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ON THE COFINALITY OF THE LEAST λ-STRONGLY COMPACT CARDINAL

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ABSTRACT. In this paper, we characterize the possible cofinalities of the least λ-strongly compact cardinal. We show that, on the one hand, for any regular cardinal, δ, that carries a λ-complete uniform ultrafilter, it is consistent, relative to the existence of a supercompact cardinal above δ, that the least λ-strongly compact cardinal has cofinality δ. On the other hand, provably the cofinality of the least λ-strongly compact cardinal always carries a λ-complete uniform ultrafilter.

1. INTRODUCTION

In [1, 2], Bagaria and Magidor introduced the notion of λ-strong compactness (see Definition 2.1), which generalized the well-known notion of strong compactness.

λ-strong compactness shares some similarities with strong compactness. For example, λ-strong compactness can be characterized in terms of compactness properties of infinitary languages, elementary embeddings, ultrafilters, etc. (see [1, 2, 13]).

It turns out that the notion of λ-strong compactness, especially for the case $\lambda = \omega_1$, provides a weaker large cardinal strength, which can be used to prove various results known to follow from strong compactness. For example, the SCH holds above the least ω_1-strongly compact cardinal (see [1]). Using this fact, recently Goldberg [8] proved a celebrated conjecture of Woodin[1] by showing that if the conjecture fails, then there exists an ω_1-strongly compact cardinal, and the conjecture holds if the SCH holds above some cardinal, and in particular, if there is an ω_1-strongly compact cardinal. Besides these consequences, λ-strong compactness also corresponds to the exact large cardinal strength of some natural properties of interest in different areas (see [1, 2]).

The least λ-strongly compact cardinal is of particular interest, because it may have very odd properties. Bagaria and Magidor [1] showed that this cardinal must be a limit cardinal. But surprisingly, it may not be weakly inaccessible. Namely, they showed that the first ω_1-strongly compact cardinal can be singular (see [2]). Also recently Gitik [7] constructed a model of ZFC, relative to the existence of a supercompact cardinal, in which the least λ-strongly compact cardinal is not strongly compact, but stays regular. He also constructed a model of ZFC, relative to the existence of two supercompact cardinals, in which the least λ-strongly compact cardinal is not a strong limit cardinal.

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[1]The conjecture states that, in second-order set theory, every two elementary embeddings $j_0, j_1 : V \to M$ into the same inner model M agree on ordinals, i.e., $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

However, there are some limitations about the cofinality of the least λ-strongly compact cardinal. By a standard argument in [1], one can see the cofinality must be greater than or equal to the least measurable cardinal.

Following the results of Bagaria-Magidor, we became very curious about the exact limitations of the cofinality of the least λ-strongly compact cardinal. This is how our work got started.

In this paper, Theorem 4.2 extends the consistency result of Bagaria-Magidor ([2, Theorem 6.1]) to λ-measurable cardinals, and Proposition 4.7 shows this result is optimal. As a corollary (Corollary 4.6), we show that relative to the existence of two supercompact cardinals, for any regular cardinal δ between them, it is consistent that δ is the cofinality of the least ω_1-strongly compact cardinal.

The structure of the paper. In Section 2 we cover some basic technical preliminaries about λ-strongly compact cardinals, Radin forcing and iterated ultrapowers. We give the main idea of the proof of our consistency result in Section 3. Finally, in Section 4 we prove the consistency result and show that it is optimal.

2. PRELIMINARIES

We use V to denote the ground model in which we work. For any ordinals $\alpha < \beta$, we use $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) for the corresponding standard interval notation. Let id_M denote the class identity function from M to M, and we will simply write id when M is clear from the context. For a sequence u, let $\mathrm{lh}(u)$ denote the length of u. For an elementary embedding $j : V \to M$ with M transitive, $\mathrm{crit}(j)$ denotes the critical point of j.

For every γ with Cantor normal form $\gamma = \omega^{\gamma_1} + \cdots + \omega^{\gamma_n}$, where $\gamma_1 \geq \cdots \geq \gamma_n$, we let $\beta_\gamma := 1 + \gamma_n$. By induction, one may easily show that if γ is a successor ordinal, then $\beta_\gamma = 1$; and if γ is a limit ordinal, then $\beta_\gamma = \limsup_{\alpha<\gamma}(\beta_\alpha + 1)$.

For a cardinal θ, a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \theta^+ \rangle$ is a $\square_{\theta,\omega}$-*sequence* if and only if whenever α is a limit ordinal with $\theta < \alpha < \theta^+$,

(1) $1 \leq |\mathcal{C}_\alpha| \leq \omega$, and
(2) for all $C \in \mathcal{C}_\alpha$,
  (a) C is a club subset of α.
  (b) C has order type at most θ.
  (c) If η is a limit point of C, then $C \cap \eta \in \mathcal{C}_\eta$.

For every A with $|A| \geq \kappa$, let $\mathcal{P}_\kappa(A) = \{x \subseteq A \mid |x| < \kappa\}$. A set $U \subseteq \mathcal{P}_\kappa(A)$ is a *measure* if it is a non-principal κ-complete ultrafilter on $\mathcal{P}_\kappa(A)$. A measure U on $\mathcal{P}_\kappa(A)$ is *fine* if for every $x \in \mathcal{P}_\kappa(A)$, $\{y \in \mathcal{P}_\kappa(A) \mid x \subseteq y\} \in U$. A measure U on $\mathcal{P}_\kappa(A)$ is *normal* if for any function $f : \mathcal{P}_\kappa(A) \to A$ with $\{x \in \mathcal{P}_\kappa(A) \mid f(x) \in x\} \in U$, there is a set in U on which f is constant.

A cardinal κ is *α-supercompact* if there exists an elementary embedding $j : V \to M$ with M transitive such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and M is closed under sequences of length α. A cardinal κ is *supercompact* if it is α-supercompact for every α. Equivalently, κ is supercompact if and only if for every $\alpha \geq \kappa$, there is a normal fine measure on $\mathcal{P}_\kappa(\alpha)$ (see [11, §22]).

2.1. λ-strongly compact cardinals.

Definition 2.1. ([1, 2]) Suppose $\delta \geq \lambda$ are uncountable cardinals.

(1) For every $\alpha \geq \delta$, δ is *λ-strongly compact up to α* if there exists a definable elementary embedding $j : V \to M$ with M transitive, such that $\mathrm{crit}(j) \geq \lambda$ and there exists a $D \in M$ such that $j''\alpha \subseteq D$ and $M \models |D| < j(\delta)$.

(2) δ is *λ-strongly compact* if δ is λ-strongly compact up to α for every $\alpha \geq \delta$.

It is easy to see that if δ is λ-strongly compact, then it is λ'-strongly compact for every uncountable cardinal $\lambda' < \lambda$, and any cardinal greater than δ is also λ-strongly compact.

We say that δ is *λ-measurable* if and only if it is λ-strongly compact up to δ.

Usuba gave a characterization of λ-strongly compact cardinals in terms of λ-complete uniform ultrafilters [13, Theorem 1.2], which generalized a result of Ketonen. The following proposition is a simple local version of the characterization.

Proposition 2.2. *Suppose $\delta \geq \lambda$ are uncountable regular cardinals. Then δ is λ-measurable if and only if δ carries a λ-complete uniform ultrafilter, i.e., there is a λ-complete ultrafilter U over δ such that every $A \in U$ has cardinality δ.*

Proof. If δ is λ-measurable, then there exists a definable elementary embedding $j : V \to M$ with M transitive, such that $\mathrm{crit}(j) \geq \lambda$ and there exists a $D \in M$ so that $j''\delta \subseteq D$ and $M \models |D| < j(\delta)$. Since δ is regular, we have $j(\delta)$ is regular in M. Hence $\sup(j''\delta) < j(\delta)$. Now we may define a λ-complete uniform ultrafilter U over δ by $X \in U$ if and only if $X \subseteq \delta$ and $\sup(j''\delta) \in j(X)$.

Conversely, if δ carries a λ-complete uniform ultrafilter, say U, then the canonical embedding $j_U : V \to M_U \cong \mathrm{Ult}(V, U)$ satisfies $\mathrm{crit}(j_U) \geq \lambda$ and $\sup(j''\delta) \leq [\mathrm{id}]_U < j(\delta)$. Thus j_U witnesses that δ is λ-measurable. □

Theorem 2.3. *([1]) The least λ-strongly compact cardinal is a limit cardinal.*

2.2. Radin forcing.

We will generally follow [2, Section 6.1] for the presentation of Radin forcing. For the sake of completeness, we also review its definition and some related basic properties, including the coherence of measure sequences (Lemma 2.5), the characterization of Radin generic objects via the geometric conditions (Theorem 2.9), and the construction of Radin generic objects via iterated ultrapowers (Theorem 2.12). For the readers' convenience, we also give proofs for some of these properties. Readers who are familiar with Radin forcing may skip these details.

We first define *measure sequences*, which are the building blocks of the Radin forcing.

Definition 2.4. A non-empty sequence $u = \langle u(\alpha) \mid \alpha < \mathrm{lh}(u)\rangle$ is a *measure sequence* if there exists a definable elementary embedding $j : V \to M$ with M transitive such that $u(0) = \mathrm{crit}(j)$, and for each α with $0 < \alpha < \mathrm{lh}(u)$, $u \restriction \alpha \in M$ and $u(\alpha) = \{A \subseteq V_{u(0)} \mid u \restriction \alpha \in j(A)\}$.

For simplicity of notation, we write $\kappa(u)$ for $u(0)$, $\mathcal{F}(u)$ for $\bigcap_{0<\alpha<\mathrm{lh}(u)} u(\alpha)$ if the length of u is greater than 1, and $\mathcal{F}(u)$ for $\{\emptyset\}$ otherwise.

The following lemma is a mild modification of a lemma of Cummings and Woodin [6, Lemma 5.1], which shows that every measure sequence u with $\mathrm{lh}(u) < \kappa(u)$ is coherent.

Lemma 2.5. *Suppose $u = \langle u(\alpha) \mid \alpha < \mathrm{lh}(u)\rangle$ is a measure sequence with $1 < \mathrm{lh}(u) < \kappa(u)$. For every α with $0 < \alpha < \mathrm{lh}(u)$, let $j_\alpha : V \to N_\alpha \cong \mathrm{Ult}(V, u(\alpha))$ be the canonical embedding. Then $u \restriction \alpha \in N_\alpha$ and the measure sequence of length $\alpha + 1$ given by j_α is exactly $u \restriction (\alpha + 1)$.*

Proof. Let $\kappa := \kappa(u)$. Since u is a measure sequence, we may find a definable elementary embedding $j : V \to N$ with N transitive such that for every α with $0 < \alpha < \mathrm{lh}(u)$, $u \restriction \alpha \in N$ and $u(\alpha) = \{A \subseteq V_\kappa \mid u \restriction \alpha \in j(A)\}$.

Fix any α with $0 < \alpha < \mathrm{lh}(u)$. For any $x \in N_\alpha$, we can find a function $f : V_\kappa \to V$ representing it, and we denote x by $[f]_\alpha$. Now we will define an embedding $k : N_\alpha \to N$. Let $k([f]_\alpha) = j(f)(u \restriction \alpha)$ for every $[f]_\alpha \in N_\alpha$. Then it is easy to see k is well-defined and elementary, and $j = k \circ j_\alpha$.

Claim 2.6. $u \restriction \alpha = [\mathrm{id}]_\alpha \in N_\alpha$.

Proof. By the definition of k, $k([\mathrm{id}]_\alpha) = j(\mathrm{id})(u \restriction \alpha) = u \restriction \alpha$. So we only need to prove that $k(u \restriction \alpha) = u \restriction \alpha$.

We will first prove $k(\kappa) = \kappa$. Note that for every $\beta < \kappa$, $k(\beta) = k(j_\alpha(\beta)) = j(\beta) = \beta$, so $\mathrm{crit}(k) \geq \kappa$. Meanwhile, $k([\mathrm{id}]_\alpha(0)) = u(0) = \kappa$ since $k([\mathrm{id}]_\alpha) = u \restriction \alpha$ with $\alpha > 0$. But since $k(\beta) = \beta < \kappa$ for every $\beta < \kappa$, it follows that $[\mathrm{id}]_\alpha(0) \geq \kappa$. Thus $[\mathrm{id}]_\alpha(0) = \kappa$. Consequently, $k(\kappa) = \kappa$ and $\mathrm{crit}(k) > \kappa$.

Now let us prove $k(u \restriction \alpha) = u \restriction \alpha$. It is easy to see $N_\alpha \cap V_{\kappa+1} = V_{\kappa+1} = N \cap V_{\kappa+1}$ and

$$\forall X \in N_\alpha \cap V_{\kappa+1} (k(X) = X). \tag{1}$$

Take any $\eta < \alpha$. Then $u(\eta) = k'' u(\eta) \subseteq k(u(\eta))$ by (1). Since $N_\alpha \cap V_{\kappa+1} = V_{\kappa+1} = N \cap V_{\kappa+1}$, and by the maximality of $u(\eta)$ as a filter, we have $k(u(\eta)) = u(\eta)$. Note also that as k fixes the length of $u \restriction \alpha$, we have $k(u \restriction \alpha) = u \restriction \alpha$. □

Now we can prove that the measure sequence of length $\alpha + 1$ obtained from j_α, say v, is exactly $u \restriction (\alpha + 1)$ by induction on β with $\beta \leq \alpha$. Obviously, $v(\beta) = \kappa = u(\beta) \in N_\alpha$ if $\beta = 0$. Now suppose inductively that $v \restriction \beta = u \restriction \beta \in N_\alpha$. For every $X \in V_{\kappa+1}$,

$$X \in v(\beta) \Leftrightarrow u \restriction \beta = v \restriction \beta \in j_\alpha(X) \Leftrightarrow u \restriction \beta = k(u \restriction \beta) \in k(j_\alpha(X)) = j(X) \Leftrightarrow X \in u(\beta).$$

The first and last " $\Leftrightarrow$ " hold by definition, the first equality holds by induction, the second " $\Leftrightarrow$ " holds by elementarity of k, and the second equality was proved above. Hence $v = u \restriction (\alpha + 1)$. □

We define next the class U_∞ of measure sequences as follows. Let $U_0 = \{u \mid u \text{ is a measure sequence}\}$, and for every $n < \omega$, let $U_{n+1} = \{u \in U_n \mid U_n \cap V_{\kappa(u)} \in \mathcal{F}(u)\}$. Finally, set $U_\infty = \bigcap_{n<\omega} U_n$. The point is that if $u \in U_\infty$, then for every α with $0 < \alpha < \mathrm{lh}(u)$, $u(\alpha)$ concentrates on $U_\infty \cap V_{\kappa(u)}$.

The class U_∞ is non-empty if there exists a $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $V_{\kappa+2} \subseteq M$, and M is closed under sequences of length κ, for then we can get a measure sequence u from j with $\mathrm{lh}(u) \geq (2^\kappa)^+$ and for every $\alpha < (2^\kappa)^+$, $u \restriction \alpha \in U_\infty$ (see [6] or [5] for details). In particular, this also holds for every α-supercompact embedding with $\alpha \geq |V_{\kappa+2}|$.

In the sequel, if we say u is a measure sequence, we mean that u is in U_∞. Given a measure sequence u of length at least 2, we may now define the Radin forcing R_u.

Definition 2.7. R_u consists of finite sequences $p = \langle (u_0, A_0), \cdots, (u_n, A_n)\rangle$, where

(1) For every $i \leq n$, $u_i \in U_\infty$, $A_i \in \mathcal{F}(u_i)$, and $A_i \subseteq U_\infty$.
(2) For every $i < n$, $(u_i, A_i) \in V_{\kappa(u_{i+1})}$.
(3) $u_n = u$.

(We say $u_0, \cdots, u_n$ occur in p).

The ordering on R_u is defined as follows. If $p = \langle (u_0, A_0), \cdots, (u_n, A_n)\rangle$ and $q = \langle (v_0, B_0), \cdots, (v_m, B_m)\rangle$ are in R_u, then $p \leq q$ if and only if

(1) $\{v_0, \cdots, v_m\} \subseteq \{u_0, \cdots, u_n\}$.
(2) For each $j \leq m$ and $i \leq n$, if $v_j = u_i$, then $A_i \subseteq B_j$.
(3) If $i \leq n$ is such that $u_i \notin \{v_0, \cdots, v_m\}$ and if $j \leq m$ is the least such that $u_i(0) < v_j(0)$, then $u_i \in B_j$ and $A_i \subseteq B_j$.

Given an R_u-generic filter G over V, let $g_G := \langle g_\alpha \mid \alpha < \mathrm{lh}(g_G)\rangle$ be the *generic sequence given by* G. Namely, g_G is the unique sequence consisting of all measure sequences w, such that $w \neq u$ and w occurs in some $p \in G$; and if $\alpha < \beta < \mathrm{lh}(g_G)$, then $\kappa(g_\alpha) < \kappa(g_\beta)$. Also let $C_G = \{\kappa(g_\alpha) \mid \alpha < \mathrm{lh}(g_G)\}$. Then C_G is a club subset of $\kappa(u)$. In addition, if $\mathrm{lh}(u) < \kappa(u)$, then there is a condition $p \in R_u$ such that p forces that the order type of C_G is $\omega^{-1+\mathrm{lh}(u)}$. (See [6] for details.)

It is not hard to see that G can be recovered from g_G, so we may view g_G as the generic object. Indeed, G consists of all $p \in R_u$ such that

(1) If v occurs in p and $v \neq u$, then $v = g_\alpha$ for some $\alpha < \mathrm{lh}(g_G)$.
(2) For every $\alpha < \mathrm{lh}(g_G)$, g_α occurs in some $q \leq p$.

Definition 2.8. Suppose M is an inner model of ZFC and δ is a limit ordinal. Let $w(\delta)$ be a measure sequence in M, and let $w = \langle w(\alpha) \mid \alpha < \delta\rangle$ be a sequence of measure sequences in M. Then w is *geometric* with respect to $w(\delta)$ and M if and only if the following holds:

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(1) The sequence $\langle \kappa(w(\alpha)) \mid \alpha \leq \delta \rangle$ is increasing continuous.
(2) For every limit $\alpha \leq \delta$ and every $A \in M \cap V_{\kappa(w(\alpha))+1}$, $A \in \mathcal{F}(w(\alpha))$ if and only if $w \restriction \alpha$ is eventually contained in A, i.e., there exists an $\alpha_A < \alpha$ such that for every γ with $\alpha_A < \gamma < \alpha$, $w(\gamma) \in A$.

The following theorem, due to W. Mitchell, characterizes Radin generic sequences in terms of the geometric condition. We follow the notation of Definition 2.8 in the statement of the next theorem.

Theorem 2.9 ([9]). *A sequence w is geometric with respect to $w(\delta)$ and M if and only if w is a Radin generic sequence given by some $R_{w(\delta)}$-generic filter over M.*

According to (2) of Definition 2.8, for the case $\delta < \kappa(w(0))$, if w is geometric w.r.t. $w(\delta)$ and M, i.e., w is a Radin generic sequence given by some $R_{w(\delta)}$-generic filter over M, then $\mathcal{F}(w(\alpha))$ concentrates on measure sequences of length less than $\mathrm{lh}(w(\alpha))$ for every $\alpha \leq \delta$. Hence, it is easily seen that $\mathrm{lh}(w(\alpha)) = 1$ if $\alpha < \delta$ is a successor ordinal, and $\mathrm{lh}(w(\alpha)) = \limsup_{\gamma<\alpha}(\mathrm{lh}(w(\gamma)) + 1)$ if $\alpha < \delta$ is a limit ordinal. In other words, $\mathrm{lh}(w(\alpha)) = \beta_\alpha$ for every $\alpha \leq \delta$, where β_α is defined at the beginning of this section.

Now we may define u-iterated ultrapowers as follows.

Definition 2.10. Suppose $\mathrm{lh}(u) < \kappa(u)$, and $\delta \leq \omega^{-1+\mathrm{lh}(u)}$ is a limit ordinal.

(1) $\langle M_\alpha, \pi_{\alpha,\alpha'} \mid \alpha \leq \alpha' \leq \delta \rangle$ is an *iterated ultrapower* if and only if
  (a) $M_0 = V$ and $\pi_{\alpha,\alpha} = \mathrm{id}_{M_\alpha}$ for every $\alpha \leq \delta$.
  (b) $M_{\alpha+1} \cong \mathrm{Ult}(M_\alpha, W_\alpha)$ is a transitive class, where $W_\alpha \in M_\alpha$ is a κ_α-complete ultrafilter over κ_α (or $M_\alpha \cap V_{\kappa_\alpha}$) for some κ_α, and the ultrapower is constructed in M_α; $\pi_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1} \cong \mathrm{Ult}(M_\alpha, W_\alpha)$ is the canonical embedding, and for every $\gamma < \alpha$, $\pi_{\gamma,\alpha+1} = \pi_{\alpha,\alpha+1} \circ \pi_{\gamma,\alpha}$.
  (c) If $\gamma \leq \delta$ is a limit ordinal, then M_γ is the direct limit of $\langle M_\alpha, \pi_{\alpha,\alpha'} \mid \alpha \leq \alpha' < \gamma \rangle$, and for every $\alpha < \gamma$, $\pi_{\alpha,\gamma} : M_\alpha \to M_\gamma$ is the corresponding embedding.

(2) $\langle M_\alpha, \pi_{\alpha,\alpha'} \mid \alpha \leq \alpha' \leq \delta \rangle$ is a *u-iterated ultrapower* if and only if it is an iterated ultrapower, and in (1b), $\kappa_\alpha = \pi_{0,\alpha}(\kappa(u))$ and $W_\alpha = \pi_{0,\alpha}(u)(\beta_\alpha)$.

For simplicity of notation, we write π_α for $\pi_{0,\alpha}$ for every $\alpha \leq \delta$, π for π_δ and M for M_δ. Here we require that the length δ of a u-iterated ultrapower is less than or equal to $\omega^{-1+\mathrm{lh}(u)}$, because β_α should be less than $\mathrm{lh}(u)$ for every $\alpha < \delta$.

Next, following the notation of the definition above, let $\langle M_\alpha, \pi_{\alpha,\alpha'} \mid \alpha \leq \alpha' \leq \delta \rangle$ be the u-iterated ultrapower of length δ, let $w = \langle \pi_\alpha(u) \restriction \beta_\alpha \mid \alpha < \delta \rangle$, and let $w(\delta) = \pi_\delta(u) \restriction \beta_\delta$.

We will use u-iterated ultrapowers to construct a Radin generic sequence over some target model. We first prove the following lemma.

Lemma 2.11. *Suppose $\theta \leq \delta$ is a limit ordinal. If $\eta < \theta$ satisfies that $\beta_\alpha < \beta_\theta$ for every α with $\eta \leq \alpha < \theta$, then for every $\bar{A} \in \mathcal{F}(\pi_\eta(u) \restriction \beta_\theta)$, we have*

$$\{w(\alpha) \mid \eta \leq \alpha < \theta\} \subseteq \pi_{\eta,\theta}(\bar{A}). \tag{2}$$

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In particular, for every limit $\theta \leq \delta$, if $A \in \mathcal{F}(\pi_\theta(u) \restriction \beta_\theta) = \mathcal{F}(w(\theta))$, then

$$w \restriction \theta \text{ is eventually contained in } A. \tag{3}$$

Proof. For every α with $\eta \leq \alpha < \theta$, since $\bar{A} \in \mathcal{F}(\pi_\eta(u) \restriction \beta_\theta)$ and $\pi_{\eta,\alpha}$ is elementary, we have $M_\alpha \models \pi_{\eta,\alpha}(\bar{A}) \in \mathcal{F}(\pi_\alpha(u) \restriction \beta_\theta)$. By our assumption, $\beta_\alpha < \beta_\theta$, so $M_\alpha \models \pi_{\eta,\alpha}(\bar{A}) \in \pi_\alpha(u)(\beta_\alpha)$. Meanwhile, by the definition of the u-iterated ultrapower $\langle M_{\alpha'}, \pi_{\alpha',\alpha''} \mid \alpha' \leq \alpha'' \leq \delta\rangle$, we have $\pi_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1} \cong \mathrm{Ult}(M_\alpha, \pi_\alpha(u)(\beta_\alpha))$. Note also that $\pi_\alpha(u)$ is a measure sequence in M_α, and the measure sequence of length $\beta_\alpha + 1$ obtained from $\pi_{\alpha,\alpha+1}$ is exactly $\pi_\alpha(u) \restriction (\beta_\alpha + 1)$ by Lemma 2.5. Hence,

$$M_{\alpha+1} \models w(\alpha) = \pi_\alpha(u) \restriction \beta_\alpha \in \pi_{\alpha,\alpha+1}(\pi_{\eta,\alpha}(\bar{A})) = \pi_{\eta,\alpha+1}(\bar{A}).$$

Since $\pi_{\alpha+1,\theta}$ is elementary, and $w(\alpha)$ is fixed by $\pi_{\alpha+1,\theta}$, i.e., $\pi_{\alpha+1,\theta}(w(\alpha)) = w(\alpha)$, we have $M_\theta \models w(\alpha) \in \pi_{\eta,\theta}(\bar{A})$. Hence $w(\alpha) \in \pi_{\eta,\theta}(\bar{A})$. So (2) holds.

Now take any limit $\theta \leq \delta$, and we prove that (3) holds. Since $A \in \mathcal{F}(\pi_\theta(u) \restriction \beta_\theta)$, we can pick a sufficiently large $\bar{\theta} < \theta$, so that $\beta_\alpha < \beta_\theta$ for every α with $\bar{\theta} \leq \alpha < \theta$, and there exists an $\bar{A} \in M_{\bar{\theta}}$ such that $\pi_{\bar{\theta},\theta}(\bar{A}) = A$. Then $\bar{A} \in \mathcal{F}(\pi_{\bar{\theta}}(u) \restriction \beta_\theta)$. Hence $\{w(\alpha) \mid \bar{\theta} \leq \alpha < \theta\} \subseteq A$, which means $w \restriction \alpha$ is eventually contained in A. □

The point of the lemma above is that by Lemma 2.5, for every $\alpha < \delta$, the measure sequence of length $\beta_\alpha + 1$ obtained from $\pi_{\alpha,\alpha+1}$ is exactly $\pi_\alpha(u) \restriction (\beta_\alpha + 1)$. So for any iterated ultrapower $\langle M_\alpha, \pi_{\alpha,\alpha'} \mid \alpha \leq \alpha' \leq \delta\rangle$, if the measure sequence of length β_α + 1 obtained from $\pi_{\alpha,\alpha+1}$ is $\pi_\alpha(u) \restriction (\beta_\alpha + 1)$ for every $\alpha < \delta$, then the lemma above also holds. For example, we may let $\pi_{\alpha,\alpha+1}$ be the ultrapower map given by $\pi_\alpha(u)(\theta)$ for every $\alpha < \delta$ in (1b) if there is a θ with $\delta < \theta < \mathrm{lh}(u)$. Then the lemma above also holds for this new iterated ultrapower.

The following theorem is essentially due to Radin [12], see also [5, Theorem 6.7.1].

Theorem 2.12 ([12]). *The sequence w is geometric with respect to $w(\delta)$ and M(=M_δ), and we have an $R_{w(\delta)}$-generic filter over M given by w.*

Proof. We only prove (2) of the geometric condition here, i.e., for every limit $\alpha \leq \delta$ and every $A \in M \cap V_{\kappa(w(\alpha))+1}$, $A \in \mathcal{F}(w(\alpha))$ if and only if $w \restriction \alpha$ is eventually contained in A.

If $A \in \mathcal{F}(w(\alpha))$, then by Lemma 2.11, $w \restriction \alpha$ is eventually contained in A.

If $A \notin \mathcal{F}(w(\alpha))$, then $A \notin \pi_\alpha(u)(\gamma)$ for some $\gamma < \beta_\alpha$. Let $B = M \cap V_{\kappa(w(\alpha))} \setminus \{x \in A \mid \mathrm{lh}(x) = \gamma\}$. Then $B \in \mathcal{F}(w(\alpha))$. Hence, $w \restriction \alpha$ is eventually contained in B. Note also that $\langle \eta < \alpha \mid \beta_\eta = \gamma\rangle$ is unbounded in α, and we have that $w \restriction \alpha$ is not eventually contained in A.

Therefore, w is geometric w.r.t. $w(\delta)$ and M, and we have an $R_{w(\delta)}$-generic filter over M obtained from w. □

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3. MAIN IDEA OF THE CONSISTENCY RESULT (THEOREM 4.2)

Suppose κ is a supercompact cardinal and $\delta < \kappa$ is a λ-measurable cardinal. Let $j : V \to M$ be a suitable supercompact ultrapower map, let $i : M \to N$ be an ultrapower map given by some λ-complete uniform ultrafilter in M, and let $\pi = i \circ j$. Let u be the measure sequence of length δ obtained from j, and let G be a suitable R_u-generic filter over V.

For the purpose of making this paper easier to read, we next give the idea of the proof of Bagaria-Magidor from [2, Theorem 6.1], as well as our idea of the proof of Theorem 4.2.

In the proof of Bagaria-Magidor they only consider the case $\lambda = \delta$, i.e., δ is measurable, and take the Radin forcing R_u to turn κ into a λ-strongly compact cardinal.

To prove the λ-strong compactness of κ, they lift the composite embedding $\pi = i \circ j$ in a $\pi(R_u)/\dot{G}_{R_{i(u)\restriction\delta}}$-generic extension of $V[G]$. Here, j grants the δ-strong compactness of κ at the end, and i makes $\langle(i(u) \restriction \delta, i(A))\rangle$ addible to $\pi(\langle(u, A)\rangle)$ for every $\langle(u, A)\rangle \in G$. Namely, $\langle(i(u) \restriction \delta, i(A)), (\pi(u), \pi(A)\rangle \in \pi(R_u)$ for every $\langle(u, A)\rangle \in G$. In addition, since R_u has a particular closure property, $i''g_G$ can generate an $R_{i(u)\restriction\delta}$-generic filter by a variation of the transfer argument (see [2], also [3, Proposition 15.1]). Thus by Silver's criterion (see [3, Proposition 9.1]), a lifting embedding of π can be obtained. However, this embedding is not definable in $V[G]$. To remedy this, Bagaria-Magidor use a closure argument, which relies not only on the closure of the Radin forcing itself, but also on the closure of N, to show that the filter generated by the lifted embedding is λ-complete. Thus κ is λ-strongly compact in $V[G]$ (here and next, actually κ is λ-strongly compact up to κ' for some κ', but a simple trick can solve this problem by lifting class-many embeddings with the same u).

In our proof, we also take the Radin forcing R_u to turn κ into a λ-strongly compact cardinal, but handle the general case (i.e., λ may not equal to δ). The major difference, or the novelty of the proof, is the lifting argument. The argument is more complicated in the general case, because δ is λ-measurable, and so it may have stronger consistency strength than measurability (however, we don't know if under the existence of a supercompact cardinal, it is possible to make some regular cardinal, for example, λ^+, into a λ-measurable cardinal). We next give more details about the lifting argument.

We start with the strategy of Bagaria-Magidor, and we still consider to lift the composite embedding π by using Silver's criterion. But there is a problem. There may exist unboundedly many $\alpha < \delta$ with $\sup(i''\alpha) < i(\alpha)$ below δ, because δ may surpass the critical point of π. Then for every such α, there will be a gap below $\pi(g_\alpha)$ to be filled in, namely, a generic object in a similar sense of $\pi(R_u)/\dot{G}_{R_{i(u)\restriction\delta}}$. But there is no place to fill in the gap below $\pi(g_\alpha)$, since there are unboundedly many $\pi(g_\gamma)$ below $\pi(g_\alpha)$. Hence, we can't lift the embedding π.

To overcome the problem, we invoke Theorem 2.12, which states that a Radin generic object can be generated by some iterated ultrapower, to fill in a gap. However, we may need to fill in many gaps. Hence, a sequence of iterated ultrapowers should be taken to fill in all these gaps (an iterated ultrapower above κ is also taken to fill in the gap above κ, i.e., the counterpart of $\pi(R_u)/\dot{G}_{R_{i(u)\restriction\delta}}$, so that we can avoid the closure argument).

So in the proof of Theorem 4.2, we may get an iteration given by a composition of embeddings: an ultrapower map given by a λ-complete uniform ultrafilter over δ, a sequence of iterated ultrapowers for filling in gaps below κ, a supercompact embedding, and an iterated ultrapower for filling in gaps above κ. This iteration lives in $V[G]$ since it is guided by the R_u-generic object G over V. Then we can build a $\pi(R_u)$-generic object H over the target model in $V[G]$. In addition, $\pi''G \subseteq H$. So we may obtain a lifting embedding of π in $V[G]$ by using Silver's criterion, which witnesses the λ-strong compactness of κ.

4. Main Results

The following proposition shows that Radin generic sequence above an ω_1-strongly compact cardinal destroys the ω_1-strong compactness of the smaller cardinal.

Proposition 4.1. *Suppose u is a measure sequence of length at least 2. Then in an R_u-generic extension of V, there is no ω_1-strongly compact cardinal below $\kappa(u)$.*

Proof. Let $\kappa := \kappa(u)$ and let G be an R_u-generic filter. Suppose, towards a contradiction, that there is a $\gamma < \kappa$ such that γ is ω_1-strongly compact. Note that there is a Prikry sequence contained in $C_G \setminus \gamma$. Then there is a $\square_{\theta,\omega}$-sequence, say $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \theta^+ \rangle$, for some $\theta \in C_G \setminus \gamma$ (see [4, Theorem 4.2]). By the ω_1-strong compactness of γ, there exists an elementary embedding $k : V[G] \to M'$ such that $\sup(k''\theta^+) < k(\theta^+)$. Then $k(\vec{\mathcal{C}})$ is a $\square_{k(\theta),\omega}$-sequence in M'. Let $\beta := \sup(k''\theta^+)$, and pick a $C' \in k(\vec{\mathcal{C}})(\beta)$. Then we have

(i) C' is a club subset of β.
(ii) C' has order type at most $k(\theta)$.
(iii) If α is a limit point of C', then $C' \cap \alpha \in \mathcal{C}_\alpha$.

It is easy to see that $k''\theta^+$ is a stationary subset of β, so there are unboundedly many $\alpha < \theta^+$, such that $k(\alpha)$ is a limit point of C'. For any such α, by (iii), we have $C' \cap k(\alpha) = k(C'_\alpha)$ for some $C'_\alpha \in \mathcal{C}_\alpha$. Hence, these C'_α are pairwise compatible, i.e., for any such $\alpha < \alpha'$, $k(C'_{\alpha'}) \cap k(\alpha) = k(C'_\alpha)$. By elementarity of k, we have $C'_{\alpha'} \cap \alpha = C'_\alpha$. So the union of these C'_α, say C, is a club subset of θ^+, and $C \cap \alpha = C'_\alpha$ for any such α. Hence, C has order type θ^+. However, any such C'_α has order type at most θ, a contradiction. Hence, there is no ω_1-strongly compact cardinal below κ. $\square$

Theorem 4.2. *Suppose κ is a supercompact cardinal, and $\delta < \kappa$ is a λ-measurable cardinal for some uncountable cardinal λ. Then in a Radin generic extension of V that preserves the λ-measurability of δ, κ is the least λ-strongly compact cardinal and has cofinality δ.*

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Proof. First let us find a measure sequence u on κ of length δ in order to obtain a suitable Radin forcing R_u.

For every $\kappa' > |V_{\kappa+2}|$, let $U_{\kappa'}$ be a normal fine measure over $\mathcal{P}_\kappa(\kappa')$, and let $j_{\kappa'} : V \to M_{\kappa'} \cong \mathrm{Ult}(V, U_{\kappa'})$ be the corresponding supercompact embedding. Let $u_{\kappa'}$ be the measure sequence of length δ obtained from $j_{\kappa'}$. Since there are at most 2^{2^κ} many such measure sequences of length δ, there exists a proper class $\mathcal{S}$ of ordinals and a measure sequence $u = \langle u(\alpha) \mid \alpha < \delta\rangle$ such that for every $\kappa' \in \mathcal{S}$, $u = u_{\kappa'}$.

Let R_u be the Radin forcing for u. Pick a condition $\langle (u, C)\rangle \in R_u$ so that $C \cap V_{\delta+1} = \emptyset$ and it forces $\mathrm{lth}(g_{\dot{G}}) = \delta$. Then C consists of measure sequences of length less than δ. Let G be an R_u-generic filter over V with $\langle (u, C)\rangle \in G$. Then G adds no new subsets of δ, and so δ is also λ-measurable in $V[G]$. Let $g_G = \langle g_\alpha \mid \alpha < \delta\rangle$ be the generic sequence given by G. Then g_G is geometric w.r.t. u and V by Theorem 2.9.

We will define a composite embedding π. Since δ is λ-measurable, by Proposition 2.2, there exists a λ-complete uniform ultrafilter over δ, say W. Let $i : V \to N_0 = \mathrm{Ult}(V, W)$ be the canonical map. W.l.o.g., we may assume $\mathrm{crit}(i) = \lambda$. Otherwise, let $\lambda' = \mathrm{crit}(i)$. Then we can prove that κ is λ'-strongly compact with the same proof. Notice that as $\lambda' \geq \lambda$, κ is also λ-strongly compact.

For simplicity of notation, let $g_\delta := u$. For every $\alpha \leq \delta$, define

$$s(\alpha) = \begin{cases} \sup(i''\alpha), & \text{if } \alpha \text{ is a limit ordinal,} \\ i(\alpha), & \text{otherwise.} \end{cases} \tag{4}$$

Then the intervals $[s(\alpha), i(\alpha)]$ for $\alpha \leq \delta$ constitute a partition of $i(\delta) + 1$ by (4). So for every $\theta \leq i(\delta)$, we may let $[\theta]$ be the unique ordinal such that $s([\theta]) \leq \theta \leq i([\theta])$.

Take any $\kappa' \in \mathcal{S}$ and let $U := U_{\kappa'}$. Then in $V[G]$, we may construct an iterated ultrapower $\langle N_\theta, \pi_{\theta,\theta'} \mid \theta \leq \theta' \leq i(\delta)\rangle$ as follows:

(1) $N_0 = V$.
(2) $\pi_{\theta,\theta} = \mathrm{id}_{N_\theta}$ for every $\theta \leq i(\delta)$.
(3) If $\theta \leq i(\delta)$ is a limit ordinal, then N_θ is the direct limit of $\langle N_{\theta_0}, \pi_{\theta_0,\theta_1} \mid \theta_0 \leq \theta_1 \leq \theta\rangle$, together with elementary embeddings $\pi_{\theta_0,\theta} : N_{\theta_0} \to N_\theta$ for all $\theta_0 < \theta$.
(4) If $\theta \leq i(\delta)$ is a successor ordinal with $s([\theta]) \leq \theta < i([\theta])$, then $N_{\theta+1}$ is the transitive class isomorphic to $\mathrm{Ult}(N_\theta, \pi_\theta(U))$ if $\theta = s(\delta)$, or isomorphic to $\mathrm{Ult}(N_\theta, \pi_\theta(g_{[\theta]})(\beta_\theta))$, otherwise; $\pi_{\theta,\theta+1} : N_\theta \to N_{\theta+1}$ is the corresponding ultrapower map, and for every $\gamma < \theta$, let $\pi_{\gamma,\theta+1} = \pi_{\theta,\theta+1} \circ \pi_{\gamma,\theta}$.
(5) If $\theta \leq i(\delta)$ is a successor ordinal with $\theta = i([\theta])$, then $N_{\theta+1} = N_\theta$; $\pi_{\theta,\theta+1} = \mathrm{id}_{N_\theta}$, and for every $\gamma < \theta$, let $\pi_{\gamma,\theta+1} = \pi_{\gamma,\theta}$.

For simplicity of notation, let $\pi_\theta = \pi_{0,\theta} \circ i$ for every $\theta \leq i(\delta)$, let $\pi = \pi_{i(\delta)}$ and $N = N_{i(\delta)}$. Let $w = \langle \pi_\theta(g_{[\theta]}) \restriction \beta_\theta \mid \theta < i(\delta)\rangle$, let $w(i(\delta)) = \pi_{i(\delta)}(g_\delta) \restriction \beta_{i(\delta)} = \pi(g_\delta)$, and let $\vec{\kappa} = \langle \kappa(w(\theta)) \mid \theta \leq i(\delta)\rangle$. Then $\vec{\kappa} = \langle \pi_\theta(\kappa(g_{[\theta]})) \mid \theta \leq i(\delta)\rangle$ since $\kappa(w(\theta)) = \kappa(\pi_\theta(g_{[\theta]})) = \pi_\theta(\kappa(g_{[\theta]}))$ for every $\theta \leq i(\delta)$.

In the iteration π, these identity class functions in (5) are used for simplicity of notation.

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The iterated ultrapower $\langle N_\theta, \pi_{\theta,\theta'} \mid \theta \leq \theta' \leq i(\delta)\rangle$ is well-founded (see [10, Theorem 19.30]), so N_θ is transitive for every $\theta \leq i(\delta)$.

Claim 4.3. *In $V[G]$, w is geometric with respect to $w(i(\delta))$ and N. That is,*

(1) The sequence $\vec{\kappa}$ is increasing continuous.
(2) For every limit $\theta \leq i(\delta)$ and every $A \in N \cap V[G]_{\kappa(w(\theta))+1}$, $A \in \mathcal{F}(w(\theta))$ if and only if $w \upharpoonright \theta$ is eventually contained in A.

Proof. For every $\alpha \leq \delta$ with $s(\alpha) < i(\alpha)$, note that $\langle N_\theta, \pi_{\theta,\theta'} \mid s(\alpha) < \theta \leq \theta' \leq i(\alpha)\rangle$ is the $\pi_{s(\alpha)+1}(g_\alpha)$-iterated ultrapower of length $i(\alpha)$ over $N_{s(\alpha)+1}$, we have $w \upharpoonright (s(\alpha), i(\alpha)]$ is geometric w.r.t. $w(i(\alpha))$ and $N_{i(\alpha)}$ by Theorem 2.12. Note also that $N_{i(\alpha)} \cap V[G]_{\kappa(w(i(\alpha)))+1} = N \cap V[G]_{\kappa(w(i(\alpha)))+1}$ and $w(i(\alpha)) \in N$, it follows that $w \upharpoonright (s(\alpha), i(\alpha)]$ is also geometric w.r.t. $w(i(\alpha))$ and N. Hence for the proof of (1) and (2), we only need to consider the case that $\theta \leq i(\delta)$ is a limit ordinal and $\theta = s([\theta])$.

Fact 4.4. *For every $\alpha \leq \delta$, if $\gamma \leq s(\alpha)$, then $\kappa(g_\alpha)$ is fixed by π_γ, i.e., $\pi_\gamma(\kappa(g_\alpha)) = \kappa(g_\alpha)$.*

Proof. If $\gamma < s(\alpha)$, then the iterated ultrapower $\langle N_\theta, \pi_{\theta,\theta'} \mid s(\alpha) < \theta \leq \theta' \leq \gamma\rangle$ is taken in $V[w \upharpoonright [\gamma]]$. Since $\kappa(g_\alpha) > \kappa(\pi_\gamma(g_{[\gamma]}))$ is inaccessible in $V[w \upharpoonright [\gamma]]$, it is fixed by π_γ.

If $\gamma = s(\alpha)$, then $\gamma = \lim_{\xi<\alpha} i(\xi)$. So for every $\eta < \pi_\gamma(\kappa(g_\alpha))$, there is a $\xi < \alpha$ and an $\bar{\eta} < \pi_{i(\xi)}(\kappa(g_\alpha))$ such that $\pi_{i(\xi),\gamma}(\bar{\eta}) = \eta$. Meanwhile, since C_G is a club subset of κ and $\pi_{i(\xi)}(\kappa(g_{\alpha'})) = \kappa(g_{\alpha'})$ for every α' with $\xi < \alpha' \leq \alpha$, we have $\{\pi_{i(\xi)}(\kappa(g_{\alpha'})) \mid \xi < \alpha' < \alpha\} = \{\kappa(g_{\alpha'}) \mid \xi < \alpha' < \alpha\}$ is also a club subset of $\pi_{i(\xi)}(\kappa(g_\alpha)) = \kappa(g_\alpha)$. Hence, $\bar{\eta} < \pi_{i(\xi)}(\kappa(g_{\alpha'}))$ for some $\alpha' < \alpha$. Then by elementarity of $\pi_{i(\xi),\gamma}$, we have $\eta = \pi_{i(\xi),\gamma}(\bar{\eta}) < \pi_\gamma(\kappa(g_{\alpha'})) = \pi_{i(\alpha')}(\kappa(g_{\alpha'})) < \pi_{i(\alpha')}(\kappa(g_\alpha)) = \kappa(g_\alpha)$. Hence, $\pi_\gamma(\kappa(g_\alpha)) = \kappa(g_\alpha)$. □

By the fact above, the sequence $\langle \kappa(w(s(\alpha))) \mid \alpha \leq \delta\rangle = \langle \kappa(g_\alpha)) \mid \alpha \leq \delta\rangle$ is increasing continuous. Meanwhile, for every $\alpha < \delta$ with $s(\alpha) < i(\alpha)$, $\vec{\kappa} \upharpoonright (s(\alpha), i(\alpha)]$ is between $\kappa(w(s(\alpha))) = \kappa(g_\alpha)$ and $\kappa(w(s(\alpha+1))) = \kappa(w(i(\alpha)+1)) = \kappa(g_{\alpha+1})$, and $\vec{\kappa} \upharpoonright (s(\delta), i(\delta)]$ is above $\kappa(g_\delta)$. Notice also that $\vec{\kappa} \upharpoonright (s(\alpha), i(\alpha)]$ is increasing continuous, it follows that (1) holds.

Take any limit $\theta \leq i(\delta)$ with $\theta = s([\theta])$.

Lemma 4.5. *For every $\eta \leq \theta$, if $B \in \mathcal{F}(\pi_\eta(g_{[\theta]}))$, then there is an $m < [\theta]$ such that*

(i) *For every α with $m \leq \alpha < [\theta]$, we have $\pi_\eta(g_\alpha) \in B$;*
(ii) *For every limit α with $m < \alpha < [\theta]$, we have $B \cap V[G]_{\pi_\eta(\kappa(g_\alpha))} \in \mathcal{F}(\pi_\eta(g_\alpha))$.*

Proof. We prove (i) and (ii) by induction on η. If $\eta = 0$, then since $\pi_\eta = i : V \to N_0 \cong \mathrm{Ult}(V, W)$, $|W| < \kappa(g_0)$, like in *Case* 2. below, we easily know that there is an m such that (i) and (ii) hold for B. Now suppose (i) and (ii) hold for every $\xi < \eta$. Then there are two cases for η (the case that η is a successor ordinal with $\eta - 1 = i([\eta - 1])$ is trivial since $\pi_{\eta-1} = \mathrm{id}$, so we omit it):

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Case 1. η is a limit ordinal. Then there is an $\bar{\eta} < \eta$ and a $\bar{B} \in N_{\bar{\eta}}$ such that $\pi_{\bar{\eta},\eta}(\bar{B}) = B$. Since $B \in \mathcal{F}(\pi_\eta(g_{[\theta]}))$ and $\pi_{\bar{\eta},\eta}$ is elementary, we have $\bar{B} \in \mathcal{F}(\pi_{\bar{\eta}}(g_{[\theta]}))$. Thus by induction, there is an $m < [\theta]$ such that (i) and (ii) hold for $\bar{\eta}$ and $\bar{B}$. Since $\pi_{\bar{\eta},\eta}$ is elementary, it follows that for such an m, (i) and (ii) hold for η and B.

Case 2. η is a successor ordinal with $s([\eta-1]) \leq \eta-1 < i([\eta-1])$. Then $\eta < \theta$, $[\eta] < [\theta]$ and $\pi_{\eta-1,\eta}$ is the ultrapower map given by $\pi_{\eta-1}(g_{[\eta-1]})(\beta_{\eta-1})$. Since $B \in \mathcal{F}(\pi_\eta(g_{[\theta]}))$ is in N_η, it can be represented by a function $f \in N_{\eta-1}$ with domain $N_{\eta-1} \cap V[G]_{\pi_{\eta-1}(\kappa(g_{[\eta-1]}))}$, and for every $x \in \operatorname{dom}(f)$, $f(x) \in \mathcal{F}(\pi_{\eta-1}(g_{[\theta]}))$. Now work in $N_{\eta-1}$, and let $\bar{B} = \bigcap_{x \in \operatorname{dom}(f)} f(x)$. Then $\bar{B} \in \mathcal{F}(\pi_{\eta-1}(g_{[\theta]}))$ since $\mathcal{F}(\pi_{\eta-1}(g_{[\theta]}))$ is $\pi_{\eta-1}(\kappa(g_{[\theta]}))$-complete and $\pi_{\eta-1}(\kappa(g_{[\theta]})) > \pi_{\eta-1}(\kappa(g_{[\eta-1]}))$, By induction, there is an $m < [\theta]$ such that (i) and (ii) hold for $\eta-1$ and $\bar{B}$.

Also, $\pi_{\eta-1,\eta}(\bar{B}) \subseteq B$ since $\bar{B} \subseteq f(x)$ for each $x \in \operatorname{dom}(f)$. Hence, since $\pi_{\eta-1,\eta}$ is elementary, it follows that for such an m, (i) and (ii) hold for η and B as well.

So, in any case, (i) and (ii) hold. Hence by induction, the lemma holds. □

Now we will prove that (2) holds. If $A \in \mathcal{F}(w(\theta)) = \mathcal{F}(\pi_\theta(g_{[\theta]}) \restriction \beta_\theta)$, then w.l.o.g., we may assume $A \in \mathcal{F}(\pi_\theta(g_{[\theta]}))$ since for every sufficient large $\xi < \theta$, the length of $w(\xi)$ is less than β_θ. Since θ is a limit ordinal, there is a $\bar{\theta} < \theta$ and an $\bar{A} \in N_{\bar{\theta}}$ such that $\pi_{\bar{\theta},\theta}(\bar{A}) = A$. Note that $A \in \mathcal{F}(\pi_\theta(g_{[\theta]}))$ and $\pi_{\bar{\theta},\theta}$ is elementary, we have $\bar{A} \in \mathcal{F}(\pi_{\bar{\theta}}(g_{[\theta]}))$.

Now by Lemma 4.5, there is an m with $[\bar{\theta}] < m < [\theta]$ such that the following holds:

(i) For every α with $m \leq \alpha < [\theta]$, $\pi_{\bar{\theta}}(g_\alpha) \in \bar{A}$;
(ii) For every limit α with $m < \alpha < [\theta]$, $\bar{A} \cap V[G]_{\kappa(\pi_{\bar{\theta}}(g_\alpha))} \in \mathcal{F}(\pi_{\bar{\theta}}(g_\alpha))$.

For every α with $m \leq \alpha < [\theta]$, since (i) holds and $\pi_{\bar{\theta},\theta}$ is elementary, it follows that $w(i(\alpha)) = \pi_{i(\alpha)}(g_\alpha) = \pi_\theta(g_\alpha) \in \pi_{\bar{\theta},\theta}(\bar{A}) = A$. Hence,

$$\{w(i(\alpha)) \mid m \leq \alpha < [\theta]\} \subseteq A. \tag{5}$$

For every limit α with $m < \alpha < [\theta]$, since (ii) holds and $\pi_{\bar{\theta},s(\alpha)}$ is elementary, we have

$$\pi_{\bar{\theta},s(\alpha)}(\bar{A}) \cap V[G]_{\kappa(\pi_{s(\alpha)}(g_\alpha))} \in \mathcal{F}(\pi_{s(\alpha)}(g_\alpha)).$$

If $s(\alpha) < i(\alpha)$, then $\beta_\eta < \beta_{i(\alpha)}$ for every η with $s(\alpha) \leq \eta < i(\alpha)$. Hence by Lemma 2.11,

$$\{w(\eta) \mid s(\alpha) \leq \eta < i(\alpha)\} \subseteq \pi_{\bar{\theta},i(\alpha)}(\bar{A}) \cap V[G]_{\pi_{i(\alpha)}(\kappa(g_\alpha))} \subseteq A. \tag{6}$$

Therefore, we have $\{w(\eta) \mid i(m) < \eta < \theta\} \subseteq A$ by (5) and (6) for some $m < [\theta]$. In other words, $w \restriction \theta$ is eventually contained in A.

If $A \notin \mathcal{F}(w(\theta))$, then $A \notin w(\theta)(\eta)$ for some $0 < \eta < \beta_\theta$. Hence $\{\gamma < \theta \mid w(\gamma) \notin A$ and $\beta_\gamma = \eta\}$ is unbounded in θ, which means that $w \restriction \theta$ is not eventually contained in A. □

In $V[G]$, let $H \subseteq \pi(R_u)$ be the filter given by w. Namely, H is the set of all $p \in \pi(R_u)$ such that

(i) If v occurs in p, then $v = w(\theta)$ for some $\theta \leq i(\delta)$;
(ii) For any $\theta \leq i(\delta)$, $w(\theta)$ occurs in some $q \leq p$.

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It follows from Claim 4.3 that H is $\pi(R_u)$-generic over N. Now we will prove that $\pi''G \subseteq H$. Take any condition $p = \langle (g_{\alpha_1}, A_1), ..., (g_{\alpha_n}, A_n)\rangle \in G$, where $\alpha_n = \delta$. We need to prove that $\pi(p) \in H$. Let $\alpha_0 = -1$ for simplicity. Note that $\pi(g_{\alpha_j}) = \pi_{i(\alpha_j)}(g_{\alpha_j}) = w(i(\alpha_j))$ and $\pi(A_j) = \pi_{i(\alpha_j)}(A_j)$ for every $1 \leq j \leq n$, so we have

$$\pi(p) = \langle (w(i(\alpha_1)), \pi_{i(\alpha_1)}(A_1)), \cdots, (w(i(\alpha_n)), \pi_{i(\alpha_n)}(A_n))\rangle.$$

Then by the definition of H, we only need to prove that for every $m \leq n$, $w(\theta) \in \pi_{i(\alpha_{m+1})}(A_{m+1})$ for every θ with $i(\alpha_m) < \theta < i(\alpha_{m+1})$. Take any such an m and a θ. Since $p \in G$, we have

$$\{g_\eta \mid \alpha_m < \eta < \alpha_{m+1}\} \subseteq A_{m+1}. \tag{7}$$

There are two cases:

Case 1. $\theta = i([\theta])$. Then $[\theta] < \alpha_{m+1}$, and $g_{[\theta]} \in A_{m+1}$. Since $\pi_{i(\alpha_{m+1})}$ is elementary,

$$w(\theta) = \pi_{i([\theta])}(g_{[\theta]}) = \pi_{i(\alpha_{m+1})}(g_{[\theta]}) \in \pi_{i(\alpha_{m+1})}(A_{m+1}).$$

Case 2. $s([\theta]) \leq \theta < i([\theta])$. Then $[\theta]$ is a limit ordinal. By (7) and the characterization of genericity, i.e., Theorem 2.9, $A_{m+1} \cap V[G]_{\kappa(g_{[\theta]})} \in \mathcal{F}(g_{[\theta]})$. Since π_θ is elementary, it follows that $\pi_\theta(A_{m+1} \cap V[G]_{\kappa(g_{[\theta]})}) \in \mathcal{F}(\pi_\theta(g_{[\theta]}))$. Note also that $\beta_\theta < \beta_{i([\theta])}$, by Lemma 2.11 (for the case $\theta = s(\delta)$, since the measure sequence $\pi_{s(\delta)}(u)$ is obtained from $\pi_{s(\delta),s(\delta)+1}$, this lemma is also true), we have

$$w(\theta) \in \pi_{i([\theta])}(A_{m+1} \cap V[G]_{\kappa(g_{[\theta]})}) = \pi_{i(\alpha_{m+1})}(A_{m+1} \cap V[G]_{\kappa(g_{[\theta]})}) \subseteq \pi_{i(\alpha_{m+1})}(A_{m+1}).$$

Hence in any case, $w(\theta) \in \pi_{i(\alpha_{m+1})}(A_{m+1})$. So $\pi''G \subseteq H$, and therefore, we may lift π and obtain an elementary embedding $\pi^+ : V[G] \to N[H]$ by using Silver's criterion.

Let $D = \pi_{s(\delta)+1,i(\delta)}(\pi''_{s(\delta),s(\delta)+1}(\pi_{s(\delta)}(\kappa')))$. Since $\pi_{s(\delta),s(\delta)+1}$ witnesses that $\pi_{s(\delta)}(\kappa)$ is $\pi_{s(\delta)}(\kappa')$-supercompact, we have $\pi''_{s(\delta),s(\delta)+1}(\pi_{s(\delta)}(\kappa')) \in N_{s(\delta)+1}$. Note also that $\pi_{s(\delta)+1,i(\delta)}$ is elementary, we have $D = \pi_{s(\delta)+1,i(\delta)}(\pi''_{s(\delta),s(\delta)+1}(\pi_{s(\delta)}(\kappa'))) \in N$. Meanwhile, $\pi''\kappa' \subseteq D$ and $N \models |D| < \pi(\kappa)$. So π^+ witnesses κ is λ-strongly compact up to κ'.

Since $\mathcal{S}$ is a proper class and $\kappa' \in \mathcal{S}$ is arbitrary, κ is λ-strongly compact in $V[G]$.

Now by Fact 4.1, we can see that κ is the least λ-strongly compact cardinal in $V[G]$ (actually, κ is the least ω_1-strongly compact cardinal).

This concludes the proof of Theorem 4.2. □

Corollary 4.6. *Suppose $\lambda < \kappa$ are supercompact cardinals. Then for every regular cardinal δ with $\lambda \leq \delta < \kappa$, there exists a generic extension of V, in which κ is the least λ-strongly compact cardinal and has cofinality δ.*

Proof. Since λ is a supercompact cardinal and δ is regular, it follows that δ is λ-measurable. By Theorem 4.2, in some Radin generic extension that adds a Radin generic sequence of length δ, κ is the least λ-strongly compact cardinal and has cofinality δ. □

The following proposition generalizes [1, Theorem 2.3] and shows that our consistency result (Theorem 4.2) is optimal.

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Proposition 4.7. *Suppose κ is the least λ-strongly compact cardinal and has cofinality δ. Then δ is λ-measurable. Namely, δ carries a λ-complete uniform ultrafilter over δ.*

Proof. We first prove that there exists a definable elementary embedding $j : V \to M$ with M transitive such that $j(\kappa) > \sup(j''\kappa)$. Since κ is the least λ-strongly compact cardinal, it follows that κ is a limit cardinal by Theorem 2.3, and for every $\gamma < \kappa$, there is an $\alpha_\gamma > \kappa$ such that γ is not λ-strongly compact up to α_γ. Let $\alpha = \sup(\{\alpha_\gamma \mid \gamma < \kappa\})^+$. Then η is not λ-strongly compact up to α for every $\eta < \kappa$.

By the λ-strong compactness of κ, there is a definable elementary embedding $j : V \to M$ with M transitive, so that $\mathrm{crit}(j) \geq \lambda$ and there is a $D \in M$ such that $j''\alpha \subseteq D$ and $M \models |D| < j(\kappa)$. If $\sup(j''\kappa) = j(\kappa)$, then there is a cardinal $\eta < \kappa$ such that $M \models |D| < j(\eta)$. Thus j witnesses that η is λ-strongly compact up to α, a contradiction. Hence, $\sup(j''\kappa) < j(\kappa)$.

For such an embedding j, we claim that $\sup(j''\delta) < j(\delta)$. If not, $j(\delta) = \sup(j''\delta)$. Since κ is a limit cardinal and has cofinality δ, there is an increasing cofinal sequence $\vec{\kappa} = \langle \kappa_\alpha \mid \alpha < \delta \rangle$ of cardinals converging to κ. By elementarity, $j(\vec{\kappa})$ is an increasing cofinal sequence on $j(\kappa)$ in M. Then in V, $\langle j(\vec{\kappa})(j(\alpha)) \mid \alpha < \delta \rangle$ is an increasing cofinal sequence on $j(\kappa)$, since $j(\delta) = \sup(j''\delta)$. By elementarity, $j(\vec{\kappa})(j(\alpha)) = j(\vec{\kappa}(\alpha)) = j(\kappa_\alpha)$. So $\langle j(\kappa_\alpha) \mid \alpha < \delta \rangle$ is an increasing cofinal sequence on $j(\kappa)$, which means that $j(\kappa) = \sup(j''\kappa)$, a contradiction. Therefore, $j(\delta) > \sup(j''\delta)$.

Hence, δ carries a λ-complete uniform ultrafilter by the proof of Proposition 2.2, and it is also λ-measurable. □

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