM, Martin's maximum, introduced by Foreman, Magidor and Shelah in 1988. Similar to PFA, it replaces the class of posets with ccc in MA by larger classes of posets, the ones preserving stationary subsets (of ω_1). Both are very strong forcing axioms, having large cardinal strength, i.e. relative to the existence of a supercompact cardinal (proofs in [38] and [134], respectively).

The ccc poset works irrespective of the widths of the tree being specialised, whereas the countability of the levels is essential in the posets used by Jensen and Shelah. These aspects could determine new approaches to Aronszajn trees without the condition of countable levels.

Theorem 5.5.46 (Todorčević)**.** *There is in ZFC an Aronszajn tree T of width continuum that cannot be specialised without adding reals.*

5.5.4 Universal tree - results

Let $\mathcal{T}_{\omega_1}$ be the class of trees of cardinality ω_1 with no uncountable branches. This class contains a great variety of trees (Aronszajn trees, special Aronszajn trees, non-special Aronszajn trees, Souslin trees, etc.). We consider this class under weak embeddings, i.e., functions that preserve the strict order, and which are not necessarily 1-1 (although they are 1-1 on branches). If we are working in a model with CH, we have $\aleph_1 = 2^{\aleph_0}$, and then we can also consider trees of cardinality $2^{\aleph_0}$ with no uncountable branches.

If κ is a singular cardinal, then there is a κ-tree without cofinal or unbounded branches. A consequence is that singular cardinals do not have the tree property. So the nontrivial case is when κ is a regular cardinal. The class of Aronszajn trees is not a first order theory since we cannot express the non-existence of an unbounded branch in first order theory, so under CH we cannot apply model theory results in tackling the universality problem for these objects.

We should start by pointing up the fact that the existence of a model in which CH holds and there is a universal Aronszajn tree (or a wide Aronszajn tree) under CH is still an open problem. But under CH, we know that there is no universal wide Aronszajn tree (σ operator). From [206], we know that if GCH holds then for every regular κ there exists an Aronszajn κ^+-tree. It is also consistent with GCH and embeddings on a club many levels that there is a universal Aronszajn tree ([1]). Recall that to say that a sentence φ (like 'Any two Aronszajn trees are isomorphic on a club') is consistent with ZFC means that 'if ZFC is consistent, then ZFC $+\varphi$ is consistent'.

However, things get complicated in the absence of GCH. Furthermore, since many statements about Aronszajn trees are independent of ZFC, we need additional axioms which, as Tatch Moore emphasizes ([153]), fit into two classes: enumeration principles and forcing axioms, with both categories coming as progressively stronger lists. The first one comprises consequences of $V = L$ and enable the construction of objects with second order properties (Souslin trees, for instance) by diagonalisation of length ω_1: CH, $\Diamond$, $\Diamond^+$. The latter, such as $MA(\aleph_1)$, PFA and $PFA(\kappa)$, are generally used "to limit diagonalisation constructions of length ω_1 to those which can be carried out in ZFC" ([153]) and they "can be viewed as postulating forms of $\sum_1$-absoluteness between V and its generic extensions" ([153], p. 4).

And another general aspect to emphasise is that there are less results about the class $(\mathcal{T}, \leq)$ of trees of size and length ω_1 with no uncountable branch then the class $(\mathcal{A}, \leq)$ of Aronszajn trees.

The σ operator

The σ-function or operation for partially ordered sets was introduced by Kurepa ([117]). His general result using this definition is the following:

Theorem 5.5.47. *For any structure (X, R) where X is a set with one binary relation R, then σX does not embed into X.*

Proof. Suppose there is an embedding $e : \sigma X \to X$. But σX is defined as the collection of all sequences $f : On \to X$, with $f(\alpha) = e(f \restriction \alpha)$. Given that the embedding preserves the relations, the function f is well-defined for all ordinals, and therefore $f \in \sigma X$. But the range of the function would form a proper class, and we would then get a contradiction with X being a set. □

A straightforward corollary, which doesn't need a proof, is the following:

Corollary 5.5.48. *For any partial order P, σP is not P-embeddable.*

Kurepa proves ([116]) the existence of a non-special tree with no uncountable branch, namely $w\mathbb{Q}$, with its variant σ, i.e., the collection of all bounded, well-ordered subsets of $\mathbb{Q}$, ordered by end-extension and so $w\mathbb{Q} = \mathfrak{c}$.

For completeness, we will also show that

Lemma 5.5.49. *σP can also be a tree.*

Proof. Suppose P is a tree whose elements $p_i \in P$ and for $i < j < i^*$, we have $p_i <_P p_j$. The elements of σP are sequences $p = \langle p_i : i < i^* \rangle$ ordered by inclusion: $\bar{p} \leq_{\sigma P} \bar{p}'$ if and only if $\bar{p} \subset \bar{p}'$. Suppose σP isn't a tree and suppose there is a strictly decreasing chain $\langle \bar{p}_i : i < \omega \rangle$ in σP. Let l_i be the length of each $\bar{p}_i$. But then $\langle l_i : i < \omega \rangle$ would constitute a strictly decreasing sequence of ordinals, which is a contradiction. □

Another corollary, when applying this operator to trees is that $\sigma\mathbb{Q}$ is not the union of countably many antichains. Given all of the above, we can state as obvious the following

Proposition 5.5.50. *The tree $\sigma\mathbb{Q}$ is universal for all normal trees having strictly increasing embeddings into the reals. We could also write $T < \sigma T$.*

Corollary 5.5.51. *There is no weak embedding $f : \sigma(T) \to T$: if T has no uncountable branch, neither is σT, but the cardinality of σT is $2^{\aleph_0}$, while the cardinality of T is ω_1.*

On one hand, the σ-function or operation represents an obstacle for the existence of a universal family for the class of all trees with no uncountable branches, $\mathcal{T}(\kappa)$. When the universal family is a set, we can apply this operator to its supremum and obtain a tree which contradicts the universality of the family. As a consequence, and one with various manifestations since Kurepa's publication of his result, is that the examination of the universality problem is restricted to various classes of trees.

Proposition 5.5.52. *Assume CH and let $\mathcal{T}_{\omega_1}$ be the class of trees of cardinality ω_1 with no uncountable branches. The σ function forbids the existence of a universal family of size $\leq \omega_1$ for $\mathcal{T}_{\omega_1}$.*

Proof. The class $\mathcal{T}_{\omega_1}$ is closed under the σ-function. The two kinds of trees have different cardinalities, but if CH holds, they are all in the same class, $\mathcal{T}_{\omega_1}$. So it follows immediately from the last corollary above that the class $\mathcal{T}$, and consequently $\mathcal{A}$, have no universal element. □

That being said, it is not always easy to find the right structure on the set $\sigma\mathcal{A}$ to produce a counterexample to universality.

Hyttinen and Väänänen ([91]) used the σT operator to define a stronger ordering of trees, a relation which is well-founded:

$$T \ll T' \text{ iff } \sigma T \leqslant T'$$

Proving $T \ll T'$ gives you directly $T' \not\leq T$. They use Ehrenfeucht-Fraïssé games.

But the σ function could also be conceived as a way of producing trees and, as emphasised by Todorčević, the σT is a tree which contains a lot of information about T. For instance, $\sigma\mathbb{Q} =_{\text{def}} \sigma((\mathbb{Q}, <))$ is an example of a Hausdorff tree, a tree where different nodes of a limit level have different sets of predecessors.

Theorem 5.5.53 (Th.8. in [219])**.** *The tree $\sigma\mathbb{Q}$ is a universal object in the class of $\mathbb{R}$-embeddable Hausdorff trees of cardinality $\leq 2^{\omega}$.*

So it is used both in the search for counterexamples to universality in different classes of structures and in finding universal elements.

Various results

Jensen proved that CH is consistent with SH. SH implies that there are no Souslin trees. Consistently, CH holds and all Aronszajn-trees are special Jensen's argument was built on an elaborate ccc forcing, a completely new iteration technique.

Another proof of this result was offered Shelah, using countable support iteration of proper posets. He strengthened this result by showing that there is a universal Aronszajn tree for all the club-isomorphic Aronszajn trees. Such a universal Aronszajn tree is special. He showed that it is consistent with CH (GCH actually) that every two Aronszajn trees contain subtrees which are isomorphic on a club (they are considered 'near' or, equivalently, some tree is embeddable on a club set into both of them) ([1]).

Gaifman and Specker had showed that there are $2^{\aleph_1}$ non-isomorphic Aronszajn tree ([70]). But every two of these non-isomorphic trees are isomorphic on a club, i.e., there is a club $C \subseteq \omega_1$ such that the restriction to the levels in C gives two isomorphic trees. And these are the only kind of isomorphic Aronszajn trees that we can.

Definition 5.5.54. A tree T_1 is embeddable on a club into T_2 if and only if there is a club $C \subseteq \omega_1$ such that there is an embedding of $T_1 \upharpoonright C$ into $T_2 \upharpoonright C$.

Theorem 5.5.55. *Assume PFA. Then every two Aronszajn trees are club-isomoprhic.*

In chapter IX in [192], Shelah shows that SH does not imply that every Aronszajn tree is special. But if the Aronszajn tree is S-special or special on a stationary set S (there is a monotonic increasing function f from $\cup T_\alpha$ to the rationals, i.e., $x < y \Longrightarrow f(x) < f(y)$) or under other forms of specialisation on a stationary set that Shelah introduces in this chapter (S-r-special and S-st-special), then T is a special Aronszajn trees and not Souslin, since every uncountable subset of these tree contains an uncountable antichain. Furthermore, if all Aronszajn trees are S-special (under other forms of specialisation on a stationary set - S-r-special and S-st-special) for some given unbounded subset of ω_1, then all of them are in fact special.

In [1], Avraham proves the consistency of 'CH + There exists a Souslin tree + There exists a special Aronszajn tree which is universal among all the Aronszajn trees which do not contain a Souslin tree' (§ 4). In such a model, the disjunction is strict: there is a Souslin tree and every Aronszajn tree either contains a Souslin tree or is a special Aronszajn tree (and a tree $\mathbb{Q}-$embeddable on a club is a special tree). Also in this model, there are $\aleph_1$ many Souslin trees up to a club.

In [40] Devlin and Shelah had showed that there is a weak form of diamond, equivalent to $2^{\aleph_0} < 2^{\aleph_1}$, partially substituting $\diamondsuit$, and following from the CH. Using this principle, Abraham and Shelah show in [1] that there are $2^{\aleph_1}$ Aronszajn trees not pairwise embeddable, and none of them embeddable on a club into the other. Furthermore, the only embedding of each tree into itself is the identity: for every club $C \subseteq \omega_1$, the only embedding of $T|C$ into $C|T$ is the identity, i.e. $T|C$ is rigid (and, therefore, the tree T is *really rigid*). This result represents a sufficient condition to construct two non-club-isomorphic Aronszajn trees.

According to another result from the same paper ([1]), $MA + \neg CH$ does not imply that any two Aronszajn trees are isomorphic on a club. But for κ regular $\geq \aleph_2$ and $\kappa^{\leq\kappa} = \kappa$, it is consistent with $MA + 2^{\aleph_0} = \kappa$ that every two Aronszajn trees are isomorphic on a club. The proof for $2^{\aleph_0} = \aleph_2$ uses proper forcing, while the one for $\kappa > \aleph_2$ involves generic reals (see [1]).

A consistency result, due to Mekler and Väänänen in [147], establishes the following:

Theorem 5.5.56 (15). *Assume CH. Let κ be a regular cardinal with $\aleph_2 \leq \kappa \leq 2^{\aleph_0}$. There is a forcing notion that preserve cardinals, cofinalities, and 2^λ for all λ and has a set $\mathcal{U}$ of trees in $\mathcal{T}_{\omega_1}$ such that*

(1) $\mathcal{U}$ has cardinality κ.

(2) If $\mathcal{A} \subset \mathcal{T}_{\omega_1}$ has cardinality $< \kappa$, then there is some $S \in \mathcal{U}$ such that $T \leq_1 S$ holds for all $T \in \mathcal{A}$.

Corollary 5.5.57 (16). *Assume CH. Let κ be a regular cardinal with $\aleph_2 \leq \kappa \leq 2^{\aleph_0}$. There is a forcing extension that preserves cardinals, cofinalities, and 2^λ for all λ and satisfies $U(\kappa)$ and $B(\kappa)$.*

$U(\kappa)$ stands for the assumption that there is a universal family of size κ for $\mathcal{T}_\omega$, while $B(\kappa)$ states that every subset of $\mathcal{T}_\omega$ of size $< \kappa$ is bounded in $\mathcal{T}_\omega$. $\mathcal{U}$ is a universal family of trees. $\leq_1$ stands for injective reduction between trees, so the result shows that both the universality number of $(\mathcal{T}, \leq)$ and the universality number of $(\mathcal{T}, \leq_1)$ - or $(\mathcal{A}, \leq)$ - are κ. This theorem offers a model "in which some subset of $\mathcal{T}_{\omega_1}$ of cardinality $\aleph_2$ is universal, hence unbounded, and another model in which every subset of $\mathcal{T}_{\omega_1}$ of cardinality $\leq \aleph_2$ is bounded" ([147], p.1059).

Another way of viewing this result is we are also introducing a cardinal invariant, the minimal size of a subfamily of structures $\mathcal{F}$ which does not embeds into any member of the family: if $\mathcal{S}_\kappa$ represents the set of structures having size κ and the complexity number $c(\mathcal{S}_\kappa) > 1$, then $b(\mathcal{S}_\kappa) =: \{min|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{S}_\kappa(\neg\exists X \in \mathcal{S}_\kappa)(\forall f \in$

$\mathcal{F}$) there is an embedding $g : f \to X\}$. Mekler and Väänänen's result show that $b(\mathcal{T}_{\aleph_1})$ and $c(\mathcal{T}_{\aleph_1})$ are both κ.

Another result, found by Džamonja and Väänänen in [43], involves the failure of CH, the presence of a club guessing at ω_1, and the use of weak embeddings that satisfy a strengthening of the Lipschitz condition ($\triangle$-preserving: $\triangle_{T_1}(x, y) = \triangle_{T_2}$ $(f(x), f(y)$, for two trees T_1, T_2):

Theorem 5.5.58. *Suppose that*

(a) there is a ladder system $\bar{C} = \langle c_\delta : \delta < \omega_1 \rangle$ *which guesses clubs, i.e. satisfies that for any club* $E \subset \omega_1$ *there are stationarily many* δ *s.t.* $c_\delta \subset E$*,*

(b) $\aleph_0 < 2^{\aleph_1}$

Then no family of size $< 2^{\aleph_1}$ *of trees of size* $\aleph_0$*, even if we allow uncountable branches, can* $\leq$*-embed all members of* $\mathcal{T}$ *in a way that preserves* $\triangle$*.*

In [44], M. Džamonja and J. Väänänen extended Scott's analysis of countable models to chain models of size κ, with κ a singular cardinal with countable cofinality in particular, and discovered that the relevant clocks of these games are κ-trees (or bounded κ-Trees). As M. Džamonja points out in [46], these κ-trees "have properties that make them similar to ordinals" (p. 3), due to a natural notion of rank. A clock is the rank attached to each pair (A, B) of non-isomorphic models, an ordinal $\alpha < \omega_1$ as the Scott watershed $S(A, B)$. Trees of size $\aleph_0$ with no uncountable branches appear in infinitary model theory as clocks of generalised EF games (they give rise to first countable ω_1-metric spaces). Using this notion, they can show that the universality number of κ-trees (bounded κ-trees) under reduction is κ^+, and that within each rank in $[1, \kappa^+)$, the universality number is ω_0. If κ is a regular cardinal, in particular $\kappa = \aleph_0$, the universality number for κ-trees (bounded κ-Trees) under reduction cannot be computed in ZFC, and the consistency of this number being equal to 1 for $\kappa = \aleph_0$ is not known.

In [217], Todorčević develops the concept of coherent trees, which are Aronszajn trees obtained from ordinal walks, and he shows that coherent trees are totally ordered by $\leq$. In [216], he defines the class of Lipschitz trees as those trees on which every weak embedding (so level-preserving) from an uncountable subset of T has an uncountable Lipschitz restriction. The Lipschitz condition on a weak embedding f stipulates the following: $\triangle_{T_1}(x, y) \geq \triangle_{T_2}(f(x), f(y))$. Furthermore, considering embeddings between Aronszajn trees into Lipschitz ones, Todorčević ([216]) proves that assuming $BPFA^{\aleph_1}$ (the bounded proper forcing axiom for partial orders of cardinality $\aleph_1$), there is no universal element in $(\mathcal{A}, \leq)$.

Theorem 5.5.59. *1 The class $\mathcal{C}$ of Lipschitz trees is not well-quasi-ordered under $\leq$.*

2 There is a family of size $2^{\aleph_1}$ of pairwise incomparable Aronszajn trees in the order $\leq$ [Theorem 3.4.].

3 Assuming the bounded proper forcing axiom for partial orders of cardinality $\aleph_1$ ($BPFA^{\aleph_1}$), the class $\mathcal{C}$ of Lipschitz trees form a chain which is coinitial and cofinal in $(\mathcal{A}, \leq)$ and it has neither a maximum nor minimum element. [Corollary 7.6].

Corollary 5.5.60. *Assuming $BPFA^{\aleph_1}$, there is no universal element in $(\mathcal{A}, \leq)$.*

Furthermore, he proves the following:

Theorem 5.5.61 ([216] Th.7.3.)**.** *Assume MA(ω_1).*

- *All coherent trees are Lipschitz trees.*
- *Every Aronszajn tree embeds into a coherent tree.*
- *There is no universal element in $(\mathcal{A}, \leq)$.*

In [51], Džamonja and Shelah offer a solution to a previous open problem (due to an error in [147]) and show (among other things) that under $MA + \neg CH$ there is no universal wide Aronszajn tree. It is a result that can be obtained using Todorčević's methods as well. They introduce a new specialising function for the forcing, one that is specialising the pair of trees simultaneously. The proof of ccc, although uses the Baumgatner, Malitz, Reinhardt Lemma, is more complicated. They first show that there is no universal Aronszajn tree. They continue by proving that every wide Aronsjan tree weakly embeds into an Aronsjan tree, in other words, under $MA + \neg CH$, the class $\mathcal{A}$ is cofinal in the class $\mathcal{T}$:

Theorem 5.5.62 (4.1)**.** *For any tree $T \in \mathcal{A}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ not weakly embeddable into T. In particular, under the assumption of $MA(\omega_1)$ there is no Aronszajn tree universal under weak embeddings.*

Theorem 5.5.63 (5.1)**.** *For every tree $T \in \mathcal{T}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ into which T weakly embeds. In particular, under the assumption of $MA(\omega_1)$ the class $\mathcal{A}$ is cofinal in the class $(T, \leq)$.*

Putting the above results together, they obtain:

Theorem 5.5.64 (6.1)**.** *Under $MA(\omega_1)$, there is no wide Aronszajn tree universal under weak embeddings.*

From [206], we know that if GCH holds, then for every regular κ there exists an Aronszajn κ^+-tree. However, not much is known in the absence of GCH and even assuming GCH, not much is known about TP for successors of singular cardinals (singular cardinals do not have the TP). Kurepa's result that a κ^+-trees exist for every κ is false.

Some difficulties in finding universal trees

The results mentioned so far show a close relation between universality and the notion of a *special* tree. But one of these difficulties involves the fact that one cannot improve the results obtained at $\aleph_1$ to $\aleph_2$ for several reasons.

When trying to apply some of the methods used in the case of ω_1 trees, the specialisation function receives a central role. As I mentioned before, it witnesses the fact that a tree T has no unbounded or cofinal branches, since that would determine an injective function from a set of size κ^+ to κ. When T is special, it remains Aronszajn in any larger model of ZFC in which κ^+ is a cardinal. Baumgartner, Malitz and Reinhardt showed that $MA + \neg CH$ implies that all tree are special. But the analogue of their lemma, used in proving that a forcing notion is ccc, is not true for $\aleph_2$-Aronszajn trees. From MA and ¬CH ($\aleph_1 < 2^{\aleph_0}$) it follows that every $\aleph_1$ tree (having cardinality $\aleph_1$) with countable chains and $\mathbb{Q}$-embeddable is consistent. The same is true for Aronszajn trees. In other words, according to the BMR Theorem ([8]), under $MA + \neg CH$, all Aronszajn trees are special. But this theorem is not true for $\aleph_2$ Aronszajn trees. We also know that PFA implies that there are no ω_2-Aronszajn trees. PFA also implies that $MA + 2^{\aleph_0} = \aleph_2$.

We already saw that if GCH holds, then there is an $\aleph_2$ Aronszajn tree. An Aronszajn-tree exists in ZFC, but for the $\aleph_2$ case, we have different consistency results and we need to assume a large cardinal axiom, in particular, a weakly compact cardinal. There are many definitions of such a cardinal, offering different perspectives adjusted to various contexts (of problems and proof methods).

Definition 5.5.65 ([94], Lemma 9.26)**.** κ is *weakly compact* if and only if it is inaccessible and has the tree property.

A cardinal κ has the *tree property* if every tree of height κ with every level having $< \kappa$ elements has a branch of length κ. As a consequence, the κ Aronszajn-tree represents a counterexample to the tree property at κ.

Assuming a large cardinal axiom, Cummings and Foreman ([29]) proved that it is consistent that no $\aleph_n$ A trees exist for any finite n other than 1.

Mitchell showed that the inexistence of a ω_2 Aronszajn-tree is equivalent to the consistency of the existence of a weakly compact cardinal ([152] and also Silver). If κ is an inaccesible cardinal, the Mitchell forcing poset adds κ Cohen reals (subsets of ω), it preserves ω_1, and collapses all cardinals between ω_1 and κ, such that κ becomes ω_2 and the poset forces that $2^\omega = \kappa$. This was a poset that was extensively used and received several variations since it was introduced in 1972. Baumgartner gave a simpler proof using proper forcing.

To resume the last paragraphs, if κ is weakly compact, then there are no κ Aronszajn trees. Conversely, if κ is inaccessible and there are no κ Aronszajn-trees, then κ is weakly compact. In other words, one can force both a model in which there are no $\aleph_2$ Aronszajn-trees and if κ is regular and there are no κ^+ Aronszajn-trees, then κ^+ is weakly compact in L.

On the other hand, by a result due to Laver and Shelah ([124]), one can force a model in which CH holds and all $\aleph_2$ Aronszajn trees are special. But in this model, constructed starting with a weakly compact cardinal, $2^{\aleph_1} > \aleph_2$ and it can be arbitrarily large.

Theorem 5.5.66 ([124]). *Con(ZFC + there is a weakly compact cardinal) implies Con(ZFC +* $2^{\aleph_0} = \aleph_1 + SH_{\aleph_2}$*).*

If we want to construct an $\aleph_2$-Aronszajn tree, the levels have size $\mathfrak{c} = 2^{\aleph_0}$, not $\aleph_1$, so the construction works if CH holds, but not in ZFC. In fact, according to Mitchell's result that we mentioned before, we can force a model in which CH fails, $\mathfrak{c} = \aleph_2$ and ω_2 is a weakly compact cardinal in L.

5.5.5 Ascent path

The concept of an *ascent path* was discovered by Laver while analysing the possibility of a model in which all $\aleph_2$-Aronszajn trees are special (see [124]). The subject was also examined by Devlin, Todorčević, Shelah, Stanley, Cummings, Brodsky, Rinot, Torres Pérez and Lücke. It could be said that it generalises the concept of cofinal branch, it can be used in the absence of GCH, and it is preserved in all cardinal-preserving forcings.

Definition 5.5.67. Let κ be a regular cardinal. A κ^+- Aronszajn tree T has an ω *ascent path* if there is a sequence $\langle x^\alpha : \alpha < \kappa^+ \rangle$ of pairwise disjoint sets such that

1. For all $\alpha < \kappa^+$, x^α is a function from ω to T_α.
2. If $\alpha < \beta < \kappa$, then $(\exists n \in \omega \forall m \geq n) x^\alpha_m < x^\beta_m$.

Laver shows that an $\aleph_2$-Aronszajn tree with an ascent path is not special. Baumgartner constructs such a tree from a $\square_{\aleph_1}$-sequence (see [37]), specifically, he shows that if $\diamondsuit_{\omega_2}$ and $\square_{\aleph_1}$ hold for a stationary set $E \subseteq \aleph_2$, then one can construct an $\aleph_2$-Souslin tree with an ω-ascent path. The construction is a variation on Jensen's construction of an $\aleph_2$-Souslin tree from the same hypotheses.

Baumgartner also showed that an ω_1-tree T has a cofinal branch if and only if it contains an ascent path of finite width ([10]). In combination with other results, the use of this notion has non-trivial consequences when one considers paths of width $\lambda \geq \omega$ for trees of height κ, with $\kappa > \lambda^+$.

An ω ascent path is not a cofinal branch per se, given that, according to definition, an Aronszajn tree does not have a cofinal branch. But its existence prevents the tree from being special.

Generalisations come in different ways. Shelah and Stanley, for instance, working on results by Laver and Todorčević, constructs Aronszajn trees with ascent paths, and describe, among other things, the connection between the existence of ascent paths and specialisation:

Theorem 5.5.68 ([201]). *If κ is uncountable, then $\square_\kappa$ implies that there is a nonspecial κ^+- Aronszajn tree.*

Theorem 5.5.69 ([201]). *If λ is an uncountable cardinal, $\kappa = \lambda^+$, and κ is not weakly compact in L, there is a κ-ascent path, with $\kappa \neq cf(\lambda)$, then there is a non-special κ- Aronszajn tree.*

Starting from a proof of Todorčević for an $\aleph_2$-Aronszajn tree (to proposition 2.3. in [218]), Lücke generalises to:

Lemma 5.5.70 (Lemma 1.6., [126]). *Let $\kappa < \lambda$ be cardinals such that λ is not a successor of a cardinal of cofinality $\leq \kappa$, $S^{\lambda}_{>\kappa}$ be the set of all limit ordinals $< \lambda$ with cofinality $> \kappa$, and S a stationary set $S \subseteq S^{\lambda}_{>\kappa}$. If S is non-stationary with respect to T, then there is no ascent path of width κ through T.*

As emphasised by Lücke (as a corollary and in accordance to lemma 3 in [201]), trees containing ascent paths are non-special in a absolute way: they remain as such in any forcing extension satisfying the conditions on the cardinals, i.e., κ and κ^+ remain cardinlas and $cf(\lambda) \neq cf(\kappa)$. He also shows that the converse of the implication also holds under certain cardinal arithmetic assumptions: if there is no ascent path of width κ through T, then T is special in a forcing extension in which the conditions on the cardinals hold. That is because the ascent paths are closely related to maximal antichains in the partial order that specialises a tree of height an uncountable regular cardinal.

5.5.6 Non-special trees again

Before going any further, there are some remarks that we would like to highlight. A Souslin tree is a more 'sensitive' object than an Aronszajn tree. Recall that a tree T is κ-Souslin if and only if it is a κ-Aronszajn trees and has no antichain of size κ. So a special κ^+ tree is a κ^+-Aronszajn tree, while a κ^+-Souslin tree is a non-special κ^+-Aronszajn tree. Special κ-trees are interesting given that they are branchless in an absolute way: they retain this property in any outer model in which κ is preserved. On the other hand, a κ-Soulsin tree can be considered a forcing poset which adds no bounded subset to κ, but adds a cofinal branch in T; in other words, a tree T is a κ-Souslin tree if the partial order induced by T satisfies κ-cc.

On one hand, Souslin trees and the SH ('there are no Souslin trees') represent central objects of study in set theory. SH is independent of ZF, but one can construct an $\aleph_1$ Souslin tree in the constructible universe with the help of the diamond principle; Jensen formulated $\diamondsuit$ in view of an earlier direct proof that there are no Souslin trees in L. But 'MA $+\neg$CH implies that SH is true. The proof for $\neg$SH in L can be generalised to larger cardinals as well. Using a combination of certain versions of $\diamondsuit$ and $\square$ in L, one can find a κ-Souslin tree for every regular, non-weakly compact κ.

In [95], Jensen proved that the existence of an $\aleph_2$-Souslin tree follows from each of the hypotheses $CH + \diamondsuit(E^{\aleph_2}_{\aleph_1})$ and $\square_{\omega_1} + \diamondsuit(E^{\aleph_2}_{\aleph_0})$ (given $m < n < \omega$, $S^n_m = \{\alpha < \aleph_n : cf(\alpha) = \aleph_m\}$).

If κ is weakly compact, then there are no κ-Aronszajn trees (κ has the TP), so SH trivially holds in this case. Generalising the proofs regarding SH's consistency to larger cardinals is rather difficult. Gregory ([76]) shows that $\aleph_2$ ('CH + $2^{\aleph_1} = \aleph_2 + \square_{\omega_1}$' imply $\neg SH(\aleph_2)$). Combining this result with a theorem by Jensen according to which if $\square_\kappa$ is false, then κ^+ is Mahlo in L, he shows that the consistency strength of 'CH + $2^{\aleph_1} = \aleph_2$ + SH($\aleph_2$)' is at least that of a Mahlo cardinal, a result that can be generalised to '$\kappa^\omega = \kappa + 2^\kappa = \kappa^+$ + SH(κ^+)', then κ^+ is Mahlo in L.

Under CH, a Souslin tree can also be constructed if $\diamondsuit(S^2_1)$ (S^2_1 represents the set $\{\alpha < \aleph_2 : cf(\alpha) = \aleph_1\}$) also holds. The tree is countably closed:[2] the antichains are guessed and solved at stages having cofinality $\aleph_1$, and the the cofinal branches are completed at stages with cofinality ω.

In [152], Mitchell (with results from Silver as well) shows that from the consistency of a weakly compact follows that there are no $\aleph_2$-Aronszajn tree, a statement implying $\neg$CH. SH($\aleph_2$) - there are no $\aleph_2$-Souslin trees - is a weaker statement than this one.

There are different ways of constructing non-special $\aleph_2$ trees using ascent paths. For instance, in [37], Devlin mentions a proof by Baumgartner (theorem 4), a

[2] A poset P is *κ-closed* if every increasing sequence of length κ or shorter in P has an upper bound. If P is *countably closed* and we work in ZFC, there are no new ω-sequences of elements of V in $V[G]$. In particular, P adds no new reals.

variation of Jensen's construction mentioned above and using the same hypotheses, which shows that given a stationary set $E \subseteq \aleph_2$ and $\diamondsuit_{\omega_2}(E)$ and $\Box_{\omega_1}(E)$ hold, one can construct an $\aleph_2$-Souslin tree with an ω-ascent path. In the context of that paper, such a tree, T^* ($\langle T^*, <^* \rangle$), of cardinality ω_2 if we assume CH, represents the reduced ω-power by a (non-principal) uniform filter on ω of an ω_2-tree T. T^* represents an extension of T and is σ-closed[3].

Cummings constructs a countably closed $\aleph_2$-Souslin tree with an ω-ascent path, but he needs stronger assumptions than the ones mentioned above regarding the construction of a $\aleph_2$-Souslin tree, i.e., CH and a stronger combinatorial principle ([26]).

Furthermore, Shelah mentions that an earlier form of the solution to the problem referred to a "somewhat weaker" form involving non-special λ^+-wide trees instead of non-special λ^+-Aronszajn trees. As such, we can reformulate the theorem above as:

Theorem 5.5.71. *Let T be a κ^+-tree with a κ-ascent path, with $\kappa \neq cf(\lambda)$, and κ is not weakly compact in L. Then T is not special.*

Remark. Let $\mathcal{T}_a(\aleph_2)$ be the class of trees of cardinality and height $\aleph_2$ with no branches of length $\aleph_2$, but having an ω-ascent path. Given the above results, we can distinguish among different types of non-special trees in $\mathcal{T}_a(\aleph_2)$: wide $\aleph_2$-trees, $\aleph_2$-Souslin, non-Souslin $\aleph_2$-Aronszajn trees. Shelah refers to a non-special κ-Aronszajn trees (the third type) as "a poor man's" κ-Souslin tree ([201], p. 888). In the terminology of [192] (chapter IX), the non-stationary trees that Shelah is constructing are λ^+-S-st-special trees, which are λ^+-Aronszajn trees. Also note that in the second case, there are different ways of constructing $\aleph_2$-Souslin.

Furthermore, in [126], Lücke generalises the construction of Souslin trees containing ascending paths of small width, using a square principle introduced

[3] Whenever $\alpha < \omega_2$ is a limit level of countable cofinality, and b is a branch of length α of $T \restriction \alpha$, there is an element of level T_α which extends b.

by Baumgartner in unpublished work, $\square^B_\kappa$, with B a subset of κ^+ and $S^{\kappa^+}_\kappa \subseteq B \subseteq$ Lim. This sequence can be added by $< \kappa$-directed closed forcings that preserve the regularity of κ^+.

Theorem 5.5.72 (Th. 5.8.). *Let κ be an uncountable regular cardinal that satisfies $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$. If there is a $\square^B_{\kappa^+}$-sequence, then there is a κ^{++}-Souslin tree that contains a κ-ascent path.*

The next question is then: what are the relations among these types of non-special trees? Is there a universal element or a universal family in the case of $\aleph_2$-non-special trees without a cofinal branch? Of course, the question can be generalised to a cardinal κ and to non-special trees in general. Finding universal non-special trees, including Souslin trees, is an interesting topic in itself, able to shed light on different, related aspects.

For a first result, we consider a non-special wide $\aleph_2$ tree T with no cofinal branches and an $\aleph_2$-Souslin tree. We show the following:

Theorem 5.5.73. *Assume V=L and $\diamondsuit(S)$, with $S \subseteq E = \{\alpha < \aleph_2 : cf(\alpha) = \aleph_1\}$. Let T be a wide $\aleph_2$-tree with no cofinal branch, and R an $\aleph_2$-Souslin tree. Then there is no embedding $f : R \to T$.*

Proof. Suppose there is such embedding. Let $S' = E \setminus S$. So S' is stationary. We enumerate S' as $\langle \gamma_\alpha : \alpha < \aleph_2 \rangle$.

R has cardinality $\aleph_2$, so for each $\alpha < \aleph_2$ there is some element t_α in R such that $ht(f(t_\alpha)) > \alpha$.

Let $S'' = \{\alpha \in S' : \gamma_\alpha = \alpha\}$. S'' is in the intersection of S' with a club, so it is stationary.

We define a regressive function $h : S'' \to \aleph_2$ by $h(\alpha) = t_\alpha(\alpha)$. It follows that it is constant on a stationary set $X \subseteq S''$.

Let $\delta, \epsilon \in X$. Then $t_\delta(\delta) = t_\epsilon(\epsilon)$. It follows that there is no embedding that would extend both t_δ and t_ϵ. As such, $\{t_\alpha : \alpha \in X\}$ is an antichain of size $\aleph_2$ in R, a contradiction with R being an $\aleph_2$-Souslin tree. □

We will now show that under GCH, the class of non-special $\aleph_2$-Souslin tree with an ω-ascent path has a maximal complexity number, $2^{\aleph_2} = \omega_3$. We will use the fact that the class $\mathcal{S}_a(\aleph_2)$ of such trees is closed under taking disjoint unions of size $\aleph_2$.

Theorem 5.5.74. *The class of non-special $\aleph_2$-Souslin trees with an ω-ascent path has a universal family of cardinality $2^{\aleph_2} = \aleph_3$.*

Proof. We will show that there is an ω_3-Souslin tree with an ω_1-ascent path in V[G,H]. As I mentioned above, there are different ways in the literature to construct non-special $\aleph_2$-Souslin tree with an ω-ascent path, but we will use the general approach offered by Lücke (op.cit.). For completeness, we will also give a full definition of a $\square^B_{\kappa^+}$-sequence.

Definition 5.5.75. Let κ be a regular cardinal and B is a subset of κ^+ with $S^{\kappa^+}_\kappa \subseteq B \subseteq \text{Lim}$. A $\square^B_{\kappa^+}$-sequence is a sequence $\langle C_\alpha : \alpha \in T\rangle$ where

1. T is a set of limit ordinals $< \kappa^+$.
2. $\{\alpha < \kappa^+ : cf(\alpha) = \kappa\} \subseteq T$.
3. For all $\alpha \in T$, C_α is a club subset of α with $o.t.(C_\alpha) \leq \kappa$.
4. If $\alpha \in T$ and $\beta \in lim(C_\alpha)$, then $\beta \in T$ and $C_\beta = C_\alpha \cap \beta$.

Such a sequence, unlike $\square_\kappa$-sequences, can be added by $< \kappa$-directed closed forcings that preserve the regularity of κ^+.

Lemma 5.5.76 ([3], Fact 2.7.)**.** *Let κ be regular. Then there exists a forcing poset P such that*

1. *P is κ-directed closed.*

2. *P is strategically closed for the game of length $\kappa+1$.*

3. $\Vdash_P$ *"$\Box^B_{\kappa^+}$ holds".*

Lücke observes that by modifying the sequence and taking B to be a *fat* stationary subset of κ^+, we obtain a sequence that avoids a stationary subset of a given stationary set.

Definition 5.5.77. A subset F of a regular cardinal κ is *fat* if and only if every club C of κ and every ordinal $\alpha < \kappa$, $F \cap C$ contains a closed copy of $\alpha + 1$.

Friedman ([68]) had shown that every stationary subset of ω_1 is fat and that it can be partitioned into ω_1 many pairwise disjoint fat subsets. For ω_2, Martin's Maximum (MM) implies the existence of $\aleph_2$-many pairwise disjoint fat subsets of $\aleph_2$. Furthermore, according to a theorem in Larson ([122]), forcing with a ω_1-directed partial order P preserves MM.

Let $P = Col(\omega_3, 2^{\omega_2})$ and G the generic set. Let $\dot{\mathrm{Q}}$ be the name for the forcing poset due to Baumgartner in the lemma above, with H the generic set. For ω_2, and in accordance with theorem 5.6.6., $\dot{\mathrm{Q}} \Vdash \Box^B_{\aleph_1}$, i.e., a non-special ω_2-Souslin tree that contains an ω-ascent path. $P* \dot{\mathrm{Q}}$ is ω_2-directed closed, so it follows (see [126] and [122]) that PFA holds in V[G,H], which implies that $2^{\omega_1} = \omega_2$ holds in V[G,H].

Given that $\dot{\mathrm{Q}}^G$ is $(\omega_2 + 1)$-strategically closed in V[G], we have $(2^{\omega_2})^{V[G,H]} = (2^{\omega_2})^{V[G]} = \omega_3^{V[G]} = \omega_3^{V[G,H]}$.

We also get a $\Box^B_{\kappa^+}$-sequence in V[G,H], and again according to theorem 5.5.6., we have an ω_3-Souslin tree with an ω_1-ascent path in V[G,H]. $\Box$

6

Aspects

Synopsis

This chapter contains some concluding remarks related to the notion of mathematical knowledge in the context provided by the universality problem. The context of such remarks is determined by the two distinctions - syntactic/semantic, abstract/concrete - and the meaning of mathematical object and method. I will finish by connecting them into an idea of method.

As a problem in set theory and model theory, universality is defined in these contexts and uses the structural and methodological instruments offered by them, but it simultaneously determines the framework of the analysis through it specificity as a mathematical object and, given that, functioning as a methodological tool. I will underline some aspects connected to the way universality can be considered as a way of interrogation and orientation in asking the *right* questions.

6.1 Syntactic - Semantics

Generally speaking, a mathematical theory can be described either *internally* or *syntactically* through axioms and theorems or *externally* or *semantically*, through its models. Gödel had connected the two approaches through his completeness theorems. The semantic method involves a 'passage' among the different models of a theory. It is used to establish negative results, unprovability results, but not only. Starting with a model of ZFC, we can construct a sub-model satisfying the CH (the constructible universe) and an extension (a forcing extension) which satisfy $\neg CH$.

That leads us to forcing, but also to results in classification theory, since, as emphasised by Džamonja, "[L]ike some other great theorems in mathematics (such as the Main Gap Theorem) the Main Theorem of Forcing states that a semantic and a seemingly unrelated syntactic notion agree. In the latter case these are the forcing and the weak forcing relations" ([54], p. 57).

In its role in set theory and model theory, universality merges both a semantic and a syntactic component. It connects. Universality, or homogeneous universality, represents an algebraic property describing a class of models, the embedding relation between models. Saturation, on the other hand, refers to one model, describing the relation between a model M and the types over its subsets (see [6], pp. 126-7). This last aspect points to a shift in model theory, from realising formulas to realising types, and this represents a syntactic aspect.

On the other hand, starting with the work of Fraíssé and then Jónsson in the 1950's (op.cit), universal domains are constructed in a semantic way. Given a theory T and the class of its models, K, a semantic approach considers only structures or models, ignoring even the background universe. Model theory evolved as an interplay between the syntactical aspect, i.e., what is there and what happens inside of a theory T, and the semantic one, that is, what can be said about K, the class of models of T. For example, we say that a first-order logic sentence is preserved by submodels (which is a semantic property) if and only of it is equivalent to a universal sentence (a syntactic property). Furthermore, I mentioned above the role of definable sets. At a very basic level, one could say that model theory is fundamentally concerned with the semantics of first order formulas, that is, models' definable sets.

But is there a tension between syntax and semantics? There are tendencies in the practice of set theory that could point to a preference for the syntactical or the semantical aspect. Shelah admitted that he is not too keen to the "syntactical flavor of the problems" ([196], p. 6), an approach characterising "the determinacy school" (the proponents of the determinacy axiom), but that he acknowledges the role of large cardinal axioms in the forcing framework.

In set theory, inner models are subclasses, so L is a proper class in V, a definable object that can be written as a syntactic formula. That is the reason for which inner models can be sometimes treated as definable classes. That being said, many of the consequences of the fine structure are not syntactical, like the *diamond* and the *square* principles. But on the other hand, these principles are not to draw the combinatorial consequences of L?

'Combinatorial' basically means *semantical*. In practice, there is a mix of these aspects and a question of personal preferences. "For Jensen, fine structure is the main point, diamonds and squares are side benefits, probably good mainly for proving to the heretics the value of the theory. Personally, I prefer to get these consequences without the fine structure, but I do not greatly appreciate the search for alternative, "pure", proofs. The question is: when we want to go further, which approach will be preferable? Of course, you will need the fine structure for syntactical statements", writes Shelah ([196], p. 6).

In set theory, logical equivalence is a syntactical notion, while equiconsistency is a semantical one. The latter might be more informative from an epistemological point of view. It involves the use of large cardinal axioms (so stronger axioms than ZFC) and, by definition, there is a sentence expressing the existence of a large cardinal that logically implies another sentence and a model where this sentence is true.

It should be noted though that the notion of *consistency* was for Skolem not a syntactic but a semantic one, referring to the existence of a structure satisfying the axioms. He thus distinguishes, as emphasised by Gaifman ([69]), between "completely formalized mathematics", amounting to the study of structures satisfying the axioms, and "ordinary mathematical practice", presumably a study of what we might call today 'the intended interpretation'. And, actually, starting with a paper by Skolem from 1934, "About the impossibility of characterizing the number sequence by means of a finite or an infinite countable number of statements involving only numeric variables", Gaifman observes some very important aspects: consistency became fundamental to allowing formalised mathematics to serve as a mediator between different foundational endeavours, "about what the intended interpretation should be. A can doubt the truth, plausibility, or factual meaningfulness of an axiom adopted by B, but, as long as it is consistent, A can make sense of what B is doing by regarding it as an investigation into the common properties of the structures that satisfy the axioms. This is possible as long as the completely formalized theory is consistent; if it is not, then those who presuppose it are not investigating anything" ([69], p. 3). But that happened to a certain price in the case of mathematical practice: "for this very reason it does not fully capture the view that underlies ordinary mathematical practice - in as much as the practice implies a particular structure that constitutes the subject matter of the inquiry, "what it is all about". If set theory is about some domain that includes uncountable sets, then any countable structure that satisfies the formalized theory must count as an unintended model. From the point of view of those who subscribe to the intended interpretation, the existence of such nonstandard models counts as a failure of the formal system to capture the semantics fully" (Ibid.). Skolem's original construction of a non-standard model of arithmetic (not a consequence of the Lowenheim-Skolem theorem) anticipates the formation of an ultrapower. Complete saturation (a property of ultraproducts, for instance) is similar to that of compactness in the case of topological spaces, for which reason it is used to ensure that continuous objects (like the ultraproducts) enjoy 'completeness' or 'compactness'.

The Completeness Theorem says that every type can be realised in an elementary extension, and an elementary embedding between models preserves the truth values of each first-order formula

Lemma 6.1.1. *Let* $A \subset M$. $T_A(M) = T_A(N) \Rightarrow S_n^M(A) = S_n^N(A)$.

So together with the method of forcing, large cardinal axioms open new possibilities as to the discoveries that incompleteness made possible in mathematics. And they were and still are both used in the presentation on different results connected to the universality problem.

6.2 Mathematical object

"For all modern mathematicians agree with Plato and Aristotle - writes Peirce - that mathematics deals exclusively with hypothetical states of things, and asserts no matter of fact whatever; and further that it is thus alone that the necessity of its conclusions is to be explained" ([161], CP 4.232). It is quite a definitive statement, not entirely true but helpful in approaching the subject, specifically, in considering the idea of a mathematical object as hypothesis. To this end, I will mention Cellucci's take on the mathematical object as hypothesis and present some further suggestions. The mathematical objects considered in this text were connected to set theory and model theory. Given what I have said so far, I will show in this section how set theory, model theory, and the universality problem in particular could offer new approaches to the notion of mathematical object.

Before going further, I would like to introduce some clarifications regarding the notion of *mathematical object*. The idea of mathematical objects as concepts can be found in the works of Bolzano (in his early mathematical writings and in [14], Frege (in [66]), or Dedekind (in [31]) [1]. I am taking *object* here in the very general sense of what the area of mathematics (pure and applied) is focusing on. It is not my intention to fully investigate the nature of mathematical object, but given that no discussion can take place in a conceptual void and also for the purposes

[1] I would like to express my gratitude for the remarks made by prof. Hourya Benis-Sinaceur

of this text, I take mathematical object to refer to the concept(s) and notion(s) that constitute the focus of mathematicians. Referring to mathematical concepts as objects does not necessarily involve any ontological commitment.

In this text, though, and as I will show bellow, the notion of *mathematical object* could be described in a relational way (in the context of and determined by the characteristics of set theory) and it also includes the concept of method. For a more detailed discussion regarding the latter, see the subsection bellow. Consequently, such a general approach regarding the notion of mathematical object includes various approaches to it: as method or heuristic device and as hypotheses (in line with Celluci's interpretation). It is also my view that even though a philosophical justification for a mathematical object is possible, it is not necessary for the mathematical practice.

The immediate intuition and the mathematical practice show that not all mathematical object are sets, but set theory does offer the possibility of a homogeneous context with objects of the same type. On his take on Von Neumann's definition of ordinals, P. Dehornoy calls these objects "pure sets", "sets that are also sets of sets, sets of sets of sets", which "if they exist, are closed under all set theoretic operations (...): reunion, intersection,(...). On the other hand, it is not a priori obvious that such sets exist. As a matter of fact, there must be at least one, that is the empty set $\varnothing$: for lack of existence, all its elements, elements of elements, etc., are sets and even pure sets. Gradually, one deduces that there is an infinity of pure examples" ([33], p. 33)

The concept of *set* points to the idea of considering a mathematical object as a kind of relation. In the mathematical practice, one is not analysing a set as a set, but a determined set, as an ordinal or as a cardinal, for instance (and in the most simple terms). Consequently, when defining a new mathematical object, we specify the object together with a mathematical structure on it, which takes the form of a certain relationship among the elements of the set. As such, every mathematical object can be represented by or as sets. Moreover, after Gödel, we cannot explicitly build a model of ZF in ZF, but given the consistency of ZF, there are different models that we can consider. Furthermore, since all mathematical objects can be represented as sets, every model of ZF will include its own version of all these objects. So a further consequence, and a very fruitful one, is that there are different perspectives in approaching the same mathematical object.

When introduced by Cantor, sets were accompanied by a certain philosophical justification, which was also developed ever since, but that is not necessary for the mathematical practice. Even Cantor started with problems related to Fourier series: given that the majority of functions are not continuous, the problem was to classify functions in a natural way. And that process finally led to the introduction transfinite numbers. Set theory brought clarity to the concept of infinity, but it also emphasises (in its current developments) how complicated the infinite is and the continuous tasks of trying to describe it. One of the difficulties connected to the notion of infinity comes from the self-similarity phenomena that it entails. An example is a theorem of Friedman from the 1970's regarding the phenomenon of self-similarity in the theory of non-standard foundational structures: every countable non-standard structure of the Peano Arithmetic (the conventional theory of arithmetic) has a proper self-embedding. But he also proved this for a fragment of ZF. Self-similar objects preserves this attribute when considered on various scales. So a framework for the study of such objects are the strong axioms of infinity, the large cardinals. That being said, the development of forcing permits us to separate objects (appearing in the forcing extension) from their names (objects belonging to the original model). Consequently, set theory offers large variety of (mathematical) objects.

This versatile nature of sets and the concepts and the methods used in their analysis offer the possibility of conceiving mathematical objects in general as heuristic devices. Such an approach is similar to Cellucci's and his view of mathematical objects as hypotheses. The purpose is knowledge, mathematical knowledge and the discovery of methods to acquire such knowledge. It is superfluous therefore to apply to them the same analysis as the one we can find in philosophy and ask about the nature of their existence, for instance, or try to justify already obtained knowledge by providing a foundation for it (in accordance with a foundationalist view). Mathematical knowledge could constitute an object of study in different philosophical approaches, but only as far as it offers new approaches to the process of interrogation.

As mentioned above, the analytic method in the philosophy of mathematics, which Cellucci is advancing, revolves around the idea of plausibility, not truth. This philosophical approach could analyse and expose inadequacies of some basic mathematical concepts and even formulate rules of discovery ([20], p. 235). The mathematical objects in such a view are considered to be hypotheses, which means that there will be no immediately justified premisses from which all knowledge is deduced. So there is no rejection of the infinite regress argument. The assumption of immediately justified knowledge, from which all knowledge is deduced from was basically founded on the rejection of the infinite regress argument. But if "the hypotheses are plausible, then there will be knowledge, albeit provisional knowledge always in need of further consideration, since new data may always emerge"; there "would be no knowledge only if the premisses, or hypotheses, occurring in the infinite series were arbitrary. But they need not be arbitrary. As in the analytic method, they must be plausible, namely the arguments for them must be stronger than the arguments against them, on the basis of the existing knowledge"([20], p.33).

Such an approach is not without benefits. Is it definitively reasonable to think that all discoveries in mathematics start with a system of axioms from which one deduces all possible consequences? Although trying to describe the entire edifice of proof (in all its aspects), logic accepts that its methods are incomplete: any reasonably strong system of logic cannot prove its own consistency. Set theory too is incomplete. The paradoxes pointed out the inability to formulate arbitrary mathematical objects as sets. But this incompleteness let open a conceptual landscape that makes discovery possible. Mathematicians respond in a unique way to mathematical problems. They formulate hypothesis, make conjectures, try different proofs etc.

Given his extraordinary work in set theory and model theory and directly connected to the universality problem, I would like to point to a description that Kanamori makes of Shelah's work and his approach to mathematics: "In set theory - writes Kanamori - Shelah was initially stimulated by specific problems. He typically makes a direct, frontal attack, bringing to bear extraordinary powers of concentration, a remarkable ability for sustained effort, an enormous arsenal of accumulated techniques, and a fine, quick memory. When he is successful on the larger problems, it is often as if a resilient, broad-based edifice has been erected, the traditional serial constraints loosened in favor of a wide, fluid flow of ideas, and the final result almost incidental to the larger structure. What has been achieved is more than a just succinctly stated theorem but rather the erection of a whole network of robust arguments. ...his insistence that his edifices be regarded as autonomous conceptual constructions. Their life is to be captured in the most general forms, and this entails the introduction of many parameters. Often, the network of arguments is articulated by complicated combinatorial principles and *transient hypotheses*, and the forward directions of the flow are rendered as elaborate transfinite inductions carrying along many side conditions. The ostensible goal of the construction, that succinctly stated result that is to encapsulate it, is often lost in a swirl of conclusions" ([99], pp. 53-54) [emphasis added].

Some problems may be more complex and difficult than others, but they represent a continuously evolving object. That means that their solution is not their end: such a solution can be generalised, can offer insights into other problems, can open new lines of research etc. The polished form of a proof makes it expressible to the community of mathematicians, but it does not tell its entire history, parts of which are only to come in a future time, through interpretation, generalisation, or reconceptualisation. That again points to the limits of axiomatisation and the possibility of describing the mathematical practice in a more heuristic way, using hypotheses.

The case of universality (and many others in set theory or even model theory) showed the limits of traditional approaches towards mathematical objects and proofs, in dealing with infinity. It determined the creation and development of new methods, new concepts and arguments, and shaped theories. Such a state of things is particularly obvious in the infinite combinatorics, but also in the complex apparatus represented by forcing. It can also be located in classification theory, which, by itself, raises questions about the nature of axiomatisation.

Objects and names The extraordinary complexity of the set theoretic concepts, terms, elaborate syntax etc. is based on the language of ZFC, made up of a single binary relation, $\in$, to which we add the first-order logic connectives. We have, of course, the tendency to consider set theoretic symbols such as $\varnothing$ and ω, $\aleph_0$, $\aleph_1$ etc. as proper names, similar to others. We might think that $\aleph_1$, for instance, refers to a unique set. But this set might be different in different models, in other words, the interpretation of the symbol '$\aleph_1$' is model-relative. When the interpretation of a name or a symbol is not model-relative, it's called absolute. Examples include the '$\varnothing$', which always picks out the same set. In every model of ZFC, $\varnothing$ is the unique set that contains no elements. And when a name is not absolute, we need to make the distinction between the name itself as a syntactic object and the set to which it refers to in a certain model of ZFC. That is the reason why one uses the notation $\aleph_1^M$ when considering the description the name '$\aleph_1$' gets in the model M.

These aspects are related to complex developments in set theory, specifically forcing. In forcing, we restrict ourselves to transitive models. Being an ordinal (a transitive set of transitive sets [2]) is absolute to transitive models. In other words, being an ordinal is a $\triangle_0$ property. This result is based on the Von Neumann's definition of ordinals and the *Foundation axiom*. One of the most important absoluteness results (given its further uses) is the following theorem. But first, we will introduce a definition:

[2]More formally: for any term t, 't is an ordinal" is an abbreviation for (t is transitive) $\wedge(\forall x) \in t(x$ is transitive).

Definition 6.2.1. If $M \models ZFC$ is transitive and $R \in M$ is a binary relation, then $M \models R$ is well-founded if and only if R is well-founded.

Theorem 6.2.2. *Well-foundedness is absolute for transitive models of ZFC.*

We should point out that well-foundedness is not absolute for transitive sets without specifying what axioms they satisfy, given that we could find a transitive set X with a binary relation $R \in X$ such that $X \models R$ is well-founded, but R is not. A function $F : Ord \to Ord^2$ is $\triangle_1$-definable, therefore it is absolute to transitive models of set theory. If X is a set of ordinals, $f(X)$ represents a set of pairs of ordinals and can be thought of as a relation on a set of ordinals. So we can just take sets of ordinals and then we can refer to functions, relations etc. Furthermore, given that such a set X can be identified by the isomorphism class of graphs connected to the membership relation below X, we could always recover X if we have enough sets of ordinals.

'ω' is a name for the first countable ordinal and it is absolute across these models. In any transitive model of ZFC, ω is the set $\mathbb{N}$ of natural numbers, also matching the description 'the intersection of all inductive sets' (the sets containing the ordinal 0 and closed under the successor function). 'ω_1' describes *the smallest uncountable ordinal* and, in ZFC, it coincide with '$\aleph_1$'. Although it represents the first uncountable cardinal, there might be models of ZFC, countable models, that make it countable. We took $\aleph_1$ as an example because it does have certain unique characteristics in set theory, and many of the results mentioned in the previous chapters corroborate this statement. Different transitive models may pick different sets for ω_1 or $\aleph_1$, but they will 'believe' that they have chosen the smallest uncountable ordinal and the smallest uncountable cardinal, respectably. Another non-absolute notion is the ordinal ω_2, the name for the first ordinal for which there's no injection into ω_1. The set ω_2^M depends on the choice of the model M. In a countable model (like the ones we use in forcing), ω_2^M is countable. And another example of a non-absolute name, one that is central in many applications in set theory, is '$P(\omega)$', describing 'the set of all subsets of ω'. In forcing, we have

Definition 6.2.3. A set is a P-name if its elements are of the form (τ, p), where τ is a P-name and $p \in P$. V^P is the class of P-names.

Using the well-foundedness of $\in$, the definition is recursive, each P-name τ is defined using the elements of τ of lower rank.

Garti ([71]) considers the relationship between mathematical objects and their names an "important philosophical issue", and forcing theory enabling "us to separate objects from their names in an accurate way" (p. 28)[3]. Although the relationship between mathematical objects and their names does constitute an important issue in philosophy, it is not necessarily the case with regard to the mathematical theory of forcing. There are many difficulties involves in forcing, but they are connected to the nature of the method, not with ambiguities related to the use of mathematical objects. The question involves again the validity and the limits of borrowing methods from one area and using them in another. Understanding the nature and the use of names in forcing doesn't imply a theory of naming applicable to the entire field of mathematics, but it is connected to this particular method. A name in the ground model M for an element in the forcing extension $M[G]$ will tell us how the element was constructed from the generic filter G. Such a name, a $\mathbb{P}$-name (with $\mathbb{P}$ the forcing notion), is a relation, a set defined by transfinite recursion, and the collection of $\mathbb{P}$-name is a proper class if $\mathbb{P} \neq 0$. In iterated forcing, for instance, the forcing notion consists of sequences of elements and each coordinate corresponds "to a name for an element of a forcing notion, not in V but in some extension of it intermediate between V and $V[G]$" ([53], p. 20).

Furthermore, regarding the way I used the notion of mathematical object in this text, in also follows that when focusing on the relation between mathematical object and method and taking into account the notion of reconceptualisation mentioned above (a process leading to mathematical knowledge and understanding, like interpretation or generalisation), the

[3] In this particular article, it is used to offer a new perspective on the self-referential component connected with Yablo's paradox.

relationship between names and objects does not have to be considered (in the case of mathematical practice) as a relationship between names and concepts.

The account of mathematical objects as hypotheses, tentatively introduced to solve mathematical problems should not be confused with fictionalism, underlines Celluci, the view that mathematical objects are as characters in fiction. A mathematical object is the hypothesis that a certain condition is satisfiable. For example, an even number x is the hypothesis that the condition $x = 2y$ is satisfiable for some integer y. If, in the course of reasoning, the condition turns out to be satisfiable, we say that the object 'exists', if it turns out to be unsatisfiable, we say that it 'does not exist'. Thus, speaking of 'existence' is just a metaphor. That the condition turns out to be unsatisfiable typically occurs in proofs by reductio ad absurdum. There is no more to mathematical existence than the fact that mathematical objects are hypotheses tentatively introduced to solve mathematical problems. Such hypotheses are in turn a problem to be solved, it will be solved by introducing other hypotheses, and so on. Thus solving a mathematical problem is a potentially infinite task (see [20], [17]).

The view that mathematical objects are hypotheses is related to Plato's view, expressed through his dialectical method, that "students of geometry, calculation, and the like" hypothesize both mathematical objects and their properties: they "hypothesize the odd and the even, the various figures, the three kinds of angles, and other things akin to these in each of their investigations, as if they knew them. They make these their hypotheses and don't think it necessary to give any account of them, either to themselves or to others, as if they were clear to everyone. And going from these first principles through the remaining steps, they arrive in full agreement" (Plato, *Republic*, VI 510 c2-d1). And for the properties of mathematical objects: "when you must give an account of your hypothesis itself you will proceed in the same way: you will assume another hypothesis, the one which seems to you best of the higher ones until you come to something acceptable, but you will not jumble the two as the debaters do by discussing the hypothesis and its consequences at the same time, if you wish to discover any truth" (Plato, *Phaedo*, 101 d4-e3). *Epistêmê* here is the ability to know what is real as it is (Republic 477b). But that which is as it is involves the Ideas/the Forms, they constitute the hightest form of knowledge and 'inhabit' the intelligible domain (*noêton*). In the analogy of the divided line (*Republic* 509d–511e), the *noêton* is subdivided into domains accessible by *diánoia* (deductive reasoning), specific to mathematics, and *noēsis* (understanding, also referred to as intellectual intuition). So the intelligible forms do constitute objects of mathematical *téchnai* (calculations and geometry), being different from the actual drawings of the geometrical shapes, which belong to the visible world. The "square itself", for instance, is still some kind of image, albeit of thought, "clear and (...) valued as such". But for the mathematical sciences, hypotheses are first principles, so the geometers proceed from hypotheses "not travelling up to a first principle, since it cannot reach beyond its hypotheses". As a result, the deductive and logical mathematical *epistêmê* is an "intermediate between opinion and understanding" (511b-511d), needing further justification. Both *noêsis* and *diánoia* the reason (*lógos*) makes use of hypotheses, but while the mathematical knowledge assumes them as first principles while moving towards final conclusions, in the case of *noêsis*, the *lógos* examines all hypotheses through dialectic (philosophy), "does not consider these hypotheses as first principles but truly as hypotheses - but as stepping stones to take off from, enabling it to reach the unhypothetical first principle of everything" (511b 3-6). Then given that for the mathematical sciences hypotheses are first principles, they never go back to "a genuine first principle (...) even though, given such a principle, they are intelligible" (511d 1-3). Through mathematics then, and in accordance to Plato's distinctions, we can only state more and more hypotheses, get better and better approximations, but never fully understanding them. So in Plato's view, the method of philosophy and the reasoning of mathematics are similar up to a point, both having an integral connection to the intelligibles.

Cellucci identifies his analytic method to Platos's method of reasoning from hypothesis, but the distinctions above point to some differences between mathematics and philosophy, which should be taken into account. The analytic method in philosophy is not the same as the analytic method in mathematics. There is not just one method in mathematics, and even the same problem, as is the case with the universality problem, is open to several of them. It's precisely the field of different perspectives in tackling the problem, therefore the possibility of discovering various hypotheses, that could offer a proof of its epistemically ampliative character. In other words, the possibility of approaching should be shaped by the mathematical context and practice. Again, adhering to any *a priori* established recipe for doing or approaching mathematics would be imitative to the heuristic value itself brought forward by considering mathematical objects as hypotheses.

Cellucci responds to the potentially infinite regress in solving a problem through the analytic method and the impossibility of knowledge that such an approach would entail by highlighting the fact that the lack of absolute justification for a hypothesis does not mean that there is no knowledge: that would happen only if the hypotheses were arbitrary, but that is not the case, since they must be plausible. The knowledge they entail is thus fallible, i.e., it may lead to error. Furthermore, new data may always emerge "with which the hypotheses on which knowledge is based may turn out to be incompatible ([19], p. 64). Given that the focus of the analytic method is plausibility and not truth, "no solution to a mathematical problem is final, every solution is revisable" (Ibid., p. 246).

There might be some difficulties in this approach. For instance, one can find revisions and developments, conceptual changes, different and new interpretations, but once accepted as correct (even if that requires a certain period of time), a mathematical result is definitive. I am taking into account here the particular case of set theory and the existence of independence results: we could find universes in which a certain result holds and other(s) in which it doesn't. In a way, we could say that there is just one solution manifested in different correct proofs. And some of these proofs point to new problems, open new interrogations, problems and solutions.

And it is the same case of the set theoretical framework and its proof methods, obvious in tackling the universality problem that show how a mathematical problem might have various sides, lead in different, sometimes seemingly opposite directions, each of which might suggest different hypotheses, different solutions, and even new relations or links to other problems, maybe even from another field. A consequence is that the connection between mathematical objects and the world is not abstraction and idealisation, but interrogation and orientation. It is about the proofs it entails and the new techniques it determines. The fact that universality represents a test problem in model theory testifies in that regard. The fundamental connection to combinatorics in set theory strengthens these aspects.

In set theory, one usually looks at sets with no other internal structure other than the membership relation. There is some additional structure in the context of PCF: the product structure and the equivalence structure arising from the filter that is used. As mentioned before, a structure is a set equipped with a collection of functions, relations, and elements, interpreted in an appropriate language. From a certain point of view, a mathematical structure represents a notion of analogy on a very abstract level, enabling structure theorems. And in this context, a universal model of T of size κ, a model in which any other model of T of cardinality κ embeds elementarily, represents, to use a notion from [215], a *critical object*, but which is *not* "almost always some canonical" member of a class S "simple to describe and visualise" (p. 43)[4].

A consequence is that the relation between models involved by universality is not determined in terms of size. The part - whole correlation has new meanings here, described in term of embeddings and orderings. And that represents another consequence related to the way we understand mathematical objects. Given a

[4]The original quote goes as follows: "Critical objects are almost always some canonical members of S simple to describe and visualize".

class K of models, for instance, expressing some 'well-behaved' theory, one doesn't find new properties moving at a certain cardinality once a universal model had been found there: a structure will be just like the others having the same size, but 'larger' in certain directions.

As I have shown above, there are different ways of approaching the universality problem, not only with respect to the area of mathematics involved, but also with regard to the methods and techniques used in proofs. The idea of extension involved by universality represents a certain kind of idealisation. This idealisation involves a certain form of abstraction, one that does not 'carry' with it the same conceptual distinctions and categories used in the finite realm. Such an idealisation implies a shift in the nature and the role of mathematical objects. Accordingly, it would also imply a change in the mathematical knowledge. Such an epistemic reconfiguration could be incorporated into our general 'knowledge', makes the thinking even more sophisticated, and returns again into a mathematical landscape determining new mathematical inquiries.

Mathematical objects are therefore always connected to a basic mathematical activity: they are part of a specific *process*. This process usually involves a transformation, a function in the most general sense, of a given *object* into another, sometimes of a quite different kind. It can be a transformation having a numerical aspect (like taking the square root), a non-numerical one, it can be *translated* into the *language* of sets etc. Universality refers to the way models are *interpreted* (to be described in different contexts) into another. In other words, any kind of generalisation should be context sensitive.

A universal object (within the framework in this text) sheds light on the objects it embeds. It is similar to various kinds of limit models and, in fact, it can be constructed as such, as the Fraïssé limit of the ultrapower construction testifies. The *process* itself constitutes a mathematical object. Consequently, the different levels and registers implied by the concept of *mathematical object* can only be meaningfully distinguished in the context of mathematical practice.

6.3 Abstract - Concrete

Mathematics defines concepts and combines them into mathematical structures (not the model theoretic notion here) and patterns described by mathematical concepts. Of course that these new patterns described by mathematics are more and more abstract and general. But that represents an external assessment to the mathematical practice. The abstract character of mathematical objects in general represents just one register of abstractisation. It is the one we could find in one of Bourkaki's statements, that "mathematics appears" as "a storehouse of abstract forms - the mathematical structures; and it so happens - without our knowing why - that certain aspects of empirical reality fit themselves into these forms, as if through a kind of pre-adaptation" (Bourbaki 1950, 231). One may ask why "some of the most intricate theories in mathematics become an indispensable tool to the modem physicist", but "fortunately for us, the mathematician does not feel called upon to answer such questions" (Bourbaki 1949, 2).

Another *level* of abstractisation is internal to the practice and methods of set theory and model theory. The idea of such a distinction, particularly the second aspect, is determined by the fact that the process of generalisation and the idea of abstraction are determined by the necessities of mathematical practice. I will return to this aspect shortly.

Within the mathematical discourse, the distinction between abstract and concrete might not take the same meaning as when one is positioning oneself in the context of the philosophy of mathematics. In the philosophy of mathematics (or in the philosophy of logic) we can talk about the abstraction principles in neo-logicism or the abstract nature of a mathematical object, one of the possible views with regard the nature of mathematical objects. It is not the purpose of this text to establish the relation of mathematical objects to the world, all these three notions being too ambiguous to obtain a definitive account.

I tend to think that Cellucci's view with regard to the nature of mathematical objects is plausible in the context of mathematical practice: mathematical objects are not obtained by abstraction from sensible things, or by idealization from our operations of collecting objects, but hypotheses we make to solve mathematical problems, several of which have an extra-mathematical origin. He presents many arguments against abstraction from *sensible things*. One of the arguments describing the relation of mathematical objects to the world as being obtained by idealization from certain operations of collecting objects belongs to Kitcher ([105], p. 12). One of the main problems with respect to such a view is that it points to an ideal subject. I think that statements asserting the existence or non-existence of a relation between mathematical objects and the *world*, that they might be obtained by abstraction from sensible things, or by idealization from our operations of collecting objects is ambiguous, unfruitful, and problematic, and not only from the point of view of mathematical practice.

In the circumstances, Kant is right when he writes that "abstraction is only the negative condition under which universal representations can be generated" ([100], p. 593). Through it, "nothing is produced, but rather left out" (Ibid., p. 487). In fact, "abstraction does not add anything", but "rather cuts off everything that does not belong to the concept" (Ibid., p. 351). As hypotheses, mathematical objects are obtained through rules determined by mathematical practice, they represent "a viewpoint" guiding the mathematical observation. Accordingly, they cannot be arbitrary, they are context related. As Hilbert emphasised, "the formation of concepts in mathematics is constantly *guided by thought and experience*, so that mathematics, on the whole, is a non-arbitrary construction" ([87], p. 5) [emphasis added].

In the case of universality, it is not only about the problem of finding a maximal element, but also about the consequences determined by the existence or non-existence of such element(s). In model theory, it enables the classification of theories, with consequences for mathematics as a whole. Another crucial challenge involving universality (mostly connected to set theory, but in this case with model theory as well) is about identifying a core in the problem, which is combinatorial. As such, it becomes a way of interrogation and of orientation in asking the *right* questions.

The second form of abstraction mentioned above is internal to mathematics and, in particular, to set theory. For instance, Moschovakis *abstracted* a property stronger than reduction, the prewellordering property, from the classical analysis of $\prod_1^1$ sets. Jensen *abstracted* his result regarding the Souslin-tree *to* (i) if $V = L$, then $\diamondsuit_\kappa$ holds for every regular cardinal $\kappa > \omega$, and (ii) if $\diamondsuit_{\omega_1}$ holds, then there is a Souslin-tree. Solovay's result was *abstracted* to κ-Kurepa trees in terms of ineffable cardinals, a new cardinal concept discovered independently by Jensen and Kunen. Shelah established an *abstract* classification theories for models.

A form of it (inherently connected to the subject of this text) was concisely emphasised by Larson in her presentation of infinite combinatorics: "[T]here is a tension in mathematics between generalization and description in concrete terms" ([121], p. 146). The extension of number into the transfinite operated by Cantor represents an example of a generalization. Concrete structures "may be sought through classification schemes, in basis problems, in universal structures, and especially in the search for and expectation of local uniformity, regions of simplicity, and inescapable structure", and this aspect can be "captured by the following quote, most often used in connection with Ramsey theory: "complete disorder is impossible"" ([121], p. 147). This formulation can be found in T. Motzkin's description of Ramsey's theory: "[W]hereas the entropy theorems of probability theory and mathematical physics imply that, in a large universe, disorder is probable, certain combinatorial theorems imply that *complete disorder is impossible*" ([156], p. 244) [emphasis added].

Universality here is an example of concrete structure. Abstraction could take place here in extension to another domain. It is actually *an extension of familiarity by abstracting to a new domain.* It can be achieved through forcing, for instance, or the methods of introducing new models in model theory. Abstraction manifests itself in the increased generalisations of the theorems that it determines and generates. And a general result means one does not get lost in having to prove each case separately, while a related consequence is that it could establish connections among different results and/or different areas of mathematics, relations that didn't seem possible before.

It is a process of determining regions of structure. And this process may find a possible analogy in the way Plato refers to the circumscribing domains in *Philebus*, as I mentioned in the first chapter. The insufficiency of some levels of analysis, of some mathematical objects, methods and solutions, and, in particular, the landscape of infinities (circumscribing the field of set theory) determines a process of epistemic unfolding, expressed in the creation of mathematical knowledge.

The use in this text of general and still problematic notions such as *knowledge* and *understanding* was related to the possibility of establishing a bridge between mathematics and philosophy in connection to the universality problem. It was not our intention to engage in an analysis of and the current debate regarding the nature of understanding and how it relates to the concept of *truth*. In connection to that, some scholars maintain that a focus on the role of understanding does not entail a preoccupation with factivity (see Kelp [103], Grimm [77] or Ross [176]).

Elgin ([57]) maintains that a strictly factive conception of understanding is empirically inadequate. Other discussions of non-factive approaches to understanding were offered by Zagzebski ([229]), Potochnik ([164]) or Rancourt ([170]). Others (see [175], for instance) point to some benefits of a strictly factive theory of understanding (such as explaining the essential role of false theories and idealisations in science).

Referring to mathematical objects as hypotheses, emphasising the distinction between mathematics and philosophy, the particular case of set theory with its different methods and problems (like universality), including the specific relationship between mathematical objects and their names (mentioned above in relation to forcing) represent different dimensions of configuring an epistemology of mathematics that does not entail an ontology (of mathematics). Plausibility also makes way for the fallibilism connected to the mathematical justification (derived from proof), no matter how this aspect is treated by different authors (see Celluci, Toffoli ([30]), Lakatos [119], Dove [41], and others).

All in all, mathematical objects display a relational character that could offer them role of hypothesis or even method (with certain contextual distinctions to be made). Once accepted in the mathematical practice, they could even be endowed with a factual character, without any ontological commitment, and caught in the transformational character of mathematical knowledge.

6.4 Method

As Agamben remarks concerning method, 'contrary to common opinion, a reflection on method usually follows practical application, rather than preceding it', being 'articulated only after extensive research', that also '[C]ontrary to public opinion, method shares with logic its inability to separate itself completely from its context' ([2], p. 7). What is more, and in accordance with Foucault archaeological perspective on the human sciences, every inquiry, including the reflection on method, 'must retrace its own trajectory back to the point where something remains obscure and unthematized' (Ibid., p. 8). Usually, the *method* or a *methodology* are considered closed systems, finished products, doctrinal recipes to follows. But in its original Greek meaning, *méthodos* (μέθοδος) is the 'pursuit of knowledge'. The purpose of method is to offer a certain direction or even several directions.

The notion of infinite has a certain role in this regard. In set theory, the infinite is never given as an ambiguous or vague concept to be prone to fuzzy digressions. It is always associated to a certain 'order' (the ordinal numbers, the uncountable linear orders), 'function', 'structure', 'size' (cardinality) etc. In other words, the infinite is always determinate in one way or another. Such determinations come from the axioms, as properties of relations. The set theoretical results show that the infinite itself is somehow associated to the idea of method. And, I would add, to the countless boundaries in our own view.

Firstly, it is not always possible to apply the same mathematical rules and intuition involved in the finite realm to the transfinite one. Given the conceptual jumps determined by the infinite, some adjustments and changes are to be made. It is already accepted that infinity can be used as a logical principle to establish certain properties of finite objects, which could otherwise have remained inaccessible (e.g. [160], but also see Roitman). Or, "I tried to show (...) that you can prove many things on infinity on the basis of the ZFC set theory if you just ask the right questions", writes Shelah ([107], p. 9).

Such a possibility is determined by the organising principles expressed in the axioms, and this represents a second aspect in tackling the idea of the infinite as a method. Even Mac Lane, one of the proponents of category theory (another alternative for the foundations of mathematics), emphasises the ability of the set theoretic axioms to reduce arguments to a few: "[T]he rich multiplicity of mathematical objects and the proofs of theorems about them can be set out formally with absolute precision on a remarkably parsimonious base" ([127], p. 358). The axiomatic method describes mathematical objects and formally prove them. It is only a method, but it is a method nonetheless, so it shouldn't be discarded as not being ampliative, as Cellucci does.

"Furthermore, the introduction of higher orders of infinity "can be likened to the introduction of an additional proof principle" ([32], p. 387) : "what is crucial now is no longer the truth of the axiom, but rather its potential richness and its proving power. And, from this viewpoint, the stronger the axiom is, and therefore the closer the contradiction, the more powerful it is likely to be in terms of applications" ([32], p. 388). Such a role for set theory goes through Cantor back to Plato. And, I think, in approaching the mathematical practice through the concept of hypothesis related to the mathematical object. The creation of different universes or the extension of models are part of the method.

With regard to universality, it represents a set theoretic and model theoretic problem. We could consider the notion of *problem* also in its original Greek meaning, *próblēma*, (πρόβλημα), as "an obstacle, anything thrown forward or projecting; as anything put before one as a defence, a headland". A mathematical problem has several sides, it can be approached from different directions, employing different hypothesis, which, at their turn, can establish relations with other problems, some in other fields. All these aspects put the initial problem into a new perspective, they create a context of analysis. The 'elements' of the universality problems present certain characteristics, both in set theory and model theory, that makes it a methodological instrument and also a form of interrogation of method. It does that through the roles it takes. In model theory, as other test problems in classification theory, it provides a context for establishing connections across different areas of mathematics.

It is inextricably connected to combinatorics in set theory. "The important ideas of combinatorics do not usually appear in the form of precisely stated theorems - writes Gowers -, but more often as general principles of wide applicability ([74], p. 68). And "[W]hile the structure is less obvious than it is in many other subjects, it is there in the form of somewhat vague general statements that allow proofs to be condensed in the mind, and therefore more easily memorized and more easily transmitted to others (Ibid., p. 72). "However, just as the true significance of a result in combinatorics is very often not the result itself, but something less explicit that one learns from the proof, so the general goals of combinatorics are not always explicitly stated" (Ibid., p. 73). The proof would offer a new technique. "In general, as one gains experience at solving problems in an area such as combinatorics, one finds that certain difficulties recur. It may not be possible to express these difficulties in the form of a precisely stated conjecture, so instead one often focuses on a particular problem which involves those difficulties. The problem then takes on an importance which goes beyond merely finding out whether the answer to it is yes or no. This explains why it was possible for so many of Erdös' problems to have hidden depths"([74], p. 75).

Set theory and the existence of a few axioms does not mean similarity in arguments. A reason for that is the way infinite combinatorics works. It often happens that the core of a set theoretic problem is combinatorial, but approaching the universality problem from both a set theoretic perspective and a model theoretic means we don't restrict ourselves to recognising the patterns in just one mathematical area. The combinatorial aspect abstracted from a certain problem is the part connected to pure calculation without taking into account the larger structure of a proof. Furthermore, a common characteristic of the combinatorial principles involved is that they are independent of the usual system of axioms in set theory. As a result, they are particularly useful in proving non-existence results regarding universality. This combinatorial core contains, from a certain point of view, the conditions that makes possible the result, the argument and the proof. It offers a certain direction in finding a solution or solutions. And, to a certain extent, in the same way in which the theorems of mathematics motivate the definitions and the definitions motivate the theorems, "just like the proof of a theorem is 'justified' by appealing to a previously given definitions" ([177], p. 172), so that there is no linear trajectory from definition to theorem.

All these aspects and the conceptual 'movements' or the various problems that it entails, shape the universality *problem* as a form of interrogating and giving directions to some methods. It offers an instrument of mapping an infinite mathematical landscape. From a chronological point of view, the analysis of universality determined the development of other central notions in model theory: saturated models, homogeneous models and, eventually, non-forking, an essential tool in classification theory. So what would a solution or the solutions to the 'universality problem' mean? Given the range of mathematical notions and results related to universality, one could look for connections. Classification theory is an answer. In this framework, connections are continuously discovered, while methods are constantly created and developed. Universality involves an analysis of models and theories, it is connected to their deeper understanding, the possibility of discovering new *relations, connections* or *orders* among them. And then some other perspectives, other approaches could follow.

6.5 And order

Various major scientific results in the 20th century have emphasised certain limits with regard to knowledge: Heisenberg's uncertainty principle for the measurement process, the constancy of the speed of light in relativity theory on the transmission of information, the sensitive dependence on initial conditions in chaos theory on the ability to predict the 'future' from less perfect measurements in the present, and others. In mathematics, we have Gödel's incompleteness theorems and Turing's theorem in computation. In addition, for Gödel, there are real objects extending beyond our capability to name or formalize them in any language. Thus, any formal system we have is incomplete, but we are also left with questions regarding the limits of naming objects, including concepts, which we nonetheless can (in some sense) think. We are left with the issue of how we come to know such objects. While the ontological presuppositions here are quite problematic, his arguments are connected to the problem of knowledge, mathematical knowledge in particular.

There are some aspects in the process of mathematical practice that cannot be quantified or brought forward without any ambiguities involved. We use notions such as intuition, imagination, reason, etc. without having a definitive response as to their meaning. They do not belong to mathematics but to the general human sphere, and they were traditionally transposed into the area of philosophy. The use of symbols does not necessarily eradicate such mundane, human tendencies. Some go further and attribute an ontological value to *what* stands behind these symbols. This "ontologising power of sign systems, that generate spaces structures within which to dream" ([212], pp. 132-133) belongs to the individual space. It might be the impetus behind extraordinary mathematical activities and results, but it points to individual values and belief. It goes the same for 'have feelings or intuition' that something is right or wrong, true or false. Mathematics is done by human beings, it represents one of the possible activities in which one could be engaged.

Besides unveiling individual values and beliefs, this *un-mathematical* element in someone's mathematical practice points to cultural elements as well. An assumption in mathematics is that mathematical theorems are necessarily true, an aspect that could lead to the assertion that mathematical objects exist by necessity (for a view on this implication, see [101], for instance). And as pointed up by Eglash, "[F]or mathematicians in the Euro-American tradition, truth is embedded in an abstract realm, and these transcendental objects are inaccessible outside of a particular symbolic analysis" ([55], p. 112).

I would contend that this vague element involves somehow the idea of *order*. This might be a place where mathematics and philosophy meet. While focusing her analysis on the overall intellectual context of the ancient Greece, Wersinger claims that the nascent 'ontology' represents one of the possible answers to a more original question concerning the harmony of things, whether of the body, of what we call 'world' or of the language which expresses it" ([226], p. 10). In set theory, a general survey of *ordering* ends up in capturing quite an impressive history: it starts with the foundational work on cardinal and ordinal numbers, fundamentals of cardinal and ordinal arithmetic, and then found ingrained in various themes throughout infinite combinatorics: Ramsey theory, trees, graphs, regressive functions, set mappings, etc. All these aspects involved interactions with other mathematical areas, such as model theory, topology, finite combinatorics.

But we can consider a more general account than set theory: "Does understanding the demonstration of a theorem consist in examining each of the syllogisms of which it is composed in succession, and being convinced that it is correct and conforms to the rules of the game? In the same way, does understanding a definition consist simply in recognizing that the meaning of all the terms employed is already known, and being convinced that it involves no contradiction? (...) Almost all are more exacting; they want to know not only whether all the syllogisms of a demonstration are correct, but why they are *linked together in one order rather than in another*. As long as they appear to them engendered by caprice, and not by an intelligence constantly conscious of the end to be attained, they do not think they have understood." (Poincaré, [163], Book II, Chapter II, p. 118) [emphasis added]. And further, "Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end" (Ibid., pp. 129–130).

Cournot made a distinction between a logical order and a rational one. I am only mentioning him in his drawing attention to this element in connection to mathematical explanation, but for an extended analysis, see [139]: "Generalizations which are fruitful because they reveal in a single general principle the rationale of a great many particular truths, the connection and common origins of which had not previously been seen, are found in all the sciences, and particularly in mathematics. Such generalizations are the most important of all, and their discovery is the work of genius. There are also sterile generalizations which consist in extending to unimportant cases what inventive persons were satisfied to establish for important cases, leaving the rest to the easily discernible indications of analogy. In such cases, further steps toward abstraction and generalization do not mean an improvement in the explanation of the order of mathematical truths and their relations, for this is not the way the mind proceeds from a subordinate fact to one which goes beyond it and explains it. (Cournot, 1851, sect. 16, Engl. trans. 1956, p. 24 quoted in [141], p. 144).

The idea of order I am referring to here has epistemological connotations and is connected to the other general notion of *mathematical knowledge*, with its different aspects mentioned in the first chapter, and also to the idea of explanation. Mathematical objects are not isolated entities to be study in isolation. They are always connected to others and part of a *space*, a mathematical conceptual space: a field, a group, a structure, a family of structure, a relation R etc. The way different mathematical objects are connected is reflected in the specific features of that conceptual space, which, at its turn, can be generalised, abstracted, or reconceptualised using strict mathematical methods. Method points to order. A good method is a way in which ideas, concepts and hypotheses are sought in a proper order.

In this context, it is not set theory *per se* that unifies mathematics, as many researchers have emphasised. There are other foundational systems, and they are not opposed, but adapted to certain aspects and problems in the vast mathematical landscape. And in doing so, each one determines, explores and give priority to certain methods and proofs, all the while creating mathematical knowledge. It so happens that a very/the most efficient way to inquire into the *universe*[5] opened by the acceptance of the strict, well-defined mathematical notion of infinity is set theory.

In a way, mathematicians, like other thinkers, are a sort of *geographers*. I am using here a distinction operated by Olsson, for whom "[T]o be a geographer is by definition to be a baptizer. (...) knowledge is an exercise in translation. The trouble is that no translation can be perfect. What, then, is knowledge? It is to say that something is something else and be believed when you do it" ([159], p. 87).

Maps point to boundaries and limits as well. And probably in a quite unintuitive way, the infinite represents the most general idea of limit. It represents the central concept around which the mathematical field of set theory is organised and, given the foundational role of set theory, it might be implied all over mathematics, although the large part of it does not use the hierarchies of infinities. But as Feynman remarked at some point, the "fact that things have common features

[5]We do not take *universe* here in the formal set theoretic definition.

turns out more and more universal".

What is it that allows us to look at a new proof or mathematical argument and conclude that it represents a sample of good reasoning? Firstly, we examine its *form* to make sure that it does not lead from true premises to a false conclusion. Then we acknowledge that the argument is *new* with regard to the content. But the second aspect involves a certain comparison and, therefore, also points to something *common* in relation to which we have something new. Such a *common thing* does not entail anything ontological.

We could then ask how are two proofs essentially the same. And how is that aspect related to *mathematical knowledge*? Tao distinguishes and exemplifies "four general ways in which one might try to capture the notion of equivalence between proof A and proof B: semantic, syntactic, algorithmic, and conceptual ([209]). The semantic or model-theoretic approach, involves the description of "the largest set of models to which proof A 'naturally' generalises, and compare that set to the corresponding set for proof B". A difficulty regarding this approach is that sometimes a proof has to be re-written or 'deconstructed' "until it becomes obvious how to extend it properly". Another one is determined by the use of model-theoretic techniques, and in this case one "may need some sort of 'second order model theory' (ugh) to properly analyse such proofs semantically". Syntactic approaches would include the use of ideas "from 'proof mining' (which are useful, for instance, in converting 'infinitary' proofs to 'finitary' ones or vice versa). One can also work in the spirit of 'reverse mathematics': declare a certain 'base theory' to be 'obvious', and then isolate the few remaining non-obvious steps in a proof which are external to that theory". For Pythagoras' theorem in Euclidian geometry, for instance, one could use as a base theory "the strictly smaller theory of affine geometry, which can handle concepts such as linear algebra and area, but not lengths, angles, and rotations", deconstruct here a proof for this theorem "and what one eventually observes is that at some point in that proof one must (implicitly or explicitly) use the fact that rotations preserve area and/or length. By isolating the one or two non-affine steps in the proof one can get a handle on the extent to which two proofs of Pythagoras are 'equivalent'". The algorithmic approach, similar to the syntactic one, works if the proof can be expressed as an algorithm. Given two constructions A and B, a "crude way to detect differences between these constructions is to look at their complexity: for instance, Construction A might be polynomial time and Construction B be exponential time". But if both are exponential time, construction B "is equivalent to Construction A if one could run Construction B in (say) polynomial time assuming that every step in Construction A could be called as an 'oracle' in O(1) time, and vice versa. This would say that, modulo polynomial errors, Construction A and Construction B use the 'same' non-trivial ingredients, though possibly in a different order. (...) That would be a convincing way to conclude that Steinitz is 'essentially a special case of' Gaussian elimination". The conceptual approach seems harder to formalise: "two different proofs of the same result (or of analogous results in different fields) are somehow 'facing the same difficulty', and 'resolving it the same way' at some high level, even if at a low level the two proofs and results are quite different (...) One could imagine in those cases that there is some formal Grothendieckian abstraction in which the two proofs could be viewed as concrete realisations of a single abstract proof, but in practice (especially when 'messy' analysis is involved) I think one has to instead proceed non-rigorously, by deconstructing each proof into vague, high-level 'strategies' and 'heuristics' and then comparing them to each other" (Ibid).

One might argue that the first and the fourth could well constitute one approach and the second and the third another one, following general lines from the philosophy of mathematics. One example of this last kind of studies could be Leitgeb ([125]), who maintains that the informal proofs differ from the formal proofs due to the fact that they are characterised by semantic and intuitive components. *Semantic* means that the terms and sentences occurring in such proofs have a meaning and are therefore open to interpretation. The intuitive aspect refers to the choices of the elementary steps of the proof and the axioms adopted. But the distinctions operated by Tao are too valuable from a mathematical practice perspective to be ignored. They highlight subtleties that should forbid a quick reduction.

These various questions and approaches point to an elusive element that is neither purely syntactic, nor purely semantic, and which could be considered as a certain kind of order in connection to the very general notion of mathematical knowledge. It is a condition of possibility for arranging the mathematical knowledge on solid basis that can only be determined by mathematical practice. It represents the *space* in which hypotheses, mathematical and logical, are conceptualised, formulated, and connected to other mathematical content.

As far as analogies go, it can be connected to Plato's notion of the *"bastard" kind of reasoning* used to introduce the *Receptacle* in *Timaeus.* First and foremost, it points to the existence of different kind of reasoning. It is the unique nature of the Receptacle which demands this specific kind of reasoning, different from the understanding used to grasp the Forms, or the opinion and sensation that apprehend the becoming of the physical objects. As many commentators have emphasized, the employment of the term bastard (*nōthos*) does not have pejorative connotations. Besides emphasizing the methodological aspects I mentioned above, it offers some new insights into the essence of metaphysical approach itself, as pointed out by Naomi Reshotko: "Plato's statement that the Receptacle is introduced through a bastard sort of reasoning should not be taken as a negative statement concerning what we ultimately find in Plato's ontology. Plato is simply confirming his conviction that, when we do metaphysics, we don't have the luxury of discovering a priori or necessary truths, nor do we have the comfort of seeing or intuiting that that which seems the best explanation for our experience must be either the true or the only explanation. Plato faces up to the fact that when we do metaphysics, we risk everything. We cannot do metaphysics without taking the chance that we are dead wrong as we somehow, by the skin of our teeth, by the seats of our pants, venture into completely uncharted territory - territory where assuming the existence of a legitimate compass is obviously and necessarily assuming far too much from the outset, and threatens to get us further lost rather than help us find our way. In using the term *nothos* Plato demonstrates his concern that to assume the existence of a legitimate compass in the form of some preordained or codifiable piece of reasoning is to degrade the integrity with which we begin our search into the question 'what is?' " ([172], pp. 134-5).

Just a small note: metaphysics here is not equivalent to ontology, quite the opposite, it offers the context of attributing truth to *objects*, making them *suches* instead of *thises*[6]. And, in this context, one could easily replace 'metaphysics' with 'mathematics'.

[6]the physical objects are not objective entities with stable identities, but fleeting properties of some further, underlying *this*).

7

Concluding remarks

We approached the universality problem in different contexts, many presented probably too concisely and not exhaustively from the point of view of references. And there are of course other suitable frameworks for analysis. But, like the current state of the universality problem in set theory and model theory, choice is also open and, once configured in the form of a question or investigation, becomes determined by formal (and informal) requirements.

Set theory and model theory represent theories able to offer valuable insights regarding the notion of mathematical knowledge and mathematical practice and even offer the possibility of further (philosophical) interrogations. The notion of universal object and the universality problem constitute a very rich conceptual space offering accordingly a good contextual framework for analysis.

We have described above some instances in which the universality problem in set theory and model theory is related to and determined the constitution and development of new mathematical proofs and techniques, and, as such, led to new ways of hypothesising about mathematical content and providing (mathematical) knowledge.

One of the central topics in model theory is represented by the classification theory for elementary classes, i.e., the classes of models of a first order theory, with the aim of finding dividing lines (by determining a structure theory for the models on one hand, and probing how complicated they are, on the other), and using good test problems, like universality. There is still work taking place in this regard, with researchers like Shelah attempting to find, like in all his endeavours, the most general theses.

Universality in set theory is, due to its nature, connected to other subjects and problems, and is inextricably determined and shapes different lines of inquiry and proof methods. The hierarchy of the infinite offers complex challenges in finding universal objects, and although there are many results regarding the first uncountable cardinal, there are less when we move to the next. One is still to find the universe in which there is a universal Aronszajn tree at $\aleph_1$ (or $\aleph_2$), for instance. There are many researchers in this area, some of them mentioned with different results throughout this text. Their work is connected to different aspects of this topic, but also to (many) others, determining a continuously expanding field of *mathematical knowledge*. It is not a unique feature of set theory or of this particular topic, but different topics and different areas of mathematics offer unique perspectives that are unavailable to others.

Consequently, it is not necessary to look for a certain kind of unification, in the form of a unique foundation for the entire mathematics, for instance. It might be more interesting to follow the movements and connections in this vast and complex area of knowledge, and maybe abstracting it to even more general forms of knowledge. Is it another question, to be asked of mathematicians and others researches as well, if such an endeavour is possible or even desirable.

8

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