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INTERSECTION OF PARABOLIC SUBGROUPS IN EVEN ARTIN GROUPS OF FC-TYPE

YAGO ANTOLÍN[1] AND ISLAM FONIQI[2]

[1] *Fac. Matemáticas, Universidad Complutense de Madrid and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Madrid, Spain* (yago.anpi@gmail.com)
[2] *Universitá degli Studi di Milano – Bicocca, Milan, Italy* (islam.foniqi@unimib.it)

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Abstract We show that the intersection of parabolic subgroups of an even finitely generated FC-type Artin group is again a parabolic subgroup.

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1. Introduction

An *Artin graph* Γ is a triple (V, E, m) where V is a set whose elements are called *vertices*, E is a set of two-element subsets of V whose elements are called *edges* and $m\colon E \to \{2, 3, 4, \dots\}$ is a function called *labelling* of the edges.

Given an Artin graph Γ, the corresponding *Artin group based on* Γ (also known as the *Artin–Tits group*) and denoted by G_Γ is the group with presentation

$$G_\Gamma \coloneqq \langle V \mid \mathrm{prod}(u,v,m(u,v)) = \mathrm{prod}(v,u,m(u,v))\ \forall \{u,v\} \in E \rangle,$$

where $\mathrm{prod}(u, v, n)$ denotes the prefix of length n of the infinite alternating word $uvuvuv\dots$.

Associated with an Artin graph, we can also construct the *Coxeter group based on* Γ which is the group with presentation

$$C_\Gamma \coloneqq \langle V \mid v^2 = 1\, \forall v \in V,\, \mathrm{prod}(u,v,m_{u,v}) = \mathrm{prod}(v,u,m_{u,v})\ \forall \{u,v\} \in E \rangle.$$

An Artin graph Γ and the corresponding group G_Γ are called *spherical type* if the associated Coxeter group C_Γ is finite.

For $S \subseteq V$, we denote by G_S to the subgroup of G_Γ generated by the vertices of S. Subgroups of this form are called *standard parabolic subgroups*, and a theorem of Van der Lek [9] shows that $G_S \cong G_\Delta$ where Δ is the Artin subgraph of Γ induced by S. An

Artin graph Γ and the corresponding group G_Γ are called of *FC-type* if every standard parabolic subgroup based on a complete subgraph is of spherical type.

A subgroup K of G_Γ is called *parabolic* if it is a conjugate of a standard parabolic subgroup. We say that K is of *spherical type* if it is conjugated to a standard parabolic subgroup that is of spherical type. It was proven by Van der Lek in [9] that the class of standard parabolic subgroups is closed under intersection and it is conjectured that the same result holds for the class consisting of all parabolic subgroups.

Let G_Γ be an Artin group and P_1, P_2 two parabolic subgroups in G_Γ. In any of the following cases, $P_1 \cap P_2$ is known to be again parabolic:

1. if G_Γ is of spherical type (see [4]),
2. if G_Γ is of FC-type and P_1 is of spherical type (see [11] which generalizes [12] where the result was obtained when both P_1 and P_2 are of spherical type),
3. if G_Γ is of *large type*, that is $m(\{u, v\}) \geq 3$ for all $\{u, v\} \in E$ (see [5]),
4. if G_Γ is a *right-angled Artin group*, that is $m(E) \subseteq \{2\}$ (see [1, 7] for a generalization to graph products),
5. if G_Γ is a *(2,2)-free two-dimensional Artin group*, i.e. Γ does not have two consecutive edges labelled by 2 and the geometric dimension of G_Γ is two [3],
6. if G_Γ is Euclidean of type $\tilde{A}_n$ or $\tilde{C}_n$ [8].

We say that an Artin graph $\Gamma = (V, E, m)$ is *even* if $m(E) \subseteq 2\mathbb{N}$. The main theorem of this article is:

Theorem 1.1. *Let $\Gamma = (V, E, m)$ be an even, finite Artin graph of FC-type. The intersection of two parabolic subgroups of G_Γ is parabolic.*

It is a standard argument to deduce from this theorem that the intersection of arbitrary many parabolic subgroups is again parabolic (see Corollary 5.3).

The class of even FC-type Artin groups includes the class of right-angled Artin groups (RAAGs for short), and they possess some similar properties. On one side, we understand well the case when Γ is a complete even FC-type Artin graph. This implies that G_Γ is a direct product of (≤ 2)-generated Artin groups (in the case of RAAGs, G_Γ is free abelian). On another side, every parabolic subgroup P of an even (FC-type) Artin group G_Γ is a *retract* i.e. there is a homomorphism $\rho\colon G \to P$ such that ρ restricted to P is the identity.

With these two properties, one can decompose even FC-type Artin groups into direct products and amalgamated free products, and in the latter case, we use the geometry of the Bass–Serre tree to deduce properties of the intersections of parabolic subgroups. In fact, we use these two facts in § 3 to reduce the proof of Theorem 1.1 to the case where the parabolic subgroups are conjugate to the same standard parabolic G_A of G_Γ and moreover, the graph Γ satisfies $\mathrm{Star}(x) = V$, for all $x \in V \backslash A$. In this setting, we deduce that the intersection of parabolic subgroups of RAAGs is parabolic and we note that this proof is different from the ones of [1, 7] which use normal forms.

We remark that in [11] the action on the Bass–Serre tree is used in a similar spirit as here.

However, for proving Theorem 1.1, we use more properties of even FC-type Artin groups. We show that under some circumstances, the kernel of the retractions of standard parabolic subgroups are again even FC-type Artin groups. Let Γ be an even FC-type Artin graph. In [2] it was shown that for every $v \in V$, the canonical retraction $\rho : G_\Gamma \to G_{V\setminus\{v\}}$, has a free kernel and they give a description of a free basis. With this result, they deduce that G_Γ is poly-free. We use these kernels and also the kernels of the retractions $\rho\colon G_\Gamma \to G_v$, which as we will show are again even FC-type Artin groups under certain conditions on $\mathrm{Link}(v)$. These are the main results of § 4, where we provide precise description of these kernels.

We prove Theorem 1.1 in § 5. We remark that in contrast with [4, 12] our proof does not make use of Garside theory. The paper is almost self-contained, we rely on the Bass–Serre theorem, the Redemiester–Schreier method and the description of kernels of [2].

We will begin setting some notation.

2. Notation

Let $\Gamma = (V, E, m)$ be an Artin graph. Note that V, E are the vertices and edges, respectively, of a simplicial graph. We will use standard terminology of graphs: for $v \in V$, the set $\mathrm{Link}(v) = \{u : \{v, u\} \in E\}$ is called the *link* of v. The set $\mathrm{Star}(v) = \mathrm{Link}(v) \cup \{v\}$ is called the *star* of v. Given a subset S of V the *subgraph induced by* S, and denoted Γ_S, is the Artin graph with vertices S, edges $E' = \{\{u, v\} \in E \mid u, v \in S\}$ and labelling that consists on restricting m to E'.

We note that the notion of being a (standard) parabolic subgroup of G_Γ depends on the presentation defined by Γ and not on the isomorphism class of G_Γ, so if needed, we will say that a subgroup is Γ-*parabolic*. This terminology will be relevant in the proof of our main theorem, as we will use that some parabolic subgroups of G_Γ are also parabolic in G_Δ, where G_Δ is an Artin subgroup of G_Γ.

For an edge $\{u, v\} \in E$ we denote $m(\{u, v\})$ by $m_{u,v}$ to simplify the notation (note that $m_{u,v} = m_{v,u}$).

Assuming that Γ is even, i.e. $m(E) \subseteq 2\mathbb{N}$, for any $S \subseteq V$ one has a retraction

$$\rho_S : G_\Gamma \longrightarrow G_S$$

defined on the generators of G_Γ as: $\rho_S(s) = s$ for $s \in S$, and $\rho_S(v) = 1$ for $v \in V\Gamma\setminus S$. When $S = \{v\}$, we might write $\rho_{\{v\}}$ as ρ_v. Moreover, as $\langle v\rangle \cong \mathbb{Z}$ via $v^n \mapsto n$, in many cases, we use $\mathbb{Z}$ as the co-domain of ρ_v without mentioning it. The use of this isomorphism should be clear from the context.

There is a simple condition for having an *even Artin graph of type FC:* m is an even labelling of E and for any triangle with edges $\{u, v\}$, $\{v, w\}$, $\{w, u\} \in E$, at least two of $m_{u,v}$, $m_{v,w}$, $m_{w,u}$ are equal to two (see [2, Lemma 3.1]).

If S, T are subsets of a group G, we write S^T to denote the set $\{tst^{-1} : t \in T, s \in S\}$. If $S = \{s\}$, we just write s^T to mean $\{s\}^T$, and similarly if $T = \{t\}$, we just write S^t instead of $S^{\{t\}}$.

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3. Even labelling and retractions

Throughout this section, $\Gamma = (V, E, m)$ is an even Artin graph. Some of the results of this section have been proved in a more general context, however, as the proof in the even case is very elementary, we have chosen to give the proof to make the paper as self-contained as possible. For example, the next lemma holds for any Artin group [9].

Lemma 3.1. *Let $A, B \subseteq V$. The following equality holds:*

$$G_A \cap G_B = G_{A\cap B}.$$

Proof. Let ρ_A, and ρ_B be the corresponding retractions for G_A, and G_B respectively. Consider the compositions $\rho_A \circ \rho_B$ and $\rho_B \circ \rho_A$. When applying them to $v \in V$, we notice that $(\rho_A \circ \rho_B)(v) = \rho_{A\cap B}(v) = (\rho_B \circ \rho_A)(v)$. Extending to morphisms on the group G_Γ, we obtain a commutative diagram of retractions, in the form: $\rho_A \circ \rho_B = \rho_B \circ \rho_A = \rho_{A\cap B}$.

As $G_{A\cap B} \subseteq G_A$ and $G_{A\cap B} \subseteq G_B$ one has $G_{A\cap B} \subseteq G_A \cap G_B$.

To show the other inclusion $G_A \cap G_B \subseteq G_{A\cap B}$, pick an element $x \in G_A \cap G_B$. One has $x \in G_A$ and $x \in G_B$, so $\rho_A(x) = \rho_B(x) = x$. Now using that retractions commute, we obtain:

$$\rho_{A\cap B}(x) = (\rho_A \circ \rho_B)(x) = \rho_A(\rho_B(x)) = \rho_A(x) = x.$$

As $\rho_{A\cap B}$ is a retraction, we have $x \in G_{A\cap B}$, as required. □

Lemma 3.2. *Let $A, B \subseteq V$ and $g, h \in G$. Then $gG_Ag^{-1} \subsetneq hG_Bh^{-1}$ implies $A \subsetneq B$.*

Proof. Conjugating by h^{-1}, we can write the proper inclusion $gG_Ag^{-1} \subsetneq hG_Bh^{-1}$ in the equivalent form $fG_Af^{-1} \subsetneq G_B$, for $f = h^{-1}g$. Applying ρ_B we obtain:

$$fG_Af^{-1} = \rho_B(fG_Af^{-1}) = \rho_B(f)G_{A\cap B}\rho_B(f)^{-1} \subsetneq G_B.$$

So, the proper inclusion $fG_Af^{-1} \subsetneq G_B$ is equivalent to the proper inclusion

$$\rho_B(f)G_{A\cap B}\rho_B(f)^{-1} \subsetneq G_B,$$

which after conjugating by $\rho_B(f)^{-1}$ becomes equivalent to $G_{A\cap B} \subsetneq G_B$, and this implies that $A \cap B \subsetneq B$.

Instead, applying ρ_A to $fG_Af^{-1} \subsetneq G_B$, we obtain

$$G_A = \rho_A(f)G_A\rho_A(f)^{-1} = \rho_A(fG_Af^{-1}) \subseteq \rho_A(G_B) = G_{A\cap B}.$$

The inclusion $G_A \subseteq G_{A\cap B}$ implies $A \subseteq A \cap B$. Ultimately $A \subseteq A \cap B \subsetneq B$, which means that $A \subsetneq B$, as required. □

In the next lemma, we reduce the problem of showing that the intersection of two parabolic subgroups is again parabolic, to deciding whether the intersection of two conjugates of a standard parabolic subgroup G_A is again parabolic. Once again, we make use of retractions.

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Lemma 3.3. *Let f, $g \in G$ and A, $B \subseteq V$. There exist $a \in G_A$ and $b \in G_B$ such that*

$$fG_Af^{-1} \cap gG_Bg^{-1} = faG_Ca^{-1}f^{-1} \cap gbG_Cb^{-1}g^{-1},$$

where $C = A \cap B$.

Proof. One has the equality

$$fG_Af^{-1} \cap gG_Bg^{-1} = f[G_A \cap (f^{-1}g)G_B(f^{-1}g)^{-1}]f^{-1}.$$

Set $h = f^{-1}g$ and consider $P = G_A \cap hG_Bh^{-1}$. Using $P \subseteq G_A$, and $G_A \cap G_B = G_{A\cap B}$ (see Lemma 3.1), we obtain:

$$\begin{aligned} P = \rho_A(P) = \rho_A(G_A \cap hG_Bh^{-1}) &\subseteq \rho_A(G_A) \cap \rho_A(hG_Bh^{-1}) \\ &= G_A \cap \rho_A(h)\rho_A(G_B)\rho_A(h^{-1}) \\ &= \rho_A(h)G_{A\cap B}\rho_A(h)^{-1}. \end{aligned}$$

Setting $a = \rho_A(h) \in G_A$ and $A \cap B = C$, we can write the inclusion above as $P \subseteq aG_Ca^{-1}$, and we notice that $aG_Ca^{-1} \subseteq G_A$. Also, $P = G_A \cap hG_Bh^{-1}$, so we have

$$\begin{aligned} P = (G_A \cap hG_Bh^{-1}) \cap aG_Ca^{-1} &= hG_Bh^{-1} \cap (G_A \cap aG_Ca^{-1}) \\ &= hG_Bh^{-1} \cap aG_Ca^{-1}. \end{aligned}$$

Multiplying the last equation by h^{-1} and denoting $P' = h^{-1}Ph$, $k = h^{-1}a$, we obtain: $P' = G_B \cap kG_Ck^{-1}$.

Applying the same procedure as for P above, we obtain:

$$\begin{aligned} P' = \rho_B(P') &= \rho_B(G_B \cap kG_Ck^{-1}) \\ &\subseteq \rho_B(G_B) \cap \rho_B(kG_Ck^{-1}) \\ &= \rho_B(k)G_{B\cap C}\rho_B(k)^{-1} \\ &= \rho_B(k)G_C\rho_B(k)^{-1}. \end{aligned}$$

Setting $b = \rho_B(k) \in G_B$ we express the inclusion above as $P' \subseteq bG_Cb^{-1} \subseteq G_B$. Putting together $P' = G_B \cap kG_Ck^{-1}$ and $P' \subseteq bG_Cb^{-1}$, we have:

$$P' = (G_B \cap kG_Ck^{-1}) \cap bG_Cb^{-1} = h^{-1}aG_ca^{-1}h \cap (G_B \cap bG_Cb^{-1}).$$

Using $G_B \cap bG_Cb^{-1} = bG_Cb^{-1}$, and $P' = h^{-1}Ph$, we ultimately have:

$$P = aG_Ca^{-1} \cap hbG_Cb^{-1}h^{-1}.$$

Turning back, we have $fG_Af^{-1} \cap gG_Bg^{-1} = fPf^{-1}$, and $h = f^{-1}g$, so we obtain:

$$fG_Af^{-1} \cap gG_Bg^{-1} = faG_Ca^{-1}f^{-1} \cap gbG_Cb^{-1}g^{-1},$$

where $C = A \cap B$, as desired. □

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The next lemma holds for any Artin group, see [11, Proposition 2.6]. The proof in the even case is much simpler.

Lemma 3.4. *Let* $g, h \in G_\Gamma$ *and* $A \subseteq V$*. If* $gG_Ag^{-1} \leqslant hG_Ah^{-1}$ *then* $gG_Ag^{-1} = hG_Ah^{-1}$*.*

Proof. We have that $gG_Ag^{-1} \leqslant hG_Ah^{-1}$ if and only if $h^{-1}gG_Ag^{-1}h \leqslant G_A$. In particular, $h^{-1}gG_Ag^{-1}h = \rho_A(h^{-1}gG_Ag^{-1}h) = G_A$. The lemma follows. □

Corollary 3.5. *Let* $A, B \subseteq V$ *and* $f, g \in G_\Gamma$*. Let* $H = fG_Af^{-1}$ *and* $K = gG_Bg^{-1}$ *be parabolic subgroups of* G_Γ*. If* $H = K$ *then* $A = B$*.*

In particular, if K is a parabolic subgroup of G_Γ, there is a unique $S \subseteq V$ such that K is conjugate to G_S. In that event, we say that K is *parabolic over* S. We note that if $K = fG_Sf^{-1}$, where $f \in G_\Gamma$, then K is also a retract of G_Γ, with the retraction homomorphism

$$\rho_K = \rho_S^f \colon G_\Gamma \longrightarrow K = fG_Sf^{-1}, \qquad \rho_S^f(g) := f\rho_S(f^{-1}gf)f^{-1}$$

for all $g \in G_\Gamma$. We will preferably use the notation ρ_K; however, we might use ρ_S^f if we want to emphasize the choice of the element in $fN_{G_\Gamma}(G_S)$, the coset of the normalizer of G_S, that we are using to conjugate.

Lemma 3.6. *Let* $A \subseteq V$ *and* $g \in G_\Gamma$*. Suppose that* $G_A \cup gG_Ag^{-1}$ *is not contained in a proper parabolic subgroup of* G *and for some* $x \in V\backslash A$*, one has that* A *is not contained in* $\mathrm{Link}(x)$*. Then* $G_A \cap gG_Ag^{-1}$ *is contained in a parabolic subgroup over a proper subset of* A*.*

Proof. Let $A \subseteq V$, $g \in G_\Gamma$, and $x \in V\backslash A$ with the property $\mathrm{Link}(x) \not\supseteq A$ be as in the hypothesis. Consider $P = G_A \cap gG_Ag^{-1}$.

If $G_{V\backslash\{x\}} = gG_{V\backslash\{x\}}$, then $g \in G_{V\backslash\{x\}}$. This means that both G_A and gG_Ag^{-1} are parabolic subgroups in $G_{V\backslash\{x\}}$, and hence $G_A \cup gG_Ag^{-1}$ is contained in the proper parabolic subgroup $G_{V\backslash\{x\}}$ of G. This contradicts the assumptions of the proposition, so suppose that $G_{V\backslash\{x\}} \neq gG_{V\backslash\{x\}}$. From the presentation, one has a splitting of G_Γ as an amalgamated free product:

$$G_\Gamma = G_{\mathrm{Star}(x)} *_{G_{\mathrm{Link}(x)}} G_{V\backslash\{x\}}.$$

Consider the Bass–Serre tree T corresponding to this splitting (see for example [6]). There are two types of vertices in T: left cosets of $G_{\mathrm{Star}(x)}$, and left cosets of $G_{V\backslash\{x\}}$ in G. Only vertices of different type can be adjacent in T. The group G acts naturally on T, without edge inversions. Moreover, the vertex stabilizers correspond to conjugates of $G_{\mathrm{Star}(x)}$ and conjugates of $G_{V\backslash\{x\}}$ for the respective type of vertices, while the edge stabilizers are conjugates of $G_{\mathrm{Link}(x)}$.

In the tree T, both $G_{V\backslash\{x\}}$ and $gG_{V\backslash\{x\}}$, are distinct vertices of the same type. Their stabilizers are $G_{V\backslash\{x\}}$, and $gG_{V\backslash\{x\}}g^{-1}$ respectively. As we are on a tree, there is a unique geodesic p in T connecting $G_{V\backslash\{x\}}$ and $gG_{V\backslash\{x\}}$.

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Since $x \notin A$, we have: $G_A \subseteq G_{V\setminus\{x\}}$ and $gG_Ag^{-1} \subseteq gG_{V\setminus\{x\}}g^{-1}$, which means that our parabolic subgroups G_A and gG_Ag^{-1} stabilize the vertices corresponding to the cosets $G_{V\setminus\{x\}}$ and $gG_{V\setminus\{x\}}$, respectively. The intersection $G_A \cap gG_Ag^{-1}$ stabilizes the geodesic p connecting those vertices, and hence it stabilizes any edge belonging to p. Since stabilizers of edges in T are conjugates of $G_{\mathrm{Link}(x)}$, we have:

$$P = G_A \cap gG_Ag^{-1} \subseteq hG_{\mathrm{Link}(x)}h^{-1}$$

for some $h \in G$. Now one can write P as:

$$P = G_A \cap gG_Ag^{-1} \cap hG_{\mathrm{Link}(x)}h^{-1} = (G_A \cap hG_{\mathrm{Link}(x)}h^{-1}) \cap (gG_Ag^{-1} \cap hG_{\mathrm{Link}(x)}h^{-1})$$

By Lemma 3.3, one can express $G_A \cap hG_{\mathrm{Link}(x)}h^{-1}$ as an intersection of two parabolic subgroups over $\mathrm{Link}(x) \cap A \subsetneq A$ (because $\mathrm{Link}(x) \not\supseteq A$), hence P is contained in a parabolic subgroup over a proper subset of A. This completes the proof. □

Lemma 3.7. *Let Δ be a subgraph of Γ, $A \subseteq V_\Delta$ and $g, t \in G_\Gamma$. If $G_A \cup gG_Ag^{-1}$ is contained in $tG_\Delta t^{-1}$, then there is $h \in G_\Delta$ such that:*

(i) *$G_A = hG_Ah^{-1}$ if and only if $G_A = gG_Ag^{-1}$.*

(ii) *$G_A \cap hG_Ah^{-1}$ is Δ-parabolic if and only if $G_A \cap gG_Ag^{-1}$ is Γ-parabolic.*

(iii) *$G_A \cap hG_Ah^{-1}$ is contained in a Δ-parabolic over a proper subset of A if and only if $G_A \cap gG_Ag^{-1}$ is Γ-parabolic over a proper subset of A.*

Proof. Suppose that the inclusion $G_A \cup gG_Ag^{-1} \subseteq tG_\Delta t^{-1}$ holds. Multiplying by t^{-1}, we obtain $t^{-1}G_At \cup t^{-1}gG_Ag^{-1}t \subseteq G_\Delta$. Applying ρ_Δ, and recalling that $A \subseteq V_\Delta$, we get:

$$t^{-1}G_At = \rho_\Delta(t^{-1})G_A\rho_\Delta(t) \text{ and } t^{-1}gG_Ag^{-1}t = \rho_\Delta(t^{-1}g)G_A\rho_\Delta(g^{-1}t).$$

Let $f_1 = \rho_\Delta(t^{-1})$ and $f_2 = \rho_\Delta(t^{-1}g) \in G_\Delta$. We have that

$$t^{-1}G_At \cap t^{-1}gG_Ag^{-1}t = f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1}.$$

Observe that $t^{-1}G_At \cap t^{-1}gG_Ag^{-1}t$ is Δ-parabolic (respectively contained in a parabolic subgroup over a subset of A) if and only if $G_A \cap gG_Ag^{-1}$ is Δ-parabolic (respectively contained in a parabolic subgroup over a subset of A).

We will take $h = f_1^{-1}f_2 \in G_\Delta$. Observe that $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1}$ is Δ-parabolic (respectively contained in a parabolic subgroup over a subset of A) if and only if $G_A \cap hG_Ah^{-1}$ is Δ-parabolic (respectively contained in a parabolic subgroup over a subset of A).

We prove (i). Note that $G_A = gG_Ag^{-1} \Leftrightarrow t^{-1}G_At = t^{-1}gG_Ag^{-1}t \overset{(*)}{\Leftrightarrow} \rho_\Delta(t^{-1}G_At) = \rho_\Delta(t^{-1}gG_Ag^{-1}t) \Leftrightarrow f_1G_Af_1^{-1} = f_2G_Af_2^{-1} \Leftrightarrow G_A = hG_Ah^{-1}$. The equivalence $(*)$ uses that $t^{-1}G_At$ and $t^{-1}gG_Ag^{-1}t$ are contained in G_Δ.

To show (ii), by the previous discussion, it is enough to show that $t^{-1}G_At \cap t^{-1}gG_Ag^{-1}t$ is Γ-parabolic if and only if $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1}$ is Δ-parabolic.

As Δ is a subgraph of Γ, any Δ-parabolic subgroup of G_Δ is a Γ-parabolic subgroup of G_Γ. Thus, if $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1}$ is Δ-parabolic, then it is also Γ parabolic. Conversely, assume there is some $B \subseteq V\Gamma$ and some $d \in G_\Gamma$ such that, $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1} =$

dG_Bd^{-1}. As $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1} \subseteq G_\Delta$, applying ρ_Δ, and noting that $B \subseteq A \subseteq \Delta$ we get that $f_1G_Af_1^{-1} \cap f_2G_Af_2^{-1} = f_3G_Bf_3^{-1}$ where $f_3 = \rho_\Delta(d)$.

A similar argument as above shows (iii). □

Suppose that $\mathcal{C}$ is a class of even Artin graphs, closed by taking subgraphs, and satisfying the following:

$$\begin{gathered}\text{for all } \Gamma \in \mathcal{C},\ A \subseteq V\Gamma,\ g \in G_\Gamma \text{ such that for all } x \in V\backslash A,\ \mathrm{Star}(x) = V \text{ one has}\\ G_A = gG_Ag^{-1} \text{ or } G_A \cap gG_Ag^{-1} \leqslant dG_Bd^{-1} \text{ for some } B \subsetneq A \text{ and } d \in G_\Gamma.\end{gathered} \tag{3.1}$$

Proposition 3.8. *If $\mathcal{C}$ is a class of even Artin graphs, closed by taking subgraphs, and satisfying (3.1), then for every $\Gamma \in \mathcal{C}$ the intersection of two Γ-parabolic subgroups of G_Γ is parabolic.*

Proof. Let $\Gamma \in \mathcal{C}$ and let P, Q be two parabolic subgroups of G_Γ. By Lemma 3.3, we can suppose that there is $A \subseteq V$ and $h_1, h_2 \in G_\Gamma$ such that $P \cap Q = h_1G_Ah_1^{-1} \cap h_2G_Ah_2^{-1}$. Therefore, $P \cap Q$ is parabolic if and only if $G_A \cap gG_Ag^{-1}$ is parabolic, where $g = h_1^{-1}h_2$.

If $G_A \cup gG_Ag^{-1}$ is contained in a proper parabolic subgroup of G_Γ, then by Lemma 3.7, we can replace Γ by a proper subgraph Δ and replace g by some $h \in G_\Delta$. Note that Δ is still in the class $\mathcal{C}$.

Therefore, we can return to the initial notation and further assume that $G_A \cup gG_Ag^{-1}$ is not contained in a proper parabolic of G_Γ.

We will show that for every $A \subseteq V$ finite and any $g \in G_\Gamma$, $G_A \cap gG_Ag^{-1}$ is parabolic. Our proof is by induction on $|A|$. If $|A| = 0$, then G_A is trivial and the result follows. We now assume that $|A| > 0$ and that for parabolic subgroups over smaller sets the result holds. We remark that the induction hypothesis is equivalent to saying that for any $B \subseteq V$, $|B| < |A|$ and any $g_1, g_2 \in G_\Gamma$, $g_1G_Bg_1^{-1} \cap g_2G_Bg_2^{-1}$ is parabolic.

If there is $x \in V\backslash A$ such that A is not contained in $\mathrm{Link}(x)$, then Lemma 3.6 implies that $G_A \cap gG_Ag^{-1} \leqslant dG_Bd^{-1}$ for some $B \subsetneq A$ and some $d \in G_\Gamma$. Therefore, by Lemma 3.3, there are $a, a' \in G_A$ and $b, b' \in G_B$ such that

$$\begin{aligned} G_A \cap gG_Ag^{-1} &= G_A \cap gG_Ag^{-1} \cap dG_Bd^{-1}\\ &= (G_A \cap dG_Bd^{-1}) \cap (gG_Ag^{-1} \cap dG_Bd^{-1})\\ &= (aG_{A\cap B}a^{-1} \cap dbG_{A\cap B}b^{-1}d^{-1}) \cap (ga'G_{A\cap B}a'^{-1}g^{-1}\\ &\quad \cap db'G_{A\cap B}(b')^{-1}d^{-1})\\ &= (aG_Ba^{-1} \cap dG_Bd^{-1}) \cap (ga'G_Ba'^{-1}g^{-1} \cap dG_Bd^{-1}). \end{aligned}$$

As $|B| < |A|$, by induction, $aG_Ba^{-1} \cap dG_Bd^{-1}$ and $aG_Ba^{-1} \cap dG_Bd^{-1}$ are parabolic subgroups of G_Γ over subsets of B. Say that $aG_Ba^{-1} \cap dG_Bd^{-1} = g_1G_{B_1}g_1^{-1}$ and $aG_Ba^{-1} \cap dG_Bd^{-1} = g_2G_{B_2}g_2^{-1}$. Thus, using again Lemma 3.3, we get that

$$G_A \cap gG_Ag^{-1} = g_1G_{B_1}g_1^{-1} \cap g_2G_{B_2}g_2^{-1} = g_1xG_Cx^{-1}g_1^{-1} \cap g_2yG_Cy^{-1}g_2^{-1}$$

where $x \in G_{B_1}$, $y \in G_{B_2}$ and $C = B_1 \cap B_2$. As $|C| < |A|$, using again induction, we get that $G_A \cap gG_Ag^{-1}$ is parabolic.

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So, we assume that for all $x \in V\backslash A$, $A \subseteq \mathrm{Link}(x)$. We now argue by induction on $N = \sharp\{x \in V\backslash A : \mathrm{Star}(x) \neq V\}$. In the case $N = 0$, as we are in the class $\mathcal{C}$ that satisfies (3.1), we have that either $G_A = gG_Ag^{-1}$ and hence $G_A \cap gG_Ag^{-1}$ is parabolic, or $G_A \cap gG_Ag^{-1} \leqslant dG_Bd^{-1}$ for some $B \subsetneq A$. As before, the latter implies that $G_A \cap gG_Ag^{-1}$ is an intersection of four parabolics over B, with $|B| < |A|$ and arguing as above, we get that $G_A \cap gG_Ag^{-1}$ is parabolic. So assume that $N > 0$ and the result is known for smaller values of N and A.

As $N > 0$ and $A \subseteq \mathrm{Link}(x)$ for all $x \in V\backslash A$, there are $x, y \in V\backslash A$ not linked by an edge. Setting $X = \mathrm{Star}(x)$, $Y = V\backslash\{x\}$, and $Z = \mathrm{Link}(x)$, we obtain an amalgamated free product:

$$G = G_X *_{G_Z} G_Y,$$

and there is an associated Bass–Serre tree T corresponding to this splitting.

Consider the edges G_Z and gG_Z on T. Let $G_Z, g_1G_Z, \ldots, g_nG_Z, gG_Z$ be a sequence of edges in the unique geodesic in T connecting G_Z and gG_Z. If $G_Z = gG_Z$ (i.e $n = 0$) and taking into account that $A \subseteq Z$, we have that $G_A \cup gG_Ag^{-1}$ is contained in G_Z, which is a proper parabolic of G_Γ and this contradicts our hypothesis. So we assume that $n \geq 1$. By the construction of T, one has either $g_i^{-1}g_{i+1} \in G_X$ or $g_i^{-1}g_{i+1} \in G_Y$, for any $i = 0, \ldots, n$ where $g_0 = 1$ and $g_{n+1} = g$. The intersection $G_A \cap gG_Ag^{-1}$ stabilizes the endpoints of the geodesic path, hence it stabilizes the whole path. As the stabilizer of a geodesic in a tree is the intersection of stabilizers of its edges, we have the equality

$$G_A \cap gG_Ag^{-1} = G_A \cap g_1G_Zg_1^{-1} \cap \ldots \cap g_nG_Zg_n^{-1} \cap gG_Ag^{-1}.$$

By Lemma 3.3, (applied to $G_A \cap g_iG_Zg_i^{-1}$), and the fact that $A \subseteq Z$ we have that there are $z_i \in G_Z$ such that $G_A \cap g_iG_Zg_i^{-1}$ is equal to $G_A \cap g_iz_iG_Az_i^{-1}g_i^{-1}$.

Note that $(g_iz_i)^{-1}(g_{i+1}z_{i+1}) = z_i^{-1}(g_i^{-1}g_{i+1})z_{i+1}$, so replacing g_iz_i by g_i we still have that $g_i^{-1}g_{i+1} \in G_X$ or $g_i^{-1}g_{i+1} \in G_Y$, for any $i = 0, \ldots, n$ where $g_0 = 1$ and $g_{n+1} = g$. Hence:

$$G_A \cap gG_Ag^{-1} = G_A \cap g_1G_Ag_1^{-1} \cap \ldots \cap g_nG_Ag_n^{-1} \cap gG_Ag^{-1}.$$

The intersections $g_iG_Ag_i^{-1} \cap g_{i+1}G_Ag_{i+1}^{-1}$ can be expressed as:

$$g_iG_Ag_i^{-1} \cap g_{i+1}G_Ag_{i+1}^{-1} = g_i[G_A \cap g_i'G_Ag_i'^{-1}]g_i^{-1},$$

where $g_i' = g_i^{-1}g_{i+1}$ is either in G_X, or in G_Y. As the number of vertices in $X\backslash A$ (respectively $Y\backslash A$) whose star is not X (respectively Y) is less than N, we can apply induction and the intersections $G_A \cap g_i'G_Ag_i'^{-1}$ are either equal to G_A or are contained in a parabolic subgroup over a proper subset B of A. If we have equality for $i = 1, \ldots, n$, then $G_A \cap gG_Ag^{-1} = G_A$. Otherwise, $G_A \cap gG_Ag^{-1}$ is contained in a parabolic subgroup over a proper subset B of A and we at the desired conclusion by induction on $|A|$. □

Corollary 3.9. *Let Γ be right-angled Artin graph. The intersection of any two parabolic subgroups of G_Γ is parabolic.*

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Proof. Let $\mathcal{C}$ be the family of finite right-angled Artin graphs. Clearly $\mathcal{C}$ is closed under subgraphs. Now take $A \subseteq V$, $g \in G_\Gamma$ such that for all $x \in V \setminus A$, $\mathrm{Star}(x) = V$. Then G_Γ is a direct product of G_A and $G_{V\setminus A}$ and thus, $G_A = gG_Ag^{-1}$. Then $\mathcal{C}$ satisfies (3.1) and the corollary follows from Proposition 3.8. □

4. Even FC-Artin labelling and kernels

Throughout this section, $\Gamma = (V, E, m)$ is an even Artin graph of FC-type.

Let $x \in V$. In this section, we describe the kernels of the retractions $\rho_{\{x\}}$ and $\rho_{V\setminus\{x\}}$. The kernel of $\rho_{V\setminus\{v\}}$ was described in [2], and turns out to be a free group, we will just recall their result. Our main contribution in this section is showing that $\ker \rho_{\{x\}}$ is isomorphic to an even FC-type Artin group G_Δ when $\mathrm{Star}(x) = V$. The construction of the Artin graph Δ and the isomorphism will be explicit and will allow us in the next section to show that certain Δ-parabolic subgroups of G_Δ are also Γ-parabolic (as subgroups of G_Γ).

4.1. Kernel of a retraction onto a vertex

Let $x \in V$ and $\rho \coloneqq \rho_{\{x\}}\colon G_\Gamma \to \langle x \rangle$ the associated retraction. We assume that at least one of the following holds:

(a) $\mathrm{Star}(x) = V$,

(b) for all $u \in L = \mathrm{Link}(x)$, $m_{u,x} = 2$.

We will see that under one of the previous conditions[*] $K = \ker \rho_x$ is isomorphic to G_Δ, where $\Delta = (V_\Delta, E_\Delta, m^\Delta)$ is an even FC-type Artin graph. Moreover, V_Δ will come with an indexing: $i\colon V_\Delta \to \mathbb{Z}$. We will say that $P \leqslant G_\Delta$ is *index parabolic* (with respect to i) if there is $n \in \mathbb{Z}$, $S \subseteq i^{-1}(n)$ and $g \in G_\Delta$ such that $P = gG_Sg^{-1}$.

Let $L = \mathrm{Link}(x) \subseteq V$ and $B = V \setminus \mathrm{Star}(x)$. For $u \in L$, let $k_u = m_{u,x}/2$. Let Δ be the graph with vertex set

$$V_\Delta = \left(\bigcup_{u\in L} \{u\} \times \{0, 1, \ldots, k_u - 1\}\right) \cup \left(\bigcup_{u\in B} \{u\} \times \mathbb{Z}\right).$$

We define the indexing $i\colon V_\Delta \to \mathbb{Z}$ as $i(v, n) = n$. For simplicity, we write a vertex (v, n) as v_n. For future use, we set the following terminology: a vertex $v_i \in V_\Delta$ is called *of type* $v \in V$ and of *index* i.

The edge set of Δ is

$$E_\Delta = \{\{u_n, v_m\}\} : u_n, v_m \in V_\Delta, \{u, v\} \in E\}.$$

That is, there is an edge between u_n and v_m in Δ if and only if there is and edge between u and v in Γ. Moreover, the label $m^\Delta_{u_n,v_m}$ of $\{u_n, v_m\}$ is the same as the label $m_{u,v}$ of $\{u, v\}$.

The labelling m^Δ of E_Δ is, by definition, even. It is also of *FC*-type. Indeed, we need to verify that any three vertices of Δ spanning a complete graph satisfy that at most one

[*] In fact, with hypothesis (b), we do not use that Γ is of FC-type

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of the labels of the edges is greater than 2. As there are no edges in Δ among vertices of the same type, if u_n, v_m, $w_l \in V_\Delta$ span a complete graph, we must have that u, v, w are three different vertices of Γ and n, m, $l \in \mathbb{Z}$. As $m^\Delta_{u_n,v_m} = m_{u,v}$, $m^\Delta_{v_m,w_l} = m_{v,w}$ and $m^\Delta_{w_l,u_n} = m_{w,u}$ and Γ is even FC-type, we get at most one of the $m^\Delta_{u_n,v_m}$, $m^\Delta_{v_m,w_l}$, $m^\Delta_{w_l,u_n}$ is greater than 2.

Lemma 4.1. *With the previous notation, $G_\Delta \cong \ker\rho$ via $v_n \mapsto x^n v x^{-n}$.*

Proof. We use the Reidemeister–Schreier procedure (see [10]) to obtain a presentation of K. Write $V = B \sqcup L \sqcup \{x\}$ where $L = \mathrm{Link}(x) \subseteq V$ and $B = V \setminus \mathrm{Star}(x)$. So the retraction map is given by:

$$\rho\colon G_\Gamma \to \mathbb{Z}, \quad x \mapsto 1,\ \forall a \in B \cup L : a \mapsto 0,$$

with $K = ker(\rho)$.

The set $T = \{x^i \mid i \in \mathbb{Z}\}$ gives a Schreier transversal for K in G_Γ. The set of generators for K is $Y = \{tv(\overline{tv})^{-1} \mid t \in T,\ v \in V,\ tv \notin T\}$ where $\overline{w}$ is the representative of w in T. Let us compute the set Y. For $v = x$ and $t = x^i$, $tv(\overline{tv})^{-1} = x^i x(\overline{x^i x})^{-1} = 1$. For $v = a$ with $a \in B \cup L$, let $a_i \coloneqq x^i a(\overline{x^i a})^{-1} = x^i a x^{-i}$. Therefore, we get that the set

$$Y = \{a_i \coloneqq x^i a x^{-i} \mid a \in L \cup B,\ i \in \mathbb{Z}\},$$

gives a set of generators for K.

Denote by R the set of relations of the defining presentation of G_Γ. To obtain relations for K, rewrite each trt^{-1} for $t \in T$ and $r \in R$ using generators in Y.

Write any $t \in T$ as x^i for some $i \in \mathbb{Z}$. We collect the relations in R into two types:

(i) relations involving only elements of $L \cup B$: i.e. of the form $r = (ab)^m (ba)^{-m}$ where a, $b \in L \cup B$,

(ii) relations involving x: i.e. of the form $r = (ax)^{k_a}(xa)^{-k_a}$ with $a \in L$.

In case (i), we have $trt^{-1} = x^i((ab)^m(ba)^{-m})x^{-i}$. Introducing $x^i x^{-i}$ between letters, and recalling that $a_i = x^i a x^{-i}$, we obtain:

$$trt^{-1} = (a_i b_i)^m (b_i a_i)^{-m}$$

which is an even Artin relation, for the pair a_i, b_i for all $i \in \mathbb{Z}$, with the same label as the Artin relation for the pair a, b.

In case (ii), we have $trt^{-1} = x^i((ax)^{k_a}(xa)^{-k_a})x^{-i}$. Again we put $x^i x^{-i}$ between letters, and use $a_i = x^i a x^{-i}$ to obtain:

$$trt^{-1} = a_i a_{i+1} \dots a_{i+k_a-1}(a_{i+1}a_{i+2}\dots a_{i+k_a})^{-1}.$$

The presentation for K is given as:

$$K = \langle Y \mid S \rangle,$$

where $Y = \{a_i = x^i a x^{-i} \mid a \in L \cup B,\ \ i \in \mathbb{Z}\}$, and the relations are described as below:

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(i) if $a, b \in L \cup B$ satisfy $(ab)^m = (ba)^m$, then for all $i \in \mathbb{Z}$: $(a_i b_i)^m = (b_i a_i)^m$

(ii) if $a \in L$ and x satisfy $(ax)^{k_a} = (xa)^{k_a}$ then for all $i \in \mathbb{Z}$: $a_i a_{i+1} \dots a_{i+k_a-1} = a_{i+1} a_{i+2} \dots a_{i+k_a}$.

We can use the type (ii) relations to simplify our presentation. If $a \in L$ and x satisfy $(ax)^{k_a} = (xa)^{k_a}$, then any a_i is a product of $a_0, a_1, \dots, a_{k_a-1}$. Indeed, if we adopt the notation $\sigma_a = a_0 a_1 \cdots a_{k_a-1}$, we obtain:

$$a_l = \sigma_a^{-q} a_r \sigma_a^q, \tag{4.1}$$

where $l = k_a \cdot q + r$ with $0 \le r < k_a$. Note that if $k_a = 1$ then $\sigma_a = a_0$ and $a_l = a_0$ for all l.

We can use Tietze transformations to eliminate all generators a_i, $i \notin \{0, 1, \dots, k_a - 1\}$ and the relations of type (ii). We obtain a new presentation with generating set

$$V_\Delta = \{a_j \mid a \in \mathrm{Link}(x),\ 0 \le j \le k_a - 1 \text{ in } \mathbb{Z}\} \cup \{b_j \mid b \in B, j \in \mathbb{Z}\}.$$

To future use, we set the following terminology.

We need to examine what happens with relations in case (i). Let us examine what is the effect of the previous Tietze transformations on $r = (a_j b_j)^k = (b_j a_j)^k$. We have several cases. Note that if $B \ne \emptyset$, then we are under hypothesis (b):

(i) $a, b \in B$. In this case, r is unaltered under the Tietze transformations as none of the generators involved are eliminated.

(ii) $a \in L$, $b \in B$. This case only can happen if we are in case (b) and thus, $k_a = 1$ and we have that $a_i = a_0$ for all $i \in \mathbb{Z}$. Thus, r becomes $(a_0 b_j)^k = (b_j a_0)^k$.

(iii) $a, b \in L$. Here we have several subcases:
- under hypothesis (b): we have that $k_a = k_b = 1$ and then r becomes $(a_0 b_0)^k = (b_0 a_0)^k$.
- under hypothesis (a): if $k > 1$, then because of the FC-condition , $k_a = k_b = 1$ and then r becomes $(a_0 b_0)^k = (b_0 a_0)^k$.
- under hypothesis (a): if $k = 1$, then because of the FC-condition, at least one of k_a and k_b is equal to 1. If both are equal to 1, then R becomes $a_0 b_0 = b_0 a_0$. If, say $k_a > 1$, then r becomes $\sigma_a^q a_s \sigma_a^{-q} b_0 = b_0 \sigma_a^q a_s \sigma_a^{-q}$ where $j = k_a \cdot q + s$ with $0 \le s < k_a$. Note that for $j \notin \{0, 1, \dots, k_a - 1\}$, r is a consequence of $a_0 b_0 = b_0 a_0, \dots, a_{k_a-1} b_0 = b_0 a_{k_a-1}$ and thus those relations can be eliminated.

It is straightforward to check that the presentation that we obtain is the presentation of the Artin group G_Δ with Δ given above. □

Assume that $B = \emptyset$. Since the relations in K come from the relations between elements of L, we obtain immediately the following corollary.

Corollary 4.2. *If G_L is free and $B = \emptyset$, then the kernel K is free as well, on $\sum_{a \in A} k_a$ generators, where $2k_a$ is the label of the edge in Γ for the pair x, a with $a \in L$.*

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The following is an important observation.

Lemma 4.3. *If P is index-parabolic in G_Δ, then P is parabolic in G_Γ.*

4.2. Kernel of a retraction onto the complement of a vertex

Let $z \in V$ and $\rho \coloneqq \rho_{V\setminus\{z\}}\colon G_\Gamma \to G_{V\setminus\{z\}}$ the associated retraction. In [2], it is shown that $K = \ker\rho$ is a free group and they give an explicit description of the basis. We shall now recall the construction in the specific case when $m_{u,z} = 2$ for all $u \in L = \mathrm{Link}(z)$, as this simplifies our description.

Following [2], a set $\mathcal{N}_L$ of normal forms for elements in G_L is described. There first a subset $L_1 = \{u \in L : m_{u,z} = 2\}$ of L is defined. Note that in our situation $L = L_1$. Then a normal form $\mathcal{N}_1$ for G_{L_1} is fixed. The set $\mathcal{N}_L$ is defined in this case as $\mathcal{N}_1$ and following the notation of [2, Paragraph before Lemma 3.6] in this case, one has that T_0^* is the empty set, $T_0 = \{1\}$, and $T = \ker\rho_L$ where $\rho_L\colon G_{V\setminus\{z\}} \to G_L$ is the canonical retraction.

Now $\{z\} \times T$ is a free basis of $\ker\rho$ (See [2, Proposition 3.16]) and we can identify $z_t \coloneqq (z, t)$ with tzt^{-1}.

5. Intersection of parabolics

The next lemma essentially proves that the intersection of parabolic subgroups on Artin groups based on graphs with two vertices is parabolic. It exemplifies some of the ideas used in the theorem of this section.

Lemma 5.1. *Let $\Gamma = (V = \{a, x\}, E = \{a, x\}, m)$ be an Artin graph with $m_{a,x} = 2k$ for some $k \geq 1$. Let $g \in G_\Gamma$. Then $\langle a\rangle \cap g\langle a\rangle g^{-1}$ is either equal to $\langle a\rangle$ or trivial.*

Proof. Let $\rho_x\colon G_\Gamma \to \langle x\rangle$. Both $\langle a\rangle$ and $g\langle a\rangle g^{-1}$ lie on $\ker\rho_x$. From § 4.1 we know that $\ker\rho_x$ is free with basis $a_0, a_1, \ldots, a_{k-1}$ where $a_i = x^i a x^{-i}$. Write $g = hx^s$ where $s = \rho_x(g)$ and $h \in \ker\rho_x$. Following Equation (4.1), we have that $x^s a x^{-s} = \sigma_a^{l(s)} a_r \sigma_a^{-l(s)}$ for some $l(s) \in \mathbb{Z}$, $0 \leq r < k$ and $\sigma_a = a_0 a_1 a_2 \cdots a_{k-1}$. In particular,

$$\langle a\rangle \cap g\langle a\rangle g^{-1} = \langle a_0\rangle \cap h\sigma_a^{l(s)}\langle a_r\rangle\sigma_a^{-l(s)}h^{-1}.$$

Now, the intersection is trivial if $r \neq 0$. If $r = 0$, as $\langle a_0\rangle$ is a malnormal subgroup (even more a free factor) of $\ker\rho_x$, the intersection is trivial unless $h\sigma_a^{l(s)} \in \langle a_0\rangle$, and in that case, the intersection is the whole $\langle a_0\rangle = \langle a\rangle$. □

The following theorem says that the class of even FC-type Artin graphs satisfies the condition of Equation (3.1).

Theorem 5.2. *Let $\Gamma = (V, E, m)$ be an even FC-type finite Artin graph. Let $A \subseteq V$, such that for all $x \in V\setminus A$, $V = \mathrm{Star}(x)$. Let $g \in G$. Then either $G_A = gG_Ag^{-1}$ or there is $B \subsetneq A$ such that $G_A \cap gG_Ag^{-1} \leqslant G_B$.*

Proof. If A is empty, then $G_A = \{1\}$ and $G_A = gG_Ag^{-1}$. So we assume that A is non-empty.

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Let N be the number of edges from A to $V\backslash A$ with label greater than 2. We will argue by induction on N.

If $N=0$, then for all $x\in V\backslash A$ and all $a\in A$, the label of $\{x,a\}$ is 2, and hence $G=G_{V\backslash A}\times G_A$ and $gG_Ag^{-1}=G_A$ for all $g\in G$.

So assume that $N>0$ and the theorem holds for smaller values of N.

Consider first the case when there is $x\in V\backslash A$ and $a\in A$ such that the label of $\{x,a\}$ is $2k_a$ with $k_a>1$ and $A\subseteq \mathrm{Star}(a)$. In this case, $\mathrm{Star}(x)=\mathrm{Star}(a)=V$, and for all $z\in Z:=V\backslash\{a,x\}$ we have that $m_{x,z}=m_{a,z}=2$, which yields $G_\Gamma=G_{\{x,a\}}\times G_Z$. Write $g=(g_1,g_2)$ with $g_1\in G_{\{x,a\}}$ and $g_2\in G_Z$. Then $G_A\cap gG_Ag^{-1}$ is equal to the direct product of the subgroup $\langle a\rangle\cap g_1\langle a\rangle g_1^{-1}$ of the direct factor $G_{\{x,a\}}$ and the subgroup $G_{A\backslash\{a\}}\cap g_2G_{A\backslash\{a\}}g_2^{-1}$ of the direct factor G_Z. By Lemma 5.1 we have that $\langle a\rangle\cap g_1\langle a\rangle g_1^{-1}$ is either trivial or equal to $\langle a\rangle$. Let $A'=A\backslash\{a\}$. Note that for all $z\in Z\backslash A'$, $\mathrm{Star}(z)=Z$ and the number of vertices $z\in Z\backslash A'$ with an edge with label >2 is less than N (as Z spans a subgraph of Γ that consists of deleting the vertices x, a and the edge $\{x,a\}$). Therefore, by induction, either $G_{A'}=g_2G_{A'}g_2^{-1}$ or $G_{A'}\cap g_2G_{A'}g_2^{-1}$ is contained in a parabolic subgroup over a proper subset of A'. The theorem follows in this case.

So let us consider the case when there is $x\in V\backslash A$ and $a\in A$ such that the label of $\{x,a\}$ is $2k_a$ with $k_a>1$ (in particular $N\geq 1$), $A\nsubseteq \mathrm{Star}(a)$ and that the theorem holds for smaller values of N. We remark that the condition $A\nsubseteq\mathrm{Star}(a)$ will not be used until Case 3.2 below.

Recall from the previous section, that there exists a finite Artin graph Δ, such that $\ker\rho_x$ is isomorphic to G_Δ. By the notation of § 4.1, $V_\Delta=\{z_0,\ldots,z_{k_z-1}:z\in V\backslash\{x\}\}$, $E_\Delta=\{\{u_i,v_j\}\subseteq V_\Delta: u_i\neq v_j \text{ and } \{u,v\}\in E\}$, and $m^{\Delta}_{u_i,v_j}=m_{u,v}$. Recall that the vertices of V_Δ are indexed. We will write A_0 to denote the vertices of type $v\in A$ and index 0, i.e. $A_0=\{b_0:b\in A\}$. Observe that the vertices y_i of $V_\Delta\backslash A_0$ such that $\mathrm{Link}(y)$ does not contain A_0 are exactly the vertices $b_1,\ldots,b_{k_b-1}$ with $b\in A$ and $k_b>1$. Indeed, if $k_b>1$, $b_0,\ldots,b_{k_b-1}$ span a subgraph with no edges of Δ and thus $b_0\notin\mathrm{Link}(b_i)$ for $i>0$. On the other hand, if $y\in V\backslash A$, as $\mathrm{Star}(y)=V$, we have that $k_y=1$ (since y, x, a form a triangle) and then in Δ we only have a vertex y_0 of type y and by definition of Δ, we have that $\mathrm{Star}(y_0)=V_\Delta$.

We note that G_{A_0} is an index-parabolic subgroup of G_Δ and it is equal to the subgroup G_A of G_Γ.

Write $g=hx^s$ where $h\in\ker\rho_x$ and $s=\rho_x(g)$. Let $Q=x^sG_Ax^{-s}\leqslant\ker\rho_x$. We note that Q might not be a parabolic subgroup of $\ker\rho_x$ although we can give a very precise description using Equation (4.1): Q is generated by $\{x^sbx^{-s}:b\in A\}$. If $k_b=1$, then $x^sbx^{-s}=b_0$. If $k_b>1$ then x^sbx^{-s} is equal to $\sigma_b^{l(s,b)}b_i\sigma_b^{-l(s,b)}$ where $i\in\{0,\ldots,k_b-1\}$, $i\equiv s \mod k_b$, $l(s,b)\in\mathbb{Z}$ and $\sigma_b=b_0b_1\ldots b_{k_b-1}$.

Now $G_A\cap gG_Ag^{-1}=G_{A_0}\cap hQh^{-1}$. Note that even if Q is not parabolic, we are reduced to show that either $G_{A_0}=hQh^{-1}$ or that $G_{A_0}\cap hQh^{-1}$ is contained in a Δ-parabolic subgroup over a proper subset of A_0. Indeed, in the latter case, as Δ-parabolics over subsets of A_0 are Γ-parabolics, we also get that $G_A\cap gG_Ag^{-1}$ is contained in a Γ-parabolic subgroup over a proper subset of A.

We consider three cases:

Case 1: $s=0$. Then $Q=G_{A_0}$ is a parabolic subgroup of G_Δ. By Lemma 3.7 we can reduce the problem to a subgraph Δ' of Δ and $h'\in G_{\Delta'}$ such that $G_{A_0}\cup h'G_{A_0}(h')^{-1}$

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is not contained in a proper parabolic subgroup of $G_{\Delta'}$. We need to show that either $G_{A_0} = h'G_{A_0}(h')^{-1}$ or $G_{A_0} \cap h'G_{A_0}(h')^{-1}$ is contained in a Δ'-parabolic subgroup over a proper subset of A_0.

If $b_i \in V_{\Delta'}$ for some $b \in A$, $k_b > 1$ and $i > 0$, then Lemma 3.6 implies that $G_{A_0} \cap h'G_{A_0}(h')^{-1}$ is contained in a Δ'-parabolic subgroup over a proper subset of A_0 and we are done. So, we can assume that $V_{\Delta'} \subseteq A_0 \cup \{y_0 : y \in V \backslash A\}$. Note that in this case, Δ' is a finite Artin graph of even FC-type, for all $y_0 \in V_{\Delta'} \backslash A_0$ we have that $\mathrm{Star}(y_0) = V_{\Delta'}$ and the number of edges from $V\Delta' \backslash A_0$ to A_0 with label > 2 is less than N (in fact, it is less than N minus the number of edges $\{x, b\}$ with label > 2). By induction, we get that either $G_{A_0} = h'Q(h')^{-1}$ or $G_{A_0} \cap h'Q(h')^{-1}$ is contained in a Δ'-parabolic subgroup over a proper subset of A_0 and we are done.

Case 2: $s \notin k_a\mathbb{Z}$**.** Then $\rho_{A_0}(hQh^{-1}) \leqslant G_{A_0\backslash\{a_0\}}$ and therefore, $G_{A_0} \cap hG_Q h^{-1} \leqslant G_{A_0\backslash\{a_0\}}$ and the lemma holds.

Case 3: $s \in k_a\mathbb{Z}$**,** $s \neq 0$**.** Recall that Q is generated by $\{x^s b x^{-s} : b \in A\}$, and the element $x^s b x^{-s}$ is equal (in G_Δ) to $\sigma_b^{l(s,b)} b_{i(s,b)} \sigma_b^{-l(s,b)}$ where $i(s, b) \equiv s \mod k_b$, $l(s, b) \in \mathbb{Z}$ and $\sigma_b = b_0 b_1 \ldots b_{k_b-1}$(see Equation(4.1)). If some $i(s, b) \neq 0$, we lie in Case 2. So we assume that $i(s, b) = 0$ for all $b \in A$.

For simplifying our notation and arguments[†] , consider the automorphism

$$\phi\colon G_\Delta \to G_\Delta \qquad \phi(v) = \begin{cases} v & v \neq a_1 \\ \sigma_a & v = a_1. \end{cases}$$

We need to show that this is well defined. By construction, the only edges of Δ adjacent to a_1 are of the form $\{z_0, a_1\}$ with $z \in \mathrm{Link}_\Gamma(a) \backslash \{x\}$. Moreover, as $\mathrm{Star}(x) = V$ and $k_a > 1$, necessarily $m_{z_0,a_1} = 2$ for $z \in \mathrm{Link}_\Gamma(a) \backslash \{x\}$. Thus, we need to check that σ_a commutes with z_0, $z \in \mathrm{Link}_\Gamma(a) \backslash \{x\}$. But this holds, as by construction $m_{z_0,a_i} = 2$ for all $i = 0, 1, \ldots, k_a - 1$ (recall that $\sigma_a = a_0 a_1 \cdot a_{k_a-1}$). Thus, ϕ is well defined. It is easy to check that ϕ is bijective.

We apply ϕ^{-1} to G_{A_0}, Q and h, and we get G_{A_0},

$$P = \langle \{a_1^{l(s,a)} a_0 a_1^{-l(s,a)}\} \cup \{\sigma_b^{l(s,b)} b_0 \sigma_b^{-l(s,b)} : b_0 \in A_0 \backslash \{a_0\}\} \rangle \tag{5.1}$$

and $f = \phi^{-1}(h)$ respectively. Note that as G_{A_0} is fixed by ϕ, we have that $G_{A_0} = hQh^{-1}$ if and only if $G_{A_0} = fPf^{-1}$ and that $G_{A_0} \cap hQh^{-1}$ is contained in a Δ-parabolic subgroup over a proper subset of A_0 if and only if the same holds for $G_{A_0} \cap fPf^{-1}$. For simplicity, we set $l = l(s, a)$. Note that as $s \neq 0$, $l \neq 0$.

Let $D = V_\Delta \backslash \{a_0\}$ and ρ_D the corresponding retraction. Then $G_\Delta = \ker \rho_D \rtimes G_D$. Let $\rho_{a_1} : G_\Delta \to \langle a_1 \rangle$ be the canonical retraction. We have now two subcases.

Case 3.1: $\rho_{a_1}(f a_1^l) \neq 0$.

We are going to show that

$$(G_{A_0} \cap \ker \rho_D) \cap (fPf^{-1} \cap \ker \rho_D) = \{1\}.$$

[†] Below a standard parabolic subgroup G_D is defined, and an advantage of using ϕ is that $\sigma_a \notin G_D$ but a_1 is in G_D .

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This implies that $G_{A_0} \cap fPf^{-1} \leqslant G_D$ and therefore, $G_{A_0} \cap fPf^{-1} \leqslant G_{A_0 \cap D}$ and we are done, as $A_0 \cap D = A_0 \backslash \{a_0\}$. Note that

$$(G_{A_0} \cap \ker \rho_D) = \langle a_0^{G_{A_0}} \rangle \quad \text{and} \quad (fPf^{-1} \cap \ker \rho_D) = \langle (fa_1^l a_0 a_1^{-l} f^{-1})^{fPf^{-1}} \rangle.$$

Let $L = \mathrm{Link}_\Delta(a_0)$. As $k_a > 1$ every vertex in L commutes with a_0. Let ρ_L be the canonical retraction $\rho_L \colon G_D \to G_L$, and $T = \ker(\rho_L)$. Recall from § 4.2 that $\ker \rho_D$ is free with free basis $\{ta_0t^{-1} : t \in T\}$. Note that if $g \in G_D$, then there are unique $g_L = \rho_L(g)$ and $g' \in \ker \rho_L$ such that $g = g'g_L$ and $ga_0g^{-1} = g'a_0g'^{-1}$.

Claim 1: $\langle a_0^{G_{A_0}} \rangle$ is free with basis $T_{A_0} = \{ta_0t^{-1} : t \in \ker \rho_L \cap G_{A_0}\}$.

Notice that $\langle T_{A_0} \rangle \leqslant \langle a_0^{G_{A_0}} \rangle$. So it is enough to show that $\langle a_0^{G_{A_0}} \rangle \leqslant \langle T_{A_0} \rangle$ to prove Claim 1. In order to show it, pick $g \in G_{A_0}$. We need to show that $ga_0g^{-1} \in \langle T_{A_0} \rangle$. Write g as $g = g_1a_0^{m_1}g_2a_0^{m_2} \ldots g_na_0^{m_n}$ where $n \geq 0$, $g_i \in (G_D \cap G_{A_0}) \backslash \{1\}$ for $i = 1, 2, \ldots, n$, $m_i \in \mathbb{Z} \backslash \{0\}$ for $i = 1, 2, \ldots, n-1$ and $m_n \in \mathbb{Z}$. We can further write g_i as c_ih_i where $h_i \in \ker \rho_D$ and $c_i \in G_L$, that is

$$g = c_1h_1a_0^{m_1}c_2h_2a_0^{m_2} \ldots c_nh_na_0^{m_n},$$

which rewriting $c_1 \cdots c_ih_ic_i^{-1} \cdots c_1$ as h_i' and using that the c_i's commute with a_0, we get that

$$g = h_1'a_0^{m_1}h_2'a_0^{m_2} \ldots h_n'a_0^{m_n}c_1c_2 \cdots c_n,$$

notice that ga_0g^{-1} is equal to $g'a_0(g')^{-1}$ where

$$g' = h_1'a_0^{m_1}h_2'a_0^{m_2} \ldots h_n'.$$

We can write $g'a_0(g')^{-1}$ as a product of elements of $T_{A_0} = \{ta_0t^{-1} : t \in \ker \rho_L \cap G_{A_0}\}$. Indeed:

$$\begin{aligned} g'a_0(g')^{-1} = {} & (h_1'a_0(h_1')^{-1})^{m_1} \cdot (h_1'h_2'a_0(h_1'h_2')^{-1})^{m_2} \\ & \cdots (h_1' \cdots h_{n-1}'a_0(h_1' \cdots h_{n-1}')^{-1})^{m_{n-1}} \cdot \\ & \cdot h_1' \cdots h_n'a_0(h_1' \cdots h_n')^{-1} \cdot \\ & \cdot (h_1' \cdots h_{n-1}'a_0(h_1' \cdots h_{n-1}')^{-1})^{-m_{n-1}} \cdots (h_1'h_2'a_0(h_1'h_2')^{-1})^{-m_2} \\ & \cdot (h_1'a_0(h_1')^{-1})^{-m_1}. \end{aligned}$$

This completes the proof of Claim 1.

Claim 2: $\langle (fa_1^la_0a_1^{-l}f^{-1})^{fPf^{-1}} \rangle$ is free with basis $T_P = \{ta_0t^{-1} : t \in f'Pa_1^l \cap \ker \rho_L\}$ where f' is the unique element of $\ker \rho_L$ such that $f = f'\rho_L(f)$.

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The proof of the claim is very similar to the previous one. One has that $\langle T_P\rangle \leqslant \langle (fa_1^l a_0 a_1^{-l} f^{-1})^{fPf^{-1}}\rangle$ so it is enough to show that for any $g \in P$ the element

$$(fgf^{-1})(fa_1^l a_0 a_1^{-l} f^{-1})(fgf^{-1}) = fga_1^l a_0 a_1^{-l} g^{-1} f^{-1}$$

lies in $\langle T_P\rangle$. Recall that from Equation (5.1) that a generating set of P is

$$\{a_1^l a_0 a_1^{-l}\} \cup \{\sigma_b^{l(s,b)} b_0 \sigma_b^{-l(s,b)} : b \in A_0 \backslash \{a_0\}\}.$$

In a similar way as before, we can write g as

$$g = c_1 h_1 (a_1^l a_0 a_1^{-l})^{m_1} c_2 h_2 (a_1^l a_0 a_1^{-l})^{m_2} \dots c_n h_n (a_1^l a_0 a_1^{-1})^{m_n}$$

where $n \geq 0$, $c_i \in P \cap G_L$ and $h_i \in P \cap \ker \rho_L$ for $i = 1, \dots n$. Let $f_L = \rho_L(f)$. Rewriting $f_L c_1 \cdots c_i h_i c_i^{-1} \cdots c_1^{-1} f_L^{-1}$ as h_i' and using that the c_i's and f_L commute with a_0, a_1, we get that

$$fg = f' h_1' (a_1^l a_0 a_1^{-l})^{m_1} h_2' (a_1^l a_0 a_1^{-l})^{m_2} \dots h_n' (a_1^l a_0 a_1^{-l})^{m_n} f_L c_1 c_2 \cdots c_n.$$

Now notice that

$$fga_1^l a_0 a_1^{-l} g^{-1} f^{-1} = f' g' a_1^l a_0 (f' g' a_1^l)^{-1}$$

where

$$g' = h_1' (a_1^l a_0 a_1^{-l})^{m_1} h_2' (a_1^l a_0 a_1^{-l})^{m_2} \dots h_n'.$$

Now we can write $f' g' a_1^l a_0 (f' g' a_1^l)^{-1}$ as a product of elements of $T_P = \{t a_0 t^{-1} : t \in \ker \rho_L \cap (f' P a_1 l)\}$.

$$\begin{aligned} f' g' a_1^l a_0 (f' g' a_1^l)^{-1} = {} & (f' h_1' a_1^l a_0 (f' h_1' a_1^l)^{-1})^{m_1} \cdot ((f' h_1' h_2' a_1^l) a_0 (f' h_1' h_2' a_1^l)^{-1})^{m_2} \cdot \\ & \cdots ((f' h_1' \cdots h_{n-1}' a_1^l) a_0 (f' h_1' \cdots h_{n-1}' a_1')^{-1})^{m_{n-1}} \cdot \\ & \cdot (f' h_1' \cdots h_n' a_1^l) a_0 (f' h_1' \cdots h_n' a_1^l)^{-1} \cdot \\ & \cdot ((f' h_1' \cdots h_{n-1}' a_1^l) a_0 (f' h_1' \cdots h_{n-1}' a_1^l)^{-1})^{-m_{n-1}} \cdot \\ & \cdots ((f' h_1' h_2' a_1^l) a_0 (f' h_1' h_2' a_1^l)^{-1})^{-m_2} \cdot ((f' h_1' a_1^l) a_0 (f' h_1' a_1^l)^{-1})^{-m_1}. \end{aligned}$$

This completes the proof of Claim 2.

Now, if $\rho_{a_1}(fa_1^l) \neq 0$, then $T_{A_0} \cap T_P = \emptyset$ and both are subsets of a free basis of $\ker \rho_D$. Therefore, $\langle T_{A_0}\rangle \cap \langle T_P\rangle = \{1\}$.

Case 3.2: $\rho_{a_1}(fa_1^l) = 0$. Note that $G_{A_0} \leqslant \ker \rho_{a_1}$ and $fPf^{-1} \leqslant \ker \rho_{a_1}$. As every $z \in \mathrm{Link}(a_1)$ commutes with a_1, we are in case (b) of § 4.1 and $\ker \rho_{a_1}$ is isomorphic to G_Λ where Λ is an even, FC-type, Artin graph (possibly infinite). Recall that

$$V_\Lambda = \{b_{i,0} : b_i \in \mathrm{Link}_\Delta(a_1)\} \cup \{z_{i,j} : z_i \in V_\Delta \backslash \mathrm{Star}_\Delta(a_1), j \in \mathbb{Z}\}$$

and there is an edge $\{v_{i,j}, u_{s,t}\}$ in Λ if and only if there is and edge $\{v_i, u_s\}$ in Δ and the label of both edges is the same.

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Let $A_{0,0} = \{b_{0,0} : b_0 \in A_0\}$ be the vertices of Λ of level 0 and type A_0. Note that $G_{A_{0,0}} \leqslant G_\Lambda$ is the subgroup G_{A_0} of G_Δ and the subgroup G_A of G_Γ.

As $\rho_{a_1}(fa_1^l) = 0$, we have that $\rho_{a_1}(f) \neq 0$ (recall that $l \neq 0$). Write f as $f'a_1^\alpha$, with $\alpha = \rho_{a_1}(f) \in \mathbb{Z}$. Consider the canonical retraction $\rho_{A_{0,0}} : G_\Lambda \to G_{A_{0,0}}$. Now, we have that fPf^{-1} is equal to $f'P'(f')^{-1}$ where P' is generated by

$$\{a_{0,0}\} \cup \{\tau_c^{l(s,c)} a_1^\alpha c_0 a_1^{-\alpha} \tau_c^{-l(s,c)} : c \in A \backslash \{a\}\}$$

where $\tau_c = a_1^\alpha \sigma_c a_1^{-\alpha}$ is some element of $\langle\{v_{i,j} : v_{i,j} \text{ of type } c_i \in V_\Delta\}\rangle$. Moreover, using Equation (4.1) in the setting of ρ_{a_1} we have that

$$a_1^\alpha c_0 a_1^{-\alpha} = \beta_c^{l(\alpha,c)} c_{0,i(\alpha,c)} \beta_c^{-l(\alpha,c)}$$

for some word $\beta_c \in \langle\{v \in V_\Lambda : \text{v of type } c_0\}\rangle$ and some $i(\alpha, c) \in \mathbb{Z}$. Observe that

$$\rho_{A_{0,0}}(\tau_c^{l(s,b)} a_1^\alpha c_0 a_1^{-\alpha} \tau_c^{-l(s,b)}) = \begin{cases} c_{0,0} & \text{if } i_{\alpha,c} = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Recall that we are assuming $A \not\subseteq \text{Link}(a)$ and therefore, there exists some $b \in A$ such that b is not linked to a. Thus, b_0 is not linked to a_i, $i = 0, 1, \ldots, k_a - 1$ in Δ. As b_0 is not linked to a_1, we have that $a_1^\alpha b_0 a_1^{-\alpha} = b_{0,\alpha}$. And we get that $\rho_{A_{0,0}}(P') \leqslant G_{A_{0,0} \backslash \{b_{0,0}\}}$. In particular $G_{A_{0,0}} \cap f'P'(f')^{-1} \leqslant \rho_{A_{0,0}}(f') G_{A_{0,0} \backslash \{b_{0,0}\}} \rho_{A_{0,0}}(f')^{-1}$. Note that $G_{A_{0,0} \backslash \{b_{0,0}\}} = G_{A_0 \backslash \{b_0\}}$. So, there is $d \in G_\Delta$ such that $\rho_{A_{0,0}}(f') G_{A_{0,0} \backslash \{b_{0,0}\}} \rho_{A_{0,0}}(f')^{-1} = dG_{A_0 \backslash \{b_0\}} d^{-1}$, and thus, $G_{A_0} \cap fPf^{-1}$ is contained in $dG_{A_0 \backslash \{b_0\}} d^{-1}$, a parabolic over a proper subset of A_0. This completes the proof in this case. □

Proof of Theorem 1.1. Let $\mathcal{C}$ be the class of finite, even, FC-type Artin graphs. Then $\mathcal{C}$ is closed under taking subgraphs and satisfies (3.1) by Theorem 5.2. The theorem now follows from Proposition 3.8. □

Corollary 5.3. *Let $\Gamma = (V, E, m)$ be an even, finite Artin graph of FC-type. Then any arbitrary intersection of parabolic subgroups in G_Γ is a parabolic subgroup.*

Proof. Let $\mathcal{P}$ be the set of parabolic subgroups in G. Note that as Γ is finite, $\mathcal{P}$ is countable. For an arbitrary indexing set I, we want to show that:

$$Q = \bigcap_{i \in I, P_i \in \mathcal{P}} P_i$$

is a parabolic subgroup. If I is finite, the claim follows from Theorem 1.1 and induction. So, we can assume that the indexing set I is countable, and we can index its elements by

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natural numbers. Write:

$$\bigcap_{i\in I} P_i = \bigcap_{n\in\mathbb{N}} \left(\bigcap_{i\leq n} P_i \right),$$

and set $Q_n = \bigcap_{i\leq n} P_i$. We know that Q_n is a parabolic subgroup for any n. Moreover, we have a chain of parabolic subgroups:

$$Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \cdots$$

where the intersection of all members Q_i of the chain above is equal to Q. We cannot have an infinite chain of nested distinct parabolic subgroups. Indeed, using Lemma 3.2, we have that $gG_Ag^{-1} \subsetneq hG_Ah^{-1}$ implies $A \subsetneq B$. Hence there are at most $|V|+1$ distinct parabolic subgroups in the chain above.

Ultimately, Q is an intersection of at most $|V|+1$ parabolic subgroups and hence it is a parabolic subgroup. □

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