DECIDABILITY FOR THE THEORY OF MODULES OVER A PRÜFER DOMAIN

LORNA GREGORY

Abstract. In this article we give elementary conditions completely characterising when the theory of modules of a Prüfer domain is decidable. Using these results, we show that the theory of modules of the ring of integer valued polynomials is decidable.

Contents

1. Introduction

In this article, we give a complete characterisation of when the theory of modules of a recursive Prüfer domain is decidable.

Decidability connects algebra and computability theory in a profound way by asking whether a given algebraic structure, or a class of algebraic structures, can algorithmically be fully understood, at least in principle. Famous instances are Hilbert's 10th problem asking whether the solvability of diophantine equations can be verified by a machine; Tarski's ground breaking result on the decidability of the real field and Ax's theorem on the decidability of the class of finite fields (showing that there is an algorithm verifying whether a first-order sentence holds in all finite fields).

Prüfer domains are a much studied class of rings, including many classically important rings and classes of rings. For example, they include Dedekind domains and hence rings of integers of number fields; Bézout domains and hence the ring of complex entire functions [Hel40, Thm. 9] and the ring of algebraic integers [Kap74, Thm. 102]; the ring of integer valued polynomials with rational coefficients [CC97, VI.1.7] and the real holomorphy rings of a formally real fields [Bec82, 2.16].

Prüfer domains have provided a rich supply of rings for which the decidability of modules can be determined. The theory of modules of a ring R is said to be decidable if there is an algorithm which decides whether a given first order sentence in the language of R-modules is true in all R-modules.

The first non-trivial example of a ring with decidable theory of modules was given by Szmielew, [Szm55], who showed that the theory of abelian groups (or equivalently $\mathbb{Z}$-modules) is decidable. This result was generalised by Eklof and Fischer, [EF72], to some Dedekind domains, among them certain rings of integers, and they showed that, for a (recursive) field k with a splitting algorithm, the theory of $k[x]$-modules is decidable.

The most recent effort to understand decidability of theories of modules over Prüfer domains started with a paper, [PPT07], of Puninski, Puninskaya and Toffalori. They showed that a recursive valuation domain with dense archimedean value group has decidable theory of modules if and only if its set of units is recursive. Proving a conjecture in [PPT07], we show in [Gre15] that an arbitrary recursive valuation domain has decidable theory of modules if and only if the radical relation $a \in \operatorname{rad} bR$ is recursive.

The theory of modules of Bézout domains of the form $D+XQ[X] \subseteq Q[X]$, where D is a principal ideal domain with field of fractions Q, is shown in [PT14] to be decidable under certain reasonable effectiveness conditions on D. In particular, it is shown that $\mathbb{Z}+X\mathbb{Q}[X]$ has decidable theory of modules. The theory of modules of the ring of algebraic integers, along with some other Bézout domains with Krull dimension 1, is shown to have decidable theory of modules in [LTP17].

Work towards characterising when a general Prüfer domain has decidable theory of modules was started in the articles [GLPT18] and [GLT19], and is finished in the present one. We will describe the results of these articles whilst describing the main result of the present article.

First a reminder of the setup for proving decidability results for theories of modules. Thanks to the Baur-Monk theorem, if R is a recursive ring then the theory of R-modules is decidable if and only if there exists an algorithm which, given pairs of pp-formulae $\varphi_1/\psi_1, \ldots, \varphi_l/\psi_l$ and intervals $[n_1, m_1], \ldots, [n_l, m_l] \subseteq \mathbb{N} \cup \{\infty\}$ where $n_i, m_i \in \mathbb{N} \cup \{\infty\}$, answers whether there exists an R-module M such that, for all $1 \leq i \leq l$, $|\varphi(M)/\psi(M)| \in [n_i, m_i]$. The existence of an algorithm answering this question when $[n_i, m_i]$ are either $[1, 1]$ or $[2, \infty]$ is equivalent to the existence of an algorithm deciding whether one Ziegler basic open set is contained in a finite union of other Ziegler basic open sets (for the definition of the Ziegler spectrum see 2.1).

We characterise when the theory of modules of a Prüfer domain R is decidable in terms of the recursivity of three sets: $\operatorname{DPR}(R)$, $\operatorname{EPP}(R)$ and $X(R)$. Each of these sets is a subset of $R^n \times \mathbb{N}_0^k$ for some $n, k \in \mathbb{N}_0$. For the sets $\operatorname{EPP}(R)$ and $X(R)$, we postpone their definitions (see 3.2.4 and 3.3.1 respectively) to section 3 and in this introduction we instead give some indication of their meaning.

For any commutative ring R, the set $\operatorname{DPR}(R)$ is defined as the set of $(a, b, c, d) \in R^4$ such that for all prime ideals $\mathfrak{p}, \mathfrak{q}$ of R with $\mathfrak{p} + \mathfrak{q} \neq R$, either $a \in \mathfrak{p}$, $b \notin \mathfrak{p}$, $c \in \mathfrak{q}$ or $d \notin \mathfrak{q}$. This set was introduced in [GLPT18] as a generalisation of the radical relation $a \in \operatorname{rad} bR$. For a recursive Bézout domain it is shown there that $\operatorname{DPR}(R)$ is recursive if and only if there is an algorithm deciding inclusions of Ziegler basic open sets. For recursive Prüfer domains, analogous sufficient conditions were given for there to exist an algorithm deciding inclusions of Ziegler basic open sets. Building heavily on those results, we extend the equivalence given in [GLPT18] for Bézout domains to all recursive Prüfer domains (see 3.1.8). As a consequence we get the following theorem.

Theorem. (See 3.1.9) *Let R be a recursive Prüfer domain such that $R/\mathfrak{m}$ is infinite for all maximal ideals $\mathfrak{m}$. The theory of R-modules is decidable if and only if* $\mathrm{DPR}(R)$ *is recursive.*

If R is a ring with a pair of pp-formulae φ/ψ and an R-module M such that $|\varphi(M)/\psi(M)|$ is finite but not equal to 1, in particular if R is a commutative ring with a finite non-zero module, then we need to do more than show that there is an algorithm deciding inclusions of Ziegler basic open sets.

For any ring R, if the theory of R-modules is decidable then the theory of modules of size n is decidable uniformly in n. In 3.2.10, we introduce a set $\mathrm{EPP}(R)$, whose recursivity, for a recursive Prüfer domain R, is equivalent to the decidability of the theory of R-modules of size n, uniformly in n. This is proved in Theorem 7.6. The main feature of $\mathrm{EPP}(R)$ is that it is often easier to check in examples that $\mathrm{EPP}(R)$ is recursive than it is to check that the theory of modules of size n is decidable uniformly in n.

The set $\mathrm{EPP}(R)$ is a generalisation of $\mathrm{PP}(R)$, which is defined in [GLT19] and inspired by the characterisation of commutative von Neumann regular rings with decidable theories of modules given in [PP88]. In [GLT19], for a recursive Bézout domain R, under the condition that for each maximal ideal $\mathfrak{m}$, $R_\mathfrak{m}$ has dense value group, it is shown that the theory of R-modules is decidable if and only if $\mathrm{DPR}(R)$ and $\mathrm{PP}(R)$ are recursive. Building heavily on [GLT19], we show, that this result also holds for Prüfer domains.

Theorem 3.2.3. *Let R be a recursive Prüfer domain such that $R_\mathfrak{m}$ has dense value group for all maximal ideals $\mathfrak{m}$. The theory of R-modules is decidable if and only if* $\mathrm{PP}(R)$ *and* $\mathrm{DPR}(R)$ *are recursive.*

It follows from this theorem that the theory of modules of a recursive Prüfer domain with dense value groups (or infinite residue fields) is decidable if and only if there is an algorithm deciding inclusions of Ziegler basic open sets and the theory of finite modules of size n is decidable uniformly in n. The same result for commutative von Neumann regular rings follows easily from [PP88]. This does not appear to be the case for arbitrary Prüfer domains.

The third set, $X(R)$, captures information about finite Baur-Monk invariants of the, in some sense intrinsically infinite modules, $R_\mathfrak{p}/I$ where $\mathfrak{p}$ is a prime ideal of R and I is an ideal of $R_\mathfrak{p}$.

Main Theorem 11.1. *Let R be a recursive Prüfer domain. The theory of R-modules is decidable if and only if the sets* $\mathrm{DPR}(R)$*,* $\mathrm{EPP}(R)$ *and* $X(R)$ *are recursive.*

Our characterisation is such that it can be easily checked for concrete rings. We illustrate this in section 12 by using our main theorem to show that the ring of integer valued polynomials with rational coefficients, $\mathrm{Int}(\mathbb{Z})$, has decidable theory of modules. In order to prove that $\mathrm{DPR}(\mathrm{Int}(\mathbb{Z}))$, $\mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$ and $X(\mathrm{Int}(\mathbb{Z}))$ are recursive, we use Ax's result, [Ax68, Thm 17], that the common theory of the p-adic valued fields $\mathbb{Q}_p$ as p varies, is decidable.

The main theorem applied to the special case of a recursive Prüfer domain R of Krull dimension 1 yields:

- The theory of R-modules is decidable if and only if $\mathrm{EPP}(R)$ is recursive and the relation $a \in \mathrm{rad}(b_1R + b_2R)$ is recursive, 11.2.
- If R is a Bézout domain, then the theory of R-modules is decidable if and only if the set of units of R and $\mathrm{EPP}(R)$ are recursive, 11.4.

Section 1 contains background material and simple preparations for the rest of the paper. Its main purpose is to make the article as accessible as possible. We postpone a guide to the proof and discussion of what is contained in each section to subsection 2.4.

When it doesn't complicate the proofs, we state some of our intermediate results for arithmetical rings, i.e. commutative rings whose localisations at prime ideals are valuation rings.

2. Preliminaries

Notation: In this article $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_n := \{m \in \mathbb{N} \mid m \geq n\}$ for $n \in \mathbb{N}$ and $\mathbb{P}$ denotes the set of prime natural numbers. For a ring R, let Mod-R denote the category of (right) modules. For R a ring, we will write $I \lhd R$ to mean I is a (right) ideal of R.

2.1. Model theory of Modules.

For general background on model theory of modules see [Pre88].

Let R be a ring. Let $\mathcal{L}_R := \{0, +, (\cdot r)_{r\in R}\}$ be the language of (right) R-modules and T_R be the theory of (right) R-modules. A (right) **pp-formula**[1] is a formula of the form

$$\exists y_1, \ldots, y_l \bigwedge_{j=1}^{m} \sum_{i=1}^{l} y_i r_{ij} + x s_j = 0$$

where $r_{ij}, s_j \in R$. For $a \in R$, we write $a|x$ for the pp-formula $\exists y\, x = ya$.

The solution set $\varphi(M)$ of a pp-formula φ in an R-module M is a subgroup of M. For φ, ψ, pp-formulae and $M \in$ Mod-R, we will write $\psi \leq_M \varphi$ to mean that $\psi(M) \subseteq \varphi(M)$. We will write $\psi \leq \varphi$ to mean that $\psi \leq_M \varphi$ for all $M \in$ Mod-R. After identifying equivalent pp-formulae, the set of pp-formulae, pp_R^1, equipped with the order $\leq$ is a lattice.

A **pp-pair** will simply mean a pair of pp-formulae and we will write φ/ψ for such pairs. For $M \in$ Mod-R, we write $\varphi/\psi(M)$ for the quotient group $\varphi(M)/\varphi(M) \cap \psi(M)$. For every $n \in \mathbb{N}$ and pp-pair φ/ψ, there is a sentence, denoted $|\varphi/\psi| \geq n$, in the language of (right) R-modules, which expresses, in every R-module M, that $\varphi/\psi(M)$ has at least n elements. We write $|\varphi/\psi| = n$ for the sentence which expresses, in every R-module M, that $\varphi/\psi(M)$ has exactly n elements.

Theorem 2.1.1 (Baur-Monk)**.** *Let R be a ring. Every sentence in $\mathcal{L}_R$ is equivalent to a boolean combination of sentences of the form $|\varphi/\psi| \geq n$ where φ/ψ is a pp-pair.*

An embedding $f : A \to B$ is **pure** if for all pp-formulae φ and $m \in A$, $f(m) \in \varphi(B)$ implies $m \in \varphi(A)$. A module N is **pure-injective** if for every pure-embedding $f : A \to B$ and homomorphism $g : A \to N$, there exists $h : B \to N$ such that

[1]This is really the definition of a pp-1-formula, i.e. a pp-formula in one variable. We only use pp-1-formulae in this article.

$h \circ f = g$. A module is **indecomposable** if it cannot be written as the direct sum of two non-zero submodules. We denote the set of isomorphism classes of indecomposable pure-injective (right) R-modules by pinj_R.

Lemma 2.1.2. [Pre88, 4.36] *Let R be a ring. For all $M \in \text{Mod-}R$, there exist indecomposable pure-injective modules $N_i \in \text{Mod-}R$ for $i \in I$ such that $\oplus_{i \in I} N_i$ is elementary equivalent to M.*

We say a pure-embedding $i : M \to N$ with N pure-injective is a **pure-injective hull** of M if for every other pure-embedding $g : M \to K$ where K is pure-injective, there is a pure-embedding $h : N \to K$ such that $h \circ i = g$. Every module M has a pure-injective hull and if $i : M \to N$ and $i' : M \to N'$ are pure-injective hulls of M then there exists $f : N \to N'$ such that $fi = i'$ (see [Pre09, 4.3.18]). We will write $H(M)$ for any module N such that the inclusion of M in N is a pure-injective hull of M. We will also often refer to such a module as the pure-injective hull of M. Every module is an elementary substructure of its pure-injective hull [Pre88, 2.27]. So, in particular every module is elementary equivalent to its pure-injective hull.

The (right) **Ziegler spectrum** of a ring R, denoted Zg_R, is a topological space whose points are isomorphism classes of indecomposable pure-injective (right) R-modules and which has a basis of compact open sets given by

$$({}^{\varphi}/_{\psi}) := \{N \in \text{pinj}_R \mid \varphi(N) \supsetneq \varphi(N) \cap \psi(N)\}$$

where ${}^{\varphi}/_{\psi}$ range over pp-pairs.

Prest gave a lattice anti-isomorphism $D : \text{pp}_R^1 \to {}_R\text{pp}^1$ (see [Pre88, 8.21]) where ${}_R\text{pp}^1$ denotes the lattice of left pp-formulae. As is standard, we denote its inverse ${}_R\text{pp}^1 \to \text{pp}_R^1$ also by D. We don't recall the full definition of D here but instead note that for all $a \in R$, $D(a|x)$ is $ax = 0$ and $D(xa = 0)$ is $a|x$.

Herzog extended this duality to an isomorphism between the lattice of open sets of the right and left Ziegler spectra of a ring [Her93, 4.4], and, to a useful bijection between the complete theories of right and left R-modules.

The following proposition is direct consequence of [Her93, 6.6].

Proposition 2.1.3. *Let R be a ring. Let $n, m \in \mathbb{N}$ be such that $n \leq m$ and for $1 \leq i \leq m$, let $N_i \in \mathbb{N}$ and let ${}^{\varphi_i}/_{\psi_i}$ be a pp-pair. For*

$$\chi := \bigwedge_{i=1}^{n} |{}^{\varphi_i}/_{\psi_i}| = N_i \wedge \bigwedge_{i=n+1}^{m} |{}^{\varphi_i}/_{\psi_i}| \geq N_i,$$

define

$$D\chi := \bigwedge_{i=1}^{n} |{}^{D\psi_i}/_{D\varphi_i}| = N_i \wedge \bigwedge_{i=n+1}^{m} |{}^{D\psi_i}/_{D\varphi_i}| \geq N_i.$$

There exists a right R-module satisfying χ if and only if there exists a left R-module satisfying $D\chi$.

A priori, duality may not appear particularly relevant to an article about commutative rings. However, its use significantly simplifies some of the proofs in this paper and, as in [Gre15], the fact that it exchanges formulae $xb = 0$ with $b|x$ allows us to reduce the number of calculations.

2.2. **Decidability and recursive Prüfer domains.** A **recursive ring** is either a finite ring or a ring R together with a bijection $\pi : \mathbb{N} \to R$ such that addition and multiplication in R induce recursive functions on $\mathbb{N}$ via π.

Note that if R is a ring and $\pi : \mathbb{N} \to R$ is a bijection then T_R is recursively axiomatisable with respect to π if and only if R together with π is a recursive ring.

When proving decidability results about theories of modules, it is common to work with an "effectively given" ring rather than just a recursive one (see for instance [PPT07, §3], [Gre15, 3.1], [GLPT18, §3] and [GLT19, §2]). Usually, a ring of a particular type is called effectively given if R is a recursive ring and the bijection π satisfies some extra conditions which are equivalent, for that particular type of ring, to Prest's condition (D) holding (see [Pre88, pg 334]). Recall that a recursive ring satisfies condition (D) if there is an algorithm which, given $\varphi, \psi \in \mathrm{pp}^1_R$ answers whether $\psi \leq \varphi$. So, in particular, if T_R is decidable then R satisfies condition (D) i.e. R is effectively given. For example, a recursive valuation domain V is said to be **effectively given** if the preimage under π of the set of units of V is recursive. A recursive Prüfer domain R is said to be **effectively given** if the preimage under π of the set of $(a, b) \in R^2$ such that $a \in bR$ is recursive. By definition, every effectively given ring is recursive and if T_R is decidable then R is effectively given, by which I mean Prest's condition (D) holds. For simplicity and generality, we choose to work with recursive rings.

We will use results from [GLPT18] and [GLT19], stated under the stronger assumption that R is an effectively given Prüfer domain. It was remarked in [GLT19, paragraph before 2.4], that the property that $a \in bR$ is recursive is never used in [GLPT18] or [GLT19]. Moreover, see [GLT19, 2.4], if R is a recursive Prüfer domain and the set $\mathrm{DPR}(R) \subseteq R^4$ is recursive then the relation $a \in bR$ is recursive. In particular, even though they are stated for effectively given Prüfer domains, the main results in [GLPT18] and [GLT19] in fact hold for recursive Prüfer domains.

The next theorem is a well-known and easy consequence of the Baur-Monk Theorem. Note that, since T_R is recursively axiomatisable when R is recursive, given a sentence χ in $\mathcal{L}_R$, we can always find, using a proof algorithm, a sentence χ' as in the statement of the Baur-Monk theorem which is T_R-equivalent to χ.

Theorem 2.2.1. *Let R be a recursive ring. The theory of R-modules is decidable if and only if there is an algorithm which, given a sentence χ of the form*

$$(\dagger) \qquad \chi := \bigwedge_{i=1}^{n} |{}^{\varphi_i}/_{\psi_i}| = E_i \wedge \bigwedge_{i=n+1}^{m} |{}^{\varphi_i}/_{\psi_i}| \geq E_i,$$

where $E_i \in \mathbb{N}$ and ${}^{\varphi_i}/_{\psi_i}$ is a pp-pair for $1 \leq i \leq m$, answers whether there exists an R-module satisfying χ.

As in the introduction we say "there is an algorithm deciding inclusions of Ziegler basic open sets" to mean that there is an algorithm, which given $n+1$ Ziegler basic open sets $({}^{\varphi_0}/_{\psi_0}), \ldots, ({}^{\varphi_n}/_{\psi_n})$, answers whether $({}^{\varphi_0}/_{\psi_0}) \subseteq ({}^{\varphi_1}/_{\psi_1}) \cup \ldots \cup ({}^{\varphi_n}/_{\psi_n})$, or not.

Remark 2.2.2. *Let R be a recursive ring. There is an algorithm deciding inclusions of Ziegler basic open sets if and only if there is an algorithm which, given a sentence*

$$\chi := \bigwedge_{i=1}^{n} |{}^{\varphi_i}/_{\psi_i}| \geq E_i \wedge \bigwedge_{j=1}^{m} |{}^{\sigma_j}/_{\tau_j}| = 1,$$

where $\varphi_i/\psi_i, \sigma_j/\tau_j$ *are pp-pairs and* $E_i \in \mathbb{N}$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*, answers whether there exists an* R*-module satisfying* χ*.*

Proof. There is a module satisfying χ as in the statement if and only if for each $1 \leq i \leq n$, there exists an indecomposable pure-injective module N_i such that $N_i \in (\varphi_i/\psi_i) \setminus \bigcup_{j=1}^{m} (\sigma_j/\tau_j)$. This is a standard argument. For the forward direction use 2.1.2. For the reverse, observe $\oplus_{i=1}^{n} N_i^{E_i}$ satisfies χ. □

2.3. **Arithmetical rings and Prüfer domains.** A commutative ring is **arithmetical**[2] if all its localisations at maximal ideals are valuation rings. Equivalently, [Jen66, Thm 1], a commutative ring R is arithmetical if its lattice of ideals is distributive. A **Prüfer domain** is an integral domain which is arithmetical.

The following lemma is a direct consequence of [Tug03, 1.3].

Lemma 2.3.1. *If* R *is an arithmetical ring then for all* $a, b \in R$*, there exist* $\alpha, r, s \in R$ *such that* $a\alpha = br$ *and* $b(\alpha - 1) = as$*.*

Note that if R is a recursive arithmetical ring then there is an algorithm which, given $a, b \in R$, finds α, r, s satisfying the above equations. We will frequently use this fact without note.

Recall that a module is called **uniserial** if its lattice of submodules is totally ordered. A module is **pp-uniserial** if its lattice of pp-definable subgroups is totally ordered. Over a commutative ring all pp-definable subgroups are submodules. Thus, all uniserial modules over a commutative ring are pp-uniserial.

The lattice of pp-formulae of a commutative ring R is distributive if and only if R is arithmetical [EH95, 3.1]. Thus, the following is a direct consequence of [Pun03, 3.3].

Lemma 2.3.2. *Let* R *be a commutative ring. All indecomposable pure-injective* R*-modules are pp-uniserial if and only if* R *is arithmetical.*

The endomorphism rings of indecomposable pure-injective modules are local [Pre09, 4.3.43]. Therefore, if R is a commutative ring and N is an indecomposable pure-injective R-module then the set, $\text{Att}N$, of $r \in R$ acting on N non-bijectively form a prime ideal. The notation "Att" stands for attached prime. Thus, if N is an indecomposable pure-injective module over a commutative ring R then N may be equipped with the structure of an $R_{\text{Att}N}$-module. Moreover, N remains indecomposable and pure-injective as an $R_{\text{Att}N}$-module. Conversely, if N is an indecomposable pure-injective $R_{\mathfrak{p}}$-module for some prime ideal $\mathfrak{p} \lhd R$ then the restriction of N to R remains indecomposable and pure-injective.

The next lemma follows easily from the fact that indecomposable pure-injective modules over arithmetical rings are pp-uniserial (a proof appears in [GLT19, 2.8]).

Lemma 2.3.3. *Let* R *be an arithmetical ring and* N *an indecomposable pure-injective* R*-module. The sets*

$$\text{Div}N := \{r \in R \mid Nr \subsetneq N\}$$

and

$$\text{Ass}N := \{r \in R \mid \text{ there exists } m \in N \backslash \{0\} \text{ such that } mr = 0\}$$

[2]This condition is often referred to as Prüfer in papers on Model Theory of Modules. However, algebraists tend to use the term Prüfer for the weaker condition that every regular ideal is invertible. To avoid confusion we choose the term with a unique definition.

are prime ideals. Moreover,

$$\mathrm{Att}N = \mathrm{Div}N \cup \mathrm{Ass}N.$$

Here, $\mathrm{Ass}N$ can be read as the assassinator of N. More dubiously, $\mathrm{Div}N$ may be read as the "divissinator" of N.

Lemma 2.3.4. *Let R be an arithmetical ring and $M \in \mathrm{Mod}\text{-}R$.*

(1) For all $\alpha \in R$, there exist $M_1, M_2 \in \mathrm{Mod}\text{-}R$ such that $M_1 \oplus M_2 \equiv M$, $|{}^{x\alpha=0}/_{x=0}(M_1)| = 1$, $|{}^{x=x}/_{\alpha|x}(M_1)| = 1$, $|{}^{x(\alpha-1)=0}/_{x=0}(M_2)| = 1$ and $|{}^{x=x}/_{(\alpha-1)|x}(M_2)| = 1$.

(2) For all $a, b \in R$, there exist $M_1, M_2 \in \mathrm{Mod}\text{-}R$ such that $M_1 \oplus M_2 \equiv M$, $|{}^{ab|x}/_{x=0}(M_1)| = 1$ and $|{}^{xa=0}/_{b|x}(M_2)| = 1$.

Proof. (1) By 2.1.2, there exist indecomposable pure-injective R-modules N_i for $i \in I$ such that $M \equiv \oplus_{i\in I} N_i$. Since $\mathrm{Att}N_i$ is a proper ideal for each N_i, for all $i \in I$, either $\alpha \notin \mathrm{Att}N_i$ or $\alpha - 1 \notin \mathrm{Att}N_i$. Let I_α be the set of $i \in I$ such that $\alpha \notin \mathrm{Att}N_i$ and let $I_{\alpha-1} = I \backslash I_\alpha$. So, for all $i \in I_{\alpha-1}$, $\alpha - 1 \notin \mathrm{Att}N_i$. For each $\beta \in R$ and N indecomposable pure-injective, $\beta \notin \mathrm{Att}N$ if and only if $|{}^{x=x}/_{\beta|x}(N)| = 1$ and $|{}^{x\beta=0}/_{x=0}(M_1)| = 1$. Therefore $|{}^{x=x}/_{\alpha|x}(\oplus_{i\in I_\alpha} N_i)| = 1$, $|{}^{x\alpha=0}/_{x=0}(\oplus_{i\in I_\alpha} N_i)| = 1$, $\left|{}^{x=x}/_{(\alpha-1)|x}(\oplus_{i\in I_{\alpha-1}} N_i)\right| = 1$ and $\left|{}^{x(\alpha-1)=0}/_{x=0}(\oplus_{i\in I_{\alpha-1}} N_i)\right| = 1$.
(2) For any $L \in \mathrm{Mod}\text{-}R$, $xb = 0 \geq_L a|x$ if and only if $ab \in \mathrm{ann}_R L$. So $|{}^{ab|x}/_{x=0}(L)| = 1$ if and only if $xb = 0 \geq_L a|x$. Let N be an indecomposable pure-injective R-module. By 2.3.2, either $xb = 0 \geq_N a|x$ or $a|x \geq_N xb = 0$. So either $|{}^{ab|x}/_{x=0}(N)| = 1$ or $|{}^{xb=0}/_{a|x}(N)| = 1$. The proof is now as in (1). □

It is easy to see that if R is a commutative ring, $\mathfrak{p} \lhd R$ is a prime ideal and M is an $R_\mathfrak{p}$-module then the restriction to R of the pure-injective hull of M as an $R_\mathfrak{p}$-module is equal to the pure-injective hull of M as an R-module.

Theorem 2.3.5. [Zie84] *Let V be a valuation domain with field of fractions Q. Every indecomposable pure-injective V-module is the pure-injective hull of a module J/I where $I \subsetneq J \subseteq Q$ are submodules of Q.*

So, in particular, over a valuation domain, all indecomposable pure-injective modules are pure-injective hulls of uniserial modules. It is not known if all indecomposable pure-injective modules over valuation rings are pure-injective hulls of uniserial modules (see [EH95, §4]).

Lemma 2.3.6. *Let R be a Prüfer domain. For any sentence $\chi \in \mathcal{L}_R$, there exists $M \in \mathrm{Mod}\text{-}R$ such that $M \models \chi$ if and only if there exist $n \in \mathbb{N}$, prime ideals $\mathfrak{p}_i \lhd R$ and uniserial $R_{\mathfrak{p}_i}$-modules U_i for $1 \leq i \leq n$ such that $\oplus_{i=1}^n U_i \models \chi$.*

Proof. For any ring R, there exists $M \in \mathrm{Mod}\text{-}R$ such that $M \models \chi$ if and only if there exist $n \in \mathbb{N}$ and indecomposable pure-injective R-modules N_i for $1 \leq i \leq n$ such that $\oplus_{i=1}^n N_i \models \chi$. The result now follows from 2.3.5. □

We will frequently use the following easy lemma.

Lemma 2.3.7. *Let V be a valuation domain, φ a pp-formula and U a uniserial V-module. If $|U/\varphi(U)|$ is finite but not equal to 1 then $U \cong V/I$ for some ideal $I \lhd V$.*

Proof. Since U is uniserial, so is $U/\varphi(U)$. Therefore, since $U/\varphi(U)$ is finite, there exists $u \in U$ such that $u + \varphi(U)$ generates $U/\varphi(U)$ as a V-module. Since $U/\varphi(U) \neq 0$, $uV \supseteq \varphi(U)$. Therefore $uV = U$. □

We finish this subsection by reviewing material about ideals of valuation domains.

For any commutative ring R, $r \in R$ and ideal $I \lhd R$, define

$$(I : r) := \{a \in R \mid ar \in I\}.$$

Note that $(I : r)$ is an ideal of R.

Definition 2.3.8. *For V a valuation domain and $I \lhd R$ a proper ideal, define $I^{\#} := \bigcup_{a \notin I}(I : a)$. By convention, we define $V^{\#}$ to be the unique maximal ideal of V.*

Note that this definition agrees with the definition given in [FS01, Ch. II §4], that is, for $I \neq 0$, $r \in I^{\#}$ if and only if $rI \subsetneq I$.

Lemma 2.3.9. *Let V be a valuation domain.*

(i) For any ideal $I \lhd V$, $I^{\#}$ is a prime ideal.
(ii) If $\mathfrak{p} \lhd V$ is a prime ideal then $\mathfrak{p}^{\#} = \mathfrak{p}$.
(iii) If $I \lhd V$ and $a \in V\backslash\{0\}$ then $(aI)^{\#} = I^{\#}$.
(iv) If $I \lhd V$ and V/I is finite then $I^{\#}$ is the unique maximal ideal of V.

Proof. The first 3 statements are in [FS01, II.4]. We prove (iv). If $I = V$ then the statement follows directly from the definition. Otherwise, $I \subseteq I^{\#}$ and hence $V/I^{\#}$ is also finite. Since $V/I^{\#}$ is a finite integral domain, it is a field. Therefore $I^{\#}$ is maximal. □

Lemma 2.3.10. *Let R be a Prüfer domain, $\mathfrak{p} \lhd R$ a prime ideal and $I \lhd R_{\mathfrak{p}}$. Then, for all $\delta, \gamma \in R$,*

(1) $|x\delta{=}0/x{=}0(R_{\mathfrak{p}}/I)| = 1$ if and only if $\delta \notin I^{\#}$ or $I = R_{\mathfrak{p}}$, and,
(2) $|x{=}x/\gamma|x(R_{\mathfrak{p}}/I)| = 1$ if and only if $\gamma \notin \mathfrak{p}$ or $I = R_{\mathfrak{p}}$.

Proof. (1) For any $\delta \in R$, $|x\delta{=}0/x{=}0(R_{\mathfrak{p}}/I)| = 1$ if and only if $(I : \delta) \subseteq I$. Now $(I : \delta) \subseteq I$ if and only if $I = R_{\mathfrak{p}}$, or, for all $a \notin I$, $\delta a \notin I$. Therefore $|x\delta{=}0/x{=}0(R_{\mathfrak{p}}/I)| = 1$ if and only if $\delta \notin I^{\#}$ or $I = R_{\mathfrak{p}}$.
(2) For any $\gamma \in R$, $|x{=}x/\gamma|x(R_{\mathfrak{p}}/I)| = 1$ if and only if $\gamma R_{\mathfrak{p}} + I = R_{\mathfrak{p}}$. This is true if and only if $\gamma \notin \mathfrak{p}$ or $I = R_{\mathfrak{p}}$. □

Remark 2.3.11. *Let R be an integral domain, $b \in R\backslash\{0\}$, $\mathfrak{p} \lhd R$ be a prime ideal and $I \lhd R_{\mathfrak{p}}$ be an ideal. If $R_{\mathfrak{p}}/I$ is finite then $|I/bI| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}|$.*

Proof. Since $b \neq 0$,

$$|R_{\mathfrak{p}}/I| \cdot |I/bI| = |R_{\mathfrak{p}}/bI| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}| \cdot |bR_{\mathfrak{p}}/bI| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}| \cdot |R_{\mathfrak{p}}/I|.$$

So, since $|R_{\mathfrak{p}}/I|$ is non-zero, $|I/bI| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}|$. □

We frequently use the following lemma which becomes particularly useful when R is a valuation ring because then for all $r \in R$ and $I \lhd R$ either $r \in I$ or $rR \supseteq I$.

Lemma 2.3.12. *Let R be a commutative ring, $r \in R$ and $I \lhd R$. Then $rR \supseteq I$ if and only if there exists $J \lhd R$ such that $I = rJ$.*

Proof. The reverse direction is clear. For the forward direction, take $J = (I : r)$. □

Lemma 2.3.13. *Let V be a valuation domain, $I, J \lhd V$ and $a \in R\backslash\{0\}$. Then $J \supseteq (I : a)$ if and only if $aJ \supseteq I$ or $J = V$.*

Proof. ($\Rightarrow$) Suppose $J \supseteq (I : a)$. Since V is a valuation domain, either $a \in I$ or $aV \supseteq I$. If $a \in I$ then $J \supseteq (I : a) = V$. So $J = V$. Suppose $aV \supseteq I$. Take $c \in I$. There exists $b \in V$ such that $ab = c$. So $b \in (I : a) \subseteq J$. Hence $c = ab \in aJ$ as required.
($\Leftarrow$) If $J = V$ then $J \supseteq (I : a)$. So, suppose $aJ \supseteq I$. Take $c \in (I : a)$. Then $ac \in I \subseteq aJ$. Since $a \neq 0$, $c \in J$. Therefore $J \supseteq (I : a)$. □

2.4. **Guide to the proof.** The reverse direction of the main theorem is proved in §3. That is, we show that if T_R is decidable then DPR(R), EPP(R) and $X(R)$ are recursive (see [GLPT18, 6.4] (or 3.1.6), 3.2.9 and 3.3.3).

The proof of the forward direction of the main theorem has 3 principal ingredients.

(A) Consequences of DPR(R), EPP(R) and $X(R)$ being recursive (§3 and §7).
(B) Syntactic reductions (§4, §5 and §9).
(C) Semantic input (§6, §8 and §10).

(A) In section 3 we introduce and analyse the sets DPR(R), EPP(R) and $X(R)$. These sets are chosen to be as simple as possible so that our theorem is as easy as possible to apply to concrete rings. For this reason, work needs to be done to obtain more elaborate consequences of them being recursive.

For each $n \in \mathbb{N}$, a set $\text{DPR}_n(R)$ was introduced in [GLPT18]. We show that, 3.1.7, for R a recursive arithmetical ring, if DPR(R) is recursive then the sets $\text{DPR}_n(R)$ are recursive (uniformly in n). Combining this with [GLPT18, 7.1], or more precisely its proof, we conclude that we can effectively decide inclusions of Ziegler basic open sets if and only if DPR(R) is recursive.

In section 7, we investigate the consequences of EPP(R) being recursive, and of EPP(R) and the radical relation being recursive (this is primarily used in section 8). We show that for R a recursive Prüfer domain the theory of R-modules of size n is decidable uniformly in n if and only if EPP(R) is recursive, 7.6.

In the proof of the forward direction of the main theorem, the set $X(R)$ is only ever used in section 10.

(B) Given a sentence χ as in (†) (from 2.2.1) we often produce a finite set S of tuples of sentences $(\chi_1, \ldots, \chi_n)$ with each χ_i having a "better" form than χ such that there exists $M \models \chi$ if and only if there exist $(\chi_1, \ldots, \chi_n) \in S$ and modules M_i for $1 \leq i \leq n$ with $M_i \models \chi_i$. This is roughly what happens in the proof of [GLT19, 4.1]. Section 4 introduces two important formalisms (and ideas) which used in combination are key to the proof. Essentially they allow us to "automate" some reductions similar to those in the proof of [GLT19, 4.1] which in this article become too complicated to perform entirely by hand.

It is shown in [GLT19, 4.1] that for arithmetical rings[3] it is enough to consider sentences as in (†) where the pp-pairs involved are of the form $d|x/x=0$ or $xb=0/c|x$. Section 5 uses the formalisms in section 4, to show that it is enough to consider sentences as in (†) where at most one conjunct of the form $|d|x/x=0| = D$ or $|d|x/x=0| \geq D$ with $D \geq 2$ occurs and where at most one conjunct of the form $|xb=0/c|x| = G$ or $|xb=0/c|x| \geq G$ with $G \geq 2$ and $b, c \neq 0$ occurs.

[3]The result is stated there only for Prüfer domains but the same proof implies the result for all arithmetical rings (see section 5).

In section 9 a notion of complexity, called the extended signature, is defined on the set W of sentences as in (†) as reduced to in section 5. The set of extended signatures is equipped with an artinian partial order. The reduction processes in section 9 terminate at expressions whose extended signatures are not reducible. Some of the sentences which are not reducible are of a form for which we can answer whether there exists a module satisfying them, because there is an algorithm deciding inclusions of Ziegler basic open sets. The remaining sentences are of a particular simple form and we deal with them in section 10.

(C) In section 6, for pp-pairs φ/ψ of the form $d|x/x=0$, $xb=0/c|x$ with $b, c \neq 0$ and $x=x/c|x$ with $c \neq 0$, we give a description of the uniserial modules U over a valuation domain V, such that $\varphi/\psi(U)$ is finite but non-zero. Unlike the descriptions of such modules in [Gre15] and [PPT07], the results we prove do not depend on whether the value group of V is dense or not. We now describe how we use semantic input to deal with sentences as in (†) (from 2.2.1) with a conjunct of the form $|d|x/x=0| = D$, or of the form $|xb=0/c|x| = G$ where $b, c \neq 0$.

For instance, if $d|x/x=0(U)$ is finite but non-zero then it is easy to show that $U \cong V/dI$ for some ideal $I \lhd V$. In view of 2.3.6, for R a Prüfer domain, this means that for χ a sentence as in (†), $D \in \mathbb{N}_2$ and $d \in R\backslash\{0\}$, there exists $M \in$ Mod-R such that $M \models |d|x/x=0| = D \wedge \chi$ if and only if there exist $h \in \mathbb{N}$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$, and, $M' \in$ Mod-R with $|d|x/x=0(M')| = 1$ such that $\bigoplus_{i=1}^{h} R_{\mathfrak{p}_i}/dI_i \oplus M' \models |d|x/x=0| = D \wedge \chi$.

In section 8, we show that if EPP(R) and the radical relation are recursive then there is an algorithm which given $D \in \mathbb{N}$, $d \in R\backslash\{0\}$ and a sentence χ as in (†), answers whether there exists a direct sum $\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/dI_i$ satisfying $|d|x/x=0| = D \wedge \chi$. This is used in 9.1.1 to produce sentences $\chi_1, \ldots \chi_n$ as in (†) such that there exists an R-module satisfying $|d|x/x=0| = D \wedge \chi$ if and only if there exists an R-module satisfying $|d|x/x=0| = 1 \wedge \chi_i$ for some $1 \leq i \leq n$. The sentences χ_i are less complex than χ in a way precisely defined in section 9.

Similar, but slightly more complicated, reductions are made for pp-pairs of the form $xb=0/c|x$ where $b, c \neq 0$.

For pp-pairs of the form $x=x/c|x$ we need to do something different. It is easy to see, 6.0.3, that if U is a uniserial module over a valuation domain V then $x=x/c|x(U)$ is finite but non-zero if and only if $U \cong V/cI$ for some $I \lhd V$ or U is finite and $c \in \text{ann}_R U$. However, it does not seem possible, in this case, to make a reduction as for $|d|x/x=0| = D$ and sums of modules of the form $R_{\mathfrak{p}}/dI$. This is the main reason that we need to make the syntactic reductions in section 5 and 9. In particular, the set of sentences that are not reducible in the sense of section 9, contains only a small number of forms of sentences with a conjunct of the form $|x=x/c|x| = C$. These sentences are considered individually in section 10.

3. Recursive sets

In this section we consider the sets DPR(R), EPP(R) and $X(R)$. In each case, we show that if T_R is decidable then they are recursive.

3.1. **The set** DPR(R)**.** In [GLPT18], a family of relations DPR$_n(R)$ were defined. Although not directly stated there, see [GLPT18, 7.1], it was shown that, for R

a recursive[4] Prüfer domain, if the sets $\mathrm{DPR}_n(R)$ are recursive (uniformly[5] in n) then there is an algorithm deciding inclusions of Ziegler basic open sets. However, it was not known if this condition was necessary for the existence of such an algorithm, or even if the decidability of the theory of modules of a Prüfer domain implied that $\mathrm{DPR}_n(R)$ is recursive for any $n > 1$. It is a consequence of 3.1.6 that the existence of an algorithm deciding inclusions of Ziegler basic open sets implies that the sets $\mathrm{DPR}_n(R)$ are recursive (uniformly in n).

For a recursive Bézout domain, it was shown that if $\mathrm{DPR}(R) := \mathrm{DPR}_1(R)$ is recursive then there is an algorithm deciding inclusions of Ziegler basic open sets. For Prüfer domains, it was not known if $\mathrm{DPR}_1(R)$ being recursive is sufficient to imply that there is an algorithm deciding inclusions of Ziegler basic open sets. We show, 3.1.7, that, for R a Prüfer domain, $\mathrm{DPR}_1(R)$ recursive implies $\mathrm{DPR}_n(R)$ is recursive uniformly in n.

Definition 3.1.1. *Let R be a commutative ring.*

- *For each $l \in \mathbb{N}$, let $\mathrm{DPR}_l(R)$ be the set of $2l+2$-tuples $(a, b_1, \ldots, b_l, c, d_1, ., d_l) \in R^{2l+2}$ such that, for all prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$ with $\mathfrak{p} + \mathfrak{q} \neq R$, either $a \in \mathfrak{p}$, $c \in \mathfrak{q}$, $b_i \notin \mathfrak{p}$ for some $1 \leq i \leq l$ or $d_i \notin \mathfrak{q}$ for some $1 \leq i \leq l$.*
- *Let $\mathrm{DPR}_*(R)$ be the set of 4-tuples (a, B, c, D), where $a, c \in R$ and $B, D \lhd R$ are finitely generated ideals, such that for all prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$ with $\mathfrak{p}+\mathfrak{q} \neq R$, either $a \in \mathfrak{p}$, $c \in \mathfrak{q}$, $B \nsubseteq \mathfrak{p}$ or $D \nsubseteq \mathfrak{q}$.*

Note that $(a, b_1, \ldots, b_l, c, d_1, \ldots, d_l) \in \mathrm{DPR}_l(R)$ if and only if

$$(a, \sum_{i=1}^{l} b_i R, c, \sum_{i=1}^{l} d_i R) \in \mathrm{DPR}_*(R).$$

The relation $\mathrm{DPR}(R)$ is referred to as the "double prime radical" relation. This is because we think of it as a generalisation of the radical relation $a \in \mathrm{rad}(bR)$ but involving 2 prime ideals instead of 1 (see also 3.1.3).

For R a commutative ring, $I \lhd R$ an ideal and $X \subseteq \mathrm{Spec}R$, let $V(I)$ denote the closed set, in the Zariski topology, of prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \supseteq I$ and $\overline{X}$ the closure of X in the Zariski topology. Note that, for any $X \subseteq \mathrm{Spec}R$, $V(\bigcap_{\mathfrak{p}\in X} \mathfrak{p}) = \overline{X}$.

Lemma 3.1.2. *Let R be a commutative ring, $a \in R$ and $B \lhd R$. Then*

$$(\mathrm{rad}\, B : a) = \bigcap_{\substack{\mathfrak{p}\in\mathrm{Spec}R \\ \mathfrak{p}\supseteq B,\ a\notin\mathfrak{p}}} \mathfrak{p}.$$

Hence,

$$V((\mathrm{rad}\, B : a)) = \overline{V(B)\backslash V(aR)}.$$

Proof. Suppose $r \in (\mathrm{rad}\, B : a)$. Then $(ra)^n \in B$ for some $n \in \mathbb{N}$. Let $\mathfrak{p} \lhd R$ be a prime ideal with $B \subseteq \mathfrak{p}$ and $a \notin \mathfrak{p}$. Then $r^n a^n = (ra)^n \in \mathfrak{p}$. Therefore $r \in \mathfrak{p}$.

Conversely, suppose that $r \notin (\mathrm{rad}\, B : a)$ i.e. $(ra)^n \notin B$ for all $n \in \mathbb{N}$. A standard argument using Zorn's lemma produces a prime ideal $\mathfrak{p} \lhd R$ such that $\mathfrak{p} \supseteq B$ and $(ra)^n \notin \mathfrak{p}$ for all $n \in \mathbb{N}$. Now $(ra)^n \notin \mathfrak{p}$ implies $r \notin \mathfrak{p}$ and $a \notin \mathfrak{p}$. So we have proved the first statement, and, consequently, the second. □

[4]Recall R recursive and $\mathrm{DPR}(R)$ recursive imply R is effectively given.

[5]Here "uniformly in n" means there is a single algorithm which given $n \in \mathbb{N}$ and a tuple $\overline{a}$ from R of length $2n+2$ answers whether $\overline{a} \in \mathrm{DPR}_n(R)$ or not.

The following statement with $\mathrm{DPR}(R) := \mathrm{DPR}_1(R)$ in place of $\mathrm{DPR}_*(R)$ is proved in [GLPT18, 6.3] for Prüfer domains. We use 3.1.2 to further extend it to all commutative rings.

Proposition 3.1.3. *Let R be a commutative ring. The following are equivalent for $a, c \in R$ and $B, D \lhd R$ finitely generated ideals.*

(1) $(a, B, c, D) \in \mathrm{DPR}_*(R)$.
(2) $1 \in (\mathrm{rad}(B) : a) + (\mathrm{rad}(D) : c)$.

Proof. We prove this proposition topologically. For all $a, c \in R$ and $B, D \lhd R$, $1 \in (\mathrm{rad}(B) : a) + (\mathrm{rad}(D) : c)$ if and only if $V((\mathrm{rad}\, B : a)) \cap V((\mathrm{rad}\, D : c)) = \emptyset$. By 3.1.2, $V((\mathrm{rad}\, B : a)) = \overline{V(B)\backslash V(aR)}$ and $V((\mathrm{rad}\, D : c)) = \overline{V(D)\backslash V(cR)}$. Thus

$$1 \in (\mathrm{rad}(B) : a) + (\mathrm{rad}(D) : c)$$

if and only if

$$\overline{V(B)\backslash V(aR)} \cap \overline{V(D)\backslash V(cR)} = \emptyset.$$

Now, by [DST19, 1.5.4 (i)],

$$\overline{V(B)\backslash V(aR)} = \bigcup_{\mathfrak{p} \in V(B)\backslash V(aR)} V(\mathfrak{p}) \quad \text{and} \quad \overline{V(D)\backslash V(cR)} = \bigcup_{\mathfrak{q} \in V(D)\backslash V(cR)} V(\mathfrak{q}).$$

So $1 \in (\mathrm{rad}(B) : a) + (\mathrm{rad}(D) : c)$ if and only if

$$(\dagger\dagger) \qquad \Big(\bigcup_{\mathfrak{p} \in V(B)\backslash V(aR)} V(\mathfrak{p})\Big) \cap \Big(\bigcup_{\mathfrak{q} \in V(D)\backslash V(cR)} V(\mathfrak{q})\Big) = \emptyset.$$

Now $(\dagger\dagger)$ holds if and only if for all prime ideals $\mathfrak{p}, \mathfrak{q}$ such that $a \notin \mathfrak{p}$, $B \subseteq \mathfrak{p}$, $c \notin \mathfrak{q}$ and $D \subseteq \mathfrak{q}$, we have $V(\mathfrak{p}) \cap V(\mathfrak{q}) = \emptyset$ (i.e. $\mathfrak{p} + \mathfrak{q} = R$). □

Remark 3.1.4. *Let R be a recursive ring. The relations $\mathrm{DPR}_*(R)$ and $\mathrm{DPR}_n(R)$ for all $n \in \mathbb{N}$ are recursively enumerable.*

Proof. By proposition 3.1.3, $(a, B, c, D) \in \mathrm{DPR}_*(R)$ if and only if $1 \in (rad(B) : a) + (rad(D) : c)$. Therefore $(a, B, c, D) \in DPR_*(R)$ if and only if there exist $u \in R$ and $n \in \mathbb{N}$ such that $(ua)^n \in B$ and $((1-u)c)^n \in D$. If R is a recursive ring then we can list all $(a, B, c, D) \in \mathrm{DPR}_*(R)$ by searching for $u \in R$ and $n \in \mathbb{N}$ witnessing that $(a, B, c, D) \in \mathrm{DPR}_*(R)$. □

We refer to the relation $a \in \mathrm{rad}\, bR$ as the **radical relation**.

Remark 3.1.5. *Let R be a commutative ring. For $a, b \in R$, $a \in \mathrm{rad}\, bR$ if and only if $(a, b, a, b) \in \mathrm{DPR}(R)$. In particular, if R is a recursive ring and $\mathrm{DPR}(R)$ is recursive then the radical relation is recursive.*

For R a commutative ring and $B, D \lhd R$ finitely generated ideals, let $xB = 0$ denote the pp-formula $\bigwedge_{i=1}^n xb_i = 0$ where $B = \sum_{i=1}^n b_iR$ and let $D|x$ denote the pp-formula $\sum_{i=1}^n d_i|x$ where $D = \sum_{i=1}^n d_iR$. Note that, up to T_R equivalence, these formulae don't depend on the choice of generators of B and D.

The next proposition with $\mathrm{DPR}_*(R)$ replaced by $\mathrm{DPR}(R) = \mathrm{DPR}_1(R)$ is proved in [GLPT18, 6.4] for Prüfer domains. A crucial ingredient in its proof is the fact that if R is an arithmetical ring then for prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$, the condition $\mathfrak{p} + \mathfrak{q} \neq R$ is equivalent to $\mathfrak{p}$ and $\mathfrak{q}$ being comparable by inclusion.

Proposition 3.1.6. *Let R be an arithmetical ring. The following are equivalent for $a, c \in R$ and $B, D \lhd R$ finitely generated ideals.*

(1) $(a, B, c, D) \in \mathrm{DPR}_*(R)$.
(2) $({}^{xB=0}/_{D|x}) \subseteq ({}^{xa=0}/_{x=0}) \cup ({}^{x=x}/_{c|x})$.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. By 3.1.3, $1 \in (\mathrm{rad}\, B : a) + (\mathrm{rad}\, D : c)$. Hence there exist $u \in R$ and $n \in \mathbb{N}$ such that $(ua)^n \in B$ and $((u-1)c)^n \in D$.

Suppose $N \in ({}^{xB=0}/_{D|x})$. Take $m \in N$ such that $mb = 0$ for all $b \in B$ and $m \notin ND$. So $mu^n a^n = 0$ and $m \notin N(u-1)^n c^n$. Since $\mathrm{Att}N$ is a proper ideal, either $u \notin \mathrm{Att}N$ or $u - 1 \notin \mathrm{Att}N$. Suppose $u \notin \mathrm{Att}N$. Then $mu^n a^n = 0$ implies $ma^n = 0$ and hence $N \in ({}^{xa^n=0}/_{x=0}) = ({}^{xa=0}/_{x=0})$ as required. Now suppose $u - 1 \notin \mathrm{Att}N$. Then $N(u-1)^n c^n = Nc^n$ and hence $N \in ({}^{x=x}/_{c^n|x}) = ({}^{x=x}/_{c|x})$.
(2) $\Rightarrow$ (1): Suppose (2) holds. Note that if $({}^{xB=0}/_{D|x}) \subseteq ({}^{xa=0}/_{x=0}) \cup ({}^{x=x}/_{c|x})$ then, applying Herzog's duality for Ziegler spectra, $({}^{xD=0}/_{B|x}) \subseteq ({}^{xc=0}/_{x=0}) \cup ({}^{x=x}/_{a|x})$.

Suppose $\mathfrak{p}, \mathfrak{q}$ are prime ideals with $\mathfrak{p} + \mathfrak{q} \neq R$. Further, suppose that $a \notin \mathfrak{p}$, $B \subseteq \mathfrak{p}$ and $D \subseteq \mathfrak{q}$. We need to show that $c \in \mathfrak{q}$.

Since R is an arithmetical ring, either $\mathfrak{p} \supseteq \mathfrak{q}$ or $\mathfrak{q} \supseteq \mathfrak{p}$. Suppose $\mathfrak{q} \supseteq \mathfrak{p}$. Then, again since R is an arithmetical ring, $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is uniserial and its pure-injective hull $N := H(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})$ is indecomposable. Since $B \subseteq \mathfrak{p}$, $xB = 0$ is equivalent to $x = x$ in $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ and hence in N. Since $D \subseteq \mathfrak{q}$, $D|x$ is not equivalent to $x = x$ in $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ and hence in N. Thus $N \in ({}^{xB=0}/_{D|x})$. Since $a \notin \mathfrak{p}$, $N \notin ({}^{xa=0}/_{x=0})$ and hence $N \in ({}^{x=x}/_{c|x})$. Therefore $c \notin \mathfrak{q}$, for otherwise $c|x$ is equivalent to $x = x$ in N. The argument when $\mathfrak{p} \supseteq \mathfrak{q}$ is very similar except this time $N := H(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p})$ and we use that $({}^{xD=0}/_{B|x}) \subseteq ({}^{xc=0}/_{x=0}) \cup ({}^{x=x}/_{a|x})$. □

Thus, for R an arithmetical ring, if there exists an algorithm deciding inclusions of Ziegler basic open sets then $\mathrm{DPR}_*(R)$ is recursive. Combining this with [GLPT18, 7.1] (and its proof) we conclude that there is an algorithm deciding inclusions of Ziegler basic open sets if and only if $\mathrm{DPR}_*(R)$ is recursive. Moreover, if $R/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m} \lhd R$ then T_R is decidable if and only if $\mathrm{DPR}_*(R)$ is recursive, i.e the sufficient conditions given in [GLPT18, 7.1] are also necessary.

Proposition 3.1.7. *Let R be a recursive arithmetical ring. If $\mathrm{DPR}(R)$ is recursive then $\mathrm{DPR}_n(R)$ is recursive uniformly in n.*

Proof. Let $n \in \mathbb{N}_2$ and $a, c, b_1, \ldots, b_n, d_1, \ldots, d_n \in R$. Suppose that $\alpha, \beta, r_1, r_2, s_1, s_2 \in R$ are such that

$$b_1\alpha = b_2 r_2, \qquad b_2(\alpha - 1) = b_1 r_1,$$
$$d_1\beta = d_2 s_2 \text{ and } d_2(\beta - 1) = d_1 s_1.$$

Claim:

$$(\star) \qquad (a, c, b_1, \ldots, b_n, d_1, \ldots, d_n) \in \mathrm{DPR}_n(R)$$

if and only if

(i) $(a\alpha, c\beta, b_2, \ldots, b_n, d_2, \ldots, d_n) \in \mathrm{DPR}_{n-1}(R)$,
(ii) $(a\alpha, c(\beta - 1), b_2, \ldots, b_n, d_1, d_3, \ldots, d_n) \in \mathrm{DPR}_{n-1}(R)$,
(iii) $(a(\alpha - 1), c\beta, b_1, b_3 \ldots, b_n, d_2, \ldots, d_n) \in \mathrm{DPR}_{n-1}(R)$ and
(iv) $(a(\alpha - 1), c(\beta - 1), b_1, b_3, \ldots, b_n, d_1, d_3, \ldots, d_n) \in \mathrm{DPR}_{n-1}(R)$.

This claim plus the fact that we can always find appropriate $\alpha, \beta, r_1, r_2, s_1, s_2 \in R$ implies the proposition.

To prove the forward direction, we show that $(\star)$ implies (i) and note that the remaining conditions (ii), (iii), (iv) are the same as (i) but with the roles of b_1 and

b_2, and of α and $\alpha-1$ interchanged (respectively of d_1 and d_2, and of β and $\beta-1$ interchanged).

Let $\mathfrak{p},\mathfrak{q}$ be prime ideals such that $\mathfrak{p}+\mathfrak{q}\neq R$. Assuming $(\star)$, we need to show that $a\alpha\in\mathfrak{p}$, $c\beta\in\mathfrak{q}$, $b_i\notin\mathfrak{p}$ for some $2\leq i\leq n$ or $d_i\notin\mathfrak{q}$ for some $2\leq i\leq n$.

Now $(\star)$ implies $a\in\mathfrak{p}$, $c\in\mathfrak{q}$, $b_i\notin\mathfrak{p}$ for some $1\leq i\leq n$ or $d_i\notin\mathfrak{q}$ for some $1\leq i\leq n$. So the only problem is when $b_1\notin\mathfrak{p}$ or $d_1\notin\mathfrak{q}$. Suppose $b_1\notin\mathfrak{p}$. If $\alpha\in\mathfrak{p}$ then $a\alpha\in\mathfrak{p}$ as required. So suppose further that $\alpha\notin\mathfrak{p}$. Then $b_1\alpha\notin\mathfrak{p}$ and hence $b_2r_2\notin\mathfrak{p}$. So $b_2\notin\mathfrak{p}$ as required. The argument when $d_1\notin\mathfrak{q}$ is the same with the roles of b_1 and d_1, of b_2 and d_2, and of α and β interchanged.

We now show that if $(\star)$ is not true then one of $(i),(ii),(iii),(iv)$ is not true. If $(\star)$ is not true then there exist prime ideals $\mathfrak{p},\mathfrak{q}$ such that $\mathfrak{p}+\mathfrak{q}\neq R$ and $a\notin\mathfrak{p}$, $c\notin\mathfrak{q}$, $b_i\in\mathfrak{p}$ for all $1\leq i\leq n$ and $d_i\in\mathfrak{q}$ for all $1\leq i\leq n$. For any proper ideal I, either $\alpha\notin I$ or $\alpha-1\notin I$ (respectively $\beta\notin I$ or $\beta-1\notin I$). Without loss of generality, we may assume that $\alpha\notin\mathfrak{p}$ and $\beta\notin\mathfrak{q}$. Hence $a\alpha\notin\mathfrak{p}$, $c\beta\notin\mathfrak{q}$, $b_i\in\mathfrak{p}$ for all $1\leq i\leq n$ and $d_i\in\mathfrak{q}$ for all $1\leq i\leq n$. So $(a\alpha,c\beta,b_2,\ldots,b_n,d_2,\ldots,d_n)\notin\mathrm{DPR}_{n-1}(R)$ as required. □

Combining this with the results in [GLPT18] we get the following.

Theorem 3.1.8. *Let R be a recursive Prüfer domain. There is an algorithm deciding inclusions of Ziegler basic open sets if and only if the relation* $\mathrm{DPR}(R)$ *is recursive.*

Corollary 3.1.9. *Let R be a recursive Prüfer domain. The relation* $\mathrm{DPR}(R)$ *is recursive if and only if there is an algorithm which, given a sentence*

$$\chi:=\bigwedge_{i=1}^{n}|{\varphi_i}/{\psi_i}|\geq E_i\wedge\bigwedge_{j=1}^{m}|{\sigma_j}/{\tau_j}|=1,$$

where ${\varphi_i}/{\psi_i},{\sigma_j}/{\tau_j}$ are pp-pairs and $E_i\in\mathbb{N}$ for $1\leq i\leq n$ and $1\leq j\leq m$, answers whether there exists an R-module M with $M\models\chi$. Moreover, if $R/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m}\lhd R$ then T_R is decidable if and only if $\mathrm{DPR}(R)$ *is recursive.*

Proof. The first statement follows from 3.1.8 and 2.2.2.

For any R-module M and pp-pair ${\varphi}/{\psi}$, ${\varphi}/{\psi}(M)$ is an R-module. By [GLT19, 3.1], if $R/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m}\lhd R$ then the only finite R-module is the zero module. So, the second statement follows from the first by a standard argument using the Baur-Monk theorem. □

Question 1. *It was shown in* [Gre15, 3.2] *that, for any commutative ring R, if T_R is recursive then the radical relation is recursive. For R an arbitrary commutative ring, does T_R decidable imply* $\mathrm{DPR}(R)$*, or more generally* $\mathrm{DPR}_*(R)$*, is recursive?*

3.2. **The sets** $\mathrm{PP}(R)$ **and** $\mathrm{EPP}(R)$**.** In [GLT19], a family of relations $\mathrm{PP}_n(R)$ were introduced. It is shown, [GLT19, 3.2], that if a recursive Prüfer domain has decidable theory of modules then $\mathrm{PP}_n(R)$ is recursive uniformly in n. Conversely, it was shown that if R is a recursive[6] Prüfer domain such that the value group of each localisation of R at a maximal ideal is dense then if $\mathrm{DPR}_n(R)$ and $\mathrm{PP}_n(R)$ are recursive uniformly in n then the theory of R-modules is decidable.

[6]It was stated there for effectively given Prüfer domains. However, recall, if R is a recursive Prüfer domain with $\mathrm{DPR}(R)$ recursive then R is effectively given.

The letters "PP" in these relations are chosen to honour Point and Prest who defined similar relations for commutative von Neumann regular rings in [PP88]. In 3.2.4, we will define another family of relations $\mathrm{EPP}_l(R)$; the letters "EPP" stand for "extended Point-Prest".

Definition 3.2.1. *Let R be a commutative ring. For $l \in \mathbb{N}$, let $\mathrm{PP}_l(R)$ consist of the tuples $(p, n, c_1, \ldots, c_l, d) \in \mathbb{P} \times \mathbb{N} \times R^{l+1}$ such that there exist positive integers $s, k_1, \ldots, k_s$ and maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ of R for which there exist $\lambda_1, \ldots, \lambda_s \in \mathbb{N}_0$ with $n = \sum_{t=1}^{s} \lambda_i k_i$ and for $1 \leq i \leq s$*

(1) $|R/\mathfrak{m}_i| = p^{k_i}$,
(2) $c_j \in \mathfrak{m}_i$ for $1 \leq j \leq l$,
(3) $d \notin \mathfrak{m}_i$.

It is clear that for R a Bézout domain, if $\mathrm{PP}_1(R) := \mathrm{PP}(R)$ is recursive then $\mathrm{PP}_l(R)$ is recursive uniformly in l. This is because $(p, n, c_1, \ldots, c_l, d) \in \mathrm{PP}_l(R)$ if and only if $(p, n, \gcd\{c_1, \ldots, c_l\}, d) \in \mathrm{PP}_1(R)$. However, with a bit more work one can show this is also true for Prüfer domains.

Proposition 3.2.2. *Let R be a recursive Prüfer domain. If $\mathrm{PP}_1(R)$ is recursive then $\mathrm{PP}_l(R)$ is recursive uniformly in l.*

Proof. We skip this proof as it is very similar to the proof of 3.2.8. □

As a direct consequence of [GLT19, 6.1], [GLT19, 3.2], [GLPT18, 6.4], 3.2.2 and 3.1.7, we get the following theorem.

Theorem 3.2.3. *Let R be a recursive Prüfer domain such that for all maximal ideals $\mathfrak{m}$, the value group of $R_\mathfrak{m}$ is dense. The theory of R-modules is decidable if and only if $\mathrm{DPR}(R)$ and $\mathrm{PP}(R)$ are recursive.*

We generalise $\mathrm{PP}_l(R)$ to $\mathrm{EPP}_l(R)$ in order to deal with Prüfer domains with maximal ideals $\mathfrak{m}$ such that $R_\mathfrak{m}$ is a valuation domain with non-dense value group.

Definition 3.2.4. *Let R be a commutative ring. For $l \in \mathbb{N}$, let $\mathrm{EPP}_l(R)$ consist of tuples*

$$(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathbb{P} \times \mathbb{N}_0 \times R^l \times R \times R \times \mathbb{N}_0$$

such that there exist $h \in \mathbb{N}_0$ and, for $1 \leq i \leq h$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ such that $\gamma \notin \mathfrak{p}_i$ and $a_1, \ldots, a_l \in I_i$ for $1 \leq i \leq h$, $|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/I_i| = p^n$ and $|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = p^m$.

We say the sequence $(\mathfrak{p}_i, I_i)_{1 \leq i \leq h}$ witnesses $(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathrm{EPP}_l(R)$. By convention, $(p, 0; a_1, \ldots, a_l; \gamma; e, 0) \in \mathrm{EPP}_l(R)$ and the empty sequence is a witness for it. We will often write $\mathrm{EPP}(R)$ for $\mathrm{EPP}_1(R)$.

Remark 3.2.5. *We may replace prime ideals with maximal ideals in 3.2.4 without changing the definition since if $|R_\mathfrak{p}/eR_\mathfrak{p}|$ is finite then either $\mathfrak{p}$ is maximal or $R_\mathfrak{p} = eR_\mathfrak{p}$ and if $|R_\mathfrak{p}/I|$ is finite then either $\mathfrak{p}$ is maximal or $I = R_\mathfrak{p}$.*

The relation $\mathrm{EPP}_l(R)$ is an extension of the relation $\mathrm{PP}_l(R)$.

Lemma 3.2.6. *Let $p \in \mathbb{P}$, $n \in \mathbb{N}$ and $c_1, \ldots, c_n, d \in R$. Then $(p, n, c_1, \ldots, c_l, d) \in \mathrm{PP}_l(R)$ if and only if $(p, n; c_1, \ldots, c_l; d; 1, 0) \in \mathrm{EPP}_l(R)$.*

Proof. Suppose $(p, n, c_1, \ldots, c_l, d) \in \mathrm{PP}_l(R)$. Let $s \in \mathbb{N}$, $k_1, \ldots, k_s \in \mathbb{N}$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be as in the definition of $\mathrm{PP}_l(R)$. Let $\lambda_1, \ldots, \lambda_s \in \mathbb{N}_0$ be such that $n = \sum_{i=1}^{s} \lambda_i k_i$. Then $c_j \in \mathfrak{m}_i$ for $1 \leq j \leq l$, $d \notin \mathfrak{m}_i$, $|\oplus_{i=1}^{s} (R_{\mathfrak{m}_i}/\mathfrak{m}_i R_{\mathfrak{m}_i})^{\lambda_i}| = p^n$ and $|\oplus_{i=1}^{s} (R_{\mathfrak{m}_i}/1 \cdot R_{\mathfrak{m}_i})^{\lambda_i}| = p^0$. So $(p, n; c_1, \ldots, c_l; d; 1, 0) \in \mathrm{EPP}_l(R)$.

Suppose $(\mathfrak{m}_j, I_j)_{1 \leq j \leq s}$ witnesses $(p, n; c_1, \ldots, c_l; d; 1, 0) \in \mathrm{EPP}_l(R)$. For $1 \leq i \leq s$, let $\lambda_i \in \mathbb{N}_0$ be such that $|R_{\mathfrak{m}_i}/I_i| = |R/\mathfrak{m}_i|^{\lambda_i}$. If $\lambda_i \neq 0$ then $I_i \subseteq \mathfrak{m}_i R_{\mathfrak{m}_i}$ and so $c_j \in \mathfrak{m}_i$ for $1 \leq j \leq l$. For $1 \leq i \leq s$, let k_i be such that $|R/\mathfrak{m}_i| = p^{k_i}$. Now $n = \sum_{i=1,\ \lambda_i \neq 0}^{s} \lambda_i k_i$ and hence $(p, n, c_1, \ldots, c_l, d) \in \mathrm{PP}_l(R)$. □

If the value group of $R_{\mathfrak{m}}$ is dense then, for all $e \in R$, $|R_{\mathfrak{m}}/eR_{\mathfrak{m}}|$ is either 1 or infinite. Moreover, $|R_{\mathfrak{m}}/eR_{\mathfrak{m}}| = 1$ if and only if $e \notin \mathfrak{m}$.

Remark 3.2.7. *Let R be a Prüfer domain such that the value group of $R_{\mathfrak{m}}$ is dense for all maximal ideals $\mathfrak{m}$. Then $(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathrm{EPP}_l(R)$ if and only if $(p, n, a_1, \ldots, a_l, \gamma \cdot e) \in \mathrm{PP}_l(R)$ and $m = 0$.*

In particular, if R is a recursive Prüfer domain such that the value group of $R_{\mathfrak{m}}$ is dense for all maximal ideals $\mathfrak{m}$ then $\mathrm{EPP}_l(R)$ is recursive if and only if $\mathrm{PP}_l(R)$ is recursive.

Proposition 3.2.8. *Let R be a recursive Prüfer domain. If $\mathrm{EPP}(R)$ is recursive then $\mathrm{EPP}_l(R)$ is recursive uniformly in l.*

Proof. We show that for all $p \in \mathbb{P}$, $n, m \in \mathbb{N}_0$, $a_1, \ldots, a_l, \gamma, e \in R$, if $\alpha, r, s \in R$ are such that $a_1\alpha = a_2 r$ and $a_2(\alpha - 1) = a_1 s$ then $(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathrm{EPP}_l(R)$ if and only if there exist $n_1, n_2 \in \mathbb{N}_0$ and $m_1, m_2 \in \mathbb{N}_0$ such that $n_1 + n_2 = n$, $m_1 + m_2 = m$, $(p, n_1; a_2, \ldots, a_l; \gamma\alpha; e, m_1) \in \mathrm{EPP}_{l-1}(R)$ and $(p, n_2, a_1, a_3, \ldots, a_l, \gamma(\alpha - 1); e, m_2) \in \mathrm{EPP}_{l-1}(R)$. This is enough since we can always effectively find appropriate $\alpha, r, s \in R$.

Suppose that $(\mathfrak{p}_j, I_j)_{1 \leq j \leq s}$ witnesses $(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathrm{EPP}_l(R)$. For all $1 \leq j \leq s$, either $\alpha \notin \mathfrak{p}_j$ or $\alpha - 1 \notin \mathfrak{p}_j$. By reordering, we may assume that $\alpha \notin \mathfrak{p}_j$ for $1 \leq j \leq t$ and $\alpha - 1 \notin \mathfrak{p}_j$ for $t + 1 \leq j \leq s$. Let $n_1 = \log_p |\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/I_i|$, $n_2 = \log_p \left|\oplus_{i=t+1}^{s} R_{\mathfrak{p}_i}/I_i\right|$, $m_1 = \log_p |\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}|$ and $m_2 = \log_p \left|\oplus_{i=t+1}^{s} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right|$. Now $\alpha\gamma \notin \mathfrak{p}_j$ and $a_2, \ldots, a_l \in I_j$ for $1 \leq j \leq t$, $|\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/I_i| = p^{n_1}$ and $|\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = p^{m_1}$. So $(p, n_1; a_2, \ldots, a_l; \gamma\alpha; e, m_1) \in \mathrm{EPP}_{l-1}(R)$. Similarly, $(p, n_2, a_1, a_3, \ldots, a_l, \gamma(\alpha - 1); e, m_2) \in \mathrm{EPP}_{l-1}(R)$.

Conversely, suppose that $n_1, n_2, m_1, m_2 \in \mathbb{N}_0$ are such that $n_1 + n_2 = n$, $m_1 + m_2 = m$, $(p, n_1; a_2, \ldots, a_l; \gamma\alpha; e, m_1) \in \mathrm{EPP}_{l-1}(R)$ and $(p, n_2, a_1, a_3, \ldots, a_l, \gamma(\alpha - 1); e, m_2) \in \mathrm{EPP}_{l-1}(R)$. Let $(\mathfrak{p}_j, I_j)_{1 \leq j \leq t}$ witness $(p, n_1; a_2, \ldots, a_l; \gamma\alpha; e, m_1) \in \mathrm{EPP}_{l-1}(R)$ and let $(\mathfrak{p}_j, I_j)_{t+1 \leq j \leq s}$ witness $(p, n_2, a_1, a_3, \ldots, a_l, \gamma(\alpha - 1); e, m_2) \in \mathrm{EPP}_{l-1}(R)$. Then

$$|\oplus_{i=1}^{s} R_{\mathfrak{p}_i}/I_i| = \left|\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/I_i\right| \cdot \left|\oplus_{i=t+1}^{s} R_{\mathfrak{p}_i}/I_i\right| = p^{n_1} p^{n_2} = p^n$$

and

$$|\oplus_{i=1}^{s} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = \left|\oplus_{i=1}^{t} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right| \cdot \left|\oplus_{i=t+1}^{s} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right| = p^{m_1} p^{m_2} = p^m.$$

For $1 \leq j \leq t$, $\gamma\alpha \notin I_j$ and hence $\gamma \notin I_j$ and $\alpha \notin I_j$. Since $a_2 \in I_j$ and $\alpha \notin \mathfrak{p}_j$, $a_1\alpha = a_2 r \in I_j$ implies $a_1 \in I_j$ for $1 \leq j \leq t$. Similarly, $\gamma \notin \mathfrak{m}_j$ and $a_2 \in I_j$ for $t + 1 \leq j \leq s$. Therefore $(\mathfrak{p}_j, I_j)_{1 \leq j \leq s}$ witnesses $(p, n; a_1, \ldots, a_l; \gamma; e, m) \in \mathrm{EPP}_l(R)$. □

Lemma 3.2.9. *Let R be a Prüfer domain. If T_R is decidable then* $\mathrm{EPP}(R)$ *(and hence* $\mathrm{EPP}_l(R)$ *is recursive uniformly in l).*

Proof. For $p \in \mathbb{P}$, $n, m \in \mathbb{N}_0$ and $a, \gamma, e \in R$, let $\Theta_{(p,n;a;\gamma;e,m)}$ be the $\mathcal{L}_R$-sentence

$$|{x=x}/{x=0}| = p^{2m+n} \wedge |{x=x}/{e|x}| = p^m \wedge |{xe=0}/{e|x}| = 1 \wedge |{x=x}/{\gamma|x}| = 1 \wedge \left|{e^2a|x}/{x=0}\right| = 1.$$

We show that, for m, n not both zero, $(p, n; a; \gamma; e, m) \in \mathrm{EPP}_1(R)$ if and only if there is an R-module satisfying $\Theta_{(p,n;a;\gamma;e,m)}$. Hence the lemma holds.

Suppose $(p, n; a; \gamma; e, m) \in \mathrm{EPP}_1(R)$. By definition, there exist prime ideals $\mathfrak{p}_1, \ldots \mathfrak{p}_s \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ such that $\gamma \notin \mathfrak{p}_j$ and $a \in I_j$ for $1 \leq j \leq s$, $| \oplus_{i=1}^s R_{\mathfrak{p}_i}/I_i| = p^n$ and $| \oplus_{i=1}^s R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = p^m$. Let $M := \oplus_{i=1}^s R_{\mathfrak{p}_i}/e^2 I_i$. Then M satisfies $\Theta_{(p,n;a;\gamma;e,m)}$.

Conversely, suppose there exists an R-module M satisfying $\Theta_{(p,n;a;\gamma;e,m)}$. Since M is finite and non-zero, there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s \lhd R$ and proper ideals $J_i \lhd R_{\mathfrak{m}_i}$ such that $\oplus_{i=1}^s R_{\mathfrak{m}_i}/J_i \cong M$. Since M satisfies $\Theta_{(p,n;a;\gamma;e,m)}$, for $1 \leq i \leq s$, $(J_i : e) \subseteq eR_{\mathfrak{m}_i} + J_i$.

By 2.3.13, either $eR_{\mathfrak{m}} + J_i = R_{\mathfrak{m}_i}$ or $J_i \subseteq e^2 R_{\mathfrak{m}_i} + eJ_i$. So, either $e \notin \mathfrak{m}_i$, $J_i \subseteq e^2 R_{\mathfrak{m}_i}$ or $J_i \subseteq eJ_i$. So, for each $1 \leq i \leq s$, there exists $I_i \lhd R_{\mathfrak{m}_i}$ with $J_i = e^2 I_i$.

Since $|{e^2a|x}/{x=0}(R_{\mathfrak{m}_i}/e^2 I_i)| = 1$, $a \in I_i$ and since $|{x=x}/{\gamma|x}(R_{\mathfrak{m}_i}/e^2 I_i)| = 1$, $\gamma \notin \mathfrak{m}_i$. Therefore $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ and $I_i \lhd R_{\mathfrak{m}_i}$ are such that $a \in I_i$ and $\gamma \notin \mathfrak{m}_i$ for $1 \leq i \leq s$, and $| \oplus_{i=1}^s R_{\mathfrak{m}_i}/I_i| = p^n$ and $| \oplus_{i=1}^s R_{\mathfrak{m}_i}/eR_{\mathfrak{m}_i}| = p^m$. Hence $(p, n; a; \gamma; e, m) \in \mathrm{EPP}_1(R)$. □

It follows from the proof of 3.2.9 that if R is a recursive Prüfer domain then $\mathrm{EPP}(R)$ is recursively enumerable. This is because if R is recursive then T_R is recursively axiomatisable and hence we can use a proof algorithm to search for the sentences of the form $\Theta_{(p,n;a;\gamma;e,m)}$, as defined in the proof of 3.2.9, which are true in all R-modules.

The following corollary is a direct consequence of the proof of 3.2.9. We will later see, 7.6, that the converse also holds.

Corollary 3.2.10. *If the theory of R-modules of size n is decidable uniformly in n then* $\mathrm{EPP}_1(R)$ *is recursive.*

3.3. **The set $X(R)$.**

Definition 3.3.1. *Let $X(R)$ be the set of $(p, n; e, \gamma, a, \delta) \in \mathbb{P} \times \mathbb{N} \times (R\backslash\{0\}) \times R^3$ such that there exist integers $h \in \mathbb{N}$ and prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right| = p^n$ and for $1 \leq i \leq h$, $\gamma \notin \mathfrak{m}_i$, and, there exists an ideal $I_i \lhd R_{\mathfrak{p}_i}$ such that $a \in I_i$ and $\delta \notin (I_i)^{\#}$.*

It is often easier to check that $X(R)$ is recursive in concrete rings using the following reformulation.

Remark 3.3.2. *Let $(p, n; e, \gamma, a, \delta) \in \mathbb{P} \times \mathbb{N} \times (R\backslash\{0\}) \times R^3$. Then $(p, n; e, \gamma, a, \delta) \in X(R)$ if and only if there exist $1 \leq h \leq n$ and maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_h$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{m}_i}/eR_{\mathfrak{m}_i}\right| = p^n$, and, for $1 \leq i \leq h$*

(1) $\gamma \notin \mathfrak{m}_i$, and,
(2) either $\delta \notin \mathfrak{m}_i$, or, there exists a prime ideal $\mathfrak{q}_i \subseteq \mathfrak{m}_i$ such that $a \in \mathfrak{q}_i$ and $\delta \notin \mathfrak{q}_i$.

Proof. Note that if $\mathfrak{p}_i$ in the definition of $X(R)$ is such that $|R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = 1$ then we may drop $\mathfrak{p}_i$ from the sequence of prime ideals witnessing $(p, n; e, \gamma, a, \delta) \in X(R)$.

Therefore, we may assume each $\mathfrak{p}_i$ is maximal and that $1 \leq h \leq n$. Now, if $a \in I_i$ and $\delta \notin (I_i)^{\#}$ then either $I_i = R_{\mathfrak{p}_i}$ and $\delta \notin \mathfrak{p}_i$, or, $I_i \lhd R_{\mathfrak{p}_i}$ is a proper ideal. If I_i is a proper ideal then $a \in I_i$ implies $a \in (I_i)^{\#}$.

Therefore, if $(p, n; e, \gamma, a, \delta) \in X(R)$ then the conditions in the statement hold with, $\mathfrak{m}_i := \mathfrak{p}_i$, and $\mathfrak{q}_i := (I_i)^{\#}$ if $\delta \in \mathfrak{m}_i$. Conversely, if the conditions in the statement hold for $(p, n; e, \gamma, a, \delta)$ then set $\mathfrak{p}_i := \mathfrak{m}_i$ and, $I_i := R_{\mathfrak{m}_i}$ if $\delta \notin \mathfrak{m}_i$ and $I_i := \mathfrak{q}_i$ otherwise. □

Proposition 3.3.3. *Let R be a Prüfer domain. If T_R is decidable then $X(R)$ is recursive.*

Proof. Let $(p, n; e, \gamma, a, \delta) \in \mathbb{P} \times \mathbb{N} \times (R \backslash \{0\}) \times R^3$. We show that $(p, n; e, \gamma, a, \delta) \in X(R)$ if and only if there exists an R-module satisfying χ defined as

$$\left|{}^{x=x}/_{e|x}\right| = p^n \wedge \left|{}^{xe=0}/_{e|x}\right| = 1 \wedge \left|{}^{e^2a|x}/_{x=0}\right| = 1 \wedge \left|{}^{x=x}/_{\gamma|x}\right| = 1 \wedge \left|{}^{x\delta=0}/_{x=0}\right| = 1.$$

First suppose that there exist $h \in \mathbb{N}$ and prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h \lhd R$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right| = p^n$ and for $1 \leq i \leq h$, $\gamma \notin \mathfrak{p}_i$, and, there exists an ideal $I_i \lhd R_{\mathfrak{p}_i}$ such that $a \in I_i$ and $\delta \notin (I_i)^{\#}$. Then $\oplus_{i=1}^h R_{\mathfrak{p}_i}/e^2 I_i \models \chi$.

Conversely, suppose there exists an R-module satisfying χ. Then, 2.3.6, there exists a finite direct sum of modules U_i such that $\oplus_{i=1}^h U_i \models \chi$ and each U_i is the restriction to R of a uniserial module over $R_{\mathfrak{p}_i}$ for some prime ideal $\mathfrak{p}_i \lhd R$. We may assume that U_i/U_ie is non-zero for each U_i, for otherwise the direct sum with U_i omitted also satisfies χ. Since U_i is uniserial as an $R_{\mathfrak{p}_i}$-module and U_i/U_ie is non-zero and finite, U_i is finitely generated over $R_{\mathfrak{p}_i}$. Therefore $U_i \cong R_{\mathfrak{p}_i}/J_i$ for some ideal $J_i \lhd R_{\mathfrak{p}_i}$. Since $\left|{}^{xe=0}/_{e|x}(U_i)\right| = 1$, $(J_i : e) \subseteq J_i + eR_{\mathfrak{p}_i}$. So, as in 3.2.9, there exists $I_i \lhd R_{\mathfrak{p}_i}$ such that $J_i = e^2 I_i$.

Now, since $\oplus_{i=1}^h U_i \models \chi$, $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}\right| = p^n$, $e^2 a \in e^2 I$ and hence $a \in I$. Moreover, by 2.3.10, $\gamma \notin \mathfrak{p}_i$ and $\delta \notin I_i^{\#}$. □

It follows from the proof of 3.3.3 that if R is a recursive Prüfer domain then, as with EPP(R) and 3.2.9, $X(R)$ is recursively enumerable.

4. Formalisms

The formalisms introduced in this section will be used throughout the paper to allow us to make reductions in the complexity of certain sets of conditions in later sections.

4.1. Sets of functions.

Let Δ be a set and $\mathcal{E}$ a set of functions from Δ to $\mathbb{N} \cup \{\infty\}$ such that if $h_1, h_2 \in \mathcal{E}$ then $h_1 \cdot h_2 \in \mathcal{E}$ and such that the function which has constant value 1 is in $\mathcal{E}$. Let $n \in \mathbb{N}$, $X, Y \subseteq \Delta$ be finite sets and let $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$.

Define $\Omega_{f,g,n}$ to be the set of all tuples of functions $(f_1, \ldots, f_n, g_1, \ldots, g_n)$ where $f_i : X \cup (Y \backslash Y_i) \to \mathbb{N}$ and $g_i : Y_i \to \mathbb{N}$ are such that $Y_i \subseteq Y$ and

- $\prod_{i=1}^n f_i(x) = f(x)$ for all $x \in X$,
- $f_i(y) < g(y)$ for all $y \in Y \backslash Y_i$,
- $g_i(y) = g(y)$ for all $y \in Y_i$, and
- for all $y \in Y$,

$$\left(\prod_{i \text{ with } y \in Y \backslash Y_i} f_i(y) \cdot \prod_{i \text{ with } y \in Y_i} g_i(y) \right) \geq g(y).$$

Note that $\Omega_{f,g,n}$ may be empty. This happens if and only if there exists $z \in X \cap Y$ such that $g(z) < f(z)$.

For $\mathcal{E}_1, \mathcal{E}_2$ sets of functions from Δ to $\mathbb{N} \cup \{\infty\}$, define $\mathcal{E}_1 \cdot \mathcal{E}_2$ to be the set of $h : \Delta \to \mathbb{N} \cup \{\infty\}$ such that there exist $h_1 \in \mathcal{E}_1$ and $h_2 \in \mathcal{E}_2$ such that $h = h_1 \cdot h_2$.

The most important instance of this set up in this paper is when Δ is a set of pp-pairs φ/ψ over a ring R and $\mathcal{E}$ is the set of R-modules M viewed as functions on Δ by setting $M(\varphi/\psi) := |\varphi/\psi(M)|$. Given finite sets $X, Y \subseteq \Delta$ and functions $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$, we want to know whether there exists an R-module $M \in \mathcal{E}$ such that $M(\varphi/\psi) = f(\varphi/\psi)$ for all $\varphi/\psi \in X$ and $M(\varphi/\psi) \geq g(\varphi/\psi)$ for all $\varphi/\psi \in Y$. Now suppose that $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are sets of R-modules with the property that all R-modules in $\mathcal{E}$ are elementary equivalent to a module of the form $M_1 \oplus \ldots \oplus M_n$ with $M_i \in \mathcal{E}_i$ for $1 \leq i \leq n$. Viewing modules as functions on Δ, this gives that $\mathcal{E} = \prod_{i=1}^n \mathcal{E}_i$ i.e. viewed as functions on Δ every module in $\mathcal{E}$ is equal to a module of form $M_1 \oplus \cdots \oplus M_n$ where $M_i \in \mathcal{E}_i$ for $1 \leq i \leq n$. The next lemma, interpreted for modules, will show that there exists an R-module $M \in \mathcal{E}$ such that $M(\varphi/\psi) = f(\varphi/\psi)$ for all $\varphi/\psi \in X$ and $M(\varphi/\psi) \geq g(\varphi/\psi)$ for all $\varphi/\psi \in Y$ if and only if, for some $(f_1, \ldots, f_n, g_1, \ldots, g_n) \in \Omega_{f,g,n}$, there exist R-modules $M_i \in \mathcal{E}_i$ for $1 \leq i \leq n$ such that for $M_i(\varphi/\psi) = f_i(\varphi/\psi)$ for all $\varphi/\psi \in X_i$ and $M(\varphi/\psi) \geq g_i(\varphi/\psi)$ for all $\varphi/\psi \in Y_i$.

Lemma 4.1.1. *Let $\mathcal{E} = \prod_{i=1}^n \mathcal{E}_i$, $X, Y \subseteq \Delta$ be finite sets and let $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$. There exists $h \in \mathcal{E}$ such that $h(x) = f(x)$ for all $x \in X$ and $h(y) \geq g(y)$ for all $y \in Y$ if and only if for some $(f_1, \ldots, f_n, g_1, \ldots, g_n) \in \Omega_{f,g,n}$ there exist $h_i \in \mathcal{E}_i$ for $1 \leq i \leq n$ such that $h_i(x) = f_i(x)$ for all $x \in X \cup (Y \backslash Y_i)$ and $h_i(y) \geq g_i(y)$ for all $y \in Y_i$.*

Proof. Let $h \in \mathcal{E}$ be such that $h(x) = f(x)$ for all $x \in X$ and $h(y) \geq g(y)$ for all $y \in Y$. Since $\mathcal{E} = \prod_{i=1}^n \mathcal{E}_i$, there exist $h_i \in \mathcal{E}_i$ for $1 \leq i \leq n$ such that $\prod_{i=1}^n h_i(x) = h(x)$ for all $x \in \Delta$. For each $1 \leq i \leq n$, let $Y_i := \{y \mid h_i(y) \geq g(y)\}$, let $f_i(x) = h_i(x)$ for all $x \in X \cup (Y \backslash Y_i)$, and let $g_i(y) = g(y)$ for all $y \in Y_i$. By definition $\prod_{i=1}^n f_i(x) = \prod_{i=1}^n h_i(x) = h(x) = f(x)$ for all $x \in X$, $f_i(y) = h_i(y) < g(y)$ for all $y \in Y \backslash Y_i$ and $g_i(y) = g(y)$ for all $y \in Y_i$. Now if $y \in Y \backslash Y_i$ then $f_i(y) = h_i(y)$ and if $y \in Y_i$ then $g_i(y) = g(y)$. Therefore, for all $y \in Y$, either $y \in Y_i$ for some $1 \leq i \leq n$ and by definition, $g_i(y) \geq g(y)$, or, $y \notin Y_i$ for all $1 \leq i \leq n$ and so $g(y) \leq h(y) = \prod_{i=1}^n h_i(y) = \prod_{i=1}^n f_i(y)$. In either case, the 4th condition in the definition of $\Omega_{f,g,n}$ holds. So $(f_1, \ldots, f_n, g_1, \ldots, g_n) \in \Omega_{f,g,n}$.

Conversely, suppose $h_1, \ldots, h_n : \Delta \to \mathbb{N}$ are such that there exists $(f_1, \ldots, f_n, g_1, \ldots, g_n) \in \Omega_{f,g,n}$ with $h_i(x) = f_i(x)$ for all $x \in X \cup (Y \backslash Y_i)$ and $h_i(y) \geq g_i(y)$ for all $y \in Y_i$. Define $h : \Delta \to \mathbb{N}$ by $h(x) = \prod_{i=1}^n h_i(x)$ for all $x \in \Delta$. Then $h(x) = f(x)$ for all $x \in X$ and $h(y) \geq g(y)$ for all $y \in Y$ as required. □

Definition 4.1.2. *Let $X, Y \subseteq \Delta$ be finite sets and let $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$. Let $G := \max\{g(y) \mid y \in Y\}$. Define $\Theta_{f,g}$ to be the set of pairs of functions (f', g') such that, for some $Y' \subseteq Y$, $f' : X \cup (Y \backslash Y') \to \mathbb{N}$, $g' : Y' \to \mathbb{N}$, $f'(x) = f(x)$ for all $x \in X$, $g'(y) = G$ for all $y \in Y'$ and $g(y) \leq f'(y) < G$ for all $y \in Y \backslash Y'$.*

As for $\Omega_{f,g,n}$, in the definition of $\Theta_{f,g}$, we don't intend that X and Y are disjoint.

Remark 4.1.3. *A function $h \in \mathcal{E}$ is such that $h(x) = f(x)$ for all $x \in X$ and $h(y) \geq g(y)$ for all $y \in Y$ if and only if there exists $(f', g') \in \Theta_{f,g}$ such that $h(x) = f'(x)$ for all $x \in X \cup (Y \backslash Y')$ and $h(y) \geq g'(y)$ for all $y \in Y'$, where Y' is the domain of the function g' as in the definition of $\Theta_{f,g}$.*

Proof. Suppose that $h \in \mathcal{E}$ is such that $h(x) = f(x)$ for all $x \in X$ and $h(y) \geq g(y)$ for all $y \in Y$. Let $G = \max\{g(y) \mid y \in Y\}$ and let $Y' = \{y \in Y \mid h(y) \geq G\}$. Set $f'(x) = h(x)$ for all $x \in X \cup (Y\backslash Y')$ and $g'(y) = G$ for all $y \in Y'$. By definition $h(x) \geq g'(y)$ for all $y \in Y'$. So, we just need to check that $(f', g') \in \Theta_{f,g}$. The first 2 conditions defining $\Theta_{f,g}$ hold automatically. By definition of Y', if $y \in Y\backslash Y'$ then $h(y) < G$. By definition of f', $f'(y) = h(y)$ for all $y \in Y\backslash Y'$. Hence, for all $y \in Y\backslash Y'$, $f'(y) < G$ and, by hypothesis on h, $g(y) \leq h(y) = f'(y)$ as required.

Suppose that $h \in \mathcal{E}$ and $(f', g') \in \Theta_{f,g}$ is such that $h(x) = f'(x)$ for all $x \in X \cup (Y\backslash Y')$ and $h(y) \geq g'(y)$ for all $y \in Y'$. Then, by definition of $\Theta_{f,g}$, $h(x) = f'(x) = f(x)$ for all $x \in X$. If $y \in Y\backslash Y'$ then, by definition of $\Theta_{f,g}$, $g(y) \leq f'(y) = h(y)$. If $y \in Y'$ then $h(y) \geq g'(y) = G \geq g(y)$. Therefore, for all $y \in Y$, $h(y) \geq g(y)$ as required. □

4.2. **Lattices generated by conditions.**
Let W be an infinite set. Let $\mathbb{W}$ be the free bounded distributive lattice[7] generated by W. We use $\sqcup$ for the supremum and $\sqcap$ for the infimum in this lattice. Any element of $\mathbb{W}$ may be expressed as $\bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij}$ where I and J_i for $i \in I$ are finite sets and $w_{ij} \in W$. Moreover, for $v_k \in W$ with $k \in K$ a finite set,

$$\bigsqcap_{k\in K} v_k \leq \bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij}$$

if and only if there exists $i \in I$ such that

$$\bigsqcap_{k\in K} v_k \leq \bigsqcap_{j\in J_i} w_{ij}$$

if and only if there exists $i \in I$ such that

$$\{v_k \mid k \in K\} \supseteq \{w_{ij} \mid j \in J_i\}.$$

We make the convention that the empty infimum is the largest element $\top$ and the empty supremum is the least element $\bot$.

We call an expression of the form $\bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij}$, where $w_{ij} \in W$, **irredundant** if for each $i \in I$, $w_{ij_1} = w_{ij_2}$ implies $j_i = j_2$ and the sets $w_i := \{w_{ij} \mid j \in J_i\}$ for $i \in I$ are pairwise incomparable by inclusion. If $\bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij}$ and $\bigsqcup_{i\in I'} \bigsqcap_{j\in J'_i} w'_{ij}$ are in irredundant form then $\bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij} = \bigsqcup_{i\in I'} \bigsqcap_{j\in J'_i} w'_{ij}$ if and only if there exist bijections $\sigma : I \to I'$ and $\sigma_i : J_i \to J'_{\sigma(i)}$ for each $i \in I$ such that $w_{ij} = w'_{\sigma(i),\sigma_i(j)}$ for all $i \in I$ and $j \in I_i$.

Given a recursive presentation of W (i.e. a bijection with $\mathbb{N}$), this presentation of W gives rise to a recursive presentation of $\mathbb{W}$ (i.e a presentation where the inclusion of W in $\mathbb{W}$ is recursive and $\sqcup$ and $\sqcap$ are recursive functions).

For any $V \subseteq W$, define $\mathbb{V}$ to be the filter generated by V in $\mathbb{W}$. Note for $w_{ij} \in W$, $\bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij} \in \mathbb{V}$ if and only if there exists $i \in I$ such that $w_{ij} \in V$ for all $j \in J_i$. So, in particular $\mathbb{V}$ is prime filter. It follows that V is a recursive subset of W if and only if $\mathbb{V}$ is a recursive subset of $\mathbb{W}$.

Suppose that $\text{clx} : W \to \alpha$ where α is a partially ordered set with the descending chain condition. Let $\underline{w} \in \mathbb{W}$ and let $\underline{w} = \bigsqcup_{i\in I} \bigsqcap_{j\in J_i} w_{ij}$ be in irredundant form. For $\beta \in \alpha$, we write $\text{clx}\underline{w} \leq \beta$ if $w_{ij} \leq \beta$ for all $i \in I$ and $j \in J_i$ and $\text{clx}\underline{w} < \beta$ if $\text{clx}w_{ij} < \beta$ for all $i \in I$ and $j \in J_i$. Note that if $\underline{w}$ is a lattice combination

[7]See [Grä11] for the definition of a free distributive lattice and add a largest and smallest element.

of elements $w_i \in W$ for $1 \leq i \leq n$ then $\text{clx} w_i \leq \beta$ (respectively $\text{clx} w_i < \beta$) for $1 \leq i \leq n$ implies $\text{clx}\underline{w} \leq \beta$ (respectively $\text{clx}\underline{w} < \beta$).

Remark 4.2.1. *Let W be an infinite recursively presented set and $V \subseteq W$. Suppose that α is an artinian recursive partially ordered set, $\text{clx} : W \to \alpha$ is recursive and $W_0 \subseteq W$ is recursive. Suppose further that there is an algorithm which given $w \notin W_0$ computes $\underline{w} \in \mathbb{W}$ such that $clx\underline{w} < clxw$ and such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$. Then V is a recursive subset of W if and only if $V \cap W_0$ is a recursive subset of W.*

The precise choice of W and V varies throughout this article.

To illustrate how this setup is used, let R be a recursive ring. Let W be the set of $\mathcal{L}_R$-sentences

$$\bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi)$$

where X, Y are finite sets of pp-pairs, $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$. Let V be the set of $w \in W$ such that there exists $M \in \text{Mod-}R$ with $M \models w$. Then, by 2.2.1, T_R is decidable if and only if V is recursive. Working with $\mathbb{W}$ and $\mathbb{V}$, rather than W and V directly, allows us to talk about more than one module at a time. For instance, for $w_1, \ldots, w_n \in W$, the condition $w_1 \sqcap \ldots \sqcap w_n \in \mathbb{V}$ says that there exist R-modules $M_i \in \text{Mod-}R$ with $M_i \models w_i$ for $1 \leq i \leq n$.

5. First syntactic reductions

Recall that, for a recursive ring R, in order to show that the theory of R-modules is decidable, it is enough to show that there is an algorithm which, given a sentence of the form

$$(\star) \qquad \bigwedge_{i=1}^{s} |\varphi_i/\psi_i| = F_i \wedge \bigwedge_{j=1}^{t} |\sigma_j/\tau_j| \geq G_j,$$

where, for $1 \leq i \leq s$ and $1 \leq j \leq t$, φ_i/ψ_i and σ_j/τ_j are pp-pairs and $F_i, G_j \in \mathbb{N}$, answers whether there exists an R-module satisfying it.

In [GLT19, 4.1], it was shown that if R is a recursive Prüfer domain then it is enough to consider sentences where the pp-pairs in $(\star)$ are all of the form $d|x/x=0$ and $xb=0/c|x$. The proof of this statement relies on 2.3.1, [PT15, 2.2] and the fact, which follows from 2.1.2 and 2.3.2, that every R-module is elementary equivalent to a direct sum of pp-uniserial modules. This is also true for arithmetical rings, and so, although not stated in [GLT19], the result, with the same proof, also holds for arithmetical rings.

Theorem 5.1. [GLT19, 4.1] *Let R be a recursive arithmetical ring. If there exists an algorithm which, given a sentence*

$$\chi := \bigwedge_{i=1}^{m} |\varphi_i/\psi_i| = G_i \wedge \bigwedge_{i=m+1}^{n} |\varphi_i/\psi_i| \geq H_i,$$

where $G_i, H_i \in \mathbb{N}$ and φ_i/ψ_i are pp-pairs of the form $d|x/x=0$ and $xb=0/c|x$ for $1 \leq i \leq n$, answers whether there exists $M \in \text{Mod-}R$ satisfying χ, then T_R is decidable.

We call any conjunction of sentences of the form

$$|d|x/x=0| = 1 \quad \text{or} \quad |xb=0/c|x| = 1$$

an **auxiliary sentence**.

Convention: In the sequel, we use the symbol $\square$ as a variable denoting either $=$ or $\geq$ when talking about conjunctions of sentences like $|\varphi/\psi|\,\square N$. It will be useful for us to extend this notation so that $\square$ can also be the symbol $\emptyset$, where $\square$ being $\emptyset$ indicates that $|\varphi/\psi|\,\square N$ is omitted from the conjunction. For instance, when $\square_1$ is $\emptyset$ and $\square_2$ is $\geq$, the sentence $|{}^{d|x}/_{x=0}|\,\square_1 D \wedge |{}^{xb=0}/_{c|x}|\,\square_2 E$ stands for $|{}^{xb=0}/_{c|x}| \geq E$.

In this section we improve [GLT19, 4.1] to prove the following.

Theorem 5.2. *Let R be arithmetical ring. If there exists an algorithm which, given a sentence χ of the form*

$$(\dagger)\qquad |{}^{d|x}/_{x=0}|\,\square_1 D \wedge |{}^{xb=0}/_{c|x}|\,\square_2 E \wedge \bigwedge_{i=1}^{m} |{}^{\varphi_i}/_{\psi_i}| = G_i \wedge \bigwedge_{i=m+1}^{n} |{}^{\varphi_i}/_{\psi_i}| \geq H_i \wedge \Xi,$$

where $\square_1, \square_2 \in \{\geq, =, \emptyset\}$, $d, c, b \in R\backslash\{0\}$, $D, E, G_i, H_i \in \mathbb{N}$, Ξ is an auxiliary sentence and ${}^{\varphi_i}/_{\psi_i}$ are pp-pairs of the form ${}^{x=x}/_{c'|x}$ and ${}^{xb'=0}/_{x=0}$ for $1 \leq i \leq n$, answers whether there exists an R-module satisfying χ, then T_R is decidable.

Definition 5.3. *Let X, Y be finite subsets of pp-pairs of the form ${}^{d|x}/_{x=0}$ or ${}^{xb=0}/_{c|x}$, and, let $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$ be functions. Define $\chi_{f,g}$ to be the sentence*

$$\bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi).$$

If X and Y are both empty then $\chi_{f,g}$ should be read as the true sentence.

For the rest of this section, let W be the set of $\mathcal{L}_R$-sentences of the form $\chi_{f,g}$ and let V be the set of $w \in W$ such that there exists $M \in$ Mod-R with $M \models w$. As in 4.2, $\mathbb{W}$ denotes the bounded distributive lattice generated by W and $\mathbb{V}$ denotes the (prime) filter in $\mathbb{W}$ generated by V.

Define $\mathrm{clx}_1\chi_{f,g}$ to be

$$\left|\{{}^{d|x}/_{x=0} \in X \mid f({}^{d|x}/_{x=0}) > 1\}\right| + \left|\{{}^{d|x}/_{x=0} \in Y \mid g({}^{d|x}/_{x=0}) > 1\}\right|$$

and $\mathrm{clx}_2\chi_{f,g}$ to be

$$\left|\{{}^{xb=0}/_{c|x} \in X \mid f({}^{xb=0}/_{c|x}) > 1 \text{ and } b, c \neq 0\}\right| + \left|\{{}^{xb=0}/_{c|x} \in Y \mid g({}^{xb=0}/_{c|x}) > 1 \text{ and } b, c \neq 0\}\right|.$$

Formally, we extend clx_1 and clx_2 to $\bot, \top \in \mathbb{W}$ by setting $\mathrm{clx}_1\bot = \mathrm{clx}_1\top = 0$ and $\mathrm{clx}_2\bot = \mathrm{clx}_2\top = 0$. We will use the notation $\mathrm{clx}_i\underline{w} \leq \mathrm{clx}_i w$ and $\mathrm{clx}_i\underline{w} < \mathrm{clx}_i w$, for $i \in \{1, 2\}$, $\underline{w} \in \mathbb{W}$ and $w \in W$ as defined in subsection 4.2.

Remark 5.4. *For all $w_1, w_2 \in W$ and $i \in \{1, 2\}$,*

$$\mathrm{clx}_i(w_1 \wedge w_2) \leq \mathrm{clx}_i(w_1) + \mathrm{clx}_i(w_2).$$

For our purposes, given $w \in W$, we may always assume that w is of the form

$$\chi_{f,g} \wedge \Xi,$$

where $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$ with X, Y finite disjoint sets of pp-pairs of the form ${}^{d|x}/_{x=0}$ or ${}^{xb=0}/_{c|x}$, and Ξ is an auxiliary sentence. This is because any $\chi_{f,g} \in W$ may be rewritten as $\chi_{f',g} \wedge \Xi$ where $f' : X' \to \mathbb{N}_2$ and Ξ is an auxiliary sentence. Moreover, for $\chi_{f,g} \wedge \Xi \in W$, let $Y' := \{\varphi/\psi \mid g(\varphi/\psi) > 1\}$ and $g' := g|_{Y'}$. Then $\chi_{f,g} \wedge \Xi \in V$ if and only if $\chi_{f,g'} \wedge \Xi \in V$. If $\varphi/\psi \in X \cap Y$ and $f(\varphi/\psi) < g(\varphi/\psi)$ then $T_R \models \neg\chi_{f,g}$. If $f(\varphi/\psi) \geq g(\varphi/\psi)$ then $\chi_{f,g} \wedge \Xi \in V$ if and only if $\chi_{f,g''} \wedge \Xi \in V$

where $g'' := g|_{Y\setminus\{\varphi/\psi\}}$. So, given $w \in W$, we can effectively decide that $w \notin V$ or compute f, g and Ξ of the required form such that, $\mathrm{clx}_1(\chi_{f,g} \wedge \Xi) \leq \mathrm{clx}_1 w$, $\mathrm{clx}_2(\chi_{f,g} \wedge \Xi) \leq \mathrm{clx}_2 w$, and, $w \in V$ if and only if $\chi_{f,g} \wedge \Xi \in V$.

Remark 5.5. *Let X, Y be disjoint finite sets of pp-pairs of the form ${}^{d|x}/_{x=0}$ or ${}^{xb=0}/_{c|x}$, $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$ functions, and Ξ an auxiliary sentence. For each $1 \leq j \leq n$, let θ_j be an auxiliary sentence. Suppose that for all $M \in \text{Mod-}R$, there exist modules $M_j \models \theta_j$ such that $M \equiv \oplus_{j=1}^n M_j$. Then $\chi_{f,g} \wedge \Xi \in V$ if and only if*

$$\bigsqcup_{(\overline{f},\overline{g})\in\Omega_{(f,g,n)}} \prod_{j=1}^{n} \chi_{f_j,g_j} \wedge \theta_j \wedge \Xi \in \mathbb{V}.$$

Moreover, for all $(\overline{f}, \overline{g}) \in \Omega_{f,g,n}$ and $1 \leq j \leq n$,

$$\mathrm{clx}_1(\chi_{f_j,g_j} \wedge \theta_j \wedge \Xi) \leq \mathrm{clx}_1(\chi_{f,g} \wedge \Xi) \text{ and } \mathrm{clx}_2(\chi_{f_j,g_j} \wedge \theta_j \wedge \Xi) \leq \mathrm{clx}_2(\chi_{f,g} \wedge \Xi).$$

The next lemma is more precise than we need in this section. However, we will need its full strength in 9.2.1 of section 9. The total order $\prec$ on the set $\{\emptyset, =, \leq\}$ is defined as $\emptyset \prec = \prec \leq$.

Lemma 5.6. *Let $\varphi/\psi, \varphi'/\psi', \sigma/\tau$ be pp-pairs and let Σ be an $\mathcal{L}_R$-sentence. Suppose that $M \models \Sigma$ implies*

$$|\varphi/\psi(M)| = |\sigma/\tau(M)| \cdot |\varphi'/\psi'(M)|$$

for all $M \in \text{Mod-}R$.

There is an algorithm which, given $\varphi/\psi, \varphi'/\psi', \sigma/\tau$, $\Box, \Box' \in \{=, \geq\}$, $E, E' \in \mathbb{N}_2$ and Σ as above, either returns $\Omega := \{\bot\}$, in which case

$$T_R \models \neg(\Sigma \wedge |\varphi/\psi| \Box E \wedge |\varphi'/\psi'| \Box' E'),$$

or, returns Ω, a finite set of tuples $(D_1, D_2, \Box_1, \Box_2) \in \mathbb{N}^2 \times \{=, \geq\}^2$ such that

$$T_R \models \Sigma \wedge |\varphi/\psi| \Box E \wedge |\varphi'/\psi'| \Box' E' \leftrightarrow \bigvee_{(D_1,D_2,\Box_1,\Box_2)\in\Omega} \Sigma \wedge |\sigma/\tau| \Box_1 D_1 \wedge |\varphi'/\psi'| \Box_2 D_2$$

and $D_1 \cdot D_2 < E \cdot E'$, $\Box_1 \preceq \Box$ and $\Box_2 \preceq \Box'$ for all $(D_1, D_2, \Box_1, \Box_2) \in \Omega$.

Proof.
Case 1: $\Box$ and $\Box'$ are both $=$.
Let $\Omega := \{\bot\}$ if E' does not divide E, otherwise $\Omega := \{(E/E', E', =, =)\}$. Note $(E/E') \cdot E' = E < E \cdot E'$.
Case 2: $\Box$ is $\geq$ and $\Box'$ is $=$.
For $x \in \mathbb{R}$, we write $\lceil x \rceil$ for the least $m \in \mathbb{Z}$ with $x \leq m$. Let $\Omega := \{(\lceil E/E' \rceil, E', \geq, =)\}$. Note

$$\lceil E/E' \rceil \cdot E' < (E/E' + 1) \cdot E' = E + E' \leq E \cdot E'.$$

Case 3: $\Box$ is $=$ and $\Box'$ is $\geq$.
Let $X := \{D \in \mathbb{N} \mid D|E \text{ and } D \geq E'\}$. Define $\Omega := \{\bot\}$ if $X = \emptyset$ and $\Omega := \{(E/D, D, =, =) \mid D \in X\}$ otherwise. Note $(E/D) \cdot D = E < E \cdot E'$.
Case 4: $\Box$ and $\Box'$ are both $\geq$.
If $E' \geq E$ then let $\Omega := \{(1, E', \geq, \geq)\}$. If $E > E'$ then set

$$\Omega := \{(\lceil E/D \rceil, D, \geq, =) \mid E > D \geq E'\} \cup \{(1, E, \geq, \geq)\}.$$

Note that $E' < E \cdot E'$, $E < E \cdot E'$ and

$$\lceil E/D \rceil \cdot D < (E/D + 1) \cdot D = E + D \leq E \cdot E'. \qquad \Box$$

The following remark is easy to prove. We record it here because we will use it frequently.

Remark 5.7. *Let R be a commutative ring. For all $a, b \in R$ and $M \in \text{Mod-}R$,*

$$\left|{}^{a|x}/_{ab|x}(M)\right| = \left|{}^{x=x}/_{xa=0+b|x}(M)\right|$$

and

$$\left|{}^{xab=0}/_{xa=0}(M)\right| = \left|{}^{a|x\wedge xb=0}/_{x=0}(M)\right|.$$

Proposition 5.8. *Let R be a recursive arithmetical ring. There is an algorithm which given $w \in W$ with $\text{clx}_1(w) > 1$ outputs $\underline{w} \in \mathbb{W}$ such that $\text{clx}_1(\underline{w}) < \text{clx}_1(w)$, $\text{clx}_2(\underline{w}) \leq \text{clx}_2(w)$, and, $w \in V$ if and only if $\underline{w} \in \mathbb{V}$.*

Proof. Let X, Y be disjoint finite sets of pp-pairs of the form ${}^{d|x}/_{x=0}$ and ${}^{xb=0}/_{c|x}$. Let $f : X \to \mathbb{N}_2$ and $g : Y \to \mathbb{N}_2$ be functions, and let Ξ be an auxiliary sentence. Let w be

$$\bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi) \wedge \Xi.$$

Suppose that there exist non-equal $a, b \in R$ such that ${}^{a|x}/_{x=0}, {}^{b|x}/_{x=0} \in X \cup Y$ i.e. $\text{clx}_1(w) > 1$. Let $\alpha, r, s \in R$ be such that $a\alpha = br$ and $b(\alpha - 1) = as$. Define

(1) Σ_1 to be $\left|{}^{x=x}/_{\alpha|x}\right| = 1 \wedge \left|{}^{rb|x}/_{x=0}\right| = 1$,
(2) Σ_2 to be $\left|{}^{x=x}/_{\alpha|x}\right| = 1 \wedge \left|{}^{xb=0}/_{r|x}\right| = 1$,
(3) Σ_3 to be$\left|{}^{x=x}/_{(\alpha-1)|x}\right| = 1 \wedge \left|{}^{as|x}/_{x=0}\right| = 1$, and
(4) Σ_4 to be$\left|{}^{x=x}/_{(\alpha-1)|x}\right| = 1 \wedge \left|{}^{xa=0}/_{s|x}\right| = 1$.

It follows directly from 2.3.4 that, for any $M \in \text{Mod-}R$, there are $M_i \models \Sigma_i$ for $1 \leq i \leq 4$ such that $M \equiv M_1 \oplus M_2 \oplus M_3 \oplus M_4$. Therefore, by 5.5, $w \in V$ if and only if

$$\bigsqcup_{(\overline{f},\overline{g}) \in \Omega_{f,g,4}} \prod_{i=1}^{4} \chi_{f_i,g_i} \wedge \Xi \wedge \Sigma_i \in \mathbb{V}.$$

For each $(\overline{f}, \overline{g}) \in \Omega_{f,g,4}$ and $1 \leq i \leq 4$, it is enough to compute $\underline{w}_i \in \mathbb{W}$ such that $\text{clx}_1(\underline{w}_i) < \text{clx}_1(\chi_{f,g} \wedge \Xi)$, $\text{clx}_2(\underline{w}_i) \leq \text{clx}_2(\chi_{f,g} \wedge \Xi)$ and $\chi_{f_i,g_i} \wedge \Xi \wedge \Sigma_i \in V$ if and only if $\underline{w}_i \in V$.

Fix $(\overline{f}, \overline{g}) \in \Omega_{f,g,4}$. For each $1 \leq i \leq 4$, let X_i be the domain of f_i and Y_i be the domain of g_i.

Case i=1: Suppose $M \models \Sigma_1$. Then $M\alpha = M$ and hence $Ma = Mbr = 0$. Therefore, if $M \models \Sigma_1$ then $\left|{}^{a|x}/_{x=0}(M)\right| = 1$.

If ${}^{a|x}/_{x=0} \in X_1$ and $f_1({}^{a|x}/_{x=0}) = 1$ then $\text{clx}_1(\chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1) < \text{clx}_1(\chi_{f,g} \wedge \Xi)$ and, by 5.5, $\text{clx}_2(\chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1) \leq \text{clx}_2(\chi_{f,g} \wedge \Xi)$. So, $\underline{w}_i := \chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1$ has the required properties.

If ${}^{a|x}/_{x=0} \in X_1$ and $f_1({}^{a|x}/_{x=0}) \neq 1$ then, by the first paragraph, $\chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1$ is not satisfied by any R-module. If ${}^{a|x}/_{x=0} \notin X_1$ then ${}^{a|x}/_{x=0} \in Y_1$ since $X \cup Y = X_1 \cup Y_1$. Moreover $g_1({}^{a|x}/_{x=0}) = g({}^{a|x}/_{x=0})$. So $g_1({}^{a|x}/_{x=0}) > 1$ and hence $\chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1$ is not satisfied by any R-module. In either case, set $\underline{w}_i := \perp$. Then $\underline{w}_i \in \mathbb{V}$ if and only if $\chi_{f_1,g_1} \wedge \Xi \wedge \Sigma_1 \in V$. By definition $\text{clx}_1(\perp) < \text{clx}_1(\chi_{f,g} \wedge \Xi)$ and $\text{clx}_2(\perp) \leq \text{clx}_2(\chi_{f,g} \wedge \Xi)$.

Case i=2: Suppose $M \models \Sigma_2$. Then $Ma = M\alpha a = Mbr$. So, since $xb = 0 \leq_M r|x$, by 5.7,

$$\left|{}^{b|x}/_{x=0}(M)\right| = \left|{}^{b|x}/_{br|x}(M)\right| \cdot \left|{}^{br|x}/_{x=0}(M)\right| = \left|{}^{x=x}/_{r|x}(M)\right| \cdot \left|{}^{a|x}/_{x=0}(M)\right|.$$

Let $X' = X_2 \backslash \{{}^{a|x}/_{x=0}, {}^{b|x}/_{x=0}\}$ and $Y' = Y_2 \backslash \{{}^{a|x}/_{x=0}, {}^{b|x}/_{x=0}\}$. Let $\square, \square' \in \{=, \geq\}$ and A, B be such that $\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2$ is

$$|{}^{a|x}/_{x=0}| \,\square A \wedge |{}^{b|x}/_{x=0}| \,\square' B \wedge \bigwedge_{\varphi/\psi \in X'} |\varphi/\psi| = f_2(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y'} |\varphi/\psi| \geq g_2(\varphi/\psi) \wedge \Xi \wedge \Sigma_2.$$

We may assume that $A, B > 1$ since otherwise $\mathrm{clx}_1(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2) < \mathrm{clx}_1(\chi_{f,g} \wedge \Xi)$. So, by 5.6, there is an algorithm which either returns $\Omega := \{\bot\}$, in which case

$$T_R \models \neg\Sigma_2 \wedge |{}^{a|x}/_{x=0}| \,\square A \wedge |{}^{b|x}/_{x=0}| \,\square' B,$$

or, a set $\Omega \subseteq \mathbb{N}^2 \times \{=, \geq\}^2$ such that

$$\Sigma_2 \wedge |{}^{a|x}/_{x=0}| \,\square A \wedge |{}^{b|x}/_{x=0}| \,\square' B$$

is equivalent, with respect to T_R, to

$$\bigvee_{(D_1,D_2,\square_1,\square_2) \in \Omega} \Sigma_2 \wedge |{}^{x=x}/_{r|x}| \,\square_1 D_1 \wedge |{}^{a|x}/_{x=0}| \,\square_2 D_2.$$

If $\Omega := \{\bot\}$ then $\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2 \in V$ if and only if $\bot \in \mathbb{V}$. By definition $\mathrm{clx}_1(\bot) < \mathrm{clx}_1(\chi_{f,g} \wedge \Xi)$ and $\mathrm{clx}_2(\bot) \leq \mathrm{clx}_2(\chi_{f,g} \wedge \Xi)$. Otherwise, for each $(D_1, D_2, \square_1, \square_2) \in \Omega$, let $u_{(D_1,D_2,\square_1,\square_2)}$ be

$$|{}^{x=x}/_{r|x}| \,\square_1 D_1 \wedge |{}^{a|x}/_{x=0}| \,\square_2 D_2 \wedge \bigwedge_{\varphi/\psi \in X'} |\varphi/\psi| = f_2(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y'} |\varphi/\psi| \geq g_2(\varphi/\psi) \wedge \Xi \wedge \Sigma_2.$$

Then $\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2$ is equivalent to

$$\bigvee_{(D_1,D_2,\square_1,\square_2) \in \Omega} u_{(D_1,D_2,\square_1,\square_2)}$$

with respect to T_R. Therefore $\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2 \in V$ if and only if

$$\bigsqcup_{(D_1,D_2,\square_1,\square_2) \in \Omega} u_{(D_1,D_2,\square_1,\square_2)} \in \mathbb{V}.$$

Moreover,

$$\mathrm{clx}_1(u_{(D_1,D_2,\square_1,\square_2)}) \leq 1 + (\mathrm{clx}_1(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2) - 2) < \mathrm{clx}_1(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2)$$

and $\mathrm{clx}_2(u_{(D_1,D_2,\square_1,\square_2)}) = \mathrm{clx}_2(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2)$. So we are done, since, by 5.5, $\mathrm{clx}_1(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2) \leq \mathrm{clx}_1(\chi_{f,g} \wedge \Xi)$ and $\mathrm{clx}_2(\chi_{f_2,g_2} \wedge \Xi \wedge \Sigma_2) \leq \mathrm{clx}_2(\chi_{f,g} \wedge \Xi)$.

The case $i = 3$ is similar to $i = 1$ and the case $i = 4$ is similar to $i = 2$. $\square$

Our task now is to show that there is an algorithm which given $w \in W$ with $\mathrm{clx}_2 w > 1$ returns $\underline{w} \in \mathbb{W}$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$, $\mathrm{clx}_1(\underline{w}) \leq \mathrm{clx}_1(w)$ and $\mathrm{clx}_2(\underline{w}) < \mathrm{clx}_2(w)$. This uses the same ideas as for clx_1 but is somewhat more complicated.

Lemma 5.9. *Let R be an arithmetical ring and let $b, c, b', c' \in R$. Let α, u, u', $\beta, \beta', s, s', r, r', \delta, \delta', \lambda, \lambda', \mu, \mu' \in R$ be such that*

$$\begin{array}{rclrclrclrcl} & & & c\alpha & = & c'u, & c'(\alpha-1) & = & cu', & & & \\ u\beta & = & b'r, & b'(\beta-1) & = & us, & u'\beta' & = & br', & b(\beta'-1) & = & u's', \\ b\delta & = & s\lambda, & s(\delta-1) & = & b\mu, & b\delta' & = & s'\lambda', & s'(\delta'-1) & = & b'\mu'. \end{array}$$

Define the sentences Λ_i and Λ'_i for $1 \leq i \leq 6$ and P_i and P'_i for $4 \leq i \leq 6$ to be the conjunction of sentences labeling the edges in the path from the root of the tree

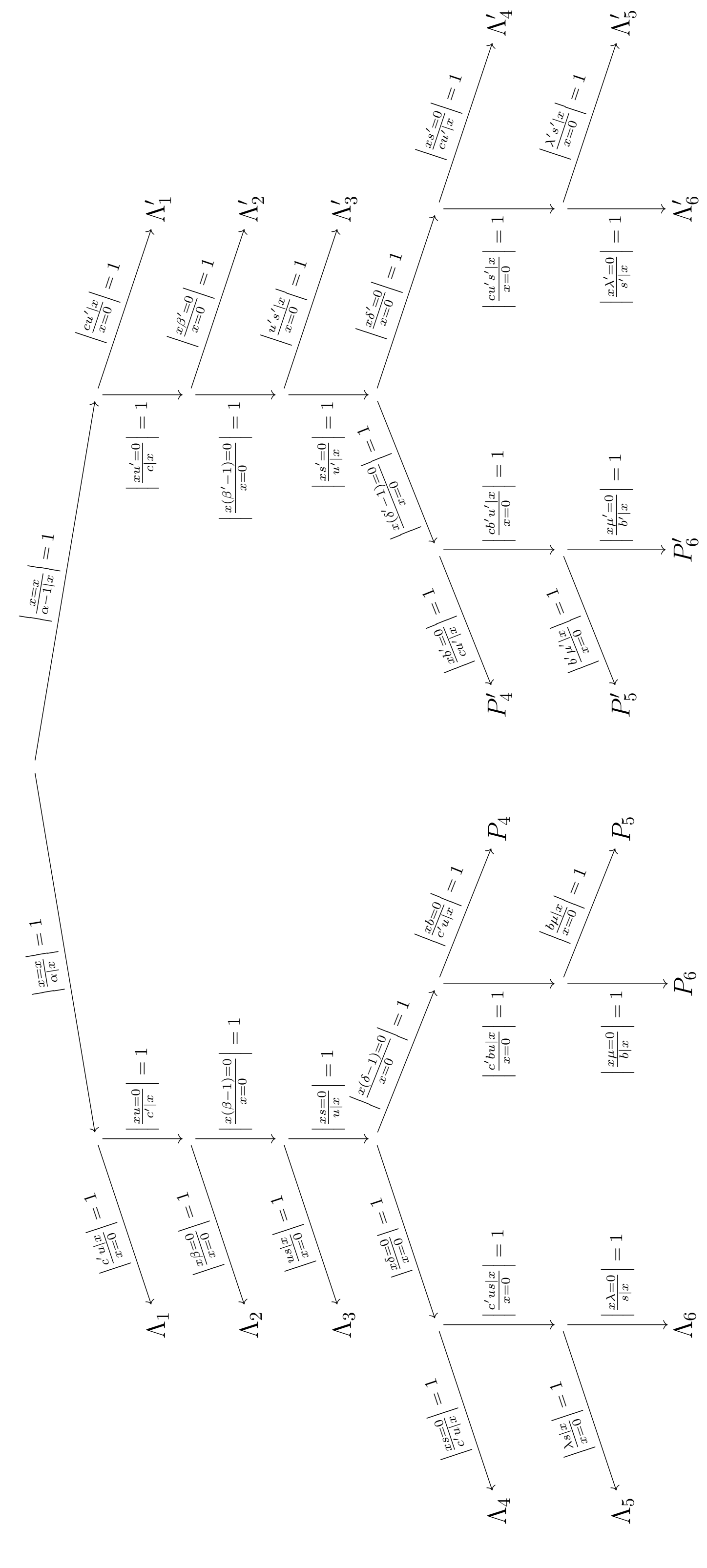
Λ_1'
Λ_2'
Λ_3'
Λ_4'
Λ_5'
Λ_6'
P_4'
P_5'
P_6'
P_4
P_5
P_6
Λ_1
Λ_2
Λ_3
Λ_4
Λ_5
Λ_6

Figure 1

in Figure 1 to the leaf of the tree with that sentence as label. Every R-module is elementary equivalent to an R-module of the form

$$(\bigoplus_{i=1}^{6} M_i \oplus \bigoplus_{i=4}^{6} N_i) \oplus (\bigoplus_{i=1}^{6} M_i' \oplus \bigoplus_{i=4}^{6} N_i')$$

where $M_i \models \Lambda_i$ and $M_i' \models \Lambda_i'$ for $1 \leq i \leq 6$ and $N_i \models P_i$ and $N_i' \models P_i'$ for $4 \leq i \leq 6$. Moreover, for all $M \in \text{Mod-}R$,

(i) $M \models \Lambda_1$ *implies* $c \in \text{ann}_R M$ *and hence* $|{}^{xb=0}/_{c|x}(M)| = |{}^{xb=0}/_{x=0}(M)|$,
(ii) $M \models \Lambda_2$ *implies* $|{}^{xb'=0}/_{c'|x}(M)| = 1$,
(iii) $M \models \Lambda_3$ *implies* $b' \in \text{ann}_R M$ *and hence* $|{}^{xb'=0}/_{c'|x}(M)| = |{}^{x=x}/_{c'|x}(M)|$,
(iv) $M \models \Lambda_4$ *implies* $|{}^{xb'=0}/_{c'|x}(M)| = 1$,
(v) $M \models \Lambda_5$ *implies* $b \in \text{ann}_R M$ *and hence* $|{}^{xb=0}/_{c|x}(M)| = |{}^{x=x}/_{c|x}(M)|$,
(vi) $M \models \Lambda_6$ *implies*

$$\left|{}^{xb=0}/_{c|x}(M)\right| = \left|{}^{xb'=0}/_{c'|x}(M)\right| \cdot \left|{}^{x\lambda=0}/_{x=0}(M)\right|,$$

(vii) $M \models P_4$ *implies* $|{}^{xb=0}/_{c|x}(M)| = 1$,
(viii) $M \models P_5$ *implies* $b' \in \text{ann}_R M$ *and hence* $|{}^{xb'=0}/_{c'|x}(M)| = |{}^{x=x}/_{c'|x}(M)|$, *and*
(ix) $M \models P_6$ *implies*

$$\left|{}^{xb'=0}/_{c'|x}(M)\right| = \left|{}^{xb=0}/_{c|x}(M)\right| \cdot \left|{}^{x\mu=0}/_{x=0}(M)\right|.$$

Similarly, the symmetry of Figure 1, gives 9 statements for Λ_i' and P_i', where c, b and c', b' are interchanged and λ, μ are replaced by λ', μ', respectively.

Proof. There are two edges coming out of each node of the tree in Figure 1. In each instance the two edges are either

(1) $|{}^{x=x}/_{\gamma|x}| = 1$ and $|{}^{x=x}/_{(\gamma-1)|x}| = 1$ for some $\gamma \in R$,
(2) $|{}^{x\gamma=0}/_{x=0}| = 1$ and $|{}^{x(\gamma-1)=0}/_{x=0}| = 1$ for some $\gamma \in R$, or
(3) $|{}^{ab|x}/_{x=0}| = 1$ and $|{}^{xa=0}/_{b|x}| = 1$.

By 2.3.4, in each case $(1), (2)$ and (3), for all modules $M \in \text{Mod-}R$, there exist M_1 satisfying the first sentence and M_2 satisfying the second sentence such that $M \cong M_1 \oplus M_2$. The first claim follows from this fact.

For any $M \in \text{Mod-}R$, $\alpha \notin \text{Div}M$ implies $c|x$ is equivalent to $c'u|x$ in M because $c\alpha = c'u$.

(i) Suppose $M \models \Lambda_1$. Since $|{}^{c'u|x}/_{x=0}(M)| = 1$, $Mc = Mc'u = 0$. So $c \in \text{ann}_R M$.

(ii) Suppose $M \models \Lambda_2$. Then $\beta \notin \text{Ass}M$ and so, since $u\beta = b'r$, $xb'r = 0$ is equivalent to $xu = 0$ in M. Since $xu = 0 \leq_M c'|x$, we conclude

$$xb' = 0 \leq_M xb'r = 0 \leq_M xu = 0 \leq_M c'|x.$$

(iii) Suppose $M \models \Lambda_3$. Then $\beta - 1 \notin \text{Ass}M$ and so, since $b'(\beta-1) = us$, $xb' = 0$ is equivalent to $xus = 0$ in M. Since $|{}^{us|x}/_{x=0}(M)| = 1$, $us \in \text{ann}_R M$ and hence $b' \in \text{ann}_R M$.

Claim: If $M \models |{}^{x=x}/_{\alpha|x}| = 1 \wedge |{}^{xu=0}/_{c'|x}| = 1 \wedge |{}^{x(\beta-1)=0}/_{x=0}| = 1 \wedge |{}^{xs=0}/_{u|x}| = 1$ then $|{}^{xb'=0}/_{c'|x}(M)| = |{}^{xs=0}/_{c'u|x}(M)|$.

First note that since $\beta - 1 \notin \text{Ass}M$ and $b'(\beta-1) = us$, $xb' = 0$ is equivalent to $xus = 0$ in M. We show that $|{}^{xus=0}/_{c'|x}(M)| = |{}^{xs=0}/_{c'u|x}(M)|$.

Consider the map $f : {}^{xus=0}/_{x=0}(M) \to {}^{xs=0}/_{c'u|x}(M)$ defined by $f(m) := mu + c'u|x(M)$ for $m \in M$ with $mus = 0$. Now f is surjective since $|{}^{xs=0}/_{u|x}| = 1$. Suppose $f(m) = m'c'u$ for some $m' \in M$. Then $(m - m'c')u = 0$. Since $|{}^{xu=0}/_{c'|x}| =$

1, $m - m'c' \in c'|x(M)$ and hence $m \in c'|x(M)$. Therefore $\ker f = c'|x(M)$. So we have proved the claim.

We now prove the statements about modules satisfying $\Lambda_4, \Lambda_5, \Lambda_6$. The statements for modules satisfying P_4, P_5 and P_6 follow similarly.

(iv) Suppose $M \models \Lambda_4$. Then $|{}^{xs=0}/_{c'u|x}(M)| = 1$ and by the claim $|{}^{xb'=0}/_{c'|x}(M)| = |{}^{xs=0}/_{c'u|x}(M)|$.

(v) Suppose $M \models \Lambda_5$. Since $\delta \notin \text{Ass}M$, $xb = 0$ is equivalent to $xs\lambda = 0$ in M. Since $|{}^{s\lambda|x}/_{x=0}(M)| = 1$, $s\lambda \in \text{ann}_R M$ and hence $b \in \text{ann}_R M$.

(vi) Suppose $M \models \Lambda_6$. Since $\delta \notin \text{Ass}M$, $xb = 0$ is equivalent to $xs\lambda = 0$ in M. Since $xs = 0 \geq_M c'u|x$,

$$|{}^{xb=0}/_{c|x}(M)| = |{}^{xs\lambda=0}/_{c'u|x}(M)| = |{}^{xs\lambda=0}/_{xs=0}(M)| \cdot |{}^{xs=0}/_{c'u|x}(M)|.$$

By 5.7, $|{}^{xs\lambda=0}/_{xs=0}(M)| = |{}^{s|x \wedge x\lambda=0}/_{x=0}(M)|$. So, since $|{}^{x\lambda=0}/_{s|x}| = 1$,

$$|{}^{xb=0}/_{c|x}(M)| = |{}^{xs\lambda=0}/_{xs=0}(M)| \cdot |{}^{xs=0}/_{c'u|x}(M)| = |{}^{x\lambda=0}/_{x=0}(M)| \cdot |{}^{xb'=0}/_{c'|x}(M)|.$$

□

Proposition 5.10. *There is an algorithm which, given $w \in W$ with $\text{clx}_2(w) > 1$, outputs $\underline{w} \in \mathbb{W}$ such that $\text{clx}_2(\underline{w}) < \text{clx}_2(w)$, $\text{clx}_1(\underline{w}) \leq \text{clx}_1(w)$, and, $w \in V$ if and only if $\underline{w} \in \mathbb{V}$.*

Proof. We start with a special case. Let $b, c, b', c' \in R\backslash\{0\}$. Let $P_6, P_6', \Lambda_6, \Lambda_6'$ be as in 5.9. Let Σ_1 be $|{}^{c|x}/_{x=0}| = 1$, Σ_2 be $|{}^{b|x}/_{x=0}| = 1$, Σ_3 be $|{}^{c'|x}/_{x=0}| = 1$, Σ_4 be $|{}^{b'|x}/_{x=0}| = 1$, Σ_5 be $|{}^{xb=0}/_{c|x}| = 1$, Σ_6 be $|{}^{xb'=0}/_{c'|x}| = 1$, Σ_7 be Λ_6, Σ_8 be P_6, Σ_9 be Λ_6' and Σ_{10} be P_6'.

Fix $1 \leq i \leq 4$ or $7 \leq i \leq 10$. Suppose w is

$$|{}^{xb=0}/_{c|x}| \,\square E \wedge |{}^{xb'=0}/_{c'|x}| \,\square' E' \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi$$

with $E, E' > 1$.

Case i=1: Let w' be

$$|{}^{xb=0}/_{x=0}| \,\square E \wedge |{}^{xb'=0}/_{c'|x}| \,\square' E' \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi.$$

Then $\text{clx}_1 w' = \text{clx}_1 w$ and $\text{clx}_2 w' < \text{clx}_2 w$. Since $T_R \models w \leftrightarrow w'$, we get $w \in V$ if and only if $w' \in V$.

Case i=2,3,4: The same argument as for $i = 1$ works.

Case i=7: By 5.9, if $M \models \Sigma_7 (:= \Lambda_6)$ then

$$|{}^{xb=0}/_{c|x}(M)| = |{}^{xb'=0}/_{c'|x}(M)| \cdot |{}^{x\lambda=0}/_{x=0}(M)|.$$

By 5.6, there is an algorithm which either returns $\Omega := \{\perp\}$, in which case

$$T_R \models \neg(|{}^{xb=0}/_{c|x}| \,\square E \wedge |{}^{xb'=0}/_{c'|x}| \,\square' E'),$$

or, a set $\Omega \subseteq \mathbb{N}^2 \times \{=, \geq\}^2$ such that

$$\Sigma_7 \wedge |{}^{xb=0}/_{c|x}| \,\square E \wedge |{}^{xb'=0}/_{c'|x}| \square' E'$$

is equivalent, with respect to T_R, to

$$\bigvee_{(D_1, D_2, \square_1, \square_2) \in \Omega} \Sigma_7 \wedge |{}^{x\lambda=0}/_{x=0}| \,\square_1 D_1 \wedge |{}^{xb'=0}/_{c'|x}| \,\square_2 D_2.$$

If $\Omega := \{\perp\}$ then $w \in V$ if and only if $\perp \in \mathbb{V}$ and by definition $\text{clx}_1 \perp \leq \text{clx}_1 w$ and $\text{clx}_2 \perp < \text{clx}_2 w$. Otherwise,

$$|{}^{xb=0}/_{c|x}| \,\square E \wedge |{}^{xb'=0}/_{c'|x}| \,\square' E' \wedge \chi_{f,g} \wedge \Sigma_7 \wedge \Xi$$

is equivalent to

$$\bigvee_{(D_1,D_2,\Box_1,\Box_2)\in\Omega} |{}^{x\lambda=0}/_{x=0}|\,\Box_1 D_1 \wedge |{}^{xb'=0}/_{c'|x}|\,\Box_2 D_2 \wedge \chi_{f,g}\wedge\Sigma_7\wedge\Xi.$$

For each $(D_1,D_2,\Box_1,\Box_2)\in\Omega$, let

$$w_{(D_1,D_2,\Box_1,\Box_2)} := |{}^{x\lambda=0}/_{x=0}|\,\Box_1 D_1 \wedge |{}^{xb'=0}/_{c'|x}|\,\Box_2 D_2 \wedge \chi_{f,g}\wedge\Sigma_7\wedge\Xi.$$

So $w\in V$ if and only if

$$\bigsqcup_{(D_1,D_2,\Box_1,\Box_2)\in\Omega} w_{(D_1,D_2,\Box_1,\Box_2)}\in\mathbb{V}.$$

For all $(D_1,D_2,\Box_1,\Box_2)\in\Omega$, $\text{clx}_1 w_{(D_1,D_2,\Box_1,\Box_2)}=\text{clx}_1 w$ and $\text{clx}_2 w_{(D_1,D_2,\Box_1,\Box_2)}<\text{clx}_2 w$.

Case i=8,9,10: The same argument as for $i=7$ works.

We now deal with the general case. Let w be

$$\bigwedge_{\varphi/\psi\in X}|\varphi/\psi| = f(\varphi/\psi)\wedge\bigwedge_{\varphi/\psi\in Y}|\varphi/\psi|\geq g(\varphi/\psi)\wedge\Xi\in W$$

where X,Y are disjoint finite sets of pp-pairs of the form ${}^{d|x}/_{x=0}$ and ${}^{xb=0}/_{c|x}$, $f:X\to\mathbb{N}_2$, $g:Y\to\mathbb{N}_2$ and Ξ is an auxiliary sentence.

Suppose that ${}^{xb=0}/_{c|x}, {}^{xb'=0}/_{c'|x}\in X\cup Y$ are distinct pp-pairs with $b,c,b',c'\in R\backslash\{0\}$. Let $\Omega\subseteq\Omega_{f,g,10}$ be such that $(\overline{f},\overline{g})\in\Omega$ if and only if ${}^{xb=0}/_{c|x}\in X_5$, ${}^{xb'=0}/_{c'|x}\in X_6$, $f_5({}^{xb=0}/_{c|x})=1$, and $f_6({}^{xb'=0}/_{c'|x})=1$. Then $w\in V$ if and only if

$$\bigsqcup_{(\overline{f},\overline{g})\in\Omega}\prod_{i=1}^{10}\chi_{f_i,g_i}\wedge\Sigma_i\wedge\Xi\in\mathbb{V}.$$

For each $(\overline{f},\overline{g})\in\Omega$ and $1\leq i\leq 10$, let $w_{i,\overline{f},\overline{g}}$ be $\chi_{f_i,g_i}\wedge\Sigma_i\wedge\Xi$. By definition of $\Omega_{f,g,10}$, for each $1\leq i\leq 10$, $\text{clx}_1 w_{i,\overline{f},\overline{g}}\leq\text{clx}_1 w$, $\text{clx}_2 w_{i,\overline{f},\overline{g}}\leq clx_2 w$.

By assumption ${}^{xb=0}/_{c|x}\in X$ and $f({}^{xb=0}/_{c|x})>1$ or ${}^{xb=0}/_{c|x}\in Y$ and $g({}^{xb=0}/_{c|x})>1$. So, since ${}^{xb=0}/_{c|x}\in X_5$ and $f_5({}^{xb=0}/_{c|x})=1$, for each $(\overline{f},\overline{g})\in\Omega_{f,g,10}$, $\text{clx}_2 w_{5,\overline{f},\overline{g}}<\text{clx}_2 w$. Replacing ${}^{xb=0}/_{c|x}$ by ${}^{xb'=0}/_{c'|x}$, the same argument gives $\text{clx}_2 w_{6,\overline{f},\overline{g}}<\text{clx}_2 w$.

Now suppose $1\leq i\leq 4$ or $7\leq i\leq 10$. If ${}^{xb=0}/_{c|x}\in X_i$ (respectively ${}^{xb=0}/_{c|x}\in Y_i$) and $f_i({}^{xb=0}/_{c|x})=1$ (respectively $g_i({}^{xb=0}/_{c|x})=1$) then $\text{clx}_2 w_{i,\overline{f},\overline{g}}<\text{clx}_2 w$. This argument together with the same argument with ${}^{xb=0}/_{c|x}$ replaced by ${}^{xb'=0}/_{c'|x}$ means that we may assume $w_{i,\overline{f},\overline{g}}$ is of the form of the special case considered at the start of the proof. Thus we may replace each $w_{i,\overline{f},\overline{g}}$ by some $\underline{w}'\in\mathbb{W}$ such that $\text{clx}_1\underline{w}'\leq\text{clx}_1 w_{i,\overline{f},\overline{g}}$ and $\text{clx}_2\underline{w}'<\text{clx}_2 w_{i,\overline{f},\overline{g}}$. □

Proof of 5.2. By 5.1, in order to show that T_R is decidable, it is enough to show that there exists an algorithm which given $w\in W$ answers whether $w\in V$ or not. Suppose that there is an algorithm which given a sentence χ as in (†) answers whether there exists an R-module satisfying χ. We may relax the assumptions on d to allow the case $d=0$ since any instance of $|{}^{0|x}/_{x=0}|$ can always be replaced by $|{}^{xb'=0}/_{x=0}|$ where $b'=1$. Therefore, by assumption, the set of $w'\in V$ with $\text{clx}_1 w'\leq 1$ and $\text{clx}_2 w'\leq 1$ is recursive. Thus the set of $\underline{w}\in\mathbb{V}$ with $\text{clx}_1\underline{w}\leq 1$ and $\text{clx}_2\underline{w}\leq 1$ is recursive.

Since $\mathbb{N}_0$ is artinian as an order, iteratively applying 5.8 and 5.10, provides an algorithm which given $w \in W$ with either $\mathrm{clx}_1 w > 1$ or $\mathrm{clx}_2 w > 1$ outputs $\underline{w} \in \mathbb{W}$ with $\mathrm{clx}_1 \underline{w}, \mathrm{clx}_2 \underline{w} \leq 1$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$. □

6. Uniserial modules with finite invariants sentences

Descriptions of the uniserial (and hence indecomposable pure-injective) modules, U, over a valuation domain which have ${\varphi}/{\psi}(U)$ finite but non-zero for a given pp-pair ${\varphi}/{\psi}$ are given for valuation domains with dense value groups in [PPT07] and for valuation domains with non-dense value groups in [Gre15]. However, we need a uniform description that works for both valuation domains with dense and non-dense value groups. This is done in Lemmas 6.0.1, 6.0.2 and 6.0.3 and used in sections 9 and 10. The rest of the section is about these modules in preparation for sections 7, 8 and 10.

Lemma 6.0.1. *Let V be a valuation domain. If $d \in V$ and U is a uniserial V-module such that ${d|x}/{x=0}(U) = Ud$ is finite and non-zero then $U \cong V/dI$ for some ideal $I \lhd V$ and $Ud \cong V/I$.*

Proof. For any module M, ${x=x}/{xd=0}(M) \cong {d|x}/{x=0}(M)$. Thus if ${d|x}/{x=0}(U) \cong {x=x}/{xd=0}(U)$ is finite but not equal to the zero module then, by 2.3.7, $U \cong V/J$ for some ideal $J \lhd V$. Since $Ud \neq 0$, $d \notin J$ and therefore $dV \supseteq J$. So $J = dI$ for some ideal $I \lhd V$. □

Note that in the assumptions of the second clause of the next lemma we are not excluding that $I = V$ or consequently that ${xb=0}/{c|x}(I/bcV) = 0$.

Lemma 6.0.2. *Let V be a valuation domain and $b, c \in V \backslash \{0\}$. If U is a uniserial V-module such that $b, c \notin \mathrm{ann}_V U$ and ${xb=0}/{c|x}(U)$ is finite but non-zero then there exists $I \lhd V$ with $b, c \in I$ such that $U \equiv I/bcV$ and ${xb=0}/{c|x}(U) \cong V/I$.*

Conversely, if $0 \neq I \lhd V$ is an ideal and $b, c \in I \backslash \{0\}$ then ${xb=0}/{c|x}(I/bcV) \cong V/I$.

Proof. Let Q be the field of fractions of V. By [Zie84, p. 168], for any non-zero uniserial module U, there exist V-submodules $K \subsetneq J \subseteq Q$ such that $U \equiv J/K$ as V-modules. Now $b, c \notin \mathrm{ann}_V J/K$ imply $(K : b) \subsetneq J$ and $(K : c) \subsetneq J$ respectively. Since V is a valuation domain, $(K : c) \subsetneq J$ implies $K \subsetneq Jc$. Therefore, since ${xb=0}/{c|x}(J/K) \neq 0$, ${xb=0}/{c|x}(J/K) = (K : b)/cJ \cong K/cbJ$. Since K/cbJ is a non-zero finite uniserial module, it has the form V/I for some proper ideal $I \lhd V$. Therefore $K = \lambda V$ for some $\lambda \in Q \backslash \{0\}$ and $\lambda^{-1} cbJ = I$. Thus $U \equiv J/K \cong I/bcV$ as required. It is easy to see that over a valuation domain there is only one uniserial module of each finite size. Therefore ${xb=0}/{c|x}(I/bcV) \cong V/I$ implies ${xb=0}/{c|x}(U) \cong V/I$. Finally, it follows from $(K : b) \subsetneq J$ that $c \in I$ and from $(K : c) \subsetneq J$ that $b \in I$.

Let $0 \neq I \lhd V$ be an ideal and $b, c \in I \backslash \{0\}$. Then ${xb=0}/{c|x}(I/bcV) = cV/cI \cong V/I$. □

Lemma 6.0.3. *Let V be a valuation domain and $c \in V$. If U is a uniserial V-module such that ${x=x}/{c|x}(U)$ is finite but non-zero then $U \cong V/K$ for some ideal $K \lhd V$. Moreover, if ${x=x}/{c|x}(U)$ is finite but non-zero then either $U \cong V/cI$ for some $I \lhd V$ and $V/cV \cong {x=x}/{c|x}(U)$, or, $c \in \mathrm{ann}_V U$ and $U \cong {x=x}/{c|x}(U)$.*

Proof. The first claim is a consequence of 2.3.7. If $c \in K$ then $c \in \mathrm{ann}_V V/K$. If $c \notin K$ then $cV \supseteq K$ and hence $K = cI$ for some $I \lhd V$. □

We avoid dealing directly with uniserial V-modules U such that ${}^{xb=0}/_{x=0}(U)$ is finite but non-zero, by using duality defined in 2.1.3.

6.1. $(\mathfrak{p}, I) \models (r, a, \gamma, \delta)$**.** For the rest of this section R will always denote a Prüfer domain.

Definition 6.1.1. *Let $r, a, \gamma, \delta \in R$, $\mathfrak{p} \lhd R$ a prime ideal and $I \lhd R_\mathfrak{p}$ an ideal. We write $(\mathfrak{p}, I) \models (r, a, \gamma, \delta)$ if $rR_\mathfrak{p} \supseteq I$, $a \in I$, $\gamma \notin \mathfrak{p}$ and $\delta \notin I^\#$.*

The task of this subsection is to show that given an auxiliary sentence Ξ and $\lambda \in R\backslash\{0\}$, we can compute $n \in \mathbb{N}$ and $(r_i, r_i a_i, \gamma_i, \delta_i)$ for $1 \leq i \leq n$ such that for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, $R_\mathfrak{p}/\lambda I \models \Xi$ if and only if $(\mathfrak{p}, I) \models (r_i, r_i a_i, \gamma_i, \delta_i)$ for some $1 \leq i \leq n$. However, in 10.5, we will additionally need that all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, $R_\mathfrak{p}/\lambda I \models D\Xi$ if and only if $(\mathfrak{p}, I) \models (r_i, r_i a_i, \delta_i, \gamma_i)$ for some $1 \leq i \leq n$.

Remark 6.1.2. *Let $r, a, \gamma, \delta, \alpha \in R$. For all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, $(\mathfrak{p}, I) \models (r, a, \gamma, \delta)$ if and only if $(\mathfrak{p}, I) \models (r, a, \gamma\alpha, \delta\alpha)$ or $(\mathfrak{p}, I) \models (r, a, \gamma(\alpha - 1), \delta(\alpha - 1))$.*

Proof. This is true because for all prime ideals $\mathfrak{p} \lhd R$, either $\alpha \notin \mathfrak{p}$ or $\alpha - 1 \notin \mathfrak{p}$ and, by definition, for all ideals $I \lhd R_\mathfrak{p}$, $I^\# \subseteq \mathfrak{p}R_\mathfrak{p}$. □

Lemma 6.1.3. *Let R be a Prüfer domain and $b, c, d \in R$ with $b \neq 0$. Let $\mathfrak{p} \lhd R$ be a prime ideal and $I \lhd R_\mathfrak{p}$ be an ideal.*

(1) Then $|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/I)| = 1$ if and only if $b \notin I^\#$, $c \notin \mathfrak{p}$, $bcR_\mathfrak{p} \supseteq I$ or $1 \in I$.
(2) Then $|{}^{d|x}/_{x=0}(R_\mathfrak{p}/I)| = 1$ if and only if $d \in I$.

Proof. (1) For any ideal $I \lhd R_\mathfrak{p}$, $|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/I)| = 1$ if and only if $cR_\mathfrak{p} + I \supseteq (I : b)$. Since $R_\mathfrak{p}$ is a valuation ring, $cR_\mathfrak{p} + I \supseteq (I : b)$ if and only if $I \supseteq (I : b)$ or $cR_\mathfrak{p} \supseteq (I : b)$. So it is enough to note that $I \supseteq (I : b)$ if and only if $b \notin I^\#$ or $1 \in I$, and, $cR_\mathfrak{p} \supseteq (I : b)$ if and only if $bcR_\mathfrak{p} \supseteq I$ or $c \notin \mathfrak{p}$.
(2) is obvious. □

Lemma 6.1.4. *Given $(r, a, \gamma, \delta), (r', a', \gamma', \delta') \in R^4$ we can compute $n \in \mathbb{N}$ and $(r_i, a_i, \gamma_i, \delta_i) \in R^4$ for $1 \leq i \leq n$ such that*

$$(\mathfrak{p}, I) \models (r, a, \gamma, \delta) \text{ and } (\mathfrak{p}, I) \models (r', a', \gamma', \delta')$$

if and only if

$$(\mathfrak{p}, I) \models (r_i, a_i, \gamma_i, \delta_i)$$

for some $1 \leq i \leq n$, and,

$$(\mathfrak{p}, I) \models (r, a, \delta, \gamma) \text{ and } (\mathfrak{p}, I) \models (r', a', \delta', \gamma')$$

if and only if

$$(\mathfrak{p}, I) \models (r_i, a_i, \delta_i, \gamma_i)$$

for some $1 \leq i \leq n$.

Proof. Let $\alpha, u_1, u_2, \beta, v_1, v_2 \in R$ be such that $r\alpha = r'u_1$, $r'(\alpha - 1) = ru_2$, $a\beta = a'v_1$ and $a'(\beta - 1) = av_2$. Then one verifies easily that

$$(\mathfrak{p}, I) \models (r, a, \gamma, \delta) \text{ and } (\mathfrak{p}, I) \models (r', a', \gamma', \delta')$$

if and only if $(\mathfrak{p}, I) \models (r, a', \alpha\beta\gamma\gamma', \alpha\beta\delta\delta')$, $(\mathfrak{p}, I) \models (r, a, \alpha(\beta - 1)\gamma\gamma', \alpha(\beta - 1)\delta\delta')$, $(\mathfrak{p}, I) \models (r', a', (\alpha - 1)\beta\gamma\gamma', (\alpha - 1)\beta\delta\delta')$ or $(\mathfrak{p}, I) \models (r', a, (\alpha - 1)(\beta - 1)\gamma\gamma', (\alpha - 1)(\beta - 1)\delta\delta')$. □

Lemma 6.1.5. *Given $(r,a,\gamma,\delta)\in R^4$ and $\lambda\in R\backslash\{0\}$, we can compute $n\in\mathbb{N}$ and $(r_j,a_j,\gamma_j,\delta_j)$ for $1\leq j\leq n$ such that, for all prime ideals $\mathfrak{p}\lhd R$ and ideals $I\lhd R_\mathfrak{p}$,*

- *$(\mathfrak{p},\lambda I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},I)\models(r_j,a_j,\gamma_j,\delta_j)$ for some $1\leq j\leq n$ and*
- *$(\mathfrak{p},\lambda I)\models(r,a,\delta,\gamma)$ if and only if $(\mathfrak{p},I)\models(r_j,a_j,\delta_j,\gamma_j)$ for some $1\leq j\leq n$.*

Proof. Let $\alpha,u,v,\beta,u',v'\in R$ be such that $r\alpha=\lambda u$, $\lambda(\alpha-1)=rv$, $a\beta=\lambda u'$ and $\lambda(\beta-1)=av'$. By 6.1.2, $(\mathfrak{p},\lambda I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},\lambda I)\models(r,a,\alpha\beta\gamma,\alpha\beta\delta)$, $(\mathfrak{p},\lambda I)\models(r,a,\alpha(\beta-1)\gamma,\alpha(\beta-1)\delta)$, $(\mathfrak{p},\lambda I)\models(r,a,(\alpha-1)\beta\gamma,(\alpha-1)\beta\delta)$ or $(\mathfrak{p},\lambda I)\models(r,a,(\alpha-1)(\beta-1)\gamma,(\alpha-1)(\beta-1)\delta)$.

It is straightforward to see that:

- If $\alpha\notin\mathfrak{p}$ then $rR_\mathfrak{p}\supseteq\lambda I$ if and only if $uR_\mathfrak{p}\supseteq I$.
- If $\alpha-1\notin\mathfrak{p}$ then $rR_\mathfrak{p}\supseteq\lambda I$.
- If $\beta\notin\mathfrak{p}$ then $a\in\lambda I$ if and only if $u'\in I$.
- If $\beta-1\notin\mathfrak{p}$ then $a\in\lambda I$ if and only if $1\in I$ and $v'\notin\mathfrak{p}$ (and hence $v'\notin I^{\#}$).

Recall that, since $\lambda\neq0$, $(\lambda I)^{\#}=I^{\#}$. Therefore, $(\mathfrak{p},\lambda I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},I)\models(u,u',\alpha\beta\gamma,\alpha\beta\delta)$, $(\mathfrak{p},I)\models(u,1,\alpha(\beta-1)v'\gamma,\alpha(\beta-1)v'\delta)$, $(\mathfrak{p},I)\models(1,u',(\alpha-1)\beta\gamma,(\alpha-1)\beta\delta)$ or $(\mathfrak{p},I)\models(1,1,(\alpha-1)(\beta-1)v'\gamma,(\alpha-1)(\beta-1)v'\delta)$. □

Lemma 6.1.6. *Given $(r,a,\gamma,\delta)\in R^4$, we can compute $n\in\mathbb{N}$ and $(r_j,r_ja_j,\gamma_j,\delta_j)$ for $1\leq j\leq n$ such that for all prime ideals $\mathfrak{p}\lhd R$ and ideals $I\lhd R_\mathfrak{p}$,*

- *$(\mathfrak{p},I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},I)\models(r_j,r_ja_j,\gamma_j,\delta_j)$ for some $1\leq j\leq n$ and*
- *$(\mathfrak{p},I)\models(r,a,\delta,\gamma)$ if and only if $(\mathfrak{p},I)\models(r_j,r_ja_j,\delta_j,\gamma_j)$ for some $1\leq j\leq n$.*

Proof. If $a=0$ then $(r,a,\gamma,\delta)=(r,r\cdot0,\gamma,\delta)$ is already of the required form. So suppose $a\neq0$.

Let $\alpha,u,v\in R$ be such that $a\alpha=ru$ and $r(\alpha-1)=av$. By 6.1.2, $(\mathfrak{p},I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},I)\models(r,a,\gamma\alpha,\delta\alpha)$ or $(\mathfrak{p},I)\models(r,a,\gamma(\alpha-1),\delta(\alpha-1))$. Since $a\alpha=ru$, $(\mathfrak{p},I)\models(r,a,\gamma\alpha,\delta\alpha)$ if and only if $(\mathfrak{p},I)\models(r,ru,\gamma\alpha,\delta\alpha)$. Since $r(\alpha-1)=av$, $(\mathfrak{p},I)\models(r,a,\gamma(\alpha-1),\delta(\alpha-1))$ if and only if $(\mathfrak{p},I)\models(av,a,\gamma(\alpha-1),\delta(\alpha-1))$. Now, since $a\neq0$, $avR_\mathfrak{p}\supseteq I$ and $a\in I$ if and only if $v\notin\mathfrak{p}$, $aR_\mathfrak{p}\supseteq I$ and $a\in I$. So $(\mathfrak{p},I)\models(av,a,\gamma(\alpha-1),\delta(\alpha-1))$ if and only if $(\mathfrak{p},I)\models(a,a,\gamma\alpha v,\delta\alpha v)$.

Therefore $(\mathfrak{p},I)\models(r,a,\gamma,\delta)$ if and only if $(\mathfrak{p},I)\models(r,rv,\gamma\alpha,\delta\alpha)$ or $(\mathfrak{p},I)\models(a,a,\gamma(\alpha-1)v,\delta(\alpha-1)v)$. The same argument shows that $(\mathfrak{p},I)\models(r,a,\delta,\gamma)$ if and only if $(\mathfrak{p},I)\models(r,rv,\delta\alpha,\gamma\alpha)$ or $(\mathfrak{p},I)\models(a,a,\delta(\alpha-1)v,\gamma(\alpha-1)v)$. □

Proposition 6.1.7. *Given an auxiliary sentence Ξ and $\lambda\in R\backslash\{0\}$, we can compute $n\in\mathbb{N}$ and $(r_j,r_ja_j,\gamma_j,\delta_j)\in R^4$ for $1\leq j\leq n$ such that, for all prime ideals $\mathfrak{p}\lhd R$ and ideals $I\lhd R_\mathfrak{p}$,*

- *$(\mathfrak{p},I)\models(r_j,r_ja_j,\gamma_j,\delta_j)$ for some $1\leq j\leq n$ if and only if $R_\mathfrak{p}/\lambda I\models\Xi$, and*
- *$(\mathfrak{p},I)\models(r_j,r_ja_j,\delta_j,\gamma_j)$ for some $1\leq j\leq n$ if and only if $R_\mathfrak{p}/\lambda I\models D\Xi$.*

Proof. Let Ξ be the sentence

$$\bigwedge_{i=1}^{l'}|{}^{d_i|x}/_{x=0}|=1\wedge\bigwedge_{i=l'}^{l}|{}^{xb_i=0}/_{c_i|x}|=1.$$

Using 2.3.10 and 6.1.3, we can compute $n_i\in\mathbb{N}$ for $1\leq i\leq l$ and $s_{ij},b_{ij},g_{ij},h_{ij}\in R$ for $1\leq i\leq l$ and $1\leq j\leq n_i$ such that for all prime ideals $\mathfrak{p}\lhd R$ and ideals $I\lhd R_\mathfrak{p}$,

- $R_{\mathfrak{p}}/I \models \Xi$ if and only if $(\mathfrak{p}, I) \models \bigwedge_{i=1}^{l} \bigvee_{j=1}^{n_i} (s_{ij}, b_{ij}, g_{ij}, h_{ij})$, and
- $R_{\mathfrak{p}}/I \models D\Xi$ if and only if $(\mathfrak{p}, I) \models \bigwedge_{i=1}^{l} \bigvee_{j=1}^{n_i} (s_{ij}, b_{ij}, h_{ij}, g_{ij})$.

Therefore, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$,

- $R_{\mathfrak{p}}/I \models \Xi$ if and only if
$$(\mathfrak{p}, I) \models \bigvee_{\substack{\sigma:\{1,\dots,l\}\to\mathbb{N}\\ \sigma(i)\le n_i}} \bigwedge_{i=1}^{l} (s_{i\sigma(i)}, b_{i\sigma(i)}, g_{i\sigma(i)}, h_{i\sigma(i)}), \text{ and}$$
- $R_{\mathfrak{p}}/I \models D\Xi$ if and only if
$$(\mathfrak{p}, I) \models \bigvee_{\substack{\sigma:\{1,\dots,l\}\to\mathbb{N}\\ \sigma(i)\le n_i}} \bigwedge_{i=1}^{l} (s_{i\sigma(i)}, b_{i\sigma(i)}, h_{i\sigma(i)}, g_{i\sigma(i)}).$$

We can use 6.1.4, to replace the conjunction $\bigwedge_{i=1}^{l} (s_{i\sigma(i)}, b_{i\sigma(i)}, g_{i\sigma(i)}, h_{i\sigma(i)})$, for each σ, by a disjunction to produce and $(s'_{\sigma k}, b'_{\sigma k}, g'_{\sigma k}, \delta'_{\sigma k})$ for $1 \le k \le m_\sigma$ such that

- $R_{\mathfrak{p}}/I \models \Xi$ if and only if $(\mathfrak{p}, I) \models (s'_{\sigma k}, b'_{\sigma k}, g'_{\sigma k}, h'_{\sigma k})$ for some $1 \le k \le m_\sigma$, and
- $R_{\mathfrak{p}}/I \models D\Xi$ if and only if $(\mathfrak{p}, I) \models (s'_{\sigma k}, b'_{\sigma k}, h'_{\sigma k}, g'_{\sigma k})$ for some $1 \le k \le m_\sigma$.

Applying 6.1.5 to each $(s'_{\sigma k}, b'_{\sigma k}, h'_{\sigma k}, g'_{\sigma k})$, we compute $m \in \mathbb{N}$ and $(r'_j, a'_j, \gamma'_j, \delta'_j)$ for $1 \le j \le m$ such that

- $R_{\mathfrak{p}}/\lambda I \models \Xi$ if and only if $(\mathfrak{p}, I) \models (r'_j, a'_j, \gamma'_j, \delta'_j)$ for some $1 \le j \le m$, and
- $R_{\mathfrak{p}}/\lambda I \models D\Xi$ if and only if $(\mathfrak{p}, I) \models (r'_j, a'_j, \delta'_j, \gamma'_j)$ for some $1 \le j \le m$.

Finally, applying 6.1.6 to each $(r'_j, a'_j, \gamma'_j, \delta'_j)$, we can compute $n \in \mathbb{N}$ and $(r_i, r_i a_i, \gamma_i, \delta_i)$ for $1 \le i \le n$ such that

- $R_{\mathfrak{p}}/\lambda I \models \Xi$ if and only if $(\mathfrak{p}, I) \models (r_i, r_i a_i, \gamma_i, \delta_i)$ for some $1 \le i \le n$, and
- $R_{\mathfrak{p}}/\lambda I \models D\Xi$ if and only if $(\mathfrak{p}, I) \models (r_i, r_i a_i, \delta_i, \gamma_i)$ for some $1 \le i \le n$. □

6.2. Simplification of $\varphi/\psi(R_{\mathfrak{p}}/\lambda I)$ and $\varphi/\psi(I/\lambda R_{\mathfrak{p}})$.

The results of this subsection are used in sections 7 and 8. In this subsection, we no longer need to worry about stability under duality.

Remark 6.2.1. *Let $a, b \in R\backslash\{0\}$ and $\alpha, u, v \in R$ be such that $a\alpha = bu$ and $b(\alpha - 1) = av$. For prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$,*

- *$a \in bI$ if and only if $(\mathfrak{p}, I) \models (1, u, \alpha, 1)$ or $(\mathfrak{p}, I) \models (1, 1, v(\alpha - 1), 1)$, and,*
- *$aR_{\mathfrak{p}} \supseteq bI$ if and only if $(\mathfrak{p}, I) \models (u, 0, \alpha, 1)$ or $(\mathfrak{p}, I) \models (1, 0, \alpha - 1, 1)$.*

Lemma 6.2.2. *Let R be a recursive Prüfer domain and $\lambda \in R\backslash\{0\}$.*

(a) *Given $d \in R$, we can compute finite sets $S_1, S_2, S_3 \subseteq R^4$, $\rho : \bigcup_{i=1}^{3} S_i \to \{1, 2, 3\}$ and $s : \bigcup_{i=1}^{3} S_i \to R$ such that for all $q \in \bigcup_{i=1}^{3} S_i$, $q \in S_{\rho(q)}$, and, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$, there exists $q \in \bigcup_{i=1}^{3} S_i$ such that $(\mathfrak{p}, I) \models q$, and*
$$|{}^{d|x}/_{x=0}(R_{\mathfrak{p}}/\lambda I)| := \begin{cases} |R_{\mathfrak{p}}/s(q)I|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 1; \\ |s(q)R_{\mathfrak{p}}/I|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 2; \\ 1, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 3. \end{cases}$$
Furthermore, if $(\mathfrak{p}, I) \models q$ for some $q \in \bigcup_{i=1}^{3} S_i$ and $\rho(q) = 1$ (respectively $\rho(q) = 2$) then $s(q) \neq 0$ (respectively $s(q)R_{\mathfrak{p}} \supseteq I$).

(b) Given $b, c \in R$, we can compute finite sets $S_1, \ldots, S_5 \subseteq R^4$, $\rho : \bigcup_{i=1}^5 S_i \to \{1, \ldots, 5\}$ and $s : \bigcup_{i=1}^5 S_i \to R$ such that for all $q \in \bigcup_{i=1}^5 S_i$, $q \in S_{\rho(q)}$, and, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, there exists $q \in \bigcup_{i=1}^5 S_i$ such that $(\mathfrak{p}, I) \models q$, and

$$\left|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/\lambda I)\right| := \begin{cases} |R_\mathfrak{p}/\lambda I|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 1; \\ |R_\mathfrak{p}/cR_\mathfrak{p}|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 2; \\ |I/bI|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 3; \\ 1, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 4; \\ |I/s(q)R_\mathfrak{p}|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 5. \end{cases}$$

Furthermore, if $(\mathfrak{p}, I) \models q$ for some $q \in \bigcup_{i=1}^5 S_i$ and $\rho(q) = 5$ then $s(q) \in I$. Moreover, if $b = 0$ then we may assume that $S_3 = S_4 = S_5 = \emptyset$.

Proof. (a) If $d = 0$ then ${}^{d|x}/_{x=0}(N) = 0$ for all R-modules N. So set $S_1 = S_2 = \emptyset$, $S_3 := \{(1, 0, 1, 1)\}$, $\rho((1, 0, 1, 1)) = 3$ and $s((1, 0, 1, 1)) = 1$.

Suppose $d \neq 0$. Let $\alpha, u, v \in R$ be such that $d\alpha = \lambda u$ and $\lambda(\alpha - 1) = dv$.

- If $(\mathfrak{p}, I) \models q_1 := (1, 0, \alpha - 1, 1)$ then
$$\left|{}^{d|x}/_{x=0}(R_\mathfrak{p}/\lambda I)\right| = |dR_\mathfrak{p}/dvI| = |R_\mathfrak{p}/vI|.$$
- If $(\mathfrak{p}, I) \models q_2 := (u, 0, \alpha, 1)$ then by definition $uR_\mathfrak{p} \supseteq I$, and,
$$\left|{}^{d|x}/_{x=0}(R_\mathfrak{p}/\lambda I)\right| = |\lambda uR_\mathfrak{p}/\lambda I| = |uR_\mathfrak{p}/I|.$$
- If $(\mathfrak{p}, I) \models q_3 := (1, u, \alpha, 1)$ then $d \in \lambda I$ and hence $\left|{}^{d|x}/_{x=0}(R_\mathfrak{p}/\lambda I)\right| = 1$.

For all prime ideals $\mathfrak{p} \lhd R$, either $\alpha \notin \mathfrak{p}$ or $\alpha - 1 \notin \mathfrak{p}$, and, for all ideal $I \lhd R_\mathfrak{p}$, either $uR_\mathfrak{p} \supseteq I$ or $u \in I$. Therefore for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, either $(\mathfrak{p}, I) \models q_1$, $(\mathfrak{p}, I) \models q_2$ or $(\mathfrak{p}, I) \models q_3$. Set $S_i := \{q_i\}$ for $1 \leq i \leq 3$. If $v = 0$ then $\alpha - 1 = 0$. Therefore if there exists $(\mathfrak{p}, I) \models q_1$ then $v \neq 0$. So we can set $s(q_1) := v$ if $v \neq 0$ and $s(q_1) := 1$ if $v = 0$. Set $s(q_2) := u$ and $s(q_3) := 1$.
(b) For all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$,

$$\left|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/\lambda I)\right| = \left|\frac{(\lambda I : b) + cR_\mathfrak{p}}{cR_\mathfrak{p} + \lambda I}\right|.$$

First suppose $b = 0$. If $c \in \lambda I$ then $\left|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/\lambda I)\right| = |R_\mathfrak{p}/\lambda I|$, and, if $cR_\mathfrak{p} \supseteq \lambda I$ then $\left|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/\lambda I)\right| = |R_\mathfrak{p}/cR_\mathfrak{p}|$. We can use 6.2.1 to compute finite sets $S_1, S_2 \subseteq R^4$ such that for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, there exists $q \in S_1$ such that $(\mathfrak{p}, I) \models q$ if and only if $c \in \lambda I$, and, there exists $q \in S_2$ such that $(\mathfrak{p}, I) \models q$ if and only if $cR_\mathfrak{p} \supseteq \lambda I$. Set $S_3 = S_4 = S_5 = \emptyset$.

Now suppose $b \neq 0$. If $bR_\mathfrak{p} \supseteq \lambda I$ then

$$\left|\frac{(\lambda I : b) + cR_\mathfrak{p}}{cR_\mathfrak{p} + \lambda I}\right| = \left|\frac{\lambda I + bcR_\mathfrak{p}}{bcR_\mathfrak{p} + b\lambda I}\right|.$$

So

$$\left|{}^{xb=0}/_{c|x}(R_\mathfrak{p}/\lambda I)\right| := \begin{cases} |R_\mathfrak{p}/\lambda I|, & \text{if } b, c \in \lambda I; \\ |R_\mathfrak{p}/cR_\mathfrak{p}|, & \text{if } b \in \lambda I \text{ and } cR_\mathfrak{p} \supseteq \lambda I; \\ |I/bI|, & \text{if } bR_\mathfrak{p} \supseteq \lambda I \text{ and } c \in \lambda I; \\ 1, & \text{if } bcR_\mathfrak{p} \supseteq \lambda I; \\ |\lambda I/bcR_\mathfrak{p}|, & \text{if } bR_\mathfrak{p} \supseteq \lambda I,\ cR_\mathfrak{p} \supseteq \lambda I \text{ and } bc \in \lambda I. \end{cases}$$

Therefore it is enough to compute:

- S_1 such that $b, c \in \lambda I$ if and only if there exists $q \in S_1$ such that $(\mathfrak{p}, I) \models q$,

- S_2 such that $b \in \lambda I$ and $cR_{\mathfrak{p}} \supseteq \lambda I$ if and only if exists $q \in S_2$ such that $(\mathfrak{p}, I) \models q$,
- S_3 such that $bR_{\mathfrak{p}} \supseteq \lambda I$ and $c \in \lambda I$ and if and only if exists $q \in S_3$ such that $(\mathfrak{p}, I) \models q$,
- S_4 such that $bcR_{\mathfrak{p}} \supseteq \lambda I$ if and only if there exists $q \in S_4$ such that $(\mathfrak{p}, I) \models q$, and,
- S_5 and for each $q \in S_5$, $s_q \in R$ such that $bR_{\mathfrak{p}} \supseteq \lambda I$, $cR_{\mathfrak{p}} \supseteq \lambda I$ and $bc \in \lambda I$ if and only if there exists $q \in S_5$ such that $(\mathfrak{p}, I) \models q$ and such that, in this situation $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |I/s_qR_{\mathfrak{p}}|$.

It is easy to compute $S_1, \dots, S_4$ using 6.2.1 and 6.1.4.

Let ϵ, r, s be such that $bc\epsilon = \lambda s$ and $\lambda(\epsilon - 1) = bcr$. By 6.2.1, $bc \in \lambda I$ if and only if $(\mathfrak{p}, I) \models (1, s, \epsilon, 1)$ or $(\mathfrak{p}, I) \models (1, 1, r(\epsilon - 1), 1)$. If $bR_{\mathfrak{p}} \supseteq \lambda I$, $cR_{\mathfrak{p}} \supseteq \lambda I$ and $(\mathfrak{p}, I) \models (1, s, \epsilon, 1)$ then

$$|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |\lambda I/bcR_{\mathfrak{p}}| = |I/sR_{\mathfrak{p}}|\,.$$

Use 6.2.1 and 6.1.4 to compute S_5' such that $(\mathfrak{p}, I) \models q$ for some $q \in S_5'$ if and only if $bR_{\mathfrak{p}} \supseteq \lambda I$, $cR_{\mathfrak{p}} \supseteq \lambda I$ and $(\mathfrak{p}, I) \models (1, s, \epsilon, 1)$.

If $(\mathfrak{p}, I) \models (1, 1, r(\epsilon - 1), 1)$ then $I = R_{\mathfrak{p}}$ and

$$|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |\lambda I/bcR_{\mathfrak{p}}| = |I/1R_{\mathfrak{p}}|\,.$$

Let $S_5'' := \{(1, 1, r(\epsilon - 1), 1)\}$ and let $S_5 := S_5' \cup S_5''$. Set $s(q) := s$ if $s \in S_5'$ and $s(q) := 1$ otherwise. Note that in both cases, by definition, $s(q) \in I$. □

Proposition 6.2.3. *Let R be a recursive Prüfer domain, $\lambda \in R\backslash\{0\}$ and Z a finite subset of pp-pairs of the form ${}^{xb=0}/_{c|x}$ and ${}^{d|x}/_{x=0}$ with $b, c, d \in R$. Let*

$$T_Z := \{\mu : Z \to \{1, \dots, 5\} \mid \text{ for all } {}^{d|x}/_{x=0} \in Z, \quad \mu({}^{d|x}/_{x=0}) \leq 3\}.$$

We can compute S_Z a finite subset of R^4, $\rho_Z : S_Z \to T_Z$ and $s_Z : S_Z \times Z \to R$ such that for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$,

(a) there exists $q \in S_Z$ such that $(\mathfrak{p}, I) \models q$;

(b) (1) if $\rho_Z(q)({}^{d|x}/_{x=0}) = 1$ then $|{}^{d|x}/_{x=0}(R_{\mathfrak{p}}/\lambda I)| = |R_{\mathfrak{p}}/s_Z(q, {}^{d|x}/_{x=0})I|$;
(2) if $\rho_Z(q)({}^{d|x}/_{x=0}) = 2$ then $|{}^{d|x}/_{x=0}(R_{\mathfrak{p}}/\lambda I)| = |s_Z(q, {}^{d|x}/_{x=0})R_{\mathfrak{p}}/I|$;
(3) if $\rho_Z(q)({}^{d|x}/_{x=0}) = 3$ then $|{}^{d|x}/_{x=0}(R_{\mathfrak{p}}/\lambda I)| = 1$;

(c) (1) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 1$ then $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |R_{\mathfrak{p}}/\lambda I|$;
(2) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 2$ then $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |R_{\mathfrak{p}}/cR_{\mathfrak{p}}|$;
(3) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 3$ then $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |I/bI|$;
(4) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 4$ then $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = 1$; and
(5) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 5$ then $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |I/s_Z(q, {}^{xb=0}/_{c|x})R_{\mathfrak{p}}|$.

Furthermore, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$, if $(\mathfrak{p}, I) \models q$ and $\rho_Z(q)({}^{d|x}/_{x=0}) = 1$ (respectively $\rho_Z(q)({}^{d|x}/_{x=0}) = 2$) then $s_Z(q, {}^{d|x}/_{x=0}) \neq 0$ (respectively $s_Z(q, {}^{d|x}/_{x=0})R_{\mathfrak{p}} \supseteq I$), and, if $(\mathfrak{p}, I) \models q$ and $\rho_Z(q)({}^{xb=0}/_{c|x}) = 5$ then $s_Z(q, {}^{xb=0}/_{c|x}) \in I$. Moreover, if $b = 0$ then we may assume $\rho_Z(q)({}^{xb=0}/_{c|x}) \in \{1, 2\}$ for all $q \in S_Z$.

Proof. We prove the proposition iteratively. Let $\lambda \in R\backslash\{0\}$ and Z be a finite set of pp-pairs of the form ${}^{xb=0}/_{c|x}$ or ${}^{d|x}/_{x=0}$. If $|Z| = 1$ then 6.2.2 gives the required result. Suppose that S_Z, ρ_Z and s_Z are as in the statement. We construct $S_{Z'}, \rho_{Z'}$ and $s_{Z'}$ for $Z' = Z \cup \{\varphi/\psi\}$ where φ/ψ is either of the form ${}^{d|x}/_{x=0}$ or ${}^{xb=0}/_{c|x}$ and $\varphi/\psi \notin Z$.

Let $S_1, \dots, S_5$, $s : \bigcup_{i=1}^5 S_i \to R$ and $\rho : \bigcup_{i=1}^5 S_i \to \{1, \dots, 5\}$ be as in 6.2.2 (1) or (2), as appropriate (if φ/ψ is of the form $d|x/x=0$ then set $S_4 = S_5 = \emptyset$). By 6.1.4, for each $q \in S_Z$ and $p \in S_i$, we can compute a finite set $S_{q,p,i} \subseteq R^4$ such that for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, $(\mathfrak{p}, I) \models q$ and $(\mathfrak{p}, I) \models p$ if and only if there exists $\hat{q} \in S_{q,p,i}$ such that $(\mathfrak{p}, I) \models \hat{q}$. Let $S_{q,i} := \bigcup_{p \in S_i} S_{q,p,i}$, let $S_q := \bigcup_{i=1}^5 S_{q,i}$ and let $S_{Z'} := \bigcup_{q \in S_Z} S_q$.

For all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, by assumption, there exists $q \in S_Z$ such that $(\mathfrak{p}, I) \models q$ and, by 6.2.2, there exists $1 \leq i \leq 5$, such that $(\mathfrak{p}, I) \models q'$ for some $q' \in S_i$. Therefore, $(\mathfrak{p}, I) \models q''$ for some $q'' \in S_{q,i} \subseteq S_{Z'}$. So (a) holds for $S_{Z'}$.

Define $\rho_{Z'} : S_{Z'} \to T_{Z'}$ by setting

$$\rho_{Z'}(q')(\sigma/\tau) := \begin{cases} \min\{\rho_Z(q)(\sigma/\tau) \mid q' \in S_q\}, & \text{if } \sigma/\tau \in Z; \\ \min\{1 \leq i \leq 5 \mid q' \in S_{q,i} \text{ for some } q \in S_Z\}, & \text{if } \sigma/\tau = \varphi/\psi. \end{cases}$$

For $\sigma/\tau \in Z$, set $s_{Z'}(q', \sigma/\tau)$ to be $s_Z(q, \sigma/\tau)$ for some $q \in S_Z$ where $q' \in S_q$ and $\rho_{Z'}(q')(\sigma/\tau) = \rho_Z(q)(\sigma/\tau)$. For $q' \in S_{Z'}$, set $s_{Z'}(q', \varphi/\psi)$ to be $s(p)$ for some $p \in S_j$ where $j = \rho_{Z'}(q')(\varphi/\psi)$ and $q' \in S_{q,p,j}$.

Now, for $\sigma/\tau \in Z$, properties (b) or (c), as appropriate, are inherited from those properties holding for ρ_Z and s_Z and for φ/ψ, properties (b) or (c), as appropriate, are inherited from ρ and s. □

Lemma 6.2.4. *Let R be a recursive Prüfer domain and $\lambda \in R\backslash\{0\}$.*

(a) Given $b, c \in R$, we can compute finite sets $S_1, \dots, S_6 \subseteq R^4$, $\rho : \bigcup_{i=1}^6 S_i \to \{1, \dots, 6\}$ and $s : \bigcup_{i=1}^6 S_i \to R$ such that for all $q \in \bigcup_{i=1}^6 S_i$, $q \in S_{\rho(q)}$, and, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$ with $\lambda \in I$, there exists $q \in \bigcup_{i=1}^6 S_i$ such that $(\mathfrak{p}, I) \models q$, and

$$|xb=0/c|x(I/\lambda R_\mathfrak{p})| := \begin{cases} |I/\lambda R_\mathfrak{p}|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 1; \\ |I/cI|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 2; \\ |R_\mathfrak{p}/bR_\mathfrak{p}|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 3; \\ 1, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 4; \\ |s(q)R_\mathfrak{p}/I|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 5; \\ |R_\mathfrak{p}/s(q)I|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 6. \end{cases}$$

Furthermore, if $\rho(q) = 5$ and $(\mathfrak{p}, I) \models q$ then $s(q)R_\mathfrak{p} \supseteq I$.

(b) Given $d \in R$, we can compute finite sets $S_1, S_2 \subseteq R^4$, $\rho : \bigcup_{i=1}^2 S_i \to \{1, 2\}$ and $s : \bigcup_{i=1}^2 S_i \to R$ such that for all $q \in \bigcup_{i=1}^2 S_i$, $q \in S_{\rho(q)}$, and, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$ with $\lambda \in I$, there exists $q \in \bigcup_{i=1}^2 S_i$ such that $(\mathfrak{p}, I) \models q$, and

$$|d|x/x=0(I/\lambda R_\mathfrak{p})| := \begin{cases} |I/s(q)R_\mathfrak{p}|, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 1; \\ 0, & \text{if } (\mathfrak{p}, I) \models q \text{ and } \rho(q) = 2. \end{cases}$$

Furthermore, if $\rho(q) = 1$ and $(\mathfrak{p}, I) \models q$ then $s(q) \in I$.

Proof. (a)Let $\alpha, r, s \in R$ be such that $\lambda\alpha = bcr$ and $bc(\alpha - 1) = \lambda s$.

Claim: If $\lambda \in I$ then

$$\left|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})\right| := \begin{cases} |I/\lambda R_{\mathfrak{p}}|, & \text{if (1) } bI \subseteq \lambda R_{\mathfrak{p}} \text{ and } cI \subseteq \lambda R_{\mathfrak{p}}; \\ |I/cI|, & \text{if (2) } bI \subseteq \lambda R_{\mathfrak{p}} \text{ and } \lambda \in cI; \\ |R_{\mathfrak{p}}/bR_{\mathfrak{p}}|, & \text{if (3) } \lambda \in bI \text{ and } cI \subseteq \lambda R_{\mathfrak{p}}; \\ 1, & \text{if (4) } \lambda \in bI,\ \lambda \in cI,\ r \in I \text{ and } \alpha \notin \mathfrak{p}; \\ |rR_{\mathfrak{p}}/I|, & \text{if (5) } \lambda \in bI,\ \lambda \in cI,\ rR_{\mathfrak{p}} \supseteq I \text{ and } \alpha \notin \mathfrak{p}; \\ |R_{\mathfrak{p}}/sI|, & \text{if (6) } \lambda \in bI,\ \lambda \in cI \text{ and } \alpha - 1 \notin \mathfrak{p}. \end{cases}$$

Note that

$$\left|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})\right| = \left|\frac{(\lambda R_{\mathfrak{p}} : b) \cap I + cI}{cI + \lambda R_{\mathfrak{p}}}\right|.$$

For all $a \in R$, $a \in \operatorname{ann}_R(I/\lambda R_{\mathfrak{p}})$ if and only if $aI \subseteq \lambda R_{\mathfrak{p}}$. Therefore, the equalities for conditions (1) and (2) hold. If $\lambda \in bI \subseteq bR_{\mathfrak{p}}$ then $b \neq 0$ since $\lambda \neq 0$ and $(\lambda R_{\mathfrak{p}} : b) \cap I/\lambda R_{\mathfrak{p}} \cong R_{\mathfrak{p}}/bR_{\mathfrak{p}}$. So the equality for condition (3) holds.

When proving the equalities for (4), (5) and (6), we may assume $b \neq 0$ and $c \neq 0$ since $\lambda \neq 0$, $\lambda \in cI$ and $\lambda \in bI$. Moreover,

$$\left|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})\right| = |(\lambda R_{\mathfrak{p}} : b) + cI/cI| = |\lambda R_{\mathfrak{p}} + bcI/bcI|.$$

If $\alpha \notin \mathfrak{p}$ and $r \in I$ then $\lambda \in bcI$. So the equality for condition (4) holds.

Suppose condition (5) holds. Then $\alpha \notin \mathfrak{p}$ implies $\lambda R_{\mathfrak{p}} = bcrR_{\mathfrak{p}}$. Since $rR_{\mathfrak{p}} \supseteq I$, $|{}^{xb=0}/_{c|x}(R_{\mathfrak{p}}/\lambda I)| = |rR_{\mathfrak{p}}/I|$. So the equality for condition (5) holds.

Suppose condition (6) holds. Since $\alpha - 1 \notin \mathfrak{p}$, $bcI = \lambda sI$. So $(\lambda R_{\mathfrak{p}} + bcI)/bcI \cong R_{\mathfrak{p}}/sI$. So the equality for condition (6) holds. So we have proved the claim.

Given a finite set of conditions of the form $\beta \notin \mathfrak{p}$, $a \in \lambda I$ or $aR_{\mathfrak{p}} \supseteq \lambda I$, using 6.1.4 and 6.2.1, we can compute a finite set $S \subseteq R^4$ such that $(\mathfrak{p}, I)$ satisfies these conditions if and only if $(\mathfrak{p}, I) \models q$ for some $q \in S$. So for each conditions (i) for $1 \leq i \leq 6$ in the claim, we can compute $S_i \subseteq R^4$ such that $(\mathfrak{p}, I)$ satisfies (i) and $\lambda \in I$ if and only if there exists $q \in S_i$ with $(\mathfrak{p}, I) \models q$. Moreover, it is easy to see that, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$, there exists $1 \leq i \leq 6$ such that $(\mathfrak{p}, I)$ satisfies (i). Let $\rho : \bigcup_{i=1}^{6} S_i \to \{1, \ldots, 6\}$ be such that $q \in S_{\rho(q)}$. Finally, set $s(q) = 1$ if $\rho(q) \leq 4$, $s(q) = r$ if $\rho(q) = 5$ and $s(q) = s$ if $\rho(q) = 6$.

(b) The case $d = 0$ is done as in 6.2.2. Suppose $d \neq 0$. Let $\alpha, r, s \in R$ be such that $d\alpha = \lambda r$ and $\lambda(\alpha - 1) = ds$. If either $\alpha \notin \mathfrak{p}$, or, $\alpha - 1 \notin \mathfrak{p}$ and $sR_{\mathfrak{p}} \supseteq I$ then $\lambda R_{\mathfrak{p}} \supseteq dI$. If $\alpha - 1 \notin \mathfrak{p}$ and $s \in I$ then $dI \supseteq dsR_{\mathfrak{p}} = \lambda R_{\mathfrak{p}}$. Since $d \neq 0$, $dI/dsR_{\mathfrak{p}} \cong I/sR_{\mathfrak{p}}$. Therefore

$$\left|{}^{d|x}/_{x=0}(I/\lambda R_{\mathfrak{p}})\right| = \left|\frac{dI + \lambda R_{\mathfrak{p}}}{\lambda R_{\mathfrak{p}}}\right| = \begin{cases} 1, & \text{if } (\mathfrak{p}, I) \models (1, 0, \alpha, 1); \\ 1, & \text{if } (\mathfrak{p}, I) \models (s, 0, \alpha - 1, 1); \\ |I/sR_{\mathfrak{p}}|, & \text{if } (\mathfrak{p}, I) \models (1, s, \alpha - 1, 1). \end{cases}$$

It is now clear how to define, S_1, S_2, ρ and s. □

Proposition 6.2.5. *Let R be a recursive Prüfer domain, $\lambda \in R\backslash\{0\}$ and Z a finite subset of pp-pairs of the form ${}^{xb=0}/_{c|x}$ and ${}^{d|x}/_{x=0}$ with $b, c, d \in R$. Let*

$$T_Z := \{\mu : Z \to \{1, \ldots, 6\} \mid \mu({}^{d|x}/_{x=0}) \leq 2\}.$$

We can compute S_Z a finite subset of R^4, $\rho_Z : S_Z \to T_Z$ and $s_Z : S_Z \times Z \to R$ such that, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$ with $\lambda \in I$, if $(\mathfrak{p}, I) \models q$ for some $q \in S_Z$ then $\lambda \in I$, and, if $\lambda \in I$ then

(a) there exists $q \in S_Z$ such that $(\mathfrak{p}, I) \models q$;

(b) (1) if $\rho_Z(q)({}^{d|x}/_{x=0}) = 1$ then $|{}^{d|x}/_{x=0}(I/\lambda R_{\mathfrak{p}})| = |I/s_Z(q, {}^{d|x}/_{x=0})R_{\mathfrak{m}}|$;

(2) if $\rho_Z(q)({}^{d|x}/_{x=0}) = 2$ *then* $|{}^{d|x}/_{x=0}(I/\lambda R_{\mathfrak{p}})| = 1$*;*

(c) (1) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 1$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = |I/\lambda R_{\mathfrak{p}}|$*;*

(2) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 2$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = |I/cI|$*;*

(3) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 3$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}|$*;*

(4) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 4$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = 1$*;*

(5) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 5$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = |s(q, {}^{xb=0}/_{c|x})R_{\mathfrak{p}}/I|$*; and*

(6) if $\rho_Z(q)({}^{xb=0}/_{c|x}) = 6$ *then* $|{}^{xb=0}/_{c|x}(I/\lambda R_{\mathfrak{p}})| = |R_{\mathfrak{p}}/s(q, {}^{xb=0}/_{c|x})I|$*.*

Furthermore, for all prime ideals $\mathfrak{p} \lhd R$ *and ideals* $I \lhd R_{\mathfrak{p}}$*, if* $\rho_Z(q)({}^{d|x}/_{x=0}) = 1$ *and* $(\mathfrak{p}, I) \models q$ *then* $s_Z(q, {}^{d|x}/_{x=0}) \in I$*, and, if* $\rho_Z(q)({}^{xb=0}/_{c|x}) = 5$ *and* $(\mathfrak{p}, I) \models q$ *then* $s_Z(q, {}^{d|x}/_{x=0})R_{\mathfrak{p}} \supseteq I$*.*

Proof. This can be proved similarly to 6.2.3 by replacing 6.2.2 by 6.2.4. The extra condition that $(\mathfrak{p}, I) \models q$ for some $q \in S_Z$ implies $\lambda \in I$ can be incorporated using 6.1.4, since $(\mathfrak{p}, I) \models (1, \lambda, 1, 1)$ if and only if $\lambda \in I$. □

7. Finite modules

In this section we investigate the consequences of EPP(R) being recursive, and of EPP(R) and the radical relation being recursive. In particular, we show that for a recursive Prüfer domain the theory of R-modules of size n is decidable uniformly in n if and only if EPP(R) is recursive, 7.6.

Observe that finite modules over a Prüfer domain R are finite direct sums of modules of the form $R_{\mathfrak{p}}/I$ where $\mathfrak{p} \lhd R$ is a prime ideal and $I \lhd R_{\mathfrak{p}}$ is an ideal. There are many ways of seeing this. If M is finite then M is pure-injective. So, by 2.1.2 and 2.3.5, there exist prime ideals $\mathfrak{p}_i \lhd R$ and uniserial $R_{\mathfrak{p}_i}$-modules U_i such that M is elementary equivalent, and hence isomorphic, to $\oplus_{i=1}^n U_i$. The desired result now follows from 2.3.7.

Let W be the set of tuples[8] $(f, g, \overline{a}, \gamma)$ where

(i) $f : X \to \mathbb{N}$ where $X := X_0 \cup \{\star\}$, X_0 is a finite subset of R and $\star \notin R$,
(ii) $g : Y \to \mathbb{N}$ where Y is a finite subset of R,
(iii) $\overline{a} := (a_1, \dots, a_m)$ is a tuple of length $m \in \mathbb{N}$ of elements of R, and $\gamma \in R$.

Let V be the set of $(f, g, \overline{a}, \gamma) \in W$ such that, for some $h \in \mathbb{N}$ and for $1 \leq i \leq h$, there exist a prime ideal $\mathfrak{p}_i$ and an ideal $I_i \lhd R_{\mathfrak{p}_i}$ such that $a_j \in I_i$ for $1 \leq j \leq m$ and $1 \leq i \leq h$, $\gamma \notin \mathfrak{p}_i$,

$$|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i| = f(\star),$$

$$|\oplus_{i=1}^h R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| = f(e) \quad \text{for} \quad e \in X_0$$

and

$$|\oplus_{i=1}^h R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}| \geq g(e) \quad \text{for} \quad e \in Y.$$

We write $(\mathfrak{p}_i, I_i)_{1 \leq i \leq h} \models (f, g, \overline{a}, \gamma)$. By convention, $\emptyset \models (f, g, \overline{a}, \gamma)$ if $f(e) = 1$ for all $e \in X$ and $g(e) = 1$ for all $e \in Y$ and in this situation, $(f, g, \overline{a}, \gamma) \in V$.

As in 4.2, $\mathbb{W}$ denotes the bounded distributive lattice generated by W and $\mathbb{V}$ denotes the (prime) filter in $\mathbb{W}$ generated by V.

Let W_0 be the subset of elements of the form $(f, \emptyset, \overline{a}, \gamma)$ with $|X_0| \leq 1$ and let W_1 be the subset of elements of the form $(f, \emptyset, \overline{a}, \gamma)$ where, in both cases, $\emptyset$ denotes the function from the empty set to $\mathbb{N}$. Let $\mathbb{W}_0$, respectively $\mathbb{W}_1$, denote the lattice generated by $W_0 \cup \{\top, \bot\}$, respectively $W_1 \cup \{\top, \bot\}$, in $\mathbb{W}$.

[8]The letter "W" is chosen to match the notation in subsection 4.2.

Lemma 7.1. *Let R be a recursive Prüfer domain. There is an algorithm which given $w \in W_1$ returns $\underline{u} \in \mathbb{W}_0$, such that $w \in V$ if and only if $\underline{u} \in \mathbb{V}$.*

Proof. Define $\operatorname{clx}(f, \emptyset, \overline{a}, \gamma) := (|X_0|, \prod_{e \in X_0} f(e))$ and order $\mathbb{N}_0 \times \mathbb{N}$ lexicographically. For any $w \in W_1$ with $|X_0| > 1$, we show how to compute $\underline{u} \in \mathbb{W}_1$ with $\operatorname{clx}\underline{u} < \operatorname{clx} w$ such that $w \in V$ if and only if $\underline{u} \in \mathbb{V}$. We can then apply the same process to those components of $\underline{u}$ which are not already in W_0; since the lexicographic order on $\mathbb{N}_0 \times \mathbb{N}$ is artinian, this is sufficient to prove the lemma.

Suppose $w = (f, \emptyset, \overline{a}, \gamma)$ where $f : X_0 \cup \{\star\} \to \mathbb{N}$ and $|X_0| > 0$. Take $e_1, e_2 \in X_0$ non-equal. Let $\alpha, r, s \in R$ be such that $e_1\alpha = e_2 r$ and $e_2(\alpha - 1) = e_1 s$. Let Ω be the set of pairs of functions (f_1, f_2) such that $f_1 : X \cup \{r\} \to \mathbb{N}$, $f_2 : X \cup \{s\} \to \mathbb{N}$ and

1. $f_1(e)f_2(e) = f(e)$ for all $e \in X$,
2. $f_1(e_1) = f_1(e_2)f_1(r)$, and
3. $f_2(e_2) = f_2(e_1)f_2(s)$.

Let $\underline{u}$ be

$$\bigsqcup_{\substack{(f_1,f_2)\in\Omega \\ f_1(e_2)\neq 1, f_2(e_1)\neq 1}} (f_1|_{(X\cup\{r\})\setminus\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha) \sqcap (f_2|_{(X\cup\{s\})\setminus\{e_2\}}, \emptyset, \overline{a}, \gamma(\alpha-1))$$

$$\sqcup \bigsqcup_{\substack{(f_1,f_2)\in\Omega \\ f_1(e_2)= 1, f_2(e_1)\neq 1}} (f_1|_{X\setminus\{e_2\}}, \emptyset, \overline{a}, e_2\gamma\alpha) \sqcap (f_2|_{(X\cup\{s\})\setminus\{e_2\}}, \emptyset, \overline{a}, \gamma(\alpha-1))$$

$$\sqcup \bigsqcup_{\substack{(f_1,f_2)\in\Omega \\ f_1(e_2)\neq 1, f_2(e_1)= 1}} (f_1|_{(X\cup\{r\})\setminus\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha) \sqcap (f_2|_{X\setminus\{e_1\}}, \emptyset, \overline{a}, e_1\gamma(\alpha-1))$$

$$\sqcup \bigsqcup_{\substack{(f_1,f_2)\in\Omega \\ f_1(e_2)= 1, f_2(e_1)= 1}} (f_1|_{X\setminus\{e_2\}}, \emptyset, \overline{a}, e_2\gamma\alpha) \sqcap (f_2|_{X\setminus\{e_1\}}, \emptyset, \overline{a}, e_1\gamma(\alpha-1)).$$

Claim: $w \in V$ if and only if $\underline{u} \in \mathbb{V}$.
For all prime ideals $\mathfrak{p} \lhd R$, $\alpha \notin \mathfrak{p}$ implies $e_1R_\mathfrak{p} = e_1\alpha R_\mathfrak{p} = e_2 r R_\mathfrak{p}$ and $\alpha - 1 \notin \mathfrak{p}$ implies $e_2R_\mathfrak{p} = e_2(\alpha-1)R_\mathfrak{p} = e_1 s R_\mathfrak{p}$. So, $\alpha \notin \mathfrak{p}$ implies

$$|R_\mathfrak{p}/e_1R_\mathfrak{p}| = |R_\mathfrak{p}/e_2R_\mathfrak{p}| \cdot |R_\mathfrak{p}/rR_\mathfrak{p}|$$

and $\alpha - 1 \notin \mathfrak{p}$ implies

$$|R_\mathfrak{p}/e_2R_\mathfrak{p}| = |R_\mathfrak{p}/e_1R_\mathfrak{p}| \cdot |R_\mathfrak{p}/sR_\mathfrak{p}|.$$

Suppose that $\mathfrak{p}_i \lhd R$ is a prime ideal and $I_i \lhd R_{\mathfrak{p}_i}$ is an ideal for $1 \le i \le h$ such that $(\mathfrak{p}_i, I_i)_{1\le i\le h} \models (f, \emptyset, \overline{a}, \gamma)$. By reordering, we may assume that $\alpha \notin \mathfrak{p}_i$ for $1 \le i \le h'$ and $\alpha - 1 \notin \mathfrak{p}_i$ for $h'+1 \le i \le h$. Let $f_1(\star) := |\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/I_i|$ and $f_2(\star) := |\oplus_{i=h'+1}^{h} R_{\mathfrak{p}_i}/I_i|$. For each $e \in X_0 \cup \{r\}$, let $f_1(e) = |\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}|$ and for each $e \in X_0 \cup \{s\}$, let $f_2(e) = |\oplus_{i=h'+1}^{h} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}|$. It follows from the first paragraph of the proof of this claim that $(f_1, f_2) \in \Omega$.

If $f_1(e_2) \neq 1$ then

$$(\mathfrak{p}_i, I_i)_{1\le i\le h'} \models (f_1|_{(X\cup\{r\})\setminus\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha)$$

and if $f_2(e_1) \neq 1$ then

$$(\mathfrak{p}_i, I_i)_{h'+1\le i\le h} \models (f_2|_{(X\cup\{s\})\setminus\{e_2\}}, \emptyset, \overline{a}, \gamma(\alpha-1)).$$

We explain the first statement. By definition of f_1, $f_1(\star) := |\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/I_i|$ and $f_1(e) = |\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/eR_{\mathfrak{p}_i}|$ for all $e \in X_0 \cup \{r\}$. Since $(\mathfrak{p}_i, I_i)_{1\leq i\leq h} \models (f, \emptyset, \overline{a}, \gamma)$, $\gamma \notin \mathfrak{p}_i$ for $1 \leq i \leq h$ and $a_1, \ldots, a_m \in I_i$ for all $1 \leq i \leq h$ where $\overline{a} = (a_1, \ldots, a_m)$. By definition of $(\mathfrak{p}_i, I_i)_{1\leq i\leq h'}$, $\alpha \notin \mathfrak{p}_i$ for $1 \leq i \leq h'$. Thus $\alpha\gamma \notin \mathfrak{p}_i$ for $1 \leq i \leq h'$. Therefore $(\mathfrak{p}_i, I_i)_{1\leq i\leq h'} \models (f_1|_{(X\cup\{r\})\backslash\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha)$.

By definition, if $f_1(e_2) = 1$, then $e_2 \notin \mathfrak{p}_i$ for $1 \leq i \leq h'$ and if $f_2(e_1) = 1$ then $e_1 \notin \mathfrak{p}_i$ for $h'+1 \leq i \leq h$. So if $f_1(e_2) = 1$ then

$$(\mathfrak{p}_i, I_i)_{1\leq i\leq h'} \models (f_1|_{X\backslash\{e_2\}}, \emptyset, \overline{a}, e_2\gamma\alpha)$$

and if $f_2(e_1) = 1$ then

$$(\mathfrak{p}_i, I_i)_{h'+1\leq i\leq h} \models (f_2|_{X\backslash\{e_1\}}, \emptyset, \overline{a}, e_1\gamma(\alpha - 1)).$$

So we have shown that if $w \in V$ then one of the components of the join defining $\underline{u}$ is in $\mathbb{V}$ and hence $\underline{u} \in \mathbb{V}$.

Conversely, take $(f_1, f_2) \in \Omega$. Suppose $f_1(e_2) \neq 1$ and

$$(\mathfrak{p}_i, I_i)_{1\leq i\leq h'} \models (f_1|_{(X\cup\{r\})\backslash\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha).$$

Since $\alpha \notin \mathfrak{p}_i$ for $1 \leq i \leq h'$,

$$\left|\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/e_1R_{\mathfrak{p}_i}\right| = \left|\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/e_2R_{\mathfrak{p}_i}\right| \cdot \left|\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/rR_{\mathfrak{p}_i}\right| = f_1(e_2)\cdot f_1(r) = f_1(e_1)$$

by definition of Ω. So $(\mathfrak{p}_i, I_i)_{1\leq i\leq h'} \models (f_1, \emptyset, \overline{a}, \gamma)$.

Suppose that $f_2(e_1) = 1$ and

$$(\mathfrak{q}_i, J_i)_{1\leq i\leq h''} \models (f_2|_{X\backslash\{e_1\}}, \emptyset, \overline{a}, e_1\gamma(\alpha - 1)).$$

Then

$$(\mathfrak{q}_i, J_i)_{1\leq i\leq h''} \models (f_2, \emptyset, \overline{a}, \gamma)$$

because $e_1 \notin \mathfrak{q}_i$ for $1 \leq i \leq h''$ implies

$$\left|\oplus_{i=1}^{h''} R_{\mathfrak{q}_i}/e_1R_{\mathfrak{q}_i}\right| = 1 = f_1(e_1).$$

So, setting $\mathfrak{p}_i := \mathfrak{q}_{i-h'}$ and $I_i := J_{i-h'}$ for $h'+1 \leq i \leq h'+h'' = h$,

$$(\mathfrak{p}_i, I_i)_{1\leq i\leq h} \models (f, \emptyset, \overline{a}, \gamma)$$

because $f_1(e)f_2(e) = f(e)$ for all $e \in X$. We leave the case $f_1(e_2) \neq 1$ and $f_2(e_1) \neq 1$, the case $f_1(e_2) = 1$ and $f_2(e_1) \neq 1$ and the case $f_1(e_2) = f_2(e_2) = 1$ to the reader.

Claim: $\text{clx}\underline{u} < \text{clx}w$

We show that each of the components, u', of the lattice combination defining $\underline{u}$ have $\text{clx}u' < \text{clx}w$. We only consider the components involving f_1; the result for those involving f_2 follows similarly.

If $f_1(e_2) = 1$ then $\text{clx}(f_1|_{X\backslash\{e_2\}}, \emptyset, \overline{a}, e_2\gamma\alpha) < \text{clx}w$ since $|X_0| > |X_0\backslash\{e_2\}|$. If $f_1(e_2) > 1$ then

$$\frac{f_1(r)}{f(e_1)} = \frac{f_1(e_2)f_1(r)}{f_1(e_2)f(e_1)} = \frac{f_1(e_1)}{f_1(e_2)f(e_1)} < 1$$

since $f_1(e_1)/f(e_1) \leq 1$. So

$$f_1(r)\cdot\prod_{x\in X_0\backslash\{e_1\}} f_1(x) \leq f_1(r)\cdot\prod_{x\in X_0\backslash\{e_1\}} f(x) = \frac{f_1(r)}{f(e_1)}\cdot\prod_{x\in X_0} f(x) < \prod_{x\in X_0} f(x).$$

Therefore $\text{clx}(f_1|_{(X\cup\{r\})\backslash\{e_1\}}, \emptyset, \overline{a}, \gamma\alpha) < \text{clx}w$. □

Lemma 7.2. *Let R be a recursive Prüfer domain. If $\mathrm{EPP}(R)$ is recursive then there is an algorithm which given $w \in W_1$ answers whether $w \in V$ or not.*

Proof. Let $w = (f, \emptyset, \overline{a}, \gamma)$ and suppose that $X_0 = \{e\}$ i.e. $w \in W_0$. Let P be the set of prime divisors of $f(\star) \cdot f(e)$. If P is empty then $f(\star) = f(e) = 1$ and hence $w \in V$. Otherwise, for each $p \in P$, let $n_p \in \mathbb{N}_0$ and $m_p \in \mathbb{N}_0$ be such that $f(\star) = \prod_{p \in P} p^{n_p}$ and $f(e) = \prod_{p \in P} p^{m_p}$. For any prime ideal $\mathfrak{p} \lhd R$, $e \in R$ and ideal $I \lhd R_\mathfrak{p}$, if $|R_\mathfrak{p}/eR_\mathfrak{p}|$ (respectively $|R_\mathfrak{p}/I|$) is finite then it is a power of $|R/\mathfrak{p}|$. Thus both $|R_\mathfrak{p}/eR_\mathfrak{p}|$ and $|R_\mathfrak{p}/I|$ are prime powers. Hence $w \in V$ if and only if $(p, n_p; \overline{a}; \gamma; e, m_p) \in \mathrm{EPP}_*(R)$ for all $p \in P$.

If $w \in W_1$ then by 7.1, we can compute $\underline{w} \in \mathbb{W}_0$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$. Therefore, by the previous paragraph there is an algorithm which, given $w \in W_1$ answers whether $w \in V$ or not. □

Corollary 7.3. *Let R be a recursive Prüfer domain with $\mathrm{EPP}(R)$ recursive. There is an algorithm which given $r, a, \gamma, \delta \in R$, $n \in \mathbb{N}$, $e_j \in R$ for $1 \leq j \leq n$, $N \in \mathbb{N}$ and $N_j \in \mathbb{N}$ for $1 \leq j \leq n$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that*

(1) $(\mathfrak{p}_i, I_i) \models (r, a, \gamma, \delta)$ *for* $1 \leq i \leq h$,
(2) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i\right| = N$, *and*
(3) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/e_j R_{\mathfrak{p}_i}\right| = N_j$ *for* $1 \leq j \leq n$.

Proof. Applying 6.1.6, we may reduce to the case where $a = ra'$. We may also assume $r \neq 0$ since $(\mathfrak{p}, I) \models (0, b', \gamma, \delta)$ implies $I = 0$ and hence $|R_\mathfrak{p}/I| = |R_\mathfrak{p}|$ which is infinite.

For any prime ideal $\mathfrak{p} \lhd R$ and ideal $I \lhd R_\mathfrak{p}$, $(\mathfrak{p}, I) \models (r, ra', \gamma, \delta)$ if and only if there exists $J \lhd R_\mathfrak{p}$ such that $I = rJ$ and $(\mathfrak{p}, J) \models (1, a', \gamma, \delta)$. Note that, because R is a domain, $|R_\mathfrak{p}/rJ| = |R_\mathfrak{p}/J| \cdot |R_\mathfrak{p}/rR_\mathfrak{p}|$.

Therefore, there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ satisfying $(1), (2)$ and (3) if and only if there exist $N', N'' \in \mathbb{N}$ with $N' \cdot N'' = N$, $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $J_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that

(a) $(\mathfrak{p}_i, J_i) \models (1, a', \gamma, \delta)$ for $1 \leq i \leq h$,
(b) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/J_i\right| = N'$, and
(c) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/rR_{\mathfrak{p}_i}\right| = N''$, $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/e_j R_{\mathfrak{p}_i}\right| = N_j$ for $1 \leq j \leq n$.

By 7.2, there is an algorithm which answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $J_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ satisfying (a),(b) and (c). □

Proposition 7.4. *Let R be a recursive Prüfer domain such that $\mathrm{EPP}(R)$ and the radical relation are recursive. There is an algorithm which, given $(f, g, \overline{a}, \gamma) \in W$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i)_{1 \leq i \leq h} \models (f, g, \overline{a}, \gamma)$.*

Proof. Suppose $(f, g, \overline{a}, \gamma) \in W$.

Case $Y = \emptyset$: This is 7.2.

Case $|Y| = 1$: Suppose that $X_0 := \{e_1, \ldots, e_n\}$ and $Y := \{e\}$.

If $\gamma \prod_{j=1}^n e_j \notin \operatorname{rad} eR$ then there exists a prime ideal $\mathfrak{p}$ such that $e_j \notin \mathfrak{p}$ for $1 \leq j \leq n$, $\gamma \notin \mathfrak{p}$ and $e \in \mathfrak{p}$. Then $|R_\mathfrak{p}/e_j R_\mathfrak{p}| = 1$ for $1 \leq j \leq n$ and $|R_\mathfrak{p}/eR_\mathfrak{p}| > 1$. So $(\mathfrak{p}_i, I_i)_{1 \leq i \leq h} \models (f, \emptyset, \overline{a}, \gamma)$ if and only if $(\mathfrak{p}_i, I_i)_{1 \leq i \leq h'} \models (f, g, \overline{a}, \gamma)$ where $h' := h + g(e)$, and, $\mathfrak{p}_i := \mathfrak{p}$ and $I_i := R_{\mathfrak{p}_i}$ for $h < i \leq g(e)$. So $(f, g, \overline{a}, \gamma) \in V$ if and only if $(f, \emptyset, \overline{a}, \gamma) \in V$. So we are now in the situation of case $Y = \emptyset$.

If $\gamma\prod_{j=1}^n e_j \in \operatorname{rad} eR$ then there exist $l \in \mathbb{N}$ and $r \in R$ such that $(\gamma\prod_{j=1}^n e_j)^l = er$. Thus, for all prime ideals $\mathfrak{p}$ with $\gamma \notin \mathfrak{p}$, $|R_\mathfrak{p}/eR_\mathfrak{p}| \leq \prod_{j=1}^n |R_\mathfrak{p}/e_jR_\mathfrak{p}|^l$. Therefore $(\mathfrak{p}_i, I_i)_{1\leq i\leq h} \models (f, g, \overline{a}, \gamma)$ if and only if there exists $f' : X \cup \{e\} \to \mathbb{N}$ where $f'(x) = f(x)$ for all $x \in X$ and $g(e) \leq f'(e) \leq \prod_{j=1}^n f(e_j)^l$ and $(\mathfrak{p}_i, I_i)_{1\leq i\leq h} \models (f', \emptyset, \overline{a}, \gamma)$. Since the set of f' is finite and computable, we are now in the situation of case $Y = \emptyset$.

Case $|Y| \geq 2$: We show how to reduce to the situation where $|Y| \leq 1$. By 4.1.3, we may assume that g is a constant function. Take $e_1, e_2 \in Y$ non-equal. Let $\alpha, r, s \in R$ be such that $e_1\alpha = e_2r$ and $e_2(\alpha - 1) = e_1s$. Since for all prime ideals $\mathfrak{p} \lhd R$, either $\alpha \notin \mathfrak{p}$ or $\alpha - 1 \notin \mathfrak{p}$, by 4.1.1, $(f, g, \overline{a}, \gamma) \in V$ if and only if

$$\bigsqcup_{(f_1,f_2,g_1,g_2)\in\Omega_{f,g,2}} (f_1, g_1, \overline{a}, \alpha\gamma) \sqcap (f_2, g_2, \overline{a}, (\alpha - 1)\gamma) \in \mathbb{V}.$$

Note that if g is constant then g_1 and g_2 are constant for all $(f_1, f_2, g_1, g_2) \in \Omega_{f,g}$.

For each $(f_1, f_2, g_1, g_2) \in \Omega_{f,g}$, either $|Y_1| < |Y|$ or $e_1, e_2 \in Y_1$. In the first case we are done. In the second, $(f_1, g_1, \overline{a}, \alpha\gamma) \in V$ if and only if $(f_1, g_1|_{Y_1\setminus\{e_1\}}, \overline{a}, \alpha\gamma) \in V$. This is because, for all prime ideals $\mathfrak{p}$ with $\alpha \notin \mathfrak{p}$, $|R_\mathfrak{p}/e_1R_\mathfrak{p}| \geq |R_\mathfrak{p}/e_2R_\mathfrak{p}|$ and hence if $|R_\mathfrak{p}/e_2R_\mathfrak{p}| \geq g_1(e_2)$ then $|R_\mathfrak{p}/e_1R_\mathfrak{p}| \geq g_1(e_2) = g_1(e_1)$. So we may replace $(f_1, g_1, \overline{a}, \alpha\gamma)$ by $(f_1, g_1|_{Y_1\setminus\{e_1\}}, \overline{a}, \alpha\gamma)$. A similar argument shows that either $|Y_2| < |Y|$ or we can replace $(f_2, g_2, \overline{a}, (\alpha - 1)\gamma)$ by $(f_2, g_2|_{Y_1\setminus\{e_2\}}, \overline{a}, (\alpha - 1)\gamma)$. □

Corollary 7.5. *Let R be a recursive Prüfer domain with the radical relation and* EPP(R) *recursive. There is an algorithm which given $r, a, \gamma, \delta \in R$, $n, n' \in \mathbb{N}$, $e_j \in R$ for $1 \leq j \leq n$, $e'_j \in R$ for $1 \leq j \leq n'$, $N \in \mathbb{N}$, $N_j \in \mathbb{N}$ for $1 \leq j \leq n$ and $N'_j \in \mathbb{N}$ for $1 \leq j \leq n'$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that*

(1) $(\mathfrak{p}_i, I_i) \models (r, a, \gamma, \delta)$ *for* $1 \leq i \leq h$,
(2) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i\right| = N$,
(3) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/e_jR_{\mathfrak{p}_i}\right| = N_j$ *for* $1 \leq j \leq n$, *and*
(4) $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/e_jR_{\mathfrak{p}_i}\right| \geq N'_j$ *for* $1 \leq j \leq n'$.

Proof. The proof is as in 7.3 but we use 7.4 in place of 7.2. □

Theorem 7.6. *Let R be a recursive Prüfer domain. The theory of R-modules of size n is decidable uniformly in n if and only if* EPP(R) *is recursive.*

Proof. The forward direction is 3.2.10.

Standard arguments using the Baur-Monk theorem and the fact that T_R is recursively axiomatisable imply that the theory of R-modules of size n is decidable uniformly in n if and only if there is an algorithm which, given $N \in \mathbb{N}$, pp-pairs φ_i/ψ_i and $A_i \in \mathbb{N}$ for $1 \leq i \leq m$, answers whether there exists $M \in \text{Mod-}R$ satisfying

$$(\star) \qquad |x{=}x/x{=}0| = N \wedge \bigwedge_{i=1}^m |\varphi_i/\psi_i| = A_i.$$

Unfortunately, we can't directly apply the statement of [GLT19, 4.1] to reduce to the case that the pp-pairs φ_i/ψ_i are of the form $d|x/x{=}0$ or $xb{=}0/c|x$ with $b, c, d \in R$. However, the proof of [GLT19, 4.1] can be easily modified to allow us to do this.

Roughly speaking, starting with a sentence χ of the form

$$(\dagger)\qquad \bigwedge_{i=1}^{l} |\varphi_i/\psi_i| = A_i \wedge \bigwedge_{i=l+1}^{m} |\varphi_i/\psi_i| \geq A_i$$

where $A_i \in \mathbb{N}$ and φ_i/ψ_i is an arbitrary pp-pair for $1 \leq i \leq m$, each step of the proof of [GLT19, 4.1] produces a finite set S of finite tuples of sentences $(\chi_1, \ldots, \chi_k)$ as in ($\dagger$), but with the form of the pp-pairs involved progressively improved, such that there is an R-module satisfying χ if and only if there exist $(\chi_1, \ldots, \chi_k) \in S$ and R-modules $M_i \in \text{Mod-}R$ with $M_i \models \chi_i$ for $1 \leq i \leq k$. In order to adapt the proof to our situation, the reader just needs to note that, for each of the steps of the proof of [GLT19, 4.1], if the initial sentence χ is as in ($\star$), then the sentences χ_i in the finite tuples $(\chi_1, \ldots, \chi_k) \in S$ are of the same form as in ($\star$).

Let $N \in \mathbb{N}$, Z be a finite set of pp-pairs of the form $d|x/x=0$ and $xb=0/c|x$, and $f : Z \to \mathbb{N}$. Let χ be the sentence

$$|x=x/x=0| = N \wedge \bigwedge_{\varphi/\psi \in Z} |\varphi/\psi| = f(\varphi/\psi).$$

Recall that every finite module over a Prüfer domain is isomorphic to a direct sum of modules of the form $R_\mathfrak{p}/I$ for some prime ideal $\mathfrak{p} \lhd R$ and ideal $I \lhd R_\mathfrak{p}$.

Let S_Z, ρ_Z and s_Z be as in 6.2.3 with $\lambda := 1$. Enumerate $S_Z := \{q_1, \ldots, q_m\}$ and let $q_i := (r_i, r_i a_i, \gamma_i, \delta_i)$ for $1 \leq i \leq m$. By definition, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$ there exists $1 \leq i \leq m$ such that $(\mathfrak{p}, I) \models q_i$. Therefore, there exists $M \in \text{Mod-}R$ satisfying χ if and only if there exist $N_i \in \mathbb{N}$ and $f_i : Z \to \mathbb{N}$ for $1 \leq i \leq m$ such that $N = \prod_{i=1}^m N_i$, for all $\varphi/\psi \in Z$, $f(\varphi/\psi) = \prod_{i=1}^m f_i(\varphi/\psi)$ and there exist $h_i \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_{ij} \lhd R$ and ideals $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq j \leq h_i$ such that, for $1 \leq i \leq m$,

(a)$_i$ $(\mathfrak{p}_{ij}, I_{ij}) \models q_i$ for $1 \leq j \leq h_i$ and

(b)$_i$
$$\oplus_{j=1}^{h_j} R_{\mathfrak{p}_{ij}}/I_{ij} \models |x=x/x=0| = N_i \wedge \bigwedge_{\varphi/\psi \in Z} |\varphi/\psi| = f_i(\varphi/\psi).$$

Fix $1 \leq i \leq m$, $N_i \in \mathbb{N}$ and f_i as above. If $r_i = 0$ and $(\mathfrak{p}, I) \models (r_i, r_i a_i, \gamma_i, \delta_i)$ then $R_\mathfrak{p}/I = R_\mathfrak{p}$, which is not finite. Hence, (a)$_i$ and (b)$_i$ holds if and only if $h_i = 0$, $N_i = 1$ and $f_i(\varphi/\psi) = 1$ for all $\varphi/\psi \in Z$. So, we may assume that $r_i \neq 0$. Note, by 2.3.9(iv), if $R_\mathfrak{p}/I$ is finite then $I^\# = \mathfrak{p}$. So, $(\mathfrak{p}, I) \models (r_i, r_i a_i, \gamma_i, \delta_i)$ if and only if $I = r_i J$ for some $J \lhd R_\mathfrak{p}$, $a \in J$, $\gamma \notin \mathfrak{p}$ and $\delta \notin \mathfrak{p}$.

Therefore $h_i \in \mathbb{N}_0$, $\mathfrak{p}_{ij} \lhd R$ and $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq j \leq h_i$ satisfy (a)$_i$ and (b)$_i$ if and only if for each $1 \leq j \leq h_i$, there exists $J_{ij} \lhd R_{\mathfrak{p}_{ij}}$ such that $I_{ij} = r_i J_{ij}$, $a \in J_{ij}$, $\gamma\delta \notin \mathfrak{p}$, $\left|\bigoplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/I_{ij}\right| = N_i$, and, for all $d|x/x=0 \in Z$,

(1) if $\rho_Z(q_i)(d|x/x=0) = 1$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, d|x/x=0) I_{ij}\right| = f_i(d|x/x=0)$;

(2) if $\rho_Z(q_i)(d|x/x=0) = 2$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, d|x/x=0) R_{\mathfrak{p}_{ij}}\right| = f_i(d|x/x=0) \cdot N_i$;

(3) if $\rho_Z(q_i)(d|x/x=0) = 3$ then $f_i(d|x/x=0) = 1$;

and, for all $xb=0/c|x \in Z$,

(1') if $\rho_Z(q_i)(xb=0/c|x) = 1$ then $f_i(xb=0/c|x) = N_i$;

(2') if $\rho_Z(q_i)(xb=0/c|x) = 2$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/c R_{\mathfrak{p}_{ij}}\right| = f_i(xb=0/c|x)$;

(3') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 3$ then $\left|\oplus_{j=1}^{h_i} I_{ij}/bI_{ij}\right| = f_i({}^{xb=0}/_{c|x})$;
(4') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 4$ then $f_i({}^{xb=0}/_{c|x}) = 1$; and
(5') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 5$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i)({}^{xb=0}/_{c|x})R_{\mathfrak{p}_{ij}}\right| = f_i({}^{xb=0}/_{c|x}) \cdot N_i$.

By definition $\rho_Z(q_i)({}^{d|x}/_{x=0}) = 1$ implies $s_Z(q_i, {}^{d|x}/_{x=0}) \neq 0$. So, if $\rho_Z(q_i)({}^{d|x}/_{x=0}) = 1$ then

$$\begin{aligned}
&\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{d|x}/_{x=0})I_{ij}\right| \\
&\quad= \left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{d|x}/_{x=0})R_{\mathfrak{p}_{ij}}\right| \cdot \left|\oplus_{j=1}^{h_i} s_Z(q_i, {}^{d|x}/_{x=0})R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{d|x}/_{x=0})I_{ij}\right| \\
&\quad= \left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{d|x}/_{x=0})R_{\mathfrak{p}_{ij}}\right| \cdot \left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/I_{ij}\right|.
\end{aligned}$$

So (1) in the first list of conditions may be replaced by

(1*) if $\rho_Z(q_i)({}^{d|x}/_{x=0}) = 1$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{d|x}/_{x=0})R_{\mathfrak{p}_{ij}}\right| = f_i({}^{d|x}/_{x=0}) \cdot N_i^{-1}$.

Since $R_{\mathfrak{p}_{ij}}/I_{ij}$ is finite, by 2.3.11, if $b \neq 0$ then $|I_{ij}/bI_{ij}| = \left|R_{\mathfrak{p}_{ij}}/bR_{\mathfrak{p}_{ij}}\right|$. By definition, $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 3$ implies $b \neq 0$. So (3') in the second list of conditions may be replaced by

(3'*) if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 3$ then $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/bR_{\mathfrak{p}_{ij}}\right| = f_i({}^{xb=0}/_{c|x})$.

Finally, the condition that $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/I_{ij}\right| = N_i$ can be replaced by the condition that there exist $N_i', N_i'' \in \mathbb{N}$ such that $N_i = N_i'N_i''$, $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/r_iR_{\mathfrak{p}_{ij}}\right| = N_i'$ and $\left|\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/J_{ij}\right| = N_i''$. The proof can now be finished using 7.3. □

8. Removing $|{}^{d|x}/_{x=0}| = D$ and $|{}^{xb=0}/_{c|x}| = G$.

This section uses results from sections 6 and 7. We show that there is an algorithm which, given $d \in R\backslash\{0\}$ and $D \in \mathbb{N}$, respectively $b, c \in R\backslash\{0\}$ and $G \in \mathbb{N}$, answers whether there exists a sum of modules $\oplus_{i=1}^h R_{\mathfrak{p}_i}/dI_i$, respectively $\oplus_{i=1}^h I_i/bcR_{\mathfrak{p}_i}$, satisfying a sentence as in the statement of Theorem 5.1 under the assumption that one of the conjuncts is $|{}^{d|x}/_{x=0}| = D$, respectively $|{}^{xb=0}/_{c|x}| = G$. These results are used in section 10 to eliminate expressions of the form $|{}^{d|x}/_{x=0}| = D$ and $|{}^{xb=0}/_{c|x}| = G$, where $D, G \geq 2$.

Proposition 8.1. *Let R be a recursive Prüfer domain such that* EPP(R) *and the radical relation are recursive. There is an algorithm which, given a sentence χ of the form*

$$|{}^{d|x}/_{x=0}| = D \wedge \bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi),$$

where $d \in R\backslash\{0\}$, $D \in \mathbb{N}$, $f : X \to \mathbb{N}$, $g : Y \to \mathbb{N}$ and X, Y are disjoint finite sets of pp-pairs of the form ${}^{xb=0}/_{c|x}$ and ${}^{a|x}/_{x=0}$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_k \lhd R$ and ideals $I_k \lhd R_{\mathfrak{p}_k}$ for $1 \leq k \leq h$ such that $\oplus_{k=1}^h R_{\mathfrak{p}_k}/dI_k \models \chi$.

Proof. Let $Z := X \cup Y$ and let S_Z, ρ_Z and s_Z be as in 6.2.3 with $\lambda := d$. Enumerate $S_Z := \{q_1, \ldots, q_m\}$ and let $q_i := (r_i, r_ia_i, \gamma_i, \delta_i)$. By definition, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$, there exists $1 \leq i \leq m$ such that $(\mathfrak{p}, I) \models q_i$.

Therefore, there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_k \lhd R$ and ideals $I_k \lhd R_{\mathfrak{p}_k}$ for $1 \leq k \leq h$ such that $\oplus_{k=1}^{h} R_{\mathfrak{p}_k}/dI_k \models \chi$ if and only if there exist $(\overline{f}, \overline{g}) \in \Omega_{f,g,m}$ and $D_i \in \mathbb{N}$ with $\prod_{i=1}^{m} D_i = D$ such that for each $1 \leq i \leq m$, there exist $h_i \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_{ij} \lhd R$ and ideals $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq j \leq h_i$ such that

(a)$_i$ $(\mathfrak{p}_{ij}, I_{ij}) \models q_i$ for $1 \leq j \leq h_i$ and
(b)$_i$ $\oplus_{j=1}^{h_j} R_{\mathfrak{p}_{ij}}/dI_{ij}$ satisfies

$$|{}^{d|x}/_{x=0}| = D_i \wedge \bigwedge_{\varphi/\psi \in X_i} |\varphi/\psi| = f_i(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y_i} |\varphi/\psi| \geq g_i(\varphi/\psi).$$

Fix $1 \leq i \leq m$, $D_i \in \mathbb{N}$ and $(\overline{f}, \overline{g}) \in \Omega_{f,g,m}$ as above. Note that

$$\left| {}^{d|x}/_{x=0}(\oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/dI_{ij}) \right| = D_i \text{ if and only if } \left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/I_{ij} \right| = D_i.$$

So, exactly as in 7.6, we may assume $r_i \neq 0$ for $1 \leq i \leq m$. Moreover, by 2.3.9(iv), $I_{ij}^{\#} = \mathfrak{p}_{ij}$. Therefore, $(\mathfrak{p}_{ij}, I_{ij}) \models q_i$ if and only if there exists $J_{ij} \lhd R_{\mathfrak{p}_i}$ such that $I_{ij} = r_i J_{ij}$, $a_i \in J_{ij}$, $\gamma_i \notin \mathfrak{p}_{ij}$ and $\delta_i \notin \mathfrak{p}_{ij}$.

Now, by 6.2.3, $h_i \in \mathbb{N}_0$, $\mathfrak{p}_{ij} \lhd R$ and $J_{ij} \lhd R_{\mathfrak{p}_{ij}}$ with $I_{ij} := r_i J_{ij}$ for $1 \leq j \leq h_i$ satisfy (b)$_i$ if and only if there exists $D_i', D_i'' \in \mathbb{N}$ with $D_i' D_i'' = D_i$ such that

$$\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/J_{ij} \right| = D_i' \text{ and } \left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/r_i R_{\mathfrak{p}_{ij}} \right| = D_i'',$$

for all ${}^{a|x}/_{x=0} \in X_i$ (respectively ${}^{a|x}/_{x=0} \in Y_i$),

(1) if $\rho_Z(q_i)({}^{a|x}/_{x=0}) = 1$ then $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{a|x}/_{x=0}) R_{\mathfrak{p}_{ij}} \right| = f_i({}^{a|x}/_{x=0}) \cdot D_i^{-1}$ (respectively $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{a|x}/_{x=0}) R_{\mathfrak{p}_{ij}} \right| \geq g_i({}^{a|x}/_{x=0}) \cdot D_i^{-1}$),
(2) if $\rho_Z(q_i)({}^{a|x}/_{x=0}) = 2$ then $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{a|x}/_{x=0}) R_{\mathfrak{p}_{ij}} \right| = f_i({}^{a|x}/_{x=0}) \cdot D_i$ (respectively $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{a|x}/_{x=0}) R_{\mathfrak{p}_{ij}} \right| \geq g_i({}^{a|x}/_{x=0}) \cdot D_i$), and,
(3) if $\rho_Z(q_i)({}^{d|x}/_{x=0}) = 3$ then $f_i({}^{a|x}/_{x=0}) = 1$ (respectively $g_i({}^{a|x}/_{x=0}) = 1$).

and for all ${}^{xb=0}/_{c|x} \in X_i$ (respectively ${}^{xb=0}/_{c|x} \in Y_i$),

(1') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 1$ then $f_i({}^{xb=0}/_{c|x}) = D_i$ (respectively $D_i \geq g_i({}^{xb=0}/_{c|x})$)
(2') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 2$ then $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/c R_{\mathfrak{p}_{ij}} \right| = f_i({}^{xb=0}/_{c|x})$ (respectively $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/c R_{\mathfrak{p}_{ij}} \right| \geq g_i({}^{xb=0}/_{c|x})$)
(3') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 3$ then $\left| \oplus_{j=1}^{h_i} I_{ij}/bI_{ij} \right| = f_i({}^{xb=0}/_{c|x})$ (respectively $\left| \oplus_{j=1}^{h_i} I_{ij}/bI_{ij} \right| \geq g_i({}^{xb=0}/_{c|x})$)
(4') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 4$ then $f_i({}^{xb=0}/_{c|x}) = 1$ (respectively $g_i({}^{xb=0}/_{c|x}) = 1$)
(5') if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 5$ then $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{xb=0}/_{c|x}) R_{\mathfrak{p}_{ij}} \right| = f_i({}^{xb=0}/_{c|x}) \cdot D_i$ (respectively $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/s_Z(q_i, {}^{xb=0}/_{c|x}) R_{\mathfrak{p}_{ij}} \right| \geq g_i({}^{xb=0}/_{c|x}) \cdot D_i$).

Exactly as in 7.6, we may replace (3') in the second list of conditions by

(3") if $\rho_Z(q_i)({}^{xb=0}/_{c|x}) = 3$ then $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/b R_{\mathfrak{p}_{ij}} \right| = f_i({}^{xb=0}/_{c|x})$ (respectively $\left| \oplus_{j=1}^{h_i} R_{\mathfrak{p}_{ij}}/b R_{\mathfrak{p}_{ij}} \right| \geq g_i({}^{xb=0}/_{c|x})$).

So we are now done by 7.5. □

Proposition 8.2. *Let R be a recursive Prüfer domain such that* EPP(R) *and the radical relation are recursive. There exists an algorithm which, given a sentence χ of the form*

$$|{}^{xb=0}/_{c|x}| = G \wedge \bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi),$$

where $b, c \in R\backslash\{0\}$, $G \in \mathbb{N}$, $f : X \to \mathbb{N}$, $g : Y \to \mathbb{N}$ and X, Y are disjoint finite sets of pp-pairs of the form ${}^{xb'=0}/_{c'|x}$ and ${}^{d|x}/_{x=0}$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I \lhd R_{\mathfrak{p}_i}$ with $b, c \in I_i$ for $1 \leq i \leq h$ such that $\oplus_{i=1}^h I_i/bcR_{\mathfrak{p}_i} \models \chi$.

Proof. The proof, which we leave to the reader, is very similar to 8.1, except we use 6.2.5 in place of 6.2.3 and the fact that $|{}^{xb=0}/_{c|x}(I/bcR_{\mathfrak{p}})| = |R_{\mathfrak{p}}/I|$ in place of $|{}^{d|x}/_{x=0}(R_{\mathfrak{p}}/dI)| = |R_{\mathfrak{p}}/I|$. □

9. Further syntactic reductions

In this section we continue work of section 5 to improve the form of the conjunction in the statement of 5.1. Some of these reductions use results in sections 7 and 8, which only apply to Prüfer domains (i.e. they do not apply to arbitrary arithmetical rings).

Let W be the set of $\mathcal{L}_R$-sentences of the form

$$(\dagger)\quad |{}^{d|x}/_{x=0}| \,\Box_1 D \wedge |{}^{xb=0}/_{c|x}| \,\Box_2 E \wedge \bigwedge_{\varphi/\psi \in X} |\varphi/\psi| = f(\varphi/\psi) \wedge \bigwedge_{\varphi/\psi \in Y} |\varphi/\psi| \geq g(\varphi/\psi) \wedge \Xi$$

where $\Box_1, \Box_2 \in \{\geq, =, \emptyset\}$, $d, c, b \in R\backslash\{0\}$, $D, E \in \mathbb{N}_2$, $f : X \to \mathbb{N}$, $g : Y \to \mathbb{N}$, X, Y are finite subsets of pp-pairs of the form ${}^{xb'=0}/_{x=0}$ and ${}^{x=x}/_{c'|x}$, and Ξ an auxiliary sentence. Let V be the set of $w \in W$ such that there is an R-module satisfying w. As in 4.2, $\mathbb{W}$ denotes the bounded distributive lattice generated by W and $\mathbb{V}$ denotes the (prime) filter in $\mathbb{W}$ generated by V.

Definition 9.0.1. *Let $w \in W$ be as in ($\dagger$). Define*

$$\begin{aligned}
z_1 &:= \left|\{{}^{xb'=0}/_{x=0} \in X \mid f({}^{xb'=0}/_{x=0}) > 1\}\right|,\\
z_2 &:= \left|\{{}^{xb'=0}/_{x=0} \in Y \mid g({}^{xb'=0}/_{x=0}) > 1\}\right|,\\
z_3 &:= \left|\{{}^{x=x}/_{c'|x} \in X \mid f({}^{x=x}/_{c'|x}) > 1\}\right|, \text{ and}\\
z_4 &:= \left|\{{}^{x=x}/_{c'|x} \in Y \mid g({}^{x=x}/_{c'|x}) > 1\}\right|.
\end{aligned}$$

*The **short signature** is defined to be the tuple $(\Box_1, \Box_2)$ and the **extended signature**,* `exsig` *w, is defined to be the tuple $((\Box_1, \Box_2), (z_1, z_2), (z_3, z_4))$.*

We equip the set $\{\geq, =, \emptyset\}$ with a total order $\succ$ by putting $\geq \succ = \succ \emptyset$. We partially order the set of short signatures $\{\geq, =, \emptyset\}^2$ by setting $(\Box_1, \Box_2) \geq (\Box_1', \Box_2')$ whenever $\Box_1 \succeq \Box_1'$ and $\Box_2 \succeq \Box_2'$. Finally, we partially order on the set of extended signatures by setting

$$((\Box_1, \Box_2), (z_1, z_2), (z_3, z_4)) \geq ((\Box_1', \Box_2'), (z_1', z_2'), (z_3', z_4'))$$

if and only if $(\Box_1, \Box_2) > (\Box_1', \Box_2')$ or $(\Box_1, \Box_2) = (\Box_1', \Box_2')$ and

- $z_1 + z_2 > z_1' + z_2'$ or $z_1 + z_2 = z_1' + z_2'$ and $z_2 \geq z_2'$, and
- $z_3 + z_4 > z_3' + z_4'$ or $z_3 + z_4 = z_3' + z_4'$ and $z_4 \geq z_4'$.

We now present various algorithms which, given $w \in W$ of a particular form, returns $\underline{w} \in \mathbb{W}$ with $\texttt{exsig}\,\underline{w} < \texttt{exsig}\,w$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$. By convention, we give both $\top \in \mathbb{W}$ and $\bot \in \mathbb{W}$ extended signature $((\emptyset, \emptyset), (0, 0), (0, 0))$.

Remark 9.0.2. *The order on the set of extended signatures is artinian.*

The next remark follows directly from 2.1.3.

Remark 9.0.3. *If $w \in W$ has extended signature $((\square_1, \square_2), (z_1, z_2), (z_3, z_4))$ then Dw has extended signature $((\square_1, \square_2), (z_3, z_4), (z_1, z_2))$ where D denotes the duality defined in 2.1.3.*

As in §5, given $w \in W$, we may always assume that w is of the form $\chi_{f,g} \wedge \Xi$ where X, Y are disjoint finite sets of pp-pairs of the form ${d|x}/{x=0}$ or ${xb=0}/{c|x}$, $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$ and Ξ is an auxiliary sentence.

Remark 9.0.4. *Let X, Y be disjoint finite subsets of pp-pairs, $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$ and Ξ an auxiliary sentence such that $\chi_{f,g} \wedge \Xi \in W$. Then, for each $(f_1, \ldots, f_n, g_1, \ldots, g_n) \in \Omega_{f,g,n}$ and $1 \le i \le n$, the sentence $\chi_{f_i,g_i} \wedge \Xi$ is in W and $\texttt{exsig}\,\chi_{f_i,g_i} \wedge \Xi \le \texttt{exsig}\,\chi_{f,g} \wedge \Xi$. Moreover, for each $(\overline{f}, \overline{g}) \in \Omega_{f,g,n}$ and $1 \le i \le n$, either $X = X_i$ and $Y = Y_i$, or, $\texttt{exsig}\,\chi_{f_i,g_i} \wedge \Xi < \texttt{exsig}\,\chi_{f,g} \wedge \Xi$.*

Proof. Fix $(\overline{f}, \overline{g}) \in \Omega_{f,g,n}$ and $1 \le i \le n$. Let $((\square_1, \square_2), (z_1, z_2), (z_3, z_4))$ be the extended signature of $\chi_{f,g} \wedge \Xi$ and $((\square_1', \square_2'), (z_1', z_2'), (z_3', z_4'))$ the extended signature of $\chi_{f_i,g_i} \wedge \Xi$. Note that since X and Y are disjoint, so are $X_i = X \cup (Y \backslash Y_i)$ and Y_i.

That the short signature of χ_{f_i,g_i} is less than or equal to the short signature of $\chi_{f,g}$ follows from the fact that $Y_i \subseteq Y$ and $X_i = X \cup (Y \backslash Y_i)$.

We show that $z_1 + z_2 \ge z_1' + z_2'$ and $z_2 \ge z_2'$. By definition, $z_1 + z_2$ is the number of ${x=x}/{c|x} \in X \cup Y$ and z_2 is the number of ${x=x}/{c|x} \in Y$. Since $g_i({x=x}/{c|x}) = g({x=x}/{c|x}) > 1$ for all ${x=x}/{c|x} \in Y_i$, we see that z_2' is the number of ${x=x}/{c|x} \in Y_i$. So $z_2' \le z_2$ because $Y_i \subseteq Y$. Since $X_i = X \cup (Y \backslash Y_i)$ and Y_i are disjoint, $z_1' + z_2'$ is equal to the number of ${x=x}/{c|x} \in X_i \cup Y_i = X \cup Y$ with either ${x=x}/{c|x} \in Y_i$, or, ${x=x}/{c|x} \in X_i$ and $f_i({x=x}/{c|x}) > 1$. So $z_1 + z_2 \ge z_1' + z_2'$, since $X \cup Y = X_i \cup Y_i$. A similar argument shows that $z_3 + z_4 \ge z_3' + z_4'$ and $z_4 \ge z_4'$. Therefore the extended signature of $\chi_{f,g}$ is less than or equal to the extended signature of χ_{f_i,g_i}.

We now prove the moreover. Since X, Y are disjoint, $X \ne X_i$ if and only if $Y \ne Y_i$. Suppose $X \ne X_i$. Then there exists ${\varphi}/{\psi} \in Y \backslash Y_i$. By assumption, $g({\varphi}/{\psi}) > 1$.

If ${\varphi}/{\psi}$ is ${d|x}/{x=0}$ then the short signature of $\chi_{f,g}$ is $(\ge, \square)$ and the short signature of χ_{f_i,g_i} is $(=, \square')$ or $(\emptyset, \square')$, and, by what we have already proved, $\square' \le \square$. So the short signature of χ_{f_i,g_i} is strictly less than the short signature of $\chi_{f,g}$. The case of ${xb=0}/{c|x}$ for $b, c \ne 0$ is similar.

If ${\varphi}/{\psi}$ is ${x=x}/{c|x}$ then

$$\left|\{{x=x}/{c'|x} \in Y_i \mid g_i({x=x}/{c'|x}) > 1\}\right| < \left|\{{x=x}/{c'|x} \in Y\}\right|.$$

So $\texttt{exsig}\,\chi_{f_i,g_i} < \texttt{exsig}\,\chi_{f,g}$. The case of ${\varphi}/{\psi}$ equal to ${xb=0}/{x=0}$ is similar. □

9.1. **Reducing the short signature.**

Proposition 9.1.1. *Let R be a recursive Prüfer domain with* $\mathrm{EPP}(R)$ *recursive. There is an algorithm which given $w \in W$ with short signature $(=, \square)$ or $(\square, =)$, for some $\square \in \{\emptyset, =, \ge\}$, outputs $\underline{w} \in \mathbb{W}$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$, and, $\underline{w}$ is a lattice combination of elements $w' \in W$ such that the short signature of w' is strictly less than the short signature of w.*

Proof. Suppose w has short signature $(=,\square)$. Then w is of the form

$$|{}^{d|x}/_{x=0}| = D \wedge \chi_{f,g} \wedge \Xi$$

where $d \in R\backslash\{0\}$, $D \in \mathbb{N}_2$, $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$, X and Y are disjoint finite subsets of $\{{}^{x=x}/_{c'|x}, {}^{xb'=0}/_{x=0} \mid c', b' \in R\} \cup \{{}^{xb=0}/_{c|x}\}$ for some $b, c \in R\backslash\{0\}$, and Ξ is an auxiliary sentence.

If $M \models w$ then, by 2.3.6 and 6.0.1, there exist $h \in \mathbb{N}$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ and $M' \in \text{Mod-}R$ such that $M \equiv \oplus_{i=1}^h R_{\mathfrak{p}_i}/dI_i \oplus M'$ and $|{}^{d|x}/_{x=0}(M')| = 1$.

Thus there exists $M \in \text{Mod-}R$ such that $M \models w$ if and only if there exist $(f_1, f_2, g_1, g_2) \in \Omega_{f,g,2}$, $h \in \mathbb{N}$, prime ideals $\mathfrak{p}_i \lhd R$ and $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$, such that $\oplus_{i=1}^h R_{\mathfrak{p}_i}/dI_i$ satisfies

$$w_{(f_1,f_2,g_1,g_2)} := |{}^{d|x}/_{x=0}| = D \wedge \chi_{f_1,g_1} \wedge \Xi$$

and M' which satisfies

$$w'_{(f_1,f_2,g_1,g_2)} := |{}^{d|x}/_{x=0}| = 1 \wedge \chi_{f_2,g_2} \wedge \Xi.$$

Let $\Omega \subseteq \Omega_{f,g,2}$ be the set of $(f_1, f_2, g_1, g_2) \in \Omega_{f,g,2}$, such that there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$, such that $\oplus_{i=1}^h R_{\mathfrak{p}_i}/dI_i$ satisfies $w_{(f_1,f_2,g_1,g_2)}$. So $w \in V$ if and only if

$$\underline{w} := \bigsqcup_{(f_1,f_2,g_1,g_2)\in\Omega} w'_{(f_1,f_2,g_1,g_2)} \in \mathbb{V}.$$

By 8.1, given w, we can compute Ω, and, so, we can compute $\underline{w}$. The short signature of each $w'_{(f_1,f_2,g_1,g_2)}$ is $(\emptyset,\square')$ for some $\square' \in \{\emptyset, =, \leq\}$ and, by 9.0.4, $\square \succeq \square'$. So $(=,\square) > (\emptyset,\square')$ as required.

Suppose w has short signature $(\square,=)$. Then w is of the form

$$|{}^{xb=0}/_{c|x}| = E \wedge \chi_{f,g} \wedge \Xi$$

where $b, c \in R\backslash\{0\}$, $E \in \mathbb{N}_2$, $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$, X and Y are disjoint finite subsets of $\{{}^{x=x}/_{c'|x}, {}^{xb'=0}/_{x=0} \mid c', b' \in R\} \cup \{{}^{d|x}/_{x=0}\}$ for some $d \in R\backslash\{0\}$, and Ξ is an auxiliary sentence.

If $M \models w$ then, by 2.3.6 and 6.0.2, there exist $M_1, M_2, M' \in \text{Mod-}R$ such that $b \in \text{ann}_R M_1$, $c \in \text{ann}_R M_2$ and $|{}^{xb=0}/_{c|x}(M')| = 1$ and there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $b, c \in I_i$ and

$$\oplus_{i=1}^h I_i/bcR_{\mathfrak{p}_i} \oplus M_1 \oplus M_2 \oplus M' \models w.$$

Therefore, $w \in V$ if and only if there exist $E_1, E_2, E_4 \in \mathbb{N}$ with $E_1E_2E_4 = E$ and $(\overline{f}, \overline{g}) \in \Omega_{f,g,4}$ such that

- w_{1,E_1,f_1,g_1}, defined as $|{}^{x=x}/_{c|x}| = E_1 \wedge \chi_{f_1,g_1} \wedge \Xi \wedge |{}^{b|x}/_{x=0}| = 1$, is in V,
- w_{2,E_2,f_2,g_2}, defined as $|{}^{xb=0}/_{x=0}| = E_2 \wedge \chi_{f_2,g_2} \wedge \Xi \wedge |{}^{c|x}/_{x=0}| = 1$, is in V,
- w_{3,f_3,g_3}, defined as $\chi_{f_3,g_3} \wedge \Xi \wedge |{}^{xb=0}/_{c|x}| = 1$, is in V, and
- there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $b, c \in I_i$ and

$$\oplus_{i=1}^h I_i/bcR_{\mathfrak{p}_i} \models |{}^{xb=0}/_{c|x}| = E_4 \wedge \chi_{f_4,g_4} \wedge \Xi.$$

Let $\mathcal{H}$ be the set of $(E_1, E_2, (\overline{f}, \overline{g})) \in \mathbb{N}^2 \times \Omega_{f,g,4}$ such that there exists E_4 with $E_1E_2E_4 = E$ and there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $b, c \in I_i$ and

$$\oplus_{i=1}^{h} I_i/bcR_{\mathfrak{p}_i} \models |xb=0/c|x| = E_4 \wedge \chi_{f_4,g_4} \wedge \Xi.$$

Then $w \in V$ if and only if

$$\underline{w} := \bigsqcup_{(E_1,E_2,(\overline{f},\overline{g}))\in\mathcal{H}} w_{1,E_1,f_1,g_1} \sqcap w_{2,E_2,f_3,g_3} \sqcap w_{3,f_3,g_3} \in \mathbb{V}.$$

By 8.2, given w, we can compute $\mathcal{H}$ and so we can compute $\underline{w}$.

Now if w has short signature $(\square, =)$ then $w_{1,E_1,f_1,g_1}, w_{2,E_2,f_3,g_3}$ and w_{3,f_3,g_3} have short signature $(\square', \emptyset)$ and by 9.0.4, $\square' \preceq \square$. So $(\square', \emptyset) < (\square, =)$, as required. $\square$

Lemma 9.1.2. *Let R be an arithmetical ring. Let $c, d \in R$ and $D \in \mathbb{N}$. For all $C \in \mathbb{N}_2$,*

$$T_R \models \left|xc^{D-1}d=0/c|x\right| = 1 \wedge |d|x/x=0| \geq D \wedge |x=x/c|x| = C$$
$$\leftrightarrow \left|xc^{D-1}d=0/c|x\right| = 1 \wedge |x=x/c|x| = C$$

and

$$T_R \models \left|c^D d|x/x=0\right| = 1 \wedge |d|x/x=0| \geq D \wedge |x=x/c|x| = C$$
$$\leftrightarrow \bigvee_{D\leq E\leq C^D} \left|c^D d|x/x=0\right| = 1 \wedge |d|x/x=0| = E \wedge |x=x/c|x| = C.$$

Proof. Suppose $M \models \left|xc^{D-1}d=0/c|x\right| = 1$. Then $xc^i d = 0 \leq_M c|x$ for $0 \leq i \leq D-1$. So, by 5.7, for $0 \leq i \leq D-1$,

$$\left|c^i d|x/c^{i+1}d|x(M)\right| = |x=x/c|x(M)|.$$

Hence

$$|d|x/x=0(M)| = |x=x/c|x(M)|^D \left|c^D d|x/x=0(M)\right|.$$

Therefore, if $C \geq 2$ then

$$T_R \models \left|xc^{D-1}d=0/c|x\right| = 1 \wedge |d|x/x=0| \geq D \wedge |x=x/c|x| = C \leftrightarrow$$
$$\left|xc^{D-1}d=0/c|x\right| = 1 \wedge |x=x/c|x| = C.$$

Now suppose that $c^D d \in \text{ann}_R M$. Then $c^D d|x$ is equivalent to $x = 0$ in M. Therefore, by 5.7,

$$|d|x/x=0(M)| = |d|x/c^D d|x(M)| = |x=x/c^D|x+xd=0(M)| \leq |x=x/c^D|x(M)| \leq |x=x/c|x|^D.$$

Hence

$$T_R \models \left|c^D d|x/x=0\right| = 1 \wedge |d|x/x=0| \geq D \wedge |x=x/c|x| = C \leftrightarrow$$
$$\bigvee_{D\leq E\leq C^D} \left|c^D d|x/x=0\right| = 1 \wedge |d|x/x=0| = E \wedge |x=x/c|x| = C. \quad \square$$

Proposition 9.1.3. *Let R be a recursive arithmetical ring. There is an algorithm, which given $w \in W$ with extended signature $((\geq, \square), (z_1, z_2), (z_3, z_4))$ with $z_1 \geq 1$ or $z_3 \geq 1$, returns $\underline{w} \in \mathbb{W}$ with $\texttt{exsig}\,\underline{w} < \texttt{exsig}\,w$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$.*

Proof. Suppose $w := \chi_{f,g} \wedge \Xi \in W$, where $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$, X, Y are disjoint finite sets of appropriate pp-pairs and Ξ is an auxiliary sentence, has extended signature $((\geq, \Box), (z_1, z_2), (z_3, z_4))$ with $z_3 \geq 1$. Then ${d|x}/{x=0} \in Y$ for some $d \in R\backslash\{0\}$ because the short signature is $(\geq, \Box)$ and ${x=x}/{c|x} \in X$ for some $c \in R$ because $z_3 \geq 1$. Let $D = g({x=x}/{c|x})$.

Then, by 2.3.4, $w \in V$ if and only if

$$\bigsqcup_{(\overline{f},\overline{g})\in\Omega_{f,g,2}} \left(\chi_{f_1,g_1} \wedge \left|{xc^{D-1}d=0}/{c|x}\right| = 1 \wedge \Xi\right) \sqcap \left(\chi_{f_2,g_2} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi\right) \in \mathbb{V}.$$

For each $(\overline{f}, \overline{g}) \in \Omega_{f,g,2}$, we define $\underline{w}_{\overline{f},\overline{g}}, \underline{w}'_{\overline{f},\overline{g}} \in \mathbb{W}$ such that

$$\texttt{exsig}\, \underline{w}_{\overline{f},\overline{g}}, \texttt{exsig}\, \underline{w}'_{\overline{f},\overline{g}} < \texttt{exsig}\, w,$$

$$\underline{w}_{\overline{f},\overline{g}} \in \mathbb{V} \text{ if and only if } \chi_{f_1,g_1} \wedge \left|{xc^{D-1}d=0}/{c|x}\right| = 1 \wedge \Xi \in V, \text{ and,}$$

$$\underline{w}'_{\overline{f},\overline{g}} \in \mathbb{V} \text{ if and only if } \chi_{f_2,g_2} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi \in V.$$

Once we have done this, the proof is complete since then

$$\underline{w} := \bigsqcup_{(\overline{f},\overline{g})\in\Omega_{f,g,2}} \underline{w}_{\overline{f},\overline{g}} \sqcap \underline{w}'_{\overline{f},\overline{g}}$$

has the properties required by the statement.

If $\texttt{exsig}\, \chi_{f_1,g_1} < \texttt{exsig}\, \chi_{f,g}$ then let $\underline{w}_{\overline{f},\overline{g}}$ be $\chi_{f_1,g_1} \wedge \left|{xc^{D-1}d=0}/{c|x}\right| = 1 \wedge \Xi$. Otherwise, by 9.0.4, $X = X_1$ and $Y = Y_1$. Further, $f_1({x=x}/{c|x}) > 1$ and, by definition of $\Omega_{f,g,2}$, $g_1({d|x}/{x=0}) = g({d|x}/{x=0}) = D$. Let $Y_1' := Y_1\backslash\{{d|x}/{x=0}\}$ and $g_1' := g_1|_{Y_1'}$. Then, by 9.1.2,

$$\chi_{f_1,g_1} \wedge \left|{xc^{D-1}d=0}/{c|x}\right| = 1 \wedge \Xi \in V$$

if and only if $\underline{w}_{\overline{f},\overline{g}}$ defined as

$$\chi_{f_1,g_1'} \wedge \left|{xc^{D-1}d=0}/{c|x}\right| = 1 \wedge \Xi \text{ is in } \mathbb{V}.$$

Moreover, $\underline{w}_{\overline{f},\overline{g}}$ has short signature $(\emptyset, \Box')$, where, by 9.0.4, $\Box' \preceq \Box$. Therefore $\texttt{exsig}\, \underline{w}_{\overline{f},\overline{g}} < \texttt{exsig}\, w$ because $(\emptyset, \Box') < (\geq, \Box)$.

If $\texttt{exsig}\, \chi_{f_2,g_2} < \texttt{exsig}\, \chi_{f,g}$ then let $\underline{w}'_{\overline{f},\overline{g}}$ be $\chi_{f_2,g_2} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi$. Otherwise, by 9.0.4, $X = X_1$ and $Y = Y_1$. Further, $f_2({x=x}/{c|x}) > 1$ and, by definition of $\Omega_{f,g,2}$, $g_2({d|x}/{x=0}) = g({d|x}/{x=0}) = D$.

Let $X_2' := X_2\backslash\{{x=x}/{c|x}\}$ and $Y_2' := Y_2\backslash\{{d|x}/{x=0}\}$. Let $f_2' := f_2|_{X_2'}$, $g_2' := g_2|_{Y_2'}$ and $C := f_2({x=x}/{c|x})$. Then $\chi_{f_2,g_2} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi$ is

$$\left|{d|x}/{x=0}\right| \geq D \wedge \left|{x=x}/{c|x}\right| = C \wedge \chi_{f_2',g_2'} \wedge \Xi \wedge \left|{c^D d|x}/{x=0}\right| = 1.$$

Let

$$\underline{w}'_{\overline{f},\overline{g}} := \bigsqcup_{D\leq E\leq C^D} \left|{d|x}/{x=0}\right| = E \wedge \left|{x=x}/{c|x}\right| = C \wedge \chi_{f_2',g_2'} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi.$$

By 9.1.2, $\underline{w}'_{\overline{f},\overline{g}} \in \mathbb{V}$ if and only if $\chi_{f_2,g_2} \wedge \left|{c^D d|x}/{x=0}\right| = 1 \wedge \Xi \in V$. The short signature of each component of the join defining $\underline{w}'_{\overline{f},\overline{g}}$ is $(=, \Box')$ where, by 9.0.4, $\Box' \preceq \Box$. Therefore $\texttt{exsig}\, \underline{w}'_{\overline{f},\overline{g}} < \texttt{exsig}\, w$ because $(=, \Box') < (\geq, \Box)$.

Suppose $w \in W$ has extended signature $((\geq, \Box), (z_1, z_2), (z_3, z_4))$ with $z_1 \geq 1$. Then Dw has extended signature $((\geq, \Box), (z_3, z_4), (z_1, z_2))$. By the previous case, we can compute $\underline{w}$, a lattice combination of elements of W with extended signatures strictly less than Dw such that $Dw \in V$ if and only if $\underline{w} \in \mathbb{V}$. Now $w \in V$ if and only if $D\underline{w} \in \mathbb{V}$ and $\texttt{exsig}\, D\underline{w} < \texttt{exsig}\, w$. $\square$

Lemma 9.1.4. *Let R be a Prüfer domain. Let $a, b, c \in R$ and $E, C \in \mathbb{N}_2$. Suppose that $r, s, \alpha \in R$ are such that $c\alpha = ar$ and $a(\alpha - 1) = cs$. Define*

(1) Σ_1 to be the formula $|x{=}x/\alpha{-}1|x| = 1$;
(2) Σ_2 to be the formula $|x{=}x/\alpha|x| = 1 \wedge |ar|x/x{=}0| = 1$;
(3) Σ_3 to be the formula $|x{=}x/\alpha|x| = 1 \wedge |xr{=}0/a|x| = 1 \wedge |rb|x/x{=}0| = 1$; and
(4) Σ_4 to be the formula $|x{=}x/\alpha|x| = 1 \wedge |xr{=}0/a|x| = 1 \wedge |xb{=}0/r|x| = 1$.

Then for all $M \in \text{Mod-}R$ there exist $M_1, \ldots, M_4 \in \text{Mod-}R$ such that $M_i \models \Sigma_i$ for $1 \leq i \leq 4$ and $M \equiv M_1 \oplus \ldots \oplus M_4$.

(i) If $i \in \{1, 4\}$ and $C < E$ then

$$T_R \models \neg(\Sigma_i \wedge |x{=}x/a|x| = C \wedge |xb{=}0/c|x| \geq E),$$

and if $i \in \{1, 4\}$ and $E \leq C$ then

$$T_R \models \Sigma_i \wedge |x{=}x/a|x| = C \wedge |xb{=}0/c|x| \geq E \leftrightarrow \bigvee_{E \leq E' \leq C} \Sigma_i \wedge |x{=}x/a|x| = C \wedge |xb{=}0/c|x| = E'.$$

(ii) In T_R the following equivalence holds.

$$\Sigma_2 \wedge |x{=}x/a|x| = C \wedge |xb{=}0/c|x| \geq E \leftrightarrow \Sigma_2 \wedge |x{=}x/a|x| = C \wedge |xb{=}0/x{=}0| \geq E.$$

(iii) In T_R the following equivalence holds.

$$\Sigma_3 \wedge |x{=}x/a|x| = C \wedge |xb{=}0/c|x| \geq E \leftrightarrow \Sigma_3 \wedge |x{=}x/a|x| = C \wedge |xb{=}0/r|x| \geq \lceil E/C \rceil.$$

Proof. The first claim follows from 2.3.4.
(i) Suppose $M \models \Sigma_1$. Then

$$|x{=}x/a|x(M)| = |x{=}x/cs|x(M)| \geq |xb{=}0/c|x(M)|.$$

Suppose $M \models \Sigma_4$. Then

$$|r|x/ar|x(M)| = |x{=}x/xr{=}0{+}a|x(M)| = |x{=}x/a|x(M)|$$

because $|xr{=}0/a|x(M)| = 1$ and hence $xr = 0 \leq_M a|x$. So

$$|xb{=}0/c|x(M)| = |xb{=}0/ar|x(M)| \leq |r|x/ar|x(M)| = |x{=}x/a|x(M)|.$$

The first equality holds because $|x{=}x/\alpha|x(M)| = 1$ and the inequality holds because $|xb{=}0/r|x| = 1$ and hence $xb = 0 \leq_M r|x$.

Therefore, if $M \models \Sigma_i$ for $i \in \{1, 4\}$ then $|xb{=}0/c|x(M)| \leq |x{=}x/a|x(M)|$. The first claim follows from this.
(ii) Suppose $M \models \Sigma_2$. Then $c|x$ is equivalent to $ar|x$ in M and $ar|x$ is equivalent to $x = 0$ in M. So $|xb{=}0/c|x(M)| = |xb{=}0/x{=}0(M)|$.
(iii) Suppose $M \models \Sigma_3$. Then

$$|xb{=}0/c|x(M)| = |xb{=}0/ar|x(M)| = |xb{=}0/r|x(M)| \cdot |r|x/ar|x(M)|.$$

Since $|xr{=}0/a|x(M)| = 1$, by 5.7, $|r|x/ar|x(M)| = |x{=}x/a|x(M)|$. The claim now follows. $\square$

Proposition 9.1.5. *Let R be a recursive Prüfer domain. There is algorithm which given $w \in W$ with extended signature $((\square, \geq), (z_1, z_2), (z_3, z_4))$ with $z_1 \geq 1$ or $z_3 \geq 1$ outputs $\underline{w} \in \mathbb{W}$ with $\texttt{exsig}\, \underline{w} < \texttt{exsig}\, w$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$.*

Proof. For $w \in W$ as in (†) (see the beginning of this section), define $\deg w = E$. Given $w \in W$ with extended signature $((\square, \geq), (z_1, z_2), (z_3, z_4))$ with $z_3 \geq 1$, we will show how to compute $\underline{w} \in \mathbb{W}$, a lattice combination of $w' \in W$, such that for each w', either $\mathtt{exsig}\, w' < \mathtt{exsig}\, w$ or $\mathtt{exsig}\, w' = \mathtt{exsig}\, w$ and $\deg w' < \deg w$, and, such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$. Since deg takes values in $\mathbb{N}$, by iterating this process we will eventually compute $\underline{w} \in \mathbb{W}$ such that $\mathtt{exsig}\, \underline{w} < \mathtt{exsig}\, w$.

Suppose w is $\chi_{f,g} \wedge \Xi \in W$, where $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$, X, Y are disjoint finite sets of appropriate pp-pairs and Ξ is an auxiliary sentence, has extended signature $((\square, \geq), (z_1, z_2), (z_3, z_4))$ with $z_3 \geq 1$. Since w has short signature $(\square, \geq)$, there exists ${}^{xb=0}/_{c|x} \in Y$. Since w has extended signature $((\square, \geq), (z_1, z_2), (z_3, z_4))$ with $z_3 \geq 1$, there exists ${}^{x=x}/_{a|x} \in X$.

Let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be as in 9.1.4. Then $w \in V$ if and only if

$$\bigsqcup_{(\overline{f},\overline{g}) \in \Omega_{f,g,4}} \prod_{i=1}^{4} \chi_{f_i,g_i} \wedge \Xi \wedge \Sigma_i \in \mathbb{V}.$$

We show that for each $(\overline{f}, \overline{g}) \in \Omega_{f,g,4}$, either the extended signature of $\chi_{f_i,g_i} \wedge \Xi$ is strictly less than the extended signature of $\chi_{f,g} \wedge \Xi$ or we will show that we can compute $\underline{w}_{i,(\overline{f},\overline{g})} \in \mathbb{W}$ such that $\underline{w}_{i,(\overline{f},\overline{g})}$ is a lattice combination of $w' \in W$ with either $\mathtt{exsig}\, w' < \mathtt{exsig}\, w$, or, $\mathtt{exsig}\, w' \leq \mathtt{exsig}\, w$ and $\deg w' < w$.

Fix $(\overline{f}, \overline{g}) \in \Omega_{f,g,4}$ and $1 \leq i \leq 4$. By 9.0.4, $\mathtt{exsig}\, \chi_{f_i,g_i} \wedge \Xi < \mathtt{exsig}\, \chi_{f,g} \wedge \Xi$ unless $X_i = X$ and $Y_i = Y$. Moreover, if $f_i({}^{x=x}/_{a|x}) = 1$ then $\mathtt{exsig}\, \chi_{f_i,g_i} \wedge \Xi < \mathtt{exsig}\, \chi_{f,g} \wedge \Xi$. So we may assume $X_i = X$, $Y_i = Y$ and $f_i({}^{x=x}/_{a|x}) > 1$.

Let $X' := X \backslash \{{}^{x=x}/_{a|x}\}$, $Y' := Y \backslash \{{}^{xb=0}/_{c|X}\}$, $f'_i := f_i|_{X'}$, $g'_i := g_i|_{Y'}$, $C := f_i({}^{x=x}/_{a|x})$ and $E := g_i({}^{xb=0}/_{c|x}) = g({}^{xb=0}/_{c|x})$. Then $\chi_{f_i,g_i} \wedge \Xi$ is

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{c|x}| \geq E \wedge \chi_{f'_i,g'_i} \wedge \Xi.$$

Case $i = 1$ **or** $i = 4$**:** If $f_i({}^{x=x}/_{a|x}) < g_i({}^{xb=0}/_{c|x})$ then, by 9.1.4,

$$T_R \models \neg(\chi_{f_i,g_i} \wedge \Xi \wedge \Sigma_i).$$

So set $\underline{w}_{i,(\overline{f},\overline{g})} := \bot$. Now suppose $f_i({}^{x=x}/_{a|x}) \geq g_i({}^{xb=0}/_{c|x})$. Then, by 9.1.4,

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{c|x}| \geq E \wedge \chi_{f'_i,g'_i} \wedge \Xi \;\in\; V$$

if and only if

$$\underline{w}_{i,(\overline{f},\overline{g})} := \bigsqcup_{E \leq E' \leq C} |{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{c|x}| = E' \wedge \chi_{f'_i,g'_i} \wedge \Xi \wedge \Sigma_i \;\in\; \mathbb{V}.$$

So we are done since the short signatures of the components of the join defining $\underline{w}_{i,(\overline{f},\overline{g})}$ are $(\square', =)$ where, by 9.0.4, $\square' \preceq \square$.

Case $i = 2$**:** By 9.1.4,

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{c|x}| \geq E \wedge \chi_{f'_2,g'_2} \wedge \Xi \wedge \Sigma_2 \;\in\; V$$

if and only if $\underline{w}_{i,(\overline{f},\overline{g})}$, defined as

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{x=0}| \geq E \wedge \chi_{f'_2,g'_2} \wedge \Xi \wedge \Sigma_2, \text{ is in } V.$$

The short signature of $\underline{w}_{i,(\overline{f},\overline{g})}$ is $(\square', \emptyset)$ where, by 9.0.4, $\square' \preceq \square$.

Case $i = 3$**:** Let $r, s, \alpha \in R$ be such that $c\alpha = ar$ and $a(\alpha - 1) = cs$. By 9.1.4,

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{c|x}| \geq E \wedge \chi_{f'_2,g'_2} \wedge \Xi \wedge \Sigma_3 \;\in\; V$$

if and only if $\underline{w}_{i,(\overline{f},\overline{g})}$, defined as

$$|{}^{x=x}/_{a|x}| = C \wedge |{}^{xb=0}/_{r|x}| \geq \lceil E/C \rceil \wedge \chi_{f_2',g_2'} \wedge \Xi \wedge \Sigma_3, \text{ is in } V.$$

Since $\lceil E/C \rceil < E$, we have $\deg \underline{w}_{i,(\overline{f},\overline{g})} < E = \deg \chi_{f,g} \wedge \Xi$.

So we have proved the lemma for the case $z_3 \geq 1$. Suppose $w \in W$ has extended signature $((\Box, \geq), (z_1, z_2), (z_3, z_4))$ with $z_1 \geq 1$. Then Dw has extended signature $((\Box, \geq), (z_3, z_4), (z_1, z_2))$. So by the version of the lemma just proved, we can compute $\underline{w} \in W$ with $\texttt{exsig}\, \underline{w} < \texttt{exsig}\, Dw$ such that $Dw \in V$ if and only if $\underline{w} \in \mathbb{V}$. Now, $w \in V$ if and only if $D\underline{w} \in \mathbb{V}$ and $\texttt{exsig}\, D\underline{w} < \texttt{exsig}\, w$, as required. $\Box$

9.2. **Reducing the extended signature.**

Lemma 9.2.1. *Let R be an arithmetical ring. Let $a, b \in R$ and $A, B \in \mathbb{N}_2$. Suppose that $r, s, \alpha \in R$ are such that $a\alpha = br$ and $b(\alpha - 1) = as$. Define*

(1) Σ_1 to be the sentence $|{}^{x=x}/_{\alpha|x}| = 1 \wedge |{}^{br|x}/_{x=0}| = 1 \wedge |{}^{b^A|x}/_{x=0}| = 1$,
(2) Σ_2 to be the sentence $|{}^{x=x}/_{\alpha|x}| = 1 \wedge |{}^{br|x}/_{x=0}| = 1 \wedge |{}^{xb^{A-1}=0}/_{b|x}| = 1$,
(3) Σ_3 to be the sentence $|{}^{x=x}/_{\alpha|x}| = 1 \wedge |{}^{xb=0}/_{r|x}| = 1$,
(4) Σ_4 to be the sentence $|{}^{x=x}/_{(\alpha-1)|x}| = 1 \wedge |{}^{as|x}/_{x=0}| = 1 \wedge |{}^{a^B|x}/_{x=0}| = 1$,
(5) Σ_5 to be the sentence $|{}^{x=x}/_{(\alpha-1)|x}| = 1 \wedge |{}^{as|x}/_{x=0}| = 1 \wedge |{}^{xa^{B-1}=0}/_{a|x}| = 1$, and
(6) Σ_6 to be the sentence $|{}^{x=x}/_{(\alpha-1)|x}| = 1 \wedge |{}^{xa=0}/_{s|x}| = 1$.

Then, for all $M \in \text{Mod-}R$, there exist $M_i \in \text{Mod-}R$ for $1 \leq i \leq 6$ such that $M \equiv \oplus_{i=1}^6 M_i$ and for $1 \leq i \leq 6$, $M_i \models \Sigma_i$.

Moreover, there is an algorithm which, given a, b, r, s, α, A, B as above, $1 \leq i \leq 6$ and $\Box, \Box' \in \{=, \geq\}$, either returns $\perp$, in which case,

$$T_R \models \neg(|{}^{x=x}/_{a|x}| \,\Box A \wedge |{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma_i)$$

or returns $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_n$ such that

$$T_R \models |{}^{x=x}/_{a|x}| \,\Box A \wedge |{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma_i \leftrightarrow \bigvee_{j=1}^{n} (\sigma_j \wedge \Sigma_i)$$

and each σ_j is either of the form

$$|{}^{x=x}/_{a'|x}| \,\Box_j A' \wedge |{}^{x=x}/_{b'|x}| \,\Box_j' B',$$

where $\Box_j, \Box_j' \in \{\emptyset, =, \geq\}$, $\Box_j \preceq \Box$, $\Box_j' \preceq \Box'$, $a', b' \in R$, $A', B' \in \mathbb{N}$ and $A'B' < AB$ or of the form

$$|{}^{x=x}/_{a'|x}| \,\Box_j A' \wedge |{}^{x=x}/_{b'|x}| \,\Box_j' B' \wedge |{}^{x=x}/_{x=0}| = N,$$

where $\Box_j, \Box_j' \in \{=, \geq\}$, $a', b' \in R$ and $A'', B'', N \in \mathbb{N}$.

Proof. Note that

$$T_R \models |{}^{x=x}/_{\alpha|x}| = 1 \wedge |{}^{br|x}/_{x=0}| = 1 \rightarrow (a|x \leftrightarrow x = 0).$$

Case 1: $\Sigma \in \{\Sigma_1, \Sigma_2\}$ and $\Box$ is $=$.
In this case, $|{}^{x=x}/_{a|x}| \,\Box A \wedge |{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma$ is equivalent to $|{}^{x=x}/_{x=0}| = A \wedge |{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma$ as required.
Case 2: $\Sigma \in \{\Sigma_1, \Sigma_2\}$ and $\Box$ is $\geq$ and $B \geq A$.
In this case $|{}^{x=x}/_{a|x}| \,\Box A \wedge |{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma$ is equivalent to $|{}^{x=x}/_{b|x}| \,\Box' B \wedge \Sigma$.
Case 3: $\Sigma = \Sigma_1$, $\Box$ is $\geq$ and $\Box'$ is $=$.

If $b^A \in \operatorname{ann}_R M$ and $\left|{}^{x=x}/_{b|x}(M)\right| = B$ then $B \leq |M| \leq B^A$. Thus $\left|{}^{x=x}/_{a|x}\right| \geq A \wedge \left|{}^{x=x}/_{b|x}\right| = B \wedge \Sigma_1$ is equivalent to

$$\bigvee_{B^A \geq A' \geq A} \left|{}^{x=x}/_{x=0}\right| = A' \wedge \left|{}^{x=x}/_{b|x}\right| = B \wedge \Sigma_1.$$

Case 4: $\Sigma = \Sigma_2$, $\square$ is $\geq$ and $\square'$ is $=$.
Suppose M satisfies $\left|{}^{xb^{A-1}=0}/_{b|x}\right| = 1$. Then $\left|{}^{b^i|x}/_{b^{i+1}|x}(M)\right| = \left|{}^{x=x}/_{b|x}(M)\right|$ for $i \geq A-1$. So $\left|{}^{x=x}/_{b^A|x}(M)\right| = \left|{}^{x=x}/_{b|x}(M)\right|^A$. Thus if $\left|{}^{x=x}/_{b|x}(M)\right| \geq 2$ then $|M| \geq A$. Therefore $\left|{}^{x=x}/_{a|x}\right| \geq A \wedge \left|{}^{x=x}/_{b|x}\right| = B \wedge \Sigma_2$ is equivalent to $\left|{}^{x=x}/_{b|x}\right| = B \wedge \Sigma_2$.
Case 5: $\Sigma \in \{\Sigma_1, \Sigma_2\}$ and $\square$ is $\geq$ and $B < A$.
If $\square'$ is $\geq$ then

$$\left|{}^{x=x}/_{a|x}\right| \geq A \wedge \left|{}^{x=x}/_{b|x}\right| \geq B \wedge \Sigma$$

is equivalent to

$$\left(\left|{}^{x=x}/_{b|x}\right| \geq A \wedge \Sigma\right) \vee \bigvee_{B \leq B' < A} \left(\left|{}^{x=x}/_{x=0}\right| \geq A \wedge \left|{}^{x=x}/_{b|x}\right| = B' \wedge \Sigma\right).$$

So we may reduce to the case where $\square'$ is $=$ and hence to either case 3 or 4 at the expense of replacing B by B' with $B \leq B' < A$. This is not a problem since in case 3 we show that σ_j has the form $\left|{}^{x=x}/_{a'|x}\right| \square_j A' \wedge \left|{}^{x=x}/_{b'|x}\right| \square'_j B' \wedge \left|{}^{x=x}/_{x=0}\right| = N$ i.e. there is no restriction on A' or B' and in case 4 we show that $\left|{}^{x=x}/_{a|x}\right| \geq A \wedge \left|{}^{x=x}/_{b|x}\right| = B' \wedge \Sigma_2$ is equivalent to $\left|{}^{x=x}/_{b|x}\right| = B' \wedge \Sigma_2$ and $B' \leq A$.
Case 6: $\Sigma = \Sigma_3$.
Suppose that M satisfies Σ_3. Then $a|x$ is equivalent to $br|x$ in M and

$$\left|{}^{x=x}/_{br|x}(M)\right| = \left|{}^{x=x}/_{b|x}(M)\right| \cdot \left|{}^{b|x}/_{br|x}(M)\right| = \left|{}^{x=x}/_{b|x}(M)\right| \cdot \left|{}^{x=x}/_{r|x}(M)\right|.$$

So, in this case, the result now follows from 5.6.

The remaining cases follow from the cases we have already covered by exchanging the roles of α and $\alpha - 1$, a and b, r and s, and A and B. $\square$

Proposition 9.2.2. *Let R be a recursive Prüfer domain with* $\mathrm{EPP}(R)$ *recursive. There is an algorithm which, given $w \in W$ with extended signature $((\square_1, \square_2), (z_1, z_2), (z_3, z_4))$ with $z_1 + z_2 > 1$ or $z_3 + z_4 > 1$, outputs $\underline{w} \in \mathbb{W}$ with* $\mathtt{exsig}\,\underline{w} < \mathtt{exsig}\,w$ *such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$.*

Proof. For $w \in W$, as in (†) (see the beginning of this section), define

$$\deg_1 w := \prod_{{}^{x=x}/_{c|x} \in X} f({}^{x=x}/_{c|x}) \cdot \prod_{{}^{x=x}/_{c|x} \in Y} g({}^{x=x}/_{c|x}).$$

Given $w \in W$ with extended signature $((\square_1, \square_2), (z_1, z_2), (z_3, z_4))$ with $z_3 + z_4 > 1$, we will show how to compute $\underline{w} \in \mathbb{W}$, a lattice combination of $w' \in W$, such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$ and such that for each w', either $\mathtt{exsig}\,w' < \mathtt{exsig}\,w$, or, $\mathtt{exsig}\,w' \leq \mathtt{exsig}\,w$ and $\deg_1 w' < \deg_1 w$. Since $\deg_1$ takes values in $\mathbb{N}$, by iterating this process, we will eventually compute $\underline{w} \in \mathbb{W}$ which is a lattice combination of $w' \in W$ such that $\mathtt{exsig}\,w' < \mathtt{exsig}\,w$.

We start with a special case. Let $a, b, \alpha, r, s \in R$ be such that $a\alpha = br$ and $b(\alpha - 1) = as$. Let Σ_i, for $1 \leq i \leq 6$, be as in 9.2.1. Suppose that, for some $1 \leq i \leq 6$, w is

$$\left|{}^{x=x}/_{a|x}\right| \square A \wedge \left|{}^{x=x}/_{b|x}\right| \square' B \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi \in W$$

where $A, B \in \mathbb{N}_2$, $\square, \square' \in \{=, \geq\}$, X and Y are finite sets of appropriate pp-pairs, $f : X \to \mathbb{N}$, $g : Y \to \mathbb{N}$ and Ξ is an auxiliary sentence.

We will compute $u = \bigsqcup_{j=1}^{n} u_j$ such that for each $1 \leq j \leq n$, either $\texttt{exsig}\, u_j < \texttt{exsig}\, w$, or, $\texttt{exsig}\, u_j \leq \texttt{exsig}\, w$ and $\deg_1 u_j < \deg_1 w$.

If the algorithm from 9.2.1 returns $\perp$, then

$$T_R \models \neg(|{}^{x=x}/{}_{a|x}|\, \square A \wedge |{}^{x=x}/{}_{b|x}|\, \square' B \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi).$$

In this case, set $u := \perp$. Otherwise, let $\sigma_1, \dots \sigma_n$ be, as in 9.2.1, such that

$$T_R \models |{}^{x=x}/{}_{a|x}|\, \square A \wedge |{}^{x=x}/{}_{b|x}|\, \square' B \wedge \Sigma_i \leftrightarrow \bigvee_{j=1}^{n} (\sigma_j \wedge \Sigma_i).$$

Therefore

$$T_R \models |{}^{x=x}/{}_{a|x}|\, \square A \wedge |{}^{x=x}/{}_{b|x}|\, \square' B \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi \leftrightarrow \bigvee_{j=1}^{n} (\sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi).$$

So $w \in V$ if and only if

$$\bigsqcup_{j=1}^{n} (\sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi) \in \mathbb{V}.$$

If σ_j is of the form

$$|{}^{x=x}/{}_{a'|x}|\square_j A' \wedge |{}^{x=x}/{}_{b'|x}|\square_j' B' \wedge |{}^{x=x}/{}_{x=0}| = N,$$

where $\square_j, \square_j' \in \{\emptyset, =, \geq\}$, $a', b' \in R$ and $A', B', N \in \mathbb{N}$ then $\sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi$ is a sentence about an R-module of fixed finite size. So, since EPP(R) is recursive, by 7.6, we can effectively decide whether $\sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi$ holds in some R-module. Set $u_j := \top$ if $\sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi$ is true in some R-module and $u_j := \perp$ otherwise.

If σ_j is of the form

$$|x = x/a'|x|\square_j A' \wedge |x = x/b'|x|\square_j' B',$$

where $\square_j, \square_j' \in \{\emptyset, =, \geq\}$, $a', b' \in R$ and $A', B' \in \mathbb{N}$ with $\square_j \preceq \square$, $\square_j' \preceq \square'$ and $A'B' < AB$ then set $u_j := \sigma_j \wedge \chi_{f,g} \wedge \Sigma_i \wedge \Xi$. The condition that $\square_j \preceq \square$ and $\square_j' \preceq \square'$ ensures that $\texttt{exsig}\, u_j \leq \texttt{exsig}\, w$. The condition that $A'B' < AB$ implies that $\deg_1 u_j < \deg_1 w$. So $w \in V$ if and only if $u := \bigsqcup_{j=1}^{n} u_j \in \mathbb{V}$ and $\deg_1 u_j < \deg_1 w$ for $1 \leq j \leq n$ as required.

We now consider the general case. Suppose w is $\chi_{f,g} \wedge \Xi$, where X and Y are disjoint finite sets of appropriate pp-pairs, $f : X \to \mathbb{N}_2$, $g : Y \to \mathbb{N}_2$ and Ξ is an auxiliary sentence, has extended signature $((\square_1, \square_2), (z_1, z_2), (z_3, z_4))$ with $z_3 + z_4 > 1$. There exist $a, b \in R$ with $a \neq b$ such that ${}^{x=x}/{}_{a|x}, {}^{x=x}/{}_{b|x} \in X \cup Y$. Then $w \in V$ if and only if

$$\bigsqcup_{(\overline{f}, \overline{g}) \in \Omega_{f,g,6}} \prod_{i=1}^{6} \Sigma_i \wedge \chi_{f_i, g_i} \wedge \Xi \in \mathbb{V}.$$

Claim: For all $(\overline{f}, \overline{g}) \in \Omega_{f,g,6}$ and $1 \leq i \leq 6$, either

$$\texttt{exsig}\, \chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi < \texttt{exsig}\, \chi_{f,g} \wedge \Xi, \text{ or, } \deg_1 \chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi < \deg_1 \chi_{f,g},$$

or, $X_i = X$, $Y_i = Y$, $f_i({}^{x=x}/{}_{c|x}) = f({}^{x=x}/{}_{c|x})$ for all ${}^{x=x}/{}_{c|x} \in X$ and $g_i({}^{x=x}/{}_{c|x}) = g({}^{x=x}/{}_{c|x})$ for all ${}^{x=x}/{}_{c|x} \in Y$.

By 9.0.4, if $X_i \neq X$ or $Y_i \neq Y$ then the extended signature of χ_{f_i,g_i} is strictly less that the extended signature of $\chi_{f,g}$. So suppose that $X = X_i$, $Y = Y_i$. By definition, $g({}^{x=x}/_{c|x}) = g_i({}^{x=x}/_{c|x})$ for all ${}^{x=x}/_{c|x} \in Y = Y_i$ and $f_i({}^{x=x}/_{c|x}) \leq f({}^{x=x}/_{c|x})$ for all ${}^{x=x}/_{c|x} \in X = X_i$. So, if $f_i({}^{x=x}/_{c|x}) < f({}^{x=x}/_{c|x})$, for some ${}^{x=x}/_{c|x} \in X$, then $\deg_1 \chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi < \deg_1 \chi_{f,g}$. So we have proved the claim.

Therefore for each $(\overline{f}, \overline{g}) \in \Omega_{f,g,6}$ and $1 \leq i \leq 6$, either $\texttt{exsig}\,\chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi < \texttt{exsig}\,\chi_{f,g} \wedge \Xi$, $\deg_1 \chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi < \deg_1 \chi_{f,g} \wedge \Xi$ or $\chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi$ is of the form of the special case. Since, by 9.0.4, we always have $\texttt{exsig}\,\chi_{f_i,g_i} \wedge \Sigma_i \wedge \Xi \leq \texttt{exsig}\,\chi_{f,g} \wedge \Xi$, we are done.

The version of the lemma with $z_1 + z_2 > 1$ follows from the one we have just proved by applying duality as in the proof of 9.1.5. □

Say $w \in W$ is **reducible** if w has

- short signature $(=, \Box)$ or $(\Box, =)$, or
- extended signature $((\geq, \Box), (z_1, z_2), (z_3, z_4))$ or $((\Box, \geq), (z_1, z_2), (z_3, z_4))$ with $z_1 \geq 1$ or $z_3 \geq 1$, or
- extended signature $((\Box, \Box'), (z_1, z_2), (z_3, z_4))$ with $z_1 + z_2 > 1$ or $z_3 + z_4 > 1$.

Remark 9.2.3. *If $w \in W$ is reducible then w is of the form required by either 9.1.1, 9.1.3, 9.1.5 or 9.2.2.*

Thus $w \in W$ is not reducible if and only if w has extended signature

- $((\emptyset, \emptyset), (z_1, z_2), (z_3, z_4))$ with $(z_1, z_2), (z_3, z_4) \in \{(1, 0), (0, 1), (0, 0)\}$ or
- $((\Box, \Box'), (z_1, z_2), (z_3, z_4))$ with $(\Box, \Box') \in \{(\geq, \emptyset), (\emptyset, \geq), (\geq, \geq)\}$ and $(z_1, z_2), (z_3, z_4) \in \{(0, 1), (0, 0)\}$.

The next remark follows directly from 3.1.9 because the condition on the extended signature of w implies that w is a conjunction of sentences of the form $|\varphi/\psi| = 1$ and $|\varphi/\psi| \geq E$ for pp-pairs φ/ψ and $E \in \mathbb{N}$.

Remark 9.2.4. *Let R be a Prüfer domain. There is an algorithm which, given $w \in W$ with extended signature $((\Box_1, \Box_2), (z_1, z_2), (z_3, z_4))$ with $(\Box_1, \Box_2) \in \{(\geq, \emptyset), (\emptyset, \geq), (\geq, \geq), (\emptyset, \emptyset)\}$ and $(z_1, z_2), (z_3, z_4) \in \{(0, 1), (0, 0)\}$, answers whether $w \in V$ or not.*

10. Algorithms for sentences which are not reducible

Lemma 10.1. *Let R be a recursive Prüfer domain with $X(R)$ recursive. There is an algorithm which, given $\lambda \in R$, $C \in \mathbb{N}$ and $(r, ra, \gamma, \delta) \in R^4$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (r, ra, \gamma, \delta)$ and $\left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/\lambda R_{\mathfrak{p}_i}\right| = C$.*

Proof. If $C = 1$ then the condition is always satisfied by taking $h = 0$, so suppose that $C \neq 1$. Let $p_1, \ldots, p_l \in \mathbb{P}$ be distinct primes and $n_1, \ldots, n_l \in \mathbb{N}$ be such that $C = \prod_{j=1}^{l} p_j^{n_j}$. For each prime ideal $\mathfrak{p} \lhd R$, if $|R_\mathfrak{p}/\lambda R_\mathfrak{p}|$ is finite then it is a prime power. Therefore, there exist prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (r, ra, \gamma, \delta)$ and $\left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/\lambda R_{\mathfrak{p}_i}\right| = C$ if and only if for each $1 \leq j \leq l$, there exist $h_j \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_{ij} \lhd R$ and ideals $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq i \leq h_j$ such that $(\mathfrak{p}_{ij}, I_{ij}) \models (r, ra, \gamma, \delta)$ and $\left|\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/\lambda R_{\mathfrak{p}_{ij}}\right| = p^{n_j}$. Thus we may reduce to the case that $C = p^n$ for some $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

We consider the cases $r = 0$ and $r \neq 0$ separately.

Case $r = 0$**:** For $\mathfrak{p} \lhd R$ prime and $I \lhd R_{\mathfrak{p}}$, $(\mathfrak{p}, I) \models (0, 0, \gamma, \delta)$ if and only if $I = 0$, $\gamma \notin \mathfrak{p}$ and $\delta \neq 0$. So, there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (0, 0, \gamma, \delta)$ and $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/\lambda R_{\mathfrak{p}_i}\right| = p^n$ if and only if $(p, n; \lambda, \gamma, 0, 1) \in X(R)$ and $\delta \neq 0$.
Case $r \neq 0$**:** If $r \neq 0$ then $(\mathfrak{p}, I) \models (r, ra, \gamma, \delta)$ if and only if $I = rJ$ for some $J \lhd R_{\mathfrak{p}}$ and $(\mathfrak{p}, J) \models (1, a, \gamma, \delta)$. So, there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (r, ra, \gamma, \delta)$ and $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/\lambda R_{\mathfrak{p}_i}\right| = p^n$ if and only if there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $J_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, J_i) \models (1, a, \gamma, \delta)$ and $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/\lambda R_{\mathfrak{p}_i}\right| = p^n$. The last statement is equivalent to $(p, n; \lambda, \gamma, a, \delta) \in X(R)$. □

Proposition 10.2. *Let R be a recursive Prüfer domain with* EPP(R) *and $X(R)$ recursive. There is an algorithm which, given $c \in R$, $C \in \mathbb{N}$ and Ξ an auxiliary sentence, answers whether there exists $M \in$* Mod-R *such that*

$$M \models |x{=}x/c|x| = C \wedge \Xi.$$

Proof. Let χ be the sentence $|x{=}x/c|x| = C \wedge \Xi$. By 2.3.6, there exists $M \models \chi$ if and only if there exist prime ideals $\mathfrak{p}_i \lhd R$ and uniserial $R_{\mathfrak{p}_i}$-modules U_i for $1 \leq i \leq l$ such that $\oplus_{i=1}^l U_i \models \chi$. Moreover, we may assume that $U_i/U_i c \neq 0$ for all $1 \leq i \leq l$. By 6.0.3, for each $1 \leq i \leq l$, either $c \in \text{ann}_R U_i$ or, for some ideal $I_i \lhd R_{\mathfrak{p}_i}$, $U_i \cong R_{\mathfrak{p}_i}/cI_i$. Thus, there exists $M \models \chi$ if and only if there exist $A, B \in \mathbb{N}$ with $AB = C$, $F \in$ Mod-R such that

$$F \models |x{=}x/x{=}0| = A \wedge |c|x/x{=}0| = 1 \wedge \Xi$$

and $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = B$ and $R_{\mathfrak{p}_i}/cI_i \models \Xi$ for $1 \leq i \leq h$. So, since EPP(R) is recursive, by 7.6, it is enough to show that there is an algorithm which, given $B \in \mathbb{N}$ and $c \in R$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = B$ and $R_{\mathfrak{p}_i}/cI_i \models \Xi$ for $1 \leq i \leq h$.

By 6.1.7, we can compute $(r_j, r_j a_j, \gamma_j, \delta_j)$ for $1 \leq j \leq n$ such that for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_{\mathfrak{p}}$, $(\mathfrak{p}, I) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for some $1 \leq j \leq n$ if and only if $R_{\mathfrak{p}}/cI \models \Xi$. Thus there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = B$ and $R_{\mathfrak{p}_i}/cI_i \models \Xi$ for $1 \leq i \leq h$ if and only if there exist $B_j \in \mathbb{N}$ for $1 \leq j \leq n$ such that $B = \prod_{j=1}^n B_j$ and for $1 \leq j \leq n$, there exist $h_j \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_{ij} \lhd R$ and ideals $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq i \leq h_j$ such that $\left|\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/cR_{\mathfrak{p}_{ij}}\right| = B_j$ and $(\mathfrak{p}_{ij}, I_{ij}) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for $1 \leq i \leq h_j$. The result now follows from 10.1. □

Corollary 10.3. *Let R be a recursive Prüfer domain with* EPP(R) *and $X(R)$ recursive. There is an algorithm which, given $b \in R$, $B \in \mathbb{N}$ and Ξ an auxiliary sentence, answers whether there exists $M \in$* Mod-R *such that*

$$M \models |xb{=}0/x{=}0| = B \wedge \Xi.$$

Proof. Apply 2.1.3 to 10.2. □

Proposition 10.4. *Let R be a recursive Prüfer domain with* EPP(R) *and $X(R)$ recursive. There is an algorithm which, given $c, b \in R$, $C, B \in \mathbb{N}$ and Ξ an auxiliary sentence, answers whether there exists $M \in$* Mod-R *such that*

$$M \models |x{=}x/c|x| = C \wedge |xb{=}0/x{=}0| = B \wedge \Xi.$$

Proof. For $\alpha \in R$, we write $\alpha \notin \text{Att}$ for the sentence $|x\alpha=0/x=0| = 1 \wedge |x=x/\alpha|x| = 1$. Recall, 2.3.4, that for all $M \in \text{Mod-}R$, there are $M_1, M_2 \in \text{Mod-}R$ such that M_1 satisfies $\alpha \notin \text{Att}$, M_2 satisfies $\alpha - 1 \notin \text{Att}$ and $M \equiv M_1 \oplus M_2$, .

Let $\alpha, u, v \in R$ be such that $c\alpha = bu$ and $b(\alpha - 1) = cv$. There exists an R-module which satisfies $|x=x/c|x| = C \wedge |xb=0/x=0| = B \wedge \Xi$ if and only if there exist $C_1, C_2, B_1, B_2 \in \mathbb{N}$ with $C = C_1C_2$ and $B = B_1B_2$ and there exists an R-module satisfying

$$|x=x/bu|x| = C_1 \wedge |xb=0/x=0| = B_1 \wedge \alpha \notin \text{Att} \wedge \Xi$$

and an R-module satisfying

$$|x=x/c|x| = C_2 \wedge |xcv=0/x=0| = B_2 \wedge (\alpha - 1) \notin \text{Att} \wedge \Xi.$$

By 2.1.3, there exists an R-module satisfying

$$|x=x/c|x| = C_2 \wedge |xcv=0/x=0| = B_2 \wedge \alpha - 1 \notin \text{Att} \wedge \Xi$$

if and only if there exists an R-module satisfying

$$|xc=0/x=0| = C_2 \wedge |x=x/cv|x| = B_2 \wedge \alpha - 1 \notin \text{Att} \wedge D\Xi.$$

Thus, in order to prove the proposition, it is enough to show that there is an algorithm which, given $b, u \in R$, $C, B \in \mathbb{N}$ and Ξ an auxiliary sentence, answers whether there exists an R-module satisfying the sentence χ defined as

$$|x=x/bu|x| = C \wedge |xb=0/x=0| = B \wedge \Xi.$$

We may assume that $bu \neq 0$, for otherwise χ is a sentence about an R-module of fixed finite size and since $\text{EPP}(R)$ is recursive, we can decide whether there exist R-modules satisfying such sentences.

By 2.3.6 and 6.0.3, there exists an R-module satisfying χ if and only if there exists $F \in \text{Mod-}R$ with $bu \in \text{ann}_R F$, $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ and $M \in \text{Mod-}R$ with $Mbu = M$ such that $F \oplus \bigoplus_{i=1}^h R_{\mathfrak{p}_i}/buI_i \oplus M$ satisfies χ. Now, this happens if and only if there exist $C_1, C_2 \in \mathbb{N}$ and $B_1, B_2, B_3 \in \mathbb{N}$ with $C = C_1C_2$ and $B = B_1B_2B_3$ such that

$$F \models |x=x/x=0| = C_1 \wedge |xb=0/x=0| = B_1 \wedge |bu|x/x=0| = 1 \wedge \Xi,$$

$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/buI_i \models |x=x/bu|x| = C_2 \wedge |xb=0/x=0| = B_2 \wedge \Xi, \text{ and}$$

$$M \models |xb=0/x=0| = B_3 \wedge |x=x/bu|x| = 1 \wedge \Xi.$$

In view of 7.6 and 10.3, it is therefore enough to show that there is an algorithm which answers whether there exists $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that

$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/buI_i \models |x=x/bu|x| = C_2 \wedge |xb=0/x=0| = B_2 \wedge \Xi.$$

By 6.1.7, we can compute $n \in \mathbb{N}$ and $(r_j, r_ja_j, \gamma_j, \delta_j)$ for $1 \leq j \leq n$ such that $R_{\mathfrak{p}}/buI \models \Xi$ if and only if $(\mathfrak{p}, I) \models (r_j, r_ja_j, \gamma_j, \delta_j)$ for some $1 \leq j \leq n$. It is therefore enough to show that there is an algorithm which, given (r, ra, γ, δ), $b, u \in R$ and $C, B \in \mathbb{N}$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i$ for $1 \leq i \leq h$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (r, ra, \gamma, \delta)$ and

$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/buI_i \models |x=x/bu|x| = C \wedge |xb=0/x=0| = B.$$

Case $r = 0$**:** In this case $(\mathfrak{p}, I) \models (r, ra, \gamma, \delta)$ implies $I = 0$. Moreover $(\mathfrak{p}, 0) \models (r, ra, \gamma, \delta)$ if and only if $\gamma \notin \mathfrak{p}$ and $\delta \neq 0$. Thus, there exist prime ideals $\mathfrak{p}_i$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, 0) \models (r, ra, \gamma, \delta)$ and $\oplus_{i=1}^h R_{\mathfrak{p}_i} \models |x=x/bu|x| = C \wedge |xb=0/x=0| = B$

if and only if $\delta \neq 0$, $B = 1$ and there exist prime ideals $\mathfrak{p}_i$ for $1 \le i \le h$ such that $\gamma \notin \mathfrak{p}_i$ and $|\oplus_{i=1}^h R_{\mathfrak{p}_i}/buR_{\mathfrak{p}_i}| = C$. Such an algorithm exists since $\mathrm{EPP}(R)$ is recursive.

Case $r \neq 0$: For all prime ideals $\mathfrak{p}$ and ideals $I \lhd R_{\mathfrak{p}}$, since $bu \neq 0$,

$$|{}^{xb=0}/{}_{x=0}(R_{\mathfrak{p}}/buI)| = |(buI : b)/buI| = |I/bI|.$$

Now, if $|I/bI|$ is finite but not equal to 1 then $I = \lambda R_{\mathfrak{p}}$ for some $\lambda \neq 0$ and $|I/bI| = |R_{\mathfrak{p}}/bR_{\mathfrak{p}}|$. Since $b \neq 0$, $|I/bI| = 1$ if and only if $b \notin I^{\#}$.

Therefore, there exist prime ideals $\mathfrak{p}_i$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \le i \le h$ such that $\oplus_{i=1}^h R_{\mathfrak{p}_i}/buI_i \models |{}^{x=x}/{}_{bu|x}| = C \wedge |{}^{xb=0}/{}_{x=0}| = B \wedge \Xi$ if and only if there exist $C', C'' \in \mathbb{N}$ with $C'C'' = C$ such that the following conditions hold.

(i) There exist prime ideals $\mathfrak{p}_i \lhd R$ and $\lambda_i \in R\backslash\{0\}$ for $1 \le i \le h$ such that $(\mathfrak{p}_i, \lambda_i R_{\mathfrak{p}_i}) \models (r, ra, \gamma, \delta)$, $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/buR_{\mathfrak{p}_i}\right| = C'$ and $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/bR_{\mathfrak{p}_i}\right| = B$.

(ii) There exist prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \le i \le h$ such that $(\mathfrak{p}_i, I_i) \models (r, ra, \gamma, \delta b)$ and $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/buR_{\mathfrak{p}_i}\right| = C''$.

Note that, if $(\mathfrak{p}, \lambda R_{\mathfrak{p}}) \models (r, ra, \gamma, \delta)$ then $(\mathfrak{p}, rR_{\mathfrak{p}}) \models (r, ra, \gamma, \delta)$. Since $(rR_{\mathfrak{p}})^{\#} = \mathfrak{p}R_{\mathfrak{p}}$, $(\mathfrak{p}, rR_{\mathfrak{p}}) \models (r, ra, \gamma, \delta)$ if and only if $\gamma\delta \notin \mathfrak{p}$. So (i) holds if and only if there exist prime ideals $\mathfrak{p}_i$ for $1 \le i \le h'$ such that $\gamma\delta \notin \mathfrak{p}_i$ and $\left|\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/buR_{\mathfrak{p}_i}\right| = C'$ and $\left|\oplus_{i=1}^{h'} R_{\mathfrak{p}_i}/bR_{\mathfrak{p}_i}\right| = B$. So, since $\mathrm{EPP}(R)$ is recursive, by 7.2, there is an algorithm which answers whether (i) holds or not.

By 10.1, since $X(R)$ is recursive, there is an algorithm which answers whether (ii) holds or not. □

The rest of this section is spent proving the following proposition.

Proposition 10.5. *Let R be a recursive Prüfer domain with* $\mathrm{EPP}(R)$ *and* $\mathrm{DPR}(R)$ *recursive. There is an algorithm which, given $c, b \in R$, $C, B \in \mathbb{N}$ and Ξ an auxiliary sentence, answers whether there exists $M \in$* Mod-R *such that*

$$M \models |{}^{x=x}/{}_{c|x}| = C \wedge |{}^{xb=0}/{}_{x=0}| \ge B \wedge \Xi.$$

We could choose the module in the following definition uniquely. For instance, one can show that when $\mathfrak{q} \supsetneq \mathfrak{p}$ the uniserial module $[\mathfrak{q}R_{\mathfrak{q}} : \lambda\mathfrak{p}R_{\mathfrak{q}}]/\mathfrak{q}R_{\mathfrak{q}}$ has the required theory where $[\mathfrak{q}R_{\mathfrak{q}} : \lambda\mathfrak{p}R_{\mathfrak{q}}]$ is the set of elements $a \in Q$, the fraction field of R, such that $a\lambda\mathfrak{q}R_{\mathfrak{q}} \subseteq \mathfrak{q}R_{\mathfrak{q}}$. However, we are only ever interested in modules up to elementary equivalence.

Definition 10.6. *Let $\lambda \in R\backslash\{0\}$ and let $\mathfrak{p}, \mathfrak{q} \lhd R$ be comparable prime ideals. If $\mathfrak{p} \supseteq \mathfrak{q}$ then define $M(\mathfrak{p}, \mathfrak{q}, \lambda)$ to be $R_{\mathfrak{p}}/\lambda\mathfrak{q}R_{\mathfrak{p}}$. If $\mathfrak{q} \supsetneq \mathfrak{p}$ then let $M(\mathfrak{p}, \mathfrak{q}, \lambda)$ be any module with theory dual to the theory of $R_{\mathfrak{q}}/\lambda\mathfrak{p}R_{\mathfrak{q}}$ in the sense of* [Her93, 6.6]*, i.e. for sentences χ as in 2.1.3, $M(\mathfrak{p}, \mathfrak{q}, \lambda) \models \chi$ if and only if $R_{\mathfrak{q}}/\lambda\mathfrak{p}R_{\mathfrak{q}} \models D\chi$.*

Note that $|{}^{x=x}/{}_{c|x}(M(\mathfrak{p}, \mathfrak{q}, \lambda))| = 1$ if and only if $c \notin \mathfrak{p}$, and, $|{}^{xb=0}/{}_{x=0}(M(\mathfrak{p}, \mathfrak{q}, \lambda))| = 1$ if and only if $b \notin \mathfrak{q}$.

Lemma 10.7. *Let R be a Prüfer domain. Suppose that $\lambda, a, \gamma, \delta \in R$ with $\lambda \neq 0$ are such that if $(\mathfrak{p}, I) \models (\lambda, \lambda a, \gamma, \delta)$ then $R_{\mathfrak{p}}/I \models \Xi$ and if $(\mathfrak{p}, I) \models (\lambda, \lambda a, \delta, \gamma)$ then $R_{\mathfrak{p}}/I \models D\Xi$. Then $\gamma \notin \mathfrak{p}$, $\delta \notin \mathfrak{q}$ and $a \in \mathfrak{p} \cap \mathfrak{q}$ implies $M(\mathfrak{p}, \mathfrak{q}, \lambda) \models \Xi$.*

Proof. Let $(\lambda, \lambda a, \gamma, \delta)$ be as in the statement. Suppose that $\gamma \notin \mathfrak{p}$, $\delta \notin \mathfrak{q}$ and $a \in \mathfrak{p} \cap \mathfrak{q}$. If $\mathfrak{p} \supseteq \mathfrak{q}$ then $(\mathfrak{p}, \lambda\mathfrak{q}R_{\mathfrak{p}}) \models (\lambda, \lambda a, \gamma, \delta)$. So $M(\mathfrak{p}, \mathfrak{q}, \lambda) \models \Xi$. Now suppose that $\mathfrak{q} \supsetneq \mathfrak{p}$. Then $(\mathfrak{q}, \lambda\mathfrak{p}R_\mathfrak{q}) \models (\lambda, \lambda a, \delta, \gamma)$. So $R_\mathfrak{q}/\lambda\mathfrak{p}R_\mathfrak{q} \models D\Xi$. Hence, by definition, $M(\mathfrak{p}, \mathfrak{q}, \lambda) \models \Xi$. □

Lemma 10.8. *Let R be a recursive Prüfer domain with* EPP(R) *and the radical relation recursive. There is an algorithm which, given $r, c \in R\backslash\{0\}$, $a, b, \gamma, \delta \in R$ and $A, B, C \in \mathbb{N}$, answers whether there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (1, a, \gamma, \delta)$, $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i\right| = A$ and*

$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/cI_i \models |{x=x}/{c|x}| = C \wedge |{xb=0}/{x=0}| \geq B.$$

Proof. Let $\mathfrak{p} \lhd R$ be a prime ideal and $I \lhd R_\mathfrak{p}$ be an ideal. Then $a \in I$ if and only if $rca \in rcI$. So $a \in I$ if and only if $|{rca|x}/{x=0}(R_\mathfrak{p}/rcI)| = 1$. If $R_\mathfrak{p}/rcI \neq 0$ then $\gamma \notin \mathfrak{p}$ if and only if $|{x=x}/{\gamma|x}(R_\mathfrak{p}/rcI)| = 1$, and, $\delta \notin I^\#$ if and only if $|{x\delta=0}/{x=0}(R_\mathfrak{p}/rcI)| = 1$. Note that

$$\left|{rc|x}/{x=0}(\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i)\right| = \left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i\right|.$$

Therefore, there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ as in the statement if and only if there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i$ satisfies χ, defined as

$$|{rc|x}/{x=0}| = A \wedge |{x=x}/{c|x}| = C \wedge |{xb=0}/{x=0}| \geq B$$
$$\wedge\, |{rca|x}/{x=0}| = 1 \wedge |{x=x}/{\gamma|x}| = 1 \wedge |{x\delta=0}/{x=0}| = 1.$$

By 8.1, there is an algorithm answering whether there exists $\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i$ satisfying χ. □

Recall that when R is a Prüfer domain, for prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$, the condition that $\mathfrak{p} + \mathfrak{q} \neq R$ is equivalent to $\mathfrak{p}$ and $\mathfrak{q}$ being comparable. So $M(\mathfrak{p}, \mathfrak{q}, r)$, as in the next proposition, is defined whenever $\mathfrak{p} + \mathfrak{q} \neq R$ and $r \neq 0$.

Proposition 10.9. *Let R be a recursive Prüfer domain with* EPP(R), DPR(R) *and $X(R)$ recursive. There is an algorithm which, given $C, B \in \mathbb{N}$, $c, b \in R$ with $c \neq 0$ and $r, a, \gamma, \delta \in R$ with $r \neq 0$, answers whether there exists $M \in$* Mod-R *satisfying*

$$|{x=x}/{c|x}| = C \wedge |{xb=0}/{x=0}| \geq B,$$

such that M is a direct sum of

- *modules of the form $R_\mathfrak{p}/rcI$ where $\mathfrak{p} \lhd R$ is a prime ideal, $I \lhd R_\mathfrak{p}$ is an ideal and $(\mathfrak{p}, I) \models (1, a, \gamma, \delta)$, and,*
- *modules of the form $M(\mathfrak{p}, \mathfrak{q}, r)$ where $\mathfrak{p}, \mathfrak{q} \lhd R$ are prime ideals such that $\mathfrak{p} + \mathfrak{q} \neq R$, $c\gamma \notin \mathfrak{p}$, $\delta \notin \mathfrak{q}$, $a \in \mathfrak{p}$ and $a \in \mathfrak{q}$.*

Proof. Recall that, 3.1.7, if DPR(R) is recursive then so is DPR$_2(R)$.

Case 1: $(c\gamma, a, a, \delta, a, b) \notin$ DPR$_2(R)$.

There exist prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$ with $\mathfrak{p} + \mathfrak{q} \neq R$ such that $c\gamma \notin \mathfrak{p}$, $\delta \notin \mathfrak{q}$, $a \in \mathfrak{p}$ and $a, b \in \mathfrak{q}$. So, $|{x=x}/{c|x}(M(\mathfrak{p}, \mathfrak{q}, r))| = 1$, since $c \notin \mathfrak{p}$, and, $|{xb=0}/{x=0}(M(\mathfrak{p}, \mathfrak{q}, r))| > 1$, since $b \in \mathfrak{q}$.

Therefore, there exists $M \in$ Mod-R as in the statement if and only if there exists $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ such that $(\mathfrak{p}, I_i) \models (1, a, \gamma, \delta)$ for $1 \leq i \leq h$ and such that

$$\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = \left|{x=x}/{c|x}(\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i)\right| = C.$$

Since $X(R)$ is recursive, we are done by 10.1.

Case 2: $(c\gamma, a, a, \delta, a, b) \in$ DPR$_2(R)$.

For all prime ideals $\mathfrak{p}, \mathfrak{q} \lhd R$ such that $\mathfrak{p} + \mathfrak{q} \neq R$, $c\gamma \notin \mathfrak{p}$, $\delta \notin \mathfrak{q}$, $a \in \mathfrak{p}$ and $a \in \mathfrak{q}$, by definition of $M(\mathfrak{p}, \mathfrak{q}, r)$ and $\mathrm{DPR}_2(R)$, we have $|{}^{xb=0}/_{x=0}(M(\mathfrak{p}, \mathfrak{q}, r))| = 1$ and $|{}^{x=x}/_{c|x}(M(\mathfrak{p}, \mathfrak{q}, r))| = 1$.

By 3.1.3, there exist $n \in \mathbb{N}$, $\epsilon, t, s_1, s_2 \in R$ such that

$$(\epsilon\gamma c)^n = at \quad \text{and} \quad ((\epsilon - 1)\delta)^n = as_1 + bs_2.$$

For all prime ideals $\mathfrak{p} \lhd R$, either $\epsilon \notin \mathfrak{p}$ or $\epsilon - 1 \notin \mathfrak{p}$. Thus, for all prime ideals $\mathfrak{p} \lhd R$ and ideals $I \lhd R_\mathfrak{p}$, $(\mathfrak{p}, I) \models (1, a, \gamma, \delta)$ if and only if $(\mathfrak{p}, I) \models (1, a, \epsilon\gamma, \delta)$ or $(\mathfrak{p}, I) \models (1, a, (\epsilon - 1)\gamma, \delta)$. Therefore, it is enough to be able to effectively answer whether there exist $C_1, C_2, B_1, B_2 \in \mathbb{N}$ with $C_1 \cdot C_2 = C$, $B_1 \cdot B_2 \geq B$ and $B_1, B_2 \leq B$, such that

(1) there is a sum of modules of the form $R_\mathfrak{p}/rcI$ where $(\mathfrak{p}, I) \models (1, a, \epsilon\gamma, \delta)$ which satisfy $|{}^{x=x}/_{c|x}| = C_1 \wedge |{}^{xb=0}/_{x=0}| \geq B_1$, and
(2) there is a sum of modules of the form $R_\mathfrak{p}/rcI$ where $(\mathfrak{p}, I) \models (1, a, (\epsilon-1)\gamma, \delta)$ which satisfy $|{}^{x=x}/_{c|x}| = C_2 \wedge |{}^{xb=0}/_{x=0}| \geq B_2$.

Suppose that $(\mathfrak{p}, I) \models (1, a, \epsilon\gamma, \delta)$ and $|{}^{x=x}/_{c|x}(R_\mathfrak{p}/rcI)| \leq C_1$. Then

$$|R_\mathfrak{p}/I| \leq |R_\mathfrak{p}/atR_\mathfrak{p}| = |R_\mathfrak{p}/(\epsilon\gamma c)^n R_\mathfrak{p}| = |R_\mathfrak{p}/c^n R_\mathfrak{p}| \leq |R_\mathfrak{p}/cR_\mathfrak{p}|^n$$

because $a \in I$ and $\epsilon\gamma \notin \mathfrak{p}$. So, $|R_\mathfrak{p}/cR_\mathfrak{p}| = |{}^{x=x}/_{c|x}(R_\mathfrak{p}/rcI)| \leq C_1$ implies $|R_\mathfrak{p}/I| \leq C_1^n$. Therefore, there exists a sum of modules as in (1) if and only if there is $A \leq C_1^n$ such that there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ with $(\mathfrak{p}_i, I_i) \models (1, a, \epsilon\gamma, \delta)$, $\left|\oplus_{i=1}^h R_{\mathfrak{p}_i}/I_i\right| = A$ and

$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i \models |{}^{x=x}/_{c|x}| = C_1 \wedge |{}^{xb=0}/_{x=0}| \geq B_1.$$

Therefore, by 10.8, there is an algorithm which answers whether (1) holds or not.

Suppose that $(\mathfrak{p}, I) \models (1, a, (\epsilon - 1)\gamma, \delta)$. Since $a \in I$, either $a \notin \mathfrak{p}$ and $I = R_\mathfrak{p}$, or, $a \in I^\#$. If $a \in I^\#$ then, since $(\epsilon - 1)\delta \notin I^\#$, $bs_2 = ((\epsilon - 1)\delta)^n - as_1 \notin I^\#$. So $b \notin I^\#$ and hence $|{}^{xb=0}/_{x=0}(R_\mathfrak{p}/rcI)| = 1$.

Thus, there exists a sum of modules as required in (2) if and only if there exist $C_2', C_2'' \in \mathbb{N}$ with $C_2 = C_2'C_2''$ such that

(i) there exist $h \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $I_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ such that $(\mathfrak{p}_i, I_i) \models (1, a, (\epsilon - 1)\delta b, \gamma)$ and
$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcI_i \models |{}^{x=x}/_{c|x}| = C_2', \text{ and}$$
(ii) there exist $h \in \mathbb{N}_0$ and prime ideals $\mathfrak{p}_i \lhd R$ for $1 \leq i \leq h$ such that $a \notin \mathfrak{p}_i$, $\gamma \notin \mathfrak{p}_i$, $\delta \notin \mathfrak{p}_i$ and
$$\oplus_{i=1}^h R_{\mathfrak{p}_i}/rcR_{\mathfrak{p}_i} \models |{}^{x=x}/_{c|x}| = C_2'' \wedge |{}^{xb=0}/_{x=0}| \geq B_2.$$

Since for $\mathfrak{p} \lhd R$ prime and $I \lhd R_\mathfrak{p}$, $|{}^{x=x}/_{c|x}(R_p/rcI)| = |R_\mathfrak{p}/cR_\mathfrak{p}|$, by 10.1, there is an algorithm which answers whether (i) holds.

To conclude the proof we need to show that we can effectively answer whether (ii) holds or not. Let $\alpha, u, v \in R$ be such that $b\alpha = rcu$ and $rc(\alpha - 1) = bv$. If $\alpha \notin \mathfrak{p}$ then

$$|{}^{x=x}/_{c|x}(R_\mathfrak{p}/crR_\mathfrak{p})| = |R_\mathfrak{p}/cR_\mathfrak{p}| \quad \text{and} \quad |{}^{xb=0}/_{x=0}(R_\mathfrak{p}/crR_\mathfrak{p})| = |R_\mathfrak{p}/crR_\mathfrak{p}|.$$

If $\alpha - 1 \notin \mathfrak{p}$ then

$$|{}^{x=x}/_{c|x}(R_\mathfrak{p}/crR_\mathfrak{p})| = |R_\mathfrak{p}/cR_\mathfrak{p}| \quad \text{and} \quad |{}^{xb=0}/_{x=0}(R_\mathfrak{p}/crR_\mathfrak{p})| = |R_\mathfrak{p}/bR_\mathfrak{p}|.$$

Since for all prime ideals $\mathfrak{p} \lhd R$, either $\alpha \notin \mathfrak{p}$ or $\alpha - 1 \notin \mathfrak{p}$, by 7.3, there is an algorithm which answers whether (ii) holds or not. $\square$

Lemma 10.10. *Let R be a recursive Prüfer domain with* $\mathrm{EPP}(R)$ *and* $\mathrm{DPR}(R)$ *recursive. There is an algorithm which, given $b, c, \gamma \in R$ and $B, C \in \mathbb{N}$, answers whether there exist $h \in \mathbb{N}_0$ and prime ideals $\mathfrak{p}_i \lhd R$ with $\gamma \notin \mathfrak{p}_i$ for $1 \leq i \leq h$ such that*

$$\oplus_{i=1}^{h} R_{\mathfrak{p}_i} \models |x{=}x/c|x| = C \wedge |xb{=}0/x{=}0| \geq B.$$

Proof. We split the proof into 3 cases. Let χ be the sentence

$$|x{=}x/c|x| = C \wedge |xb{=}0/x{=}0| \geq B.$$

Case $b \neq 0$: Then $|xb{=}0/x{=}0(R_{\mathfrak{p}})| = 1$ for all prime ideals $\mathfrak{p} \lhd R$. So $\oplus_{i=1}^{h} R_{\mathfrak{p}_i} \models \chi$ if and only if $B = 1$ and $\left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = C$. Since $\mathrm{EPP}(R)$ is recursive, we are done by 7.3.

Case $b = 0$ and $C > 1$: Then $\left|xb{=}0/x{=}0(\oplus_{i=1}^{h} R_{\mathfrak{p}_i})\right| = \left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}\right|$. So $\left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = C > 1$ implies $\left|xb{=}0/x{=}0(\oplus_{i=1}^{h} R_{\mathfrak{p}_i})\right|$ is infinite. So $\oplus_{i=1}^{h} R_{\mathfrak{p}_i} \models \chi$ if and only if $\left|\oplus_{i=1}^{h} R_{\mathfrak{p}_i}/cR_{\mathfrak{p}_i}\right| = C$. So, since $\mathrm{EPP}(R)$ is recursive, we are done by 7.3.

Case $b = 0$ and $C = 1$: If $B = 1$ then the zero module satisfies χ i.e. $h = 0$. Otherwise, if $\oplus_{i=1}^{h} R_{\mathfrak{p}_i} \models \chi$ and $\gamma \notin \mathfrak{p}_i$ for $1 \leq i \leq h$ then $h \geq 1$ and $c\gamma \notin \mathfrak{p}_i$ for all $1 \leq i \leq h$. So there exists a prime ideal $\mathfrak{p} \lhd R$ such that $\gamma c \notin \mathfrak{p}$. Conversely, if $\mathfrak{p} \lhd R$ is a prime ideal such that $c\gamma \notin \mathfrak{p}$ then $R_{\mathfrak{p}} \models \chi$. There exists a prime ideal $\mathfrak{p} \lhd R$ such that $\gamma c \notin \mathfrak{p}$ if and only if $(\gamma c, 1, 0, 0) \notin \mathrm{DPR}(R)$. □

Proof of 10.5. We may assume that $c \neq 0$ since if $c = 0$ then $|x{=}x/c|x| = C \wedge |xb{=}0/x{=}0| \geq B \wedge \Xi$ is a statement about an R-module of a fixed finite size and in this case we know such an algorithm exists, by 7.6, since $\mathrm{EPP}(R)$ is recursive.

By 6.1.7, we can compute $n \in \mathbb{N}$ and $(r_j, r_j a_j, \gamma_j, \delta_j) \in R^4$ for $1 \leq j \leq n$ such that $R_{\mathfrak{p}}/cI \models \Xi$ if and only if $(\mathfrak{p}, I) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for some $1 \leq j \leq n$ and such that $R_{\mathfrak{p}}/cI \models D\Xi$ if and only if $(\mathfrak{p}, I) \models (r_j, r_j a_j, \delta_j, \gamma_j)$ for some $1 \leq j \leq n$.

Claim: There exists $M \in \text{Mod-}R$ such that

$$M \models |x{=}x/c|x| = C \wedge |xb{=}0/x{=}0| \geq B \wedge \Xi$$

if and only if there exist $C_j \in \mathbb{N}$ for $0 \leq j \leq n$ and $B_j \in \mathbb{N}$, $B_j \leq B$ for $0 \leq j \leq n+1$ with $\prod_{j=0}^{n} C_j = C$ and $\prod_{j=0}^{n+1} B_j \geq B$, satisfying the following conditions.

(1) There exists $F \in \text{Mod-}R$ such that

$$F \models |x{=}x/x{=}0| = C_0 \wedge |xb{=}0/x{=}0| \geq B_0 \wedge |c|x/x{=}0| = 1 \wedge \Xi.$$

(2) There exists $M' \in \text{Mod-}R$ such that

$$M' \models |x{=}x/c|x| = 1 \wedge |xb{=}0/x{=}0| \geq B_{n+1} \wedge \Xi.$$

(3) For $1 \leq j \leq n$,

(a)$_j$ if $r_j = 0$ then there exist $h_j \in \mathbb{N}_0$ and prime ideals $\mathfrak{p}_{ij} \lhd R$ for $1 \leq i \leq h_j$ such that $\gamma_j \notin \mathfrak{p}_{ij}$, $\delta_j \neq 0$ and

$$M_j := \oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}} \models |x{=}x/c|x| = C_j \wedge |xb{=}0/x{=}0| \geq B_j, \text{ and}$$

(b)$_j$ if $r_j \neq 0$ then there exist $h_j, k_j \in \mathbb{N}_0$, prime ideals $\mathfrak{p}_j, \mathfrak{q}_j, \mathfrak{p}_{ij} \lhd R$ and ideals $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ for $1 \leq i \leq h_j$ such that $(\mathfrak{p}_{ij}, I_{ij}) \models (1, a_j, \gamma_j, \delta_j)$ for $1 \leq i \leq h_j$, $\gamma_j \notin \mathfrak{p}_j$, $\delta_j \notin \mathfrak{q}_j$, $a \in \mathfrak{p}_j$, $a \in \mathfrak{q}_j$, and

$$M_j := M(\mathfrak{p}_j, \mathfrak{q}_j, r_j)^{k_j} \oplus \bigoplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/r_j c I_{ij} \models |x{=}x/c|x| = C_j \wedge |xb{=}0/x{=}0| \geq B_j.$$

Proof of claim. (⇒) By 6.0.3, if U is a uniserial module with ${x=x}/{c|x}(U)$ finite but non-zero then either $c \in \mathrm{ann}_R U$ or $U \cong R_\mathfrak{p}/cI$ for some prime ideal $\mathfrak{p} \lhd R$ and ideal $I \lhd R_\mathfrak{p}$. Therefore, by 2.3.6, there exists

$$M \models |{x=x}/{c|x}| = C \wedge |{xb=0}/{x=0}| \geq B \wedge \Xi$$

if and only if there exists $F \in \text{Mod-}R$ with $c \in \mathrm{ann}_R F$, prime ideals $\mathfrak{p}_i \lhd R$ and ideals $J_i \lhd R_{\mathfrak{p}_i}$ for $1 \leq i \leq h$ and $M' \in \text{Mod-}R$ with $|{x=x}/{c|x}(M')| = 1$ such that

$$F \oplus M' \oplus \bigoplus_{i=1}^{h} R_{\mathfrak{p}_i}/cJ_i \models |{x=x}/{c|x}| = C \wedge |{xb=0}/{x=0}| \geq B \wedge \Xi.$$

Since $R_\mathfrak{p}/cJ \models \Xi$ if and only if $(\mathfrak{p}, J) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for some $1 \leq j \leq n$, we may rewrite $\oplus_{i=1}^h R_{\mathfrak{p}_i}/cJ_i$ as $\oplus_{j=1}^n \oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/cJ_{ij}$ where $(\mathfrak{p}_{ij}, J_{ij}) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for $1 \leq j \leq n$ and $1 \leq i \leq h_j$.

If $r_j = 0$ then $(\mathfrak{p}_{ij}, J_{ij}) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ if and only if $J_{ij} = 0$, $\delta_j \neq 0$ and $\gamma_j \notin \mathfrak{p}_{ij}$. If $r_j \neq 0$ then $(\mathfrak{p}_{ij}, J_{ij}) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ if and only if there exists $I_{ij} \lhd R_{\mathfrak{p}_{ij}}$ such that $J_{ij} = rI_{ij}$ and $(\mathfrak{p}_{ij}, I_{ij}) \models (1, a_j, \gamma_j, \delta_j)$.

Let $C_0 := |{x=x}/{c|x}(F)| = |F|$ and $B_0 := \min\{|{xb=0}/{x=0}(F)|, B\}$. Let $B_{n+1} := \min\{|{xb=0}/{x=0}(M')|, B\}$. If $r_j = 0$ then let $C_j := \left|{x=x}/{c|x}(\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}})\right|$ and $B_j := \min\{\left|{xb=0}/{x=0}(\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}})\right|, B\}$. If $r_j \neq 0$ then let $C_j := \left|{x=x}/{c|x}(\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/r_j cI_{ij})\right|$ and $B_j := \min\{\left|{xb=0}/{x=0}(\oplus_{i=1}^{h_j} R_{\mathfrak{p}_{ij}}/r_j cI_{ij})\right|, B\}$. Now, setting $k_j = 0$ for $1 \leq j \leq n$, we are done.

(⇐) Fix $1 \leq j \leq n$. Suppose $r_j = 0$. Then $(\mathfrak{p}_{ij}, 0) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for each $1 \leq i \leq h_j$ and hence $R_{\mathfrak{p}_{ij}} \models \Xi$. Suppose $r_j \neq 0$. Then $(\mathfrak{p}_{ij}, r_j I_{ij}) \models (r_j, r_j a_j, \gamma_j, \delta_j)$ for each $1 \leq i \leq h_j$ and hence $R_{\mathfrak{p}_{ij}}/r_j cI_{ij} \models \Xi$.

By 10.7, $\gamma_j \notin \mathfrak{p}_j$, $\delta_j \notin \mathfrak{q}_j$, $a \in \mathfrak{p}_j$ and $a \in \mathfrak{q}_j$ implies that $M(\mathfrak{p}_j, \mathfrak{q}_j, r_j) \models \Xi$. Thus

$$F \oplus M' \oplus \bigoplus_{j=1}^{n} M_j \models \Xi.$$

Therefore

$$F \oplus M' \oplus \bigoplus_{j=1}^{n} M_j \models |{x=x}/{c|x}| = \prod_{j=0}^{n} C_i \wedge |{xb=0}/{x=0}| \geq \prod_{j=0}^{n+1} B_i \wedge \Xi.$$

Since $C = \prod_{j=0}^n C_i$ and $\prod_{j=0}^{n+1} B_i \geq B$, we are done.

The set of $C_j \in \mathbb{N}$ for $0 \leq j \leq n$ and $B_j \in \mathbb{N}$ with $B \geq B_j$ for $0 \leq j \leq n+1$ such that $C = \prod_{j=0}^n C_j$ and $\prod_{j=0}^{n+1} B_i \geq B$ is finite. Therefore it is enough to show that for fixed $C_j \in \mathbb{N}$ for $0 \leq j \leq n$ and $B_j \in \mathbb{N}$ for $0 \leq j \leq n+1$, there are algorithms answering whether (1), (2) and (3) hold. By 7.6, since $\mathrm{EPP}(R)$ is recursive, there is an algorithm which answers whether (1) holds. By 3.1.9, since $\mathrm{DPR}(R)$ is recursive there is an algorithm which answers whether (2) holds. Since $\mathrm{DPR}(R), \mathrm{EPP}(R)$ and $X(R)$ are recursive, by 10.9, if $r_j \neq 0$ then there is an algorithm which answers whether $(\mathrm{b})_j$ holds. Since $\mathrm{DPR}(R)$ and $\mathrm{EPP}(R)$ are recursive, by 10.10, if $r_j = 0$ then there is an algorithm which answers whether $(\mathrm{a})_j$ holds. Thus, there is an algorithm which answers whether (3) holds. □

11. The main theorem

Theorem 11.1. *Let R be a Prüfer domain. The theory of R-modules is decidable if and only if* $\mathrm{DPR}(R)$*,* $\mathrm{EPP}(R)$ *and* $X(R)$ *are recursive.*

Proof. The forward direction follows from [GLPT18, 6.4] (or 3.1.6), 3.2.9 and 3.3.3.

By 5.2, in order to show that T_R is decidable, it is enough to show that there is an algorithm which, given a sentence χ of the form

$$|{}^{d|x}/{}_{x=0}|\,\square_1 D \wedge |{}^{xb=0}/{}_{c|x}|\,\square_2 E \wedge \chi_{f,g} \wedge \Xi,$$

where $\square_1, \square_2 \in \{\geq, =, \emptyset\}$, $d, c, b \in R\backslash\{0\}$, $D, E \in \mathbb{N}$, $f : X \to \mathbb{N}$, $g : Y \to \mathbb{N}$, X, Y are finite sets of pp-pairs of the form ${}^{xb'=0}/{}_{x=0}$ or ${}^{x=x}/{}_{c'|x}$ and Ξ is an auxiliary sentence, answers whether there exists an R-module which satisfies χ or not.

Let W and V be as in §9. By 9.1.1, 9.1.3, 9.1.5 and 9.2.2, there is an algorithm which, given $w \in W$ reducible, returns $\underline{w} \in \mathbb{W}$ such that $w \in V$ if and only if $\underline{w} \in \mathbb{V}$, and, $\texttt{exsig}\,\underline{w} < \texttt{exsig}\,w$. Since the set of extended signatures is artinian, by 4.2.1, it is enough to show that there is an algorithm which, given $w \in W$ not reducible, answers whether $w \in V$ or not. By 9.2.4 and the statement just before that, it is enough to show that there is an algorithm which, given $w \in W$ with extended signature in

$$\{((\emptyset,\emptyset),(z_1,z_2),(z_3,z_4)) \mid z_1+z_2 \leq 1 \text{ and } z_3+z_4 \leq 1\},$$

answers whether $w \in V$ or not. Now, $w \in V$ if and only if $Dw \in V$. So, by 9.0.3, we can reduce the set of extended signatures we need to consider further to

$$S := \{((\emptyset,\emptyset),(1,0),(1,0)),((\emptyset,\emptyset),(1,0),(0,0)),((\emptyset,\emptyset),(1,0),(0,1)),$$
$$((\emptyset,\emptyset),(0,1),(0,1))\}.$$

By 9.2.4, 10.2, 10.4 and 10.5, for each $w \in S$ such an algorithm exists. □

We now consider the consequences of our theorem for Prüfer and Bézout domains of Krull dimension 1.

Corollary 11.2. *Let R be a recursive Prüfer domain of Krull dimension* 1*. The theory of R-modules is decidable if and only if* $\mathrm{EPP}(R)$ *is recursive and the relation* $a \in \mathrm{rad}(b_1R + b_2R)$ *is recursive.*

Proof. It is easy to see, using 3.1.3, that $(a, b_1, b_2, 1, 0, 0) \in \mathrm{DPR}_2(R)$ if and only if $a \in \mathrm{rad}(b_1R + b_2R)$. So, since $\mathrm{DPR}(R)$ recursive implies $\mathrm{DPR}_2(R)$ recursive, the forward direction follows from 11.1. The reverse direction is a direct consequence of 11.1 and claims 1 and 2 below.

Claim 1: $(a, b, c, d) \in \mathrm{DPR}(R)$ if and only if the following 3 conditions hold:

(i) $ac \in \mathrm{rad}(bR + dR)$.
(ii) $c \in \mathrm{rad}(dR)$ or $a = 0$ or $b \neq 0$.
(iii) $a \in \mathrm{rad}(bR)$ or $c = 0$ or $d \neq 0$.

Since R has Krull dimension 1, if $\mathfrak{p}, \mathfrak{q} \lhd R$ are prime ideals with $\mathfrak{p} + \mathfrak{q} \neq R$ then either $\mathfrak{p} = \mathfrak{q}$, $\mathfrak{p} = 0$ or $\mathfrak{q} = 0$. Therefore $(a, b, c, d) \in \mathrm{DPR}(R)$ if and only if

(i') for all prime ideals $\mathfrak{p}$, either $a \in \mathfrak{p}$, $c \in \mathfrak{p}$, $b \notin \mathfrak{p}$ or $d \notin \mathfrak{p}$;
(ii') for all prime ideals $\mathfrak{p}$, either $a = 0$, $c \in \mathfrak{p}$, $b \neq 0$ or $d \notin \mathfrak{p}$; and
(iii') for all prime ideals $\mathfrak{q}$, either $a \in \mathfrak{q}$, $c = 0$, $b \notin \mathfrak{q}$ or $d \neq 0$.

The claim now follows since (i) is equivalent to (i'), (ii) is equivalent to (ii') and (iii) is equivalent to (iii').

Claim 2:

(1) $(p, n; e, \gamma, 0, \delta) \in X(R)$ if and only if $\delta \neq 0$ and $(p, 0; 0; \gamma; e, n) \in \mathrm{EPP}(R)$.
(2) If $a \neq 0$ then $(p, n; e, \gamma, a, \delta) \in X(R)$ if and only if $(p, 0; 0; \gamma\delta; e, n) \in \mathrm{EPP}(R)$.

Note that, by 3.2.5 and the definition of $\mathrm{EPP}(R)$, for $p \in \mathbb{P}$, $n \in \mathbb{N}$ and $\gamma, e \in R$, $(p, 0; 0; \gamma; e, n) \in \mathrm{EPP}(R)$ if and only if there exist $h \in \mathbb{N}$ and maximal ideals $\mathfrak{m}_i \lhd R$ with $\gamma \notin \mathfrak{m}_i$ for $1 \leq i \leq h$ such that $|\oplus_{i=1}^{h} R_{\mathfrak{m}_i}/eR_{\mathfrak{m}_i}| = p^n$. The equivalence (1) now follows from 3.3.2.

Now consider (2). Since R has Krull dimension 1, if $\mathfrak{q} \lhd R$ is a prime ideal and $a \in \mathfrak{q}$ for some $a \neq 0$ then $\mathfrak{q}$ is maximal. Thus, by 3.3.2, if $a \neq 0$ then $(p, n; e, \gamma, a, \delta) \in X(R)$ if and only if there exists $h \in \mathbb{N}$ and maximal ideals $\mathfrak{m}_i \lhd R$ for $1 \leq i \leq h$ such that $|\oplus_{i=1}^{h} R_{\mathfrak{m}_i}/eR_{\mathfrak{m}_i}| = p^n$ and for $1 \leq i \leq h$, $\gamma \notin \mathfrak{m}_i$ and $\delta \notin \mathfrak{m}_i$. So (2) now follows from the characterisation of $(p, 0; 0; \gamma\delta; e, n) \in \mathrm{EPP}(R)$ given in the previous paragraph. □

The next lemma is essentially taken from [LTP17].

Lemma 11.3. *Let R be a Bézout domain with Krull dimension 1. For all $a, b \in R$ with $b \neq 0$, $a \notin \operatorname{rad} bR$ if and only if there exists $c \in R$ such that $1 \in aR + cR$ and $1 \notin bR + cR$. Moreover, if R is recursive and the set of units of R is recursive then the radical relation is recursive.*

Proof. The first statement is contained in the proof of [LTP17, 3.3]. The second statement is part of [LTP17, 3.3] but our assumptions are a priori weaker than the assumptions there.

If $a \in \operatorname{rad}(bR)$ then $a^n = br$ for some $n \in \mathbb{N}$ and $r \in R$. Therefore, since R is recursive, we can effectively list the pairs $(a, b) \in R^2$ such that $a \in \operatorname{rad}(bR)$.

Since R is a recursive Bézout domain, given $a, c \in R$, we can effectively find $d \in R$ such that $dR = aR + cR$. Therefore we can effectively list the pairs (a, c) such that $1 \in aR + cR$ and since the set of units of R is recursive, we can effectively list the pairs $(b, c) \in R^2$ such that $1 \notin bR + cR$. Thus, by the first statement, we can effectively list the pairs $(a, b) \in R^2$ such that $a \notin \operatorname{rad}(bR)$. □

The next corollary generalises [GLPT18, 6.7], which in turn generalised the main theorem of [LTP17] (i.e. 3.4 therein).

Corollary 11.4. *Let R be a recursive Bézout domain with Krull dimension 1. The theory of R-modules is decidable if and only if the set of units of R and $\mathrm{EPP}(R)$ are recursive.*

Proof. For any ring R, if T_R is decidable then the set of units of R is recursive because $r \in R$ is a unit if and only if $|x{=}x/r|x| = 1$ holds in all R-modules. The forward direction now follows from 11.1.

For any recursive Bézout domain R, given $b_1, b_2 \in R$ we can effectively find $b \in R$ such that $bR = b_1R + b_2R$. Thus, the reverse direction follows from 11.2 and 11.3. □

12. Integer-valued polynomials

We use our main theorem, 11.1, to show that the theory of modules over the ring of integer valued polynomials with rational valued coefficients, $\mathrm{Int}(\mathbb{Z})$, is decidable.

First we fix some notation: For all $p \in \mathbb{P}$, $\mathbb{Z}_{(p)}$ denotes $\mathbb{Z}$ localised at the ideal generated by p, $\mathbb{Q}_p$ denotes the field of p-adic numbers, $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ denotes the p-adic valuation on $\mathbb{Q}_p$ and $\widehat{\mathbb{Z}}_p$ denotes the p-adic integers.

The ring $\text{Int}(\mathbb{Z})$ is the subring of $\mathbb{Q}[x]$ consisting of all polynomials $a \in \mathbb{Q}[x]$ such that $a(\mathbb{Z}) \subseteq \mathbb{Z}$. Recall, [CC97, I.1.1], that the polynomials

$$\left(\begin{array}{c} x \\ n \end{array} \right) := \frac{x(x-1)\dots(x-(n-1))}{n!}$$

are a basis for $\text{Int}(\mathbb{Z})$ as a $\mathbb{Z}$-module. This readily gives us a recursive presentation of $\text{Int}(\mathbb{Z})$. The ring $\text{Int}(\mathbb{Z})$ is a Prüfer domain [CC97, VI.1.7].

Note, [CC97, I.2.1], that, for all $p \in \mathbb{P}$, if $f \in \mathbb{Q}[x]$ and $f(\mathbb{Z}) \subseteq \mathbb{Z}$ then $f(\mathbb{Z}_{(p)}) \subseteq \mathbb{Z}_{(p)}$. Further, for any $p \in \mathbb{P}$, by continuity of polynomials over $\mathbb{Q}_p$, $f(\mathbb{Z}_{(p)}) \subseteq \mathbb{Z}_{(p)}$ implies $f(\widehat{\mathbb{Z}_p}) \subseteq \widehat{\mathbb{Z}_p}$.

The prime spectrum of $\text{Int}(\mathbb{Z})$ is described in [CC97, V.2.7]. We recall the information we need.

- For any $p \in \mathbb{P}$, the prime ideals of $\text{Int}(\mathbb{Z})$ containing p are in bijective correspondence with the elements of $\widehat{\mathbb{Z}_p}$ by mapping $\alpha \in \widehat{\mathbb{Z}_p}$ to

  $$\mathfrak{m}_{p,\alpha} := \{f \in \text{Int}(\mathbb{Z}) \mid f(\alpha) \in p\widehat{\mathbb{Z}_p}\}.$$

  The prime ideals $\mathfrak{m}_{p,\alpha}$ are exactly the maximal ideals of $\text{Int}(\mathbb{Z})$ and the quotient $\text{Int}(\mathbb{Z})/\mathfrak{m}_{p,\alpha}$ has size p.
- The non-zero prime ideals $\mathfrak{p}$ of $\text{Int}(\mathbb{Z})$ such that $\mathbb{Z} \cap \mathfrak{p} = \{0\}$ are in bijective correspondence with the monic irreducible polynomials $q \in \mathbb{Q}[x]$ via the mapping

  $$q \mapsto \mathfrak{p}_q := q\mathbb{Q}[x] \cap \text{Int}(\mathbb{Z}).$$

  Note that $\mathfrak{p}_q \subseteq \mathfrak{m}_{p,\alpha}$ if and only if $q(\alpha) = 0$ in $\mathbb{Q}_p$.

It will sometimes be useful to have an alternate notation for the non-maximal prime ideals. For $\alpha \in \widehat{\mathbb{Z}_p}$, let

$$\mathfrak{p}_\alpha := \begin{cases} \mathfrak{p}_q, & \text{if } \alpha \text{ is algebraic and } q \in \mathbb{Q}[x] \text{ is its monic minimal polynomial;} \\ \{0\}, & \text{if } \alpha \text{ is transcendental.} \end{cases}$$

This notation has the disadvantage that $\mathfrak{p}_\alpha = \mathfrak{p}_\beta$ does not imply $\alpha = \beta$. However, it allows us to work with $\alpha \in \widehat{\mathbb{Z}_p}$ algebraic and transcendental uniformly in the following ways: Firstly, for $a \in \text{Int}(\mathbb{Z})$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a \in \mathfrak{p}_\alpha$ if and only if $a(\alpha) = 0$. Secondly, for $\mathfrak{q} \lhd R$ a prime ideal and $\alpha \in \widehat{\mathbb{Z}_p}$, $\mathfrak{q} \subseteq \mathfrak{m}_{p,\alpha}$ if and only if $\mathfrak{q} = \mathfrak{m}_{p,\alpha}$, $\mathfrak{q} = \mathfrak{p}_\alpha$ or $\mathfrak{q} = \{0\}$

By 11.1, we need to show that $\text{DPR}(\text{Int}(\mathbb{Z}))$, $\text{EPP}(\text{Int}(\mathbb{Z}))$ and $X(\text{Int}(\mathbb{Z}))$ are recursive. In order to do this, we use the fact, [Ax68, Thm 17], that the common theory T_{adic} of the valued fields $\mathbb{Q}_p$, as p varies, is decidable. We shall work in a two-sorted language $\mathcal{L}_{val}$ of valued fields with a sort for the field K, a sort Γ for the value group extended by ∞ and a function symbol $v : K \to \Gamma$ which is interpreted as v_p in each $\mathbb{Q}_p$. For convenience, we add a constant symbol 1 to the value group sort Γ, which for each valued field $\mathbb{Q}_p$, is interpreted as the least strictly positive element of the value group.

Let $\mathcal{L}^0_{val}$ be the set of sentences in $\mathcal{L}_{val}$. The sets

$$T_{adic} := \{\varphi \in \mathcal{L}^0_{val} \mid \text{ for all } p \in \mathbb{P}, \ \mathbb{Q}_p \models \varphi\}$$

and

$$\{\varphi \in \mathcal{L}_{val}^0 \mid \text{ there exists } p \in \mathbb{P} \text{ such that } \mathbb{Q}_p \models \varphi\}$$

are recursive. Hence, since $\mathbb{Q}_p \models \varphi$ if and only if $\mathbb{Q}_q \models v(p) = 0 \vee \varphi$ for all $q \in \mathbb{P}$, the set

$$\{(p, \varphi) \in \mathbb{P} \times \mathcal{L}_{val}^0 \mid \mathbb{Q}_p \models \varphi\}$$

is recursive.

Proposition 12.1. *The set* $DPR(\text{Int}(\mathbb{Z}))$ *is recursive.*

Proof. Let $a, b, c, d \in \text{Int}(\mathbb{Z})$. Then $(a, b, c, d) \in \text{DPR}(\text{Int}(\mathbb{Z}))$ if and only if

(1) for all $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a \in \mathfrak{m}_{p,\alpha}$, $b \notin \mathfrak{m}_{p,\alpha}$, $c \in \mathfrak{m}_{p,\alpha}$ or $d \notin \mathfrak{m}_{p,\alpha}$;
(2) for all $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a \in \mathfrak{m}_{p,\alpha}$, $b \notin \mathfrak{m}_{p,\alpha}$, $c \in \mathfrak{p}_\alpha$ or $d \notin \mathfrak{p}_\alpha$;
(3) for all $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a \in \mathfrak{m}_{p,\alpha}$, $b \notin \mathfrak{m}_{p,\alpha}$, $c = 0$ or $d \neq 0$;
(4) for all $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a \in \mathfrak{p}_\alpha$, $b \notin \mathfrak{p}_\alpha$, $c \in \mathfrak{m}_{p,\alpha}$ or $d \notin \mathfrak{m}_{p,\alpha}$;
(5) for all $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$, $a = 0$, $b \neq 0$, $c \in \mathfrak{m}_{p,\alpha}$ or $d \notin \mathfrak{m}_{p,\alpha}$;
(6) for all $q \in \mathbb{Q}[x]$ irreducible and monic, $a \in \mathfrak{p}_q$, $b \notin \mathfrak{p}_q$, $c \in \mathfrak{p}_q$ or $d \notin \mathfrak{p}_q$;
(7) for all $q \in \mathbb{Q}[x]$ irreducible and monic, $a \in \mathfrak{p}_q$, $b \notin \mathfrak{p}_q$, $c = 0$ or $d \neq 0$;
(8) for all $q \in \mathbb{Q}[x]$ irreducible and monic, $a = 0$, $b \neq 0$, $c \in \mathfrak{p}_q$ or $d \notin \mathfrak{p}_q$; and
(9) $a = 0$ or $b \neq 0$ or $c = 0$ or $d \neq 0$.

One sees this by considering all possible pairs of comparable prime ideals of $\text{Int}(\mathbb{Z})$. The case where one of the prime ideals is maximal is discussed just under the definition of $\mathfrak{p}_\alpha$. If $\mathfrak{p}$ is a non-maximal non-zero prime ideal then $\mathfrak{p} = \mathfrak{p}_q$ for some monic irreducible $q \in \mathbb{Q}[x]$. Now, if $\mathfrak{p}_q \supseteq \mathfrak{p}_{q'}$ for monic and irreducible $q, q' \in \mathbb{Q}[x]$ then $q \in q'\mathbb{Q}[x]$. Hence $q = q'$ and so $\mathfrak{p}_q = \mathfrak{p}_{q'}$.

Define $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5 \in \mathcal{L}_{val}^0$ to be

$$\chi_1 := \forall x\ (v(x) < 0 \vee v(a(x)) \geq 1 \vee v(b(x)) = 0 \vee v(c(x)) \geq 1 \vee v(d(x)) = 0),$$
$$\chi_2 := \forall x\ (v(x) < 0 \vee v(a(x)) \geq 1 \vee v(b(x)) = 0 \vee c(x) = 0 \vee d(x) \neq 0),$$
$$\chi_3 := \forall x\ (v(x) < 0 \vee v(a(x)) \geq 1 \vee v(b(x)) = 0),$$
$$\chi_4 := \forall x\ (v(x) < 0 \vee a(x) = 0 \vee b(x) \neq 0 \vee v(c(x)) \geq 1 \vee v(d(x)) = 0), \text{ and}$$
$$\chi_5 := \forall x\ (v(x) < 0 \vee v(c(x)) \geq 1 \vee v(d(x)) = 0).$$

Claim: $(a, b, c, d) \in \text{DPR}(\text{Int}(\mathbb{Z}))$ if and only if

(i) $\chi_1, \chi_2, \chi_4 \in T_{adic}$,
(ii) either $c = 0$, $d \neq 0$ or $\chi_3 \in T_{adic}$,
(iii) either $a = 0$, $b \neq 0$ or $\chi_5 \in T_{adic}$,
(iv) $ac \in \text{rad}_{\mathbb{Q}[x]}(b\mathbb{Q}[x] + d\mathbb{Q}[x])$,
(v) $a \in \text{rad}_{\mathbb{Q}[x]}(b\mathbb{Q}[x])$ or $c = 0$ or $d \neq 0$,
(vi) either $c = 0$, $d \neq 0$ or $c \in \text{rad}_{\mathbb{Q}[x]}(d\mathbb{Q}[x])$, and
(vii) either $a = 0$, $b \neq 0$, $c = 0$ or $d \neq 0$.

Recall that $a \in \mathfrak{m}_{p,\alpha}$ if and only if $v_p(a(\alpha)) \geq 1$ and $a \in \mathfrak{p}_\alpha$ if and only if $a(\alpha) = 0$. Therefore, for $j \in \{1, 2, 4\}$, (j) holds if and only if $\chi_j \in T_{adic}$, (3) holds if and only if $\chi_3 \in T_{adic}$, $c = 0$ or $d \neq 0$, and, (5) holds if and only if $\chi_5 \in T_{adic}$, $a = 0$ or $b \neq 0$.

The statement that, for all prime ideals $\mathfrak{p} \lhd \text{Int}(\mathbb{Z})$ with either $\mathfrak{p} = 0$ or $\mathfrak{p} = \mathfrak{p}_q$ for some monic irreducible $q \in \mathbb{Q}[x]$, either $a \in \mathfrak{p}$, $c \in \mathfrak{p}$, $b \notin \mathfrak{p}$ or $d \notin \mathfrak{p}$ is equivalent to $ac \in \text{rad}_{\mathbb{Q}[x]}(b\mathbb{Q}[x] + d\mathbb{Q}[x])$. So (6) and (9) hold if and only if (iv) holds. Similarly, (7) and (9) holds if and only if (v) holds and, (8) and (9) holds if and only if (vi) holds. Finally (9) holds if and only if (vii) holds. So the claim holds.

Since T_{adic} is decidable, we can effectively decide whether $(i), (ii)$ and (iii) hold. If $a, b_1, b_2 \in \mathbb{Q}[x]$ then $a \in \mathrm{rad}_{\mathbb{Q}[x]}(b_1\mathbb{Q}[x] + b_2\mathbb{Q}[x])$ if and only if for all $q \in \mathbb{Q}[x]$ irreducible, q divides b_1 and q divides b_2 implies q divides a. Since $\mathbb{Q}$ has a splitting algorithm, there is an algorithm which, given $a, b_1, b_2 \in \mathbb{Q}[x]$, decides whether $a \in \mathrm{rad}_{\mathbb{Q}[x]}(b_1\mathbb{Q}[x] + b_2\mathbb{Q}[x])$. Therefore, we can effectively decide whether (iv)-(vi) holds. It is obvious that we can effectively decide whether (vii) holds. □

In order to analyse $\mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$, we need to understand the valuation overrings of $\mathrm{Int}(\mathbb{Z})$.

- For each $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$ transcendental, define $v_{p,\alpha} : \mathbb{Q}(x) \to \mathbb{Z} \cup \{\infty\}$ by setting $v_{p,\alpha}(f/g) = v_p(f(\alpha)/g(\alpha))$.
- For each $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$ algebraic with monic minimal polynomial $q \in \mathbb{Q}[x]$, define $v_{p,\alpha} : \mathbb{Q}(x) \to \mathbb{Z}\times\mathbb{Z}\cup\{\infty\}$ by setting $v_{p,\alpha}(h) = (k, v_p(f(\alpha)/g(\alpha)))$ where $h = q^k \cdot f/g$, $f(\alpha) \neq 0$ and $g(\alpha) \neq 0$.

By [CC97, VI.1.9], $\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha}}$ is the valuation ring of $v_{p,\alpha}$. Let $e \in \mathrm{Int}(\mathbb{Z})$ and $N \in \mathbb{N}_0$.

- For $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$ transcendental, $v_{p,\alpha}(e) = v_p(e(\alpha)) = N$ if and only if
$$|\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha}}/e\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha}}| = p^N.$$
- For $p \in \mathbb{P}$ and $\alpha \in \widehat{\mathbb{Z}_p}$ algebraic, $v_p(e(\alpha)) = N$ if and only if $v_{p,\alpha}(e) = (0, N)$ if and only if
$$|\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha}}/e\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha}}| = p^N.$$

For the first equivalence of the second bullet point, note that the minimal polynomial q of α divides $e \in \mathrm{Int}(\mathbb{Z})$ if and only if $v_p(e(\alpha)) = \infty$. Thus $v_p(e(\alpha)) = N$ implies $v_{p,\alpha}(e) = (0, N)$. Conversely, if $v_{p,\alpha}(e) = (0, N)$ then $e = q^0 \cdot e$ and $v_p(e(\alpha)) = N$.

Proposition 12.2. *The set* $\mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$ *is recursive.*

Proof. **Claim:** $(p, M; a; \gamma; e, N) \in \mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$ if and only if there exist $h \in \{1, \ldots, N + M\}$, $N_i, M_i \in \mathbb{N}_0$ for $1 \leq i \leq h$ with $\sum_{i=1}^h N_i = N$ and $\sum_{i=1}^h M_i = M$ such that for each $1 \leq i \leq h$,

$$(\dagger) \qquad \mathbb{Q}_p \models \exists x\ (v(x) \geq 0 \wedge v(e(x)) = N_i \wedge v(a(x)) \geq M_i \wedge v(\gamma(x)) = 0).$$

Suppose $(p, M; a; \gamma; e, N) \in \mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$. There exists $\alpha_1, \ldots, \alpha_h$ and $l_1, \ldots, l_h \in \mathbb{N}_0$ such that $\gamma \notin \mathfrak{m}_{p,\alpha_i}$, $a \in \mathfrak{m}_{p,\alpha_i}^{l_i}$,

$$p^{\sum_{i=1}^h l_i} = \prod_{i=1}^h |\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}/\mathfrak{m}_{p,\alpha_i}^{l_i}| = p^M$$

and

$$p^{\sum_{i=1}^h v_p(e(\alpha_i))} = \prod_{i=1}^h |\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}/e\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}| = p^N.$$

We may assume that $h \leq N + M$ since the size of the set of $1 \leq i \leq h$ such that $v_p(e(\alpha_i)) > 0$ or $l_i > 0$ is at most $N + M$. Set $N_i := v_p(e(\alpha_i))$ and $M_i := l_i$. Then $N = \sum_{i=1}^h N_i$ and $M = \sum_{i=1}^h M_i$. Since $a \in \mathfrak{m}_{p,\alpha_i}^{M_i}$, $v_p(a(\alpha_i)) \geq M_i$ and since $\gamma \notin \mathfrak{m}_{p,\alpha_i}$, $v_p(\gamma(\alpha_i)) = 0$, as required.

For the converse, suppose that for each $1 \leq i \leq h$, α_i witness the truth of the sentence (†). Set $\mathfrak{p}_i := \mathfrak{m}_{p,\alpha_i}$ and $I_i := \mathfrak{m}_{p,\alpha_i}^{M_i}$ for $1 \leq i \leq h$. Then $v_p(a(\alpha_i)) \geq M_i$ implies $a \in \mathfrak{m}_{p,\alpha_i}^{M_i}$. Since $v_p(\gamma(\alpha_i)) = 0$, $\gamma \notin \mathfrak{m}_{p,\alpha_i}$ and $v_p(e(\alpha_i)) = N_i$ implies

$$|\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}/e\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}| = p^{N_i}.$$

So $(p, M; a; \gamma; e, N) \in \mathrm{EPP}(\mathrm{Int}(\mathbb{Z}))$ as required.

The proposition now follows from the claim since the set of $(p, \varphi) \in \mathbb{P} \times \mathcal{L}^0_{val}$ such that $\mathbb{Q}_p \models \varphi$ is recursive. □

Proposition 12.3. *The set $X(\mathrm{Int}(\mathbb{Z}))$ is recursive.*

Proof. Recall, 3.3.2, $(p, n; e, \gamma, a, \delta) \in X(\mathrm{Int}(\mathbb{Z}))$ if and only if there exist $h \in \mathbb{N}$ and maximal ideals $\mathfrak{m}_{p,\alpha_i}$ for $1 \leq i \leq h$ such that

$$p^{\sum_{i=1}^h v_p(e(\alpha_i))} = |\oplus_{i=1}^h \mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}/e\mathrm{Int}(\mathbb{Z})_{\mathfrak{m}_{p,\alpha_i}}| = p^n,$$

$\gamma \notin \mathfrak{m}_{p,\alpha_i}$ and for each $1 \leq i \leq h$, either $\delta \notin \mathfrak{m}_{p,\alpha_i}$ or there exists $\mathfrak{q}_i$ a prime ideal such that $\mathfrak{q}_i \subsetneq \mathfrak{m}_{p,\alpha_i}$, $a \in \mathfrak{q}_i$ and $\delta \notin \mathfrak{q}_i$. As in 12.2, we may assume that $1 \leq h \leq n$.

If $\mathfrak{q}$ is a prime ideal strictly contained in $\mathfrak{m}_{p,\alpha}$ then $\mathfrak{q} = \mathfrak{p}_\alpha$ or $\mathfrak{q} = \{0\}$. Thus there exists $\mathfrak{q} \subsetneq \mathfrak{m}_{p,\alpha}$ such that $a \in \mathfrak{q}$ and $\delta \notin \mathfrak{q}$ if and only if $a(\alpha) = 0$ and $\delta(\alpha) \neq 0$, or, $a = 0$ and $\delta \neq 0$. For $\alpha \in \widehat{\mathbb{Z}_p}$, $\gamma \notin \mathfrak{m}_{p,\alpha}$ if and only if $v_p(\gamma(\alpha)) = 0$. Therefore, $(p, n; e, \gamma, a, \delta) \in X(\mathrm{Int}(\mathbb{Z}))$ if and only if there exist $1 \leq h \leq n$ and $N_i \in \mathbb{N}$ for $1 \leq i \leq n$ such that $\sum_{i=1}^h N_i = n$ and for $1 \leq i \leq h$,

$$\mathbb{Q}_p \models \exists x\ (v(x) \geq 0 \wedge v(\gamma(x)) = 0 \wedge v(e(x)) = N_i \\ \wedge [v(\delta(x)) = 0 \vee (a(x) = 0 \wedge \delta(x) \neq 0) \vee (a = 0 \wedge \delta \neq 0)]).$$

So, since the set of $(p, \varphi) \in \mathbb{P} \times \mathcal{L}^0_{val}$ such that $\mathbb{Q}_p \models \varphi$ is recursive, we are done. □

Theorem 12.4. *The theory of modules of the ring of integer valued polynomials with rational coefficients is decidable.*

Proof. This follows from 11.1, 12.1, 12.2 and 12.3. □

Acknowledgement This work was completed while the author was employed by Università degli Studi della Campania "Luigi Vanvitelli" and supported by PRIN 2017-Mathematical Logic: models, sets, computability.

I wish to thank the anonymous referees for carefully reading this article and for many suggestions which have greatly improved the text.

References

[Ax68] J. Ax. The elementary theory of finite fields. *Ann. of Math. (2)*, 88:239–271, 1968.

[Bec82] E. Becker. Valuations and real places in the theory of formally real fields. In *Real algebraic geometry and quadratic forms (Rennes, 1981)*, volume 959 of *Lecture Notes in Math.*, pages 1–40. Springer, Berlin-New York, 1982.

[CC97] P.-J. Cahen and J.-L. Chabert. *Integer-valued polynomials*, volume 48 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[DST19] M. Dickmann, N. Schwartz, and M. Tressl. *Spectral spaces*, volume 35 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2019.

[EF72] P. C. Eklof and E. R. Fischer. The elementary theory of abelian groups. *Ann. Math. Logic*, 4:115–171, 1972.

[EH95] P. C. Eklof and I. Herzog. Model theory of modules over a serial ring. *Ann. Pure Appl. Logic*, 72(2):145–176, 1995.

[FS01] L. Fuchs and L. Salce. *Modules over non-Noetherian domains*, volume 84 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[GLPT18] L. Gregory, S. L'Innocente, G. Puninski, and C. Toffalori. Decidability of the theory of modules over Prüfer domains with infinite residue fields. *J. Symb. Log.*, 83(4):1391–1412, 2018.

[GLT19] L. Gregory, S. L'Innocente, and C. Toffalori. Decidability of the theory of modules over Prüfer domains with dense value groups. *Ann. Pure Appl. Logic*, 170(12):102719, 23, 2019.

[Grä11] G. Grätzer. *Lattice theory: foundation*. Birkhäuser/Springer Basel AG, Basel, 2011.

[Gre15] L. Gregory. Decidability for theories of modules over valuation domains. *J. Symb. Log.*, 80(2):684–711, 2015.

[Hel40] O. Helmer. Divisibility properties of integral functions. *Duke Math. J.*, 6:345–356, 1940.

[Her93] I. Herzog. Elementary duality of modules. *Trans. Amer. Math. Soc.*, 340(1):37–69, 1993.

[Jen66] C. U. Jensen. Arithmetical rings. *Acta Math. Acad. Sci. Hungar.*, 17:115–123, 1966.

[Kap74] I. Kaplansky. *Commutative rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[LTP17] S. L'Innocente, C. Toffalori, and G. Puninski. On the decidability of the theory of modules over the ring of algebraic integers. *Ann. Pure Appl. Logic*, 168(8):1507–1516, 2017.

[PP88] F. Point and M. Prest. Decidability for theories of modules. *J. London Math. Soc. (2)*, 38(2):193–206, 1988.

[PPT07] G. Puninski, V. Puninskaya, and C. Toffalori. Decidability of the theory of modules over commutative valuation domains. *Ann. Pure Appl. Logic*, 145(3):258–275, 2007.

[Pre88] M. Prest. *Model theory and modules*, volume 130 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[Pre09] M. Prest. *Purity, spectra and localisation*, volume 121 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009.

[PT14] G. Puninski and C. Toffalori. Decidability of modules over a Bézout domain $D+XQ[X]$ with D a principal ideal domain and Q its field of fractions. *J. Symb. Log.*, 79(1):296–305, 2014.

[PT15] G. Puninski and C. Toffalori. Some model theory of modules over Bézout domains. The width. *J. Pure Appl. Algebra*, 219(4):807–829, 2015.

[Pun03] G. Puninski. The Krull-Gabriel dimension of a serial ring. *Comm. Algebra*, 31(12):5977–5993, 2003.

[Szm55] W. Szmielew. Elementary properties of Abelian groups. *Fund. Math.*, 41:203–271, 1955.

[Tug03] A. A. Tuganbaev. Distributive rings, uniserial rings of fractions, and endo-Bezout modules. *J. Math. Sci. (N.Y.)*, 114(2):1185–1203, 2003. Algebra, 22.

[Zie84] M. Ziegler. Model theory of modules. *Ann. Pure Appl. Logic*, 26(2):149–213, 1984.

School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK
Email address: `Lorna.Gregory@gmail.com`