Diversities and the Generalized Circumradius

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Abstract

The *generalized circumradius* of a set of points $A \subseteq \mathbb{R}^d$ with respect to a convex body K equals the minimum value of $\lambda \geq 0$ such that a translate of λK contains A. Each choice of K gives a different function on the set of bounded subsets of $\mathbb{R}^d$; we characterize which functions can arise in this way. Our characterization draws on the theory of *diversities*, a recently introduced generalization of metrics from functions on pairs to functions on finite subsets. We additionally investigate functions which arise by restricting the generalized circumradius to a finite subset of $\mathbb{R}^d$. We obtain elegant characterizations in the case that K is a simplex or parallelotope.

Keywords Diversity · Generalized circumradius · Generalized Minkowski spaces · Metric geometry · Convex geometry

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The *circumradius* of a set of points in the plane is the radius of the smallest circle containing them. The concept is key to optimal containment and facility location

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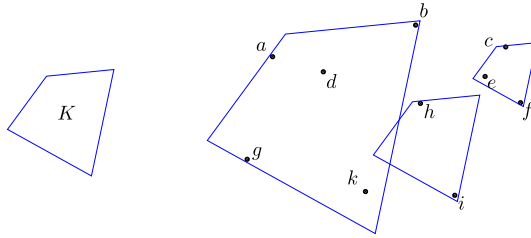K.Huber@uea.ac.uk

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Fig. 1 An example of the generalized circumradius. In this example $R(\{a, b, d, g, k\}, K) = 2$ since $2K$ is the smallest scaled version of K which can be translated to cover $\{a, b, d, g, k\}$. Similarly, $R(\{c, e, f\}, K) = 0.6$, since $0.6\,K$ is the smallest scaled version of K which can be translated to cover $\{c, e, f\}$, and $R(\{h, i\}, K) = 1$

problems, including a classic problem studied by Sylvester [21, 32], since the center of the smallest enclosing circle minimizes the maximum distance to any of the points.

The *generalized circumradius* replaces the plane with $\mathbb{R}^d$ and the circle or ball with a general *convex body*, that is a compact, convex set with non-empty interior. For a convex body K in $\mathbb{R}^d$ and bounded $A \subseteq \mathbb{R}^d$ we say that the generalized circumradius of A with respect to K is

$$R(A, K) = \inf \{\lambda \geq 0 : A \subseteq \lambda K + x \text{ for some } x \in \mathbb{R}^d\}.$$

In other words, $R(A, K)$ equals the minimal amount that K must be scaled so that a translate covers A (see Fig. 1). The set K is called the *kernel*. See [2, 3, 16, 22] for properties and inequalities related to the generalized circumradius and [4, 17–19] for computational results.

Our motivation for studying the generalized circumradius comes from connections with *diversity theory*. A (mathematical) diversity is an extension of the idea of a metric space. Instead of assigning values just to pairs of objects, a diversity assigns values to all finite sets of objects. More formally, a *diversity* is a pair (X, δ) where X is a set and δ a function from finite subsets of X to $\mathbb{R}_{\geq 0}$ such that, for A, B, C finite,

(D1) $\delta(A) = 0$ if and only if $|A| \leq 1$;
(D2) if $B \neq \emptyset$ then $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$.

Diversities were introduced in [9]. A consequence of (D1) and (D2) is that diversities are *monotonic*, that is, if $A \subseteq B$ then $\delta(A) \leq \delta(B)$. Furthermore, δ restricted to pairs satisfies the definition of a metric; we call this the metric *induced* by (X, δ). We say that a diversity (X, δ) is *finite* if X is finite. Note that on occasion we use the term 'diversity' to refer to the function δ rather than the pair (X, δ).

Many well-known functions on sets are diversities. Examples include

1. the diameter of a set,
2. the length of a shortest Steiner tree connecting a set,
3. the mean width of a set,
4. the length of the shortest traveling salesman tour through a set,
5. the L_1 diversity (in $\mathbb{R}^d$),
$$\delta(A) = \delta_1(A) = \sum_{i=1}^{d} \max \{|a_i - a_i'| : a, a' \in A\},$$

  6. the circumradius of a set.

We show that the generalized circumradius is also a diversity:

Theorem 2.1 *Let K be a convex body in $\mathbb{R}^d$. If we define $\delta(A) = R(A, K)$ for all finite A then $(\mathbb{R}^d, \delta)$ is a diversity.*

In reference to the concept of a *Minkowski norm* we say that a diversity $(\mathbb{R}^d, \delta)$ is a *Minkowski diversity* if there is a convex body K such that $\delta(A) = R(A, K)$ for all finite $A \subset \mathbb{R}^d$.

Theorem 2.1 connects the generalized circumradius to a growing and varied literature on diversity theory. The first diversity paper [9] described diversity analogs to the metric tight span and metric hyperconvexity, leading to new results in analysis and fixed point theory; see [23, Chapter 15] and [26, 27]. It was shown in [10] that results of [25] and others on embedding of finite metrics in L_1 can be extended to diversities, with potential algorithmic gains. There is a direct analog of the Urysohn metric space [7] for diversities and work on diversity theory within model theory [8, 20], lattice theory [6, 33], image analysis [15], machine learning [24], and phylogenetics [5, 9, 30] and [31, Chapter 6].

In this paper we are mainly concerned with characterizations and embeddings for Minkowski diversities—what characterizes these diversities and which finite diversities can be embedded into a Minkowski diversity. Such embeddings (possibly with distortion) should in future provide valuable graphical representations of diversities in addition to algorithmic and computational tools.

Regarding characterization we prove the following result. A real-valued function f on subsets of $\mathbb{R}^d$ is said to be *sublinear* if

$$f(A + B) \le f(A) + f(B) \qquad \text{for all } A, B,$$
$$f(\lambda A) = \lambda f(A) \qquad \text{for all } A \text{ and } \lambda \ge 0,$$

where $A + B$ denotes the Minkowski sum.

Theorem 2.4 *Let $(\mathbb{R}^d, \delta)$ be a diversity. Then δ is a Minkowski diversity if and only if δ is sublinear and for all finite A, B there are $a, b \in \mathbb{R}^d$ such that*

$$\delta((A + a) \cup (B + b)) \le \max \{\delta(A), \delta(B)\}.$$

We also show that the last result extends beyond diversities to functions defined on bounded subsets of $\mathbb{R}^d$.

Theorem 3.2 *Let f be a function on bounded subsets of $\mathbb{R}^d$. Then there is a convex body K such that $f(A) = R(A, K)$ for all bounded A if and only if f is sublinear, monotonic, and f restricted to finite subsets is a Minkowski diversity.*

Having characterized which diversities on $\mathbb{R}^d$ are Minkowski diversities, we turn to the more difficult problem of characterizing which diversities can be isometrically embedded into Minkowski diversities. An *isometric embedding* of a diversity (X, δ_1) into a diversity $(\mathbb{R}^d, \delta_2)$ is a map $\phi \colon X \to \mathbb{R}^d$ such that $\delta_1(A) = \delta_2(\phi(A))$ for all finite

$A \subseteq X$. If there is an isometric embedding from a diversity (X, δ_1) into a Minkowski diversity $(\mathbb{R}^d, \delta_K)$ for some d and some convex body $K \subseteq \mathbb{R}^d$ then we say that (X, δ_1) is *Minkowski-embeddable*.

We provide a complete characterization of Minkowski-embeddability for diversities which are finite and *symmetric*, meaning that the diversity of a set is determined by a function of its cardinality.

Theorem 4.3 *Let (X, δ) be a finite symmetric diversity. Then (X, δ) is Minkowski-embeddable if and only if*

$$\frac{\delta(A \setminus \{a\})}{\delta(A)} \geq \frac{|A| - 2}{|A| - 1} \tag{1}$$

for all $A \subseteq X$ with $|A| \geq 2$, $a \in A$.

A consequence of the theorem is that, even though any diversity on three elements is Minkowski-embeddable, there exist diversities on four elements which are *not*.

We then investigate which finite diversities (X, δ_1) can be embedded into the diversity $(\mathbb{R}^d, \delta_2)$ when δ_2 is a Minkowski diversity with kernel K restricted to a particular class. A diversity (X, δ) is a *diameter diversity* if $\delta(A) = \max \{\delta(\{a, a'\}) : a, a' \in A\}$ for all finite $A \subseteq X$, see [9]. The following characterization for when K is a cube (or non-degenerate transform of a cube) follows from an observation of [2].

Theorem 5.1 *A finite diversity (X, δ) can be embedded in a Minkowski diversity with kernel equal to some parallelotope if and only if (X, δ) is a diameter diversity.*

The case when K is a simplex is more complex. We say that a finite diversity (X, δ) is of *negative type* if

$$\sum_{A \neq \emptyset} \sum_{B \neq \emptyset} x_A x_B \delta(A \cup B) \leq 0$$

for all zero-sum vectors x indexed by non-empty subsets of X. Diversities of negative type are analogous to metrics of negative type, and several of the properties of metrics of negative type extend to diversities of negative type, see [35]. For negative-type diversities we prove the following characterization.

Theorem 5.2 *A finite diversity (X, δ) can be embedded in a Minkowski diversity with kernel equal to some simplex if and only if (X, δ) has negative type.*

Significantly, the set of diversities with negative type is quite large, with the same dimension as the set of diversities on X. The result shows that the class of Minkowski-embeddable diversities is even larger, potentially opening up possibilities for quite general theoretical and algorithmic results.

1 The Generalized Circumradius

In this section we collect together a number of fundamental results about the generalized circumradius. We begin with several observations from [21, Proposition 3.2]. Let conv(A) denote the convex hull of A.

Proposition 1.1 *Let A, A', B be bounded subsets of $\mathbb{R}^d$, let K, K' be convex bodies in $\mathbb{R}^d$. Then*

(a) *if $A \subseteq A'$ and $K' \subseteq K$ then $R(A, K) \leq R(A', K')$;*
(b) *if $\mathrm{conv}(A) = \mathrm{conv}(A')$ then $R(A, K) = R(A', K)$;*
(c) *$R(A + B, K) \leq R(A, K) + R(B, K)$ with equality if $B = \alpha K$ for some $\alpha \geq 0$;*
(d) *$R(A + x, K + y) = R(A, K)$ for all $x, y \in \mathbb{R}^d$;*
(e) *$R(\alpha A, \beta K) = (\alpha/\beta)R(A, K)$ for all $\alpha, \beta > 0$.*

An indirect consequence of Helly's theorem (see e.g. [13, Sect. 6.2]) is that for bounded $A \subseteq \mathbb{R}^d$ and a convex body K we can find a small subset $A' \subseteq A$ such that $|A'| \leq d + 1$ and $R(A', K) = R(A, K)$. The following more general result forms one part of [2, Theorem 1.2].

Proposition 1.2 *Suppose that $A \subset \mathbb{R}^d$ is bounded and K is a convex body. For all $\epsilon \geq 0$ there exists $A' \subseteq A$ such that $|A'| \leq \lceil d/(1 + \epsilon) \rceil + 1$ and*

$$R(A', K) \leq R(A, K) \leq (1 + \epsilon)R(A', K).$$

Note that for particular choices of K there can be much smaller bounds on $|A'|$. For example, when K is the Euclidean ball in $\mathbb{R}^d$ we have for all bounded $A \subset \mathbb{R}^d$ and $\epsilon > 0$ that there is a subset $A' \subseteq A$ such that $R(A, K) \leq (1 + \epsilon) R(A', K)$ and

$$|A'| \leq \left\lceil \frac{1}{2\epsilon + \epsilon^2} \right\rceil + 1.$$

This bound is independent of the dimension d.

We will make use of the following general property for Minkowski addition which is established during the proof of [2, Theorem 4.1]. Let $A \subseteq \mathbb{R}^d$ be any set with cardinality $k + 1$, $k \geq 2$, and zero sum. Then

$$A \subseteq \frac{k}{(k + 1)(k - 1)} \sum_{a \in A} \mathrm{conv}\,(A \setminus \{a\}).$$

Combining this observation with (d), (b), and (c) of Proposition 1.1, we have

Proposition 1.3 *Suppose $A \subset \mathbb{R}^d$, $|A| = k + 1$, and K is a convex body. Then*

$$R(A, K) \leq \frac{k}{(k + 1)(k - 1)} \sum_{a \in A} R(A \setminus \{a\}, K).$$

For $C \subseteq \mathbb{R}^p \times \mathbb{R}^q$ we define the two projection operators

$$\pi_1(C) = \{c_1 \in \mathbb{R}^p : (c_1, c_2) \in C \text{ for some } c_2\},$$
$$\pi_2(C) = \{c_2 \in \mathbb{R}^q : (c_1, c_2) \in C \text{ for some } c_1\}.$$

The following result will be useful for questions regarding embeddings.

Proposition 1.4 *Let A be a bounded subset of $\mathbb{R}^p \times \mathbb{R}^q$. If B and C are convex bodies in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, then*

$$R(A, B \times C) = \max \{R(\pi_1(A), B), R(\pi_2(A), C)\}.$$

Proof Suppose $\lambda = R(A, B \times C)$ and that

$$A + (a_1, a_2) \subseteq \lambda(B \times C).$$

Applying π_1 and π_2 to both sides gives

$$\pi_1(A) + a_1 \subseteq \lambda B \quad \text{and} \quad \pi_2(A) + a_2 \subseteq \lambda C.$$

Hence $\max \{R(\pi_1(A), B), R(\pi_2(A), C)\} \leq R(A, B \times C)$. Conversely, if $\lambda = \max \{R(\pi_1(A), B), R(\pi_2(A), C)\}$, then there are a_1, a_2 such that

$$\pi_1(A) + a_1 \subseteq \lambda B, \qquad \pi_2(A) + a_2 \subseteq \lambda C.$$

We then have

$$
\begin{aligned}
A + (a_1, a_2) &\subseteq \pi_1(A) \times \pi_2(A) + (a_1, a_2) \\
&= (\pi_1(A) + a_1) \times (\pi_2(A) + a_2) \subseteq \lambda(B \times C),
\end{aligned}
$$

so that $R(A, B \times C) \leq \max \{R(\pi_1(A), B), R(\pi_2(A), C)\}$. □

2 Characterization of Minkowski Diversities

We begin this section by proving the first main result connecting the theory of generalized circumradii with diversity theory.

Theorem 2.1 *Let K be a convex body in $\mathbb{R}^d$. If $\delta(A) = R(A, K)$ for all finite A, then $(\mathbb{R}^d, \delta)$ is a diversity.*

Proof Clearly $\delta(A) \geq 0$ for all A and $\delta(A) = R(A, K) = 0$ if and only if $|A| \leq 1$. Hence (D1) holds. By Proposition 1.1 (a), $R(A, K)$ is monotonic in A. Suppose A, B are finite subsets of $\mathbb{R}^d$ and $x \in A \cap B$. Then $A - x = (A - x) + 0 \subseteq (A - x) + (B - x)$ and $B - x = 0 + (B - x) \subseteq (A - x) + (B - x)$. Hence

$$(A - x) \cup (B - x) \subseteq (A - x) + (B - x),$$

and so by Proposition 1.1, (d) and (c),

$$
\begin{aligned}
\delta(A \cup B) = \delta((A \cup B) - x) &= \delta((A - x) \cup (B - x)) \\
&\leq \delta(A - x) + \delta(B - x) = \delta(A) + \delta(B).
\end{aligned}
$$

This fact, together with monotonicity, implies the diversity triangle inequality (D2).

□

In the rest of this section we focus on characterizing which diversities on $\mathbb{R}^d$ can be obtained in this way (Theorem 2.4). Let $\mathfrak{B}$ denote the Euclidean unit ball in $\mathbb{R}^d$. The *Hausdorff distance* on (bounded) subsets of $\mathbb{R}^d$ is given by

$$d_{\mathrm{H}}(A, B) = \inf\{\lambda : A \subseteq B + \lambda\mathfrak{B} \text{ and } B \subseteq A + \lambda\mathfrak{B}\},$$

see [29]. We note that d_{H} becomes a metric when restricted to compact sets.

A diversity $(\mathbb{R}^d, \delta)$ is *sublinear* if δ is a sublinear function, $\delta(A + B) \le \delta(A) + \delta(B)$ and $\delta(\lambda A) = \lambda\delta(A)$ for all finite A, B and non-negative λ. Note that finite sublinear diversities are closely related to *set-norms* [12, Definition 2.1]. The L_1 diversity $(\mathbb{R}^d, \delta_1)$, as introduced earlier, is an example of a sublinear diversity. Here,

$$\delta_1(A) = \sum_{i=1}^{d} \max\{|a_i - a_i'| : a, a' \in A\},$$

and δ_1 is sublinear since, given $\lambda \ge 0$ and finite subsets $A, B \subseteq \mathbb{R}^d$ we have

$$\delta_1(\lambda A) = \sum_{i=1}^{d} \max\{|\lambda a_i - \lambda a_i'| : a, a' \in A\} = \lambda\delta_1(A)$$

and

$$\delta_1(A + B) = \sum_{i=1}^{d} \max\{|a_i + b_i - a_i' - b_i'| : a, a' \in A, b, b' \in B\}$$

$$\le \sum_{i=1}^{d} \max\{|a_i - a_i'| : a, a' \in A\} + \max\{|b_i - b_i'| : b, b' \in B\}$$

$$= \delta_1(A) + \delta_1(B).$$

Sublinearity has several important consequences for diversities.

Proposition 2.2 *Let* $(\mathbb{R}^d, \delta)$ *be a sublinear diversity. Then*

(a) δ *is translation invariant:* $\delta(A + x) = \delta(A)$ *for all* x;
(b) δ *is determined by the convex hull: if* A, B *finite in* $\mathbb{R}^d$ *and* $\mathrm{conv}(A) = \mathrm{conv}(B)$, *then* $\delta(A) = \delta(B)$;
(c) δ *is Lipschitz continuous with respect to the Hausdorff metric.*

Proof (a) By (D1) and sublinearity we have

$$\delta(A + x) \le \delta(A) + \delta(\{x\}) = \delta(A) \quad \text{and}$$
$$\delta(A) \le \delta(A + x) + \delta(\{-x\}) = \delta(A + x).$$

(b) Let $B' \subseteq B$ be a maximal subset of B such that $\delta(A \cup B') = \delta(A)$. Suppose that there is $b \in B \setminus B'$. As $b \in \text{conv}(B) = \text{conv}(A)$ there are non-negative $\{\lambda_a\}_{a \in A}$ with unit sum such that

$$b = \sum_a \lambda_a a$$

and hence

$$b \in \sum_a \lambda_a A.$$

As

$$A \cup B' \subseteq \sum_a \lambda_a (A \cup B'),$$

we have

$$\delta(A \cup B' \cup \{b\}) \leq \delta\left(\sum_a \lambda_a (A \cup B')\right) \leq \sum_a \lambda_a \delta(A \cup B') = \delta(A \cup B') = \delta(A).$$

This contradicts the choice of B'. Hence $B' = B$ and $\delta(A \cup B) = \delta(A)$. Exchanging A and B in this argument gives $\delta(A \cup B) = \delta(B)$. Hence $\delta(A) = \delta(B)$.

(c) Let V denote any finite set with convex hull $\text{conv}(V)$ containing the unit ball $\mathfrak{B}$. One example is $V = \{-1, 1\}^d$. Let $\kappa = \delta(V)$ and suppose that $d_{\mathrm{H}}(A, B) = \lambda$. From the definition of d_{H} we have $A \subseteq B + \lambda\mathfrak{B}$ and $B \subseteq A + \lambda\mathfrak{B}$ so there are finite sets $A', B' \subseteq \mathfrak{B}$ such that

$$A \subseteq B + \lambda A' \quad \text{and} \quad B \subseteq A + \lambda B'.$$

As $\mathfrak{B} \subseteq \text{conv}(V)$, we have $\text{conv}(A' \cup V) = \text{conv}(B' \cup V) = \text{conv}(V)$. By sublinearity, monotonicity of δ and part (b) we therefore have

$$\delta(A) \leq \delta(B) + \lambda\delta(V) \quad \text{and} \quad \delta(B) \leq \delta(A) + \lambda\delta(V)$$

giving $|\delta(A) - \delta(B)| \leq \kappa d_{\mathrm{H}}(A, B)$. $\qquad \square$

Our next theorem gives a complete characterization of Minkowski diversities. A key idea in the proof of the theorem is that we can extend a sublinear diversity $(\mathbb{R}^d, \delta)$ to a function on convex bodies in $\mathbb{R}^d$. More specifically, given a sublinear diversity δ, define the function $\tilde{\delta}$ on the set of convex bodies in $\mathbb{R}^d$ by setting

$$\tilde{\delta}(P) = \delta(\text{Vert}(P))$$

for all polytopes P with vertex set $\mathrm{Vert}(P)$ and

$$\tilde{\delta}(K) = \lim_{n\to\infty} \tilde{\delta}(P_n)$$

for any convex body K and sequence $P_1, P_2, \ldots$ of polytopes converging under the Hausdorff metric to K.

Lemma 2.3 *Given a sublinear diversity δ, the function $\tilde{\delta}$ is well defined and Lipschitz continuous.*

Proof Since $\tilde{\delta}$ is Lipschitz on the set of polytopes, it is uniformly continuous on the same set, and so can be uniquely extended to a continuous function on the closure of that set [28, Problem 13, Chap. 4], the convex bodies. An expression for $\tilde{\delta}$ can then be obtain for convex bodies using the limits of sequences of polytopes as above, and this gives that the extension has the same Lipschitz constant. □

Using this observation, we now prove the main theorem for this section.

Theorem 2.4 *Let $(\mathbb{R}^d, \delta)$ be a diversity. Then $(\mathbb{R}^d, \delta)$ is a Minkowski diversity if and only if δ is sublinear and for all finite A, B there are $a, b \in \mathbb{R}^d$ such that*

$$\delta((A + a) \cup (B + b)) \leq \max\{\delta(A), \delta(B)\}. \tag{2}$$

Proof Suppose that $(\mathbb{R}^d, \delta)$ is a Minkowski diversity, so that there is a convex body $K \subseteq \mathbb{R}^d$ such that $\delta(A) = R(A, K)$ for all finite $A \subseteq \mathbb{R}^d$. Sublinearity is given by Proposition 1.1, parts (c) and (e). Given finite A and B, there are $a, b \in \mathbb{R}^d$ such that $A \subseteq \delta(A)\, K - a$ and $B \subseteq \delta(B)\, K - b$. Hence

$$(A + a) \cup (B + b) \subseteq \max\{\delta(A), \delta(B)\}\, K$$

and

$$\begin{aligned}
\delta((A + a) \cup (B + b)) &= R((A + a) \cup (B + b), K) \\
&\leq R(\max\{\delta(A), \delta(B)\}K, K) \\
&= \max\{\delta(A), \delta(B)\}\, R(K, K) = \max\{\delta(A), \delta(B)\}.
\end{aligned}$$

Now suppose that $(\mathbb{R}^d, \delta)$ is sublinear and satisfies (2) for all finite A, B. Let

$$\rho = \max\left\{\frac{\|x\|}{\delta(\{0, x\})} : x \neq 0\right\} = \max\left\{\frac{1}{\delta(\{0, x\})} : \|x\| = 1\right\}.$$

Then $0 < \rho < \infty$ and $\|x - y\| \leq \rho\delta(\{x, y\})$ for all x, y. (Here and below we use $\|\cdot\|$ to denote the Euclidean norm).

We show that for any convex bodies $L, K \subseteq \mathbb{R}^d$ and the function $\tilde{\delta}$ in Lemma 2.3 there is some $x \in \mathbb{R}^d$ such that

$$\tilde{\delta}(\mathrm{conv}(K \cup (L + x))) \leq \max\{\tilde{\delta}(K), \tilde{\delta}(L)\}. \tag{3}$$

Suppose that $K_1, K_2, \ldots$ is a sequence of finite subsets of K such that $\mathrm{conv}(K_n) \to K$ and $L_1, L_2, \ldots$ is a sequence of finite subsets of L such that $\mathrm{conv}(L_n) \to L$. Applying (2) and translation invariance of δ we have that for each $n \in \mathbb{N}$ there is an $x_n \in \mathbb{R}^d$ such that

$$\delta(K_n \cup (L_n + x_n)) \leq \max\{\delta(K_n), \delta(L_n)\}.$$

Hence, for all n,

$$\tilde{\delta}(\mathrm{conv}\,(K_n \cup (L_n + x_n))) \leq \max\{\tilde{\delta}(\mathrm{conv}(K_n)), \tilde{\delta}(\mathrm{conv}(L_n))\}. \tag{4}$$

We show that the sequence x_n has a convergent subsequence. First note that since K and L are bounded, convergence of $\mathrm{conv}(K_n)$ and $\mathrm{conv}(L_n)$ to them in the Hausdorff metric implies that the union of all these sets is bounded, and hence the sequence $\max\{\delta(K_n), \delta(L_n)\}$ is bounded. Choose $k_n \in K_n$ and $\ell_n \in L_n$ for each n, which are then bounded over all n. We then have

$$\|x_n - (k_n - \ell_n)\| = \|(x_n + \ell_n) - k_n\| \leq \rho\delta(K_n \cup (L_n + x_n)) \leq \max\{\delta(K_n), \delta(L_n)\}.$$

So the set $\{x_1, x_2, x_3, \ldots\}$ is bounded. Let $x_{i_1}, x_{i_2}, \ldots$ be a convergence subsequence and let $x \in \mathbb{R}^d$ be its limit. Since $\mathrm{conv}(L_n) \to L$ and $x_n \to x$, $\mathrm{conv}\,(L_n + x_n) \to L + x$. So, $\mathrm{conv}(K_n) \cup \mathrm{conv}\,(L_n + x_n) \to K \cup (L + x)$ and

$$\mathrm{conv}\,(K_n \cup (L_n + x_n)) = \mathrm{conv}\,(\mathrm{conv}(K_n) \cup \mathrm{conv}(L_n + x_n)) \to \mathrm{conv}\,(K \cup (L + x)).$$

Taking the limit as $n \to \infty$ of (4) and using the continuity of $\tilde{\delta}$ gives (3), which proves the claim.

Let $\mathfrak{C}$ denote the set of convex bodies

$$\mathfrak{C} = \{A \subseteq \mathbb{R}^d : \tilde{\delta}(A) \leq 1\}.$$

The set $\mathfrak{C}$ is closed under the Hausdorff metric and both volume and $\tilde{\delta}$ are continuous with respect to the Hausdorff metric [29, Sect. 1.8]. It follows that there is some $K \in \mathfrak{C}$ such that the volume of K is at least as large as any other element in $\mathfrak{C}$. The convex body K is necessarily inclusion-maximum: if K was a proper subset of some $K' \in \mathfrak{C}$ then the volume of K' would be strictly greater than the volume of K.

We claim that $R(A, K) = 1$ for all finite A such that $\delta(A) = 1$. Take such an A. By (3) there is $x \in \mathbb{R}^d$ such that

$$\tilde{\delta}(\mathrm{conv}\,(K \cup (\mathrm{conv}(A) + x))) \leq \max\{\tilde{\delta}(K), \tilde{\delta}(\mathrm{conv}(A))\} = 1.$$

We therefore have $\mathrm{conv}\,(K \cup (\mathrm{conv}(A) + x)) \in \mathfrak{C}$. As K is set inclusion-maximum in $\mathfrak{C}$ and $K \subseteq \mathrm{conv}\,(K \cup (\mathrm{conv}(A) + x))$ we have

$$K = \mathrm{conv}\,(K \cup (\mathrm{conv}(A) + x))$$
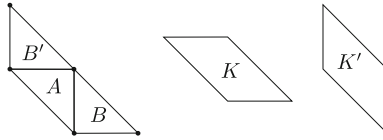
Fig. 2 The sets A, B, B' in the proof of Proposition 2.5 with (translated) kernels K and K'

and so $A + x \subseteq K$ and $R(A, K) \leq 1$. On the other hand, if there is some $\lambda \geq 0$ and $z \in \mathbb{R}^d$ such that $A \subseteq \lambda K + z$ then

$$\delta(A) = \tilde{\delta}(\text{conv}(A)) \leq \tilde{\delta}(\lambda K + z) = \lambda \tilde{\delta}(K) = \lambda,$$

showing that $R(A, K) \geq 1$. Hence $R(A, K) = 1$, as claimed. It follows that $\delta(A) = R(A, K)$ when $\delta(A) = 1$.

More generally, suppose $\delta(A) = d$. The case $d = 0$ is straightforward, as then $|A| \leq 1$. If $d > 0$ then, by sublinearity, $\delta(A) = d\delta((1/d)A) = dR((1/d)A, K) = R(A, K)$. □

We note that there are diversities on $\mathbb{R}^d$ which are sublinear, but do not satisfy property (2) in Theorem 2.4 (and are hence not Minkowski diversities). For example, consider the L_1 diversity in the plane $(\mathbb{R}^2, \delta_1)$. We saw above that L_1 diversities are sublinear but if

$$A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

then for any $a, b \in \mathbb{R}^2$ we have

$$\delta_1((A + a) \cup (B + b)) \geq 2 > 1 = \max \{\delta_1(A), \delta_1(B)\}.$$

The set of diversities on $\mathbb{R}^d$, and indeed the the set of sublinear diversities on $\mathbb{R}^d$, are both convex. Condition (2) in Theorem 2.4 suggests that the set of Minkowski diversities on $\mathbb{R}^d$ is not convex, as we now confirm.

Proposition 2.5 *The set of Minkowski diversities on $\mathbb{R}^d$, $d \geq 2$, is not convex.*

Proof We first establish this for $d = 2$. Let

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad B' = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}.$$

Let $K = \text{conv}(A \cup B)$ and $K' = \text{conv}(A \cup B')$. Let (X, δ) be the diversity on $\mathbb{R}^2$ given by $\delta(Y) = R(Y, K)/2 + R(Y, K')/2$. We will show that for all $a, b \in \mathbb{R}^2$

$$\delta((A + a) \cup (B + b)) > \max \{\delta(A), \delta(B)\},$$

from which it follows by Theorem 2.4 that (X, δ) is not a Minkowski diversity (Fig. 2).

First note that since δ is translation invariant, we can assume $a = 0$. Now, note that

$$R(A, K) = R(A, K') = R(B, K) = R(B, K') = 1,$$

so $\max\{\delta(A), \delta(B)\} = 1$. If $b = 0$ then $R(A \cup (B + b), K) = 1$, otherwise $R(A \cup (B + b), K) > 1$. If

$$b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

then $R(A \cup (B + b), K') = 1$, otherwise $R(A \cup (B + b), K') > 1$. Hence we have

$$\delta((A + a) \cup (B + b)) = \frac{R(A \cup (B + b), K)}{2} + \frac{R(A \cup (B + b), K')}{2} > 1,$$

even though $\max\{\delta(A), \delta(B)\} = 1$.

For $d > 2$, in the above argument we replace K and K' with their product with the unit hypercube in $\mathbb{R}^{d-2}$, and append $d - 2$ zeros to the elements in A and B and to x. □

3 Characterizing the General Circumradius

In this section, we characterize functions f for which there is a convex body K such that $f(A) = R(A, K)$ for all bounded A, noting that in the previous section we only considered finite A. The main idea behind the proof is to show that a sublinear, monotonic function is Hausdorff continuous, after which the result follows almost immediately from Theorem 2.4.

Lemma 3.1 *Let f be a sublinear, monotonic function on bounded subsets of $\mathbb{R}^d$. Then f is Hausdorff continuous.*

Proof Let $\mathfrak{B}$ denote the Euclidean unit ball in $\mathbb{R}^d$, let $b = f(\mathfrak{B})$ and let $\epsilon > 0$. Then for any bounded A, B such that $d_H(A, B) \leq \epsilon/b$ we have

$$f(A) \leq f\left(B + \frac{\epsilon}{b}\mathfrak{B}\right) \leq f(B) + \epsilon \quad \text{and}$$

$$f(B) \leq f\left(A + \frac{\epsilon}{b}\mathfrak{B}\right) \leq f(A) + \epsilon. \qquad \square$$

Theorem 3.2 *Let f be a function on bounded subsets of $\mathbb{R}^d$. Then there is a convex body K such that $f(A) = R(A, K)$ for all bounded A if and only if f is sublinear, monotonic, and f restricted to finite subsets is a Minkowski diversity.*

Proof Necessity follows from the arguments used for Theorem 2.4. Suppose that f is sublinear and monotonic, and f restricted to finite subsets is a Minkowski diversity. By Lemma 3.1, f is Hausdorff continuous. Let A be a bounded subset of $\mathbb{R}^d$. Then $f(\overline{A}) = f(A)$, where $\overline{A}$ denotes the topological closure of A.

By Theorem 2.4 there is a convex body K such that $f(B) = R(B, K)$ for any finite B. Given any natural number $n \geq 1$ there is a finite cover of $\overline{A}$ by balls of radius $1/n$. Let A_n denote the set of centers of those balls, so that $A_n \to \overline{A}$. We then have

$$f(A) = f(\overline{A}) = \lim_{n \to \infty} f(A_n) = \lim_{n \to \infty} R(A_n, K) = R(A, K). \qquad \square$$

4 Embedding Finite Diversities

In this section we consider an (isometric) embedding problem: when can a given finite diversity (X, δ) be embedded in a Minkowski diversity? There is a long history in mathematics regarding embedding of finite *metrics* into standard spaces. Perhaps best known is the characterization due to Cayley and Menger of when a finite metric can be embedded into Euclidean space [1, 14]. The theory of metric embeddings forms the basis of many methods for multi-dimensional scaling, an approximate low-dimensional embedding designed specifically for data reduction and representation. Approximate embeddings have proven exceptionally useful for algorithm design and approximations (e.g. [25]), work that has a direct analog in the mathematics of diversities [10].

To discuss embeddings, it is convenient to consider a slight generalization of diversities. A *semimetric* is a bivariate, symmetric map d on X that vanishes on the diagonal and satisfies the triangle inequality, but where we allow $d(x, y) = 0$ even when $x \neq y$, $x, y \in X$ (so, in particular a metric is a semimetric). Similarly, a pair (X, δ) is a *semidiversity* if it satisfies (D2) and the following slightly weaker version of (D1),

(D1') $\delta(A) = 0$ if $|A| \leq 1$.

We say that a (semi)diversity (X, δ) is *Minkowski-embeddable* if for some d there is a map $\phi \colon X \to \mathbb{R}^d$ and a convex body K in $\mathbb{R}^d$ such that

$$\delta(A) = R(\phi(A), K)$$

for all finite $A \subseteq X$.

For the rest of this section we shall focus on the embedding problem for symmetric diversities, where a (semi)diversity (X, δ) is *symmetric* if $\delta(A) = \delta(B)$ whenever $|A| = |B|$, $A, B \subseteq X$, that is, the value of δ on a set depends only upon the cardinality of the set, see [11]. We shall characterize when a finite symmetric diversity is Minkowski-embeddable. As a corollary we also show that not every diversity is Minkowski-embeddable.

We start with some utility results on embeddings. For convenience, for the rest of this section we shall assume that X is a finite set. Note that if (X, δ_1) and (X, δ_2) are two semidiversities then $(X, \delta_1 \vee \delta_2)$ denotes the semidiversity with

$$(\delta_1 \vee \delta_2)(A) = \max \{\delta_1(A), \delta_2(A)\}$$

for all $A \subseteq X$. To see that this is a semidiversity, note that for all A, B, C with $B \neq \emptyset$ we have without loss of generality,

$$(\delta_1 \vee \delta_2)(A \cup C) = \delta_1(A \cup C)$$

so that

$$(\delta_1 \vee \delta_2)(A \cup C) \leq \delta_1(A \cup B) + \delta_1(B \cup C) \leq (\delta_1 \vee \delta_2)(A \cup B) + (\delta_1 \vee \delta_2)(B \cup C).$$

Proposition 4.1 (a) *Let (X, δ_1) and (X, δ_2) be Minkowski-embeddable semidiversities, and $\lambda > 0$. Then both $(X, \lambda\delta_1)$ and $(X, \delta_1 \vee \delta_2)$ are Minkowski-embeddable.*
(b) *Suppose that K is a convex body in $\mathbb{R}^d$ and $\phi \colon \mathbb{R}^d \to \mathbb{R}^d$ is a non-degenerate affine map. Then for all A we have $R(\phi(A), \phi(K)) = R(A, K)$. Hence if there is an isometric embedding from (X, δ) into $(\mathbb{R}^d, \delta_K)$ then there is also an isometric embedding from (X, δ) into $(\mathbb{R}^d, \delta_{\phi(K)})$.*
(c) *If (X, δ) is Minkowski-embeddable and $|A| = k + 1$, $k \geq 2$, then*

$$\delta(A) \leq \frac{k}{(k+1)(k-1)} \sum_{a \in A} \delta(A \setminus \{a\}). \tag{5}$$

Proof (a) There are maps $\phi_1 \colon X \to \mathbb{R}^p$ and $\phi_2 \colon X \to \mathbb{R}^q$ and convex bodies $K_1 \subset \mathbb{R}^p$ and $K_2 \subset \mathbb{R}^q$ such that $\delta_1(A) = R(\phi_1(A), K_1)$ and $\delta_2(A) = R(\phi_2(A), K_2)$ for all $A \subseteq X$. Then for all $A \subseteq X$ we have

$$\lambda\delta_1(A) = R(\phi_1(A), \lambda^{-1}K_1)$$

and by Proposition 1.4,

$$(\delta_1 \vee \delta_2)(A) = R(\phi_1(A) \times \phi_2(A), K_1 \times K_2).$$

(b) If there are $\lambda \geq 0$ and x such that $A + x \subseteq \lambda K$ then $\phi(A) + \phi(x) \subseteq \lambda\phi(K)$ so $R(\phi(A), \phi(K)) \leq R(A, K)$. Applying the inverse map gives equality.

(c) There is a map $\phi \colon X \to \mathbb{R}^d$ and a convex body K such that $R(\phi(A), K) = \delta(A)$. Applying Proposition 1.3 gives the result. □

We now consider Minkowski-embeddability for a few key examples of symmetric diversities.

Proposition 4.2 (a) *The diversity (X, δ) with $\delta(A) = 1$ for all $A \subseteq X$ with $|A| > 1$ is Minkowski-embeddable.*
(b) *The diversity (X, δ) with $\delta(A) = |A| - 1$ for all non-empty $A \subseteq X$ is Minkowski-embeddable.*
(c) *Any diversity (X, δ) with $X = \{a, b, c\}$ and $\delta(\{a, b\}) = \delta(\{a, c\}) = \delta(\{b, c\}) = 1$ is Minkowski-embeddable.*

Proof (a) Let $n = |X| - 1$. Let K be the simplex with vertex set V given by the standard basis vectors in $\mathbb{R}^n$ together with 0. Then for non-singleton subset $V' \subseteq V$ we have $R(V', K) = 1$. Hence any bijection from X to V gives an isometric embedding.

(b) Let K be the same simplex as in (a) and now let V be the vertex set of $-K$. The proof of [2, Theorem 4.1] then gives $R(V', K) = |V'| - 1$ for all non-empty subsets of $V' \subseteq V$. The result follows.

(c) Let $x = \delta(\{a, b, c\})$. As (X, δ) is a diversity, $1 \leq x \leq 2$. Let (X, δ_1) be the diversity on X with $\delta_1(A) = 1$ for all non-singleton $A \subseteq X$ and let (X, δ_2) be the diversity on X with $\delta_2(A) = |A| - 1$ for all non-empty $A \subseteq X$. Then (X, δ_1) and (X, δ_2) are Minkowski-embeddable, and by Proposition 4.1 (a) so is

$$(X, \delta) = (X, \delta_1 \vee ((x/2)\delta_2)).$$ □

We now give an exact characterization for when a finite symmetric diversity is Minkowski-embeddable.

Theorem 4.3 *Let (X, δ) be a finite symmetric diversity. Then (X, δ) is Minkowski-embeddable if and only if*

$$\frac{\delta(A \setminus \{a\})}{\delta(A)} \geq \frac{|A| - 2}{|A| - 1} \tag{6}$$

for all $A \subseteq X$ with $|A| \geq 2$, $a \in A$.

Proof Suppose that (X, δ) is Minkowski-embeddable, that $A \subseteq X$, $|A| \geq 2$ and $a \in A$. If $|A| = 2$ then (6) holds trivially. If $|A| > 2$ then by Proposition 4.1 (c),

$$\delta(A) \leq \frac{|A| - 1}{|A|(|A| - 2)} \sum_{x \in A} \delta(A \setminus \{x\}) = \frac{|A| - 1}{|A|(|A| - 2)} |A| \delta(A \setminus \{a\}),$$

where the identity follows since δ is symmetric. Hence (6) holds. Conversely, suppose that (6) holds for all $A \subseteq X$ such that $|A| \geq 2$ and $a \in A$. As (X, δ) is symmetric, δ is monotonic and there is an increasing function $f : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ such that $\delta(A) = f(|A| - 1)$ for all non-empty $A \subseteq X$. Note that for each $k = 2, \ldots, |X|$, choosing an $A \subseteq X$ with $|A| = k$ and substituting A into (6) gives

$$\frac{f(k - 2)}{f(k - 1)} \geq \frac{k - 2}{k - 1}, \tag{7}$$

a fact that we shall use later on in the proof.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and for each $m \leq n$ define $X_m = \{x_1, x_2, \ldots, x_m\}$. We will use induction on m to show that for each $m \leq n$ the symmetric diversity (X, δ) restricted to X_m is Minkowski-embeddable. This is clearly the case when $m \leq 2$. Suppose that $2 < m \leq n$ and that (X, δ) restricted to X_{m-1} is Minkowski-embeddable. Then there is a map $\phi : X_{m-1} \to \mathbb{R}^d$ for some d and a convex body $K \subseteq \mathbb{R}^d$ such that

$$\delta_K(\phi(A)) = \delta(A)$$

for all $A \subseteq X_{m-1}$. We now define a collection $\delta^{(i)}$ of (semi)diversities on X_m, $1 \leq i \leq m$.

First, for each $i = 1, 2, \ldots, m-1$ define the map $\phi^{(i)} : X_m \to \mathbb{R}^d$ by $\phi^{(i)}(x) = \phi(x)$ if $x \in X_{m-1}$ and $\phi^{(i)}(x_m) = \phi(x_i)$. We then define $(X_m, \delta^{(i)})$ by

$$\delta^{(i)}(A) = \delta_K(\phi^{(i)}(A)).$$

Since $(\mathbb{R}^d, \delta_K)$ is a diversity, $(X_m, \delta^{(i)})$ satisfies (D1′) and (D2), and so $(X_m, \delta^{(i)})$ is a Minkowski-embeddable semidiversity. Moreover, the definitions of ϕ and $\phi^{(i)}$ give

$$\delta^{(i)}(A) = \begin{cases} f(|A| - 2) & \text{if } x_i, x_m \in A, \\ f(|A| - 1) & \text{otherwise.} \end{cases}$$

Second, let $(X_m, \delta^{(m)})$ be the diversity defined by setting

$$\delta^{(m)}(A) = \frac{|A| - 1}{m - 1} f(m - 1)$$

for all non-empty $A \subseteq X_m$. This is Minkowski-embeddable by Propositions 4.2 (b) and 4.1 (a).

We now claim that

$$\delta(A) = \max\{\delta^{(i)}(A) : i = 1, \ldots, m\} \tag{8}$$

holds for all $A \subseteq X_m$. Proving this claim will complete the proof of the theorem by induction since each (semi)diversity $\delta^{(i)}$ is Minkowski-embeddable, and hence by Proposition 4.1 (a), (X_m, δ) is Minkowski-embeddable.

When $|A| \leq 1$, (8) holds trivially, as all relevant quantities are zero. Suppose $|A| \geq 2$. Three cases may hold:

- $x_i \notin A$ for some $i = 1, \ldots, m-1$; then $\delta^{(i)}(A) = f(|A| - 1)$.
- $x_m \notin A$; then $\delta^{(i)}(A) = f(|A| - 1)$ for all $i = 1, \ldots, m-1$.
- $A = X_m$; then $\delta^{(i)}(A) = f(|A| - 2)$ for all $i = 1, \ldots, m-1$.

Hence

$$\max\{\delta^{(i)}(A) : i = 1, 2, \ldots, m-1\} = \begin{cases} f(|A| - 1) & \text{if } A \neq X_m, \\ f(|A| - 2) & \text{if } A = X_m. \end{cases}$$

We now consider $\delta^{(m)}$ and $A \subseteq X_m$. If $A = X_m$ then

$$\delta^{(m)}(A) = \frac{(|A| - 1)}{(m - 1)} f(m - 1) = f(m - 1).$$

Otherwise suppose $1 < |A| < m$ and so

$$\delta^{(m)}(A) = \frac{(|A| - 1)}{(m - 1)} f(m - 1)$$

$$= \frac{(|A| - 1)}{(m - 1)} \left(\prod_{k=|A|+1}^{m} \frac{f(k-1)}{f(k-2)} \right) f(|A| - 1)$$

$$\leq \frac{(|A| - 1)}{(m - 1)} \left(\prod_{k=|A|+1}^{m} \frac{k-1}{k-2} \right) f(|A| - 1) \qquad \text{(by (7))}$$

$$= f(|A| - 1).$$

Hence if $A \neq X_m$,

$$\max \{\delta^{(i)}(A) : i = 1, 2, \ldots, m\} = \max \{f(|A| - 1), \delta^{(m)}(A)\} = f(|A| - 1),$$

while if $A = X_m$ then since f is increasing

$$\max \{\delta^{(i)}(A) : i = 1, 2, \ldots, m\} = \max \{f(|A| - 2), \delta^{(m)}(A)\} = f(|A| - 1).$$

We conclude that $\delta(A) = \max \{\delta^{(i)}(A) : i = 1, \ldots, m\}$ for all $A \subseteq X_m$ which completes the proof of (8) and also the theorem. □

By considering the diversity on $X = \{a, b, c, d\}$ with

$$\delta(A) = \begin{cases} 2, & |A| = 4, \\ 1, & 2 \leq |A| \leq 3, \\ 0, & \text{otherwise}, \end{cases}$$

and taking $A = \{b, c, d\}$ in Theorem 4.3 we immediately obtain

Corollary 4.4 *There exists a diversity a set of four elements that is not Minkowski-embeddable.*

We show later (Corollary 5.3) that *every* diversity on three elements is Minkowski-embeddable.

5 Parallelotopes and Simplices

We have shown that not every diversity is Minkowski-embeddable, and so the question now becomes one of characterizing which diversities are. In this section we characterize when we can embed the diameter and negative-type diversities defined in the introduction in terms of Minkowski diversities having kernels equal to parallelotopes and simplices, respectively.

We first consider diameter diversities.

Theorem 5.1 *A finite diversity (X, δ) can be embedded in a Minkowski diversity with kernel equal to some parallelotope if and only if (X, δ) is a diameter diversity.*

Proof First note that if there is some such embedding then (X, δ) is a diameter diversity by [2, Proposition 3.4]. Conversely, suppose that $X = \{x_1, x_2, \ldots, x_n\}$ and (X, δ) is a diameter diversity. Let $\phi \colon X \to \mathbb{R}^n$ be the standard Fréchet embedding

$$\phi \colon X \to \mathbb{R}^n, \quad y \mapsto (d(x_1, y), d(x_2, y), \ldots, d(x_n, y)),$$

of the metric d induced by δ. Then $d(x, y) = \|\phi(x) - \phi(y)\|_\infty$ for all $x, y \in X$. Let K be the unit cube in $\mathbb{R}^n$. For all $A \subseteq X$ we have

$$\begin{aligned}
\delta(A) = \max_{a,b \in A} d(a, b) &= \max_{a,b \in A} \|\phi(a) - \phi(b)\|_\infty \\
&= \max_i \max_{a,b \in A} |\phi(a)_i - \phi(b)_i| = R(\phi(A), K).
\end{aligned}$$

□

We now consider finite diversities of negative type, the diversity analog of metrics of negative type [14]. Note that the cone of all diversities on a set X of cardinality n has dimension $2^n - (n + 1)$, the number of subsets $A \subset X$ with $|A| \geq 2$, see [35]. The set of diversities of negative type forms a cone of the same dimension, indicating that an appropriately chosen 'random' diversity could have negative type with non-zero probability.

Theorem 5.2 *A finite diversity (X, δ) can be embedded in a Minkowski diversity with kernel equal to some simplex if and only if (X, δ) has negative type.*

Proof Let $n = |X|$. From [35, Theorem 7], (X, δ) is negative-type if and only if it can be embedded in $(\mathbb{R}^d, \delta_{\text{neg}})$ for some d and

$$\delta_{\text{neg}}(A) = \sum_{i=1}^{d} \max\{a_i : a \in A\} - \min\left\{\sum_{i=1}^{d} a_i : a \in A\right\}.$$

Define the polytope

$$K = -\,\text{conv}\,(0, e_1, e_2, \ldots, e_d) = \left\{x \leq 0 : \sum_{i=1}^{d} x_i \geq -1\right\}.$$

Suppose $\lambda = R(A, K)$, so there is z such that $A \subseteq \lambda K + z$. Then for all $a \in A$ and all i we have $a_i - z_i \leq 0$. Hence

$$z_i \geq \max\{a_i : a \in A\}.$$

Let $a^* \in A$ minimize $\sum_{i=1}^{d} a_i^*$. Then $a^* \in \lambda K + z$ implies $\sum_{i=1}^{d}(a_i^* - z_i) \geq -\lambda$ and so

$$\lambda \geq \sum_{i=1}^{d} z_i - \sum_{i=1}^{d} a_i^* \geq \sum_{i=1}^{d} \max\{a_i : a \in A\} - \min_{a \in A} \sum_{i=1}^{d} a_i = \delta_{\text{neg}}(A).$$

Now suppose $\lambda = \delta_{\mathrm{neg}}(A)$. Let $z_i = \max\{a_i : a \in A\}$ so that $a_i - z_i \leq 0$ for all $a \in A$ and all $i = 1, \ldots, d$. Furthermore,

$$-\lambda = -\sum_{i=1}^{d} z_i + \min\left\{\sum_{i=1}^{d} a_i : a \in A\right\} \leq -\sum_{i=1}^{d} z_i + \sum_{i=1}^{d} a_i$$

for all $a \in A$, so that $\sum_{i=1}^{d}(a_i - z_i) \geq -\lambda$ and $a - z \in \lambda K$ for all $a \in A$. Hence $R(A, K) \leq \delta_{\mathrm{neg}}(A)$.

We have shown that (X, δ) is of negative type if and only if it can be embedded into a Minkowski diversity with kernel equal to the particular simplex K. The theorem now follows from Proposition 4.1 (b) and the fact that every simplex in $\mathbb{R}^d$ can be transformed into another by a non-degenerate affine map. □

The last theorem immediately implies that two further classes of diversity are Minkowski-embeddable.

Corollary 5.3 *If δ is diversity on three elements or a finite diversity that can be embedded in L_1, then δ is Minkowski-embeddable.*

Proof All three-element diversities and L_1-embeddable diversities have negative type [35]. □

6 Open Problems

We have characterized diversities and functions defined by the generalized circumradius $R(A, K)$, and established preliminary results on embedding finite diversities into these diversities. Our results suggest several avenues for further investigation.

First, is there a complete characterization of when a finite diversity is Minkowski-embeddable? Indeed it is not even obvious which finite diversities can be embedded into *sublinear* diversities in $\mathbb{R}^d$.

A second related question is algorithmic in nature: Are there efficient algorithms for determining whether or not a finite diversity can be embedded in some dimension? Interestingly, we note that for the classical case of a circumradius, even though we do not know a characterization for Minkowski-embeddability, we are able to give an efficient algorithm for deciding embeddability (for bounded dimension):

Proposition 6.1 *Let (X, δ) be a finite diversity such that $\delta(A) = \max\{\delta(A') : A' \subseteq A, |A'| \leq d+1\}$ for all $A \subseteq X$. For fixed d, there is an algorithm which runs in polynomial time in $n = |X|$ to determine if (X, δ) is Minkowski-embeddable in $\mathbb{R}^d$ with kernel equal to the unit ball $\mathfrak{B}$.*

Proof We begin with a useful observation. Suppose that there is an (unknown) embedding $\phi\colon X \to \mathbb{R}^d$ such that $\delta(A) = \delta_{\mathfrak{B}}(\phi(A))$ for all $A \subseteq X$. Note that since the metric induced by $\delta_{\mathfrak{B}}$ is Euclidean so is the one induced by δ. Let $\psi\colon X \to \mathbb{R}^d$ be any map which preserves the metrics induced by δ and $\delta_{\mathfrak{B}}$. In addition, let f be the (unknown) isometry from $\psi(X)$ to $\phi(X)$ given by $f(\psi(x)) = \phi(x)$ for all $x \in X$.

As $\mathbb{R}^d$ is a finite dimensional Hilbert space, f can be extended to an isometry on the whole space $\mathbb{R}^d$ (see, e.g., [34, Theorem 11.4]). Moreover, for any $A \subseteq X$ we have

$$\begin{aligned}
\delta_{\mathfrak{B}}(\psi(A)) &= \inf\{\sup\{\|\psi(a) - x\|_2 : a \in A\} : x \in \mathbb{R}^d\} \\
&= \inf\{\sup\{\|f(\psi(a)) - f(x)\|_2 : a \in A\} : x \in \mathbb{R}^d\} \\
&= \inf\{\sup\{\|\phi(a) - y\|_2 : a \in A\} : y \in \mathbb{R}^d\} \\
&= \delta_{\mathfrak{B}}(\phi(A)) = \delta(A).
\end{aligned}$$

Hence, if (X, δ) is Minkowski embeddable into $(\mathbb{R}^d, \delta_{\mathfrak{B}})$, then the map ψ gives one embedding.

We now present an algorithm for deciding whether or not (X, δ) is embeddable in $(\mathbb{R}^d, \delta_{\mathfrak{B}})$:

1. Decide whether or not the metric induced by δ on X is Euclidean. If not, then (X, δ) cannot be embedded in $(\mathbb{R}^d, \delta_{\mathfrak{B}})$. Else, compute a (metric) embedding ψ of X in $\mathbb{R}^d$ which preserves the induced metrics.
2. If $\delta_{\mathfrak{B}}(\psi(A)) = \delta(A)$ for all A with $|A| \leq d + 1$ then (X, δ) can be embedded in $(\mathbb{R}^d, \delta_{\mathfrak{B}})$, otherwise (X, δ) cannot be embedded in $(\mathbb{R}^d, \delta_{\mathfrak{B}})$.

The correctness of this algorithm follows by the observation above. To see that it also runs in polynomial time in n (for fixed d), note that step 1. can be computed in polynomial time in n by the results in e.g. [14, Sect. 6.2] or [34, Theorem 2.1], and that for step 2. the definition of δ and Proposition 1.2 imply that to determine whether $\delta_{\mathfrak{B}}(\psi(A)) = \delta(A)$ for all $A \subseteq X$ we need only check subsets A with $|A| \leq d + 1$. □

Another question is how to extend the embedding results to include distortion. Let (X_1, δ_1) and (X_2, δ_2) be two diversities. We say that a map $\phi \colon X_1 \to X_2$ has *distortion* c if there are $c_1, c_2 > 0$ such that $c = c_1 c_2$ and

$$\frac{\delta_1(A)}{c_1} \leq \delta_2(\phi(A)) \leq c_2 \delta_1(A)$$

for all finite $A \subseteq X$. Continuing the program of [25], it is shown in [10] that bounds on the distortion of embeddings from diversities into L_1-diversities provide approximation algorithms for hypergraph generalizations of sparsest cut. It was shown in [35] that there are finite diversities of metric type which cannot be embedded into L_1 without at least $\Omega(\sqrt{\log|X|})$ distortion. This bound therefore holds for Minkowski-embeddable diversities. In general, questions concerning distortion seem intricately connected with core sets of the generalized circumradius [2].

Apart from potential algorithmic gains it would be good to explore embeddings with distortion for diversities into Minkowski diversities with low dimension, simply for their use in visualization and modelling of diversity type data.

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