KOSTANT'S PROBLEM FOR FULLY COMMUTATIVE PERMUTATIONS

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ABSTRACT. We give a complete combinatorial answer to Kostant's problem for simple highest weight modules indexed by fully commutative permutations. We also propose a reformulation of Kostant's problem in the context of fiab bicategories and classify annihilators of simple objects in the principal birepresentations of such bicategories generalizing the Barbasch–Vogan theorem for Lie algebras.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Kostant's problem, as defined and popularized by Joseph in [Jo80], is a famous open problem in representation theory of Lie algebras. It asks for which simple modules L over a semi-simple complex finite dimensional Lie algebra the universal enveloping algebra surjects onto the space of adjointly finite linear endomorphisms of L. Although several (both positive and negative) results are known, see Subsection 3.6 for a historical overview, the general case is wide open even for simple highest weight modules in the principal block of BGG category $\mathcal{O}$.

Knowing the answer to Kostant's problem for a particular simple module is important for potential applications, for example, understanding the structure of induced modules, see [KhM04, MS08], using the approach of establishing equivalences of categories for Lie algebra modules proposed in [MiSo97].

As described in Subsection 3.6, the existing results on Kostant's problem can be divided into four classes:

- there are some classes of simple modules for which the answer is known to be positive;

- there are some classes of simple modules for which the answer is known to be negative;

- there are some results that relate the answer for one simple module to the answer for another simple module;

- there are some complete classification results for Lie algebras of small rank.

To the best of our knowledge, none of the existing results provides a complete solution to Kostant's problem for some natural general class of simple modules which contains both simple modules with positive and negative answers. The main result of the present paper is the first result of this kind.

We look at a natural family of permutations (elements of the symmetric group S_n) which are called *fully commutative*. The latter means that any two reduced expressions for such an element can be transformed one into another just by using the commutativity relations for simple reflections. The number of fully commutative elements in S_n is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Under the Robinson–Schensted correspondence, fully commutative elements are exactly those permutations which correspond to two-row partitions. From the point of view of the Kazhdan–Lusztig combinatorics in type A, it is well known that fully commutative elements form a union of two-sided Kazhdan–Lusztig

cells and that, for general n, the value of Lusztig's $\mathbf{a}$-function on fully commutative elements in S_n can be arbitrarily large. This demonstrates that the class of fully commutative permutations is quite large. Fully commutative permutations naturally index a basis in the Temperley-Lieb algebra, which is a certain quotient of the Hecke algebra of S_n.

The main result of the present paper, Theorem 5, gives a complete combinatorial classification of the fully commutative permutations such that Kostant's problem for the corresponding simple modules in the principal block of BGG category $\mathcal{O}$ has positive answer (we call such elements *Kostant positive*). The answer is both quite beautiful and rather non-trivial. We first define certain special fully commutative involutions. These are permutations of the following form:
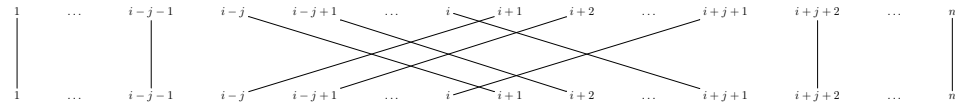
For a special involution, we call the area where this involution acts non-trivially its *support*. In Theorem 5, we show that a fully commutative involution w is Kostant positive if and only if it is a product of distant special involutions, in the sense that the supports of any two special factors in w are separated in the diagram of w by at least one vertical line.

Our approach to Kostant's problem for fully commutative elements is crucially based on the availability of the following three things:

- the explicit description of the Serre subcategory of category $\mathcal{O}$ generated by the simple objects indexed by fully commutative elements and the action of projective functors on this category, due to Brundan and Stroppel, see [BS10];

- Kåhrström's conjectural combinatorial reformulation of Kostant's problem, see [KMM20, Conjecture 1.2];

- the 2-representation theoretic approach to Kåhrström's conjecture in relation to Kostant's problem as developed by Ko, Mrđen and the second author in [KMM20, Theorem B].

The idea behind the proof is to use Kåhrström's combinatorial reformulation of Kostant's problem to reduce it to the question of solvability of certain equations in the Temperley-Lieb algebra. Then we establish, in the appropriate cases, the solvability of these equations by explicitly giving the solutions. In the remaining cases, we establish the impossibility of solving these equations by a rather technical case-by-cases analysis.

Our answer to Kostant's problem for fully commutative elements is explicit enough to compute its asymptotic behavior. More precisely, in Theorem 12, we prove the following:

- asymptotically, the answer to Kostant's problem for fully commutative elements is almost surely negative;

- asymptotically, the answer to Kostant's problem for fully commutative involutions is almost surely negative;

- asymptotically, the answer to Kostant's problem for fully commutative elements (or involutions) of a fixed $\mathbf{a}$-value is almost surely positive.

From its very definition, Kostant's problem is intimately connected with the problem of understanding the annihilators of simple modules over semi-simple Lie algebras. These

annihilators are classified combining two classical theorems of Duflo and Barbasch–Vogan. In the last section of the paper, Section 7, we provide an analogue of the Barbasch–Vogan Theorem for a class of bicategories known as fiab bicategories. We also propose a reformulation of Kostant's problem in this more general context.

2. Category $\mathcal{O}$ preliminaries

2.1. **Setup.** We work over $\mathbb{C}$.

Let $\mathfrak{g}$ denote a semi-simple finite dimensional Lie algebra with a fixed triangular decomposition

$$\text{(1)} \qquad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Here $\mathfrak{h}$ is a Cartan subalgebra. We denote by W the corresponding Weyl group, by S the set of all simple reflections in W associated to the triangular decomposition (1) and by w_0 the longest element of W.

2.2. **Category $\mathcal{O}$.** Consider the BGG category $\mathcal{O}$ associated to the triangular decomposition (1), see [BGG76, Hu08]. Explicitly, this is the full subcategory of the category of all $\mathfrak{g}$-modules whose objects are finitely generated, diagonalizable with respect to the action of $\mathfrak{h}$ and locally finite with respect to the action of $\mathfrak{n}^+$.

2.3. **Principal block.** Let $\mathcal{O}_0$ denote the principal block of $\mathcal{O}$, that is the direct summand containing the trivial $\mathfrak{g}$-module. Up to isomorphism, the simple objects in $\mathcal{O}_0$ are the simple highest weight modules $L_w := L(w \cdot 0)$, where $0 \in \mathfrak{h}^*$ is the zero weight and $w \in W$. Here $\cdot$ is the usual dot-action of W on $\mathfrak{h}^*$ defined, for $\lambda \in \mathfrak{h}^*$, via $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is half of the sum of all positive roots.

For $w \in W$, we denote by Δ_w the Verma cover and by P_w the indecomposable projective cover of L_w. Let A be the opposite of the endomorphism algebra of the direct sum of the P_w, taken over all $w \in W$. Then A is a finite dimensional associative algebra and the category A-mod of finite dimensional A-modules is equivalent to $\mathcal{O}_0$.

2.4. **Projective functors.** For $w \in W$, we denote by θ_w the unique, up to isomorphism, indecomposable projective endofunctor of $\mathcal{O}_0$, in the sense of [BG80], determined by the property $\theta_w P_e \cong P_w$.

We denote by $\mathscr{P}$ the monoidal category of all projective functors acting on $\mathcal{O}_0$. Thanks to Soergel's combinatorial description, $\mathscr{P}$ is equivalent to the monoidal category of Soergel bimodules over the coinvariant algebra associated to W.

2.5. **Graded lift.** The algebra A is Koszul, in particular, it has the corresponding Koszul $\mathbb{Z}$-grading, see [So90, St90]. We denote by $\mathcal{O}_0^{\mathbb{Z}}$ the category of all finite dimensional $\mathbb{Z}$-graded A-modules.

All structural modules in $\mathcal{O}_0$ admit graded lifts. For indecomposable modules, these graded lifts are unique, up to isomorphism and grading shift. For indecomposable structural modules, there is a preferred graded lift, called standard lift.

Abusing notation, we will denote the ungraded objects in $\mathcal{O}_0$ by the same symbols as their standard graded lifts. We use a similar convention for the graded version of projective functors, see [St90], and denote by $\mathscr{P}^{\mathbb{Z}}$ the monoidal category of all graded projective functors acting on $\mathcal{O}_0^{\mathbb{Z}}$.

2.6. **Hecke algebra combinatorics.** Let $\mathbf{H} = \mathbf{H}(W, S)$ be the Hecke algebra of W over $\mathbb{Z}[v, v^{-1}]$ (in the normalization of [So07]). The algebra $\mathbf{H}$ has the standard basis $\{H_w : w \in W\}$ and the Kazhdan–Lusztig basis $\{\underline{H}_w : w \in W\}$, see [KL79].

The Grothendieck group $\mathbf{Gr}[\mathcal{O}_0^{\mathbb{Z}}]$ is isomorphic to $\mathbf{H}$ by sending $[\Delta_w]$ to H_w. This isomorphism sends $[P_w]$ to $\underline{H}_w$, see [BB81, BK81].

The split Grothendieck group $\mathbf{Gr}_\oplus[\mathscr{P}^{\mathbb{Z}}]$ is isomorphic to $\mathbf{H}$ by sending $[\theta_w]$ to $\underline{H}_w$. The action of $\mathscr{P}^{\mathbb{Z}}$ on $\mathcal{O}_0^{\mathbb{Z}}$ decategorifies to the right regular $\mathbf{H}$-module, see [So92].

We denote by $\leq_L$, $\leq_R$ and $\leq_J$ the Kazhdan–Lusztig left, right and two-sided pre-orders on W, respectively. The associated equivalence classes are called left, right and two-sided cells. Each left and right cell contains a distinguished involution, also called the Duflo involution. In type A, all involutions are Duflo involutions.

2.7. **Indecomposability conjecture.** The following conjecture is proposed in [KiM16, Conjecture 2].

Conjecture 1. *If $\mathfrak{g} \cong \mathfrak{sl}_n$, then, for any $x, y \in W$, the module $\theta_x L_y$ is either indecomposable or zero.*

We will denote by $\mathbf{KM}(x, y) \in \{\texttt{true}, \texttt{false}\}$ the value of the claim "the module $\theta_x L_y$ is either indecomposable or zero". We also denote by $\mathbf{KM}(*, y)$ the conjunction of the $\mathbf{KM}(x, y)$ over all $x \in W$.

3. Kostant's problem

3.1. **Annihilators of simple modules.** Let L be a simple $\mathfrak{g}$-module. By Dixmier–Schur's Lemma, L has a central character, say $\chi : Z(\mathfrak{g}) \to \mathbb{C}$, where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Assume that L is such that χ coincides with the central character of the trivial $\mathfrak{g}$-module.

Then Duflo's Theorem [Du77] asserts that there exists $w \in W$ such that the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(L)$ of L in $U(\mathfrak{g})$ coincides with $\mathrm{Ann}_{U(\mathfrak{g})}(L_w)$.

Further, a result of Barbasch and Vogan [BV82, BV83] says that, given $x, y \in W$, we have $\mathrm{Ann}_{U(\mathfrak{g})}(L_x) \subset \mathrm{Ann}_{U(\mathfrak{g})}(L_y)$ if and only if $x \geq_L y$.

3.2. **Harish-Chandra bimodules.** Recall, see [Ja83, Kapitel 6], that a $\mathfrak{g}$-$\mathfrak{g}$-bimodule is called a Harish-Chandra bimodule if it is finitely generated and, additionally, if the adjoint action of $\mathfrak{g}$ on it is locally finite and has finite multiplicities.

A typical example of a Harish-Chandra bimodule is the bimodule $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M)$, for $M \in \mathcal{O}_0$.

Here is another example: let M, N be objects in $\mathcal{O}_0$, then the space $\mathcal{L}(M, N)$ of linear maps from M to N on which the adjoint action of $\mathfrak{g}$ is locally finite is a Harish-Chandra bimodule. If $M = N$, then $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \subset \mathcal{L}(M, M)$.

3.3. **Classical Kostant's problem.** Kostant's problem, as advertised in [Jo80], is formulated as follows:

Kostant's Problem. For which $w \in W$ is the embedding

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(L_w) \hookrightarrow \mathcal{L}(L_w, L_w)$$

an isomorphism?

We will denote by $\mathbf{K}(w) = \mathbf{K}_{\mathfrak{g}}(w) \in \{\texttt{true}, \texttt{false}\}$ the logical value of the claim "the embedding $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(L_w) \hookrightarrow \mathcal{L}(L_w, L_w)$ is an isomorphism".

3.4. **Kåhrström's conjecture.** The following conjecture is due to Johan Kåhrström, see [KMM20, Conjecture 1.2].

Conjecture 2. *Assume that we are in type A_{n-1}, so $W \cong S_n$, the symmetric group. Let $d \in S_n$ be an involution. Then the following assertions are equivalent:*

(1) $\mathbf{K}(d) = \texttt{true}$.

(2) For any $x, y \in W$ such that $x \neq y$ and $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, we have $\theta_x L_d \not\cong \theta_y L_d$.

(3) For any $x, y \in W$ such that $x \neq y$ and $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, we have $[\theta_x L_d] \neq [\theta_y L_d]$ in $\mathbf{Gr}[\mathcal{O}_0^{\mathbb{Z}}]$.

(4) For any $x, y \in W$ such that $x \neq y$ and $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$, we have $[\theta_x L_d] \neq [\theta_y L_d]$ in $\mathbf{Gr}[\mathcal{O}_0]$.

3.5. **Kostant's problem vs Kåhrström's conjecture and the indecomposability conjecture.** One of the main results of [KMM20] is [KMM20, Theorem B], which asserts that Conjecture 2(1) is equivalent to the conjunction of Conjecture 2(2) with $\mathbf{KM}(*, d)$.

3.6. **Known results on Kostant's problem.** Here is a (possibly incomplete) list of known results on Kostant's problem, in particular, a list of the special cases in which the answer to Kostant's problem is known.

- In type A, the value $\mathbf{K}(w)$ is constant on Kazhdan–Lusztig left cells, see [MS08, Theorem 61].

- Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$. Let $W^{\mathfrak{p}}$ be the corresponding parabolic subgroup of W and $w_0^{\mathfrak{p}}$ be the longest element in $W^{\mathfrak{p}}$. Then $\mathbf{K}(w_0^{\mathfrak{p}} w_0) = \texttt{true}$, see [GJ81, Theorem 4.4] and [Ja83, Section 7.32].

- Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and $s \in W^{\mathfrak{p}}$ a simple reflection. Then $\mathbf{K}(s w_0^{\mathfrak{p}} w_0) = \texttt{true}$, see [Ma05, Theorem 1].

- Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and $w \in W^{\mathfrak{p}}$. Let $\mathfrak{a}$ be the semi-simple part of the Levi quotient of $\mathfrak{p}$. Then we have $\mathbf{K}_{\mathfrak{a}}(w) = \texttt{true}$ if and only if $\mathbf{K}_{\mathfrak{g}}(w w_0^{\mathfrak{p}} w_0) = \texttt{true}$, see [Ka10, Theorem 1.1].

- [KaM10, Theorem 1] gives a module-theoretic characterization of the statement $\mathbf{K}(w) = \texttt{true}$.

- A complete answer to Kostant's problem for $\mathfrak{sl}_n$, where $n = 2, 3, 4, 5$, is given in [KaM10, Section 4]. There one can also find a partial answer for $\mathfrak{sl}_6$. Some further cases for $\mathfrak{sl}_6$ are dealt with in [Ka10, Section 6]. A complete answer to Kostant's problem for $\mathfrak{sl}_6$ is given in [KMM20, Subsection 10.1].

4. Fully commutative permutations and the Temperley-Lieb algebra

4.1. **Fully commutative permutations.** Consider the symmetric group S_n as a Coxeter group in the usual way, that is, fixing the elementary transpositions $s_i := (i, i+1)$, for $i = 1, 2, \ldots, n-1$, as the set of simple reflections. The classical presentation of S_n with respect to these generators has the following relations:

$$\begin{align}
(2) \qquad\qquad s_i^2 &= e; \\
(3) \qquad\qquad s_i s_j &= s_j s_i, \text{ if } |i - j| \neq 1; \\
(4) \qquad\qquad s_i s_{i\pm 1} s_i &= s_{i\pm 1} s_i s_{i\pm 1}.
\end{align}$$

The classical result of Matsumoto [Mat64] asserts that, for $w \in W$, any reduced expression of w can be transformed into any other reduced expression of w by using only the relations in (3) and (4). An element $w \in S_n$ is said to be *fully commutative* provided that any reduced expression of w can be transformed into any other reduced expression of w by only using relation (3).

The classical Robinson–Schensted correspondence, see [Sc61] and [Sa01, Section 3.1],

$$\mathbf{RS} : S_n \to \coprod_{\lambda \vdash n} \mathbf{SYT}_\lambda \times \mathbf{SYT}_\lambda,$$

assigns to an element of S_n a pair of standard Young tableaux of the same shape (that shape is a partition of n). An element $w \in S_n$ is fully commutative provided that the shape of the tableaux in $\mathbf{RS}(w)$ is a partition with at most two parts. Alternatively, an element w is fully commutative provided that it avoids the $(3, 2, 1)$-pattern, that is, the longest decreasing subsequence for w has length at most two. We refer to [Fa95, Fa96] for details.

4.2. **Temperley-Lieb algebra.** Let $\mathbb{A}$ be a commutative ring and $\delta \in \mathbb{A}$. For $n \geq 1$, the corresponding Temperley-Lieb algebra $\mathbf{TL}_n(\mathbb{A}, \delta)$ is an $\mathbb{A}$-algebra which is free as an $\mathbb{A}$-module with a basis consisting of all planar (i.e. non-crossing) pairings of $2n$ points in a plane. Composition is defined via concatenation of diagrams, straightening the outcome to a new diagram and, finally, multiplication with δ^k, where k is the number of closed loops removed during the straightening procedure. We refer to [Mar91, Chapter 6] for details. Here is an example:
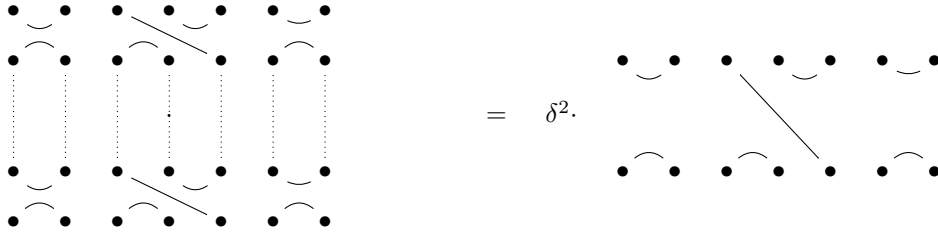
For $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$ and $\delta = v + v^{-1}$, one can give the following alternative description of $\mathbf{TL}_n(\mathbb{A}, \delta)$. It is the quotient of the Hecke algebra $\mathbf{H}_n$ for S_n modulo the ideal generated by all $\underline{H}_w$, where w is not fully commutative. We note that the latter ideal is just the $\mathbb{Z}[v, v^{-1}]$-span of all such $\underline{H}_w$. By construction, $\mathbf{TL}_n(\mathbb{Z}[v, v^{-1}], v + v^{-1})$ has a basis given by all $\underline{H}_w$, for w fully commutative. Setting $\mathsf{e}_i := \underline{H}_{s_i}$, for $i = 1, 2, \ldots, n-1$, we have the following presentation for $\mathbf{TL}_n(\mathbb{Z}[v, v^{-1}], v + v^{-1})$:

$$\begin{align}
(5) \qquad\qquad \mathsf{e}_i^2 &= (v + v^{-1})\mathsf{e}_i; \\
(6) \qquad\qquad \mathsf{e}_i \mathsf{e}_j &= \mathsf{e}_j \mathsf{e}_i, \text{ if } |i - j| \neq 1; \\
(7) \qquad\qquad \mathsf{e}_i \mathsf{e}_{i\pm 1} \mathsf{e}_i &= \mathsf{e}_i.
\end{align}$$

Here e_i corresponds to the diagram

where the horizontally paired points are i and $i+1$.

For a fully commutative element $w \in W$, we denote by e_w the image of $\underline{H}_w$ in $\mathbf{TL}_n(\mathbb{Z}[v, v^{-1}], v + v^{-1})$. Taking any reduced expression $w = s_1 s_2 \ldots s_k$, we have $\mathsf{e}_w = \mathsf{e}_{s_1} \mathsf{e}_{s_2} \ldots \mathsf{e}_{s_k}$, see [Fa95].

The number of cups (or caps) in e_w coincides with the value of Lusztig's $\mathbf{a}$-function on w. Also, the cups and caps determine the corresponding Kazhdan–Lusztig right and left pre-orders, respectively. For example, $x \leq_L y$ if and only each cap of e_x is a cap of e_y. For details, see [FG97, Section 3].

Remark 3. Let $x, y, z \in S_n$ be fully commutative elements such that $\mathsf{e}_x \mathsf{e}_y = f(v)\mathsf{e}_z$, for some $f(v) \in \mathbb{Z}[v, v^{-1}]$. Then the number of cups (resp. caps) in e_z is greater than or equal to the maximum of the numbers of cups (resp. caps) in e_x and e_y.

Remark 4. Let $x \in S_n$ be a fully commutative element. Then $\mathsf{e}_{x^{-1}}$ is obtained from e_x by reflection in a horizontal line.

4.3. **Results of Brundan and Stroppel.** Given two arbitrary fully commutative elements $x, y \in S_n$, the corresponding module $\theta_x L_y$ in the category $\mathcal{O}$ for $\mathfrak{sl}_n$ is either indecomposable or zero. If y belongs to a Kazhdan–Lusztig right cell containing an element of the form $w_0^{\mathfrak{p}} w_0$, for some parabolic $\mathfrak{p}$, this follows by combining [BS10, Theorem 4.11] and [BS11, Theorem 1.1]. Indeed, [BS11, Theorem 1.1] asserts, among other things, that, in type A, the principal block of the parabolic category $\mathcal{O}$ corresponding to a maximal parabolic subalgebra is equivalent to the module category of a certain diagrammatic algebra studied in [BS10]. For the latter algebra, the claim of [BS10, Theorem 4.11(iii)] translates exactly into the statement that $\theta_x L_y$ is either zero or has simple top. Using [MS08, Proposition 35], we can remove this assumption on the right cell of y. In particular, this implies that $\mathbf{KM}(*, w) = \mathtt{true}$ for any fully commutative $w \in S_n$.
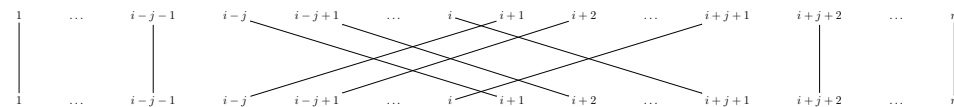
Note that, by [Ge06, Theorem 5.1], the two-sided order on S_n is given by the dominance order on partitions. Consequently, if $x \in S_n$ is not fully commutative while $y \in S_n$ is, then $\theta_x L_y = 0$ by [MM11, Lemma 12]. Combined with [KMM20, Theorem B], we obtain equivalence of assertions (1) and (2) in Conjecture 2 in the case of fully commutative elements $x, y \in S_n$.

5. Main result

5.1. **Special fully commutative involutions.** For $i \in \{1, 2, \ldots, n-1\}$, set $\sigma_{i,0} := s_i$. Also, for $j \in \{1, \ldots, \min(i-1, n-1-i)\}$, consider the following element of S_n:

$$\sigma_{i,j} = s_i(s_{i-1}s_{i+1})(s_{i-2}s_i s_{i+2})\cdots(s_{i-j}s_{i-j+2}\cdots s_{i+j})\cdots(s_{i-2}s_i s_{i+2})(s_{i-1}s_{i+1})s_i.$$

As a permutation, the element $\sigma_{i,j}$ has the following diagram:

It is readily seen that this element is $(3, 2, 1)$-avoiding and hence fully commutative (cf. [BJS93, Theorem 2.1]). The corresponding Temperley-Lieb diagram is:
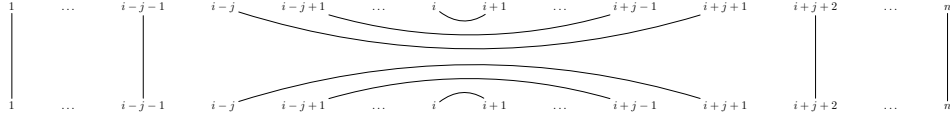
We will call the set $\{i - j, i - j + 1, \ldots, i + j + 1\}$ the *support* of $\sigma_{i,j}$ and the set $\{i-j-1, i-j, i-j+1, \ldots, i+j+1, i+j+2\}$ the *extended support* of $\sigma_{i,j}$. Elements of the form $\sigma_{i,j}$ will be called *special*.

We will say that $\sigma_{i,j}$ and $\sigma_{i',j'}$ are *distant* provided that their extended supports intersect in at most one element. For example, $s_1 = \sigma_{1,0}$ and $s_4 = \sigma_{4,0}$ are distant since $\{0,1,2,3\} \cap \{3,4,5,6\} = \{3\}$, while $s_1 = \sigma_{1,0}$ and $s_3 = \sigma_{3,0}$ are not distant since $\{0,1,2,3\} \cap \{2,3,4,5\} = \{2,3\}$ is neither empty nor a singleton. We note that, in the latter example, the supports of $\sigma_{1,0}$ and $\sigma_{3,0}$ do not have common elements.

5.2. **Formulation of the main result.**

Theorem 5. *Conjecture 2 is true for all fully commutative involutions in S_n. Moreover, if $d \in S_n$ is a fully commutative involution, then $\mathbf{K}(d) = \mathtt{true}$ if and only if d is a product of pairwise distant special elements.*

In other words, for a fully commutative involution d, we have $\mathbf{K}(d) = \mathtt{true}$ if and only if any two non-nested cups (or caps) in the Temperley-Lieb diagram of d are separated by at least one vertical line. We will call such fully commutative involutions *Kostant involutions*.

Corollary 6. *If $w \in S_n$ is a fully commutative element, then $\mathbf{K}(w) = \mathtt{true}$ if and only if any two caps in e_w are either nested or separated by at least one propagating line.*

Proof. This follows from Theorem 5, the first bullet in Subsection 3.6, the paragraph before Remark 3, the last paragraph of Section 2.6, as well as the fact that Lusztig's a-function is constant on left cells.   □

The two directions of Theorem 5 will be proved in Subsections 5.4 and 5.5.

5.3. **Auxiliary lemmata.**

Lemma 7. *Let x and y be two fully commutative elements in S_n. Then $\theta_x L_y \neq 0$ is equivalent to the condition that each cup of $\mathrm{e}_{x^{-1}}$ is a cup of e_y.*

Proof. By [MM11, Lemma 12], the condition $\theta_x L_y \neq 0$ is equivalent to the condition $x^{-1} \leq_L y$. Note that $(\theta_x)^* \cong \theta_{x^{-1}}$ and that the diagram for x^{-1} is the flip of the diagram for x. As recalled in the text above Remark 3, the condition $x^{-1} \leq_L y$ holds if and only if the cups of $e_{x^{-1}}$ form a subset of those of e_y. By Remark 3, this implies the result.   □

Lemma 8. *Let x, y, z be three fully commutative elements in S_n such that*

$$\mathrm{e}_x \mathrm{e}_y = f(v)\mathrm{e}_z, \qquad \text{for some } f(v) \in \mathbb{Z}[v, v^{-1}].$$

Then $\theta_x \theta_y$ is isomorphic to $\theta_z^{f(1)} \oplus \theta$, where $\theta L_w = 0$ for any fully commutative element $w \in S_n$.

Proof. This follows from the realization of the Temperley-Lieb algebra as a quotient of the Hecke algebra and the action of the latter on $\mathcal{O}_0$ via projective functors. Recall from Section 2.6 that, for $w \in \{x, y, z\}$, the Grothendieck class $[\theta_w] \in \mathbf{Gr}_\oplus[\mathscr{P}^{\mathbb{Z}}]$ corresponds to the Kazhdan-Lusztig basis element $\underline{H}_w \in \mathbf{H}$, which, in turn, is mapped to the Temperley-Lieb diagram $e_w \in \mathbf{TL}_n(\mathbb{Z}[v, v^{-1}], v + v^{-1})$. Therefore, the combinatorics of the θ_w corresponds precisely to the combinatorics of the e_w (after putting $v = 1$). All this is, of course, up to higher order Kazhdan-Lusztig basis elements. These latter elements correspond to the additional summand θ in the formulation and are killed in the Temperley-Lieb quotient. In particular, we also have $\theta L_w = 0$, for all fully commutative w. $\qquad \square$

5.4. **Positive answer.** Let d be a fully commutative involution which is a product of pairwise distant special elements. We are going to prove that d has the property described in Conjecture 2(4).

Recall that, for fully commutative elements, assertions (1) and (2) in Conjecture 2 are equivalent. We also have the obvious implications (4)$\Rightarrow$(3)$\Rightarrow$(2). Hence $\mathbf{K}(d) = \mathtt{true}$ for any fully commutative involution d that is a product of pairwise distant special elements.

Let $x, y \in S_n$ be two different elements such that $\theta_x L_d \neq 0$ and $\theta_y L_d \neq 0$. In particular, this implies that x and y are fully commutative, see Section 4.3. To prove that d has the property described in Conjecture 2(4), it is enough to find some $u, v \in S_n$ such that $\dim \mathrm{Hom}_{\mathfrak{g}}(\theta_u P_v, \theta_x L_d) \neq \dim \mathrm{Hom}_{\mathfrak{g}}(\theta_u P_v, \theta_y L_d)$. By adjunction, we have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(\theta_u P_v, \theta_x L_d) &\cong \mathrm{Hom}_{\mathfrak{g}}(\theta_{x^{-1}} \theta_u P_v, L_d), \\ \mathrm{Hom}_{\mathfrak{g}}(\theta_u P_v, \theta_y L_d) &\cong \mathrm{Hom}_{\mathfrak{g}}(\theta_{y^{-1}} \theta_u P_v, L_d). \end{aligned}$$

Note that, for a projective module P, the dimension of $\mathrm{Hom}_{\mathfrak{g}}(P, L_d)$ equals the multiplicity of P_d as a direct summand of P. Therefore, we need to find u and v such that the multiplicity of θ_d as a summand of $\theta_{x^{-1}} \theta_u \theta_v$ is different from the multiplicity of θ_d as a summand of $\theta_{y^{-1}} \theta_u \theta_v$.

Since θ_d is self-adjoint and both u and v are arbitrary, we can reformulate this as follows: find u and v such that the multiplicity of θ_d as a summand of $\theta_v \theta_u \theta_x$ differs from the multiplicity of θ_d as a summand of $\theta_v \theta_u \theta_y$.

For future use, we record the following technical lemma.

Lemma 9. *In the notation from above, the multiplicity of θ_d as a summand of $\theta_d \theta_{x^{-1}} \theta_x$ equals 2^{2a}.*

Proof. The diagram $\mathsf{e}_{x^{-1}} \mathsf{e}_x$ is self-dual (i.e. symmetric with respect to reflection in a horizontal line), has a cups and a caps, and also a circles in the middle before straightening. By Lemma 7, each cup of this diagram corresponds to a cap of e_d. Therefore, the number of circles removed when straightening the product $\mathsf{e}_d \mathsf{e}_{x^{-1}} \mathsf{e}_x$ equals $2a$. This implies the claim. $\qquad \square$

Without loss of generality, we may assume that the number a of caps in e_x is greater than or equal to the number b of caps in e_y.

In most cases below, we will see that the choice $v = d$ and $u = x^{-1}$ does the job.

Case 1. Let us assume that $a > b$, so there is at least one cup of e_x which is not a cup of e_y. Set $u = x^{-1}$ and $v = d$.

By Lemma 9, the multiplicity of θ_d as a summand of $\theta_d \theta_{x^{-1}} \theta_x$ equals 2^{2a}.

If the underlying diagram of $e_d e_{x^{-1}} e_y$ is not e_d, then, by Lemma 8, the multiplicity of θ_d as a summand of $\theta_d \theta_{x^{-1}} \theta_y$ equals 0 and we are done. If the underlying diagram of $e_d e_{x^{-1}} e_y$ is e_d, we need to compute the number of closed loops removed when straightening the product $e_d e_{x^{-1}} e_y$. Each closed loop contains at least one cap. The total number of original caps in $e_d e_{x^{-1}} e_y$ before straightening is $a + b + c$, where c is the number of caps in e_d. As $c \geq a > b$ by Lemma 7, the resulting diagram has at least c caps, see Remark 3. Therefore, the number of closed loops removed during the straightening procedure is at most $a + b < 2a$. Consequently, the multiplicity of θ_d as a summand of $\theta_d \theta_{x^{-1}} \theta_y$ is at most $2^{a+b} < 2^{2a}$. This completes Case 1.

Case 2. Let us assume that $a = b$ and e_x and e_y have exactly the same cups. Set $u = x^{-1}$ and $v = d$.

Since x and y are different by assumption, there should be at least one cap of e_x which is not a cap of e_y. This implies that the underlying diagrams e_q of $e_{x^{-1}} e_x$ and e_p of $e_{x^{-1}} e_y$ are different.

Now we claim that the underlying diagram for $e_d e_{x^{-1}} e_x$ is e_d, while the underlying diagram for $e_d e_{x^{-1}} e_y$ is different from e_d. This, of course, will complete the present case. The underlying diagram of $e_d e_{x^{-1}} e_x$ being e_d follows directly from the fact that each cap of e_x is also a cap of e_d, see Lemma 8. Here is an example:
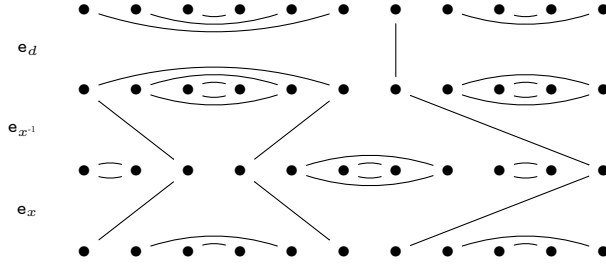
Let us now look at the underlying diagram for $e_d e_{x^{-1}} e_y$. Recall that:

- e_p and e_q have exactly the same cups;

- not all caps in e_p and e_q are the same;

- any cap of e_p and any cap of e_q is also a cap of e_d;

- $d = d_1 d_2 \cdots d_r$ is a product of pairwise distant special elements d_i.

This implies that there exists a special factor d_i of d such that the number of caps e_{d_i} shares with e_q is different from the number of caps e_{d_i} shares with e_p.

Take the leftmost such factor. Due to this assumption, to the left of this factor in e_d, the diagrams $e_d e_{x^{-1}} e_x$ and $e_d e_{x^{-1}} e_y$ fully agree. We have two subcases.

Subcase 2a. The number of caps e_{d_i} shares with e_q is smaller than the number of caps e_{d_i} shares with e_p.

In this case, the additional caps of e_{d_i}, when multiplied with e_p, are moved to the right. Here is a fairly generic example:
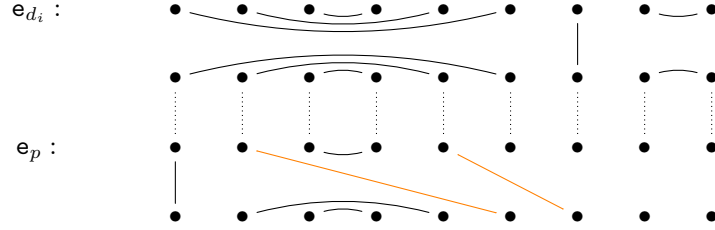
This makes the resulting diagram different from e_d, as claimed.

Subcase 2b. The number of caps e_{d_i} shares with e_q is greater than the number of caps e_{d_i} shares with e_p.

In this case, we have some additional points corresponding to propagating lines to the left of the nested caps in e_p. The rightmost of these points either hits a cap in e_d or it hits a propagating line in e_d. In the former case, we obtain a cap in $e_d e_p$ which is not a cap in e_d. Here is a fairly generic example:
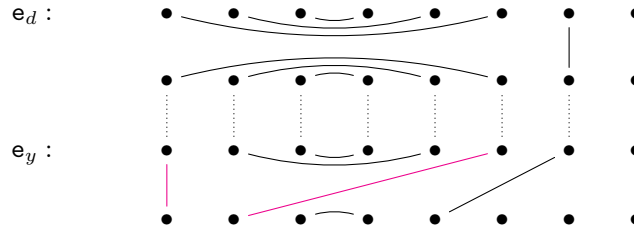
This makes the resulting diagram different from e_d.

In the latter case, the resulting diagram has a non-vertical propagating line. Here is a fairly generic example:
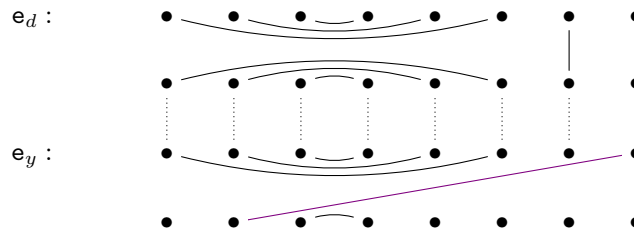
This makes the resulting diagram again different from e_d and completes Case 2.

Case 3. Let us assume that $a = b$ and e_x and e_y have exactly the same caps. Set $u = x^{-1}$ and $v = d$ as before.

Consider first the situation when the underlying diagrams of $e_{x^{-1}}e_x$ and $e_{x^{-1}}e_y$ coincide, let us call this diagram e_z. Note that e_z has a caps. As e_x and e_y have the same caps but are assumed to be different, not all of their cups can coincide. In particular, the multiplicity of e_z in $e_{x^{-1}}e_y$ is strictly smaller than the multiplicity 2^a of e_z in $e_{x^{-1}}e_x$. Consequently, if we now multiply with e_d on the left, we obtain e_d as the underlying diagram in both $e_d e_{x^{-1}}e_x$ and $e_d e_{x^{-1}}e_y$, but with different multiplicities, and we are done with this situation.

Now consider the situation when the underlying diagrams e_q of $e_{x^{-1}}e_x$ and e_p of $e_{x^{-1}}e_y$ are different. Let us assume, for a contradiction, that

$$e_d e_{x^{-1}} e_x = e_d e_{x^{-1}} e_y.$$

Note that $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_x = 2^{2a}\mathsf{e}_d$ by Lemma 9. The total number of original caps in $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$ is $2a + r$, where r is the number of caps in e_d. The underlying diagram of $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$ is e_d, which accounts for r caps. The only way to get 2^{2a} as the multiplicity of e_d in $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$ is to have a bijection between the set of original caps in $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$ and the union of the set of all caps in e_d with the set of all closed loops removed during the straightening procedure. In particular, each such closed loop consists of exactly one cup and one cap.

By the previous paragraph, each propagating line in e_d must hit a propagating line in e_p. Since all propagating lines in e_d are vertical, the same has to be true for the corresponding propagating lines in e_p by Lemma 7. In other words, the set of propagating lines in e_d is a subset of the set of propagating lines in e_p.

Take now a cap C in e_d which is not nested inside any other cap in e_d. Due to our assumptions on d, the immediate outside neighbors of C are propagating lines. Therefore the endpoints of C correspond to either a cup or two propagating lines in e_p. In the former case, all caps nested inside C correspond to cups of e_p. In the latter case, both of these propagating lines have to be vertical, for otherwise there would exist some extra caps in e_p which are not caps of e_d.

We proceed by induction on the number $\mathbf{k}$ of nested caps contained inside C which do not correspond to any cups in e_p, to show that e_p and e_q coincide in the corresponding regions which interact with C and all inner points of C during the multiplications $\mathsf{e}_d\mathsf{e}_p$ and $\mathsf{e}_d\mathsf{e}_q$. This implies $\mathsf{e}_p = \mathsf{e}_q$, which contradicts our assumption. If $\mathbf{k} = 0$, the above argument shows that both endpoints of C hit vertical propagating lines in e_p. Since e_p and e_q have the same caps and all propagating lines of e_q are vertical by construction, these two propagating lines of e_p are also propagating lines of e_q, and we are done. If $\mathbf{k} > 0$, there is a unique outermost cap C' nested inside C by our assumption that d is a product of pairwise distant special elements. The argument that we just applied to C applies to C'. Proceeding inductively we obtain that e_p and e_q coincide at all parts that hit the endpoints and all inner points of C. This completes Case 3.

Case 4. Let us assume that $a = b$ and that some of the caps and some of the cups in e_x and e_y are different. Set $u = x^{-1}$ and $v = d$ as before. In most situations, it is possible to adapt the argument we used in Case 3.

If $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_x \neq \mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$, then we are done. So, let us assume $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_x = \mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$. Note that $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_x = 2^{2a}\mathsf{e}_d$ by Lemma 9.

As before, if the underlying diagrams of $\mathsf{e}_{x^{-1}}\mathsf{e}_x$ and $\mathsf{e}_{x^{-1}}\mathsf{e}_y$ are the same, the multiplicity of e_d in $\mathsf{e}_d\mathsf{e}_{x^{-1}}\mathsf{e}_y$ is strictly smaller than 2^{2a}. Therefore, it remains to consider the situation when the underlying diagram e_q of $\mathsf{e}_{x^{-1}}\mathsf{e}_x$ is different from the underlying diagram e_p of and $\mathsf{e}_{x^{-1}}\mathsf{e}_y$. If e_q and e_p have the same caps, we can use the argument from Case 3. In particular, we may assume that not all caps in e_q and e_p agree. In this case we will need to construct a modification $\mathsf{e}_{d'}$ of e_d which will take the place of e_v.

The same argument as in Case 3 shows that the set of propagating lines of e_d is a subset of both the set of propagating lines of e_p and the set of propagating lines of e_q. Furthermore, the sets of caps of e_p and e_q are (different) subsets of the set of caps of e_d.

Let us consider some full collection $\mathcal{F}$ of nested caps in e_d. Let α and β be the numbers of caps of e_p, respectively, e_q contained in this collection.

Note that e_q has a caps while e_p has at least a caps. Therefore, since not all caps in e_p and e_q agree, we can assume that $\alpha > \beta$ for the chosen $\mathcal{F}$. The argument from Case 3

implies that each cap in $\mathcal{F}$ either hits a cup or two vertical lines in e_p. This means that the parts of e_p and e_q corresponding to $\mathcal{F}$ are as in the following example:

Consider the element d' such that $\mathsf{e}_{d'}$ is the same as e_d except $\mathcal{F}$ is adjusted as follows:
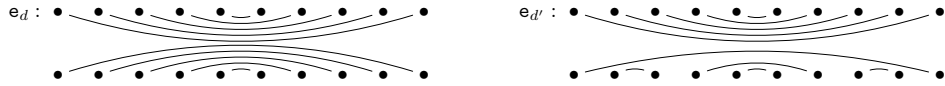
Then $\mathsf{e}_{d'}\mathsf{e}_p$ is a multiple of e_d while $\mathsf{e}_{d'}\mathsf{e}_q$ is not. This completes the proof for the positive answer.

5.5. **Negative answer.** Let d be a fully commutative involution which is not a product of pairwise distant special elements. We are going to prove that d does not have the property described in Conjecture 2(2). More explicitly, we will find two different fully commutative elements x and y such that $\theta_x L_d$ is isomorphic to $\theta_y L_d$.

Recall that, for fully commutative elements, assertions (1) and (2) in Conjecture 2 are equivalent. We also have the obvious implications $\neg(2) \Rightarrow \neg(3) \Rightarrow \neg(4)$ of the other assertions. The above therefore implies $\mathbf{K}(d) = \texttt{false}$ and, moreover, the rest of Conjecture 2 for the involution d.

If d is not a product of pairwise distant special elements, the diagram e_d has two adjacent non-nested caps. Let us fix a pair A and B of such adjacent non-nested caps (with A on the left). We may assume that they are not nested in some other cap which itself has an adjacent non-nested cap. Let A' and B' be the corresponding cups.

Define the element e_x by changing e_d as follows:

- remove B and B';
- if applicable, remove all caps in which A and B are nested;
- remove all cups corresponding to the latter caps;
- replace all the removed cups and caps by propagating lines.

The latter process is unique due to the non-intersection condition.

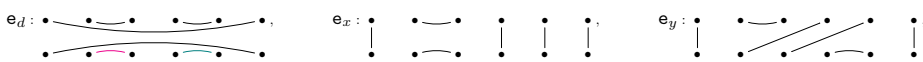Define the element e_y by changing e_d as follows:

- remove A and B';
- if applicable, remove all caps in which A and B are nested;
- remove all cups corresponding to the latter caps;
- replace all the removed cups and caps by propagating lines.

Again, the latter process is unique due to the non-intersection condition. Here is an example, with A and B colored:

Let c denote the number of caps in e_d and a the number of caps in e_x. Then $a \leq c-1$, by construction, and also a equals the number of caps in e_y. We are going to prove that $\theta_x L_d$ and $\theta_y L_d$ are isomorphic as graded modules. For this we need some combinatorial

preparation for estimates of graded shifts. The following statements can probably be deduced from the results of [BS10], but it is easier to prove them directly.

Lemma 10. *Let u and w be two fully commutative permutations such that $\theta_u L_w \neq 0$. Let k be the minimum of the numbers of caps in e_u and in e_w. Then $\theta_u L_w$ is a graded self-dual module and, for $|i| > k$, the graded component $(\theta_u L_w)_i$ is zero.*

Proof. The module $\theta_u L_w$ is self-dual as it is the image of a self-dual module L_w under a projective functor. In order to prove the rest of the lemma, it is enough to argue that, for $i < -k$, the graded component $(\theta_u P_w)_i$ is zero. Since the algebra of $\mathcal{O}_0$ is positively graded, all standard graded lifts of projectives live in non-negative degrees. When computing the product $\mathsf{e}_u \mathsf{e}_w$, the number of closed loops removed in the straightening procedure is at most k. This gives the scalar $(v + v^{-1})^m$, where $m \leq k$. Therefore the maximal graded shift of a projective in $\theta_u P_w$ is bounded by k. The claim follows. □

Corollary 11. *Both modules $\theta_x L_d$ and $\theta_y L_d$ have simple tops, which live in degree $-a$.*

Proof. Both modules $\theta_x L_d$ and $\theta_y L_d$ have simple tops by [BS10, Theorem 4.11]. We prove the second claim for $\theta_x L_d$. For $\theta_y L_d$ the arguments are similar. Lemma 10 implies that the simple top of $\theta_x L_d$ lives in some degree $i \geq -a$.

Let us now look more closely at the proof of Lemma 10. Note that, when computing the product $\mathsf{e}_x \mathsf{e}_d$, we need to remove exactly a circles. This means that the degree $-a$ component of $\theta_x P_d$ is non-zero.

At the same time, any simple subquotient L_w in the radical of P_d lives in a strictly positive degree. Therefore, using Lemma 10 and an appropriate shift of grading in the positive direction, the module $\theta_x L_w$ lives in degrees that are strictly bigger than $-a$. Thus the degree $-a$ component of $\theta_x L_d$ is indeed non-zero. Due to the positivity of the grading, this component is the top of $\theta_x L_d$. □

We want to prove that two graded modules $\theta_x L_d$ and $\theta_y L_d$ which have simple tops that live in the same degree are isomorphic. Due to the positivity of the grading, it is enough to show that there is a non-zero degree zero morphism from $\theta_x L_d$ to $\theta_y L_d$. By adjunction, this is equivalent to the existence of a degree zero morphism from L_d to $\theta_{x^{-1}} \theta_y L_d$.

By construction, e_x and e_y have the same cups. Therefore, during the straightening of the product $\mathsf{e}_{x^{-1}} \mathsf{e}_y$, there are a loops to remove. Let e_p be the underlying diagram of $\mathsf{e}_{x^{-1}} \mathsf{e}_y$ and note that it has exactly the same caps as e_y. By an analogue of Corollary 11 for e_p (which works verbatim since e_p and e_y have the same caps), the module $\theta_p L_d$ is a self-dual indecomposable module with simple top in degree $-a$. Since θ_p appears exactly once with shift a in the decomposition of $\theta_{x^{-1}} \theta_y$, it follows that L_d appears, as a graded module, in the socle of $\theta_{x^{-1}} \theta_y L_d$. This completes the proof of the negative answer and the proof of Theorem 5.

5.6. **Sanity check: comparison to previously known results.** As already mentioned in Subsection 3.6, it is known that all elements of the form $w_0^{\mathfrak{p}} w_0$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}_n$, are Kostant positive.

If we take $\mathfrak{p}$ to be a maximal parabolic subalgebra, that is, one for which the semisimple part of the Levi factor equals $\mathfrak{sl}_i \oplus \mathfrak{sl}_{n-i}$, for $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, the element $w_0^{\mathfrak{p}} w_0$ turns

out to be fully commutative. The Temperley-Lieb diagram of the element $w_0^{\mathfrak{p}} w_0$ is as follows, where i is the number of caps:
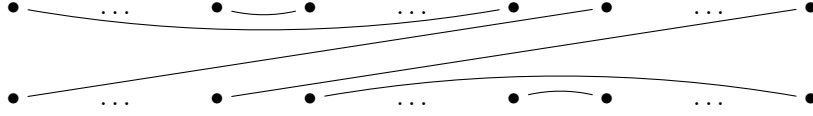
If $i = \frac{n}{2}$, this element is an involution, in fact, it is $\sigma_{a,b}$, where $a = \frac{n}{2}$ and $b = \frac{n}{2} - 1$. If $i < \frac{n}{2}$, the above element belongs to the left Kazhdan–Lusztig cell of $\sigma_{a,b}$, where $a = i$ and $b = i - 1$. Therefore, for such elements, our Theorem 5 agrees with the previous results.

Moreover, all known results in small ranks mentioned in Subsection 3.6 indeed agree with Theorem 5.

We also note that Theorem 5, combined with [Ka10, Theorem 1.1], gives a lot of new full answers to Kostant's problem even for not necessarily fully commutative permutations.

5.7. **Problems to extend outside fully commutative elements.** The fact that the answers to Kostant's problem for the elements $s_1 s_2 s_1$ and $s_2 s_3 s_2$ of S_5 are different, see [KaM10, Section 4], suggests that it will not be straightforward to extend Theorem 5 outside the set of fully commutative elements.

6. Asymptotic results

6.1. **Various sequences.** For $n \in \mathbb{Z}_{\geq 1}$, we denote

- by $\mathbf{ki}_n$ the number of fully commutative Kostant involutions in S_n;
- by $\mathbf{k}_n$ the number of fully commutative $w \in S_n$ for which $\mathbf{K}(w) = \mathtt{true}$;
- by $\mathbf{mi}_n$ the number of fully commutative involutions in S_n;
- by $\mathbf{m}_n$ the number of fully commutative elements in S_n.

For $a \in \mathbb{Z}_{\geq 1}$ and $\mathbf{x} \in \{\mathbf{ki}_n, \mathbf{k}_n, \mathbf{mi}_n, \mathbf{m}_n\}$, we denote by $\mathbf{x}^a$ the number of elements in the family $\mathbf{x}$ with exactly a caps.

It is very well-known that $\mathbf{m}_n$ equals the n-th Catalan number

$$C_n := \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}.$$

6.2. **Main asymptotic results.**

Theorem 12.

(a) *We have* $\displaystyle\lim_{n\to\infty} \frac{\mathbf{ki}_n}{\mathbf{mi}_n} = 0$.

(b) *We have* $\displaystyle\lim_{n\to\infty} \frac{\mathbf{k}_n}{\mathbf{m}_n} = 0$.

(c) *For any fixed* $a \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, *we have* $\displaystyle\lim_{n\to\infty} \frac{\mathbf{ki}_n^a}{\mathbf{mi}_n^a} = 1$.

The remainder of this section is devoted to the proof of this theorem. In Subsections 6.3, 6.4 and 6.5, we first establish some explicit formulae for the enumeration of the main protagonists defined in the previous subsection. We are sure that some of the combinatorial arguments and results presented in this section are not new and can be

found in or derived from the existing literature. However, we feel that is would be more difficult to find appropriate references than to prove these results. When working on the proofs, the Online Encyclopedia of Integer Sequences was really helpful.

6.3. **Fully commutative Kostant involutions.** Recall the family of Fibonacci polynomials $F_n(x)$, where $n \geq 0$, given by the following recursion:

$$F_0(x) = 1, \qquad F_1(x) = x, \qquad F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2.$$

Here are some initial members of this family:

$n:$	0	1	2	3	4	5
$F_n(x):$	1	x	$x^2 + 1$	$x^3 + 2x$	$x^4 + 3x^2 + 1$	$x^5 + 4x^3 + 3x$

The evaluation $F_n(1)$ is exactly the n-th Fibonacci number.

Proposition 13. *We have $F_n(x) = \displaystyle\sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \mathbf{ki}_n^a \cdot x^{n-2a}$.*

Proof. This is easy to check for $n = 1, 2$. Therefore, it suffices to show that the numbers $\mathbf{ki}_n^a$ satisfy the same recursion as the coefficients of the Fibonacci polynomials. Observe that $\mathbf{ki}_n^a = 0$ for $a > \lfloor \frac{n}{2} \rfloor$.

Let d be a Kostant involution and look at the strand in $\mathbf{e}_d$ which starts at the top point 1. If this strand is vertical, removing it yields a Kostant involution for $n - 1$ with the same number of caps as $\mathbf{e}_d$. If this strand is a cup, removing it together with the corresponding cap produces a Kostant involution for $n - 2$ with one fewer cap than $\mathbf{e}_d$. This defines a bijection between the set of all Kostant involutions for n with a caps and the union of the set of all Kostant involutions for $n - 1$ with a caps and the set of all Kostant involutions for $n - 2$ with $a - 1$ caps. This implies $\mathbf{ki}_n^a = \mathbf{ki}_{n-1}^a + \mathbf{ki}_{n-2}^{a-1}$, which establishes the necessary recursion. $\qquad\square$

Corollary 14. *The number $\mathbf{ki}_n$ is the n-th Fibonacci number.*

Proof. Evaluate the equality in Proposition 13 at 1. $\qquad\square$

Corollary 15. *We have $\mathbf{ki}_n^a = \binom{n-a}{a}$.*

Proof. Again, this is easy to verify for small values of n, so we only need to check that the binomial coefficients on the right hand side satisfy the same recursion as $\mathbf{ki}_n^a$. By the usual Pascal triangle formula, we have:

$$\binom{n-a}{a} = \binom{(n-1)-a}{a} + \binom{n-1-a}{a-1} = \binom{(n-1)-a}{a} + \binom{(n-2)-(a-1)}{a-1}.$$

This implies the claim. $\qquad\square$

6.4. **Fully commutative Kostant elements.** Under the Robinson-Schensted correspondence, two-sided Kazhdan–Lusztig cells in type A are in bijection with partitions of n, and the left cells in a given two-sided cell corresponding to $\lambda \vdash n$ are in bijection with the standard tableaux of shape λ. More specifically, the Robinson-Schensted correspondence gives a bijection between the elements w in a left cell and pairs $(P(w), Q(w))$ of standard Young tableaux of shape λ for which $Q(w)$ is fixed, by [KL79, Theorem 1.4] (for the present formulation of that result and a more elementary proof, see [A00, Theorem A]). In particular, two-sided cells of fully commutative permutations correspond to partitions with at most two rows and the value of the $\mathbf{a}$-function is given by the length of the second row. This follows from, for example, [MT22, Lemma 6.5].

Corollary 16. *We have* $\mathbf{k}_n^a = \binom{n-a}{a} \frac{n!(n-2a+1)!}{a!(n-2a)!(n-a+1)!}$.

Proof. As recalled above, the two-sided cell which corresponds to our a is indexed by the partition $(n-a, a)$. Since every Kazhdan–Lusztig left cell contains a unique involution, the number $\mathbf{k}_n^a$ is the product of $\mathbf{ki}_n^a$ with the size of this left cell. The latter equals the number of standard Young tableaux $P(w)$ of shape $(n-a, a)$ under the Robinson-Schensted correspondence. Now the claim of our corollary follows from Corollary 15 and the Hook Formula. $\square$

6.5. **Fully commutative involutions.**

Proposition 17. *We have* $\mathbf{mi}_n = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. This claim is easy to check for small values of n, so we need to show that both sides satisfy the same recursion.

Let d be a fully commutative involution. Consider the strand of $\mathbf{e}_d$ connected to the upper point 1. If it is vertical, removing it results in a fully commutative involution for $n-1$. If the strand is a cup X connecting 1 to some point $2i$, there is a crossingless pairing of $2i-2$ points inside this cup. Removing X and all cups contained inside it together with the corresponding caps, we obtain a fully commutative involution for $n-2i$. This implies that we have the following recursion for the left hand side of our formula:

$$\mathbf{mi}_n = \mathbf{mi}_{n-1} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} C_{i-1} \mathbf{mi}_{n-2i}.$$

We claim the the middle binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ satisfy the same recursion. Indeed, consider the Pascal triangle with the point (x, y) corresponding to $\binom{x}{y}$, for all appropriate x and y. Then $\binom{n}{i}$ is exactly the number of shortest paths between $(0,0)$ and (n, i) in this triangle.

Assume first that $n = 2k$. Then we need to prove that

$$\binom{2k}{k} = \binom{2k-1}{k-1} + \sum_{i=1}^{k} C_{i-1} \binom{2k-2i}{k-i}.$$

Equivalently, we need to show that

$$\binom{2k-1}{k} = \sum_{i=1}^{k} C_{i-1} \binom{2k-2i}{k-i}.$$

For $i = 1, 2, \ldots, k$, let $\mathcal{P}_i$ be the set of all shortest paths between $(0,0)$ and $(2k-1, k)$ such that the path goes through the point $(2(k-i), k-i)$ and i is minimal with this property. The set of all paths between $(0,0)$ and $(2k-1, k)$ is the disjoint union of the $\mathcal{P}_i$. There are $\binom{2k-2i}{k-i}$ shortest paths between $(2(k-i), k-i)$ and $(0,0)$. The classical interpretation of Catalan numbers as paths in a square that do not cross the diagonal implies that there are exactly C_{i-1} shortest paths between $(2k-1, k)$ and $(2(k-i), k-i)$ satisfying the condition for the minimality of i. This establishes the necessary recursion formula for the right hand side.

The case $n = 2k + 1$ is similar and left to the reader. $\square$

Darij Grinberg informed us that the formula in Proposition 17 can be found in [SS84, Proposition 3] with a different proof.

6.6. **Proof of Theorem 12**(c). As remarked at the beginning of Section 6.4, the partition corresponding to a is $(n-a, a)$, hence $\mathbf{mi}_n^{\bullet a} = \frac{n!(n-2a+1)!}{a!(n-2a)!(n-a+1)!}$ by the Hook Formula. Using Corollary 15, we have

$$\frac{\mathbf{ki}_n^a}{\mathbf{mi}_n^a} = \frac{(n-a)!a!(n-2a)!(n-a+1)!}{a!(n-2a)!n!(n-2a+1)!} = \frac{(n-a+1)(n-a)\dots(n-2a+2)}{n(n-1)\dots(n-a+1)}.$$

Here both the numerator and the denominator are polynomials in n of degree a and with leading coefficient 1. The claim of Theorem 12(c) follows.

6.7. **Proof of Theorem 12**(a). By Corollary 14, the number $\mathbf{ki}_n$ is the n-th Fibonacci number. It is given by the formula

$$\frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}.$$

Since the absolute value of $\frac{1-\sqrt{5}}{2}$ is less than 1, it follows that F_n grows as $(\frac{1+\sqrt{5}}{2})^n$.

At the same time, if $n = 2k$, the central binomial coefficient $\binom{2k}{k}$ is not smaller than $\frac{4^k}{2k+1}$, as follows directly from

$$4^k = 2^{2k} = (1+1)^{2k} = \sum_{i=0}^{2k} \binom{2k}{i}.$$

If $n = 2k + 1$, the coefficient $\binom{2k+1}{k}$ is not smaller than $\binom{2k}{k}$. This implies that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ grows at least as fast as $\frac{2^n}{n}$. Since $\frac{1+\sqrt{5}}{2} < 2$, the claim of Theorem 12(a) follows.

6.8. **Proof of Theorem 12**(b). Using Corollary 16, we need to show that

$$(8) \qquad \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)\binom{n-a}{a}}{\binom{2n}{n}} \cdot \frac{n!(n-2a+2)!}{a!(n-2a)!(n-a+1)!} \to 0, \quad n \to \infty.$$

We rewrite the expression in (8) as

$$\sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)(n-a)!n!(n-2a+1)!n!n!}{a!(n-2a)!a!(n-2a)!(n-a+1)!(2n)!}$$

and then, further, as

$$\sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)(n-a)!(n-a)!n!(n-2a+1)!n!n!a!}{a!(n-2a)!a!(n-2a)!(n-a+1)!(2n)!a!(n-a)!}$$

and, finally, as

$$(9) \qquad \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)\binom{n-a}{a}\binom{n-a}{a}\binom{n}{a}}{\binom{n-a+1}{a}\binom{2n}{n}}.$$

Note that $\binom{n-a}{a} \leq \binom{n-a+1}{a}$ and hence the expression in (9) is bounded from above by

$$(10) \qquad \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)\binom{n-a}{a}\binom{n}{a}}{\binom{2n}{n}}.$$

The Fibonacci coefficient $\binom{n-a}{a}$ is bounded by the n-the Fibonacci number and hence grows at most as $(\frac{1+\sqrt{5}}{2})^n$. The coefficient $\binom{n}{a}$ is bounded by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ and hence grows as 2^n up to some factor of at most polynomial growth. At the same time, $\binom{2n}{n}$ grows as 4^n up to some factor of at most polynomial growth. As the number of summands

is linear in n, it follows that the whole expression (10) tends to 0, when $n \to \infty$. This proves Theorem 12(b).

6.9. **Conjectures.** Taking Theorem 12 into account, we conjecture the following for general elements of S_n:

- Among all involutions in S_n, the proportion of those for which the answer to Kostant's problem is positive is asymptotically 0.

- The proportion of elements in S_n for which the answer to Kostant's problem is positive is asymptotically 0.

- Fix a partition λ of some m and consider, for $n > m$, the partition $\lambda^{(n)}$ of n obtained from λ by increasing the first part by $n - m$. Then, among the elements in S_n belonging to the two-sided cell indexed by the partition $\lambda^{(n)}$, the proportion of those for which the answer to Kostant's problem is positive is asymptotically 1.

We note that the recent results in [MSr23, Subsection 4.10] support, in some mild sense, these conjectures.

7. KOSTANT'S PROBLEM AND BARBASCH–VOGAN THEOREM FOR FIAB BICATEGORIES

7.1. **Fiat** 2-**categories.** Let $\mathscr{C}$ be a fiab bicategory in the sense of [M-Z21], i.e. a finitary bicategory with weak involution $\star$ and adjunction morphisms. Consider the left, right and two-sided pre-orders $\leq_L, \leq_R$ and $\leq_J$ on the set $\mathcal{S}(\mathscr{C})$ of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$. In particular, we have $\mathrm{F} \leq_L \mathrm{G}$ provided that there exists a 1-morphism H such that G is isomorphic to a direct summand of HF. The other pre-orders are defined similarly, see [MM14, Section 3] for details.

The associated equivalence classes are called cells (left, right and two-sided, respectively). Each left and each right cell contains a unique special 1-morphism called the Duflo 1-morphism, see [MM11, Subsection 4.5].

7.2. **Kostant's problem for fiat bicategories.** Fix a left cell $\mathcal{L}$ in $\mathscr{C}$. Then there is an object $\mathtt{i} \in \mathscr{C}$ such that all elements of $\mathcal{L}$ have $\mathtt{i}$ as a domain. Consider the abelianization $\overline{\mathbf{P}}_\mathtt{i}$ of the principal birepresentation $\mathbf{P}_\mathtt{i}$ of $\mathscr{C}$ in the sense of [M-T19, Section 3].

Denote by $\hat{\mathcal{L}}$ the set of all $\mathrm{F} \in \mathcal{S}(\mathscr{C})$ such that $\mathrm{F} \leq_L \mathcal{L}$. The collection of Serre subcategories of the $\mathbf{P}_\mathtt{i}(\mathtt{j})$, for $\mathtt{j} \in \mathscr{C}$, generated by all simple objects L_F, where $\mathrm{F} \in \hat{\mathcal{L}}$, is invariant under the action of $\mathscr{C}$. We denote the corresponding (abelian) birepresentation of $\mathscr{C}$ by $\overline{\mathbf{M}}_\mathcal{L}$ and its (finitary) restriction to projective objects by $\mathbf{M}_\mathcal{L}$.

On the other hand, let $\mathrm{D}_\mathcal{L}$ be the Duflo 1-morphism in $\mathcal{L}$ and consider the finitary birepresentation $\mathbf{K}_\mathcal{L}$ given by the action of $\mathscr{C}$ on the additive closure of $\mathscr{C} L_{\mathrm{D}_\mathcal{L}}$. The Yoneda morphism from $\mathbf{P}_\mathtt{i}$ to $\mathbf{K}_\mathcal{L}$ sending $\mathbb{1}_\mathtt{i}$ to $L_{\mathrm{D}_\mathcal{L}}$ factors through $\mathbf{M}_\mathcal{L}$ by construction. We will say that $\mathrm{D}_\mathcal{L}$ is *Kostant positive* provided that the induced morphism of birepresentations from $\mathbf{M}_\mathcal{L}$ to $\mathbf{K}_\mathcal{L}$ is an equivalence.

Let us now look what happens in the case of the bicategory $\mathscr{P}$. The identity object of $\mathscr{P}$ is given by the quotient of the universal enveloping algebra modulo the annihilator of $\mathcal{O}_0$. By [BG80, Theorem 5.9], $\mathcal{O}_0$ itself is equivalent to the subcategory of the principal birepresentation of $\mathscr{P}$ generated by the quotient of the universal enveloping algebra by the central character of $\mathcal{O}_0$. Hence, we can view a simple module $L_w \in \mathcal{O}_0$ as a simple object of the principal birepresentation of $\mathscr{P}$. Take now $w = d$ to be the Duflo involution in some Kazhdan-Lusztig left cell $\mathcal{L}$. Then the corresponding

birepresentation $\mathbf{M}_{\mathcal{L}}$ as defined above is generated, as a birepresentation of $\mathscr{P}$, by the quotient of the universal enveloping algebra by the annihilator of L_d. At the same time, in [KMM20, Proposition 7.2] it is shown that the corresponding birepresentation $\mathbf{K}_{\mathcal{L}}$ as defined above is generated by $\mathcal{L}(L_p, L_p)$. Therefore, the fact that that the natural embedding of $\mathbf{M}_{\mathcal{L}}$ into $\mathbf{K}_{\mathcal{L}}$ is an equivalence is, indeed, equivalent to the fact that the answer to Kostant's problem for L_d is positive, see [KMM20, Corollary 7.6].

Note that the kernel of the above Yoneda morphism from $\mathbf{P}_i$ to $\mathbf{K}_{\mathcal{L}}$ is exactly the annihilator of $L_{\mathrm{D}_{\mathcal{L}}}$ in $\mathscr{C}$, i.e. the left biideal of 2-morphisms α in $\mathscr{C}$ such that $\mathbf{P}_i(\alpha)_{L_{\mathrm{D}_{\mathcal{L}}}} = 0$. This connects our reformulation of Kostant's problem in this more general setup to the problem of studying annihilators of simple objects in principal birepresentations. This naturally leads to an analogue of the classical Barbasch–Vogan theorem for fiab bicategories, presented in the next subsection.

7.3. **Barbasch–Vogan theorem for $\mathscr{C}$.** Note that the abelianization $\overline{\mathbf{P}}_i$ of $\mathbf{P}_i$ is an abelian birepresentation of $\mathscr{C}$. For $j \in \mathscr{C}$ and an indecomposable 1-morphism $\mathrm{F} \in \mathscr{C}(i, j)$, denote by $\mathscr{J}_{\mathrm{F}}$ the annihilator of L_{F} in $\mathscr{C}$. Then $\mathscr{J}_{\mathrm{F}}$ is a left biideal of $\mathscr{C}$.

Theorem 18. *For two indecomposable 1-morphism F and G in $\mathscr{C}$, the following conditions are equivalent:*

(a) $\mathrm{F} \leq_R \mathrm{G}$.

(b) $\mathscr{J}_{\mathrm{G}} \subseteq \mathscr{J}_{\mathrm{F}}$

Proof. In this proof, we will use the word "module" instead of "birepresentation". The direct sum of all principal $\mathscr{C}$-modules has the obvious structure of a $\mathscr{C}$-$\mathscr{C}$-bimodule, the regular $\mathscr{C}$-$\mathscr{C}$-bimodule.

Given an object M of this regular $\mathscr{C}$-$\mathscr{C}$-bimodule and any 1-morphism $\mathrm{F} \in \mathscr{C}$ which acts on M on the right, we have

$$\mathrm{Ann}_{\mathscr{C}}(M) \subset \mathrm{Ann}_{\mathscr{C}}(M\mathrm{F})$$

of left annihilators. Also, if some L_{G} is a subquotient of M, we have

$$\mathrm{Ann}_{\mathscr{C}}(M) \subset \mathrm{Ann}_{\mathscr{C}}(L_{\mathrm{G}}).$$

Let F and G be two 1-morphisms in $\mathscr{C}$. Then $\mathrm{F} \leq_R \mathrm{G}$ if and only if there exists H in $\mathscr{C}$ such that G is a summand of FH. By adjunction, this is equivalent to

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}}(\mathrm{FH}, L_{\mathrm{G}}) \cong \mathrm{Hom}_{\overline{\mathscr{C}}}(\mathrm{F}, L_{\mathrm{G}}\mathrm{H}^{\star}).$$

In other words, $\mathrm{F} \leq_R \mathrm{G}$ is equivalent to the existence of H such that L_{F} is a subquotient of $L_{\mathrm{G}}\mathrm{H}^{\star}$. The previous paragraph now yields that $\mathrm{F} \leq_R \mathrm{G}$ implies $\mathscr{J}_{\mathrm{G}} \subset \mathscr{J}_{\mathrm{F}}$.

For the converse, note that if $\mathrm{F} \not\leq_R \mathrm{G}$, then either $\mathrm{F} >_R \mathrm{G}$, or F and G are not comparable in the right order.

If $\mathrm{G} <_R \mathrm{F}$, then all elements in the two-sided cell of F annihilate L_{G} by [MM11, Lemma 12]. By the same lemma, there are elements in the two-sided cell of F that do not annihilate L_{F}. Therefore $\mathscr{J}_{\mathrm{F}} \subsetneq \mathscr{J}_{\mathrm{G}}$.

If F and G are not comparable in the right order, then the two-sided cells of F and G annihilate L_{G} and L_{F}, respectively, again by [MM11, Lemma 12]. On the other hand, the two sided cell of F does not annihilate L_{F}, and similarly that of G does not annihilate L_{G}. Thus the annihilators are not comparable.

This completes the proof. $\square$

We can also give slightly more detailed information.

Proposition 19. *In the setup of Theorem 18, if* $\mathrm{F} <_R \mathrm{G}$ *and* $\mathtt{i}$ *is the codomain for both* F *and* G, *then there is* $\mathrm{H} \in \mathscr{C}(\mathtt{i},\mathtt{i})$ *such that*

$$\dim \mathrm{Hom}_{\mathscr{C}/\mathscr{J}_{\mathrm{G}}}(\mathrm{H}, \mathbb{1}_{\mathtt{i}}) \neq \dim \mathrm{Hom}_{\mathscr{C}/\mathscr{J}_{\mathrm{F}}}(\mathrm{H}, \mathbb{1}_{\mathtt{i}}).$$

Proof. We take H to be the Duflo 1-morphism in the right cell of G. Then $\mathrm{H}L_{\mathrm{F}} = 0$ by [MM11, Lemma 12] and hence the evaluation of any element of $\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}, \mathbb{1}_{\mathtt{i}})$ at L_{F} is the zero morphism. At the same time, the evaluation of the morphism $\mathrm{H} \to \mathbb{1}_{\mathtt{i}}$ which defines H as a Duflo 1-morphism (see [MM11, Subsection 4.5]) at L_{G} is non-zero. The claim follows. $\square$

7.4. **Classical Barbasch–Vogan theorem.** The above proposition implies the following classical result due to Barbasch and Vogan, see [BV82, BV83].

Corollary 20. *Let* $\mathfrak{g}$ *be a semi-simple finite dimensional complex Lie algebra with Weyl group* W. *Then, for* $x, y \in W$, *we have* $\mathrm{Ann}_{U(\mathfrak{g})}(L_x) \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(L_y)$ *if and only if* $y \leq_L x$, *where* $\leq_L$ *is the Kazhdan–Lusztig left order on* W.

The appearance of the left order in Corollary 20, compared to the right order in Theorem 18, is due to the right nature of the action of the bicategory of projective functors on category $\mathcal{O}$.

Proof. Consider the bicategory $\mathscr{P}$ of projective functors acting on $\mathcal{O}_0$. The latter is, naturally, a subbirepresentation of the abelianized principal birepresentation $\overline{\mathbf{P}}$ and is equivalent to a certain category of Harish-Chandra bimodules for $\mathfrak{g}$ by [BG80, Theorem 5.9]. This equivalence matches the indecomposable projective object P_{θ_e} corresponding to the identity with the quotient of $U(\mathfrak{g})$ modulo the trivial central character.

Sending P_{θ_e} to the dominant Verma module in $\mathcal{O}_0$ using the Yoneda Lemma, defines a morphism of birepresentations from $\overline{\mathbf{P}}$ to $\mathcal{O}_0$ which sends simple objects to simple objects. By Theorem 18, $y \leq_L x$ implies $\mathrm{Ann}_{\mathscr{P}}(L_x) \subset \mathrm{Ann}_{\mathscr{P}}(L_y)$.

In particular,

$$\mathrm{Hom}_{\mathrm{Ann}_{\mathscr{P}}(L_x)}\Big(\bigoplus_{w \in W} \theta_w, \theta_e\Big) \subseteq \mathrm{Hom}_{\mathrm{Ann}_{\mathscr{P}}(L_y)}\Big(\bigoplus_{w \in W} \theta_w, \theta_e\Big) \subseteq \mathrm{Hom}_{\mathscr{P}}\Big(\bigoplus_{w \in W} \theta_w, \theta_e\Big),$$

where the latter corresponds to the quotient of $U(\mathfrak{g})$ modulo the ideal generated by the trivial central character under the equivalence from [BG80, Theorem 5.9]. Hence $\mathrm{Ann}_{U(\mathfrak{g})}(L_x) \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(L_y)$.

Since $\mathscr{P}$ has only one object, the fact that $y <_L x$ implies $\mathrm{Ann}_{U(\mathfrak{g})}(L_x) \subsetneq \mathrm{Ann}_{U(\mathfrak{g})}(L_y)$ follows directly from Proposition 19. As in the proof of Theorem 18, if x and y are not comparable in the left order, their annihilators are incomparable, which completes the proof. $\square$

REFERENCES

[A00] Ariki, S. Robinson-Schensted correspondence and left cells. Advanced Studies in Pure Mathematics 28, 2000, Combinatorial Methods in Representation Theory pp. 1–20.

[BV82] Barbasch, D.; Vogan, D. Primitive ideals and orbital integrals in complex classical groups. Math. Ann. **259** (1982), no. 2, 153–199.

[BV83] Barbasch, D.; Vogan, D. Primitive ideals and orbital integrals in complex exceptional groups. J. Algebra **80** (1983), no. 2, 350–382.

[BB81] Beilinson, A.; Bernstein, J. Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Ser. I Math. **292** (1981), no. 1, 15–18.

[BGG76] Bernstein, I.; Gelfand, I.; Gelfand, S. A certain category of 𝔤-modules. (Russian) Funkcional. Anal. i Prilozen. **10** (1976), no. 2, 1–8.

[BG80] Bernstein, J.; Gelfand, S. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245–285.

[BJS93] Billey, S.; Jockusch, W.; Stanley, R. Some combinatorial properties of Schubert polynomials. J. Algebraic Combin. **2** (1993), no. 4, 345–374.

[BS10] Brundan, J.; Stroppel, C. Highest weight categories arising from Khovanov's diagram algebra. II. Koszulity. Transform. Groups **15** (2010), no. 1, 1–45.

[BS11] Brundan, J.; Stroppel, C. Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$. Represent. Theory **15** (2011), 170–243.

[BK81] Brylinski, J.-L.; Kashiwara, M. Kazhdan–Lusztig conjecture and holonomic systems. Invent. Math. **64** (1981), no. 3, 387–410.

[Du77] Duflo, M. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. Ann. of Math. (2) **105** (1977), no. 1, 107–120.

[GJ81] Gabber, O.; Joseph, A. On the Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula. Compositio Math. **43** (1981), no. 1, 107–131.

[Ge06] Geck, M. Kazhdan–Lusztig cells and the Murphy basis. Proc. London Math. Soc. (3) **93** (2006), no. 3, 635–665.

[Hu08] Humphreys, J. Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. Graduate Studies in Mathematics, **94**. American Mathematical Society, Providence, RI, 2008. xvi+289 pp.

[Fa95] Fan, C. K. A Hecke Algebra Quotient and Properties of Commutative Elements of a Weyl Group, Ph.D. thesis, MIT, 1995.

[Fa96] Fan, C. K. A Hecke algebra quotient and some combinatorial applications. J. Algebraic Combin. **5** (1996), no. 3, 175–189.

[FG97] Fan, C. K.; Green, R. M. Monomials and Temperley-Lieb algebras. J. Algebra **190** (1997), 498–517.

[Ja83] Jantzen, J. Einhüllende Algebren halbeinfacher Lie-Algebren. (German) Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **3**. Springer-Verlag, Berlin, 1983. ii+298 pp.

[Jo80] Joseph, A. Kostant's problem, Goldie rank and the Gelfand-Kirillov conjecture. Invent. Math. **56** (1980), no. 3, 191–213.

[KL79] Kazhdan, D.; Lusztig, G. Representations of Coxeter groups and Hecke algebras. Invent. Math. **53** (1979), no. 2, 165–184.

[KhM04] Khomenko, O.; Mazorchuk, V. Structure of modules induced from simple modules with minimal annihilator. Canad. J. Math. **56** (2004), no. 2, 293–309.

[KiM16] Kildetoft, T.; Mazorchuk, V. Parabolic projective functors in type A. Adv. Math. **301** (2016), 785–803.

[Ka10] Kåhrström, J. Kostant's problem and parabolic subgroups. Glasg. Math. J. **52** (2010), no. 1, 19–32.

[KaM10] Kåhrström, J.; Mazorchuk, V. A new approach to Kostant's problem. Algebra Number Theory **4** (2010), no. 3, 231–254.

[KMM20] Ko, H.; Mazorchuk, V.; Mrđen, R. Some homological properties of category $\mathcal{O}$, V. Preprint arXiv:2007.00342, to appear in IMRN.

[M-T19] Mackaay, M.; Mazorchuk, V.; Miemietz, V.; Tubbenhauer, D.; Simple transitive 2-representations via (co-)algebra 1-morphisms. Indiana Univ. Math. J. **68** (2019), no. 1, 1–33.

[M-Z21] Mackaay, M.; Mazorchuk, V.; Miemietz, V.; Tubbenhauer, D.; Zhang, X. Finitary birepresentations of finitary bicategories. Forum Math. **33** (2021), no. 5, 1261–1320.

[Mar91] Martin, P. Potts models and related problems in statistical mechanics. Series on Advances in Statistical Mechanics, **5**. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. xiv+344 pp.

[Mat64] Matsumoto, H. Générateurs et relations des groupes de Weyl généralisés. C. R. Acad. Sci. Paris **258** (1964), 3419–3422.

[Ma05] Mazorchuk, V. A twisted approach to Kostant's problem. Glasg. Math. J. **47** (2005), no. 3, 549–561.

[MM11] Mazorchuk, V.; Miemietz, V. Cell 2-representations of finitary 2-categories. Compos. Math. **147** (2011), no. 5, 1519–1545.

[MM14] Mazorchuk, V.; Miemietz, V. Additive versus abelian 2-representations of fiat 2-categories. Mosc. Math. J. **14** (2014), no. 3, 595–615.

[MSr23] Mazorchuk, V.; Srivastava, S. Kostant's problem for parabolic Verma modules. Preprint arXiv:2301.07090.

[MS08] Mazorchuk, V.; Stroppel, C. Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219** (2008), no. 4, 1363–1426.

[MT22]   Mazorchuk, V.; Tenner, B. E. Intersecting principal Bruhat ideals and grades of simple modules. Comb. Theory **2** (2022), no. 1, Paper No. 14, 31 pp.

[MiSo97] Miličić, D.; Soergel, W. The composition series of modules induced from Whittaker modules. Comment. Math. Helv. **72** (1997), no. 4, 503–520.

[Sa01]   Sagan, B. The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, 2001. xvi+238 pp.

[Sc61]   Schensted, C. Longest increasing and decreasing subsequences. Canadian J. Math. **13** (1961), 179–191.

[SS84]   Simion, R.; Schmidt, F. Restricted permutations. Proceedings of the fifteenth Southeastern conference on combinatorics, graph theory and computing (Baton Rouge, La., 1984). Congr. Numer. **45** (1984), 323–325.

[So90]   Soergel, W. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.

[So92]   Soergel, W. The combinatorics of Harish-Chandra bimodules. J. Reine Angew. Math. **429** (1992), 49–74.

[So07]   Soergel, W. Kazhdan–Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. (German) J. Inst. Math. Jussieu **6** (2007), no. 3, 501–525.

[St90]   Stroppel, C. Category $\mathcal{O}$: gradings and translation functors. J. Algebra **268** (2003), no. 1, 301–326.

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