Problem solving in the digital era: Examples from the work of mathematics students on a divisibility problem

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This paper draws on the discussion around the impact of digital resources on problem-solving activities and presents findings from the analysis of eight undergraduate mathematics students' responses to a problem on divisibility that was part of their summative assessment through a portfolio of learning outcomes. The analysis indicates that the availability of digital resources impacts students' problem-solving activity: digital resources provide information useful for problem-solving; they provide answers to the problem; and, they facilitate hypothesis building or execution of time-consuming procedures. A digital resource might be used mainly for a procedural performance of a repetitive task (e.g., trial and error) or it may also include modelling (e.g. programming an algorithm underpinning a repetitive task).

Keywords: Students' practices, problem-solving, digital resources, exploration, divisibility.

PROBLEM-SOLVING AND RESOURCES

Problem-solving is an activity related to tasks students do not know how to approach in advance. What makes a task a 'problem' depends on the context in which this task is encountered, on the available tools and on solvers' precedent experiences (Bosch & Winsløw, 2015; Schoenfeld, 1992). A routine task for secondary students may become a problem for exploration for primary school students. Problem-solving appears in all human activities. Especially in mathematics, problems might be *pure mathematical* problems within a mathematical theory (e.g., proving a conjecture that will lead to a new theorem); *application* problems that are related to real-life situations (e.g., calculating the volume of a 3D shape); or, *modelling* problems that are application problems, in which a transformation of a real-life situation to the mathematical structure is required (e.g., modelling the spread of a virus) (Verschaffel et al. 2014).

Problem-solving activity has been seen through approaches that can guide and organise mathematical explorations – see, for example, the four problem-solving steps proposed by George Pólya (1945): *understand the problem*, *devise a plan*, *carry out the plan* and *look back on your work*. In mathematics education, problem-solving is seen also as a vehicle for students' learning. Teaching through problem-solving opens opportunities for mathematical learning as well as for appreciation of mathematics and its value (e.g., Schoenfeld, 1992; Liljedahl et al., 2016). Problem-solving activities have been also connected to mathematical intuition and affect (Liljedahl et al., 2016). Also, problem-solving activities trigger, or are followed by, discussions and reflection *about* problem-solving. Such discussions have been seen by researchers as: a metacognitive activity (Verschaffel et al. 2014); a *synthesis* of selection, organisation and connection of results obtained during problem-solving that will be useful for future problems (Bosch & Winsløw, 2015); or, as an opportunity for meta-level learning (Sfard, 2008). In the work I am discussing in this paper, I am interested in the last of these, as I see problem-solving activity as an opportunity for reflection on solvers' ways to engage with exploration, conjecturing and verification.

Of relevance to the discussion in this paper is the role of resources in problem-solving activities. Problem-solving is strongly related to the tools that are used and the environment in which the problem-solving takes place (Bosch & Winsløw, 2015). Amongst these tools we can consider analogue tools (e.g., rulers, textbooks, physical models etc.) as well as digital resources such as educational, or other general use software (e.g., Dynamic Geometry Software - DGS, online blogs, etc.) Students and teachers can access platforms with affordances to seek information, to participate in discussions, to ask questions or to experiment with ideas (Santos-Trigo, 2020). The use of digital resources has enhanced the range of mathematical investigations with quick and accurate calculations, reliable drawings, dynamic manipulations of objects and affordances for modelling. Also, digital resources offer more opportunities, in comparison to the analogue world, for conjecturing (abductive reasoning) through exploration of a wide range of cases (e.g., through experimentations in DGS environments or trials with repeated calculations in spreadsheets) before proceeding to a deductive proof of what looks like a plausible response to the problem. Bosch and Winsløw (2015) discuss the dialectic relationship between *questions* and *answers* as an essential component of knowledge development. Answers to questions, when established, become resources for the investigation of further questions "through a variety of *media* (books, journal articles, conference talks, teachers, web tutorials and so on)" (p. 363). Media are not seen in abstract: they are part, and contribute to, knowledge development, in interaction with the institutional context (*milieu*) where questioning and answering are taking place (ibid). Problem-posing and problem-solving that consider media together with "an appropriate experimental milieu" (ibid, p. 21) are essential for students' self-sustaining work with questions and answers.

Recently, some university mathematics programmes have introduced programming courses in which students investigate mathematical ideas, solve problems, and discuss real-life applications of mathematics (e.g., Buteau et al. 2019). In those courses, programming is a means for mathematical investigation as well as for mathematical learning. Gueudet et al (2020) report that programming mediates mathematical enquiry activity in the social context of those who are involved. Very often, programming is one of the range of resources available to solvers that mediate problem-solving. Thus, it is plausible to claim that such spread of available resources (mostly digital) has changed our way of solving problems. This interaction of the problem-solving activity with the available (digital) resources is the focus of the investigation presented in this paper.

For this investigation, I draw on the documentational approach (Gueudet et al., 2014) that has been developed to discuss the interaction of the resources with teachers. In this paper, the attention is on the interaction of problem solvers, and not necessarily teachers, with available resources when they deal with the problem which is not familiar to them. A resource can be anything that informs problem-solving activity, it can be an online blog, a piece of software, a textbook or interactions with others (Trouche et al., 2019; Kayali & Biza, 2021). In contrast to other studies that discuss problem-solving in the mediation with a specific digital technology, here I do not refer to any specific type of digital technology. Problem-solvers would use any resource at their discretion for their investigation. Thus, the choice of the resources, their use and appropriation to the problem-solving activity, as well as the mediation of these resources to the problem-solving discourse are seen together and in interaction. With this conceptual frame in mind, in this paper, I investigate the question: *How does the use of digital resources influence the problem-solving work on an unfamiliar divisibility problem*? I do so through the analysis of undergraduate students' written work on a problem of divisibility, with a particular focus on their use of resources (digital or not). I now present the context of the study, the participants and the problem before discussing examples from students' work.

CONTEXT, PROBLEM, PARTICIPANTS AND METHODS

The examples I discuss in this paper are from the work of eight students who attended a Mathematics Education course for Mathematics (also, occasionally Engineering or Science) undergraduate students. The course is offered as optional to finalist (Year 3) students of Bachelor of Science courses in a research-intensive university in the UK.

The aim of the course (entitled *The Learning and Teaching of Mathematics*) is to introduce students to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum (Biza & Nardi, 2020). The learning objectives of the course include: to become familiar with Research in Mathematics Education (RME) theories; to be able to critically appraise RME literature and use it to compose arguments regarding the learning and teaching of mathematics; to become familiar with the requirements (professional, curricular and other) for teaching mathematics; to engage with findings from research into the use of digital resources in the learning and teaching of mathematics; and, to practise problem-solving. Contact time is four hours per week (two for lectures and two for seminars) for twelve weeks. Lectures are teacher-led and partly interactive. Seminars are student-led (see details about the course in Biza & Nardi, in press; Nardi & Biza, in press).

"Problem-Solving" is one of the topics discussed in the sessions. Students are introduced to literature on problem-solving (e.g., Verschaffel et al. 2014; Pólya, 1945) in the lectures. Also, students have the opportunity to practise with mathematical problems and reflect on their solution in the seminars. The course is assessed through a *Portfolio of Learning Outcomes* that involves: nutshell accounts of RME theoretical constructs; reflection on students' own learning experiences in mathematics; solving a mathematical problem and reflecting on the problem-solving approach; and, responding to fictional classroom situations (see Biza & Nardi, in press). The examples presented in this paper are from students' responses to the problem-solving item of the portfolio (Figure 1) and their reflections on their problem-solving approach.

If possible, construct a 10-digit number, which is divisible by all natural numbers up to 18, including 18, by using ALL digits 0, 1, 2, … , 9 only ONCE.

Figure 1: A divisibility problem

The problem in Figure 1, is an adaptation of similar problems on divisibility found online in a blog for mathematics teachers and students (https://www.algebra.com/). The problem was chosen because it can be approached with different methods, it requires simple divisibility rules and does not require a known algebraic approach. Also, the problem requires a level of exploration of what the target number might be, without knowing whether such a number exists or not. Such exploration can be done through the use of divisibility rules (e.g., the digits of a number divisible by 9 add up to a number which is divisible by 9 and vice versa), finding the Lower Common Multiple (LCM) of all the divisors of the target number (LCM of 1, 2, …, 18 is 12252240) and, then, finding a multiple of LCM that has ALL the 0, 1, 2, … , 9 digits only ONCE, if this number exists. The last step involves the time-consuming process of checking all the multiples of 12252240 with 10-digits. In fact, there are four numbers that satisfy the conditions of the problem: 2438195760, 3785942160, 4753869120 and 4876391520. Any of those numbers is a sufficient response to the problem that asks to "construct a 10-digit number". As the description in the portfolio indicates, students had the liberty to follow their own way with the problem and use any available resources (including digital tools):

> Any mathematically correct and accurately justified response will receive full marks. In your investigation, you may consider using digital tools (e.g., computer or scientific calculators) and software (e.g., Excel, MATLAB®[1], etc.). In addition to your solution to the problem, you will attach your working on the problem. This is not going to be marked […] It does not need to be tidy or correct; a scanned version of your handwriting suffices.

Data include students' solutions to the problem, their working on the problem and their reflection on their problem-solving approach. Although there was no access to the actual problem-solving activity of the students, I analyse the submitted responses as evidence of what the students chose to report and how they self-reported their approaches to the problem.

EXAMPLES OF STUDENTS' WORK ON THE PROBLEM

Of the eight responses I discuss here, only Student H (for simplicity S-H), followed a deductive approach to the problem. S-H wrote that he accessed the divisibility rules from the *Brilliant* platform of resources for STEM[2]. They named the target number *ABCDEFGHIJ* (where each letter represents a digit of the number) and they applied the divisibility rules to create a set of simultaneous equations, see an excerpt from the

[1] MATLAB®, https://uk.mathworks.com/products/matlab.html
[2] Brilliant, https://brilliant.org/wiki/divisibility-rules/

response (not the entire response) in Figure 2. A logical fault in the steps led to a contradiction that made S-H to conclude that such a number does not exist.

> Suppose we had a 10-digit number ABCDEFGHIJ consisting of 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly once each. It must be divisible by every natural number from 1 to 18.
>
> All integers are divisible by 1, so ABCDEFGHIJ is divisible by 1. Also, as it is divisible by 10, the number ends in a 0.
>
> The divisibility rules state that all multiples of 3 have the total sum of their digits equal to a multiple of 3, and ditto with multiples of 9.
>
> 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45, which is divisible by both 3 and 9, so we can construct a number divisible by both 3 and 9 regardless of arrangement.
>
> All multiples of 10 are divisible by both 2 and 5, so ABCDEFGHIJ is divisible by both 2 and 5.
>
> **[…]**
>
> Divisibility rules for 7, 11, 13 and 17 are a little more complicated. The divisibility rule for 14 is dependent on divisibility by 7.
>
> Anything divisible by both n and 10 divided by 10 is divisible by n, so it is sufficient to find ABCDEFGHI, which is divisible by 7, 11, 13 and 17.
>
> According to Brilliant (the source used), the following must be true at the same time:
>
> - $ABCDEFGH - 2I = 7n$
> - $A - B + C - D + E - F + G - H + I = 11k$
> - $ABCDEFGH + 4I = 13p$
>
> **[…]**

Figure 2: An excerpt from Student H's response to the problem

Another student, S-C, calculated how many numbers with 10 different digits exist ("We start with 10! different numbers") and started narrowing down the choices of numbers:

> We start with 10! different numbers. We can identify that the final digit of the number must be 0, otherwise the number would not be divisible by 10. This now leaves 9! different numbers.
>
> To be divisible by 4, the last 2 digits must form a number that is also divisible by 4. This gives us the choices of 2,4,6,8 as possibilities for the 9th digit. By checking through each possibility for the 8th digit and eliminating repeat numbers, we can reduce this to $7! \times (32)$. (S-C's response, original copy)

Then, S-C created a Java program that uses the Heap algorithm (Heap, 1963) to produce and check 10-digit numbers that satisfy the conditions of the problem:

> Having narrowed the choices down and being unable to get any further with the problem, I made a computer program that used Heap's algorithm to check every possibility for the 10-digit number that met the constraints of the problem. This gave four solutions: 4876391520, 4753869120, 3785942160, 2438195760. (S-C's response, original copy)
>
> Since the question only asks for one solution, I chose 4876391520 and checked it was divisible by each number 1-18 manually. This was indeed a solution to the problem. (S-C's response, original copy)

Although S-C started the exploration by narrowing down the range of 10-digit numbers, they did not manage to produce a small enough set of numbers. As a result, their course of action changed and they programmed an algorithm that produces and checks all the 10-digit numbers (10^{10}).

The remaining students calculated the LCM and then tried to find the appropriate multiple of the LCM that satisfies the condition of the problem. One of them, S-A, identified the LCM correctly but could not work out an approach, other than trial and error in a range of numbers as they describe below:

> I must admit that I was unable to come across this number on my own mathematical ability alone as I could not work out a way, other than a simple trial and error approach, to complete the problem without assistance. […]
>
> From here I looked to find what ballpark number [roughly estimated number] would be needed to multiply my LCM to get a 10-digit number. It was clear that some value in between roughly 100 and 1000 would give me the required result. Other than plugging some very random values into my calculator, this is where I hit a wall. I eventually crumbled and resorted to researching online to find a method or some sort of answer by anyone who had done [on a] similar problem [sic]. After some searching, I found a website in which people submit different problem solving questions and people try and give their answers. Someone had already submitted this question [the problem in Figure 1] and people had gone about it in a similar way to myself. One person had written a computer program which gave back several 10 digit numbers constructed from the digits 0,…, 9 which supposedly were divisible by the natural numbers up to and including 18.
>
> I checked that this number, **2438195760** [their emphasis], was divisible by my LCM, which it was meaning that this 10-digit number is indeed divisible by the natural up to and including 18 using each digit only once. (S-A's response, original copy with my additions in square brackets)

S-A found the LCA, but "hit a wall" in their effort to find the right number. They could not see any option other than "plugging some very random values". So, they felt that they cannot solve the problem on their "own mathematical ability" and sought help from somebody who has solved a similar problem. So, with appropriate search, they found a webpage[3] that includes a discussion on, and a proposed solution to, the problem. In this webpage, S-A found a response to the problem by somebody who had "written a computer program". It is not clear whether S-A attempted the computer program or not and how they ended up with the right number (which they then checked whether it was divisible by the LCM). If S-A took the number from the website, as the outcome of the work somebody else "had done [on a] similar problem", their role as problem-solver was to verify whether this number satisfies the given conditions or not.

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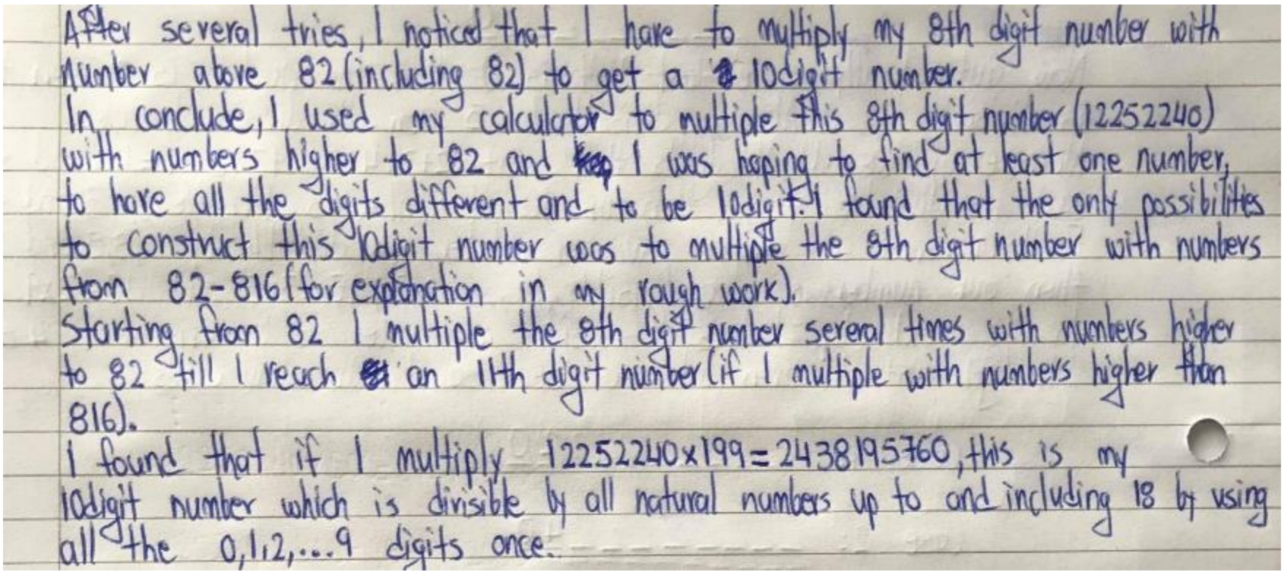
[3] The website S-A mentions in their response is: https://www.algebra.com/algebra/homework/word/misc/Miscellaneous_Word_Problems.faq.question.58446.html

So, instead of exploring whether a number with certain properties exists, S-A ended up confirming whether a number found by somebody else has these properties or not.

S-D, S-F and S-G calculated the LCM as well and then identified the target number with trial and error. S-D, for example, used the (ANS+NUMBER) functionality of the calculator and checked the answers one by one:

> Solving the LCM as 12,252,240 I then used trial and error on my calculator (ANS + 12,252,240) and visually checked each answer for one that met the conditions of the problem. (S-D's response, original copy)

S-F and S-G identified a range where the multiplier might be located (in the interval 82-816 for S-F and in the interval 101-199 for S-G) before performing their trials (Figure 3).



After several tries, I noticed that I have to multiply my 8th digit number with number above 82 (including 82) to get a 10digit number.
In conclude, I used my calculator to multiple this 8th digit number (12252240) with numbers higher to 82 and I was hoping to find at least one number, to have all the digits different and to be 10digit. I found that the only possibilities to construct this 10digit number was to multiple the 8th digit number with numbers from 82-816 (for explanation in my rough work).
Starting from 82 I multiple the 8th digit number several times with numbers higher to 82 till I reach an 11th digit number (if I multiple with numbers higher than 816).
I found that if I multiply 12252240x199=2438195760, this is my 10digit number which is divisible by all natural numbers up to and including 18 by using all the 0,1,2,...9 digits once.

Figure 3: An excerpt from Student F's response to the problem

The choice of 199 as the upper boundary by S-G (199 is the first multiplier that gives a target number) sounds quite precise. This boundary might have chosen retrospectively after S-G had found the target number, but this is a speculation that cannot be verified. However, it seems that S-G is not convinced that the trial and error is the best approach to the problem, as they acknowledge:

> Although I am happy that I found the correct solution, I feel that my approach was not the most efficient. If I had a better comprehension of [a] mathematical programming tool such as MatLab I could have produced a code that would have eliminated a lot of the tedious calculation that took up a lot of time. (S-G's response, original copy)

S-B and S-E overcame the tedious part of calculating possible numbers by using a spreadsheet. As S-B wrote:

> To begin I found that the lowest common multiple of all of these numbers is 12252240. Then, as the number must be divisible by 10, it must end in zero. This means the smallest and largest possible numbers are 1234567890 and 9876543210, respectively, so dividing both by 12252240 give about 100 and 708. I then made a spreadsheet of the multiples of

> 12252240 from 100 and 708. I then checked all of these numbers to see if they satisfied the requirements. I found the numbers 2438195760, 3785942160, 4753869120 and 4876391520. (S-B's response, original copy)

S-E took a creative step by narrowing down possible numbers in the spreadsheet:

> I used Excel to calculate multiples of the LCM in the required range. Some blocks that could be ignored were easily identifiable and shaded out (those with the first and second digits the same, and with a 0 as second digit as well as last). I then scanned the remaining numbers and ignored those with digits repeated. (S-E's response, original copy)

It seems that the spreadsheet facilitated the generation of LCM multiples, similarly to the repeated additions (or multiplications) other students did with the calculator. However, in the spreadsheet, the whole range of numbers was provided, instead of producing one number after another in a calculator. In a spreadsheet, the identification of patterns is easier, as is the elimination process – exactly as S-E did.

DISCUSSION

Findings presented in this paper aim to contribute to the discussion around the impact the availability of digital resources may have on problem-solving work. Specifically, I draw on the work of eight undergraduate mathematics students on a problem to investigate the question: *How does the use of digital resources influence the problem-solving work on an unfamiliar divisibility problem*? The examples indicate three observations.

First, online resources might be used as a source of information (e.g., definitions, rules, etc.) that feeds the problem-solving activity (e.g., S-H search online to find divisibility rules). Such resources become *documents* (Gueudet et al., 2014) for solvers and influence their approach to the problem. The accuracy of those resources, and whether such accuracy was checked by students, is not discussed in this paper. However, personal experience has indicated that uncritical use of information may mislead problem-solving activity. For example, one result of a Google search for *what a polynomial is* might be the inaccurate statement: "an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s)".

Second, an online search may aim to identify responses to a problem provided by others; searching for *answers* to questions (Bosch & Winsløw, 2015). This is well connected to everyday practices of seeking responses to enquiries through a search to the web for what other people have done in a similar situation (e.g., Yeoman et al., 2017). Finding what other solvers have done to a similar problem shifts the nature of problem-solving activity from explorative to confirmatory (e.g., S-A confirmed whether the number they found online meets the criteria instead of identifying such number). Solvers search with appropriate keywords, interpret a solution they have found and confirm that the proposed solution is right. Thus, exploratory routines of problem-solving activity – for example, conjecturing and testing – change to routines such as: unpacking the problem for search purposes; interpreting others' work; or, accepting the work of others, sometimes after verification or sometimes uncritically.

Third, digital resources might facilitate hypothesis building or execution of time-consuming procedures. This may lead to a less productive engagement with procedural performance of a repetitive task (e.g., pressing the button in a calculator) or to creative engagement with mathematical modelling (e.g., programming an algorithm that can produce and examine range of cases effectively).

I note that the students worked on the problem for the purpose of summative assessment with the liberty of using any resource available to them. The examples discussed in this paper draw on students' self-reported responses and not on the observation of students working on the problem. As a result, the examples reflect what students have chosen to report. For example, students might have found the right number through an online search and then constructed a narrative about the process through which they reached a solution retrospectively. Future research should draw on the observation of students' actual activity with consideration of the resources that are available and the context in which this activity takes place (media-milieu interaction, Bosch & Winsløw, 2015).

In conclusion, as the availability of digital resources impacts problem-solving activity, further research should provide more insight into such impact first, and then propose problem design that factors in this impact. It is plausible to assume that solvers will keep seeking help from digital resources and keep looking for what others have done in similar situations. A question is how we make sure that solvers are prepared to manage such abundance of resources productively and to their learning benefit.

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