Boolean Images Of Connected Spaces

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Abstract

This thesis investigates an interesting generalisation of the concept of continuous functions, namely the notion of Boolean image. This type of image does not preserve connectedness, but otherwise has many of the properties of continuous images. We analyse this notion on many different kinds of topological spaces, deepening our understanding of combinatorial methods from set theory. The main contributions of this thesis can be summarized as follows. In chapter 3, we prove that every compact subspace of 2^κ with a finite support is a Boolean image of a connected space. This main result is followed by some applications in chapters 4 and 5 to different spaces, such as Eberlein compact spaces and Radon-Nikodým spaces, respectively. Chapter 6 centres around the connection between Banach spaces of continuous functions $C(K)$ and $C(L)$, in the case that the spaces L and K are both compact and zero-dimensional. In particular, we prove that if L is a a bijective Boolean image of a compact zero-dimensional space K, then $C(L)$ is isometric to $C(K)$. Moreover, we prove that if L is a Boolean image of K, then $C(L)$ is isometric to a subspace of $C(K)$. On the other hand, we prove that if the Banach spaces $C(K)$ and $C(L)$ are isomorphic, where the spaces K and L are zero-dimensional, then there is a subspace K' in K and a subspace L'

in L such that K' is a Boolean image of L'. In chapter 7, we examine some cardinal functions in terms of the possibility of being transferred via a Boolean image. In this respect, we show that weight and countable density are preserved by Boolean images.

Contents

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1

Introduction, preliminaries and background

This chapter is devoted to an introduction and some definitions and results used in this thesis.

1.1 Introduction

Traditional research in Banach spaces is concentrated on separable spaces, namely spaces with a countable dense subset. In the last 15 or so years much progress has been made in the context of non-separable spaces, where such a countable dense set is not available. This progress was made possible by the use of methods from set theory. We concentrate on one such method, which is the method of Boolean images. It has been shown particularly useful for answering various questions from Banach space theory, as we now explain. The method that we present here works for separable and non-separable spaces.

An important class of Banach spaces are the spaces of the form $C(K)$, i.e., the set of all continuous functions from a compact space K to the set of real numbers $\mathbb{R}$, endowed with the topology of uniform convergence. A question that can be asked is to what extent the space $C(K)$ determines the space K, knowing for example that all uncountable separable metric compact spaces K have the same space $C(K)$, up to isomorphism; this is a theorem of Miljutin (see [31]). With this motivation in mind, Banach space theory is also interested in the class of continuous images of a given compact space, which brings the research back to topology, where, of course, continuous mappings form the basic tool of research—one main reason being of course that continuous mappings preserve various properties of the space, among which is connectedness.

We shall study an interesting generalisation of the concept of continuous image, namely the notion of Boolean image. This type of image does not preserve connectedness, but otherwise has many of the properties of continuous images.

We shall give partial solutions to the following open questions:

Question 1.1.1. *(1) Suppose that κ is an infinite cardinal. Let K be a compact space that is a subset of 2^κ and such that every x from K has support of size at most 3. Is such a space a Boolean image of a connected space?*

(2) Is every Eberlein compact space a Boolean image of a connected space?

(3) Is every Radon-Nikodým compact space a Boolean image of a connected space?

The notion of *support of a point* x in 2^κ is defined by

$$\mathrm{supt}(x) = \{\alpha < \kappa : \, x(\alpha) \neq 0\}.$$

Avilés and Plebanek proved in [11] that every compact subspace of 2^κ which consists of points of support of size ≤ 2 is a Boolean image of a connected space.

Let us now define other notions mentioned in this question. We use the concept of Eberlein compact space, which is defined as follows:

Definition 1.1.2. A compact Hausdorff space is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space (see Definition 1.2.17).

We shall not really use the weak topology, so we do not define it. In fact, we prefer the following combinatorial characterisation of Eberlein compacta. Namely, they are the compact subspaces of a product of the form $\mathbb{R}^\Gamma$, for some index set Γ, where for each $\delta > 0$, there is only a finite number of points whose norm is bigger than δ (see [20]). This characterisation shows the connection between parts (1) and (2) of Question 1.1.1.

The notion of Radon-Nikodým space is defined as follows:

Definition 1.1.3. [29] A compact topological space is called *Radon-Nikodým compact* if it is homeomorphic to a weak* compact subset of the dual of an Asplund space.

Again, we shall not need the weak topology or Asplund spaces, so we prefer a combinatorial characterisation. Radon-Nikodým spaces are the compacta fragmented by some lower semi-continuous metric ([4]). Every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space ([29]), from which it follows that an Eberlein compact space is a Radon-Nikodým compact space. This last aspect represents a connection between parts (2) and (3) of Question 1.1.1.

In the second part of the thesis, we study the connections between Banach spaces of continuous functions $C(K)$ and $C(L)$, in the case that the compact space L is a Boolean image of the compact space K. We will give partial answers to the following two questions:

Question 1.1.4. *(1) If the compact space L is a Boolean image of a compact space K, does this imply that $C(L)$ is isomorphic to $C(K)$?*

(2) If the Banach space $C(L)$ is isomorphic to the Banach space $C(K)$, does this imply that L is a Boolean image of K?

In the third part, we study some cardinal functions. The focus is on how cardinal invariants might be transferred from one topological space to another via a Boolean image.

The thesis is focused on investigating the concept of *Boolean image*. First, in chapter 3, we will prove that every compact subspace of 2^κ with a finite support is a Boolean image of a connected space.

This main result will be followed by some applications to different spaces, such as Eberlein compact spaces and Radon-Nikodým spaces. In chapter 4 we will prove that a strong Eberlein compact space is a Boolean image of a connected space. We will continue by showing that every scattered Eberlein compact space is a Boolean image of a connected space. In chapter 5, we will show that every scattered compact Corson R-N space is a Boolean image of a connected space.

Chapter 6 will contain a proof of the fact that if spaces L and K are both compact zero-dimensional and L is a a bijective Boolean image of K, then $C(L)$ is isometric to $C(K)$. Moreover, we prove that if L is a Boolean image of K, then $C(L)$ is isometric to a subspace of $C(K)$. On the other hand, we will prove that if K and L are zero-dimensional, and $C(K)$ and $C(L)$

are isomorphic as Banach spaces, then there are subspaces K' and L' of K and L, respectively, such that K' is a Boolean image of L'.

The final chapter will examine cardinal functions. We will show that weight and countable density are preserved by Boolean images.

1.2 Preliminaries and background

All topological spaces we consider throughout the thesis are going to be Hausdorff spaces, in most cases compact. No other property is assumed unless explicitly stated.

1.2.1 Some general topology

In this section, we present the classical notions that will be used throughout our work.

Definition 1.2.1. A space X is said to be *compact* if every open covering A of X contains a finite subcollection that also covers X.

Theorem 1.2.2 (see [26], Theorem 26.2)**.** *Every closed subspace of a compact space is compact.*

Definition 1.2.3. A topological space X is *Hausdorff* if for any $x, y \in X$ with $x \neq y$ there exist open sets U containing x and V containing y such that $U \cap V = \varnothing$.

Definition 1.2.4. Let X be a topological space. A *separation of* X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there is no separation of X.

Theorem 1.2.5 ([26], Theorem 23.5)**.** *The image of a connected space under a continuous map is connected.*

Definition 1.2.6. [26] Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be path connected if every pair of points of X can be joined by a path in X.

Definition 1.2.7. A *connected component* of a space X is the maximal connected subset of X, i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X.

Definition 1.2.8. [12] A subset A of a topological space (X, τ) is called a *cozero set* if there is a continuous real-valued function f on X such that $A = \{x \in X : f(x) \neq 0\}$.

Separability is one of the principal topological properties that is used to characterize different topological spaces.

Definition 1.2.9. [33] A topological space X is *separable* if there is some countable subset of X which is dense in X.

Definition 1.2.10. [11] We say that K is *separably connected* if every two points of K are contained in a connected separable subspace of K.

Definition 1.2.11. [30] A space X is called *zero-dimensional* if it is nonempty and has a base consisting of clopen (both open and closed) sets, i.e., if for every point $x \in X$ and for every neighborhood N of x there exists a clopen subset $A \subseteq X$ such that $x \in A \subseteq N$.

Definition 1.2.12. A space is said to be *totally disconnected* if the only connected nonempty sets are the singletons.

Definition 1.2.13. [26] Let X and Y be topological space; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a *homeomorphism*.

Definition 1.2.14. [27] A continuous map $f : X \to Y$ is said to be a *perfect map* if f is closed, surjective, and each fiber $f^{-1}(y)$ is compact in X.

Definition 1.2.15. [17] Let $\mathbb{R}$ denote the real line, Γ an index set, and $\mathbb{R}^\Gamma$ the usual product of $|\Gamma|$ lines. We set

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma \, : \, |\{\alpha \, : \, x(\alpha) \neq 0\}| \leq \omega\}.$$

A compact space X is *Corson compact* if and only if X is homeomorphic to a compact subspace of $\Sigma(\mathbb{R}^\Gamma)$ for some Γ.

1.2.2 Banach spaces

Definition 1.2.16. [22] (Normed space, Banach space). A *normed space* X is a vector space with a norm defined on it. A Banach space is a complete normed space (complete in the metric defined by the norm as follows). Here a norm on a (real or complex) vector space X is a real valued function on X whose value at an $x \in X$ is denoted by

$$\| x \|$$

(read norm of x) and which has the properties

(N1) $\| x \| \geq 0$

(N2) $\| x \| = 0 \Leftrightarrow x = 0$

(N3) $\| \alpha x \| = |\alpha| \, \| x \|$

(N4) $\| x + y \| \leq \| x \| + \| y \|$ (Triangle inequality);

here, x and y are arbitrary vectors in X and α is any scalar.

A norm X defines a metric d on X which is given by

$$d(x, y) = \| x - y \| \qquad (x, y \in X)$$

and is called the metric *induced* by the norm. The normed space just defined is denoted by $(X, \| \, . \, \|)$ or simply by X.

Definition 1.2.17. [19] A closed convex subset C of a Banach space B is *weakly compact* if and only if each continuous linear functional on B attains a maximum on C.

Definition 1.2.18. [38] A Banach space is an *Asplund space* if and only if every separable subspace Y of X has separable dual Y^*.

Definition 1.2.19. [10] Let X be a topological space and let $d : X \times X \to [0, \infty)$ be a metric on X.

- We say that d *fragments* X if for every nonempty subset $Y \subset X$ and for every $\varepsilon > 0$ there exists a nonempty relatively open $V \subset Y$ of d-diameter less than ε, that is,

$$\sup\{d(x,y) \,:\, x, y \in V\} < \varepsilon$$

- We say that d is *lower semicontinuous* if for every $r > 0$, the set

$$\{(x,y) \in X \times X \,:\, d(x,y) \leq r\}$$

  is closed.

A compact space K is *fragmentable* if there exists a metric that fragments it, while it is called *Radon-Nikodým (R-N) compact* if there exists a lower semicontinuous metric that fragments it.

2

The concept of Boolean image

2.1 Motivation

I found an interesting statement in my research journey pointing to the fact that "Topology is really just the study of continuous functions". The concept of continuity is a central idea in topology. If we wish to be able to deform a topological space into another, we need to start considering the relationships between spaces by using the notion of continuous functions. This special type of functions between topological spaces is defined in such a way that some features of the topological structure of the domain space are also preserved in the co-domain space. Continuous functions, also called mappings, provide a means to detect whether two given topological spaces are "topologically equivalent" from the point of view of the topological structure.

2.2 A generalisation of a continuous mapping

In our work, we study a generalisation of the concept of continuous mappings, which is represented by the idea of Boolean image. The notion of Boolean image was first defined by Avilés and Plebanek in [11]. They state that a Boolean image allows to 'lose connectedness'. In a sense, we construct a Boolean image by twisting the definition of a continuous image so as to allow disconnected images of connected spaces. By this we mean that although the Boolean mapping does not preserve connectedness, it has many properties of the continuous mappings. Of course, studying different types of mappings is helpful when considering the relationship between spaces and the possibility of deforming one space into another.

We shall now define the basic objects. We note that the definition below differs from the one in [11] in that we require 'closed under complements' in item (2) of the main definition. This simplifies certain proofs and is equivalent to the original definition. We also need to define preliminary concepts.

Definition 2.2.1. [11]

1. A family $\mathcal{G}$ of clopen sets in some space L *separates the points of* L (or is *point-separating*) if for every two different points $x, y \in L$ there exists $c \in \mathcal{G}$ which contains exactly one among x, y.

2. A *pseudoclopen* in a compact space K is a pair $a = (a^-, a^+)$ such that a^- is a closed subset of K, a^+ is an open subset of K and $a^- \subseteq a^+$.

Definition 2.2.2. [11] Given a family of sets $\mathcal{G}$ and a bijection ϕ from $\mathcal{G}$ onto a family of pseudoclopens of a compact space K, we say that ϕ is an *isomorphism* if for every m, n and any distinct $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{G}$,

$$\text{if } \bigcap_{i\leq n} a_i \setminus \bigcup_{j\leq m} b_j = \varnothing \text{ then } \bigcap_{i\leq n} \phi(a_i)^+ \setminus \bigcup_{j\leq m} \phi(b_i)^- = \varnothing;$$

$$\text{if } \bigcap_{i\leq n} \phi(a_i)^- \setminus \bigcup_{j\leq m} \phi(b_j)^+ = \varnothing \text{ then } \bigcap_{i\leq n} a_i \setminus \bigcup_{j\leq m} b_j = \varnothing.$$

Now for the main definition:

Definition 2.2.3. 1. We define the set-theoretic operations on the family of pseudoclopen pairs by naturally extending the operations from $\mathcal{P}(K)$; for example,

$$(a^-, a^+) \cap (b^-, b^+) = (a^- \cap b^-, a^+ \cap b^+),$$

and similarly for the operation $\cup$. We define the complement by

$$(a^-, a^+)^c = ((a^+)^c, (a^-)^c).$$

2. We say that a compact zero-dimensional space L is a *Boolean image* of a compact space K if there is a family of clopens of L that separates the points of L and which is isomorphic to a family of pseudoclopens of K with the above introduced operations of $\cap, \cup$ and ^{c}, under an isomorphism which assigns (K, K) to L.

Note that if c is a clopen set in a compact space K, then (c, c) is a pseudoclopen. Therefore, the idea of pseudoclopen pairs generalises the concept of clopen sets. We identify a clopen c with the pseudoclopen pair (c, c) that it generates.

We may also invert isomorphisms.

Lemma 2.2.4. *Suppose that a zero-dimensional space L is a Boolean image of a compact space K by an isomorphism φ. Then the inverse of φ is an isomorphism from the clopens of L to the pseudoclopens of K.*

Proof. Since L is a Boolean image of K, there is an isomorphism φ from a family $\{(a_i^-, a_i^+) : i \in I\}$ of pseudoclopens of K to a point-separating family $\{a_i : i \in I\}$ of clopens of L such that $\varphi : (a_i^-, a_i^+) \mapsto a_i$.

So φ preserves the set-theoretic operations on the family of pseudoclopen pairs as follows:

Let (a_1^-, a_1^+), (a_2^-, a_2^+) be pseudoclopens. We have

$\varphi[(a_1^-, a_1^+) \cap (a_2^-, a_2^+)] = \varphi(a_1^-, a_1^+) \cap \varphi(a_2^-, a_2^+)$;

$\varphi[(a_1^-, a_1^+) \cup (a_2^-, a_2^+)] = \varphi(a_1^-, a_1^+) \cup \varphi(a_2^-, a_2^+)$;

$\varphi[(a_1^-, a_1^+)^c] = (\varphi(a_1^-, a_1^+))^c$.

Since φ is a bijection, its inverse $\varphi^{-1} : a_i \mapsto (a_i^-, a_i^+)$ exists and is bijective.

Now we want to prove that φ^{-1} respects the set-theoretic operations on the family of clopens: Let a_1, a_2 be clopens in L.

First, we show that φ^{-1} is closed under intersection:

$\varphi^{-1}(a_1 \cap a_2) = \varphi^{-1}(\varphi(a_1^-, a_1^+) \cap \varphi(a_2^-, a_2^+))$

$= \varphi^{-1}[\varphi((a_1^-, a_1^+) \cap (a_2^-, a_2^+))]$

$= (a_1^-, a_1^+) \cap (a_2^-, a_2^+)$

$= \varphi^{-1}(a_1) \cap \varphi^{-1}(a_2)$

Then, φ^{-1} is closed under union:

$\varphi^{-1}(a_1 \cup a_2) = \varphi^{-1}(\varphi(a_1^-, a_1^+) \cup \varphi(a_2^-, a_2^+))$

$= \varphi^{-1}[\varphi((a_1^-, a_1^+) \cup (a_2^-, a_2^+))]$

$= (a_1^-, a_1^+) \cup (a_2^-, a_2^+)$

$= \varphi^{-1}(a_1) \cup \varphi^{-1}(a_2)$.

φ^{-1} is closed under complement:

$\varphi^{-1}(L \setminus a_1) = \varphi^{-1}(a_1^c) = \varphi^{-1}(\varphi((a_1^c)^-, (a_1^c)^+)))$

$= ((a_1^c)^-, (a_1^c)^+))$. □

The following is given in the paper [11] as Remark 1.6, with a sketch of the proof. We give a detailed proof.

Lemma 2.2.5. *Let the compact space K be zero-dimensional. Then a compact zero-dimensional space L is a Boolean image of K if and only if L is a continuous image of K.*

Proof. $\Longleftarrow$: Suppose that $g \colon K \to L$ is continuous onto function. We want to produce L as a Boolean image which matches the clopens of L to the pseudoclopens of K.

If c is clopen in L, then $g^{-1}(c)$ is, by continuity, clopen in K.
So, $(g^{-1}(c), g^{-1}(c))$ is a pseudoclopen in K. Define

$$\varphi : c \mapsto (g^{-1}(c), g^{-1}(c)).$$

Then $\mathrm{Clop}(L) \approx \{(a, a) \colon a \text{ is clopen in } K\}$. We need to show that φ is an isomorphism. We have, for all c, d that are clopen in L:

φ is closed under intersection:

$\varphi(c \cap d) = (g^{-1}(c \cap d), g^{-1}(c \cap d)) = (g^{-1}(c) \cap g^{-1}(d), g^{-1}(c) \cap g^{-1}(d)) = \varphi(c) \cap \varphi(d)$,

φ is closed under union:

$\varphi(c \cup d) = (g^{-1}(c \cup d), g^{-1}(c \cup d)) = (g^{-1}(c) \cup g^{-1}(d), g^{-1}(c) \cup g^{-1}(d)) = \varphi(c) \cup \varphi(d)$,

φ is closed under complement:

$\varphi(L \setminus c) = (g^{-1}(L \setminus c), g^{-1}(L \setminus c)) = (K \setminus g^{-1}(c), K \setminus g^{-1}(c))$, which is by definition the complement of the pseudoclopen pair $\varphi(c)$.

Clearly the family of all clopen sets in L separates the points of L, so we are done.

$\Longrightarrow$: Suppose that L is a Boolean image of K. We want to show that L is a continuous image of K. Suppose that we are given a family $\mathcal{G}$ of clopens of L which separates the points of L and which is isomorphic by an isomorphism φ to a family $\mathcal{G}'$ of pseudoclopens of K that contains (K, K). Without loss of generality, $\mathcal{G}'$ is closed under finite intersections, since if it is not, we can replace it by its closure under finite intersections and complements. It follows that also $\mathcal{G}$ is closed under these operations. For $d \in \mathcal{G}$ denote $\varphi(d) = (\varphi(d)^-, \varphi(d)^+)$.

Since K is zero-dimensional, for every pseudoclopen pair $\bar{a} = (a^-, a^+)$ there exists a clopen set c such that $a^- \subseteq c \subseteq a^+$. We choose one such set c and call it $c(\bar{a})$, and we make this choice so as to respect the complements, that is $c(\bar{a}^c) = c(\bar{a})^c$. We now claim that there is a continuous onto function $g : K \to L$ such that for every $d \in \mathcal{G}$ we have that $g^{-1}(d) = c^*$ for the unique c^* such that $c(\varphi(d)) = c^*$.

To see this, let us define $g(x) = y$ if y is the unique point in $\bigcap_{d\in\mathcal{G},x\in c(\varphi(d))} d$. We shall show that g is indeed a well-defined continuous function from K to L. First, let us see that for every $x \in K$ there is indeed a point in $\bigcap_{d\in\mathcal{G},x\in c(\varphi(d))} d$. We show that there must be $d \in \mathcal{G}$ such that $x \in c(\varphi(d))$. Otherwise, there would certainly be a $d \in \mathcal{G}$, so by the assumption $x \notin c(\varphi(d))$. Therefore, $x \in c(\varphi(d))^c = c(\varphi(d^c))$ (which is well defined since $\mathcal{G}$ is closed under complements), a contradiction with the assumption. Hence the family $\{d \in \mathcal{G}, x \in c(\varphi(d))\}$ is non-empty and we can take the intersection. Now, if this intersection is empty, then for every $y \in L$ there is $d = d_y \in \mathcal{G}$ such that $y \notin d$ but $x \in c(\varphi(d))$. Consequently, the family $\{d_y^c : y \in L\}$ of clopen sets covers L and hence there must be a finite subset $\{y_0, \ldots, y_n\}$ such that $L = \bigcup_{i\leq n} d_{y_i}^c$. But then, since φ is an isomorphism, we have that $\varphi(\bigcup_{i\leq n} d_{y_i}^c) = (K, K)$ and hence $\varphi(\bigcap_{i\leq n} d_{y_i}) = (\varnothing, \varnothing)$, and so $c(\bigcap_{i\leq n} d_{y_i}) = \varnothing$. Yet, by the choice of $d_y's$ we must have $x \in c(\bigcap_{i\leq n} d_{y_i}) = \bigcap_{i\leq n}(c(d_{y_i}))$. Contradiction.

Now let us show that for any x, the intersection $\bigcap_{d\in\mathcal{G},x\in c(\varphi(d))} d$ can only contain one point. Suppose, for a contradiction, that it contains two different points y and z. Since $\mathcal{G}$ separates points of L, there is $d \in \mathcal{G}$ such that exactly one of the points y, z belongs to d. By the choice of y, z we cannot have that $x \in c(\varphi(d))$ and, therefore, $x \in c(\varphi(d))^c = c(\varphi(d^c))$ (this is well defined since $\mathcal{G}$ is closed under complements). Therefore $y, z \in d^c$, which is a contradiction. So g is a well-defined function.

Now we show that g is continuous. For this, suppose that O is an open subset of L and consider

$$g^{-1}(O) = \{x \in K : g(x) \in O\}.$$

We need to show that $g^{-1}(O)$ is open. Suppose that $x \in g^{-1}(O)$ and let $g(x) = y$. Hence $y \in O$ and, by the choice of g, there is $d^* \in \mathcal{G}$ such that $y \in d^*$ and $x \in c(\varphi(d^*))$. Moreover, by intersecting with further elements d of $\mathcal{G}$ that satisfy $x \in c(\varphi(d))$ if needed and using the fact that O is open in a compact space L, we can find a finite intersection of such elements which is a subset of O. By the closure of $\mathcal{G}$ under finite intersections, we have that such an intersection is itself a member of $\mathcal{G}$: let us abuse the notation and still call it d^*. We claim that $c^* = c(\varphi(d^*)) \subseteq g^{-1}(O)$. Indeed, if $z \in c^*$, then d^* is such that $z \in c(\varphi(d^*))$. Therefore $\bigcap_{d\in\mathcal{G}, z\in c(\varphi(d))} d \subseteq d^* \subseteq O$ and so $g(z) \in O$.

Finally, we show that g is onto. For this, we check that for every $y \in L$,

$$K(y) = \bigcap\{\varphi(d)^- : d \in \mathcal{G}, y \in d\} \setminus \bigcup\{\varphi(d)^+ : d \in \mathcal{G}, y \notin d\} \neq \varnothing.$$

Taking $x \in K(y)$ we have $g(x) = y$, so g is a surjection. □

2.3 More facts about Boolean images

In this section we consider a compact zero-dimensional space L and an arbitrary compact space K.

Proposition 2.3.1. [11] *If L is a Boolean image of K and K is a continuous image of K_0, then L is a Boolean image of K_0. In particular, if L is a Boolean image of K, then L is a continuous image of every zero-dimensional compact space that maps continuously onto the space K.*

Proof. If $f : K_0 \longrightarrow K$ is the continuous onto map and $\{(a_i^-, a_i^+) : i \in I\}$ is a family of pseudoclopens of K that is isomorphic to a point-separating subfamily of $\text{Clop}(L)$, then simply consider the family $\{(f^{-1}[a_i^-], f^{-1}[a_i^+]) \colon i \in I\}$ of pseudoclopens in K_0. The second statement follows from Lemma 2.2.5. □

Boolean images and continuous functions are linked in a way that the existence of the former points to the existence of the latter. Avilés and Plebanek prove in [11] that if there is a Boolean image between topological spaces L and K, then there is a continuous map from a closed subspace K' of K onto L. Their theorem is the following.

Proposition 2.3.2. [11] *If L is a Boolean image of K, then L is a continuous image of a closed subspace of K.*

Proof. Let $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ be an isomorphism of a point-separating family $\mathcal{G} \subset \text{Clop}(L)$ with a family $\mathcal{F}$ of pseudoclopens of K. Consider

$$K_0 = \bigcap_{a \in \mathcal{G}} (\phi(a)^- \cup (K \setminus \phi(a)^+)),$$

and define $f : K_0 \longrightarrow L$ by declaring $f(x)$ to be the only point in

$$L(x) = \bigcap \{a \in \mathcal{G} : x \in \phi(a)^-\} \setminus \bigcup \{a \in \mathcal{G} \, : \, x \notin \phi(a)^+\}.$$

To check that f is well defined, note that we cannot have two different elements in $L(x)$; indeed, take any $y_1, y_2 \in L, y_1 \neq y_2$. Because the family $\mathcal{G}$ separates the points of L we can assume, by symmetry, that $y_1 \in a$ and $y_2 \notin a$ for some $a \in \mathcal{G}$. Since $x \in K_0$, we have either $x \in \phi(a)^-$, giving $L(x) \subset a$ and $y_2 \notin L(x)$, or $x \notin \phi(a)^+$, giving $L(x) \cap a = \varnothing$ and $y_1 \notin L(x)$.

To see that the set $L(x)$ must be nonempty, note that the family

$$\{a \in \mathcal{G} : x \in \phi(a)^- \text{ or } x \notin \phi(a)^+\},$$

has the finite intersection property, as ϕ is an isomorphism. Therefore, by compactness, such a family has a nonempty intersection. Otherwise, by compactness, we would have a finite subset $\mathcal{G}_0 \subset \mathcal{G}$ such that

$$\bigcap\{a \in \mathcal{G}_0 \,:\, x \in \phi(a)^-\} \setminus \bigcup\{a \in \mathcal{G}_0 \,:\, x \notin \phi(a)^+\} = \varnothing.$$

But since ϕ is an isomorphism, that would mean that

$$\bigcap\{\phi(a)^- : a \in \mathcal{G}_0, x \in \phi(a)^-\} \setminus \bigcup\{\phi(a)^+ \,:\, a \in \mathcal{G}_0, x \notin \phi(a)^+\} = \varnothing,$$

which is absurd, since x belongs to that set.

In the same manner we check that for every $y \in L$,

$$K(y) = \bigcap\{\phi(a)^- \,:\, a \in \mathcal{G}, y \in a\} \setminus \bigcup\{\phi(a)^+ \,:\, a \in \mathcal{G}, y \notin a\} \neq \varnothing.$$

Taking $x \in K(y)$ we have $f(x) = y$, so f is a surjection.

Finally, note that if $a \in \mathcal{G}$, then

$$f^{-1}[a] = \phi(a)^- \cap K_0 = \phi(a)^+ \cap K_0$$

is a clopen subset of K_0. Since $\mathcal{G}$ separates the points of L, $\mathcal{G}$ generates $\text{Clop}(L)$ and therefore $f^{-1}[b]$ is clopen for every $b \in \text{Clop}(L)$. Hence f is continuous.

□

Similarly, if we change the definition of Boolean image to be applied to Lindelöf spaces and we require that the countable intersection is preserved by the isomorphism ϕ, then we can make a similar observation and obtain the following proposition:

Proposition 2.3.3. *Suppose that K and L are Hausdorff Lindelöf spaces such that L is a Boolean image of K that preserves countable intersection. Then L is a continuous image of a closed subspace of K.*

Proof. Let $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ be an isomorphism of a point-separating family $\mathcal{G} \subset \mathrm{Clop}(L)$ with a family $\mathcal{F}$ of pseudoclopens of K. Consider

$$K_0 = \bigcap_{a \in \mathcal{G}} (\phi(a)^- \cup (K \setminus \phi(a)^+)),$$

and define $f : K_0 \longrightarrow L$ by declaring $f(x)$ to be only in

$$L(x) = \bigcap\{a \in \mathcal{G} : x \in \phi(a)^-\} \setminus \bigcup\{a \in \mathcal{G} \,:\, x \notin \phi(a)^+\}.$$

To check that f is well defined note first that we can not have two different elements in $L(x)$; indeed, take any $y_1, y_2 \in L$, $y_1 \neq y_2$. Because the family $\mathcal{G}$ separates the points of L we can assume, by symmetry, that $y_1 \in a$ and $y_2 \notin a$ for some $a \in \mathcal{G}$. Since $x \in K_0$, we have either $x \in \phi(a)^-$, giving $L(x) \subset a$ and $y_2 \notin L(x)$, or $x \notin \phi(a)^+$, giving $L(x) \cap a = \varnothing$ and $y_1 \notin L(x)$.

To see that the set $L(x)$ must be nonempty, note that the family

$$\{a \in \mathcal{G} : x \in \phi(a)^- \text{ or } x \notin \phi(a)^+\}$$

has the countable intersection property, as ϕ is an isomorphism, because otherwise, by Lindelöfness, we would have a countable subset $\mathcal{G}_0 \subset \mathcal{G}$ such that

$$\bigcap\{a \in \mathcal{G}_0 \,:\, x \in \phi(a)^-\} \setminus \bigcup\{a \in \mathcal{G}_0 \,:\, x \notin \phi(a)^+\} = \varnothing.$$

But since ϕ is an isomorphism, that would mean that

$$\bigcap\{\phi(a)^- \in \mathcal{G}_0 \, : \, x \in \phi(a)^-\} \setminus \bigcup\{\phi(a)^+ \in \mathcal{G}_0 \, : \, x \notin \phi(a)^+\} = \varnothing,$$

which is a contradiction since x belongs to that set.

In the same manner we check that for every $y \in L$,

$$K(y) = \bigcap\{\phi(a)^- \, : \, a \in \mathcal{G}, y \in a\} \setminus \bigcup\{\phi(a)^+ \, : \, a \in \mathcal{G}, y \notin a\} \neq \varnothing.$$

Taking $x \in K(y)$ we then have $f(x) = y$, so f is onto.

Finally, note that if $a \in \mathcal{G}$, then

$$f^{-1}[a] = \phi(a)^- \cap K_0 = \phi^+(a) \cap K_0$$

is a clopen subset of K_0. Since $\mathcal{G}$ separates the points of L, $\mathcal{G}$ generates $\mathrm{Clop}(L)$ and therefore $f^{-1}[b]$ is clopen for every $b \in \mathrm{Clop}(L)$. Hence f is continuous.

□

However, since the main focus of this study is on compact spaces, we are not pursuing this direction any further.

Proposition 2.3.2 is a pivotal result in our work. By analysing it, we notice that this proposition can be improved in the following way.

Corollary 2.3.4. *If L is a Boolean image of K, then there is a closed subspace K_0 of K that maps onto L by a perfect mapping* (see Definition 1.2.14).

To prove this corollary, we need the following well known proposition.

Proposition 2.3.5. *Any continuous map from a compact space onto a Hausdorff space is a perfect map.*

Proof. First, we need to prove that every continuous map from a compact space to a Hausdorff space is closed: Let $f : X \to Y$ be a continuous function such that X is a compact topological space and Y is a Hausdorff topological space. We need to show that if $A \subset X$ is a closed subset of X, then $f(A) \subset Y$ is a closed subset of Y. Now, since a closed subset of a compact space is compact, it then follows that $A \subset X$ is also compact. Since a continuous images of a compact space is compact, it follows that $f(A) \subset Y$ is compact. Since a compact subspace of a Hausdorff space is closed, it follows that $f(A)$ is also closed in Y.

Then, for a map to be perfect we need to show that the inverse images of points are compact: Since f is continuous and every point is closed, the inverse image of any point is closed and hence compact due to the fact that every closed subset of a compact subspace is compact. □

The next proposition is proved in [11], but we give a detailed proof here.

Proposition 2.3.6. *If L_i is a Boolean image of K_i for $i \in I$, then $L = \prod_{i\in I} L_i$ is a Boolean image of $K = \prod_{i\in I} K_i$.*

Proof. For every i, consider a point-separation family $\mathcal{G}_i \subset \mathrm{Clop}(L_i)$ and a family $\mathcal{F}_i$ of pseudoclopens of K_i that is isomorphic to $\mathcal{G}_i$. Write for every $i \in I$, $\pi_i \colon L \longrightarrow L_i$ and $\mathfrak{p}_i \colon K \longrightarrow K_i$ for the projections, and put

$$\mathcal{G} = \bigcup_{i\in I} \{\pi_i^{-1}[a] \colon a \in \mathcal{G}_i\}, \ \ \mathcal{F} = \bigcup_{i\in I} \{(\mathfrak{p}_i^{-1}[b^-], \mathfrak{p}_i^{-1}[b^+]) \colon b \in \mathcal{F}_i\}.$$

It is sufficient to show that for every $i \in I$, the set $\{\pi_i^{-1}[a] \colon a \in \mathcal{G}_i\}$ is isomorphic, by an isomorphism φ, to the set $\{(\mathfrak{p}_i^{-1}[b^-], \mathfrak{p}_i^{-1}[b^+]) \colon b \in \mathcal{F}_i\}$.

For every c, d that are clopen in $\{\pi_i^{-1}[a] \colon a \in \mathcal{G}_i\}$, let $c = \pi_i^{-1}[a_1]$, $d = \pi_i^{-1}[a_2]$, for some $a_1, a_2 \in \mathcal{G}_i$.

φ is closed under intersection:

$\varphi(c \cap d) = \varphi(\pi_i^{-1}[a_1] \cap \pi_i^{-1}[a_2]) = \varphi(\pi_i^{-1}[a_1]) \cap \varphi(\pi_i^{-1}[a_2]) = \varphi(c) \cap \varphi(d)$

φ is closed under union:

$\varphi(c \cup d) = \varphi(\pi_i^{-1}[a_1] \cup \pi_i^{-1}[a_2]) = \varphi(\pi_i^{-1}[a_1]) \cup \varphi(\pi_i^{-1}[a_2]) = \varphi(c) \cup \varphi(d)$

φ is closed under complement:

$\varphi(L \setminus c) = \varphi(L \setminus \pi_i^{-1}[a_1]) = \varphi(L) \setminus \varphi(\pi_i^{-1}[a_1]) = K \setminus (\mathfrak{p}_i^{-1}[b^-], \mathfrak{p}_i^{-1}[b^+])$, which is by definition the complement of a pseudoclopen pair. □

The following corollary was stated without a proof in [11]. So we give the proof.

Corollary 2.3.7. *For every infinite set* Γ *the space* 2^Γ *is a Boolean image of* $[0,1]^\Gamma$.

Proof. This follows since $2 = \{0,1\}$ is a Boolean image of $[0,1]$. We need to find a family of pseudoclopens in $[0,1]$ which is isomorphic to $\{\{0\},\{1\}\}$ (these are all clopens of $\{0,1\} = 2$, and it is of course a point-separating family).

So we want to find pseudoclopens $(a^-, a^+), (b^-, b^+)$ in $[0,1]$. We choose $a^- = [0, \frac{1}{3}]$ as a closed subset of $[0,1]$ and $a^+ = [0, \frac{1}{2})$ as an open subset of $[0,1]$ and such that $a^- \subseteq a^+$. Also, we choose $b^- = [\frac{1}{2}, 1]$, which is a closed subset of subset of $[0,1]$, and $b^+ = (\frac{1}{3}, 1]$, which is an open subset such that $b^- \subseteq b^+$.

Let $\varphi : (a^-, a^+) \to \{0\}$

$$(b^-, b^+) \to \{1\}$$

We can close φ under intersection, union and complement, so it becomes an isomorphism. □

Theorem 2.3.8. [11] *If K is separably connected and L is a Boolean image of K, then either L is Corson compact or L maps continuously onto 2^{ω_1}.*

Corollary 2.3.9. [11] *If K is a separably connected space that does not map onto $[0,1]^{\omega_1}$, then every Boolean image of K is Corson compact.*

Proof. Otherwise, K has a Boolean image that maps onto 2^{ω_1}. By Proposition 2.3.2, this implies that K has a closed subspace that maps continuously onto 2^{ω_1}, hence also onto $[0,1]^{\omega_1}$. By Tietze's extension theorem: (continuous functions on a closed subset of a normal topological space can be extended to the entire space), we have that K itself maps continuously onto $[0,1]^{\omega_1}$. □

Corollary 2.3.10. [11] Let L be a zero-dimensional compact space which is a continuous image of a zero-dimensional compact line (compact linearly ordered topological space) L^*. Then L is a Boolean image of a compact connected space.

2.4 A few more examples

1. An adequate space:

   The notion of an adequate family of subsets was introduced in [1].

   Definition 2.4.1. Let Γ be a non-empty set, a family $\mathcal{A}$ of subsets of Γ is called *adequate* if:

(i) If $A \in \mathcal{A}$, and $B \subset A$, then $B \in \mathcal{A}$.

(ii) If $B \subset \Gamma$ is such that all finite subsets of B belongs to $\mathcal{A}$, then $B \in \mathcal{A}$.

Every adequate family of subsets of Γ can be viewed as a closed subset of $\{0,1\}^\Gamma$. Such a family may be seen as a compact space $L = \{x \in 2^\Gamma : \{\gamma : x_\gamma = 1\} \in \mathcal{A}\}$ and is therefore called an *adequate compact space.*

Proposition 2.4.2. [11] *Every space L defined by an adequate family is a Boolean image of a path-connected compact space.*

2. Tree spaces:

Tree spaces constitute another interesting example. First, the definition of a *tree* should be recalled:

Definition 2.4.3. [11] A tree is a partially ordered set T with a minimum element (its root) and in which each initial segment

$$\{t \in T : t < s\}$$

is well ordered.

Remark 2.4.4. [11] *Let $\bar{T}$ be a family of subsets of T that are either initial segments or full branches in T. Then $\bar{T}$ is again a tree when ordered by inclusion. In a sense, it represents the completion of T.*

The tree space is a classical construction. The definition can be found in [11]:

Definition 2.4.5. A *tree space associated to T* is the compact subspace of 2^T made of the characteristic functions of subsets $A \in \bar{T}$ (for some $\bar{T}$).

Proposition 2.4.6. [11] Let T be a tree and let $L \subset 2^T$ be the tree space defined by T, that is, for every $x \in L$ we have that $\{t \in T : x_t = 1\} \in \bar{T}$. Then L is a Boolean image of a compact connected space. If all branches of T are countable, then L is a Boolean image of a path-connected space.

3. Boolean images of convex spaces:

   Avilés and Plebanek in [11] give a result on the class of compact spaces defines by an n-adequate family. They prove that such spaces are Boolean images of convex compacta.

   We need to recall some definitions as follows:

Definition 2.4.7. [11] Given a natural number n, an adequate family $\mathcal{A}$ of sets is *n-adequate* if a set belongs to $\mathcal{A}$ if and only if all of its subsets of cardinality at most n belong to $\mathcal{A}$.

Definition 2.4.8. A set C is *convex* if the line segment between any two points in C lies in C, i.e. $\forall x_1, x_2 \in C, \forall\, \phi \in [0,1]$,

$$\phi\, x_1 + (1-\phi)\, x_2 \in C.$$

We can generalize the definition of a convex set above from two points to any number of points n. A convex combination of points $x_1,\, x_2, \ldots, x_k \in C$ is any point of the form

$$\phi_1\, x_1 + \phi_2\, x_2 + \ldots + \phi_k\, x_k,$$

where $\phi_i \geq 0$, for $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \phi_i = 1$. Then a set C is convex iff any convex combination of points in C is in C.

Proposition 2.4.9. [11] *Every compact space L defined by an*

n-adequate family for some n is a Boolean image of a compact convex set.

2.5 A partial answer to a problem posed by Avilés and Plebanek

The following problems are posed by Avilés and Plebanek in [11].

Problem 2.5.1. *(i) Suppose that L is a continuous image of every zero-dimensional space that maps continuously onto K. Does this imply that L is a Boolean image of K ?*

(ii) Let L be a Boolean image of K and suppose that L' is a continuous image of L. Is L' a Boolean image of K ?

We are able to answer the question 2.5.1(ii) under the additional assumption that L' is the image of L by a bijective continuous mapping. This implies that the spaces are homeomorphic, since they are both compact. Hence the answer is not surprising, but it is still not completely trivial since being a Boolean image is not something that is given by the existence of a function between spaces, so we cannot just compose such a function with a homeomorphism.

Proposition 2.5.2. *Let L be a Boolean image of K and suppose that L' is a bijective continuous image of L. Then L' is a Boolean image of K.*

Proof. Since L is a Boolean image of K, let $\{(a_i^-, a_i^+) : i \in I\}$ be a family of pseudoclopens of K that is isomorphic by an isomorphism ψ to a family $\{a_i : i \in I\}$ of clopens of L and that separates the points of L so that $\psi : (a_i^-, a_i^+) \mapsto a_i$.

To show that L' is a zero-dimensional space:

Let $f : L \to L'$ be a continuous bijective function. Since L is a compact space and f is continuous, L' is compact.

We know that any bijective continuous function between compact spaces is a homeomorphism. Thus, f is a homeomorphism, which means that it is closed and open. Since closed and open continuous functions preserve zero-dimensionality, L' is a zero-dimensional space.

To prove that L' is a Boolean image of the space K, we need to find clopen sets in L' which separate the points of L' and are isomorphic to the pseudoclopens in K. To do so, we will take the sets $f(a_i)$. These sets are clopen, as we shall now show:

We know that a_i is clopen in L. Since a_i is a closed subset of a compact space L and the function f is continuous, the image $f(a_i)$ is closed. Since the set a_i is open and the function f is open, the set $f(a_i)$ is open. So, $f(a_i)$ is clopen in L'.

Now, we want to prove that there is an isomorphism φ between the family $\{f(a_i) : i \in I\}$ of clopen sets in L' and the family (a_i^-, a_i^+) of pseudoclopens of K. Define $\varphi = f\psi$, which takes the psedoclopen pair (a_i^-, a_i^+) in K to the clopen sets $f(a_i)$ in L'.

For any pseudoclopens c and d in K such that $c = (a_1^-, a_1^+)$ and $d = (a_2^-, a_2^+)$, we have the following:

φ is closed under intersection:

$\varphi(c \cap d) = \varphi[(a_1^-, a_1^+) \cap (a_2^-, a_2^+)]$

$= f\psi[(a_1^-, a_1^+) \cap (a_2^-, a_2^+)]$

$= f(\psi(a_1^-, a_1^+) \cap \psi(a_2^-, a_2^+))$

$= f\psi(a_1^-, a_1^+) \cap f\psi(a_2^-, a_2^+)$

$= \varphi(a_1^-, a_1^+)] \cap \varphi(a_2^-, a_2^+))$

φ is closed under union:

$\varphi(c \cup d) = \varphi[(a_1^-, a_1^+) \cup (a_2^-, a_2^+)]$

$= f\psi[(a_1^-, a_1^+) \cup (a_2^-, a_2^+)]$

$= f(\psi(a_1^-, a_1^+) \cup \psi(a_2^-, a_2^+))$

$= f\psi(a_1^-, a_1^+) \cup f\psi(a_2^-, a_2^+)$

$= \varphi(a_1^-, a_1^+)] \cup \varphi(a_2^-, a_2^+))$

φ is closed under complement:

$\varphi(K \setminus c) = \varphi(K \setminus (a_i^-, a_i^+)) = f\psi(K \setminus (a_i^-, a_i^+)) = f\psi(K) \setminus f\psi(a_i^-, a_i^+) = L' \setminus f(a_i)$, which is by definition the complement of the clopen pair $\varphi(c)$.

Next, we want to prove that the family $\{f(a_i), i \in I\}$ of clopen sets of L' separates the points of L': this means that for every two points x, y in L', there exists $c \in \{f(a_i) : i \in I\}$ which contains exactly one among x, y.

Let $x \neq y$ in L'. Since $f : L \to L'$ is onto, there are z, $w \in L$ such that $f(z) = x$ and $f(w) = y$.

Let a_j be a clopen set and suppose that $z \in a_j$ and $w \notin a_j$ (since $(a_i)_{i \in I}$ separates the points in L). Since f is a bijective function, we have $z \in a_j \Rightarrow f(z) \in f(a_j) \Rightarrow x \in f(a_j)$.

But $y \notin f(a_j)$ because f is one-to-one and $y = f(w)$.

Therefore, for every two different points $x \neq y \in L'$, there exists $f(a_j) \in \{f(a_i) : i \in I\}$ that contains exactly one among x, y. □

Problem 2.5.1(i) similarly has a positive solution under the same assumptions, even dropping the requirement 'for every 0-dimensional space' and replacing it by 'for some 0-dimensional space'. This again is not surprising and we omit the proof. So we have the following proposition:

Proposition 2.5.3. *If the zero-dimensional space L is a continuous image of every zero-dimensional compact space that maps continuously by a bijective map onto a compact space K, then L is a Boolean image of K.*

Proof. Suppose that K_j are zero-dimensional spaces that map continuously onto a compact K. Since the spaces K_j are zero-dimensional and L is a continuous image of every K_j, using Lemma 2.2.5 we have that L is a Boolean image of every K_j.

Consequently, for every j there is a family $\{a_{i_j} : i_j \in I_j\}$ of clopens of L that separates the points of L and which is isomorphic to a family $\{(a^-_{i_j}, a^+_{i_j}) : i \in I_j\}$ of pseudoclopens in the space K_j by an isomorphism $\varphi_j : (a^-_{i_j}, a^+_{i_j}) \to a_{i_j}$.

Now let $f_j : K_j \to K$ be a continuous bijective function for every j. Since the sets $a^-_{i_j}$ are closed in the compact space K_j, f_j is continuous, so $f_j(a^-_{i_j})$ are closed in K.

Since the sets $a^+_{i_j}$ are open in K_j, $f_j(a^+_{i_j})$ is open in K. Also, $f_j(a^-_{i_j}) \subseteq f_j(a^+_{i_j})$ (since $a^-_{i_j} \subseteq a^+_{i_j}$). Therefore, the pair $(f(a^-_{i_i}), f(a^+_{i_i}))$ is a pseudoclopen in the compact space K.

So we want to prove that there is an isomorphism ψ between all families $\{a_{i_j} : i \in I_j\}$ of clopens of L that separate the points of L and the families $\{(f_j(a^-_{i_j}), f_j(a^+_{i_j})) : i \in I_j\}$ of pseudoclopens of K. Define $\psi = \varphi_j f_j^{-1}$, which takes the pseudoclopen pair $(f_j(a^-_{i_j}), f_j(a^+_{i_j}))$ in K to the clopen a_{i_j} in L. We have, for $(f_j(a^-_{i_1}), f_i(a^+_{i_1}))$ and $(f_j(a^-_{i_2}), f_j(a^+_{i_2}))$ that are pseudoclopens in K,

ψ is closed under intersection:

$\psi[(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cap (f_j(a_{i_2}^-), f_j(a_{i_2}^+))]$

$= \varphi_j f_j^{-1}[(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cap (f_j(a_{i_2}^-), f_j(a_{i_2}^+))]$

$= \varphi_j f_j^{-1}(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cap \varphi_j f_j^{-1}(f_j(a_{i_2}^-), f_j(a_{i_2}^+))$

$= \psi(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cap \psi(f_j(a_{i_2}^-), f_j(a_{i_2}^+))$.

ψ is closed under union:

$\psi[(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cup (f_j(a_{i_2}^-), f_j(a_{i_2}^+))]$

$= \varphi_j f_j^{-1}[(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cup (f_j(a_{i_2}^-), f_j(a_{i_2}^+))]$

$= \varphi_j f_j^{-1}(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cup \varphi_j f_j^{-1}(f_j(a_{i_2}^-), f_j(a_{i_2}^+))$

$= \psi(f_j(a_{i_1}^-), f_j(a_{i_1}^+)) \cup \psi(f_j(a_{i_2}^-), f_j(a_{i_2}^+))$.

ψ is closed under complement:

$\psi(K \setminus (f_j(a_{i_1}^-), f_j(a_{i_1}^+))) = \varphi_j f_j^{-1}(K \setminus (f_j(a_{i_1}^-), f_i(a_{i_1}^+)))$

$= \varphi_j(f_j^{-1}(K) \setminus f_j^{-1}(f_j(a_{i_j}^-), f_j(a_{i_j}^+))) = \varphi_j(k_j \setminus ((a_{i_1}^-), (a_{i_1}^+))$

$= L \setminus \varphi_j((a_{i_1}^-), (a_{i_1}^+)) = L \setminus (a_{i_1})$, which is by definition the complement of a clopen set.

It remains to prove that $\psi(K, K) = L$:

$\psi(K, K) = \psi(f_j[(a_{i_j}^-, a_{i_j}^+) \cup (a_{i_j}^-, a_{i_j}^+)^c)]$ such that $(a_{i_j}^-, a_{i_j}^+) = \varnothing, (a_{i_j}^-, a_{i_j}^+)^c = K$

$= \varphi_j f_j^{-1} f_j[((a_{i_j}^-), (a_{i_j}^+)) \cup ((a_{i_j}^-), (a_{i_j}^+))^c]$

$= \varphi_j[((a_{i_j}^-), (a_{i_j}^+)) \cup ((a_{i_j}^-), (a_{i_j}^+))^c]$

$= \varphi_j[K_j]$

$= L.$

□

3

Function spaces with finite support

The concept of Boolean image represents a generalisation of a continuous image in the sense that there can be a zero-dimensional space constituting a Boolean image of a connected one. This cannot happen under continuous mappings. In fact, a Boolean image is not a function between the points of one space and the points of another space, which makes it different from a continuous map. But, as the results in this project show, Boolean images seem to behave quite like continuous maps.

This research develops a relatively new concept which has emerged from the Avilés and Plebanek paper [11]. They discuss the Boolean images of different kinds of spaces, among them being the compact zero-dimensional subspaces of

$$\sigma_n(\Gamma) = \{x \in 2^\Gamma : |\{\gamma : x_\gamma = 1\}| \leq n\} \text{ for } n = 1, 2, 3, \ldots$$

for an uncountable set Γ. A question arises if such spaces are Boolean images of connected spaces. Avilés and Plebanek prove this for $n = 2$ in the following proposition:

Proposition 3.0.1. [11] *If L is a compact subspace of $\sigma_2(\Gamma)$, then L is a Boolean image of a compact connected space.*

No study to date has examined the case when $n = 3$. To do that, we first reprove that for a compact subspace K of 2^κ such that every x from K has support of size equal to 1 or 2 is a Boolean image of a connected space. Then we prove the main result of this chapter stating that every subspace of 2^κ with a finite support n is a Boolean image of a connected space, which is given in the following theorem:

Theorem 3.0.2. *Let $n = 1, 2, 3, \ldots$, κ is an uncountable set and $X_n = \{f \in 2^\kappa : |supt(f)| \leq n\}$. Then X_n is a Boolean image of $[0, 1]^\kappa$.*

Section 3.1 is devoted to that.

3.1 An answer to another question posed by Avilés and Plebanek

Along with the aim announced in the previous section, the aim of this section is to study the spaces $C(K)$ by studying the spaces K where K is a compact Hausdorff space. A basic example of a compact space is 2^κ, where

$$2^\kappa = \{f : \kappa \to 2 = \{0, 1\}\}$$

and where κ is an infinite cardinal, along with its closed subspaces. Of special interest are the subspaces that have a property of every point having finite support. This means that every point (function) is equal to zero except at finitely many $\gamma \in \kappa$. First, we will prove that these spaces are closed subspaces of 2^κ, implying that they are compact. Then, we will show that these spaces are Boolean images of connected spaces.

The first question to ask concerns the space Y_1, which is the subspace of 2^κ consisting of all $x \in 2^\kappa$ with support of size 1.

Problem 3.1.1. *Is the subspace Y_1 of 2^κ consisting of functions $f \in 2^\kappa$ with $|supt(f)| = 1$ compact ?*

Observation 3.1.2. *Let κ be an infinite cardinal and let Y_1 be the subspace of 2^κ consisting of all functions f_α with $|supt(f_\alpha)| = 1$. We can write it as the set of all functions f_α, for $\alpha < \kappa$, such that* $f_\alpha(\beta) = \begin{cases} 0 & \text{if } \beta \neq \alpha. \\ 1 & \text{if } \beta = \alpha. \end{cases}$

We claim that this space is not compact: recall that Y_1 is compact if every open cover of it has a finite subcollection that also covers Y_1. Fix $\alpha < \kappa$, and note that $[(\alpha, 1)] \cap Y_1 = \{f \in Y_1 : f(\alpha) = 1\} = \{f_\alpha\}$, so this is a relatively open set in Y_1 and $\mathcal{O} = \{[(\alpha, 1)] \cap Y_1 : \alpha < \kappa\}$ is an open cover of Y_1. Suppose that $\mathcal{F} \subseteq \mathcal{O}$ is finite, so
$\mathcal{F} = \{[(\alpha_0, 1)] \cap Y_1, [(\alpha_1, 1)] \cap Y_1, \ldots, [(\alpha_n, 1)] \cap Y_1\}$ for some $\alpha_0, \alpha_1, \ldots, \alpha_n < \kappa$. If $\alpha < \kappa$ and $\alpha \notin \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, then $f_\alpha \notin \mathcal{F}$. Hence $\mathcal{F}$ is not an open cover of Y_1. This argument shows that Y_1 is not compact and hence it is not a closed subspace of 2^κ.

Now, let X_1 be the set of functions in 2^κ which take value zero except at *at most* one point.

Proposition 3.1.3. *The space $X_1 = Y_1 \cup \{0\} = \{f \in 2^\kappa : |\text{supt}(f)| \leq 1\}$ is a compact subspace of 2^κ.*

Proof. To prove that X_1 is a compact subspace, it suffices to show that it is a closed subspace of 2^κ . We show that its complement $2^\kappa \backslash X_1$ is open.

$2^\kappa \backslash X_1$ is the set of all functions that are equal to 1 at least twice,

$$2^\kappa \backslash X_1 = \{f \in 2^\kappa : \exists \alpha_1 \neq \alpha_2, f(\alpha_1) = f(\alpha_2) = 1\}.$$

It is open because for every point $x \in 2^\kappa \backslash X_1$, there is an open set: $[(\alpha_1, 1), (\alpha_2, 1)] = \{g \in 2^\kappa : g(\alpha_1) = g(\alpha_2) = 1\}$ containing x and contained in $2^\kappa \backslash X_1$. Therefore the space X_1 is a closed subspace of 2^κ, so it is compact.

□

Similarly, if we take Y_2 to be a subspace of 2^κ such that every point x from Y_2 has support exactly equal to 2, then Y_2 is not a compact space. Namely, $Y_2 = \{f \in 2^\kappa : |\mathrm{supt}(f)| = 2\} = \{f \in 2^\kappa : \exists \alpha_1 \neq \alpha_2, f(\alpha_1) = f(\alpha_2) = 1\}$. Fix $\alpha_1, \alpha_2 < \kappa$. Let $[(\alpha_1, 1), (\alpha_2, 1)] \cap Y_2 = \{f \in Y_2 : f(\alpha_1) = f(\alpha_2) = 1\} = \{f_{\alpha_1,\alpha_2}\}$, so this is a relatively open set in Y_2. Then $\mathcal{O} = \{[(\alpha_1, 1), (\alpha_2, 1)] \cap Y_2 : \alpha_1, \alpha_2 < \kappa\}$ is an open cover of Y_2. Now suppose that $\mathcal{F} \subseteq \mathcal{O}$ is finite such that $\mathcal{F} = \{[(\beta_1^1, 1), (\beta_2^1)] \cap Y_2, \ldots, [(\beta_1^n, 1), (\beta_2^n, 1)] \cap Y_2\}$ for some $\beta_1, \ldots, \beta_n < \kappa$. If $\alpha_1, \alpha_2 < \kappa$, and $\alpha_1, \alpha_2 \notin \{\beta_1, \ldots, \beta_n\}$, then $\{f_{\alpha_1,\alpha_2}\} \notin \mathcal{F}$, which gives a point in Y_2 which is not covered by $\mathcal{F}$. So, Y_2 is not a compact subspace of 2^κ.

But again if we choose our space to contain all the functions that have support of size *at most* 2, we will obtain a compact subspace of 2^κ. Therefore, let X_2 be the set of functions that take value zero except at most two points.

Proposition 3.1.4. *The space* $X_2 = Y_2 \cup Y_1 \cup \{0\} = \{f \in 2^\kappa : |\mathrm{supt}(f)| \leq 2\}$ *is a closed subspace of* 2^κ.

Proof. to prove that X_2 is a closed subspace, we prove that its complement is open. We have that $2^\kappa \backslash X_2$ is the set of all functions that are equal to 1 at least three times:

$$2^\kappa \backslash X_2 = \{f \in 2^\kappa : \exists \alpha_1 \neq \alpha_2 \neq \alpha_3, f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = 1\}.$$

This space is open because for every point $x \in 2^\kappa \backslash X_2$, there is an open set: $[(\alpha_1, 1), (\alpha_2, 1), (\alpha_3, 1)] = \{g \in 2^\kappa : g(\alpha_1) = g(\alpha_2) = g(\alpha_3) = 1\}$ containing x and contained in $2^\kappa \backslash X_2$. Therefore, the space X_2 is a closed subspace of 2^κ, so it is compact. □

Arguing as above, we conclude that for each n,

$$X_n = Y_n \cup Y_{n-1} \cup \ldots \cup Y_1 \cup \{0\} = \{f \in 2^\kappa : |\mathrm{supt}(f)| \leq n\}$$

is a closed subspace of 2^κ, where Y_n is defined as the set of all points in 2^κ which have support consisting of exactly n points.

Theorem 3.1.5. *Each X_n is a Boolean image of a connected space.*

We first deal with the case $n = 1$. We recall:

Definition 3.1.6. (1) Let $X_1 = \{x \in 2^\kappa : |\{\alpha : x(\alpha) \neq 0\}| \leq 1\}$.

(2) For each $\alpha < \kappa$, let $[\langle \alpha, 1 \rangle] = \{x \in 2^\kappa : x(\alpha) = 1\}$.

Lemma 3.1.7. *X_1 is a Boolean image of $[0,1]^\kappa$.*

Proof. We observe that X_1 is a zero-dimensional compact space. Let x_{-1} be the identically 0 function and x_α the unique point in the intersection $X_1 \cap [\langle \alpha, 1 \rangle]$. Then $X_1 = \{x_\alpha : \alpha \geq -1\}$.

Let $\mathcal{C} = \{X_1 \cap [\langle \alpha, 1 \rangle] : \alpha < \kappa\} \cup \{X_1 \cap [\langle 0, 0 \rangle]\}$. It is easily checked that $\mathcal{C}$ is a family of clopens of X_1 which separates the points of X_1. Let $\mathcal{C}^*$ be the closure of $\mathcal{C}$ under $\cap, \cup$ and ^{c}.

For $\alpha < \kappa$ define $a_\alpha^- = \{x \in [0,1]^\kappa : x(\alpha) = 1\}$ and let $a_\alpha^+ = [0,1]^\kappa$. Then (a_α^-, a_α^+) is pseudoclopen in $[0,1]^\kappa$. Also, we define $a^- = \{x \in [0,1]^\kappa : x = 0\}$ and let $a^+ = [0,1]^\kappa$. Then (a^-, a^+) is pseudoclopen in $[0,1]^\kappa$. We define $\varphi((a_\alpha^-, a_\alpha^+)) = X_1 \cap [\langle \alpha, 1 \rangle]$ and $\varphi((a^-, a^+)) = X_1 \cap [\langle 0, 0 \rangle]$.

Then we extend this to an isomorphism by defining

$$\varphi((a_\alpha^-, a_\alpha^+) \cap (a_\beta^-, a_\beta^+)) = \varphi(a_\alpha^- \cap a_\beta^-, a_\alpha^+ \cap a_\beta^+)$$

and similarly for $\cup$ and ^{c} and (a^-, a^+). This is a mapping onto $\mathcal{C}^*$.

It remains to prove that $\varphi([0,1]^\kappa, [0,1]^\kappa) = X_1$.
By the definition of the extension using complements, we have $\varphi([0,1]^\kappa, [0,1]^\kappa) = \varphi((a_\alpha^-, a_\alpha^+) \cup (a_\alpha^-, a_\alpha^+)^c \cup (a^-, a^+) \cup (a^-, a^+)^c)$

$= X_1 \cap [\langle \alpha, 1 \rangle] \cup (X_1 \cap [\langle \alpha, 1 \rangle])^c \cup X_1 \cap [\langle 0, 0 \rangle] \cup (X_1 \cap [\langle 0, 0 \rangle])^c = X_1$. □

Now we are able to prove that every compact subspace of 2^κ where every point has support of size at most 2 is a Boolean image of a connected space

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Definition 3.1.8. (1) Let $X_2 = \{x \in 2^\kappa : |\{\alpha : x(\alpha) \neq 0\}| \leq 2\}$.

(2) For each $\alpha_1, \alpha_2 < \kappa$, let $[\langle \alpha_1, 1 \rangle, \langle \alpha_2, 1 \rangle] = \{x \in 2^\kappa : x(\alpha_1) = x(\alpha_2) = 1\}$.

Lemma 3.1.9. *X_2 is a Boolean image of $[0,1]^\kappa$.*

Proof. We observe that X_2 is a zero-dimensional compact space. Let x_{-1} be the identically 0 function and x_{α_1,α_2} the unique point in the intersection $X_2 \cap [\langle \alpha_1, 1 \rangle, \langle \alpha_2, 1 \rangle]$. Then

$$X_2 = X_1 \cup \bigcup_{\alpha_1,\alpha_2<\kappa} \{x \in 2^\kappa : x(\alpha_1) = x(\alpha_2) = 1, \, x(\alpha) = 0 \text{ if } \alpha \neq \alpha_1, \alpha_2\}$$

Let $\mathcal{C} = \{X_2 \cap [\langle\alpha_1, 1\rangle, \langle\alpha_2, 1\rangle] : \alpha_1, \alpha_2 < \kappa\} \cup \{X_2 \cap [\langle 0, 0\rangle]\}$ (α_1, α_2 might be equal). It is easily checked that $\mathcal{C}$ is a family of clopens of X_2 which separates the points of X_2. Let $\mathcal{C}^*$ be the closure of $\mathcal{C}$ under $\cap, \cup$ and ^{c}.

For α_1, $\alpha_2 < \kappa$, define $a^-_{\alpha_1,\alpha_2} = \{x \in [0,1]^\kappa : x(\alpha_1) = x(\alpha_2) = 1\}$ and let $a^+_{\alpha_1,\alpha_2} = [0,1]^\kappa$. Then $(a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2})$ is pseudoclopen in $[0,1]^\kappa$. Also, we define $a^- = \{x \in [0,1]^\kappa : x = 0\}$ and let $a^+ = [0,1]^\kappa$. Then (a^-, a^+) is pseudoclopen in $[0,1]^\kappa$.

We define $\varphi((a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2})) = X_2 \cap [\langle\alpha_1, 1\rangle, \langle\alpha_2, 1\rangle]$ and $\varphi((a^-, a^+)) = X_2 \cap [\langle 0, 0\rangle]$. Then we extend this to an isomorphism by defining

$$\varphi((a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2}) \cap (a^-_{\beta_1,\beta_2}, a^+_{\beta_1,\beta_2})) = \varphi(a^-_{\alpha_1,\alpha_2} \cap a^-_{\beta_1,\beta_2}, a^+_{\alpha_1,\alpha_2} \cap a^+_{\beta_1,\beta_2})$$

and similarly for $\cup$ and ^{c} and (a^-, a^+). This is a mapping onto $\mathcal{C}^*$.

It remains to prove that $\varphi([0,1]^\kappa, [0,1]^\kappa) = X_2$. By the definition of the extension using complements, we have

$$\varphi([0,1]^\kappa, [0,1]^\kappa) = \varphi((a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2}) \cup (a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2})^c \cup (a^-, a^+) \cup (a^-, a^+)^c)$$

$$= \varphi(a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2}) \cup \varphi(a^-_{\alpha_1,\alpha_2}, a^+_{\alpha_1,\alpha_2})^c \cup \varphi(a^-, a^+) \cup \varphi(a^-, a^+)^c$$

$$= X_2 \cap [\langle\alpha_1, 1\rangle, \langle\alpha_2, 1\rangle] \cup (X_2 \cap [\langle\alpha_1, 1\rangle, \langle\alpha_2, 1\rangle])^c \cup X_2 \cap [\langle 0, 0\rangle] \cup (X_2 \cap [\langle 0, 0\rangle])^c$$

$$= X_2.$$ □

Let us consider now, for $n = 1, 2, 3, \ldots$ and an uncountable set κ, the zero-dimensional compact space $X_n = \{f \in 2^\kappa : |\mathrm{supt}(f)| \leq n\}$. We will show that any closed subspace of 2^κ with a finite support is a Boolean image of a connected space.

Definition 3.1.10. (1) Let $X_n = \{x \in 2^\kappa : |\{\alpha : x(\alpha) \neq 0\}| \leq n\}$.

(2) For each $\alpha_1, \alpha_2, \ldots, \alpha_n < \kappa$, let $[\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle] = \{x \in 2^\kappa : x(\alpha_1) = x(\alpha_2) = \ldots = x(\alpha_n) = 1\}$.

Theorem 3.1.11. *X_n is a Boolean image of $[0,1]^\kappa$.*

Proof. We observe that X_n is a zero-dimensional compact space. Let x_{-1} be the identically 0 function and $x_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ the unique point in the intersection $X_n \cap [\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle]$.

Then $X_n = X_{n-1} \cup \bigcup_{\alpha_1,\alpha_2,\ldots,\alpha_n<\kappa} \{x \in 2^\kappa : x(\alpha_1) = x(\alpha_2) = \ldots = x(\alpha_n) = 1,\ x(\alpha) = 0 \text{ if } \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Let $\mathcal{C} = \{X_n \cap [\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle] : \alpha_1, \alpha_2, \ldots, \alpha_n < \kappa\} \cup \{X_n \cap [\langle 0, 0\rangle]\}$ ($\alpha_1, \alpha_2, \ldots, \alpha_n$ might be equal). It is easily checked that $\mathcal{C}$ is a family of clopens of X_n which separates the points of X_n. Let $\mathcal{C}^*$ be the closure of $\mathcal{C}$ under $\cap$, $\cup$ and ^{c}.

For $\alpha_1, \alpha_2, \ldots, \alpha_n < \kappa$, define

$a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n} = \{x \in [0,1]^\kappa : x(\alpha_1) = x(\alpha_2) = \ldots = x(\alpha_n) = 1\}$ and let $a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n} = [0,1]^\kappa$. Then $(a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n})$ is pseudoclopen in $[0,1]^\kappa$. Also, we define $a^- = \{x \in [0,1]^\kappa : x = 0\}$ and let $a^+ = [0,1]^\kappa$. Then (a^-, a^+) is pseudoclopen in $[0,1]^\kappa$.

We define $\varphi((a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n})) = X_n \cap [\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle]$ and $\varphi((a^-, a^+)) = X_n \cap [\langle 0, 0\rangle]$. Then we extend this to an isomorphism by defining

$$\varphi((a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \cap (a^-_{\beta_1,\beta_2,\ldots,\beta_n}, a^+_{\beta_1,\beta_2,\ldots,\beta_n}))$$

as

$$\varphi(a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n} \cap a^-_{\beta_1,\beta_2,\ldots,\beta_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n} \cap a^+_{\beta_1,\beta_2,\ldots,\beta_n}),$$

and similarly for $\cup$ and ^{c} and (a^-, a^+). This is a mapping onto $\mathcal{C}^*$.

It remains to prove that $\varphi([0,1]^\kappa, [0,1]^\kappa) = X_n$. By the definition of the extension using complements, we have $\varphi([0,1]^\kappa, [0,1]^\kappa) = \varphi((a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \cup (a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n})^c \cup (a^-, a^+) \cup (a^-, a^+)^c)$

$= \varphi(a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \cup \varphi(a^-_{\alpha_1,\alpha_2,\ldots,\alpha_n}, a^+_{\alpha_1,\alpha_2,\ldots,\alpha_n})^c \cup \varphi(a^-, a^+) \cup \varphi(a^-, a^+)^c)$

$= X_n \cap [\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle] \cup (X_n \cap [\langle \alpha_1, 1\rangle, \langle \alpha_2, 1\rangle, \ldots, \langle \alpha_n, 1\rangle])^c \cup X_n \cap [\langle 0, 0\rangle] \cup (X_n \cap [\langle 0, 0\rangle])^c = X_n$. $\square$

4

Eberlein compact spaces

This chapter deals with Eberlein compact spaces. The main references for this section are [13] and [20]. The motivation of this chapter comes from the result given by Y. Benyamini, M.E. Rudin and M. Wage in [13] showing that the continuous image of an Eberlein compact space is also Eberlein compact. With this motivation in mind, and with respect to the fact that Boolean images are a generalisation of continuous mappings, we study Eberlein compacta in terms of the possibility of being Boolean images of connected spaces.

In section 4.1 we give a brief introduction to the theory of Eberlein compact spaces, followed by some definitions. In section 4.2 we recall the definition of strong Eberlein compact spaces, and some of their properties. We also prove in Theorem 4.2.5 that strong Eberlein compacta are Boolean images of connected spaces. Section 4.3 is dedicated to definitions and properties of scattered spaces, Corson spaces and W-spaces. Then, we show in Theorem 4.3.15 that every scattered Eberlein compact space is a Boolean image of a connected space.

4.1 Introduction

The weakly compact subspaces of Banach spaces were first studied by William Frederick Eberlein of hence the name 'Eberlein compact spaces'. His results showed that these compact spaces are very important and have many applications in different areas of mathematics such as topological algebra, functional analysis, and topology.

Definition 4.1.1. [13] A compact Hausdorff space is called *Eberlein compact* (E-C) if it is homeomorphic to a weakly compact subset of a Banach space.

As we mentioned earlier, we shall not need the weak topology, so we do not define it and prefer the combinatorial characterisation of Eberlein compacta. The main structure theorem for the combinatorial definition of Eberlein compact is due to Amir and Lindenstrauss [3]: a compact space is Eberlein compact if and only if it can be embedded into a Σ_*-product of real lines where $\Sigma_*(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : \forall \varepsilon > 0 \ \{\gamma \in \Gamma \, : \, x(\gamma) > \varepsilon\}$ is finite $\}$.

By the mentioned result, Kalenda in [20] defines Eberlein compact spaces as follows:

Definition 4.1.2. A compact space is Eberlein if and only if it is homeomorphic to a subset of the space

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \forall \varepsilon > 0 \ \{\gamma \in \Gamma \, : \, x(\gamma) > \varepsilon\} \text{ is finite } \}$$

for some set Γ.

Equivalently, due to H. P. Rosenthal [32], Eberlein compact spaces can be characterised using the covering property. Let us first recall the following definition.

Definition 4.1.3. [13] A family $\mathscr{C}$ is called point-finite if each $x \in X$ belongs to at most finitely many sets in $\mathscr{C}$. It is called σ-point-finite if $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}_n$, where each $\mathscr{C}_n$ is point finite.

Proposition 4.1.4. [35] *A compact space X is an Eberlein compact if and only if there exists a σ-point-finite system $\mathscr{C} \subset Coz(X)$ which weakly separates points of X, i.e., for any two different x, $y \in X$ there is a $C \in \mathscr{C}$ such that $\{x, y\} \cap C \neq \varnothing$ and $\{x, y\} - C \neq \varnothing$.*

Benyamini, Rudin and Wage [13] proved in 1977 that the class of Eberlein compacts is stable with respect to continuous images. Moreover, they state that a closed subset of an Eberlein compact space is Eberlein compact, and a countable product of Eberlein compact spaces is Eberlein compact. Nonetheless, an uncountable product of non-trivial Eberlein compact spaces is never Eberlein compact [23].

When dealing with Eberlein compact spaces, we come across a special type of these spaces called strong Eberlein. Strong Eberlein compact spaces are much easier to handle and thus we devote section 4.2 to the study of such spaces.

4.2 Strong Eberlein compact spaces

At the beginning of this section, we recall the definition of strong Eberlein compacta and collect some known results related to these spaces in an attempt to understand and analyse them. Subsequently, we prove that every strong Eberlein compact space is a Boolean image of a connected space.

Definition 4.2.1. [13] An Eberlein compact space X is called *strong* if it embeds in $c_0(\Gamma)$ in such a way that $x(\gamma) = 0$ or $x(\gamma) = 1$ for all $x \in X$ and $\gamma \in \Gamma$.

Equivalently, P. Simon in [35] state that strong Eberlein compacts can be defined by using a covering property:

Definition 4.2.2. A compact space X will be called *strong Eberlein compact* if there exists a point-finite system $\mathscr{C} \subset Coz(X)$, weakly separating points of X.

Here are some results related to the strong Eberlein compact spaces. One of the main theorems is that strong Eberlein compacts are stable under continuous images.

Theorem 4.2.3. [35] *A continuous image of a strong Eberlein compact space is also strong Eberlein compact.*

Now we want to prove that every strong Eberlein compact space is a Boolean image of a connected compact space. To this end, we need the following proposition to show that every strong Eberlein compact space is a closed subspace of the zero-dimensional space 2^{Γ}, where Γ is an infinite cardinal.

Proposition 4.2.4. [35] *Every strong Eberlein compact space can be embedded into 2^{Γ} for some set of indices Γ in such a manner that for every $x \in X$, the set $\{\gamma \in \Gamma \mid x(\gamma) = 1\}$ is finite .*

Proof. Let $\mathscr{C} \subset \text{Clopen}(X)$ be a point-finite system, which weakly separates points of X. For $C \in \mathscr{C}$ let $f_C : X \to 2$ be the mapping which maps C onto 1 and maps $X \setminus C$ onto 0. Let $\psi : X \to 2^{\mathscr{C}}$ be defined by the rule $\psi(x)(C) = f_C(x)$. Then the mapping ψ is the desired embedding. □

Now, we prove the main result of this section.

Theorem 4.2.5. *Every strong Eberlein compact space is a Boolean image of a connected space.*

Proof. We know from Proposition 4.2.4 that every strong Eberlein compact space is a closed subspace of the function space 2^{Γ}. Also, from Theorem 3.1.11 we know that every compact subspace of 2^{Γ} with a finite support is a Boolean image of a connected space. Therefore, every strong Eberlein compact space is a Boolean image of a connected compact space. □

4.3 Scattered Eberlein spaces

In this section we prove a theorem in which we give some necessary and sufficient condition for Eberlein compact spaces to be a Boolean image of a connected space.

Before starting to prove our theorem, we recall some definitions and fundamental results that will help us complete the proof.

4.3.1 Scattered spaces

This section is dedicated to the definition of scattered spaces and some of their topological properties.

Definition 4.3.1. [2] A topological space X is said to be *scattered* if every non-empty subset S of X contains at least one point which is isolated in S (that is, X is scattered if and only if it contains no non-empty subset which is dense-in-itself).

Remark 4.3.2. [2] *Every subspace of a scattered space is itself scattered.*

Proposition 4.3.3. [2] *Let X be a T_0-space and S_1, S_2 be two scattered subspaces of X. Then the following statements hold:*

(1) *$S_1 \cap S_2$ is a scattered subset of X.*

(2) *$S_1 \cup S_2$ is a scattered subset of X.*

Proof. (1) Straightforward.

(2) Since S_1 is a scattered subspace of X, there exists $x \in S_1$ and an open neighborhood O of x such that $S_1 \cap O = \{x\}$. If $O \cap S_2 = \varnothing$, then $O \cap (S_1 \cup S_2) = \{x\}$; so that x is an isolated point of $S_1 \cup S_2$.

Suppose that $O \cap S_2 \neq \varnothing$. Since $O \cap S_2$ is a subset of the scattered subspace S_2 of X, there exists $y \in O \cap S_2$ and an open neighborhood V of y such that $(O \cap S_2) \cap V = \{y\}$. Hence $O \cap V = \{y\}$ or $O \cap V = \{x, y\}$. If $O \cap V = \{y\}$, then y is an isolated point of $S_1 \cup S_2$. Since X is a T_0-space, there exists an open set W of X such that $x \in W$ and $y \notin W$ or $y \in W$ and $x \notin W$. Hence $(O \cap V \cap W) \cap (S_1 \cup S_2) = \{x\}$ or $(O \cap V \cap W) \cap (S_1 \cup S_2) = \{y\}$. Thus $S_1 \cup S_2$ has an isolated point. Therefore $S_1 \cup S_2$ is a scattered subspace of X. □

Corollary 4.3.4. [2] *Let X be a topological space and $\{S_i \,|\, i \in I\}$ be a finite collection of scattered subspaces of X. Then $\bigcup_{i\in I} S_i$ is a scattered subspace of X.*

Simon [35] found the connection between strong Eberlein compact spaces and scattered spaces in the following proposition.

Proposition 4.3.5. [35] *Every strong Eberlein compact space is scattered.*

Consequently, we think it is important to answer the following question: Is it true that each scattered Eberlein compact is strong? This was an open question in Simon's paper [35], and before giving the proof we proceed by recalling two strong notions of topological spaces: Corson compacts and W-spaces. Then, we investigate some properties of these spaces followed by some main results.

4.3.2 Corson compact spaces

Σ-products and their subspaces were studied by many mathematicians since the 1950s. We will consider the class of Σ-products of the real line and their compact subspaces, which became interesting because it contains all Eberlein compact spaces. Benyamini, Rudin and Wage in [13] proved that not all compact spaces that are lying in $\Sigma(\Gamma)$ are Eberlein, for some Γ, by giving a consistent example of non-Eberlein compact subspace of $\Sigma(\Gamma)$. As a result, E. Michael and M.E. Rudin in [25] introduced the concept of Corson compact spaces that are lying in $\Sigma(\Gamma)$.

Corson compact spaces have been intensively studied by many authors because of their distinctive topological properties. Also, they have various connections to many different areas in functional analysis, which makes them an interesting subject.

Definition 4.3.6. [17] Let $\mathbb{R}$ denote the real line, Γ an index set, and $\mathbb{R}^\Gamma$ the usual product of $|\Gamma|$ lines. We set

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\alpha : x(\alpha) \neq 0\}| \leq \omega\}.$$

A compact space X is *Corson compact* if and only if X is homeomorphic to a compact subspace of $\Sigma(\mathbb{R}^\Gamma)$ for some Γ.

Corson compact spaces behave nicely in terms of some topological operations. According to Kalenda in [20], numerous functional-analytic properties of Corson compact were established in the 1980s. These studies had two orientations. The first direction was to study the common properties shared by Eberlein compact and Corson compact spaces. The second direction was to study the gap between Corson and Eberlein compacta. It is apparent that every Eberlein compact is Corson compact. But the converse is not true. Therefore, we obtain the following chain:
Strong Eberlein compact $\Rightarrow$ Eberlein compact $\Rightarrow$ Corson compact space.

A topological property of Corson compact spaces can be seen in the next theorem.

Theorem 4.3.7. [17] *The closed image of a Corson compact space is Corson compact.*

The following theorem shows a stronger result on stability of Corson compact spaces.

Theorem 4.3.8. [25] *Every continuous image of a Corson compact space is Corson compact.*

Next, we present a different class of topological spaces, called W-spaces. We show that every Corson compact space is a W-space. Before starting to prove our theorem, we recall the definition of these spaces and present some more results.

4.3.3 W-spaces

The class of W-spaces was first introduced by Gruenhage in [16], as a generalization of first countable spaces. In order to define these spaces, we consider first the notion of W-sets. This is defined in terms of a simple two-person infinite game: at the n^{th} play, O (for 'open') chooses an open $U_n \supset H$, and then P (for 'point') chooses a point $x_n \in U_n$. We say that O *wins* if $x_n \to H$ (*i.e.*, every neighbouhood of H contains all but finitely many x_n). We call H is a W-set in X if O has a winning strategy in this game. According to G. Gruenhage in [17], if $H \subset X$ has a countable neighbourhood base in X, then H is a W-set in X.

Definition 4.3.9. [16] We call a space X in which each point of X is a W-set a *W-space.*

Theorem 4.3.10. [16] *Every subspace of a W-space is a W-space.*

Theorem 4.3.11. [16] *The countable product of W-spaces is a W-space.*

Theorem 4.3.12. [16] *Every Σ-subspace of a product of W-spaces is a W-space.*

Gruenhage in [16] defines the notions of Σ-product as follows: For each $\alpha \in \Gamma$, X_α a space, and $x_\alpha \in X_\alpha$, the subspace

$$\{x \in \prod_{\alpha\in\Gamma} X_\alpha : \{\alpha : x(\alpha) \neq x_\alpha\} \text{ is countable}\}$$

of $\prod_{\alpha\in\Gamma} X_\alpha$ is called a Σ-product of the X_α's. Then he proved that a Σ-product of W-spaces is a W-space. In particular, Corson compact spaces are W-spaces.

As a next step, we want to demonstrate that every scattered Corson compact space is strong Eberlein. To accomplish this we first need to recall the following result showing that a compact scattered space X is strong Eberlein compact if and only if X is a W-space.

Theorem 4.3.13. [17] *The following are equivalent:*

(a) X is strong Eberlein compact.

(b) X is a compact scattered W-space.

Now we show that every scattered Eberlein compact is strong Eberlein, which lead us to the proof of our theorem. We give the proof here as the following proposition:

Proposition 4.3.14. *Every scattered Eberlein space is strong Eberlein compact.*

Proof. Let X be a scattered Eberlein compact space. Since every Eberlein compact space is Corson compact, X is scattered and Corson. We know that Corson compact spaces are W-spaces, which means that X is scattered and W-space. Then, by Theorem 4.3.13, we have that X is a strong Eberlein compact space. □

Now we prove the main result of this chapter, namely.

Theorem 4.3.15. *Every scattered Eberlein compact is a Boolean image of a connected space.*

Proof. We know from Proposition 4.3.14 that every scattered Eberlein space is strong Eberlein compact. And from Theorem 4.2.5, every strong Eberlein compact space is a Boolean image of a connected space. Therefore, we obtain that every scattered Eberlein compact space is a Boolean image of a connected space. □

5

Radon-Nikodým spaces and more related classes

This chapter is divided into three sections. Section 5.1 is dedicated to Radon-Nikodým (R-N) spaces. We show in Theorem 5.1.4 that every zero-dimensional compact that is a scattered Corson R-N space is a Boolean image of a connected space. In Theorem 5.1.5 we prove that if L is a Boolean image of a zero-dimensional R-N compact space and $C(K)$ is isomorphic to $C(L)$, then K is also a Boolean image of a zero-dimensional R-N compact space. In section 5.2 we recall the definition of *quasi Radon-Nikodým spaces* and study the nature of these spaces. Then we show in Theorem 5.2.6 that if L is a zero-dimensional quasi Radon-Nikodým space that is scattered Corson, then it is a Boolean image of a connected space. Section 5.3 is devoted to *fragmentable compact lines*. We prove in Theorem 5.3.4 that every scattered Corson fragmentable compact line is a Boolean image of a connected space.

5.1 Radon-Nikodým spaces

The class of Radon-Nikodým compacta has been investigated by many authors. The main references for this section are [10] and [28].

Definition 5.1.1. [10] A compact topological space is called *Radon-Nikodým (R-N) compact* if there exists a lower semicontinuous metric that fragments it (see Definition 1.2.19).

We showed in the previous chapter in Theorem 4.3.15 that every scattered Eberlein space is a Boolean image of a connected space. Every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space ([29]), from which it follows that an Eberlein compact space is a Radon-Nikodým compact space, since every reflexive space is Asplund (this follows from the conjunction of the fact that every reflexive Banach space admits a Frechet differentiable norm [40] and the fact that every Banach space admitting a Frechet differentiable norm is an Asplund space [7]).

Therefore, the fact that all Eberlein compacts are among the Radon-Nikodým spaces leads us to an interesting question: Is every Radon-Nikodým space a Boolean image of a connected space? We are not able to answer this question in its full generality but we do show in Theorem 5.1.4 that every zero-dimensional compact that is a scattered Corson R-N space is a Boolean image of a connected space. To approach that theorem, we need to understand these spaces by recalling some important results from the literature.

5.1.1 Properties of Radon-Nikodým spaces

In this section, we summarize some properties of Radon-Nikodým compact spaces. As will be seen, these properties, in most cases, are analogous to the corresponding properties of Eberlein compact spaces.

Theorem 5.1.2. [28] (*a*) *A closed subset of an R-N compact space is R-N compact.*

(*b*) *The product of countably many R-N compact spaces is again R-N compact.*

As we stated above, every Eberlein compact is a Radon-Nikodým compact space. Consequently, it is important to recall the following theorem which offers a necessary and sufficient condition for a Radon-Nikodým compact space to be an Eberlein compact.

Theorem 5.1.3. [18] *A compact space is Eberlein compact if and only if it is both Radon-Nikodým and Corson.*

Now our previous Theorem 4.3.15 can be rephrased in the following way:

Theorem 5.1.4. *Every scattered Corson R-N space is a Boolean image of a connected space.*

Proof. Let X be a scattered Corson R-N space, so it is a scattered Eberlein space. Since every scattered Eberlein compact space is a Boolean image of a connected space (Theorem 4.3.15), X is a Boolean image of a connected space. □

Theorem 5.1.4 allows us to reformulate Theorem 4.3.15 in terms of Radon-Nikodým spaces.

5.1.2 Radon-Nikodým spaces and Banach spaces

The class of Radon-Nikodým compact spaces plays an important role in Banach space theory. The fact that this class can be characterized through its Banach spaces of continuous functions $C(K)$, allows one to establish a significant connection.

Avilés and Koszmider in [9] explain this connection by stating that both classes of R-N compact spaces and their continuous images are stable under taking isomorphisms of their space of continuous functions, meaning that:

1. If L is an R-N compact space and $C(K)$ is isomorphic to $C(L)$, then K is also R-N compact.
2. If L is a continuous image of an R-N compact space and $C(K)$ is isomorphic to $C(L)$, then K is also a continuous image of an R-N compact space.

Now we want to find the connection between the Radon-Nikodým space K and the Banach space $C(K)$ in terms of Boolean images.

Theorem 5.1.5. *If L is a Boolean image of a zero-dimensional R-N space and K is a compact Hausdorff space such that $C(K)$ is isomorphic to $C(L)$, then K is a continuous image of an R-N space.*

Proof. If L is a Boolean image of a zero-dimensional space L', then by Lemma 2.2.5, L is a continuous image of L', since L is itself zero-dimensional. Since $C(K)$ is isomorphic to $C(L)$, from the above characterization 2, it follows that there is an R-N compact space K' that maps continuously onto K. □

5.2 Quasi Radon-Nikodým spaces

The analysis of Radon-Nikodým spaces in terms of Boolean images leads us to study a closely related class of compact spaces called quasi Radon-Nikodým spaces, introduced by A. D. Arvanitakis [6]. The class of quasi Radon-Nikodým compact includes Radon-Nikodaim compact and subsequently Eberlein compact spaces (every Eberlein space is R-N compact). This follows from the fact that quasi Radon-Nikodým compact spaces are fragmented by a metric [6] and from the definition of Radon-Nikodým spaces 5.1.1. Therefore, we obtain the following chain: Eberlein compact $\Rightarrow$ Radon-Nikodým compact $\Rightarrow$ quasi Radon-Nikodým compact.

In this section we first recall the definition of quasi Radon-Nikodým spaces and some important results. This will be followed by an examination of the necessary conditions for a quasi Radon-Nikodým space to be a Boolean image of a connected space.

Definition 5.2.1. [6] We call a compact space K *quasi Radon-Nikodým* if there exists a lower semicontinuous (as a function on $K \times K$) fragmenting quasimetric defined on K. By a quasimetric, we mean a "metric" which may fail to satisfy the triangle inequality, i.e. a function $f : K \times K \to [0, 1]$ for which:

1. For all $x, y \in K, f(x, y) = 0 \Leftrightarrow x = y$.
2. For all $x, y \in K, f(x, y) = f(y, x)$.

Proposition 5.2.2. [6] *The continuous image of a quasi Radon-Nikodým compact space is also quasi Radon-Nikodým.*

Proposition 5.2.3. [6] *A closed subspace of a quasi Radon-Nikodým compact space is also quasi Radon-Nikodým. The Cartesian product of countably many quasi Radon-Nikodým compact spaces is quasi Radon-Nikodým.*

We know that every Radon-Nikodým compact space is quasi Radon-Nikodým compact. Actually the following theorem yields that all totally disconnected quasi Radon-Nikodým compacta are Radon-Nikodým spaces.

Theorem 5.2.4. [6] *A totally disconnected quasi Radon-Nikodým compact space is a closed subset of a countable product of scattered compact spaces, and hence is Radon-Nikodým.*

Now, as mentioned above, every Eberlein compact is a quasi Radon-Nikodým compact space. The following theorem gives us a necessarily condition for a quasi Radon-Nikodým compact space to be Eberlein.

Theorem 5.2.5. [6] *Let K be a quasi Radon-Nikodým compact space. If K is Corson compact, then it is already Eberlein.*

The main result of this section is the following theorem which improves Theorem 5.1.4.

Theorem 5.2.6. *Let L be a quasi Radon-Nikodým space that is scattered Corson. Then it is a Boolean image of a connected space.*

Proof. Since L is a Corson quasi Radon-Nikodým space, and from Theorem 5.2.5, L is Eberlein compact. And from the assumption, L is scattered Eberlein, which means, from Theorem 4.3.15, that L is a Boolean image of a connected space. □

5.3 Fragmentable compact lines

Fragmentable compact lines were introduced in [8]. This class of compact spaces is larger than the classes of Radon-Nikodým and quasi Radon-Nikodým spaces.

In the last two sections we used topological characterisations of quasi Radon-Nikodým compact spaces and Radon-Nikodým compact spaces, as follows: those spaces admitting a lower semicontinuous fragmenting metric and those spaces admitting a lower semicontinuous quasi fragmenting metric (respectively). These characterisation give rise to the study of fragmentable compact lines in terms if Boolean images.

Before proceeding with the main result, we must first recall some definitions.

Definition 5.3.1. [10] A compact space K is *fragmentable* if there exists a metric that fragments it (see Definition 1.2.19) .

Definition 5.3.2. [10] A *compact line* is a linearly ordered space that is a compact space in the topology generated by the base of open intervals.

We shall use the following theorem, proved in [10], in order to pave the way for the main conclusion of this section.

Theorem 5.3.3. [10] *If K is a fragmentable compact line, then K is Radon-Nikodým compact.*

This leads directly to the final result:

Theorem 5.3.4. *Every scattered Corson fragmentable compact line is a Boolean image of a connected space.*

Proof. Let K be a scattered Corson fragmentable compact line. From Theorem 5.3.3, we have that K is a scattered Corson R-N compact space. We know from Theorem 5.1.3 that every Corson R-N space is Eberlein. Thus K is a scattered Eberlein compact space, which means that K a Boolean image of a connected space (Theorem 4.3.15). □

6

Banach spaces of continuous functions

This chapter is dedicated to Banach spaces of continuous functions. We show in Theorem 6.1.4 that if spaces L and K are both compact zero-dimensional and L is a a bijective Boolean image of K, then $C(L)$ is isometric to $C(K)$. Moreover, we show in Proposition 6.1.5 that if L is a Boolean image of K, then $C(L)$ is isometric to a subspace of $C(K)$. On the other hand, we prove in Theorem 6.1.7 that if K and L are zero-dimensional, and $C(K)$ and $C(L)$ are isomorphic as Banach spaces, then there are subspaces K' and L' of K and L, respectively, such that K' is a Boolean image of L'.

6.1 The birth of Banach space theory

Banach spaces were named after the Polish mathematician Stefan Banach. He introduced the notion of *complete normed linear spaces*, which are now known as Banach spaces. These spaces originally emanated from the study of function spaces. The study of Banach spaces and their applications forms the core of linear and nonlinear functional analysis.

6.1.1 The Banach spaces C(K)

Given a compact space K, we denote by $C(K)$ the Banach space of real-valued continuous functions with the usual supremum norm (Marciszewski and Plebanek in [24]). As pointed out for example by Haskell P. Rosenthal (see [32]), the special Banach spaces of the form $C(K)$ play a fundamental role in the study of general Banach spaces. They admit beautiful characterizations singling them out from the general theory. Their structure is particularly rich. Moreover, every Banach space is isometrically isomorphic to a closed subspace of $C(K)$ for some compact Hausdorff space K [14]. Also, a number of results for general Banach spaces obtained from some of those initially established and developed for the special spaces $C(K)$ [32]. In fact, if we want to prove some general results about all Banach spaces, it might be sufficient to prove that result for the spaces $C(K)$.

In the earlier sections of this thesis we analysed the compact spaces K with the aim of paving the way for understanding the Banach spaces $C(K)$. Now we are going to shift our attention from studying the functions between compact Hausdorff spaces to the possibility of the existence of an isomorphic embedding between two Banach spaces of continuous functions.

A celebrated result of Miljutin from 1966 states that if K and L are uncountable compact metric spaces, then $C(K)$ is isomorphic to $C(L)$ as Banach spaces. For this reason, we are only interested in spaces K which are compact (so $C(K)$ makes sense) but not metric. Such spaces K are not Banach spaces, but $C(K)$ is.

The goal of this chapter is to show the connection between compact Hausdorff spaces K and L, related by a Boolean image, and the isomorphism between their Banach spaces $C(K)$ and $C(L)$. Therefore, our question is: If L is a Boolean image of a compact space K, are $C(K)$ and $C(L)$ isomorphic as Banach spaces?

We are aware that our question may not be solvable if we work on an arbitrary compact Hausdorff topological space. Consequently, we choose our spaces to be zero-dimensional. First, we prove that if a compact zero-dimensional space L is a bijective Boolean image of a compact zero-dimensional space K, then $C(L)$ is isometric to $C(K)$. Moreover, we prove that if L is a Boolean image of K, then $C(L)$ is isometric to a subspace of $C(K)$.

Probably the best-known result that we shall use to answer our question is that if X and Y are homeomorphic topological spaces, then $C(X)$ and $C(Y)$ are isomorphic as vector spaces, as order spaces, and as rings, etc. More precisely, a homeomorphism $\tau : X \to Y$ induces a bijection $f \mapsto f \circ \tau$ of $C(Y)$ onto $C(X)$ that preserves all kinds of structure [15].

We start by introducing the following known results, which will help us to prove our theorem:

Lemma 6.1.1. [4] *If (X, τ_0) and (Y, τ_1) are compact Hausdorff spaces and $f : (X, \tau_0) \to (Y, \tau_1)$ is a continuous mapping, then f is a closed mapping.*

Proof. We need to show that if $U \subseteq X$ is a closed subset of X, then also $f(U) \subseteq Y$ is a closed subset of Y. Now, since closed subsets of compact spaces are compact, it follows that $U \subseteq X$ is also compact. Since continuous images of compact spaces are compact, it then follows that $f(U) \subseteq Y$ is compact. Since compact subspaces of Hausdorff spaces are closed, it finally follows that $f(U)$ is also closed in Y. □

Lemma 6.1.2. [34] *If the bijection $f : X \to Y$ is closed, then the inverse* *map $g : Y \to X$ is continuous.*

Proof. We know that g is continuous if and only if the inverse image $g^{-1}(U)$ is closed for all $U \subseteq X$. We have that $g^{-1}(U) = f(U)$ due to the fact that g is the inverse of the bijection f. Since f is a closed map by assumption, $g^{-1}(U)$ is closed and hence g is continuous. □

Proposition 6.1.3. [36] *Any continuous bijection between compact Hausdorff spaces is a homeomorphism.*

Proof. Let f be a continuous bijection between compact Hausdorff spaces X and Y. By Lemma 6.1.1, f is a closed mapping, which means, by Lemma 6.1.2, that the inverse map $g = f^{-1}$ is continuous too. And if both the function f and the inverse function f^{-1} are continuous, then f is a homeomorphism. So, we are done. □

Theorem 6.1.4. *If a compact zero-dimensional space L is a bijective Boolean image of a compact zero-dimensional space K, then $C(L)$ is isometric to $C(K)$.*

Proof. Since the spaces L and K are compact zero-dimensional and L is a Boolean image K, L is a continuous image of K (Lemma 2.2.5). From the assumption, there is a continuous bijection between L and K as compact Hausdorff spaces, which means that K and L are homeomorphic (Proposition 6.1.3). Therefore, by the Banach-Stone theorem, saying that two compact Hausdorff spaces K and L are homeomorphic if and only if $C(K)$ and $C(L)$ are isometric, we obtain that $C(K)$ and $C(L)$ are isometric. □

Proposition 6.1.5. *If a compact zero-dimensional space L is a Boolean image of a compact zero-dimensional space K, then $C(L)$ is isometric to a* *subspace of $C(K)$.*

To prove this proposition, we present the next result:

Proposition 6.1.6. [21] *Suppose K an L are compact spaces. If L is a continuous image of K, then $C(L)$ is isometric to a subspace of $C(K)$.*

Now we prove Remark 6.1.5.

Proof. Since the spaces L and K are compact zero-dimensional and L is a Boolean image K, L is a continuous image of K (Lemma 2.2.5).

Therefore, by Proposition 6.1.6, we have that $C(L)$ is isometric to a subspace of $C(K)$. □

On the other hand, the properties of the Banach spaces of continuous functions $C(K)$ reflect those properties of the compact spaces K themselves. That means that some information regarding the class of spaces of the form $C(K)$, such as being isomorphic as Banach spaces, might help to analyse the nature of the spaces K and the topological relations between them.

Now, we know that if $C(K)$ and $C(L)$ are merely isomorphic as Banach spaces, then K and L may be far from being homeomorphic. Can we nonetheless find a topological relation between these spaces?

To that end, we want to investigate the possibility of existence of a Boolean image between the spaces K and L using which the Banach space $C(K)$ is isomorphically mapped into a Banach space $C(L)$.

Theorem 6.1.7. *Let L and K be compact zero-dimensional spaces. If $C(K)$ is isomorphic to $C(L)$, then there is a subspace $K' \subset K$ and a subspace*

$L' \subset L$ such that K' is a Boolean image of L'.

Before starting to prove our theorem, we need to present the following proposition:

Proposition 6.1.8. [31] *Let K and L be compact spaces such that $C(K)$ is isomorphic to $C(L)$. Then for every nonempty open set $U \subseteq K$ there exists a nonempty open set $V \subseteq U$ such that $\overline{V}$ is a continuous image of some compact subspace of L.*

Now we prove Theorem 6.1.7:

Proof. From Theorem 6.1.8 it follows that if K and L are compact spaces such that $C(K)$ is isomorphic to $C(L)$, then for every nonempty open set $U \subseteq K$ there exists a nonempty set $V \subseteq U$ such that $\overline{V}$ is a continuous image of some compact subspace L' of L.

So choosing $K' = \overline{V}$ will give that K' is a continuous image of a compact subspace L'. Since K' and L' are zero-dimensional (as they are subspaces of zero-dimensional spaces), K' is a Boolean image of L'. $\square$

7

Cardinal Functions

This chapter deals with cardinal functions. We examine some cardinal functions in terms of the possibility of being transferred via a Boolean image. We show in Theorem 7.2.5 that weight is preserved by Boolean images. Then we show in Theorem 7.3.2 that If L is a Boolean image of a compact connected space K and K is separable, then L is separable. Therefore, countable density is also preserved by Boolean images.

7.1 Introduction

What are cardinal functions and why are they useful?

Cardinal functions represent a classical topic in topology and are widely used for extending different topological properties such as separability, countability of the base, and first countability to higher cardinality. Cardinal invariants then help one to prove, formulate, and generalize some results related to different topological properties. In addition, these cardinal characteristics allow us to make precise quantitative comparisons between certain topological properties. For example, it is well known that a space with a countable base (second-countable) has a countable dense set (i.e., is separable). A celebrated result by Arhangelskii in [5] is a converse of this result which states that a regular space with a countable dense set has a base of cardinality $\leq 2^\omega$. Therefore, cardinal invariants play a vital role in general topology. In fact, these invariants describe the local behaviour of a given space in different ways. Also, they are used to reveal some specific features of the space.

Definition 7.1.1. [37] A function ϕ which assigns a cardinal $\phi(X)$ to each topological space X is called a *cardinal function* if ϕ is a topological invariant, i.e., if we have $\phi(X) = \phi(Y)$ whenever X and Y are homeomorphic.

When we consider any kind of mappings, it is very important to know which of the cardinal invariants are preserved by these maps. We will study some cardinal functions and will focus on how these invariants transfer to other spaces via Boolean images.

7.2 The weight of a topological space

The most important cardinal function is the weight $w(X)$ of X.

Definition 7.2.1. [37] The *weight* $w(X)$ *of* X, is defined by

$$w(X) = \min\{|B| : B \text{ is a base for } X\} + \omega.$$

Here ω is added to make the value infinite.

Definition 7.2.2. [37] A space X *satisfies the second axiom of countability*, or is *second countable*, if $w(X) = \omega$, in other words, if it has a countable base.

Theorem 7.2.3. [26] *A subspace of a second countable space is second countable.*

Proof. Suppose X is second countable and that $\mathcal{B}$ is a countable basis for X. Then for any $A \subset X$, $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for the subspace A and A is second countable. $\square$

Theorem 7.2.4. *If the compact space L is a Boolean image of a compact connected space K and the weight of K is ω, then the weight of L is also ω.*

Proof. If the weight of K is ω, then K is a second countable. Since L is a Boolean image of K, by Proposition 2.3.2 we have that L is a continuous image of a closed subspace K_0 of K. Since K_0 is a closed subspace of a second countable space, it is compact and second countable.

We know that if the space is compact, then its weight is preserved by a continuous onto map. Therefore, L is a second countable space. $\square$

The same proof works for any cardinal κ in place of ω.

Theorem 7.2.5. *If the compact space L is a Boolean image of a compact connected space K, and $w(K) \leq \kappa$, for some cardinal κ, then the $w(L) \leq \kappa$.*

Proof. If the weight $w(K) \leq \kappa$, then every subspace K_0 of K has weight $\leq \kappa$. Since L is a Boolean image of K, and by Proposition 2.3.2, L is a continuous onto image of a closed subspace K_0 of K.

We know that if the space is compact, then its weight is preserved by a continuous onto map. Therefore, $w(L) \leq \kappa$. $\square$

7.3 The density of a topological space

Density is one of the most significant cardinal characteristics.

Definition 7.3.1. [37] Let X be an infinite space. Define the density $d(X)$ of X by

$$d(X) = \min\{|D| : D \text{ is a dense subset of } X\}.$$

If $d(X) = \omega$, we say that X is separable.

Theorem 7.3.2. *If L is a Boolean image of a compact connected space K and K is separable, then L is separable.*

Before giving the proof we recall a few basic notions and some results which will be relevant to our discussion.

7.3.1 Stone-Cech compactification

Definition 7.3.3. [41] A *compactification* of a topological space X is a compact space K together with an embedding $e : X \to K$ with $e[X]$ dense in K.

Remark 7.3.4. [41] *We will usually identify X with $e[X]$ and consider X as a subspace of K.*

Theorem 7.3.5. [26] *Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous map $f : X \to K$, with K compact, extends uniquely to a continuous map of Y into K.*

Definition 7.3.6. [26] For each completely regular space X, a compactification of X satisfying the extension condition in Theorem 7.3.5 will be called a *Stone-Cech compactification* and will be denoted by βX.

Remark 7.3.7. [26] *The Stone-Cech compactification is characterised by the fact that any continuous map $f : X \to K$ of X into a compact Hausdorff space K extends uniquely to a continuous map $g : \beta X \to K$.*

7.3.2 Stone-Cech compactification of a discrete space

First, we consider $\beta\omega$, the Stone-Cech compactification of the discrete space of natural numbers ω. We will use the function extension property (Remark 7.3.7) that characterises the Stone-Cech compactification to prove a well known fact about $\beta\omega$ stating that every separable space is a continuous image of the space $\beta\omega$.

Theorem 7.3.8. *Let ω be the discrete space of natural numbers. Then $\beta\omega$ maps continuously onto any separable compact Hausdorff space.*

Proof. Suppose K is a separable compact Hausdorff space, with countable dense subset A. Let $f : \omega \to A$ be a bijection from ω onto A. Then f is necessarily continuous since ω is discrete (for any open subset V in A, $f^{-1}(V)$ is in $\mathcal{P}(\omega)$ and hence is open in ω).

Let $F : \beta\omega \to K$ be a continuous extension of f (Remark 7.3.7).
Note that $F(\beta\omega) \supseteq F(\omega) = f(\omega) = A$, hence $F(\beta\omega)$ is dense in K. Then $\overline{F(\beta\omega)} = K$. Also, since the space $\beta\omega$ is compact, $F(\beta\omega)$ is a compact subspace of Hausdorff space K, which means that $F(\beta\omega)$ is closed in K. So, we have $F(\beta\omega) = \overline{F(\beta\omega)}$. It follows that $F(\beta\omega) = K$. Therefore F maps $\beta\omega$ continuously onto K. □

Definition 7.3.9. [39] A space is called *extremally disconnected* (or ED for short) if it is regular and the closure of every open set is open.

Remark 7.3.10. [42] *A topological space X is extremally disconnected if and only if the Stone-Cech compactification βX of X is extremally disconnected.*

Proposition 7.3.11. *The space $\beta\omega$ is zero-dimensional.*

Proof. Since the space ω is discrete, it is extremally disconnected. From Remark 7.3.10, we have that $\beta\omega$ is extremally disconnected, which means that the closure of any open set is open in $\beta\omega$. Then the clopen sets form an open base for $\beta\omega$, thus the space is zero-dimensional. □

Now we prove Theorem 7.3.2:

Proof. K is compact and separable, so it is a continuous image of $\beta\omega$ (Theorem 7.3.8). Hence, by Propositions 2.3.1, if L is a Boolean image of K and K is a continuous image of $\beta\omega$, L is Boolean image of $\beta\omega$. Since $\beta\omega$ is the Stone-Cech compactification of a discrete space, by Proposition 7.3.11 it is is zero-dimensional. By Lemma 2.2.5, L is a Boolean image of a compact zero-dimensional space K if and only if L is a continuous image of K. Thus, L is a continuous image of the space $\beta\omega$. $\beta\omega$ is separable (ω is a countable dense set in it), and separability is preserved by continuous onto maps, so we have that L is separable. We conclude that separability is preserved by Boolean images. □

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