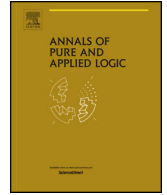


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ABSTRACT

We study the model theory of vector spaces with a bilinear form over a fixed field. For finite fields this can be, and has been, done in the classical framework of full first-order logic. For infinite fields we need different logical frameworks. First we take a category-theoretic approach, which requires very little set-up. We show that linear independence forms a simple unstable independence relation. With some more work we then show that we can also work in the framework of positive logic, which is much more powerful than the category-theoretic approach and much closer to the classical framework of full first-order logic. We fully characterise the existentially closed models of the arising positive theory. Using the independence relation from before we conclude that the theory is simple unstable, in the sense that dividing has local character but there are many distinct types. We also provide positive version of what is commonly known as the Ryll-Nardzewski theorem for ω -categorical theories in full first-order logic, from which we conclude that bilinear spaces over a countable field are ω -categorical.

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1. Introduction

Vector spaces with a bilinear form, or *bilinear spaces* as we will call them, appear in many different places in mathematics. Examples include inner product spaces (such as Hilbert spaces) and symplectic spaces arising from symplectic geometry. The model theory of bilinear spaces has been studied using various approaches. One approach is to study K -bilinear spaces over some fixed finite field K [15,6]. As K is finite, its elements can simply be named in the signature (e.g. as constants). The arising theory turns out to be simple unstable. In [9] another approach is taken to study bilinear spaces over infinite fields. Their set-up is to consider a bilinear space as a two-sorted structure, with one vector space sort V and one field sort K . They prove that the arising theory is non-simple. Later it was shown in [7] that the theory is NSOP₁ when K is algebraically closed.

When studying the model theory of vector spaces we generally take a one-sorted approach, where scalar multiplication is coded by introducing a unary function symbol for each field element. The arising theory is well known to be very well-behaved (i.e. stable), regardless of the field. In contrast, when taking a two-sorted approach the arising theory will be at least as complicated as the theory of the field sort. So by making the field part of the language, and thus fixing it, it no longer complicates the arising theory. The main idea of this paper is to do something similar for bilinear spaces, namely fix the field and then study its model theory.

The main problem with the classical first-order approach to vector spaces with a bilinear form $[\cdot, \cdot]$ over some infinite field K , such as the approach in [9], is the strength of compactness. If $[x, y] = \lambda$ is definable for every $\lambda \in K$, as it should be in any reasonable signature, then the set $\{[x, y] \neq \lambda : \lambda \in K\}$ has a realisation. This means that either the field needs to vary with the models, as in the approach in [9], or there are models where the bilinear form $[\cdot, \cdot]$ is incomplete. In this paper we sidestep these issues by working in different logical frameworks. We first take a category-theoretic approach. This is a very general framework that requires very little set-up. Then we consider the more powerful setting of positive logic, which is much closer to the classical framework of full first-order logic, but still allows us to fix the field K to be any field we like.

Independence relations are a central tool in determining where a theory belongs in Shelah's stability hierarchy, at least in the class of NSOP theories. General theory for simple independence relations has been developed for positive logic [17,3,4] and for the category-theoretic approach [13]. These results are roughly of the form "a given theory/category can only have one nice enough independence relation (the *canonical independence relation*), which reveals its place in the stability hierarchy". In [11] such tools are used to study the positive theory of exponential fields. In this paper we will, in a similar way, employ these tools to show that linear independence is the canonical independence relation in bilinear spaces over a fixed field, and that this implies simplicity and non-stability.

Main results. We study the category \mathbf{Bil}_K of K -bilinear spaces with bilinear monomorphisms (injective linear maps that respect the bilinear form). We write \mathbf{Bil}_K^s and \mathbf{Bil}_K^a for the full subcategories of symmetric and alternating K -bilinear spaces respectively, and we write \mathbf{Bil}_K^* when we mean any of these three categories. To study the model-theoretic behaviour of bilinear spaces using a category-theoretic approach we use the framework of *abstract elementary categories*, or *AECats*, from [13], see section 2.2 for the relevant details.

Theorem 1.1. *The category \mathbf{Bil}_K^* is an AECat with the amalgamation property that has a canonical simple unstable independence relation \perp given by linear independence.*

We then move to positive logic and define a theory T_K for K -bilinear spaces, again with the notation T_K^s and T_K^a for the symmetric and alternating cases respectively and T_K^* for any of these theories. The signature \mathcal{L}_K will be the standard signature for K -vector spaces, i.e. where we have a unary function symbol for scalar

multiplication for each $\lambda \in K$, a symbol \neq for inequality and the bilinear form is coded by binary relation symbols of the form $[x, y] = \lambda$ for every $\lambda \in K$. The important part is that $[x, y] \neq \lambda$ will not be positively definable when K is infinite, resulting in the fact that the e.c. (existentially closed) models are K -bilinear spaces. In fact, we fully characterise the e.c. models.

Theorem 1.2. *The following are equivalent for an \mathcal{L}_K -structure V :*

- (i) V is an e.c. model of T_K^* ,
- (ii) V is a finitely injective K -bilinear space,
- (iii) V is an infinite dimensional non-degenerate finitely injective K -bilinear space.

If the theory T_K^* in (i) is T_K^s or T_K^a then the conditions in (ii) and (iii) should be further restricted to symmetric or alternating K -bilinear spaces respectively.

We then prove some results about T_K^* that are useful for technical reasons. We prove that the quantifier-free part of a type determines the entire type (Theorem 4.9), but T_K^* has positive quantifier elimination precisely when K is finite (Theorem 4.13). If K is finite then T_K^* is Boolean (every full first-order formula is equivalent to a positive formula), so we are essentially back in the classical first-order case mentioned earlier in this introduction. However, when K is infinite then T_K^* cannot be Boolean (not even *Hausdorff*, see Definition 2.23), but we still have that equality of types is type-definable (i.e. *semi-Hausdorff*, see Definition 2.23). Putting everything together we can cast Theorem 1.1 in the setting of positive logic.

Theorem 1.3. *The theory T_K^* is simple unstable, and non-dividing coincides with linear independence. That is, for every \bar{a}, \bar{b} and C in some e.c. model M we have $\bar{a} \downarrow_C^M \bar{b}$ iff $\text{tp}(\bar{a}/C\bar{b})$ does not divide over C .*

In the above \downarrow is the relation given by linear independence, see Definition 3.1. Dividing, simplicity, unstability and the type $\text{tp}(\bar{a}/C\bar{b})$ should all be read in the sense of positive logic, see Definition 2.20 and the definitions at the end of section 4.

We also extend [10, Theorem 6.5] to Theorem 5.8, resulting in a characterisation of ω -categorical positive theories (i.e. theories where there is exactly one countable e.c. model, up to isomorphism). This includes a positive variant of being an isolated type. Using this reformulation we then conclude that T_K^* is ω -categorical (for countable K), see Corollary 5.9.

Overview. We start with some preliminaries in section 2 about bilinear spaces, the category-theoretic framework of AECats and positive logic. Then in section 3 we establish the properties that make linear independence a simple unstable independence relation in bilinear spaces (over a fixed field). In section 4 we study the positive theory of bilinear spaces over a fixed field. In the stand-alone section 5 we prove a theorem characterising ω -categorical theories. Finally, in section 6 we discuss two other model-theoretic approaches to certain bilinear spaces, and compare them to our approach: Hilbert spaces in continuous logic and the approach of [9] to bilinear spaces over an infinite field.

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2. Preliminaries

We discuss some preliminaries about bilinear spaces and the logical frameworks of AECats and positive logic. All claims made in this section are well-known and are either trivial or can be found in the references given at the start of each subsection.

Lowercase letters such as a, b, c and x, y, z will generally denote single elements or variables. We write \bar{a} to mean a (possibly empty, possibly infinite) tuple. The exact length of a tuple often does not matter, so we write $\bar{a} \in A$ instead of $\bar{a} \in A^n$. We write unions in juxtaposition, so AB means $A \cup B$.

2.1. Bilinear spaces

Throughout this subsection we fix a field K . We will drop the K from any names and terms, e.g. we write “vector space” instead of “ K -vector space”.

Definition 2.1. Let V be a vector space. A *bilinear form* is a map $[\cdot, \cdot] : V \times V \rightarrow K$ that is linear in each argument. That is, for all $x, y, z \in V$ and $\lambda \in K$:

- $[x, y + z] = [x, y] + [x, z]$ and $[x, \lambda y] = \lambda[x, y]$,
- $[x + y, z] = [x, z] + [y, z]$ and $[\lambda x, y] = \lambda[x, y]$.

A *bilinear space* is a vector space equipped with a bilinear form.

For any $A \subseteq V$ we write $\langle A \rangle$ for the linear span of A . If V is a bilinear space we naturally view $\langle A \rangle$ as a bilinear space by restricting the bilinear form.

Definition 2.2. Let V be a bilinear space, we call V or the bilinear form on V :

- *symmetric* if $[x, y] = [y, x]$ for all $x, y \in V$;
- *alternating* if $[x, x] = 0$ for all $x \in V$.

Note that in an alternating bilinear space V we always have $[x, y] = -[y, x]$ for all $x, y \in V$, this is easily seen by expanding $[x + y, x + y]$.

Example 2.3. We give some common examples of bilinear spaces.

- (i) For any field K we can consider the n -dimensional vector space K^n with the dot product as bilinear form, which yields a symmetric K -bilinear space.
- (ii) A real Hilbert space H is a real vector space with an inner product such that the associated metric makes it into a complete space. In particular, H together with the inner product is a symmetric real bilinear space.
- (iii) The standard symplectic space \mathbb{R}^{2n} : let $x_1, \dots, x_n, y_1, \dots, y_n$ be the standard basis, then the bilinear form is determined by $[x_i, y_i] = -[y_i, x_i] = 1$ and 0 for any other combination of basis vectors.

We will use the following straightforward fact implicitly throughout this paper.

Fact 2.4. Let V be a vector space with basis B . To specify a bilinear form on V it suffices to specify the value for $[a, b]$ for every pair $a, b \in B$ (extend linearly in each argument). In fact, any bilinear form is uniquely determined by the values $[a, b]$ for all $a, b \in B$. It is symmetric precisely when $[a, b] = [b, a]$ for all $a, b \in B$ and it is alternating precisely when $[a, a] = 0$ and $[a, b] = -[b, a]$ for all $a, b \in B$.

Definition 2.5. We call a bilinear space V *non-degenerate* if $[x, y] = 0$ for all $y \in V$ implies $x = 0$ and, symmetrically, $[x, y] = 0$ for all $x \in V$ implies $y = 0$.

All examples in Example 2.3 are non-degenerate. One might normally only be interested in non-degenerate spaces. However, we will not worry about this so much as every degenerate space embeds into a non-degenerate one. This last claim follows from Theorem 1.2 and Fact 2.19, but we give a sketch of a direct proof here. Start with some bilinear space V_0 . We extend V_0 to V_1 by adding a new basis vector y_x for each $x \in V_0$ and extend the bilinear form so that x and y_x have non-zero bilinear product. We repeat this process ω times and then take the union of the resulting chain of bilinear spaces.

Definition 2.6. Let K be a field. We write \mathbf{Vec}_K for the category of K -vector spaces with injective linear maps as arrows.

Definition 2.7. A *bilinear homomorphism* is a linear map between bilinear spaces that respects the bilinear forms. That is, a linear map $f : V \rightarrow W$ where V and W are bilinear spaces, such that $[x, y] = [f(x), f(y)]$ for all $x, y \in V$. A *bilinear monomorphism* is an injective bilinear homomorphism.

Definition 2.8. Let K be a field. We write \mathbf{Bil}_K for the category of K -bilinear spaces with bilinear monomorphisms. Furthermore, we write \mathbf{Bil}_K^s and \mathbf{Bil}_K^a for the full subcategory of symmetric and alternating K -bilinear spaces respectively. We will write \mathbf{Bil}_K^* if something applies to all of \mathbf{Bil}_K , \mathbf{Bil}_K^s and \mathbf{Bil}_K^a .

2.2. Abstract Elementary Categories (AECats)

We first recall the basic notions concerning accessible categories. A great reference for this is [1].

Definition 2.9. Let \mathcal{C} be a category and let λ be a regular cardinal. An object X in \mathcal{C} is called λ -presentable if whenever $Y = \text{colim}_{i \in I} Y_i$ is a λ -directed colimit then every arrow $X \rightarrow Y$ factors essentially uniquely as $X \rightarrow Y_i \rightarrow Y$ for some $i \in I$. Equivalently: $\text{Hom}(X, -)$ preserves λ -directed colimits.

Definition 2.10. A category \mathcal{C} is called λ -accessible if:

- (i) \mathcal{C} has λ -directed colimits;
- (ii) there is a set \mathcal{A} of λ -presentable objects, such that every object in \mathcal{C} can be written as a λ -directed colimit of objects in \mathcal{A} .

In the case where $\lambda = \omega$ we say that \mathcal{C} is finitely accessible. A category is called *accessible* if it is λ -accessible for some λ .

In [13, Definition 2.5] an AECat is defined as a pair of categories $(\mathcal{C}, \mathcal{M})$ satisfying some properties. We will only be interested in the case where $\mathcal{C} = \mathcal{M}$, so when we say “ \mathcal{C} is an AECat” we will actually mean “ $(\mathcal{C}, \mathcal{C})$ is an AECat”. This also allows us to simplify the definition.

Definition 2.11. An accessible category \mathcal{C} is called an *AECat* if it has all directed colimits and all arrows are monomorphisms.

Definition 2.12. A category \mathcal{C} is said to have the *amalgamation property* if any span of arrows $Y_1 \leftarrow X \rightarrow Y_2$ can be completed to a commuting square.

There are many examples of AECats with the amalgamation property, such as the motivating example: any category of models of some first-order theory with elementary embeddings. In this paper we will just be interested in \mathbf{Vec}_K and \mathbf{Bil}_K^* . Note that the arrows are injective maps in both cases, because all arrows in an AECat have to be monomorphisms. The following is straightforward to check.

Fact 2.13. For any K , the categories \mathbf{Vec}_K and \mathbf{Bil}_K^* are finitely accessible. An object V in any of these categories is λ -presentable precisely when $\dim(V) < \lambda$. In particular, each of these categories is an AECat.

It is easily seen that \mathbf{Vec}_K has the amalgamation property. Later, we will see that \mathbf{Bil}_K^* also has the amalgamation property (Proposition 3.9). This is mainly relevant for the notion of Galois type, which we define below. The amalgamation property ensures that “having the same Galois type” is indeed transitive and thus an equivalence relation. Once again, we simplify the definition for our specific case.

Definition 2.14. Let \mathcal{C} be one of \mathbf{Vec}_K or \mathbf{Bil}_K^* . Let V and V' be objects in \mathcal{C} , $\bar{a} \in V$, $\bar{a}' \in V'$ and let $B \subseteq V, V'$ be a shared subset. Then we say that \bar{a} and \bar{a}' have the same *Galois type over B* , and we write

$$\text{gtp}(\bar{a}/B; V) = \text{gtp}(\bar{a}'/B; V'),$$

if there are arrows $V \xrightarrow{f} W \xleftarrow{g} V'$ in our category \mathcal{C} that agree on B and are such that $f(\bar{a}) = g(\bar{a}')$. If $B = \emptyset$ we drop it from the notation altogether.

We can make sense of arbitrary elements and subsets of the objects in \mathbf{Vec}_K and \mathbf{Bil}_K^* because any tuple $\bar{a} \in V$ uniquely determines a subspace $\langle \bar{a} \rangle \subseteq V$, which is then again an object in our category. Furthermore, as \bar{a} generates $\langle \bar{a} \rangle$ arrows with domain \bar{a} are in one-to-one correspondence with arrows with domain $\langle \bar{a} \rangle$.

2.3. Positive logic

For a more extensive treatment we refer to [2,18].

Definition 2.15. Fix a signature \mathcal{L} . A *positive existential formula* in \mathcal{L} is one that is built using atomic formulas and $\wedge, \vee, \top, \perp$ and \exists . An *h-inductive sentence* is a sentence of the form $\forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$, where $\varphi(\bar{x})$ and $\psi(\bar{x})$ are positive existential formulas. A *positive theory* is a set of h-inductive sentences.

Convention 2.16. Whenever we say “formula” or “theory” we will mean “positive existential formula” and “positive theory” respectively, unless explicitly stated otherwise. This also means that every formula and theory we consider will be implicitly assumed to be positive (existential).

In full first-order logic we consider elementary embeddings because they preserve and reflect truth of all first-order formulas. We do not have negation in positive logic, so there is a difference between preserving and reflecting truth of formulas.

Definition 2.17. A function $f : M \rightarrow N$ between \mathcal{L} -structures is called a *homomorphism* if it preserves truth of formulas. That is, if for every $\varphi(\bar{x})$ and every $\bar{a} \in M$ we have

$$M \models \varphi(\bar{a}) \implies N \models \varphi(f(\bar{a})).$$

We call f an *immersion* if the converse implication also holds.

Definition 2.18. We call a model M of T an *existentially closed model* or an *e.c. model* if the following equivalent conditions hold:

- (i) every homomorphism $f : M \rightarrow N$ with $N \models T$ is an immersion;
- (ii) for every $\bar{a} \in M$ and $\varphi(\bar{x})$ such that $M \not\models \varphi(\bar{a})$ there is $\psi(\bar{x})$ with $T \models \neg \exists \bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}))$ and $M \models \psi(\bar{a})$.

Positive model theory generally studies the e.c. models of a theory. Every model can be completed to an e.c. model anyway, see the fact below.

Fact 2.19. *Let M be a model of some theory T then there is a homomorphism $f : M \rightarrow N$ such that N is an e.c. model of T .*

Definition 2.20. Let M be an e.c. model, $B \subseteq M$ and $\bar{a} \in M$. Then the *type of \bar{a} over B* is defined as:

$$\text{tp}(\bar{a}/B) = \{\varphi(\bar{x}) \text{ with parameters in } B : M \models \varphi(\bar{a})\}.$$

In other words, it is a maximal consistent set of formulas with parameters in B . A *partial type* over B is just any consistent set of formulas over B .

Definition 2.21. We call an e.c. model M of some theory T κ -saturated if any partial type with $< \kappa$ variables and $< \kappa$ parameters from M that is finitely satisfiable in M has a realisation in M .

Because e.c. models are generally not the same as just models of some theory there can be h-inductive sentences that are true in all e.c. models, but fail in some models. Such sentences can be added to the theory without changing the class of e.c. models. It will be useful to have some notation for this.

Definition 2.22. Let T be a theory. The *Kaiser hull* of T is defined as:

$$T^{\text{ec}} = \{\chi \text{ an h-inductive sentence} : M \models \chi \text{ for every e.c. model } M \text{ of } T\}.$$

The following definitions, except for being Boolean, are taken from [4], and are very useful for developing (neo)stability theory for positive logic.

Definition 2.23. Let T be a positive theory. We call T :

- *Boolean* if every formula in full first-order logic is equivalent to some positive existential formula modulo T , or equivalently: for every positive existential formula $\varphi(\bar{x})$ there is a positive existential $\psi(\bar{x})$ that is equivalent to $\neg\varphi(\bar{x})$ modulo T ;
- *Hausdorff* if for any two distinct types $p(\bar{x})$ and $q(\bar{x})$ there are $\varphi(\bar{x}) \notin p(\bar{x})$ and $\psi(\bar{x}) \notin q(\bar{x})$ such that $T^{\text{ec}} \models \forall \bar{x}(\varphi(\bar{x}) \vee \psi(\bar{x}))$;
- *semi-Hausdorff* if equality of types is type-definable, so there is a partial type $\Omega(\bar{x}, \bar{y})$ such that for any \bar{a}, \bar{b} in some e.c. model M we have $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$ if and only if $M \models \Omega(\bar{a}, \bar{b})$;

The reason for the name Hausdorff is that this corresponds to the type spaces being Hausdorff, where formulas correspond to closed sets.

Fact 2.24. *Boolean implies Hausdorff implies semi-Hausdorff.*

The reader that is familiar with the above terminology might be missing one term: *thickness*, which asserts that being an indiscernible sequence is type-definable and which is again weaker than being semi-Hausdorff. However, we will have no use for that notion here, so we leave it at an honourable mention.

Boolean theories are essentially the classical full first-order theories. Through a process called *positive Morleyisation* [18, section 2.3] we can view any theory in full first-order logic as a Boolean positive theory, and we will implicitly do so.

The following fact is useful for proving or disproving that a theory is Hausdorff.

Fact 2.25 ([18, Theorem 8]). *The following are equivalent for a theory T :*

- (i) T is Hausdorff;
- (ii) any model of T^{ec} is an amalgamation base, so any span of homomorphisms $M_1 \leftarrow M \rightarrow M_2$ between models of T (not necessarily e.c.) with $M \models T^{\text{ec}}$ can be amalgamated to $M_1 \rightarrow N \leftarrow M_2$ where $N \models T$.

The following is the positive analogue of being complete in full first-order logic.

Definition 2.26. A theory T has the *joint embedding property*, or *JEP*, if for any two e.c. models M_1 and M_2 there is $N \models T$ with homomorphisms $M_1 \rightarrow N \leftarrow M_2$.

3. The independence relation

Throughout this section we again fix a field K and drop it from any names and terms (except for the names of our categories).

The notion of linear independence can be formulated as a ternary relation on subsets of vector spaces. We give two equivalent formulations.

Definition 3.1. Let V be a vector space and let $A, B, C \subseteq V$. We say that A is (linearly) independent from B over C , and write $A \downarrow_C^V B$, if the following equivalent statements hold:

- (i) $\langle AC \rangle \cap \langle BC \rangle = \langle C \rangle$;
- (ii) given a basis C_0 for $\langle C \rangle$ and A_0 and B_0 such that $A_0 C_0$ is a basis for $\langle AC \rangle$ and $B_0 C_0$ is a basis for $\langle BC \rangle$, we have that $A_0 B_0 C_0$ is a linearly independent set.

Notation: any of A , B or C can be replaced by a tuple enumerating them. For example, if \bar{a} enumerates A then $\bar{a} \downarrow_C^V B$ just means $A \downarrow_C^V B$.

The relation \downarrow really is a ternary relation on subobjects of objects in \mathbf{Vec}_K (again using the idea that an arbitrary subset A is essentially the same as $\langle A \rangle$). This independence relation is known to have many desirable properties in \mathbf{Vec}_K . Most of these properties are immediate from the definition, for the remainder see for example [14, Fact 2.1.4].

Fact 3.2. *The independence relation \downarrow has the following properties in \mathbf{Vec}_K .*

INVARIANCE: For any arrow $f : V \rightarrow W$ we have $A \downarrow_C^V B$ iff $f(A) \downarrow_{f(C)}^W f(B)$.

MONOTONICITY: If $A \downarrow_C^V B$ and $B' \subseteq B$ then $A \downarrow_C^V B'$.

BASE MONOTONICITY: If $A \downarrow_C^V B$ and $C \subseteq C' \subseteq B$ then $A \downarrow_{C'}^V B$.

TRANSITIVITY: If $A \downarrow_B^V C$ and $A \downarrow_C^V D$ with $B \subseteq C \subseteq D$ then $A \downarrow_B^V D$.

SYMMETRY: If $A \downarrow_C^V B$ then $B \downarrow_C^V A$.

EXISTENCE: We always have $A \downarrow_C^V C$.

FINITE CHARACTER: If $A \downarrow_C^V B'$ for all finite $B' \subseteq B$ then $A \downarrow_C^V B$.

LOCAL CHARACTER: For any $A, B \subseteq V$ there is $B' \subseteq B$ with $\dim(\langle B' \rangle) \leq \dim(\langle A \rangle)$ such that $A \downarrow_{B'}^V B$.

EXTENSION: If $\bar{a} \downarrow_C^V B$ then for any $D \subseteq V$ there is an extension $V \subseteq W$ with some \bar{a}' in W such that $\text{gtp}(\bar{a}/BC; V) = \text{gtp}(\bar{a}'/BC; W)$ and $\bar{a}' \downarrow_C^W BD$.

STATIONARITY: If $\text{gtp}(\bar{a}/C; V) = \text{gtp}(\bar{a}'/C; V)$ then $\bar{a} \downarrow_C^V B$ and $\bar{a}' \downarrow_C^V B$ implies $\text{gtp}(\bar{a}/BC; V) = \text{gtp}(\bar{a}'/BC; V)$.

Definition 3.3. An independence relation \perp satisfying the properties in Fact 3.2 is called a *stable independence relation*.

The fact that \mathbf{Vec}_K has a stable independence relation means that it is model-theoretically very well-behaved. The situation in \mathbf{Bil}_K^* turns out to be a little bit more complicated: we lose the STATIONARITY property.

Proposition 3.4. *The STATIONARITY property fails for \perp over every C in \mathbf{Bil}_K^* . That is, for any C in \mathbf{Bil}_K^* there is an extension $C \subseteq V$ with $a, a', b \in V$ such that $\text{gtp}(a/C; V) = \text{gtp}(a'/C; V)$, $a \perp_C^V b$ and $a' \not\perp_C^V b$, while $\text{gtp}(a/Cb; V) \neq \text{gtp}(a'/Cb; V)$.*

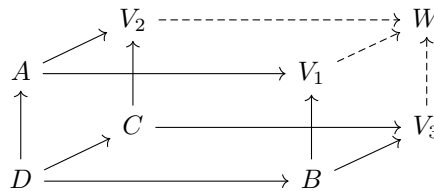
Proof. Let C be a bilinear space and introduce new linearly independent vectors a, a', b and set $V = \langle Ca a'b \rangle$. By construction we then have $a \perp_C^V b$ and $a' \perp_C^V b$. We make V into a bilinear space by setting $[x, c] = [c, x] = 0$ for $x \in \{a, a', b\}$ and $c \in C$. We set $[a', b] = [b, a'] = 1$ (in the alternating case we set $[b, a'] = -1$), for the remainder of the pairs in $\{a, a', b\}$ we take their bilinear product to be 0.

Define $f : \langle Ca \rangle \rightarrow V$ to be the identity on C and $f(a) = a'$, and extend linearly. Then f is a bilinear monomorphism. So we can amalgamate $V \supseteq \langle Ca \rangle \xrightarrow{f} V$ using Proposition 3.9 to find $V \xrightarrow{g} W \xleftarrow{h} V$ such that $g|_{\langle Ca \rangle} = hf$. So g and h agree on C and $g(a) = h(a')$, and thus $\text{gtp}(a/C; V) = \text{gtp}(a'/C; V)$.

Suppose for a contradiction that $\text{gtp}(a/Cb; V) = \text{gtp}(a'/Cb; V)$, then there are $V \xrightarrow{g} W \xleftarrow{h} V$ that agree on Cb and $g(a) = h(a')$. But then $0 = [a, b] = [g(a), g(b)] = [h(a'), h(b)] = [a', b] = 1$. \square

The independence relation on \mathbf{Bil}_K^* is still reasonably nice. We just have to replace STATIONARITY with some weaker property, namely 3-AMALGAMATION. Once again, we give a simplified definition for our situation.

Definition 3.5. An independence relation \perp has 3-AMALGAMATION if the following holds. Suppose that we have a commuting diagram as below, but without the dashed arrows and without W . We view all the arrows as inclusions. Suppose furthermore that $A \perp_D^{V_1} B$, $B \perp_D^{V_3} C$ and $C \perp_D^{V_2} A$. Then we can find the dashed arrows and W , such that $A \perp_D^W V_3$ and the resulting diagram commutes.



Definition 3.6. An independence relation \perp that satisfies the properties in Fact 3.2, except possibly STATIONARITY, and also satisfies 3-AMALGAMATION is called a *simple independence relation*.

The possible failure of STATIONARITY is precisely what distinguishes a simple independence relation from a stable one. This is because we get 3-AMALGAMATION from STATIONARITY, modulo the rest of the properties.

Fact 3.7 ([13, Proposition 6.16]). *Any stable independence relation also satisfies 3-AMALGAMATION. So every stable independence relation is also simple.*

Simple independence relations are always canonical, in the sense that there can only be one on a given AECat, see the fact below. So \perp , being stable and thus in particular simple, is the canonical independence

relation on \mathbf{Vec}_K . Even though the formulation of the properties is slightly different from the formulation in [13], it is an easy exercise to see that they are equivalent.

Fact 3.8 ([13, Theorem 1.1]). *Let \mathcal{C} be an AECat with the amalgamation property. If \perp and \perp' are simple independence relations on \mathcal{C} then $\perp = \perp'$.*

The remainder of this section is now devoted to proving that \perp is a simple independence relation on \mathbf{Bil}_K^* , and is thus also the canonical independence relation.

Proposition 3.9 (Independent amalgamation). *Let V, W_1, W_2 be bilinear spaces. Let $W_1 \xleftarrow{f_1} V \xrightarrow{f_2} W_2$ be bilinear monomorphisms. Then there is a bilinear space U and bilinear monomorphisms $W_1 \xrightarrow{g_1} U \xleftarrow{g_2} W_2$, such that $g_1 f_1 = g_2 f_2$ and $g_1(W_1) \perp_{g_1 f_1(V)}^U g_2(W_2)$. Furthermore, if V, W_1, W_2 are symmetric/alternating then we can choose U to be symmetric/alternating.*

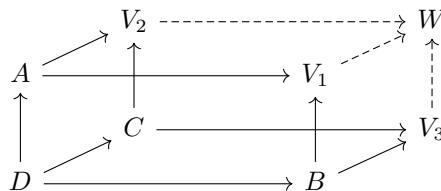
Proof. After renaming elements we may assume $V \subseteq W_1, W_2$ and $W_1 \cap W_2 = V$. So we can take $U = \langle W_1 W_2 \rangle$ and take g_1 and g_2 to be the relevant embeddings. So we have $W_1 \perp_V^U W_2$ and we are left to extend the bilinear form to all of U . Let $V' \subseteq V$ be a basis for V and for $i \in \{1, 2\}$ let $W'_i \subseteq W_i$ be such that $V' W'_i$ is a basis for W_i . Since $W_1 \perp_V^U W_2$ we have that $V' W'_1 W'_2$ is a linearly independent set, and hence a basis for U . So we can set $[w_1, w_2] = [w_2, w_1] = 0$ for all $w_1 \in W'_1$ and $w_2 \in W'_2$, and extend linearly in each argument. The final claim about the symmetric/alternating property then immediately follows. \square

Proposition 3.10. *The independence relation \perp on \mathbf{Bil}_K^* satisfies EXTENSION.*

Proof. Let $\bar{a} \perp_C^V B$. We will prove that there is some extension $V \subseteq W$ and a tuple \bar{a}' in W such that $\bar{a}' \perp_C^W V$ and $\text{gtp}(\bar{a}/BC; V) = \text{gtp}(\bar{a}'/BC; W)$. Consider the span of inclusions $\langle BC\bar{a} \rangle \supseteq \langle BC \rangle \subseteq V$ and use Proposition 3.9 to find $\langle BC\bar{a} \rangle \xrightarrow{f} W \xleftarrow{g} V$ completing this span to a commuting square, where we may assume g to be an inclusion, such that $f(\langle BC\bar{a} \rangle) \perp_{\langle BC \rangle}^W V$. We thus have the required extension $V \subseteq W$, and we set $\bar{a}' = f(\bar{a})$. Then $\text{gtp}(\bar{a}'/BC; W) = \text{gtp}(\bar{a}/BC; \langle BC\bar{a} \rangle) = \text{gtp}(\bar{a}/BC; V)$. It follows that $\bar{a}' \perp_C^W \langle BC \rangle$ and applying MONOTONICITY to $f(\langle BC\bar{a} \rangle) \perp_{\langle BC \rangle}^W V$ yields $\bar{a}' \perp_{\langle BC \rangle}^W V$. So by TRANSITIVITY we get $\bar{a}' \perp_C^W V$, as required. \square

We prove a stronger version of 3-AMALGAMATION, where the resulting cube satisfies two extra instances of independence.

Theorem 3.11 (3-AMALGAMATION). *Suppose that we have a commuting diagram as below of bilinear monomorphisms (which we view as inclusions), but without the dashed arrows and without W . Suppose furthermore that $A \perp_D^{V_1} B$, $B \perp_D^{V_3} C$ and $C \perp_D^{V_2} A$. Then there is a bilinear space W , together with the dashed bilinear monomorphisms, such that $A \perp_D^W V_3$, $B \perp_D^W V_2$ and $C \perp_D^W V_1$ and the resulting diagram commutes.*



Furthermore, if V_1, V_2, V_3 are all symmetric/alternating then we can choose W to be symmetric/alternating.

Proof. First pick a basis D' of D and let A', B', C' be such that $A'D', B'D'$ and $C'D'$ are bases of AD, BD and CD respectively. Because $A \downarrow_D^{V_1} B$ we have that $A'B'D'$ is a linearly independent set, so we extend it to a basis for V_1 . So let V'_1 such that $A'B'D'V'_1$ is a basis for V_1 . Similarly we find V'_2 and V'_3 . We may assume that V'_1, V'_2, V'_3 are such that $D'A'B'C'V'_1V'_2V'_3$ is a basis for $W = \langle D'A'B'C'V'_1V'_2V'_3 \rangle$. This induces canonical inclusions $V_i \subseteq W$. So in particular $V_1 \cap V_2 = A, V_1 \cap V_3 = B$ and $V_2 \cap V_3 = C$, as subspaces of W . From this the independence relations in the conclusions easily follow. For example: $A \cap V_3 = A \cap V_1 \cap V_3 = A \cap B = D$ hence $A \downarrow_D^W V_3$, where we used $A \downarrow_D^{V_1} B$ in the final equality.

We are left to define a bilinear form on W . For any $e, e' \in D'A'B'C'V'_1V'_2V'_3$ with $e, e' \in V_i$ for some $1 \leq i \leq 3$ the choice for $[e, e']$ is forced, and these choices are compatible by commutativity of the original diagram. For the remainder of the pairs e, e' we set $[e, e'] = 0$ and extend linearly in each argument. The final claim about the symmetric/alternating property then follows by Fact 2.4. \square

Theorem 1.1, repeated. *The category \mathbf{Bil}_K^* is an AECat with the amalgamation property that has a canonical simple unstable independence relation \downarrow given by linear independence.*

Proof. From Fact 2.13 we know that \mathbf{Bil}_K^* is an AECat. We get the amalgamation property from Proposition 3.9. That \downarrow then forms a simple independence relation comes down to checking all the required properties. We only need to verify INVARIANCE, EXTENSION and 3-AMALGAMATION, as the remaining properties do not depend on the category we are working in, so we get them directly from Fact 3.2. As any bilinear monomorphism is an injective linear map, INVARIANCE is immediate. The two remaining properties are exactly Proposition 3.10 and Theorem 3.11. In Proposition 3.4 we saw that STATIONARITY fails for \downarrow on \mathbf{Bil}_K^* , and so \downarrow is not stable. Finally, canonicity follows from Fact 3.8. \square

4. Bilinear spaces in positive logic

We define and study the positive theory of K -bilinear spaces.

Definition 4.1. Let K be some field. We define the signature \mathcal{L}_K as follows: \mathcal{L}_K includes the standard signature for K -vector spaces, with a function symbol for scalar multiplication for each $\lambda \in K$ and a symbol \neq for inequality. Furthermore, for every $\lambda \in K$ we have a binary relation symbol $[x, y] = \lambda$, which will express that the bilinear product of x and y is λ .

We naturally view every K -bilinear space as an \mathcal{L}_K -structure.

Definition 4.2. Let K be some field. We define the following \mathcal{L}_K -theories:

- T_K is the common h-inductive theory of all K -bilinear spaces,
- T_K^s is the common h-inductive theory of all symmetric K -bilinear spaces,
- T_K^a is the common h-inductive theory of all alternating K -bilinear spaces.

We write T_K^* if a statement applies to all of T_K, T_K^s and T_K^a .

Note that having a symbol for inequality forces the homomorphisms between models of T_K^* to be injective. Any positive existential formula $\varphi(\bar{x})$ is equivalent to one of the form

$$\bigvee_{i=1}^n \exists \bar{y}_i \psi_i(\bar{x}, \bar{y}_i),$$

where each $\psi_i(\bar{x}, \bar{y}_i)$ is a conjunction of atomic formulas. We will implicitly use this fact in what follows by assuming all formulas are of this form, and we recall some relevant terminology.

Definition 4.3. A *regular formula* (sometimes also called *pp-formula*) is one of the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$ where $\psi(\bar{x}, \bar{y})$ is a conjunction of atomic formulas.

Lemma 4.4. Let $\lambda \in K$ and let $\psi(x, y)$ be a regular formula that contains no linear equations such that $T_K^* \models \neg \exists xy(\psi(x, y) \wedge [x, y] = \lambda)$. Then there is some $\lambda' \in K$ such that $T_K^* \models \forall xy(\psi(x, y) \rightarrow [x, y] = \lambda')$. Similarly, for a regular formula $\psi(x)$ that contains no linear equations such that $T_K^* \models \neg \exists xy(\psi(x) \wedge [x, x] = \lambda)$ there is some $\lambda' \in K$ such that $T_K^* \models \forall x(\psi(x) \rightarrow [x, x] = \lambda')$.

Proof. The proof below is for the case of T_K and a formula $\psi(x, y)$ in two variables, at the end we discuss how to make the same proof work for a single variable and in T_K^s and T_K^a . We can write $\psi(x, y)$ as

$$\exists \bar{z}(\chi(x, y, \bar{z}) \wedge \bigwedge_{i \in I} [v_i, w_i] = \mu_i),$$

where $\chi(x, y, \bar{z})$ is a conjunction of linear inequalities (as $\psi(x, y)$ contains no linear equations) and each v_i and w_i is a linear combination of x, y and \bar{z} .

We now claim that there is some λ' such that for any bilinear space V and any $c, d \in V$ such that $V \models \psi(c, d)$ we have $V \models [c, d] = \lambda'$. This is enough, because then $\forall xy(\psi(x, y) \rightarrow [x, y] = \lambda')$ is in T_K , as required.

To prove the claim we argue by contradiction. Suppose there are distinct λ_1 and λ_2 and bilinear spaces V_1 and V_2 with $c_1, d_1 \in V_1$ and $c_2, d_2 \in V_2$ such that $V_1 \models \psi(c_1, d_1) \wedge [c_1, d_1] = \lambda_1$ and $V_2 \models \psi(c_2, d_2) \wedge [c_2, d_2] = \lambda_2$. We rename the variables appearing in the quantifier-free part of ψ as follows: x becomes z_1, y becomes z_2 and then we can enumerate \bar{z} as z_3, \dots, z_n for some n . We introduce a variable u_{ij} for all $1 \leq i, j \leq n$, so letting u_{ij} represent $[z_i, z_j]$ we can view $\bigwedge_{i \in I} [v_i, w_i] = \mu_i$ as a system S of linear equations in variables $(u_{ij})_{1 \leq i, j \leq n}$. This determines an affine space

$$A = \{(\alpha_{ij})_{1 \leq i, j \leq n} \in K^{n^2} : (\alpha_{ij})_{1 \leq i, j \leq n} \text{ is a solution to } S\}.$$

By our assumption about the existence of V_1 and V_2 we know that there must be solutions $(\alpha_{ij})_{1 \leq i, j \leq n}$ and $(\beta_{ij})_{1 \leq i, j \leq n}$ in A with $\alpha_{1,2} = \lambda_1$ and $\beta_{1,2} = \lambda_2$. The projection of an affine space onto one coordinate must be either a point or all of K . So since λ_1 and λ_2 are distinct, the projection of A on the coordinate indexed by $(1, 2)$ must be all of K . We thus find a solution $(\gamma_{ij})_{1 \leq i, j \leq n}$ with $\gamma_{1,2} = \lambda$. Let V be a vector space with basis e_1, \dots, e_n and make it into a bilinear space by setting $[e_i, e_j] = \gamma_{ij}$ for all $1 \leq i, j \leq n$. We now have $V \models \psi(e_1, e_2)$, where the existential quantifier over \bar{z} is satisfied by e_3, \dots, e_n . The χ part is then satisfied because it only contains linear inequalities and e_1, \dots, e_n are linearly independent, while the part $\bigwedge_{i \in I} [v_i, w_i] = \mu_i$ is satisfied by our choice of $[e_i, e_j]$. At the same time we have $V \models [e_1, e_2] = \lambda$, but this contradicts $T_K \models \neg \exists xy(\psi(x, y) \wedge [x, y] = \lambda)$.

The case where $\psi(x)$ has only one variable is a simpler version of the argument above. We replace any occurrence of the y variable by the x variable, and we will just have that $c_1 = d_1, c_2 = d_2$ and $e_1 = e_2$.

To make the above argument work for T_K^s we restrict ourselves to symmetric bilinear spaces in the entire argument. To make sure the V we construct is also symmetric, we can just add equations $u_{ij} = u_{ji}$ to our system S of linear equations. Since the V_1 and V_2 we find are then assumed to be symmetric bilinear spaces, they still yield solutions to S . A similar trick works for T_K^a . \square

Lemma 4.5. Every e.c. model of T_K^* is a K -bilinear space.

Proof. Let M be an e.c. model of T_K^* . Clearly T_K^* will already specify that M is a K -vector space. We need to prove that the binary relations $[x, y] = \lambda$ encode a bilinear form on M . For that we only need to check that for every $a, b \in M$ there is some $\lambda \in K$ such that $M \models [a, b] = \lambda$. Then T_K^* will guarantee that this λ is unique and that the binary function $M \times M \rightarrow K$ encoded by the relation symbols in \mathcal{L}_K is actually a bilinear form. In fact, it is enough to check this for linearly independent a and b , and when $a = b$. Then we can take any basis B of M and since $[\cdot, \cdot]$ will then be defined on any $a, b \in B$ it extends to all of M , which is forced by T_K^* .

If $M \models [a, b] = 0$ then we are done. So suppose that $M \not\models [a, b] = 0$. Then as M is e.c. there must be some $\varphi(x, y)$ such that $T_K^* \models \neg \exists xy(\varphi(x, y) \wedge [x, y] = 0)$ and $M \models \varphi(a, b)$. We can write $\varphi(x, y)$ as a disjunction of regular formulas. Let $\psi(x, y)$ be a disjunct so that $M \models \psi(a, b)$, then we have $T_K^* \models \neg \exists xy(\psi(x, y) \wedge [x, y] = 0)$. We can write $\psi(x, y)$ as $\exists \bar{z} \chi(x, y, \bar{z})$, where $\chi(\bar{z})$ is a conjunction of atomic formulas. We may assume that $\chi(x, y, \bar{z})$ does not contain any linear equations. This is because we assumed a and b to be linearly independent, or $a = b$ and then we just drop the y variable, and any $z \in \bar{z}$ that is linearly dependent on the remainder of the variables can be eliminated from the quantifier by replacing it by the appropriate linear combination of variables. By Lemma 4.4 there is then some $\lambda \in K$ such that $T_K^* \models \forall xy(\psi(x, y) \rightarrow [x, y] = \lambda)$. We thus have $M \models [a, b] = \lambda$, as required. \square

The following is the analogue of being complete for first-order theories, and in fact together with Proposition 4.14 this implies that T_K^* is complete for finite K .

Corollary 4.6. *The theory T_K^* has JEP.*

Proof. Use that e.c. models are K -bilinear spaces (Lemma 4.5) to amalgamate (Proposition 3.9) over the trivial bilinear space. \square

Proposition 4.7. *For every $n \geq 1$ there is a formula $\theta_n(x_1, \dots, x_n)$ such that for every a_1, \dots, a_n in any e.c. model M of T_K^* we have that $M \models \theta_n(a_1, \dots, a_n)$ iff a_1, \dots, a_n are linearly independent.*

Proof. We will build the formula $\theta_n(x_1, \dots, x_n)$ by induction on n . For $\theta_1(x_1)$ we take $x_1 \neq 0$. Having built $\theta_n(x_1, \dots, x_n)$ we define $\theta_{n+1}(x_1, \dots, x_n, x_{n+1})$ as

$$\theta_n(x_1, \dots, x_n) \wedge \exists yz \left(\bigwedge_{i=1}^n ([y, x_i] = 1 \wedge [z, x_i] = 1) \wedge [y, x_{n+1}] = 1 \wedge [z, x_{n+1}] = 0 \right).$$

Let M be an e.c. model of T_K^* and let $a_1, \dots, a_{n+1} \in M$. Suppose that $M \models \theta_{n+1}(a_1, \dots, a_{n+1})$. Then by the inductive hypothesis a_1, \dots, a_n are linearly independent. So it remains to be shown that a_{n+1} is not a linear combination of a_1, \dots, a_n . Suppose for a contradiction that $a_{n+1} = \lambda_1 a_1 + \dots + \lambda_n a_n$. Let b and c be realisations for the y and z variables respectively. Then as $[b, a_i] = 1$ for all $1 \leq i \leq n$ we must have that $1 = [b, a_{n+1}] = \lambda_1 + \dots + \lambda_n$. By similar reasoning we also get $0 = [c, a_{n+1}] = \lambda_1 + \dots + \lambda_n$, and thus $1 = 0$. So we arrive at a contradiction and conclude that a_1, \dots, a_{n+1} must indeed be linearly independent.

Conversely, suppose that a_1, \dots, a_{n+1} are linearly independent. Then by the induction hypothesis $M \models \theta_n(a_1, \dots, a_n)$. Extend a_1, \dots, a_{n+1} to a basis B of M and add two new independent vectors b and c . Let $V = \langle Bbc \rangle$, so it extends M . We extend the bilinear form on M to V as follows. We set $[b, a_i] = [c, a_i] = 1$ for all $1 \leq i \leq n$, and we set $[b, a_{n+1}] = 1$ and $[c, a_{n+1}] = 0$. We set the values for $[a_i, b]$ and $[a_i, c]$ for $1 \leq i \leq n + 1$ according to whether we work in T_K^* or T_K^a (in the case of just T_K it does not matter). Then $V \models \theta_{n+1}(a_1, \dots, a_{n+1})$ because b and c are realisations for the y and z variables. So by existential closedness $M \models \theta_{n+1}(a_1, \dots, a_{n+1})$. \square

We can now give a full characterisation of the e.c. models of T_K^* , which will use the following definition (borrowing some category-theoretic terminology).

Definition 4.8. We call a K -bilinear space V *finitely injective* if for any finite dimensional K -bilinear spaces $A \subseteq B$ and any bilinear monomorphism $f : A \rightarrow V$ we can find a bilinear monomorphism $g : B \rightarrow V$ such that g extends f . In the symmetric/alternating case we require all the spaces involved to be symmetric/alternating.

Theorem 1.2, repeated. *The following are equivalent for an \mathcal{L}_K -structure V :*

- (i) V is an e.c. model of T_K^* ,
- (ii) V is a finitely injective K -bilinear space,
- (iii) V is an infinite dimensional non-degenerate finitely injective K -bilinear space.

If the theory T_K^* in (i) is T_K^s or T_K^a then the conditions in (ii) and (iii) should be further restricted to symmetric or alternating K -bilinear spaces respectively.

Proof. (ii) \Leftrightarrow (iii). The direction (iii) \Rightarrow (ii) is trivial, so we prove the other direction. Let V be a finitely injective K -bilinear space. Then V is infinite dimensional, as we can embed spaces of arbitrarily large (finite) dimension in it. For any $a \in V$ let $A = \langle a \rangle$ and define $B = \langle ab \rangle$ where b is some new vector linearly independent from a . Make B into a K -bilinear space by setting $[a, b] = 1$ and also $[b, a] = \pm 1$ (the exact value depends on whether we are in the symmetric or alternating case). We can then extend the embedding $A \subseteq V$ to some bilinear monomorphism $g : B \rightarrow V$, and so we have $[a, g(b)] \neq 0$ and $[g(b), a] \neq 0$. Since a was arbitrary we conclude that V is non-degenerate.

(i) \Rightarrow (ii). By Lemma 4.5 we already know that every e.c. model is in fact a K -bilinear space. Let $A \subseteq B$ be finite dimensional K -bilinear spaces and let $f : A \rightarrow V$ be a bilinear monomorphism. Let \bar{a} be a basis for A and extend it to a basis $\bar{a}\bar{b}$ for B . Let \bar{x} and \bar{y} be variables matching \bar{a} and \bar{b} respectively. Let $\chi(\bar{x}, \bar{y})$ be a conjunction of binary relation symbols $[\cdot, \cdot] = \lambda$ capturing all the bilinear products in $\bar{a}\bar{b}$. Amalgamate $V \xleftarrow{f} A \subseteq B$ to $V \xrightarrow{h} W \supseteq B$, using Proposition 3.9. We may assume that W is an e.c. model (otherwise complete it to one). Consider the formula $\varphi(\bar{x}, \bar{y})$ given by $\theta_n(\bar{x}, \bar{y}) \wedge \chi(\bar{x}, \bar{y})$, where θ_n is the formula from Proposition 4.7 capturing linear independence. Then $W \models \varphi(\bar{a}, \bar{b})$ and thus $W \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$. As $\bar{a} = hf(\bar{a})$ and $h : V \rightarrow W$ is an immersion we have $V \models \exists \bar{y} \varphi(f(\bar{a}), \bar{y})$. We thus find $\bar{c} \in V$ with $V \models \varphi(f(\bar{a}), \bar{c})$. We can now extend f to $g : B \rightarrow V$ by setting $g(\bar{b}) = \bar{c}$ and extend linearly.

(ii) \Rightarrow (i). Let $V \subseteq W$ be an extension and $W \models \varphi(\bar{a})$ for some $\bar{a} \in V$. We may assume that W is an e.c. model (otherwise complete it to one), so in particular it is a K -bilinear space. Write $\varphi(\bar{x})$ as $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is quantifier-free. Let $\bar{b} \in W$ be such that $W \models \psi(\bar{a}, \bar{b})$. Define $A = \langle \bar{a} \rangle$ and $B = \langle \bar{a}\bar{b} \rangle$. So $A \subseteq V$ and $A \subseteq B$ are finite dimensional K -bilinear spaces. As V is finitely injective we find $g : B \rightarrow V$ extending the inclusion $A \subseteq V$. As ψ is quantifier-free we have $B \models \psi(\bar{a}, \bar{b})$ and thus $V \models \psi(\bar{a}, g(\bar{b}))$. So $V \models \varphi(\bar{a})$, as required. \square

Theorem 4.9. *In T_K^* all types are determined by their quantifier-free part.*

Proof. Let \bar{a} be some tuple in some e.c. model M and let \bar{b} be a tuple in some e.c. model N , such that $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$. Define a function $f : \langle \bar{a} \rangle \rightarrow N$ by $f(\bar{a}) = \bar{b}$ and extend linearly. Then f is a bilinear monomorphism because $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$. So we can amalgamate, using Proposition 3.9, to find bilinear monomorphisms $M \xrightarrow{g} V \xleftarrow{h} N$ such that $g(\bar{a}) = h(\bar{b})$. As M and N are e.c. models, g and h are immersions, so $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$ follows. \square

Corollary 4.10. *Modulo $(T_K^*)^{ec}$ every type in finitely many variables without parameters is equivalent to a formula. If K is finite, this formula can be taken to be quantifier-free.*

Proof. Let \bar{a} be a finite tuple in an e.c. model M . By Theorem 4.9 it is enough to construct a formula $\varphi(\bar{x})$ that is equivalent to $\text{qftp}(\bar{a})$ (modulo $(T_K^*)^{ec}$). Let \bar{a}' be a maximal linearly independent subtuple of \bar{a} . Let $\psi(\bar{x})$ be a conjunction of linear equations capturing how the remainder of \bar{a} depends on \bar{a}' . Let $\chi(\bar{x})$ be a conjunction of binary relation symbols $[\cdot, \cdot] = \lambda$ capturing the bilinear products in \bar{a}' . Then the formula $\theta_n(\bar{x}') \wedge \psi(\bar{x}) \wedge \chi(\bar{x}')$ is the $\varphi(\bar{x})$ we are looking for, where θ_n is the formula from Proposition 4.7 capturing linear independence. The final claim follows from the fact that linear independence over a finite field can be expressed by a quantifier-free formula, so we can replace θ_n by this quantifier-free formula. \square

Remark 4.11. In full first-order logic we have that if all types are determined by their quantifier-free part, then the theory has quantifier elimination, see for example [12, Theorem 8.4.1]. In positive logic this is no longer true. As we will see below, T_K^* for infinite K is an example. By Theorem 4.9 we do have that in T_K^* every type is determined by its quantifier-free part. However, Theorem 4.13 shows that we do not have quantifier elimination.

Definition 4.12. We say that a theory T has *positive quantifier elimination* if for every formula $\varphi(\bar{x})$ there is some quantifier-free formula $\psi(\bar{x})$ that is equivalent to $\varphi(\bar{x})$ modulo T^{ec} .

Theorem 4.13. *The theory T_K^* has positive quantifier elimination iff K is finite.*

Proof. We first prove the right to left direction. The quantifier-free type of any n -tuple is fully determined by any linear dependencies and the bilinear products in that tuple. As K is finite, there are only finitely many possibilities for this for a fixed n . So there are only finitely many quantifier-free n -types, and hence finitely many n -types by Theorem 4.9.

Let now $\varphi(\bar{x})$ be some formula and write $[\varphi(\bar{x})]$ for the set of all types that contain $\varphi(\bar{x})$. By the above discussion $[\varphi(\bar{x})]$ is finite and, by Corollary 4.10, for each $p \in [\varphi(\bar{x})]$ there is some quantifier-free $\chi_p(\bar{x})$ that is equivalent to p . It then immediately follows that $\varphi(\bar{x})$ is equivalent to $\bigvee_{p \in [\varphi(\bar{x})]} \chi_p(\bar{x})$, modulo $(T_K^*)^{ec}$.

We now prove the contrapositive of the converse. So let K be infinite, we prove that for any $n \geq 2$ the formula θ_n from Proposition 4.7 is not equivalent to a quantifier-free formula, modulo $(T_K^*)^{ec}$. We give a proof for T_K and at the end of the proof we describe how to make the proof work for T_K^s and T_K^a . Let $n \geq 2$ and suppose that $\theta_n(x_1, \dots, x_n)$ is equivalent (modulo T_K^{ec}) to some quantifier free formula $\psi(x_1, \dots, x_n)$. We can write ψ as a disjunction $\bigvee_{\ell=1}^k \psi_\ell(x_1, \dots, x_n)$, where each ψ_ℓ is a conjunction of atomic formulas.

Let $1 \leq \ell \leq k$. For each $1 \leq i, j \leq n$ introduce a variable u_{ij} that will represent $[x_i, x_j]$. The atomic formulas in ψ_ℓ of the form $[t, s] = \lambda$, where $\lambda \in K$ and t and s are linear combinations of x_1, \dots, x_n , will then determine a linear system of equations S_ℓ in variables $(u_{ij})_{1 \leq i, j \leq n}$. We define the affine space

$$A_\ell = \{(\alpha_{ij})_{1 \leq i, j \leq n} \in K^{n^2} : (\alpha_{ij})_{1 \leq i, j \leq n} \text{ is a solution to } S_\ell\}.$$

We distinguish two cases.

- (1) There is some ℓ such that A_ℓ is the entire space K^{n^2} . This means that ψ_ℓ is (equivalent to) some formula purely in the language of vector spaces. Then as linear independence is not definable in vector spaces over an infinite field, there is some vector space V with $v_1, \dots, v_n \in V$ such that $V \models \psi_\ell(v_1, \dots, v_n)$ while v_1, \dots, v_n are not linearly independent. We make V into a bilinear space (the choice of the bilinear form does not matter) and then extend it to an e.c. model M . This process does not invalidate the truth of $\psi_\ell(v_1, \dots, v_n)$. So $M \models \psi_\ell(v_1, \dots, v_n)$ and hence $M \models \psi(v_1, \dots, v_n)$. However, we do not have $M \models \theta_n(v_1, \dots, v_n)$, so θ_n cannot be equivalent to ψ in all e.c. models.

(2) Every A_ℓ is a proper subspace of K^{n^2} . As K is infinite, $A = \bigcup_{1 \leq \ell \leq k} A_\ell$ is still a proper subset of K^{n^2} . Pick some $(\alpha_{ij})_{1 \leq i, j \leq n} \in K^{n^2} - A$. Let $V = \langle a_1, \dots, a_n \rangle$ be an n -dimensional vector space with basis a_1, \dots, a_n . Make V into a bilinear space by setting $[a_i, a_j] = \alpha_{ij}$ for all $1 \leq i, j \leq n$. Extend V to some e.c. model $M \supseteq V$. We now have $M \models \theta_n(a_1, \dots, a_n)$, as a_1, \dots, a_n are linearly independent. However, for every $1 \leq \ell \leq k$ we have that $([a_i, a_j]_{ij})_{1 \leq i, j \leq n} = (\alpha_{ij})_{1 \leq i, j \leq n}$ is not a solution to S_ℓ , so $M \not\models \psi_\ell(a_1, \dots, a_n)$. Hence $M \not\models \psi(x_1, \dots, x_n)$, which contradicts that $\theta_n(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are equivalent in all e.c. models.

A similar proof works for T_K^s , we just restrict the variables $(u_{ij})_{1 \leq i, j \leq n}$ and the corresponding systems S_ℓ and affine subspaces A_ℓ to those i and j with $i \leq j$. This suffices as they will encode all the necessary information for a symmetric bilinear form. Similarly, for T_K^a we restrict things to those i and j where $i < j$. \square

Proposition 4.14. *If K is finite then T_K^* is Boolean. If K is infinite then T_K^* is semi-Hausdorff, but not Hausdorff.*

Proof. Suppose that K is finite. Then $[x, y] \neq \lambda$ is equivalent to $\bigvee_{\lambda' \neq \lambda} [x, y] = \lambda'$. As we also have a symbol for inequality, we have that for every atomic formula $\chi(\bar{x})$ there is a positive (quantifier-free) formula that is equivalent to $\neg\chi(\bar{x})$. The same is then true for quantifier-free formulas. We conclude that T_K^* is Boolean by positive quantifier elimination (Theorem 4.13).

We now move on to the case where K is infinite. We first prove that T_K^* is semi-Hausdorff. By Theorem 4.9 we only need to prove that having the same quantifier-free type is type-definable. This is clearly true when we restrict to just the language of K -vector spaces. So it suffices to show that $[x, y] = [x', y']$ is definable. Consider the formula $\varphi(x, y, x', y')$ given by:

$$\exists z z' ([x, y - z] = 0 \wedge [x - z', z] = 0 \wedge [z', z - y'] = 0 \wedge [z' - x', y'] = 0).$$

Let M be an e.c. model and let $a, b, a', b' \in M$ with $M \models \varphi(a, b, a', b')$. Let $c, c' \in M$ be such that $[a, b - c] = [a - c', c] = [c', c - b'] = [c' - a', b'] = 0$. We thus get $[a, b] - [a, c] = [a, c] - [c', c] = [c', c] - [c', b'] = [c', b'] - [a', b'] = 0$, and so $[a, b] = [a, c] = [c', c] = [c', b'] = [a', b']$. So $\varphi(x, y, x', y')$ does indeed imply $[x, y] = [x', y']$.

Conversely, let $a, b, a', b' \in M$ such that $[a, b] = [a', b'] = \lambda$. Define $A = \langle aba'b' \rangle$ and introduce two new linearly independent vectors c, c' and form $B = \langle aba'bc'c' \rangle$. Make B into a K -bilinear space by setting $[a, c] = [c', c] = [c', b'] = \lambda$. Pick anything for the remainder of the bilinear products (respecting the form having to be symmetric/alternating). As M is e.c. it is finitely injective (Theorem 1.2), so the inclusion $A \subseteq M$ extends to a bilinear monomorphism $g : B \rightarrow M$. Then $g(c)$ and $g(c')$ are realisations for z and z' respectively in φ , so $M \models \varphi(a, b, a', b')$.

Now we prove that T_K^* is not Hausdorff. Let M be an e.c. model of T_K^* . Using the assumption that K is infinite, we can use compactness for full first-order logic to find an elementary extension N of M with linearly independent $a, b \in N$ such that $N \not\models [a, b] = \lambda$ for all $\lambda \in K$. We claim that there are extensions N_1 and N_2 of N , both models of T_K^* , such that $N_1 \models [a, b] = 0$ and $N_2 \models [a, b] = 1$. This shows that T_K^* is not Hausdorff by Fact 2.25, as $N_1 \supseteq N \subseteq N_2$ cannot be amalgamated.

Suppose for a contradiction that one of these extensions, say N_1 , does not exist. Then by compactness there is a finite conjunction $\varphi(a, b, \bar{c})$ of atomic formulas that are true in N such that $T_K^* \models \neg \exists xy (\exists \bar{z} \varphi(x, y, \bar{z}) \wedge [x, y] = 0)$. We may assume φ contains no linear equations since a and b are linearly independent and we can replace any $c \in \bar{c}$ that is linearly dependent on a, b and the remainder of \bar{c} by replacing it by the appropriate linear combination. We can thus apply Lemma 4.4 to obtain some $\lambda \in K$ such that $T_K^* \models \forall xy (\exists \bar{z} \varphi(x, y, \bar{z}) \rightarrow [x, y] = \lambda)$. However, this would imply that $N \models [a, b] = \lambda$, a contradiction. \square

We close out this section by recalling the definitions of simplicity (in the sense of [3]) and stability for positive theories, and prove that T_K^* is simple unstable.¹

Definition 4.15. Let M be some e.c. model and let $\bar{a}, \bar{b} \in M$, $C \subseteq M$. We say that a type $p(x, \bar{b}) = \text{tp}(\bar{a}/C\bar{b})$ divides over C if there is an extension $M \subseteq N$ with a C -indiscernible sequence $(\bar{b}_i)_{i < \omega}$ in N such that $\bigcup_{i < \omega} p(x, \bar{b}_i)$ is inconsistent.

Definition 4.16. A theory T is called *simple* if dividing has *local character*. That is, there is some cardinal μ such that for any finite \bar{a} in any e.c. model M and any $B \subseteq M$ there is $B_0 \subseteq B$ with $|B_0| \leq \mu$ and $\text{tp}(\bar{a}/B)$ does not divide over B_0 .

Definition 4.17. A theory T is called *stable* if there is some cardinal μ such that for all $A \subseteq M$, where M is an e.c. model and $|A| \leq \mu$, there are at most μ different types over A (possibly realised in extensions of M).

The following fact is one half of a Kim-Pillay style theorem for positive logic, which allows us to characterise simple positive theories based on the existence of a simple independence relation. We simplified the statement for our setting and only mentioned one half as that is what we need, the original theorem is much stronger.

Fact 4.18 ([3, Theorem 1.51]). *Let T be a positive theory. If there is a simple independence relation \perp on subsets of the e.c. models of T then T is simple and \perp coincides with non-dividing.*

Theorem 1.3, repeated. *The theory T_K^* is simple unstable, and non-dividing coincides with linear independence. That is, for every \bar{a}, \bar{b} and C in some e.c. model M we have $\bar{a} \perp_C^M \bar{b}$ iff $\text{tp}(\bar{a}/C\bar{b})$ does not divide over C .*

Proof. We get simplicity and the claim about \perp coinciding with non-dividing directly from applying Fact 4.18 to \perp , as all the necessary properties have been verified in section 3. To translate between the framework of AECats and the framework of positive logic, we note that Galois types coincide with positive types. That is, if B is a common subset of e.c. models M_1 and M_2 and $\bar{a}_1 \in M_1$ and $\bar{a}_2 \in M_2$ then we have $\text{tp}(\bar{a}_1/B) = \text{tp}(\bar{a}_2/B)$ if and only if $\text{gtp}(\bar{a}_1/B; M_1) = \text{gtp}(\bar{a}_2/B; M_2)$ in \mathbf{Bil}_K^* . The left to right direction is a straightforward exercise using compactness and the method of diagrams. For the right to left direction we let $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ witness the equality of Galois types. Then indeed $\text{tp}(\bar{a}_1/B) = \text{tp}(f_1(\bar{a}_1)/B) = \text{tp}(f_2(\bar{a}_2)/B) = \text{tp}(\bar{a}_2/B)$, because the bilinear monomorphisms f_1 and f_2 are \mathcal{L}_K -homomorphisms and thus immersions since M_1 and M_2 are e.c. models.

We could prove non-stability using the fact that STATIONARITY fails, as in Proposition 3.4, because in stable positive theories we must have STATIONARITY over certain sets [3, Theorem 2.8]. However, we will give a direct proof for Definition 4.17. Let μ be any infinite cardinal, and let M be an e.c. model with at least μ many linearly independent vectors $(a_i)_{i < \mu}$. Then for any $\chi : \mu \rightarrow \{0, 1\}$ we define the partial type $\Sigma_\chi(x) = \{[x, a_i] = \chi(i) : i < \mu\}$. Then each $\Sigma_\chi(x)$ can be extended to a type $p_\chi(x)$ over $(a_i)_{i < \mu}$ (i.e. it will have a realisation in some extension of M). This yields an injection from 2^μ into the space of types over $(a_i)_{i < \mu}$, so there are more than μ many types over $(a_i)_{i < \mu}$. \square

We named real Hilbert spaces as an example of bilinear spaces (Example 2.3(ii)). Complex Hilbert spaces are not bilinear spaces, because they are required to be conjugate symmetric, that is $[x, y] = \overline{[y, x]}$. Together with linearity in the first argument this means that the form is conjugate linear in the second argument:

¹ Having JEP allows us to work in a monster model, but we have so far not used monster models and it seems unnecessary to introduce them just for the final result.

$[x, \alpha y + \beta z] = \bar{\alpha}[x, y] + \bar{\beta}[x, z]$. This is an example of a *Hermitian space*, which is a generalisation of bilinear spaces. Even more general are *sesquilinear spaces*. It seems likely that similar techniques can be used to study such spaces. We thank Jan Dobrowolski for asking the following question.

Question 4.19. Can we use positive logic to study sesquilinear or Hermitian spaces over a fixed base field (or even: division ring)? Is the arising theory still simple?

5. ω -Categoricity in positive logic

In this section we provide a positive version of what is commonly known as the Ryll-Nardzewski theorem in full first-order logic, extending [10, Theorem 6.5]. This will then be used to conclude that, for countable K , the theory T_K^* of (symmetric/alternating) K -bilinear spaces is ω -categorical, see Corollary 5.9.

Definition 5.1. Let κ be a cardinal. A theory T is called κ -categorical if it has only one e.c. model of cardinality κ , up to isomorphism.

The following definition is taken from [10]. More precisely, [10, page 844] gives a topological definition of what it means for a set of formulas to be supported. We translate that to a logical property, where we restrict our attention to types (remember, for us these are maximal types, in contrast to [10]). This results in Definition 5.2(ii). We also provide an equivalent property, which is similar to the usual notion of isolated type. In fact, in full first-order logic supported is the same as isolated. In positive logic the latter would not be a good term, as it no longer corresponds to a type being an isolated point in the type space.

Definition 5.2. A type $p(\bar{x})$ in finitely many variables is called *supported* if there is a formula $\varphi(\bar{x}) \in p(\bar{x})$ such that the following equivalent conditions hold:

- (i) for all $\chi(\bar{x}) \in p(\bar{x})$ we have $T^{\text{ec}} \models \forall \bar{x}(\varphi(\bar{x}) \rightarrow \chi(\bar{x}))$,
- (ii) for all $\psi(\bar{x}) \notin p(\bar{x})$ we have $T \models \neg \exists \bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}))$.

In this case we call $\varphi(\bar{x})$ the *support* of $p(\bar{x})$.

Lemma 5.3. *The conditions in Definition 5.2 are indeed equivalent.*

Proof. (i) \Rightarrow (ii) Let $\psi(\bar{x}) \notin p(\bar{x})$ and assume for a contradiction that there is some model M of T with $\bar{a} \in M$ such that $M \models \varphi(\bar{a}) \wedge \psi(\bar{a})$. We may assume M to be e.c. Write $q(\bar{x}) = \text{tp}(\bar{a})$, so we have $\varphi(\bar{x}) \in q(\bar{x})$ and hence $p(\bar{x}) \subseteq q(\bar{x})$ by our assumption on φ . By maximality of types we then have $p(\bar{x}) = q(\bar{x})$, and hence $\psi(\bar{x}) \in q(\bar{x}) = p(\bar{x})$, which is a contradiction.

(ii) \Rightarrow (i) Let $\chi(\bar{x}) \in p(\bar{x})$. If $T^{\text{ec}} \not\models \forall \bar{x}(\varphi(\bar{x}) \rightarrow \chi(\bar{x}))$ then by definition of T^{ec} there must be an e.c. model M with $\bar{a} \in M$ such that $M \models \varphi(\bar{a})$ and $M \not\models \chi(\bar{a})$. So there is a negation $\psi(\bar{x})$ of $\chi(\bar{x})$ such that $M \models \psi(\bar{a})$. As $\chi(\bar{x}) \in p(\bar{x})$ we must have $\psi(\bar{x}) \notin p(\bar{x})$, so by our assumption on $\varphi(\bar{x})$ we have $T \models \neg \exists \bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}))$. However, this contradicts $M \models \varphi(\bar{a}) \wedge \psi(\bar{a})$. \square

Definition 5.4. Let M be an e.c. model. We call M *atomic* if it only realises supported types. We call M *prime* if every e.c. model N is an extension of M .

Fact 5.5 ([10, Proposition 6.3]). *Let M be an e.c. model of a countable theory T with JEP. Then M is prime if and only if it is countable and atomic.*

Lemma 5.6. *If every n -type is supported then every e.c. model is ω -saturated.*

Proof. Let M be an e.c. model and let $\Sigma(\bar{x}, \bar{b})$ be finitely satisfiable in M , where \bar{x} and $\bar{b} \in M$ are finite. Then there is a realisation \bar{a} in some e.c. model N that is an extension of M . Set $p(\bar{x}, \bar{y}) = \text{tp}(\bar{a}, \bar{b})$ and let $\varphi(\bar{x}, \bar{y})$ be the support of $p(\bar{x}, \bar{y})$. Then $N \models \exists \bar{x} \varphi(\bar{x}, \bar{b})$ so because M is e.c. we find $\bar{a}' \in M$ with $M \models \varphi(\bar{a}', \bar{b})$. As φ supports p we have that $M \models p(\bar{a}', \bar{b})$ and hence $M \models \Sigma(\bar{a}', \bar{b})$. \square

Remark 5.7. Theorem 5.8 below provides several equivalent characterisations of being ω -categorical for positive theories. However, compared to the analogous theorem for full first-order logic one important characterisation is missing: namely that the space of n -types is finite, for every $n < \omega$. As pointed out in [10, Example 6.6] this is simply no longer equivalent to being ω -categorical in positive logic. In fact, one easily sees that having finite type spaces is equivalent to being ω -categorical and Boolean, where being Boolean follows because the complement of any positively definable set is positively definable using a finite disjunction.

The counterexample from [10] is quite simple, so we repeat it here. Consider the theory T with constants $\{c_i\}_{i < \omega}$, asserting that $c_i \neq c_j$ for all $i \neq j$. Then T has a unique e.c. model consisting of only interpretations for the constants. We see that T is ω -categorical, but each constant yields a different type, so we have infinitely many 1-types.

Theorem 5.8. *Let T be a countable theory with JEP. Then the following are equivalent:*

- (i) T is ω -categorical,
- (ii) every n -type is supported,
- (iii) all e.c. models are atomic,
- (iv) all countable e.c. models are atomic,
- (v) every e.c. model is ω -saturated,
- (vi) there is a saturated prime e.c. model.

Proof. The equivalence between (i), (ii), (iii) and (iv) is [10, Corollary 6.4 and Theorem 6.5]. Lemma 5.6 yields (ii) \Rightarrow (v), while (v) \Rightarrow (i) easily follows from back-and-forth. We are left to prove that (vi) is equivalent to properties (i) to (v).

(vi) \Rightarrow (iii) Let M be prime and saturated. By Fact 5.5 M is atomic. Let p be a type that is realised in some e.c. model N . By JEP and saturation p is also realised in M and is thus supported. As p and N were arbitrary we conclude that indeed all e.c. models are atomic.

(i)–(v) \Rightarrow (vi) Let M be the unique countable model. By (iii) M is atomic and by (v) M is ω -saturated, and hence saturated. Finally, by Fact 5.5 M is prime. \square

Corollary 5.9. *Let K be any field. Every e.c. model of T_K^* is ω -saturated. If K is at most countable then T_K^* is ω -categorical.*

Proof. By Corollary 4.10 every type in finitely many variables is supported and Corollary 4.6 gives us JEP. So Lemma 5.6 and Theorem 5.8 apply. \square

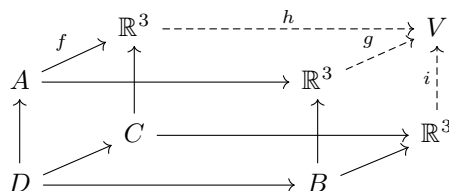
6. Comparison to different approaches

We consider two other model-theoretic approaches to certain bilinear spaces: Hilbert spaces and the two-sorted approach in full first-order logic. The former is known to be stable, while the latter is known to be non-simple (but is NSOP₁). This is in contrast to our main results Theorem 1.1 and Theorem 1.3, which claim simplicity and non-stability. In each of these two cases we point out precisely where the difference lies, in terms of the canonical independence relation.

6.1. Hilbert spaces

From [5, section 15] we know that Hilbert spaces (over the real numbers), studied in the framework of continuous logic, are stable. The canonical independence relation is given by orthogonality. So linear independence cannot be a simple independence relation on the category of Hilbert spaces, as it would have to coincide with orthogonality by canonicity. It is then natural to ask: what property fails? The answer to this question turns out to be 3-AMALGAMATION, as we will show in Example 6.1.

Example 6.1. We write \perp for linear independence (as in Definition 3.1), and we will show that 3-AMALGAMATION fails for \perp in the category of Hilbert spaces. We work in the Hilbert space \mathbb{R}^3 with the usual inner product as bilinear form. Let $a = (1, 0, 0)$, $a' = (0, 1, 0)$, $b = (1, 0, 1)$, $c = (0, 1, 1)$ and $d = (\frac{1}{2}, \frac{1}{2}, 2)$. Set $A = \langle ad \rangle$, $A' = \langle a'd \rangle$, $B = \langle bd \rangle$, $C = \langle cd \rangle$ and $D = \langle d \rangle$. So we have a commuting diagram as below (ignoring the dashed arrows), where every arrow is an inclusion, except for f , which is defined by $f(d) = d$ and $f(a) = a'$, and then extend linearly.



Furthermore, we have $A \perp_D^{\mathbb{R}^3} B$, $B \perp_D^{\mathbb{R}^3} C$ and $A' \perp_D^{\mathbb{R}^3} C$. Noting that $A' = f(A)$ 3-AMALGAMATION would give us the dashed arrows such that everything commutes. We will view i as a genuine inclusion and write $a^* = g(a) = h(a')$.

Set $v = 2(b + c - d)$, so in \mathbb{R}^3 this is just $(1, 1, 0)$. We calculate:

$$\begin{aligned}
 [a^* - v, a^* - v] &= [a^*, a^* - v] - [v, a^* - v] \\
 &= ([a^*, a^*] - [a^*, v]) - ([v, a^*] - [v, v]) \\
 &= (1 - ([a^*, 2b] + [a^*, 2c] - [a^*, 2d])) - ([v, a^*] - [v, v]) \\
 &= (1 - (2 + 2 - 1)) - ([2b, a^*] + [2c, a^*] - [2d, a^*] - 2) \\
 &= -2 - (2 + 2 - 1 - 2) \\
 &= -3,
 \end{aligned}$$

here we have used the definition of a^* and commutativity of the above diagram multiple times for simplifications like $[a^*, 2b] = [g(a), i(2b)] = [g(a), g(2b)] = [a, 2b] = 2$.

So we have found an element of V , namely $a^* - v$, such that $[a^* - v, a^* - v] = -3$. This means that the form on V is not positive definite. So V cannot be a Hilbert space and we conclude that 3-AMALGAMATION fails for linear independence \perp in the category of Hilbert spaces.

Of course, we could apply Theorem 3.11 to the diagram in Example 6.1 to find V together with the dashed arrows. We will just get a bilinear form that is not positive definite. In fact, we can give an explicit description of V and a^* (which completely determines the diagram). We take $V = \mathbb{R}^4$, and let the bilinear form be defined by $[(x, y, z, w), (x', y', z', w')] = xx' + yy' + zz' - ww'$. Now we can take $a^* = (\frac{1}{2}, 1\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{15}}{2})$. It is then straightforward to verify that this does indeed form a solution to the 3-AMALGAMATION problem.

6.2. Failure of simplicity when the field varies

Bilinear spaces over some infinite field K have been studied in the framework of full first-order logic by considering a two-sorted theory T_∞^K : one sort for the vector space and one for the field, which has to be elementarily equivalent to K in the language of rings. This comes with two disadvantages: the theory is going to be at least as complicated as the theory of the field and the field varies between different models of the theory.

Even when we restrict ourselves to algebraically closed fields, for which we write T_∞^{ACF} , the resulting theory will be non-simple, as was established in [9, Proposition 7.4.1]. Later, in [7, Corollary 6.4], it was shown that T_∞^{ACF} is NSOP₁. In terms of independence relations this means that the canonical independence relation has all properties that a simple independence relation has, except for BASE-MONOTONICITY. We give an example of how precisely BASE MONOTONICITY fails.

Definition 6.2. Let T_∞^{ACF} be the full first-order theory of an infinite dimensional non-degenerate bilinear space, either symmetric or alternating, over an algebraically closed field of characteristic other than 2. We have two sorts, V and K , for the vector space and the field respectively. The sort V has the language of abelian groups on it, and K has the language of rings on it. Furthermore, we have a function symbol $K \times V \rightarrow V$ for scalar multiplication and a function symbol $[\cdot, \cdot] : V \times V \rightarrow K$ for the bilinear form.

We use T_∞^{ACF} to refer both to the symmetric and alternating version, as it makes no difference in what follows. We introduce some notation. For any set A in some model M we write $K(A)$ for the restriction of A to the field sort, so $K(A) = A \cap K(M)$. Similarly $V(A) = A \cap V(M)$. As the field can now vary with the models we need to include it in the notation of the linear span as well: for a field $K_0 \subseteq K(M)$ we write $\langle A \rangle_{K_0}$ for the K_0 -linear span of $V(A)$.

The canonical independence relation in T_∞^{ACF} was first described in [16, Proposition 9.37], where the base was restricted to models. In [8, Corollary 8.13] some corrections were made and the independence relation was extended to arbitrary sets, resulting in the following fact. We write $\text{dcl}(X)$ for the model-theoretic definable closure of X and \downarrow^{ACF} is algebraic independence.

Fact 6.3. The canonical independence relation \downarrow^K (the K here stands for Kim-independence) for T_∞^{ACF} is as follows: for $A, B, C \subseteq M$ we have $A \downarrow_C^{K, M} B$ if and only if $\langle AC \rangle_{K(M)} \cap \langle BC \rangle_{K(M)} = \langle C \rangle_{K(M)}$ and $K(\text{dcl}(AC)) \downarrow_{K(\text{dcl}(C))}^{\text{ACF}, M} K(\text{dcl}(BC))$.

Example 6.4. We show that BASE MONOTONICITY fails for \downarrow^K in T_∞^{ACF} . The relation \downarrow^K consists of two parts: linear independence in the vector space sort and algebraic independence in the field sort. The failure will take place in the algebraic independence, and comes from taking the definable closure of AC .

Fix some model $M = (V_0, K_0)$. Let $M \preceq N$ with $v, w \in V(N)$ be such that:

- (i) v and w are $K(N)$ -linearly independent over V_0 ,
- (ii) $[v, w] = b$ is transcendental over K_0 ,
- (iii) $[v, v] = [w, w] = [v, x] = [x, v] = [w, x] = [x, w] = 0$ for all $x \in V_0$.

Let K_b be the smallest algebraically closed field containing K_0b . Set $A = (\langle V_0v \rangle_{K_0}, K_0)$, $B = (\langle V_0w \rangle_{K_b}, K_b)$ and $C = (\langle V_0w \rangle_{K_0}, K_0)$. Each of A, B, C is algebraically closed. We quickly see that $A \downarrow_M^{K, N} B$ and $M \subseteq C \subseteq B$. We also have that $b = [v, w] \in \text{dcl}(AC)$ and so $K(\text{dcl}(AC)) \not\downarrow_{K(\text{dcl}(C))}^{\text{ACF}, N} K(\text{dcl}(BC))$, because b is transcendental over $K_0 = K(\text{dcl}(C))$ and by construction $b \in K(\text{dcl}(BC))$. We thus conclude that $A \not\downarrow_C^{K, N} B$, so BASE MONOTONICITY fails.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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