Generalization of k-Uniform Hypergraphons

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Abstract

This thesis studies a generalization of the k-uniform hypergraphons by working with an arbitrary space. In particular, we offer two different generalizations: k-uniform hypergraphons and k-uniform mixed hypergraphons. Motivated by results on graphons, we reduce every k-uniform hypergraphon to a twin-free separable k-uniform hypergraphon. Moreover, we prove that every k-uniform hypergraphon is weakly isomorphic to a twin-free separable k-uniform hypergraphon. However, we find a counterexample which shows the notion of k-uniform hypergraphon does not satisfy the purity conditions. Therefore, we construct a new generalization, the k-uniform mixed hypergraphon, that helps us to prove every twin-free separable k-uniform mixed hypergraphon is weakly ismomorphic to a pure k-uniform mixed hypergraphon. Furthermore, we show that every twin-free separable k-uniform mixed hypergraphon is isomorphic, up to a null set, to a pure k-uniform mixed hypergraphon.

Contents

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Introduction

Many graphs are so large that it is impossible to define or study them by classical means, especially since they may be repeatedly changing, as in the example of the internet graph. There are many ways to approximate a large graph and one of them is by constructing an increasing sequence of graphs that will get us close to our large graph. We can examine how the graphs are similar or close to each other using *homomorphism densities* [13].

Over the past decades, Lovász and Szegedy introduced a new concept called *graphons* as limits of sequences of graphs [14]. The new concept opens new doors for studying large graphs not only in graph theory, but also in many other fields of mathematics such as measure theory, functional analysis and probability. Therefore, we introduced in the beginning of Chapter 2 the basic notions and results of graphs, metric space, and measure space. Moreover, we gave in section 2.4 the definition of *graphon* and we explained how a sequence of graphs convergent to a limit, a graphon. We closed this section by investigated some notions of distance such as *cut distance* for graphs and graphons.

Lovász and Szegedy also proved that every graphon is *weakly isomorphic* to a *pure graphon*. What does pure graphon mean? and how can we purify graphons? We will answer those questions in Chapter 3 using the main references [12], [13] and [15]. In section 3.2, we presented some examples for the purification of graphons that will help the readers to understand in detail the way how can we find a pure graphon of a given graphon.

A notion of k-*uniform hypergraphon* is a fascinating and useful object to study. They have been investigated by Elek and Szegedy [8], and Zhao [22]. We started Chapter 4 by introducing a brief history of two k-uniform hypergraphon definitions that made by Elek and Szegedy in 2012 and Zhao in 2015, and then we explored the relation between those definitions.

In section 4.2, we gave a generalization of a k-uniform hypergraphon (Definition 4.10) by working with an arbitrary measure space. Thus, instead of working only with the space $[0, 1]$, we worked with an arbitrary space.

The main result of this chapter is in section 4.3. We reduced a given k-uniform hypergraphon $\mathcal{H}$ as defined in Definition 4.10 on a measure space $(J, \mathcal{A}, \pi)$ to a twin-free separable k-uniform hypergraphon. Our proof is inspired by the original work of Borge, Chayes and Lovász with graphons [6]. We constructed a new concept of *twin* for a k-uniform hypergraphon $\mathcal{H}$. Then we transformed $\mathcal{H}$ into a twin-free k-uniform hypergraphon through several steps. First, we made $\mathcal{H}$ *strong* by changing its value on a set of measure zero. Second, we showed that $\mathcal{H}$ is measurable since there is a *countably generated* σ-algebra $\mathcal{A}'$ of $\mathcal{A}$. Third, if $\mathcal{H}$ is countably generated, then we can see that a *separating* k-uniform hypergraphon $\mathcal{H}/\mathcal{P}_{[\mathcal{A}]}$ is countably generated too. Fourth, we showed that the completion of $\mathcal{H}$ can be *embedded* into a separable k-uniform hypergraphon. The last step is that if we have a strong k-uniform hypergraphon $\mathcal{H}$ and $\mathcal{P}$ is the partition into the twin-classes of $\mathcal{H}$, then we see that $\mathcal{H}/\mathcal{P}$ is twin-free.

We concluded section 4.3 by another result which is every k-uniform hypergraphon is weakly isomorphic to a twin-free separable k-uniform hypergraphon.

In section 4.4, we defined a *neighbourhood distance* $r_{\mathcal{H}}$. The aim of this definition is measuring the distance that we want in order to define purity of k-uniform hypergraphon.

Motivated by the purification of graphons [13], we tried to purify the k-uniform hypergraphon that we generalized in Chapter 4. However, as we show by example, it is not straightforward to purify a k-uniform hypergraphon as defined in Definition 4.10. Therefore, we proposed a new generalization which is called a *k-uniform mixed hypergraphon*. Our new definition allowed us to purify a *twin-free separable k-uniform mixed hypergraphon*. Then we proved the main theorem in Chapter 5 that says every twin-free separable k-uniform mixed hypergraphon is weakly isomorphic to a pure k-uniform mixed hypergraphon. Furthermore, we show that for every twin-free separable k-uniform mixed hypergraphon there is a pure k-uniform mixed hypergraphon isomorphic, up to a null set, to it.

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Background Materials

In this chapter, we will investigate the theory of *graphons*. In order to introduce graphons it is first necessary to clarify some notions and results from several areas of mathematics such as graph theory, measure theory, and probability. Lovász's book [13] and [21] will be the main resource for sections 2.1 and 2.4 while we use [19],[20],[6], and [4] for sections 2.2 and 2.3.

2.1 Graphs and Graph Homomorphisms

A *graph* G is an ordered pair $G = (V, E)$ where V is a finite set of elements (vertices) and E is a set of ordered pairs of size 2 (edges) of V. If $e = (v, u)$ is an edge, we say that e joins v and u, or e connects v and u; we then say that v and u are adjacent. Also, if an edge e joins a vertex v to itself, then we call it a *loop*. However, if there are two or more edges that join the same two vertices, we say that G has *multiple edges*. Moreover, if the graph G does not have loops or multiple edges, we name it a *simple graph*. We always work with undirected graphs, so if (v, u) is an edge, so is (u, v).

Any simple graph that has n vertices with an edge between every pair of vertices is called a *complete graph*, denoted by K_n. For example,

K_2  K_3

If we have graph $G = (V, E)$ and a subset of vertices of G, say H then we say that $G[H]$ is *an induced subgraph* if for any $u, v \in H$, u and v are adjacent in $G[H]$ if and only if they are adjacent in G.

There are some special graphs such as a *path* P_n. We define this to be a simple graph which has vertices that can be ordered in a sequence $v_1, v_2, \ldots, v_n$ and edges $E(P_n) = \big\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\big\}$. In addition, we define a *cycle* C_n as a simple graph whose vertices can be ordered in a cyclic sequence $(v_1, v_2, \ldots, v_n)$ and edges

$$E(C_n) = \big\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\big\}$$

Erdös, Renyi and Gilbert in 1959 defined a *random graph* as follows: Let n be a positive integer and $0 \leqslant p \leqslant 1$. Then, the *random graph* $G(n,p)$ on n vertices is generated by taking n vertices, and connecting any two of them with probability p, independently of previous choices.

A *weighted graph* G is a looped-simple graph, that is, a finite graph with no multiple edges in which any subset of the vertices can have a loop, with a positive real weight $\alpha_i(G)$ connected with each vertex i and a real weight $\beta_{i,j}(G)$ connected with each edge (i,j).

There are many large graphs, and it is hard to store or define them in the traditional way. Moreover, some of those graphs are changing continually such as the internet graph. We can approximate them by smaller graphs or find another way to represent them which is easier to study. If we want to approximate a large graph by another, possibly smaller, graph we need to know how similar or close two graphs are. This similarity is measured by *homomorphisms.*

Let us consider two simple graphs G and G' with vertex sets $V(G)$ and $V(G')$. Then

- A *graph homomorphism* from G to G' is a mapping $\varphi : V(G) \to V(G')$ such that $\big(\varphi(u), \varphi(v)\big) \in E(G')$ whenever $(u,v) \in E(G)$.
- A *graph isomorphism* between G and G' is a bijective map $\varphi : V(G) \to V(G')$ such that

$$(u,v) \in E(G) \iff (\varphi(u), \varphi(v)) \in E(G')$$

for all $u, v \in V(G)$.

In the next section, we will use homomorphisms to define certain numbers which will describe how closely related two graphs are.

2.1.1 Homomorphism numbers

For any two simple graphs G and G', we define the following:

- $\mathrm{Hom}(G, G')$: the set of homomorphisms from G to G'.
- $\mathrm{hom}(G, G')$: the number of homomorphisms from G to G'.
- $\mathrm{inj}(G, G')$: the number of injective homomorphisms of G into G'.
- $\mathrm{ind}(G, G')$: the number of embeddings of G into G' as an induced subgraph.

If G and G' are *multigraphs*, loops and multiple edges are allowed and then the definition of $\mathrm{hom}(G, G')$ can be extended to this context. We use the idea that a homomorphism must indicate which edge goes on which edge as well as which vertex goes on which vertex. Let $i, j \in V(G)$ be two vertices which are connected by $a_{(i,j)}$ edges, and $u, v \in V(G')$ that are connected by $b_{(u,v)}$ edges. Then, if i maps to u and j maps to v, there are $b_{(u,v)}^{a_{(i,j)}}$ ways of mapping the i-j edges to u-v edges. That means a *vertex-and-edge homomorphism* from G to G' is defined as a pair of maps $\varphi_1 : V(G) \to V(G')$ and $\varphi_2 : E(G) \to E(G')$ such that if e connects i and j in $E(G')$ then we see that $\varphi_1(i)$ and $\varphi_1(j)$ are connected by $\varphi_2(e)$.

Furthermore, we can extend the definition of homomorphism numbers for weighted graphs. Let G' be a weighted graph with vertex weights $\alpha_v(G')$ and edge weights $\beta_{(u,v)}(G')$. For every map $\psi : V(G) \longrightarrow V(G')$, we have the weights

$$\alpha_\psi = \prod_{u \in V(G)} \alpha_{\psi(u)}(G')$$

and

$$\hom_\psi(G, G') = \prod_{(u,v) \in E(G)} \beta_{(\psi(u),\psi(v))}(G'),$$

where we take $\beta(x, y) = 0$ if (x, y) is not an edge of G'. Note that if ψ is not a homomorphism then $\hom_\psi(G, G') = 0$. Thus we define

$$\hom(G, G') = \sum_\psi \alpha_\psi \hom_\psi(G, G')$$

where the sum is over all maps $\psi : V(G) \to V(G')$,

and

$$\mathrm{inj}(G, G') = \sum_\psi \alpha_\psi \hom_\psi(G, G')$$

where the sum is over all maps ψ where ψ is injective.

If G is a multigraph and G' is a weighted graph, then we may define $\hom(G, G')$ as above. However, what if G' is an unweighted multigraph? Then there is a weighted simple graph F in which each edge is weighted by its multiplicity in G'. We define

$$\hom(G, G') = \hom(G, F).$$

2.1.2 Homomorphism densities

Let G and G' be simple graphs. To obtain the *homomorphism densities*, we normalize their homomorphism number by setting

$$t(G, G') = \frac{\hom(G, G')}{\big(v(G')\big)^{v(G)}}$$

where $\big(v(G')\big)^{v(G)}$ is the total number of functions from $V(G)$ to $V(G')$. Thus, $t(G,G')$ is the probability that a random map of $V(G)$ into $V(G')$ is a homomorphism. Also we define

$$t_{\text{inj}}(G,G') = \frac{\text{inj}(G,G')}{\big(v(G')\big)_{v(G)}}$$

where $\big(v(G')\big)_{v(G)}$ is the number of injective maps from $V(G')$ to $V(G)$. This is the probability that a random injective map from $V(G)$ to $V(G')$ is a homomorphism. We also define

$$t_{\text{ind}}(G,G') = \frac{\text{ind}(G,G')}{\big(v(G')\big)_{v(G)}}$$

the probability that a random injective maps from $V(G)$ to $V(G')$ preserves both adjacency and non-adjacency.

Example 2.1. Let $G = K_2$ and $G' = K_3$. Then the homomorphism density for G and G' is

$$t(G,G') = \frac{\text{hom}(G,G')}{|v(G')|^{|v(G)|}} = \frac{6}{9} = \frac{2}{3}$$

Also, $t_{\text{ind}}(G,G') = \frac{6}{6} = 1$ since every injective map is a homomorphism.

2.2 Metric Spaces

Let (J,τ) be a topological space. A *neighbourhood* of a point $x \in J$ is a subset S of J that includes an open subset that contains x, that is $x \in O \subseteq S$ for some open set O.

Suppose that J_1 and J_2 are topological spaces, and let $f : J_1 \to J_2$. Then

- f is a *continuous* function if $f^{-1}(O)$ is an open set in J_1 for every open set O in J_2.

- f is a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous.

Let (X, d) be a metric space. If $x \in X$ and $\epsilon > 0$, define

$$B_\epsilon(x) := B_\epsilon(x, d) := \{y \in X : d(x, y) < \epsilon\}.$$

If S is a subset of a metric space X, then the closure of S, denoted by $\overline{S}$, defined by

$$\overline{S} = \Big\{x \in X : B_\epsilon(x) \cap S \neq \varnothing \quad \text{for every} \quad \epsilon > 0\Big\}.$$

The following is a well known theorem.

Theorem 2.2. Suppose X is a metric space. Let τ be the set of open sets of X. Then (X, τ) is a topological space.

Suppose that (X, d) is a metric space. Then

- a set S is called *dense* in X if for every $x \in X$ and $\epsilon > 0$, we have $s \in S$ such that $0 < d(x, s) < \epsilon$.
- The metric space (X, d) is called *separable* if it has a countable subset S, which is dense (i.e. $\bar{S} = X$).
- (X, d) is said to be *complete* if every Cauchy sequence in X converges to some point in X.

2.3 Measure Spaces

This section presents some of the main concepts and results in measure theory, following [20] and [19].

Definition 2.3. Let J be a set. A nonempty collection $\mathcal{A}$ of subsets of J said to be σ-algebra on J if:

(i) $J \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ where A^c stands for the complement of A.

(iii) If $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

A *measurable space* is a tuple $(J, \mathcal{A})$ consisting of a set J and a σ-algebra $\mathcal{A}$ on J. Furthermore, if M is an element of the σ-algebra $\mathcal{A}$, then we call M a *measurable set.*

Definition 2.4. Suppose that $\mathcal{A}$ is a σ-algebra on a set J_1 and $\mathcal{B}$ is a σ-algebra on a set J_2. We say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as σ-algebras if there exists a bijective map $\varphi : \mathcal{A} \to \mathcal{B}$ such that for all $A_1, A_2 \in \mathcal{A}$ we have

$$\varphi(A_1) \subseteq \varphi(A_2) \iff A_1 \subseteq A_2.$$

Suppose that $\mathcal{U}$ is a collection of subsets of J. We define $\sigma(\mathcal{U})$ to be the minimal σ-algebra containing $\mathcal{U}$, which is called the *σ-algebra generated* by $\mathcal{U}$ (i.e. the intersection of all σ-algebras containing $\mathcal{U}$). We need to show that such a $\sigma(\mathcal{U})$ exists.

Lemma 2.5. Let $\mathcal{U}$ be a collection of subsets of J. Then, there exists a minimal σ-algebra containing $\mathcal{U}$.

Proof. First of all, let us note the following fact: Consider $\mathcal{C}$ to be any nonempty collection of σ-algebras of subsets of J. Then,

$$\bigcap \mathcal{C} = \{B \in J : B \in \mathcal{G} \quad \text{for every} \quad \mathcal{G} \in \mathcal{C}\}$$

contains of all sets B which belong to each σ-algebra $\mathcal{G}$ of $\mathcal{C}$. We see that $\bigcap \mathcal{C}$ is a σ-algebra of subsets of J.

Suppose $\mathcal{U}$ is a collection of subsets of J. Define $\mathcal{C}_{\mathcal{U}}$ to be the collection of all σ-algebras containing all the sets of $\mathcal{U}$. Note that $\mathcal{C}_{\mathcal{U}} \neq \emptyset$ since $P(J) \in \mathcal{C}_{\mathcal{U}}$. Therefore, $\bigcap \mathcal{C}_{\mathcal{U}}$ is a σ-algebra which include all the sets of $\mathcal{U}$. Furthermore if $\mathcal{G}$ is a σ-algebra such that $\mathcal{U} \subseteq \mathcal{G}$ then $\bigcap \mathcal{C}_{\mathcal{U}} \subseteq \mathcal{G}$ so $\bigcap \mathcal{C}_{\mathcal{U}}$ is the minimal σ-algebra containing $\mathcal{U}$. □

We have shown that if $\mathcal{U}$ is a collection of subsets of J then there is a minimal σ-algebra containing $\mathcal{U}$. We denote this σ-algebra by $\sigma(\mathcal{U})$ and call it the σ-algebra generated by $\mathcal{U}$.

Remark 2.6. If $\mathcal{U}$ is itself a σ-algebra, then $\sigma(\mathcal{U}) = \mathcal{U}$.

Definition 2.7 (Borel σ-algebra [20])**.** Let (J, τ) be a topological space. The smallest σ-algebra $\mathcal{B}$ that contains τ is called the Borel σ-algebra of (J, τ). The elements of $\mathcal{B}$ are called Borel sets.

The topological space (J, τ) is said to be a Polish space if it is homeomorphic to a separable complete metric space. A standard Borel space is a measurable space isomorphic to the Borel σ-algebra over a dense in itself Polish space.

After we gave the definition of the measurable space, it is important to know that when a function between two measurable spaces is measurable.

Definition 2.8. [19] Let $(J_1, \mathcal{A}_1)$ and $(J_2, \mathcal{A}_2)$ be two measurable spaces. The function $\varphi : J_1 \to J_2$ is said to be *measurable* if $\varphi^{-1}(A) \in \mathcal{A}_1$ for each $A \in \mathcal{A}_2$.

It is important to state the definition of a *step function*.

Definition 2.9 (Step Function [20])**.** Let J be a set. A function $f : J \to \mathbb{R}$ is called a *step function* if it takes only finitely many values, i.e. the image $f(J)$ is a finite subset of $\mathbb{R}$.

Definition 2.10. [20] A function $\pi : \mathcal{A} \to [0, \infty]$ on a measurable space $(J, \mathcal{A})$ is called a (positive) *measure* if:

- $\pi(\varnothing) = 0$,
- π is σ-additive, i.e., Let I be a countable set. Then,

$$\pi\Big(\bigcup_{i\in I} A_i\Big) = \sum_{i\in I} \pi(A_i)$$

for any pairwise disjoint sequence $\{A_i\}_{i\in I}$ in $\mathcal{A}$.

A *measure space* is a triple $(J, \mathcal{A}, \pi)$ consisting of a nonempty set J, a σ-algebra $\mathcal{A}$ on J, and a measure π on $\mathcal{A}$.

Example 2.11.
- We consider a measure space $(\mathbb{R}, \mathcal{A}, \pi)$. Let I be the set of intervals $\{(a, b) : a < b\}$ and let $\mathcal{A}$ be the σ-algebra generated by I. The measure π is determined by setting $\pi\big((a, b)\big) = b - a$ for all $(a, b) \in I$. Then $(\mathbb{R}, \mathcal{A}, \pi)$ is a measure space.
- Let $J = \{0, 1\}$. Define $\mathcal{A}'$ to be the power set

$$P(\{0, 1\}) = \big\{\varnothing, \{0\}, \{1\}, \{0, 1\}\big\},$$

so that $\mathcal{A}'$ is a σ-algebra. Define the measure π' by $\pi'(\varnothing) = 0$, $\pi'(\{0\}) = \pi'(\{1\}) = \frac{1}{2}$, and $\pi'(\{0, 1\}) = 1$. Then $(J, \mathcal{A}', \pi')$ is a measure space.

These measure spaces are not isomorphic since $\mathcal{A}$ is infinite and $\mathcal{A}'$ is finite.

Example 2.12. Let $J = [0, 1]$ we give two different σ-algebras on the interval $[0, 1]$.

- Let $\mathcal{A}_1 = \{\varnothing, [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, 1]\}$. Define a measure π_1 by $\pi_1(\varnothing) = 0, \pi_1([0, \frac{1}{2}]) = \pi_1([\frac{1}{2}, 1]) = \frac{1}{2}$, and $\pi_1([0, 1]) = 1$. Note that for example $\pi_1\big((\frac{1}{3}, \frac{2}{3})\big)$ is not defined.
- Set $I = \{(a, b) : 0 \leqslant a < b \leqslant 1\}$. Then $\mathcal{A}_2$ is the σ-algebra generated by I, and π_2 is the measure determined by setting $\pi_2\big((a, b)\big) = b - a$ for all $(a, b) \in I$.

In the next sections when we talk about the measure space $J = [0, 1]$, we will use the second of these σ-algebras, that is we will mean $(J, \mathcal{A}_2, \pi_2)$.

Definition 2.13. [20] Let $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ be measure spaces. A function $\varphi : J_1 \to J_2$ is called measure preserving if:

- $\varphi^{-1}(A) \in \mathcal{A}_1$ for every $A \in \mathcal{A}_2$,
- $\pi_1(\varphi^{-1}(A)) = \pi_2(A)$ for every $A \in \mathcal{A}_2$.

The next theorem gives an outline of the basic properties of measures.

Theorem 2.14. [20, p. 18] Let $(J, \mathcal{A}, \pi)$ be a measure space. Then the following holds.

(i) $\pi(\varnothing) = 0$.

(ii) If $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{A}$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$ then

$$\pi(A_1 \cup \cdots \cup A_n) = \pi(A_1) + \cdots + \pi(A_n).$$

(iii) If $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subseteq A_2$ then $\pi(A_1) \leqslant \pi(A_2)$.

(iv) Let (A_i) be a sequence of elements of $\mathcal{A}$ such that $A_i \subseteq A_{i+1}$ for all i. Then

$$\pi\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \pi(A_i).$$

(v) Let (A_i) be a sequence of elements of $\mathcal{A}$ such that $A_i \supseteq A_{i+1}$ for all i. Then

$$\pi(A_1) < \infty \implies \pi\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \pi(A_i).$$

Now we are going to state the definitions of *outer measure* and *Lebesgue measure*.

Definition 2.15. [20] Let E be a nonempty set and $\mu : P(E) \to [0, \infty]$, where $P(E)$ is the power set of E. The function μ is said to be an *outer measure* if:

(i) $\mu(\varnothing) = 0$;

(ii) If $E_1 \subseteq E_2 \subseteq E$ then $\mu(E_1) \leqslant \mu(E_2)$;

(iii) For all sequences $(E_n)_{n=1}^{\infty}$ of subsets of E, $\mu(\bigcup\limits_{n=1}^{\infty} E_n) \leqslant \sum\limits_{n=1}^{\infty} \mu(E_n)$.

Let A be a subset of E. Then, A is called *μ-measurable* if

$$\mu(S) = \mu(S \cap A) + \mu(S \backslash A)$$

for every $S \subseteq E$.

Definition 2.16. If $I = (a, b)$ is an interval of $\mathbb{R}$, set $l(I) = b - a$. For each subset E of $\mathbb{R}$, we define the *Lebesgue outer measure* $\mu^*(E)$ of E by

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : \{I_n\} \quad \text{a sequence of intervals with} \quad E \subset \bigcup_{n=1}^{\infty} I_n \right\}.$$

Furthermore, the set E is called a *Lebesgue measurable* set if, for any $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E).$$

In addition, if E is a Lebesgue measurable set, then we define the Lebesgue measure of E as $\mu(E) = \mu^*(E)$.

Now, we can express the definition of *Lebesgue integral* of a nonnegative measurable function. However, we need to define the notion of a *characteristic function.*

Definition 2.17 (Characteristic Function)**.** Let J be a set and let $A \subseteq J$.

Then, the *characteristic function* $\chi_A : J \to \mathbb{R}$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Definition 2.18 (Lebesgue Integral [20])**.** Let $(J, \mathcal{A}, \pi)$ be a measure space and let $S \in \mathcal{A}$ be a measurable set.

- Let $\varphi : J \to [0, \infty)$ be a measurable step function of the form

  $$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

  where $a_i \in [0, \infty)$ and $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots, n$. The Lebesgue integral of φ over S is the number $\int_S \varphi d\pi \in [0, \infty]$ defined by

  $$\int_S \varphi d\pi := \sum_{i=1}^{n} a_i \pi(S \cap A_i).$$

- Let $\psi : J \to [0, \infty]$ be a measurable function. The Lebesgue integral of ψ over S is the number $\int_S \psi d\pi \in [0, \infty]$ defined by

  $$\int_S \psi d\pi := \sup_{\varphi \leqslant \psi} \int_S \varphi d\pi,$$

  where the supremum is taken over all measurable step functions $\varphi : J \to [0, \infty)$ that satisfy $\varphi(x) \leqslant \psi(x)$ for all $x \in J$.

We are going to state the basic properties of the Lebesgue integral. For more details see [20, p. 20].

Theorem 2.19. Suppose we have a measure space $(J, \mathcal{A}, \pi)$ and measurable functions $\varphi, \psi : J \to [0, \infty]$. Let $S \in \mathcal{A}$. Then

(i) if $\varphi \leqslant \psi$ on S then $\int_S \varphi d\pi \leqslant \int_S \psi d\pi$.

(ii) $\int_S \varphi d\pi = \int_J \varphi \chi_S d\pi$.

(iii) if $\varphi(x) = 0$ for all $x \in S$ then $\int_S \varphi d\pi = 0$.

(iv) if $\pi(S) = 0$ then $\int_S \varphi d\pi = 0$.

(v) if $A \in \mathcal{A}$ and $S \subseteq A$ then $\int_S \varphi d\pi \leqslant \int_A \varphi d\pi$.

(vi) if $c \in [0, \infty)$ then $\int_S c\varphi d\pi = c \int_S \varphi d\pi$.

Definition 2.20 (L^p spaces)**.** Suppose that $(J, \mathcal{A}, \pi)$ is a measure space. If $1 \leqslant p < \infty$, we define the space $L^p(J, \mathcal{A}, \pi)$, simply $L^p(J)$, as

$$L^p(J) = \left\{ f : J \to \mathbb{R} \quad : f \quad \text{measurable, and} \quad \int_J |f|^p d\pi < \infty \right\}.$$

Define the L^p-norm of $f \in L^p(J)$ by

$$||f||_{L^p} = \left(\int_J |f|^p d\pi \right)^{1/p}.$$

Now we want to give some important notions of measure theory from [5] and [20] that we will use later.

Let $(J, \mathcal{A}, \pi)$ be a measure space, and let $\mathcal{E}$ be a set of subsets of J. Then we call $\sigma(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$. If E is a countable subset of the σ-algebra $\mathcal{A}$ such that $\sigma(E) = \mathcal{A}$, then we call $\mathcal{A}$ *countably generated.* Moreover, we say that $\mathcal{E} \subseteq \mathcal{A}$ is a *basis* for $(J, \mathcal{A}, \pi)$ if $\sigma(E)$ is dense in $\mathcal{A}$.

Suppose that $(J, \mathcal{A}, \pi)$ is a measure space. Let B be a measurable set. Then B said to be a *null set* if $\pi(B) = 0$. Let $\mathcal{Q}$ be a property of points in J. We say that $\mathcal{Q}$ holds *almost everywhere* if there is a set $B \subseteq J$ of measure zero such that every $b \in J \backslash B$ has the property $\mathcal{Q}$. Note that the set of all points $b \in J$ which have the property $\mathcal{Q}$ need not to be measurable.

A measure space $(J, \mathcal{A}, \pi)$ is said to be *complete* if any subset A of any $B \in \mathcal{A}$ with $\pi(B) = 0$ is also in $\mathcal{A}$. For every measure space $(J, \mathcal{A}, \pi)$ there is a unique completion $(J, \overline{\mathcal{A}}, \overline{\pi})$ that is the smallest complete measure space such that $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and $\overline{\pi}|_{\mathcal{A}} = \pi$.

Theorem 2.21. [20, p. 39] Suppose that $(J, \mathcal{A}, \pi)$ is a measure space. Define

$$\mathcal{N} = \{A \in \mathcal{A} : \pi(A) = 0\}$$

to be the collection of its null sets. Then,

$$\overline{\mathcal{A}} = \left\{A \cup B : A \in \mathcal{A} \quad \text{and} \quad B \subseteq N \quad \text{for some} \quad N \in \mathcal{N}\right\}$$

is a σ-algebra. Furthermore, there is a unique measure $\overline{\pi} : \overline{\mathcal{A}} \to [0, \infty]$ such that $\overline{\pi}|_{\mathcal{A}} = \pi$. The complete measure space $(J, \overline{\mathcal{A}}, \overline{\pi})$ is called the completion of $(J, \mathcal{A}, \pi)$.

Let us consider any set S of J. Then S *separates* two distinct points $x_1, x_2 \in J$ if $x_1 \in S$ and $x_2 \notin S$ or $x_1 \notin S$ and $x_2 \in S$. A set $\mathcal{E}$ of subsets of J separates x_1 and x_2 if there is $S \in \mathcal{E}$ which separates x_1 and x_2. This defines a partition $\mathcal{P}_{[\mathcal{E}]}$ of J by setting two points in the same class if and only if they are not separated by $\mathcal{E}$. Then $\mathcal{E}$ is said to be separating if it separates any two points of J.

For example, if $(J, \mathcal{A}, \pi)$ is a measure space, we will consider the partition $\mathcal{P}_{[\mathcal{A}]}$ in Chapter 5 and elsewhere.

Lemma 2.22. Let $\mathcal{A}$ be a σ-algebra on J and let $\sim_{\mathcal{A}}$ be the equivalence relation on J defined by $x \sim y$ if

$$\{A \in \mathcal{A} : x \in A\} = \{A \in \mathcal{A} : y \in A\}$$

for all $x, y \in J$. Let $[x] := \{y \in J : y \sim x\}$.

Let J' be the set of equivalence classes of J under $\sim$ and set $\mathcal{B}$ to be the σ-algebra of J' consisting of elements of the form

$$[A] = \{[x] : x \in A\}$$

for $A \in \mathcal{A}$. Then the function $\varphi : A \mapsto [A]$ for $A \in \mathcal{A}$ is an isomorphism of

σ-algebras.

Proof. It is clear that the map $\varphi : \mathcal{A} \to \mathcal{B}$ which sends $A \mapsto [A]$ is a bijection. Suppose that $A_1, A_2 \in \mathcal{A}$.

- Assume that $A_1 \subseteq A_2$. Let $[x] \in \varphi(A_1)$. Then, $x \in A_1$ which implies that $x \in A_2$. That means $[x] \in [A_2] = \varphi(A_2)$. Hence, $\varphi(A_1) \subseteq \varphi(A_2)$.
- Assume that that $\varphi(A_1) \subseteq \varphi(A_2)$. Let $x \in A_1$. Then, $[x] \in \varphi(A_1)$ which implies that $[x] \in \varphi(A_2)$. That means $x \in A_2$. Thus, $A_1 \subseteq A_2$.

□

Suppose that $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ are two measure spaces, and we have a measure preserving map $\varphi : J \to J'$. We say that φ is an *isomorphism* if it is one-to-one and onto, and both φ and φ^{-1} are measurable and measure preserving. Moreover, φ is said to be an *isomorphism mod* 0 if there are null sets $N \in \mathcal{A}$ and $N' \in \mathcal{A}'$ with $\pi(N) = \pi'(N') = 0$ such that the restriction of φ to $J\backslash N$ is an isomorphism between $J\backslash N$ and $J'\backslash N'$. In particular, we can say that two measure spaces $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ are isomorphic mod 0.

A measure space $(J', \mathcal{A}', \pi')$ is said to be a *full subspace* of $(J, \mathcal{A}, \pi)$ if $J' \subseteq J$ of outer measure $\pi(J)$, $\mathcal{A}' = \{A \cap J' : A \in \mathcal{A}\}$, and $\pi'(A \cap J') = \pi(A)$ for all $A \in \mathcal{A}$. Moreover, a map $\varphi : (J^*, \mathcal{A}^*, \pi^*) \to (J, \mathcal{A}, \pi)$ between two measure spaces is said to be an *embedding* if it is an isomorphism between $(J^*, \mathcal{A}^*, \pi^*)$ and a full subspace of $(J, \mathcal{A}, \pi)$.

We close this section by introducing the notion of *conditional expectation.*

Consider a measure space $(J, \mathcal{A}, \pi)$ and a bounded measurable function $f : J \to \mathbb{R}$ with respect to $\mathcal{A}$, and let $\mathcal{A}'$ be a sub-σ-algebra of $\mathcal{A}$. For every function $f \in L^1(\mathcal{A})$, *the conditional expectation* of f with respect to $\mathcal{A}'$ is the unique function $E(f|\mathcal{A}') \in L^1(\mathcal{A})$ such that

$$\int_{A'} E(f|\mathcal{A}') \quad d\pi = \int_{A'} f \quad d\pi \tag{2.3.1}$$

for all $A' \in \mathcal{A}'$.

A special case of a measure space is a *probability space*. A probability space is a measure space $(J, \mathcal{A}, \pi)$ such that $\pi(J) = 1$. The measure $\pi : \mathcal{A} \to [0, 1]$ is called a probability measure.

The product σ-algebra and the product measure

Suppose we have two measurable spaces $(J_1, \mathcal{A}_1)$ and $(J_2, \mathcal{A}_2)$. Then we say that a *rectangle* in $J_1 \times J_2 = \{(x, y) : x \in J_1 \quad \text{and} \quad y \in J_2\}$ is a set of the form $A_1 \times A_2$ where $A_1 \subseteq J_1$ and $A_2 \subseteq J_2$.

The product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ is defined to be the smallest σ-algebra on $J_1 \times J_2$ that contains

$$\mathcal{R} = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}.$$

We say that $\mathcal{R}$ is the collection of measurable rectangles. Moreover, if $E \subseteq J_1 \times J_2$ then we say that E is *elementary* if it is the union of finitely many pairwise disjoint subsets of the form $A_1 \times A_2$ where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Lemma 2.23. [20, p. 209] For any $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, let E_x and E^y for $x \in J_1$ and $y \in J_2$ be defined by

$$E_x = \{y \in J_2 : (x, y) \in E\}$$

and

$$E^y = \{x \in J_1 : (x, y) \in E\}$$

Then, $E_x \in \mathcal{A}_2$ and $E^y \in \mathcal{A}_1$.

We say that a measure π on a measurable space $(J, \mathcal{A})$ is *σ-finite* or that the measure space $(J, \mathcal{A}, \pi)$ is σ-finite if and only if there exists a sequence $A_1, A_2, \ldots$ such that $\bigcup A_i = \mathcal{A}$ and $\pi(A_i) < \infty$ for all i. Furthermore, the measure $\pi : \mathcal{A} \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is called a *signed measure* on J if

- π takes at most one of the values $+\infty$ or $-\infty$.
- $\pi(\varnothing) = 0$.
- $\pi(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} \pi(A_n)$ for every sequence $A_1, A_2, \ldots$ of disjoint measurable sets.

Definition 2.24. [20] Let π and μ be two σ-finite measures on a measurable space $(J, \mathcal{A})$. Then we say that

(i) μ is an *absolutely continuous* with respect to π, denoted as $\mu << \pi$, if $A \in \mathcal{A}$ then $\pi(A) = 0$ implies that $\mu(A) = 0$.

(ii) μ is a *singular* with respect to π, denoted as $\mu \perp \pi$, if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0$ and $\pi(A^c) = 0$.

The idea of a *product measure* on the product σ-algebra is based on the following theorem.

Theorem 2.25. [20, p. 214] Let $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ be σ-finite measure spaces and let $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then the functions

$$\varphi : J_1 \underset{x \mapsto \pi_2(E_x)}{\longrightarrow} [0, \infty], \quad \psi : J_2 \underset{y \mapsto \pi_1(E^y)}{\longrightarrow} [0, \infty]$$

are measurable and

$$\int_{J_1} \pi_2(E_x)d\pi_1 = \int_{J_2} \pi_1(E^y)d\pi_2. \tag{2.3.2}$$

In order to prove Theorem 2.25, we shall state some useful definitions and theorems, without proof.

Definition 2.26 (Monotone Class [3])**.** Suppose we have a nonempty set J and a set of subsets of J, say $\mathcal{M}$. Then $\mathcal{M}$ is called a *monotone class* if:

- If $A_1 \subseteq A_2 \subseteq \ldots$ is an increasing sequence of sets in $\mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$,
- If $A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing sequence of sets in $\mathcal{M}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

Since σ-algebras are closed under arbitrary countable unions and intersections, then it is obvious that every σ-algebra is a nonempty monotone class.

Recall that if we have a collection $\mathcal{U}$ of subsets of a set J then we define $\sigma(\mathcal{U})$ as the minimal σ-algebra containing $\mathcal{U}$. We showed in Lemma 2.5 that such $\sigma(\mathcal{U})$ exists.

Remark 2.27. In particular, if $\mathcal{M}$ is a class of functions from a given measure space J into $\mathbb{R}$, and

$$\mathcal{U} = \{f^{-1}\big((a,b)\big) : f \in \mathcal{M}, \quad a \leqslant b \in \mathbb{R}\}$$

then the σ-algebra $\sigma(\mathcal{U})$ gives the smallest σ-algebra of subsets of J such that every function in $\mathcal{M}$ is measurable with respect to this algebra.

Theorem 2.28 (Monotone class theorem for functions [11])**.** Suppose that $\mathcal{M}$ is a class of functions mapping a measure space J into $\mathbb{R}$. Suppose that

$\mathcal{M}$ is closed under multiplication (i.e. $\{f, g\} \subseteq \mathcal{M} \implies fg \in \mathcal{M}$). Further, suppose that $\mathcal{V}$ is a vector space of functions with $\mathcal{M} \subset \mathcal{V}$ containing the constant functions and such that whenever $(f_n)_{n \geqslant 1}$ is a sequence in $\mathcal{V}$ satisfying $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots$, then if $f = \lim_{n\to\infty} f_n$ is bounded, it follows that $f \in \mathcal{V}$.

Let $\mathcal{A}$ denote the smallest σ-algebra on J such that all of the functions in $\mathcal{M}$ are measurable with respect to J (such $\mathcal{A}$ exists by Remark 2.27). Then $\mathcal{V}$ contains all bounded functions measurable with respect to $\mathcal{A}$.

Theorem 2.29. (Lebesgue Monotone Convergence Theorem [20, p. 23]) Let $(J, \mathcal{A}, \pi)$ be a measure space and let $f_n : J \to [0, \infty]$ be a sequence of measurable functions such that

$$f_n(x) \leqslant f_{n+1}(x)$$

for all $x \in J$ and all $n \in \mathbb{N}$.

Define $f : J \to [0, \infty]$ by $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in J$. Then, f is measurable and

$$\lim_{n\to\infty} \int_J f_n d\pi = \int_J f d\pi.$$

We note that we use the convention that if $(f_n(x))_n$ diverges to ∞, then $f(x) = \infty$ and hence if $f(x) = \infty$ almost everywhere, then $\int_J f dx = \infty$.

Definition 2.30. (Lebesgue Integrable Functions [20]) Let $(J, \mathcal{A}, \pi)$ be a measure space. A function $\varphi : J \to \mathbb{R}$ is called (Lebesgue) integrable if φ is measurable and $\int_J |\varphi| d\pi < \infty$.

Theorem 2.31. (Lebesgue Dominated Convergence Theorem [20, p. 32]) Let $(J, \mathcal{A}, \pi)$ be a measure space, let $g : J \to [0, \infty)$ be an integrable function, and let $f_n : J \to \mathbb{R}$ be a sequence of integrable functions satisfying

$$|f_n(x)| \leqslant g(x)$$

for all $x \in J$ and all $n \in \mathbb{N}$, and converging pointwise to $f : J \to \mathbb{R}$ (That means $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in J$).

Then, f is integrable and for every $E \in \mathcal{A}$, we have

$$\int_E f d\pi = \lim_{n\to\infty} \int_E f_n d\pi.$$

Now we can prove Theorem 2.25, see [20, Theorem 7.9].

***Proof of Theorem** 2.25.* Recall that $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ are σ-finite measure spaces. For $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$ define functions φ_E and ψ_E by

$$\varphi_E : \underset{x\mapsto\pi_2(E_x)}{J_1 \to [0,\infty]}, \quad \psi_E : \underset{y\mapsto\pi_1(E^y)}{J_2 \to [0,\infty]}.$$

Let

$$\Omega = \Big\{ E \in \mathcal{A}_1 \otimes \mathcal{A}_2 : \varphi_E : J_1 \to [0,\infty] \quad \text{and} \quad \psi_E : J_2 \to [0,\infty]$$
$$\text{are measurable and satisfy (2.3.2)} \Big\}.$$

We are going to show that $\Omega = \mathcal{A}_1 \otimes \mathcal{A}_2$.

First of all, we want to show that if we have $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, then $E = A_1 \times A_2 \in \Omega$. By assumption $E_x = A_2$ if $x \in A_1$ and $E_x = \varnothing$ otherwise. Similarly $E^y = A_1$ if $y \in A_2$ and $E^y = \varnothing$ otherwise.

Define $\varphi : J_1 \to [0,\infty]$ by $\varphi(x) = \pi_2(E_x) = \pi_2(A_2)\chi_{A_1}(x)$ for $x \in J_1$, where

$$\chi_{A_1} = \begin{cases} 1 & \text{if} \quad x \in A_1 \\ 0 & \text{if} \quad x \notin A_1, \end{cases}$$

and also $\psi : J_2 \to [0,\infty]$ by $\psi(y) = \pi_1(E^y) = \pi_1(A_1)\chi_{A_2}(y)$ for $y \in J_2$, where

$$\chi_{A_2} = \begin{cases} 1 & \text{if} \quad y \in A_2 \\ 0 & \text{if} \quad y \notin A_2. \end{cases}$$

Therefore, by Lemma 2.23 the functions φ and ψ are measurable and it follows that

$$\int_{J_1} \varphi d\pi_1 = \pi_1(A_1)\pi_2(A_2) = \int_{J_2} \psi d\pi_2.$$

Thus, $E \in \Omega$.

Second, we want to show that if $E_1, E_2 \in \Omega$ and their intersection is empty then $E = E_1 \cup E_2 \in \Omega$. Let us define $\varphi_i(x) = \pi_2((E_i)_x)$, $\varphi(x) = \pi_2(E_x)$ and $\psi_i(y) = \pi_2((E_i)^y)$, $\psi(y) = \pi_2(E^y)$ for $x \in J_1$ and $y \in J_2$, where $i = 1, 2$. Then, $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$. Since $E_i \in \Omega$ we can see that for $i = 1, 2$,

$$\int_{J_1} \varphi_i d\pi_1 = \int_{J_2} \psi_i d\pi_2 \implies \int_{J_1} \varphi d\pi_1 = \int_{J_2} \psi d\pi_2.$$

Thus, $E \in \Omega$.

Third, suppose that $E_i \in \Omega$ for all $i \in \mathbb{N}$ and $E_i \subseteq E_{i+1}$. Then we want to prove that $E = \bigcup_{i=1}^{\infty} E_i \in \Omega$. We define $\varphi_i, \varphi : J_1 \to [0, \infty]$ and $\psi_i, \psi : J_2 \to [0, \infty]$, for all i as above. Since $E_x = \bigcup_{i=1}^{\infty} (E_i)_x$ and $E^y = \bigcup_{i=1}^{\infty} (E_i)^y$ where $(E_i)_x \in \mathcal{A}_2$ and $(E_i)^y \in \mathcal{A}_1$ for all i, we have

$$\varphi(x) = \pi_2(E_x) = \lim_{i\to\infty} \pi_2((E_i)_x) = \lim_{i\to\infty} \varphi_i(x) \quad \text{for all} \quad x \in J_1$$
$$\psi(y) = \pi_2(E^y) = \lim_{i\to\infty} \pi_2((E_i)^y) = \lim_{i\to\infty} \psi_i(y) \quad \text{for all} \quad y \in J_2.$$

By the Lebesgue Monotone Convergence Theorem 2.29,

$$\int_{J_1} \varphi d\pi_1 = \lim_{i\to\infty} \int_{J_1} \varphi_i d\pi_1 = \lim_{i\to\infty} \int_{J_2} \psi_i d\pi_1 = \int_{J_2} \psi d\pi_1.$$

Therefore, $E \in \Omega$.

Fourth, suppose $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then $\pi_1(A_1) < \infty$ and $\pi_2(A_2) < \infty$.

If $E_i \in \Omega$ for all $i \in \mathbb{N}$ such that $A_1 \times A_2 \supseteq E_1 \supseteq E_2 \supseteq \dots$ then $E = \bigcap\limits_{i=1}^{\infty} E_i \in \Omega$, because $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras. Since $(E_i)_x \subseteq A_2$ and $\pi_2(A_2) < \infty$ then φ_i converge pointwise to φ. Furthermore, $\varphi_i \leqslant \pi_2(A_2)\chi_{A_1}$, for all i and then the function $\pi_2(A_2)\chi_{A_1} : J_1 \to [0, \infty)$ is integrable since $\pi_1(A_1) < \infty$ and $\pi_2(A_2) < \infty$. By the Lebesgue Dominated Convergence Theorem 2.31, we have

$$\int\limits_{J_1} \varphi\pi_1 = \lim_{i\to\infty} \int\limits_{J_1} \varphi_i d\pi_1.$$

Similarly for $\int\limits_{J_2} \psi d\pi_1 = \lim_{i\to\infty} \int\limits_{J_2} \psi_i d\pi_1$. Since $E_i \in \Omega$ for all i, then we have that $\int\limits_{J_1} \varphi d\pi_1 = \int\limits_{J_2} \psi d\pi_1$ and thus $E \in \Omega$.

Finally, by assumption we know that $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ are σ-finite. Then there are sequences of measurable sets $(J_1)_n \in \mathcal{A}_1$ and $(J_2)_n \in \mathcal{A}_2$ such that $(J_1)_n \subseteq (J_1)_{n+1}$ and $(J_2)_n \subseteq (J_2)_{n+1}$ where $\pi_1((J_1)_n) < \infty$ and $\pi_2((J_2)_n) < \infty$ for all $n \in \mathbb{N}$ such that $J_1 = \bigcup\limits_{n=1}^{\infty} (J_1)_n$ and $J_2 = \bigcup\limits_{n=1}^{\infty} (J_2)_n$. Now let

$$\mathcal{M} = \Big\{ E \in \mathcal{A}_1 \otimes \mathcal{A}_2 : E \cap \big((J_1)_n \times (J_2)_n \big) \in \Omega \quad \text{for all} \quad n \in \mathbb{N} \Big\}.$$

So, $\mathcal{M}$ is a monotone class. Then clearly $\mathcal{M} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2$. Since

$$E \cap \Big((J_1)_n \times (J_2)_n \Big) \in \Omega$$

for all $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, we have

$$E = \bigcup_{n=1}^{\infty} \Big(E \cap ((J_1)_n \times (J_2)_n) \Big) \in \Omega$$

for all $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Therefore, $\mathcal{A}_1 \otimes \mathcal{A}_2 \subseteq \Omega \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2$ which implies $\mathcal{A}_1 \otimes \mathcal{A}_2 = \Omega$. □

The following is a well known theorem.

Theorem 2.32. [2] Suppose we have two measurable spaces $(J_1, \mathcal{A}_1, \pi_1)$

and $(J_2, \mathcal{A}_2, \pi_2)$, where π_1 and π_2 are σ-finite measures. Then, there exists a unique σ-finite measure π on $(J_1 \times J_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ such that $\pi(A_1 \times A_2) = \pi_1(A_1)\pi_2(A_2)$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Proof. We start our proof by showing that any measure π satisfying the condition above must be σ-finite. Since π_1 and π_2 are σ-finite, then we have $(A_n)_{n=1}^{\infty} \in \mathcal{A}_1$ and $(B_n)_{n=1}^{\infty} \in \mathcal{A}_2$ such that $\bigcup\limits_{n=1}^{\infty} A_n = J_1$ and $\bigcup\limits_{n=1}^{\infty} B_n = J_2$, thus $\pi_1(A_n)$ and $\pi_2(B_n)$ are finite for all n. Suppose that $\bigcup\limits_{(i,j)\in\mathbb{N}^2} A_i \times B_j$, and for any $(x, y) \in J_1 \times J_2$ we have i, j such that $x \in A_i$ and $y \in B_j$. Thus $(x, y) \in A_i \times B_j$ which implies that $\bigcup\limits_{(i,j)\in\mathbb{N}^2} A_i \times B_j = J_1 \times J_2$. For $(i, j) \in \mathbb{N}^2$ we have $\pi(A_i \times B_j) = \pi_1(A_i)\pi_2(B_j) < \infty$. Since $\mathbb{N}^2$ is a countable set then π is σ-finite.

Now, let us consider σ-finite measures π and π' which satisfy the condition above. We define collection of measurable rectangles $\mathcal{R}$ by

$$\mathcal{R} = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}.$$

$\mathcal{R}$ is closed under finite intersection. Moreover, the two σ-finite measures π and π' agree on this $\mathcal{R}$. That means they also agree on the generated σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$, so $\pi = \pi'$. Thus the measure must be unique.

Suppose that $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$, and let $\pi(A) = \int\limits_{J_1} \pi_2(A_x) d\pi_1(x)$ where

$$A_x = \{y \in J_2 : (x, y) \in J_1 \times J_2\}.$$

By Theorem 2.25, π is a measure. Now, let us consider that $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, then

$$\begin{aligned}\pi(A_1 \times A_2) = \int\limits_{J_1} \pi_2((A_1 \times A_2)_x) d\pi_1(x) &= \int\limits_{J_1} 1_{A_1} \pi_2(A_2) d\pi_1(x) \\ &= \pi_2(A_2) \int\limits_{J_1} 1_{A_1} d\pi_1(x) = \pi_1(A_1)\pi_2(A_2).\end{aligned}$$

□

We call the unique measure that satisfies

$$\pi(A_1 \times A_2) = \pi_1(A_1)\pi_2(A_2) \quad \text{for all} \quad A_1 \in \mathcal{A}_1 \quad \text{and} \quad A_2 \in \mathcal{A}_2$$

the product measure. We write it as $\pi = \pi_1 \otimes \pi_2$. The product measure satisfies

$$(\pi_1 \otimes \pi_2)(A_1 \times A_2) = \pi_1(A_1)\pi_2(A_2)$$

for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Suppose that $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ be measure spaces. Let φ is a function from J_1 to J_2. Define $\varphi \otimes \varphi : J_1^2 \to J_2^2$ by

$$\varphi \otimes \varphi(x, y) = (\varphi(x), \varphi(y)).$$

We want to show that if the function φ is measurable then $\varphi \otimes \varphi$ is measurable.

Lemma 2.33. If the function $\varphi : J_1 \to J_2$ is measurable, then the function $\varphi \otimes \varphi : J_1^2 \to J_2^2$ is measurable. Further, if φ is measure preserving then so is $\varphi \otimes \varphi$.

Proof. Suppose that we have two measure spaces $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$. By the definition of product spaces, the σ-algebra of J_1^2 is $\sigma(\{A \times B : A, B \in \mathcal{A}_1\})$. Similarly, the σ-algebra on J_2^2 is $\sigma(\{A \times B : A, B \in \mathcal{A}_2\})$. To show $\varphi \otimes \varphi$ is measurable it is enough to show that for $A, B \in \mathcal{A}_2$, we have

$$(\varphi \otimes \varphi)^{-1}(A, B) \in \mathcal{A}_1 \otimes \mathcal{A}_1.$$

Now $\varphi^{-1}(A) \in \mathcal{A}_1$ and $\varphi^{-1}(B) \in \mathcal{A}_1$, since φ is measurable. Therefore,

$$(\varphi^{-1}(A), \varphi^{-1}(B)) \in \mathcal{A}_1 \otimes \mathcal{A}_1$$

Then, we get $(\varphi \otimes \varphi)^{-1}(A, B) \in \mathcal{A}_1 \otimes \mathcal{A}_1$, so $\varphi \otimes \varphi$ is measurable.
Now, let $A, B \in \mathcal{A}_2$, and let $\pi_1 \otimes \pi_1$ be the product measure of J_1. If $(\varphi \otimes \varphi)^{-1}(A, B) = \varphi^{-1}(A) \times \varphi^{-1}(B)$, then

$$\begin{aligned}
\pi_1 \otimes \pi_1((\varphi \otimes \varphi)^{-1}(A, B)) &= \pi_1 \otimes \pi_1(\varphi^{-1}(A) \times \varphi^{-1}(B)) \\
&= \pi_1(\varphi^{-1}(A)) \cdot \pi_1(\varphi^{-1}(B)) \\
&= \pi_2(A) \cdot \pi_2(B) \qquad \text{(since } \varphi \text{ is measure preserving)} \\
&= \pi_2 \otimes \pi_2(A \times B)
\end{aligned}$$

□

2.4 Graphons

In the previous sections we explored the idea that two graphs G and G' are close together if the homomorphism densities $t(F, G)$ and $t(F, G')$ are close to each other, for all graphs F. We can look for sequences of graphs that will get the graphs closer and closer to each other.

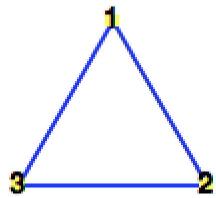
The rational sequence $1, 1.4, 1.41, \ldots$ is a Cauchy sequence but it is convergent to irrational number $\sqrt{2}$. So we see that a sequence of rational numbers can converge to the irrational number. Therefore, it is not unexpected if we have a sequence of graphs that converges to something other than a graph. *Graphons* are the limits of Cauchy sequences of graphs. They are symmetric, Lebesgue measurable functions from $[0, 1]^2$ to $[0, 1]$.

In this section we introduce one of the important objects in graphs. An *adjacency matrix* is an essential tool where we can visualize the pixel picture of any graph and then its limit function.

Suppose we have a simple graph $G = (V, E)$ with $V = [n] = \{1, \ldots, n\}$. The *adjacency matrix* $A = (A_G)$ is the $n \times n$ symmetric matrix defined by

$$A_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.1}$$

Example 2.34. Suppose the graph A is

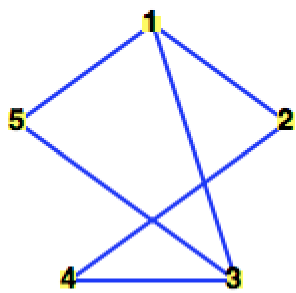


Then the adjacency matrix of A is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The graphon corresponding to a graph is closely related to the graph's adjacency matrix. We can build the adjacency matrix of graph by labeling the graph's vertices $[n] = \{1, 2, \ldots, n\}$. Then, we defined the adjacency matrix to be the $n \times n$ matrix of 0's and 1's as in (2.4.1).

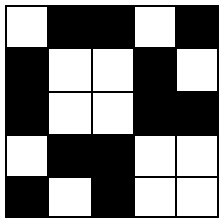
Example 2.35. Let G be the following graph:



The adjacency matrix of G is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We can see the graphon for G by making the adjacency matrix a *stepfunction* on the unit square. Thus, the graphon for G is



As we see in Example 2.35 a graphon can be represented as a pixelation of the adjacency matrix of the graph.

Now we see an example how sequence of graphs tends to a graphon.

Example 2.36. Let H_n be *the half graph*, that is, the bipartite graph on $2n$ vertices $\{1, \ldots, n, 1', \ldots, n'\}$ where the edge (i, j') is present if $i \leqslant j'$. Then we can see the sequence of graphs H_n converges to a limit W as in Figure 2.4.1.

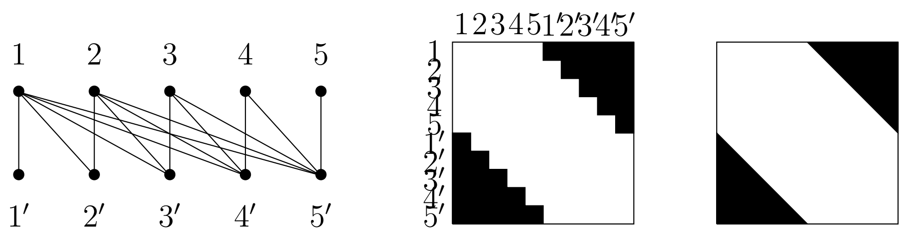


Figure 2.4.1: The half-graph H_5, its pixel picture W_{H_5}, and its limit $W = \lim_{n\to\infty} W_{H_n}$.

Note that as $n \to \infty$ limit of the graph sequence is a map $W : [0,1] \times [0,1] \to [0,1]$. This map is a graphon.

Now we want to introduce graphons in a formal way and find how a sequence of graphs converges to a graphon. Thus, we should consider graphons as generalization of graphs.

Definition 2.37. [13, p. 115] Let $\mathscr{W}$ be the space of all bounded symmetric measurable functions $W : [0,1]^2 \longrightarrow \mathbb{R}$. The elements of the space $\mathscr{W}$ will be called *kernels*. Also, define the subspace

$$\mathscr{W}_\circ := \{W \in \mathscr{W} : 0 \leqslant W(x,y) \leqslant 1, \quad \text{for all} \quad x, y \in [0,1]\}.$$

Then we call any $W \in \mathscr{W}_\circ$ a *graphon*.

Given a graph G, we now explain how to construct a graphon W_G.

Definition 2.38. [13, p. 116] Let G be any graph and $V(G) = \{1, 2, \ldots, v(G)\}$. For $x \in \left[\frac{i-1}{v(G)}, \frac{i}{v(G)}\right)$ and $y \in \left[\frac{j-1}{v(G)}, \frac{j}{v(G)}\right)$ we define

$$W_G(x,y) := \begin{cases} 1 & \text{if} \quad (i,j) \in E(G), \\ 0 & \text{if} \quad (i,j) \notin E(G). \end{cases} \tag{2.4.2}$$

Recall that we defined homomorphism densities for graphs in section 2.1.2. We want to expand the homomorphism densities in graphs to homomorphism densities in graphons, generally in kernels. We should notice that we are moving from discrete to continuous objects so we should use integrals instead of counting and sums.

Suppose that W is a kernel ($W \in \mathscr{W}$), and let $F = (V, E)$ be a multigraph with no loops. Then, we define the homomorphism density of F into W as

$$t(F, W) := \int_{[0,1]^V} \prod_{(i,j) \in E} W(x_i, x_j) \prod_{s \in V} dx_s.$$

If $W = W_G$ is a kernel corresponding to a graph G then $t(F, W)$ measures the homomorphism density between F and G.

Proposition 2.39. [13, p. 116] If G and H are graphs, then

$$t(G, H) = t(G, W_H).$$

Example 2.40. In this example Steele in [21] shows that

$$t(K_2, H_n) = t(K_2, W_{H_n}).$$

First of all, we calculate $t(K_2, H_n)$ using the definition from section 2.1.2:

$$t(K_2, H_n) = \frac{2e(H_n)}{v(H_n)^{v(K_2)}} = \frac{2(\frac{n(n+1)}{2})}{(2n)^2} = \frac{n+1}{4n},$$

where $e(H_n)$ is the number of edges in H_n, and we denote the number of vertices in H_n and K_2 by $v(H_n)$ and $v(K_2)$. Now, we will calculate $t(K_2, W_{H_n})$:

$$\begin{aligned} t(K_2, W_{H_n}) &= \sum_{i=1}^{2n}\sum_{j=1}^{2n}\int_{\frac{i-1}{2n}}^{\frac{i}{2n}}\int_{\frac{j-1}{2n}}^{\frac{j}{2n}} W(x_1, x_2)dx_1dx_2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\int_{\frac{i-1}{2n}}^{\frac{i}{2n}}\int_{\frac{j-1}{2n}}^{\frac{j}{2n}} W(x_1, x_2)dx_1dx_2 \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\left(\frac{i}{2n} - \frac{i-1}{2n}\right)\left(\frac{j}{2n} - \frac{j-1}{2n}\right) \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n+1-i}\frac{1}{4n^2} \\ &= \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n+1-i} 1 \\ &= \frac{1}{2n^2}\sum_{i=1}^{n} n+1-i \\ &= \frac{1}{2n^2}\left((n+1)n - \frac{(n+1)n}{2}\right) \\ &= \frac{n+1}{4n}. \end{aligned}$$

We notice that $\frac{n+1}{4n}$ goes to $\frac{1}{4}$ as n goes to ∞. That means the homomorphism densities of K_2 in H_n converge as n goes to ∞.

We mentioned that a sequence of graphs convergent to a limit which is graphon. However, we need to know what it means for a sequence of graphs be convergent. Thus, we define convergence by taking a sample, k vertices, from graphs in the sequence and then if this sampling converges that means the sequence of graphs also converges.

Definition 2.41. [13, p. 173] Let (G_n) be a sequence of graphs where the number of vertices of G_n goes to infinity, $v(G_n) \to \infty$. Then (G_n) is convergent if the induced subgraph densities $t_{\text{ind}}(F, G_n)$ converge for all finite graphs F.

We can also define convergence by using the homomorphism densities $t(F, G_n)$. It is the same as defining convergence as above. Referring to the relationship between homomorphism numbers in ([13], section 5.2.3), $t_{\text{inj}}(F, G_n)$ can be expressed as a linear combination of $t_{\text{ind}}(F, G_n)$ and vice versa, thus $t_{\text{inj}}(F, G_n)$ tends to a limit as $n \longrightarrow \infty$ if and only if $t_{\text{ind}}(F, G_n)$ does. Moreover,

$$t(F, G) - t_{\text{inj}}(F, G)$$

tends to zero as $v(G) \longrightarrow \infty$. Therefore, $t(F, G_n)$ tends to a limit as $n \longrightarrow \infty$ if and only if $t_{\text{inj}}(F, G_n)$ does.

Theorem 2.42. [13, p. 173] A sequence (G_n) of simple graphs with $v(G_n) \longrightarrow \infty$ is convergent if and only if $t(F, G_n)$ is convergent for every finite graph F.

We will see later (Theorem 2.58) that every convergent sequence of graphs converges to a graphon.

Since we are talking about convergent sequences, we define some notions of distance.

2.4.1 Cut distance of graphs

Let G be a bipartite graph. That means we can partition its vertices $V(G)$ into two sets, say G_1 and G_2, where all edges in $E(G)$ are between G_1 and G_2. Then we call (G_1, G_2) a cut of G.

We can ask ourselves that how close, approximately, a graph G is being bipartite. We are going to measure how close to a cut we obtain when we partition $V(G)$ by means of various (G_1, G_2). Then we can get a cut distance of these pieces by measuring how good their cut is.

We can represent the cut distance between arbitrary graphs as a measure of their similarity.

Now, let us review the notion of *cut distance* for graphs and graphons of Chapter 8 in [13].

Definition 2.43 (Cut norms of graphs)**.** Suppose we have a graph G with $V(G) = [n]$. Let $S, T \subseteq [n]$, we need not assume that $S \cap T = \varnothing$. If we want to study how close (S, T) is to being a cut, we need to count the number of edges between S and T compared to the total number of edges. We say that (S, T) will be the closest to the cut if it has the largest value between all possible pairs. We represent a density parameter of the graph G as follows:

$$||G||_{\square} = \frac{1}{n^2} \max_{S,T \subseteq [n]} \Big| \{(i,j) \in E(G) : i \in S, j \in T\} \Big|.$$

We can rewrite this in terms of the adjacency matrix of G, denoted by A_G. Then, we have

$$||G||_{\square} = \frac{1}{n^2} \max_{S,T \subseteq [n]} \Big| \sum_{i \in S, j \in T} (A_G)_{ij} \Big|.$$

Frieze and Kannan in [9] generalized this to any $n \times n$ matrix, say A, to define a *cut norm* as a following:

$$||A||_{\square} = \frac{1}{n^2} \max_{S,T \subseteq [n]} \left| \sum_{i \in S, j \in T} A_{ij} \right|.$$

Definition 2.44. (Two graphs on the same set of vertices [13]) If we have two graphs G_1 and G_2 on the same set of vertices, we can define their distance through the norm of the difference of the adjacency matrices of the cut norm above.

Let A and A' be $n \times n$ matrices, then

$$d_{\square}(A, A') = ||A - A'||_{\square} = \frac{1}{n^2} \max_{S,T \subseteq [n]} \left| \sum_{i \in S, j \in T} A_{ij} - A'_{ij} \right|.$$

By adapting this to graphs case we get a distance between two graphs on the same set of vertices. Suppose G_1 and G_2 are simple graphs with $V(G_1) = V(G_2) = [n]$. Define

$$d_{\square}(G_1, G_2) = \max_{S,T \subseteq [n]} \frac{|e_{G_1}(S,T) - e_{G_2}(S,T)|}{n^2} = ||A_{G_1} - A_{G_2}||_{\square},$$

where $e_{G_i}(S,T)$ for $i = 1, 2$ is the number of edges in G_i with one vertex in S and the other in T.

Example 2.45. Suppose we have two graphs $G_1 = K_3$ and G_2 is an empty graph on 3 vertices. For any $S, T \subseteq [3]$, we get $e_{G_2}(S,T) = 0$. We realize that if $|T| = \varnothing$ or $|S| = \varnothing$, then $e_{G_1}(S,T) = 0$. However, if $|T| = |S| = 3$, then $e_{G_1}(S,T) = 6$ is maximal. Therefore, $d_{\square}(G_1, G_2) = \frac{6}{9} = \frac{2}{3}$.

Let G_1 and G_2 be isomorphic graphs. Then, let us ask ourselves that what will we get if we relabel the vertices of G_1 and G_2? Clearly, they are the same graph. Thus, their $d_{\square}$-distance should be zero. So, we need to define the following:

Definition 2.46. [13] Let G_1 and G_2 be graphs on n vertices. Then, their cut distance is

$$\hat{\delta}_\square(G_1, G_2) = \min_{\hat{G_1}, \hat{G_2}} d_\square(\hat{G_1}, \hat{G_2})$$

where $\hat{G_1}$ and $\hat{G_2}$ are range over all possible labelings of the vertices of G_1 and G_2.

Previously, we explained the way how to find the distance between graphs that have the same number of vertices. However, here we need to define the distance of two arbitrary graphs. Therefore, we need to introduce a new notion called *blow-up*.

Definition 2.47. [13] Let G be a graph and $m \in \mathbb{N}$. Then, we get the graph $G(m)$ from G by replacing each vertex of G by m vertices $v(m)$ and connecting all vertices in $v(m)$ with all vertices in $v'(m)$ if and only if there is an edge in G between v and v'.

Example 2.48. In this example we can see how a graph G blowing up to $G(2)$.

Now we can define the distance of two arbitrary graphs.

Definition 2.49. (Two arbitrary graphs [13]) Let us consider two graphs G_1 and G_2, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with two sets of vertices $V_1 = [n_1]$ and $V_2 = [n_2]$. By using the blow-up operation, we have $G_1(n_2)$ and $G_2(n_1)$. More generally, we have $G_1(kn_2)$ and $G_2(kn_1)$ for $k \in \mathbb{N}$, so we define the distance between G_1 and G_2 by using the $\hat{\delta}_\square$ distance as

$$\delta_\square(G_1, G_2) = \lim_{k \longrightarrow \infty} \hat{\delta}_\square(G_1(kn_2), G_2(kn_1))$$

Note that if $n_1 = n_2$, then it is not necessarily true that

$$\delta_\square(G_1, G_2) = \hat{\delta}_\square(G_1, G_2).$$

However,

$$\delta_\square(G_1, G_2) \leqslant \hat{\delta}_\square(G_1, G_2).$$

In fact $\delta_\square$ is not a metric, it is only a *pseudometric* since $\delta_\square(G_1, G_2)$ might be zero for different graphs G_1 and G_2.

2.4.2 Cut norm and cut distance of kernels

Previously, we explored the cut distance for graphs and now we want to extend this to kernels.

Definition 2.50. (Cut norm of kernels [13]) We define a *cut norm* of kernel W as

$$||W||_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S\times T} W(x,y)dxdy \right|$$

where the supremum is taken over all measurable subsets S and T. Since $S \times T$ and $W(x,y)$ are bounded, the supremum exists.

The cut metric is defined as

$$d_\square(U, W) = ||U - W||_\square ,$$

where U and W are kernels.

Representing a graph as a kernel depends on the labelling of the graph. Actually, we label the kernel by intervals $[\frac{k}{n}, \frac{k+1}{n}]$ through copying the labelling of the graph. Just as for graphs, we need to represent an "unlabeled" version of the cut norm. We let $S_{[0,1]}$ to be the set of all invertible measure preserving maps $\varphi : [0,1] \longrightarrow [0,1]$. Since the inverse of the map φ is a measure preserving as well, $S_{[0,1]}$ is a group.

Definition 2.51. (Cut distance of kernels [13]) For two kernels U, W and

$\varphi \in S_{[0,1]}$, define the cut distance of them as

$$\delta_{\square}(U, W) = \inf_{\varphi \in S_{[0,1]}} d_{\square}(U, W^{\varphi})$$

where $W^{\varphi}(x, y) = W\big(\varphi(x), \varphi(y)\big)$.

Moreover, let $\bar{S}_{[0,1]}$ be the set of measure preserving maps $[0,1] \longrightarrow [0,1]$ and then

$$\delta_{\square}(U, W) = \inf_{\varphi \in S_{[0,1]}} d_{\square}(U^{\varphi}, W) = \inf_{\varphi \in \bar{S}_{[0,1]}} d_{\square}(U, W^{\varphi}) = \inf_{\psi, \varphi \in \bar{S}_{[0,1]}} d_{\square}(U^{\psi}, W^{\varphi})$$

Note that $\delta_{\square}(U, W)$ is a pseudometric since the different kernels U and W can have distance zero.

The next theorem shows that the $\delta_{\square}$ distance does not change when we replace a graph G by W_G.

Theorem 2.52. For any two weighted graphs G_1 and G_2 we have

$$\delta_{\square}(G_1, G_2) = \delta_{\square}(W_{G_1}, W_{G_2})$$

Proof. See Lemma 8.9 in [13]. □

Now we can define what it means for a sequence of graphs to be convergent through the cut distance and homomorphism densities by stating some results of Lovász. However, we need to state the Counting Lemma and the Inverse Counting Lemma. For more details see [[13] Chapter 10].

Lemma 2.53. [13, p. 167] (Counting Lemma for Graphs). For any three simple graphs G_1, G_2 and G_3

$$|t(G_1, G_2) - t(G_1, G_3)| \leqslant e(G_1)\delta_{\square}(G_2, G_3).$$

The lemma extends to graphons:

Lemma 2.54. [13, p. 167] (Counting Lemma for Graphons). Let G be a simple graph, and let W_1 and W_2 be graphons. Then

$$|t(G, W_1) - t(G, W_2)| \leqslant e(G)\delta_{\square}(W_1, W_2).$$

Lemma 2.55. [13, p. 169] (Inverse Counting Lemma). Let $n \in \mathbb{Z}^+$. Suppose we have two graphons W_1 and W_2, and assume that for every simple graph G on n vertices, we have

$$|t(G, W_1) - t(G, W_2)| \leqslant 2^{-n^2}.$$

Then

$$\delta_{\square}(W_1, W_2) \leqslant \frac{50}{\sqrt{\log n}}.$$

Lemma 2.56. Suppose that W_1 and W_2 are graphons. Then for all finite simple graphs F,

$$t(F, W_1) = t(F, W_2) \iff \delta_{\square}(W_1, W_2) = 0.$$

Proof. Follows from Lemmas 2.54 and 2.55. □

The following theorem is proved using the Counting Lemma for graphs 2.53 and the Inverse Counting Lemma 2.55.

Theorem 2.57. [13, p. 174] A sequence (G_n) of simple graphs with $v(G_n) \longrightarrow \infty$ is convergent with respect to induced subgraph density if and only if it is a Cauchy sequence in the metric $\delta_{\square}$.

The next two theorems tell us that a graphon is the limit to every convergent sequence of graphs.

Theorem 2.58. [13, p. 180] For any convergent sequence (G_n) of simple graphs there exists a graphon W such that $t(G, G_n) \longrightarrow t(G, W)$ for every simple graph G. The graphon W is the limit of the graph sequence, and write $G_n \longrightarrow W$.

Theorem 2.58 has three different methods of proof. It was first proved by Lovász and Szegedy in 2006. They used Szemerédi partitions and the Martingale Convergence Theorem. We sketch their proof below.

The second method of proof was given by Diaconis and Janson in 2008 [7]. The proof used results of Aldous [1] and Hoover [10] on exchangeable random variables, identifying a basic connection to probability theory.

The last way of proving Theorem 2.58 was given by Elek and Segedy in 2012 [8]. They gave a different proof by using an *ultraproduct*, and they extended their work to many other structures such as *hypergraphs* and *hypergraphons*.

Recall that

$$\mathscr{W}_\circ := \{W \in \mathscr{W} : 0 \leqslant W(x,y) \leqslant 1, \quad \text{for all} \quad x, y \in [0,1]\}$$

is a set of graphons. We can identify two graphons whose cut distance is zero to get the set $\widetilde{\mathscr{W}_\circ}$ of unlabeled graphons.

Theorem 2.59. [16] The space $(\widetilde{\mathscr{W}_\circ}, \delta_\square)$ is compact.

The proof of Theorem 2.58 below depends on this theorem.

Proof of Theorem 2.58. [13, Theorem 11.21]. From Theorem 2.59 we know that $(\widetilde{\mathscr{W}_\circ}, \delta_\square)$ is compact. Then, the sequence $(W_n = W_{G_n} : n = 1, 2, \dots)$ has a convergent subsequence $(W_{n_j} : j = 1, 2, \dots)$ with limit $W \in \widetilde{\mathscr{W}_\circ}$. By using the Counting lemma for graphons Lemma 2.54, we have

$$|t(G, W_{n_j}) - t(G, W)| \leqslant e(G)\delta_\square(W_{n_j}, W) \to 0, \quad \text{when} \quad j \to \infty,$$

for every simple graph G. Thus, $t(G, W_{n_j}) = t(G, G_{n_j}) \to t(G, W)$. Since

$(t(G, G_n)), n = 1, 2, \ldots,$ is a Cauchy sequence, then $t(G, G_n) \to t(G, W)$ for every simple graph G. $\square$

The next theorem show that how the distance function can be used to describe the convergence to the limit object.

Theorem 2.60. [13, Theorem 11.22]. For a sequence (G_n) of graphs with $v(G_n) \longrightarrow \infty$ and a graphon W, we have $G_n \longrightarrow W$ if and only if $\delta_\square(W_{G_n}, W) \longrightarrow 0$.

Proof. Assume that $G_n \to W$. Then $t(G, G_n) \to t(G, W)$ for every simple graph G. By using Theorem 2.55, for every fixed m we have

$$|t(G, W_{G_n}) - t(G, W)| \leqslant 2^{-m^2}.$$

Then

$$\delta_\square(W_{G_n}, W) \leqslant \frac{20}{\sqrt{\log m}}$$

if n is large enough. Thus, $\delta_\square(W_{G_n}, W) \longrightarrow 0$.

Conversely, assume that $\delta_\square(W_{G_n}, W) \longrightarrow 0$. Then by using the Counting Lemma, Lemma 2.54, we have

$$|t(G, W_{G_n}) - t(G, W)| \leqslant e(G)\delta_\square(W_{G_n}, W) \to 0$$

for every simple graph G. Thus,

$$t(G, W_{G_n}) = t(G, G_n) \to t(G, W) \implies G_n \to W.$$

$\square$

3

The Purification of Graphons

In the previous section, we worked with graphons over the space $[0, 1]$. In general, it is convenient to define a graphon on an arbitrary space rather than $[0, 1]$. Let us consider a probability space $J = (J, \mathcal{A}, \pi)$. A graphon on J is a symmetric measurable function $W : J \times J \to [0, 1]$.

In this chapter we explore a significant notion which is called a *pure graphon*. The main references in this chapter are [13] and [12]. We first need to investigate two essential concepts which are a *pull-back* of a graphon and a *weak isomorphism* between two graphons.

Let us consider two probability spaces $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$, and a measure preserving function $\varphi : (J_1, \mathcal{A}_1, \pi_1) \to (J_2, \mathcal{A}_2, \pi_2)$. Let W and W' be graphons on $(J_1, \mathcal{A}_1, \pi_1)$ and $(J_2, \mathcal{A}_2, \pi_2)$ respectively. Then a *pull-back* $(W')^{\varphi}$ can be defined as

$$(W')^{\varphi}(x, y) = W'(\varphi(x), \varphi(y))$$

for all $x, y \in J_1$. In addition, the function φ above is said to be a *weak isomorphism* from W to W' if φ is a measure preserving from $\overline{\mathcal{A}_1}$ to $\mathcal{A}_2$ and $W = (W')^{\varphi}$ almost everywhere.

Furthermore, we say that W and W' are *weakly isomorphic* if there is a third graphon W'' and weak isomorphisms from W and W' to W''.

3.1 Pure graphons

We introduce a *pure* graphon. This pure version of graphon is uniquely determined up to a permutation of $[0, 1]$. Before we state the definition of *pure* graphon, we need to describe an important notion, the notion of *twins*.

Given a graphon W on a probability space $(J, \mathcal{A}, \pi)$, we say following [13] that two points x and x' in J are twins if $W(x, y) = W(x', y)$ for almost all

y in J. However, if no two distinct points in J are twins then we call W a twin-free graphon.

Moreover, we say that the graphon W is almost twin-free if there exists a null set $N \subseteq J$ such that there are no twins x and x' in $J \backslash N$ with $x \neq x'$.

Example 3.1. Let $W : [0,1] \times [0,1] \to [0,1]$. Then

- If $W(x,y) = xy$ where $x, y \in [0,1]$, then W is a twin-free graphon.
- If

$$W(x,y) = \begin{cases} 1 & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2}, \quad 0 \leqslant y \leqslant \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

  then we can see that W is not twin-free graphon since for example $\frac{1}{3}$ and $\frac{1}{4}$ are twin points for W. In fact, x and x' are twin points for W if and only if $x, x' \in [0, \frac{1}{2}]$ or $x, x' \in (\frac{1}{2}, 1]$.

Theorem 3.2. [13, p. 219] For every graphon W on $(J, \mathcal{A}, \pi)$ there is a twin-free graphon W' on a probability space $(J', \mathcal{A}', \pi')$, and a measure preserving map $\varphi : J \to J'$ such that $W = (W')^{\varphi}$ almost everywhere.

Proof. Let W be a graphon on $(J, \mathcal{A}, \pi)$. Define $\mathcal{F}$ to be the sub σ-algebra of $\mathcal{A}$ consisting of sets F such that if $x, y \in J$ are twins for W then F does not separate x and y. We get a new probability space $(J, \mathcal{F}, \pi)$.

Now, let us define $U = E(W|\mathcal{F} \times \mathcal{F})$. Then we want to show that $W = U$ almost everywhere. That means we need to show that for all $A, B \in \mathcal{A}$,

$$\int_{A \times B} W d\pi \times d\pi = \int_{A \times B} U d\pi \times d\pi.$$

Suppose that x and y are twin points and $A \in \mathcal{A}$. Then we define a function

$$X_A = \int_A W(\cdot, y) d\pi(y)$$

that is measurable with respect to $\mathcal{F}$. Similarly, we can define a measurable functions $g_A = E(1_A|\mathcal{F})$ where 1_A is the indicator (characteristic) function, and $Y_A = \int W(\cdot, y)g_A(y)d\pi(y)$. By using the measurable functions X_A, g_A and Y_A, we have

$$\begin{aligned}
\int_{A\times B} W d\pi \times d\pi &= \int X_A 1_B d\pi = \int X_A g_B d\pi \\
&= \int Y_B 1_A d\pi = \int Y_B g_A d\pi \\
&= \int W g_{AB} d\pi \times d\pi = \int U g_{AB} d\pi \times d\pi \\
&= \int U 1_{A\times B} d\pi \times d\pi = \int_{A\times B} U d\pi \times d\pi.
\end{aligned}$$

Thus, $W = U$ almost everywhere.

Now, suppose that J' is the set of equivalence classes of being twins on J, and for $x \in J$ let $\varphi(x)$ be the equivalence class containing x. Define

$$\mathcal{A}' = \{\varphi(C) : C \in \mathcal{F}\},$$

and for $C \in \mathcal{A}'$ we define $\pi'(C) = \pi(\varphi^{-1}(C))$. Then we get a new probability space $(J', \mathcal{A}', \pi')$.

Let $S_1, S_2 \in J'$. Then we define

$$W'(S_1, S_2) = U(x, y) = W(x, y)$$

for any $x \in S_1$ and $y \in S_2$. Since $\varphi : J \to J'$ and $W' : J' \times J' \to [0,1]$, we can see that

$$J \times J \xrightarrow{\varphi\times\varphi} J' \times J' \xrightarrow{W'} [0,1].$$

Therefore, $(W')^{\varphi} : J \times J \to [0,1]$, where $(W')^{\varphi} = U = W$ almost everywhere. □

Lovász and Szegedy in [15] define a *neighbourhood distance* as follows. Let

W be a graphon on a probability space $(J, \mathcal{A}, \pi)$. For any $x, y \in J$,

$$r_W(x,y) = \|W(x,\cdot) - W(y,\cdot)\|_1 = \int_J |W(x,z) - W(y,z)| d(\pi)z.$$

Definition 3.3. [15] A graphon W on a probability space $(J, \mathcal{A}, \pi)$ is *pure* if r_W is a metric and the metric space (J, r_W) is complete and separable, and π has full support.

Let us consider a probability space $(J, \mathcal{A}, \pi)$. A measurable set A of J is called an *atom* if $\pi(A) > 0$, and for every measurable subset B of A we have $\pi(A) = \pi(B)$ or $\pi(B) = 0$. If $(J, \mathcal{A}, \pi)$ has no atom, we call it *atomless*. If the measure space $(J, \mathcal{A}, \pi)$ is atomless, then it is said to be standard if it is isomorphic to $[0,1]$ with the Lebesgue measure, modulo null sets.

Suppose that we have a graphon $W : J \times J \to [0,1]$ on a probability space J. For each $x \in J$, we have a function $W_x : J \to [0,1]$ defined by

$$W_x(y) = W(x,y), \quad \text{for all} \quad y \in J.$$

That means W_x is a measurable function from J to $[0,1]$ and

$$\int W_x(y)dy = \int W(x,y)dxdy \quad < \infty.$$

Thus, $W_x \in L^1(J)$.

Define a function $\varphi_W : J \to L^1(J)$ by $\varphi_W(x) = W_x$. By a standard argument in the monotone class theorem for functions, φ_W is measurable (for more details [11]).

We define $\pi_W(A) = \pi(\varphi_W^{-1}(A))$ for $A \subseteq L^1(J)$, so that $\pi_W(A)$ is the measure on $L^1(J)$ induced by the measure π on J.

Now we define

$$J_W = \left\{ f \in L^1(J) : \quad \text{for every open set } O \text{ that contains } f, \quad \pi_W(O) > 0 \right\}$$

to be the support of π_W. Thus, $J_W \subseteq L^1(J)$.

Lemma 3.4. [12, p. 34] For a graphon W on $(J, \mathcal{A}, \pi)$, the function φ_W from J to J_W is injective if and only if W is twin-free.

Proof. Assume that $\varphi_W : J \to J_W$ is injective. Then, for any $x_1, x_2 \in J$ we have $r_W(x_1, x_2) > 0$ if and only if $x_1 \neq x_2$. That means J has no twins, thus W is twin-free. Conversely, let W be a twin-free graphon. That implies $r_W(x_1, x_2) > 0$ for $x_1, x_2 \in J$, then φ_W is injective. □

Lemma 3.5. A graphon W on $(J, \mathcal{A}, \pi)$ is pure if and only if $\varphi_W : J \to J_W$ is a bijection.

Proof. Suppose that W is a pure graphon. Then by Definition 3.3, (J, r_W) is a complete separable metric space and π_W has full support. That means J is twin-free, then by Lemma 3.4 $\varphi_W : J \to J_W$ is injective. If we have a set $\{x \in J : \varphi_W^{-1}(x) = \varnothing\}$, then its measure is zero. That means φ_W is bijective.

Conversely, suppose that $\varphi_W : J \to J_W$ is a bijection. If φ_W is measurable and $\pi_W(X) = \pi(\varphi_W^{-1}(X))$ for each $X \subseteq L^1(J)$, then φ_W is measure preserving bijection. Since $J_W \subseteq L^1(J)$ then J_W is a complete separable metric space and π_W has full support. Hence, W is a pure graphon. □

Note that if we consider two equivalent graphons W and W_1, where W is pure, then it is not necessarily true that W_1 is pure. However, we can say that W_1 is pure if for every x, not just for almost every x,

$$\pi(\{y : W(x, y) \neq W_1(x, y)\}) = 0.$$

Theorem 3.2 showed that for every graphon W there is a twin-free graphon W' and a measure preserving map $\varphi : J \to J'$ such that $W = (W')^{\varphi}$ almost everywhere. Now, we are going to see how Lovász in [13] proves that for every twin-free graphon we can find a pure graphon weakly isomorphic to it.

Theorem 3.6. [13, p. 223] For every twin-free graphon W there is a pure graphon W' such that W and W' are weakly isomorphic.

Proof. Suppose we have a twin-free graphon W. For each $x \in J$, we define $W_x : J \to [0,1]$ by $W_x(y) = W(x,y)$ for all $y \in J$. Note that $W_x \in L^1(J)$.

Now, let us define a measurable function $\varphi_W : J \to L^1(J)$ by $\varphi_W(x) = W_x$. For $A \subseteq L^1(J)$, we define $\pi_W(A) = \pi(\varphi_W^{-1}(A))$ where $\pi_W(A)$ is measurable on $L^1(J)$. Define

$$J_W = \{f \in L^1(J) : \quad \text{for every open set } O \text{ that contains } f, \quad \pi_W(O) > 0\}$$

to be the support of π_W, then $J_W \subseteq L^1(J)$. Note that J_W is a separable Banach space and π_W has full support on J_W since $J_W \subseteq L^1(J)$.

Suppose that $J' = \{x \in J : W_x \in J_W\}$, and let $J'_W = \{W_x : x \in J'\}$. Then, we define a map $\varphi' : J' \to J'_W$ by $x \mapsto W_x$ which is bijective since W has no twins. We need to show that the measure of $J \backslash J'$ is zero.

If $l \in L^1(J) \backslash J_W$, then there is a neighbourhood U_l of l such that the intersection of U_l and J_W is empty and the measure π_W of U_l is zero. Then, $U_l \subseteq L^1(J) \backslash J_W$ and $\pi(\{x \in J : W_x \in U_l\}) = 0$.

Suppose that U is the union of the neighbourhood U_l, where $l \notin J_W$. Then, U equals the union of some countable subfamily $\{U_{l_i} : i \in \mathbb{N}\}$ if $L^1(J)$ is separable, thus $\pi(\{x \in J : W_x \in U\}) = 0$ by countably additive of π. Therefore, we can see that the measure π of $J \backslash J'$ is zero due to the fact that $J \backslash J' \subseteq U$.

We know that J_W is a complete separable metric space and π_W has full support. Then $\pi(J\backslash J') = 0$ implies that $\pi'(J_W\backslash J'_W) = 0$, where $\pi' = \pi \circ (\varphi')^{-1}$.

Now, we can define a new graphon $\widetilde{W} : J_W \times J_W \to [0,1]$ as follows. Let $f_1, f_2 \in J_W$.

- If $f_1 \in J'_W$ then $f_1 = W_{x_1}$ for some $x_1 \in J$. We define

$$\widetilde{W}(f_1, f_2) = f_2(x_1).$$

- If $f_2 \in J'_W$ then $f_2 = W_{x_2}$ for some $x_2 \in J$. We define

$$\widetilde{W}(f_1, f_2) = f_1(x_2).$$

- If $f_1, f_2 \in J'_W$ then

$$\widetilde{W}(f_1, f_2) = f_1(x_2) = f_2(x_1) = W(x_1, x_2).$$

- If $f_1, f_2 \notin J'_W$ then $\widetilde{W}(f_1, f_2) = 0$.

Now, we want to show that the graphon $\widetilde{W}$ is pure by proving that $r_{\widetilde{W}}$ and the L^1 norm are equal. That means they will have the same properties.

For any $f_1, f_2 \in J'_W \cong J'$ there are x_1 and x_2 such that $f_1 = W_{x_1}$ and $f_2 = W_{x_2}$. Then,

$$\begin{aligned} r_{\widetilde{W}}(f_1, f_2) &= \int_{J'_W} |\widetilde{W}_{f_1}(g) - \widetilde{W}_{f_2}(g)| \quad d\pi'(g) \\ &= \int_{J'_W} |\widetilde{W}(f_1, g) - \widetilde{W}(f_2, g)| \quad d\pi'(g) \\ &= \int_{J} |\widetilde{W}(x_1, y) - \widetilde{W}(x_2, y)| \quad d\pi(y) \\ &= r_W(x_1, x_2). \end{aligned}$$

Therefore,

$$r_{\widetilde{W}}(f_1, f_2) = r_W(x_1, x_2) = ||W_{x_1} - W_{x_2}||_1 = ||f_1 - f_2||_1 .$$

Since $r_{\widetilde{W}}$ and the L^1 norm are equal, then we can see that $\widetilde{W}$ is pure.

Note that $\widetilde{W}$ is a pullback of W and we see that

$$\widetilde{W}(f_1, f_2) = \widetilde{W}(\varphi_W(x_1), \varphi_W(x_2)) = (\widetilde{W})^{\varphi_W}(x_1, x_2) = W(x_1, x_2).$$

Thus, W and $\widetilde{W}$ are weakly isomorphic. □

3.2 Some examples of graphon's purification

We would like to give some examples which shows the purification of graphons. In the beginning, we need to check whether a given graphon W is pure. If so, then we are done. If not, then we have to find a pure graphon which is weakly isomorphic to W.

Example 3.7. In this example we study the purification of the following graphon

$$W(x, y) = \begin{cases} x + y, & 0 \leqslant x \leqslant \frac{1}{2}, \quad 0 \leqslant y \leqslant \frac{1}{2} \\ (x - \frac{1}{2}) + y, & \frac{1}{2} < x \leqslant 1, \quad 0 \leqslant y \leqslant \frac{1}{2} \\ x + (y - \frac{1}{2}), & 0 \leqslant x \leqslant \frac{1}{2}, \quad \frac{1}{2} < y \leqslant 1 \\ (x - \frac{1}{2}) + (y - \frac{1}{2}), & \frac{1}{2} < x \leqslant 1, \quad \frac{1}{2} < y \leqslant 1, \end{cases}$$

where $W : [0, 1] \times [0, 1] \to [0, 1]$.

The first step is to check whether r_W, the neighbourhood distance function, defines a metric on W. We have three cases to study by using the definition of the neighborhood distance:

Case 1: If $x, y \leqslant \frac{1}{2}$, then

$$\begin{aligned} r_W(x,y) &= \int_0^{\frac{1}{2}} |(x+z)-(y+z)|dz + \int_{\frac{1}{2}}^1 |(x+z-\frac{1}{2})-(y+z-\frac{1}{2})|dz \\ &= \int_0^{\frac{1}{2}} |x+z-y-z|dz + \int_{\frac{1}{2}}^1 |x+z-\frac{1}{2}-y-z+\frac{1}{2}|dz \\ &= (x-y)z\Big|_0^{\frac{1}{2}} + (x-y)z\Big|_{\frac{1}{2}}^1 \\ &= \frac{|x-y|}{2} + |x-y| - \frac{|x-y|}{2} \\ &= |x-y|. \end{aligned}$$

Case 2: If $x, y \geqslant \frac{1}{2}$, then we have

$$r_W(x,y) = |x-y|.$$

Case 3: If $x \geqslant \frac{1}{2}$ and $y \leqslant \frac{1}{2}$, then

$$\begin{aligned} r_W(x,y) &= \int_0^{\frac{1}{2}} |(x+z-\frac{1}{2})-(y+z)|dz + \int_{\frac{1}{2}}^1 |(x+z-1)-(y+z-\frac{1}{2})|dz \\ &= \int_0^{\frac{1}{2}} |x+z-\frac{1}{2}-y-z|dz + \int_{\frac{1}{2}}^1 |x+z-1-y-z+\frac{1}{2}|dz \\ &= \int_0^{\frac{1}{2}} |x-y-\frac{1}{2}|dz + \int_{\frac{1}{2}}^1 |x-y-\frac{1}{2}|dz \\ &= (x-y-\frac{1}{2})z\Big|_0^{\frac{1}{2}} + (x-y-\frac{1}{2})z\Big|_{\frac{1}{2}}^1 \\ &= \frac{|x-y-\frac{1}{2}|}{2} + |x-y-\frac{1}{2}| - \frac{|x-y-\frac{1}{2}|}{2} \\ &= |x-y-\frac{1}{2}|. \end{aligned}$$

Therefore,

$$r_W(x,y) = \begin{cases} |x-y| & \text{if} \quad 0 \leqslant x,y \leqslant \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < x,y \leqslant 1 \\ |x-y-\frac{1}{2}| & \quad \text{otherwise.} \end{cases} \tag{3.2.1}$$

The graphon W is not twin-free and therefore not pure. However, W is weakly isomorphic to a pure graphon. Thus, we are going to find this pure graphon.

First of all, let us modify the σ-algebra of the probability space $J = (J, \mathcal{A}, \pi)$. Suppose $\mathcal{C}$ is the collection of all Borel sets of $(0, \frac{1}{2}]$. Then we define a new σ-algebra

$$\mathcal{A}' = \left\{ c \cup (c + \frac{1}{2}) \cup \{0\}, c \cup (c + \frac{1}{2}) : c \in \mathcal{C} \right\}.$$

Thus, we have another probability space $J' = (J, \mathcal{A}', \pi)$.

Now, let $W' = E(W|\mathcal{A}' \times \mathcal{A}')$. Then, $W = W'$ is $\mathcal{A}'$-measurable, so $W = W'$ almost everywhere. Define J_1 to be the set of equivalence classes of being twins on J. That is

$$J_1 = \{0\} \cup \left\{ \{x, x + \frac{1}{2}\} : 0 < x \leqslant \frac{1}{2} \right\}.$$

If $x \in [0, \frac{1}{2}]$, then $\varphi(x)$ is the equivalence class containing $x \in J$. Let $\mathcal{A}_1 = \{\varphi(X) : X \in \mathcal{A}'\}$ and define $\pi_1(X) = \pi(\varphi^{-1}(X))$ for $X \in \mathcal{A}_1$.

For instance, take $X = (\frac{1}{4}, \frac{1}{3}) \cup (\frac{3}{4}, \frac{5}{6}) \in \mathcal{A}_1$. Then

$$\pi_1(X) = \pi(\varphi^{-1}(X)) = (\frac{1}{4}, \frac{1}{3}) \cup (\frac{3}{4}, \frac{5}{6}) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.$$

We get a probability space $J_1 = (J_1, \mathcal{A}_1, \pi_1)$.

For $S, T \in J_1$, we define $W_1(S, T) = W'(x, y) = W(x, y)$ for any $x \in S, y \in T$. Now, $\varphi : J \to J_1$ and $W_1 : J_1 \times J_1 \to [0, 1]$.

For example, consider $S = \{\frac{1}{4}, \frac{3}{4}\}$ and $T = \{\frac{1}{3}, \frac{5}{6}\}$. If we choose any $x \in S$ and $y \in T$, we will have the same result.

$$W_1\left(\{\frac{1}{4}, \frac{3}{4}\}, \{\frac{1}{3}, \frac{5}{6}\} \right) = W(\frac{1}{4}, \frac{1}{3}) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.$$

Also,

$$W_1\Big(\{\frac{1}{4},\frac{3}{4}\},\{\frac{1}{3},\frac{5}{6}\}\Big) = W(\frac{1}{4},\frac{5}{6}) = \frac{1}{4}+(\frac{5}{6}-\frac{1}{2}) = \frac{7}{12}.$$

Now we can say that $[0,1]\times[0,1] \xrightarrow{\varphi\times\varphi} J_1 \times J_1 \xrightarrow{W_1} [0,1]$ then we get $W_1^{\varphi} : J \times J \to [0,1]$ where $W_1^{\varphi} = W$ is the pullback of W_1. Thus, we have $W_1^{\varphi} = W' = W$ almost everywhere. Now, we have a twin-free graphon $W_1 : J_1 \times J_1 \to [0,1]$ and we want to see that is W_1 pure? To study the purifying of W_1 we need to show that (J_1, r_{W_1}) is a complete separable metric space and π_1 has full support.

We know that for $S,T \in J_1$, we define $W_1(S,T) = W'(x,y) = W(x,y)$ for almost all $x \in S, y \in T$. That implies $r_{W_1}(S,T) = r_W(x,y)$ for $x \in S, y \in T$. Let us again consider $S = \{\frac{1}{4},\frac{3}{4}\}$ and $T = \{\frac{1}{3},\frac{5}{6}\}$ then

$$r_{W_1}(S,T) = r_W(\frac{1}{4},\frac{1}{3}) = \frac{1}{12}.$$

We notice that $r_{W_1}(S,T) = 0$ if and only if $S = T$. Therefore, r_{W_1} is a metric. Is the metric space (J_1, r_{W_1}) complete and separable? Recall that J_1 is the set of equivalence classes,

$$J_1 = \{0\} \cup \Big\{\{x, x+1/2\} : 0 < x \leqslant \frac{1}{2}\Big\},$$

then let $S_x = \{x, x+\frac{1}{2}\}$ and $S_0 = \{0\}$. Thus we can rewrite J_1 as

$$J_1 = \{S_x : 0 \leqslant x \leqslant \frac{1}{2}\},$$

and also we have the formula $r_{W_1}(S_{x_1}, S_{x_2}) = |x_1 - x_2|$.

Take a Cauchy sequence in (J_1, r_{W_1}). The distance formula is the same as a Cauchy sequence in $[0,1]$. Let X be a set of representatives of J_1, i.e. for every equivalence class in J_1, we choose an element. Consider $S_1, S_2 \in J_1$, then let x_1 and x_2 be the respective representatives. Then,

$$r_{W_1}(S_1, S_2) = r_W(x_1, x_2) = \int_z |W(x_1, z) - W(x_2, z)| dz.$$

Now suppose we have a Cauchy sequence in (J_1, r_{W_1}), say $(S_1, S_2, S_3, \dots)$, we want to show that for some $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $i, j > N$ we have

$$r_{W_1}(S_i, S_j) < \epsilon.$$

That is,

$$\int_z |W(x_i, z) - W(x_j, z)| dz < \epsilon$$

for all $i, j > N$.

We want to know if this Cauchy sequence converges to an element of (J_1, r_{W_1}). i.e. is there $S \in (J_1, r_{W_1})$ such that $(S_1, S_2, \dots) \to S$? Suppose that $(S_{x_1}, S_{x_2}, S_{x_3}, \dots)$ is a Cauchy sequence where $0 \leqslant x_i \leqslant \frac{1}{2}$. Since the Cauchy sequence $(x_1, x_2, x_3, \dots)$ converges to x in the standard metric space then $(S_{x_1}, S_{x_2}, S_{x_3}, \dots)$ converges to S_x in (J_1, r_{W_1}). That means (J_1, r_{W_1}) is complete.

Now we want to show that (J_1, r_{W_1}) is separable metric space. That means we need to find a countable dense subset of J_1. Let $J_1^{\mathbb{Q}} = \{S_x : 0 \leqslant x \leqslant \frac{1}{2}, x \in \mathbb{Q}\}$. Let $T \in J_1$ and take $\epsilon > 0$. Consider the open set $L = \{S \in J_1 : r_{W_1}(S, T) < \epsilon\}$, we show that $L \cap J_1^{\mathbb{Q}} \neq \varnothing$.

Suppose x is the representative of T. Given $\epsilon > 0$, then there exists $y \in \mathbb{Q}$ such that $|x - y| < \epsilon$. Then $S_y = \{y, y + \frac{1}{2}\} \in J_1^{\mathbb{Q}}$ and

$$|S_y - T| = |x - y| < \epsilon.$$

Hence the metric space (J_1, r_{W_1}) is separable.

Finally, we want to show that π_1 has full support (i.e. we need to show that every open set has a positive measure zero). Take any point x of an open set A in J_1. Then, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. Thus

$$\pi(A) \geqslant \pi(B_\epsilon(x)) = 2\epsilon > 0 \implies 0 < \pi(A).$$

Since (J_1, r_{W_1}) is a complete separable metric space and π_1 has full support, W_1 is a pure graphon which is weakly isomorphic to W.

Example 3.8. Let $W(x, y)$ be the graphon $W(x, y) = x^2y^2$ where $x, y \in [0, 1]$. We show that $W(x, y)$ is twin-free. We want to find $r_W(x, y)$ and then study whether the metric space $([0, 1], r_W)$ is complete and separable.

First of all, let us find r_W.

$$\begin{aligned} r_W(x, y) &= \int_z |W(x, z) - W(y, z)| dz \\ &= \int_0^1 |x^2z^2 - y^2z^2| dz \\ &= \frac{z^3}{3}|x^2 - y^2| \Big|_0^1 \\ &= \frac{1}{3}|x^2 - y^2|. \end{aligned}$$

To show $([0, 1], r_W)$ is a complete metric space, we need to show $([0, 1], r_W)$ is compact. Suppose we have two metric spaces the standard metric space $X = ([0, 1], d)$ and $Y = ([0, 1], r_W)$. Note that if $\varphi : X \to Y$ is a continuous function, then $\varphi[K]$ is compact in Y for every compact subset $K \subseteq X$.

Claim. Let $\varphi : X \to Y$ be the identity function. We claim that φ is continuous. In particular, every open set in Y is also open in X.

<u>Proof:</u> Let $B_{r_W}(x, \epsilon) = \{y \in [0, 1] : r_W(x, y) < \epsilon\}$ be an arbitrary open ball in Y. We want to show that $B_{r_W}(x, \epsilon)$ is an open in X. Indeed, notice that $B_{r_W}(x, \epsilon) = (\sqrt{x^2 - 3\epsilon}, \sqrt{3\epsilon + x^2})$ is an open interval in X where $x \neq 0$ and $x \neq 1$.

- If $x = 0$, then $B_{r_W}(0, \epsilon) = [0, \sqrt{3\epsilon})$ is open in X.

- If $x = 1$, then $B_{r_W}(1, \epsilon) = (\sqrt{1-3\epsilon}, 1]$ is also open in X.

We can conclude that φ is continuous and thus $Y = \varphi[X]$ is compact since $X = ([0,1], d)$ is compact. ■

Now, we can say that $Y = ([0,1], r_W)$ is complete since it is a compact metric space.

The metric space $([0,1], r_W)$ is separable since it is homeomorphic to the standard topology of $([0,1], d)$.

Now we want to show that $([0,1], r_W)$ has full support. Let A be any open set in $[0,1]$. Then, there exists $\epsilon > 0$ and $x \in A$ such that $B_\epsilon(x) \subseteq A$. We want to show that $\pi(B_\epsilon(x)) > 0$.

$$\pi(B_\epsilon(x)) = 2\epsilon > 0 \implies \pi(A) > 0.$$

Thus, every open set of $([0,1], r_W)$ has positive measure and therefore W is pure.

Example 3.9. Let $W : [0,1] \times [0,1] \to [0,1]$ be the graphon defined by

$$W(x,y) = \begin{cases} 1 & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2}, \quad 0 \leqslant y \leqslant \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < x \leqslant 1, \quad \frac{1}{2} \leqslant y \leqslant 1 \\ 0 & \quad \text{otherwise.} \end{cases} \tag{3.2.2}$$

First of all, we want to see if the graphon W is pure. Therefore, the first thing to do is to see if r_W is a metric. Let $0 \leqslant x, y \leqslant \frac{1}{2}$, then by the definition of the neighborhood distance we get that

$$r_W(x,y) = \int_0^{\frac{1}{2}} |1-1|dz + \int_{\frac{1}{2}}^1 |0-0|dz = 0.$$

Similarly, if $\frac{1}{2} < x, y \leqslant 1$ then $r_W(x,y) = 0$. Also, let $0 \leqslant x \leqslant \frac{1}{2}$ and

$\frac{1}{2} < y \leqslant 1$, thus

$$r_W(x,y) = \int_0^{\frac{1}{2}} |1-0|dz + \int_{\frac{1}{2}}^1 |0-1|dz = 1.$$

In the same way, if $0 \leqslant y \leqslant \frac{1}{2}$ and $\frac{1}{2} < x \leqslant 1$, then $r_W(x,y) = 1$.

Therefore,

$$r_W(x,y) = \begin{cases} 0 & \text{if} \quad 0 \leqslant x, y \leqslant \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < x, y \leqslant 1 \\ 1 & \quad \text{otherwise.} \end{cases} \tag{3.2.3}$$

Thus, r_W is not a metric since we can have $r_W(x,y) = 0$ for $x \neq y$. That means the graphon W is not pure. However, W is weakly isomorphic to a pure graphon. Thus, we are going to find this pure graphon.

First of all, let us modify the σ-algebra of the probability space $J = (J, \mathcal{A}, \pi)$. Define the new σ-algebra $\mathcal{A}' = \{\varnothing, [0, \frac{1}{2}], (\frac{1}{2}, 1], [0,1]\}$ and let $J' = (J, \mathcal{A}', \pi)$. Define $W' = E(W|\mathcal{A}' \times \mathcal{A}')$. In fact, $W = W'$ is $\mathcal{A}'$-measurable so we can have $W' = W$ almost everywhere. Let J_1 be the set of equivalence classes of being twins on J. That is $J_1 = \{[0, \frac{1}{2}], (\frac{1}{2}, 1]\}$. If $x \in [0,1]$, then $\varphi(x)$ is the equivalence class containing $x \in J$. Let

$$\mathcal{A}_1 = \Big\{\varnothing, \{[0, \frac{1}{2}]\}, \{[\frac{1}{2}, 1]\}, \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}\Big\},$$

and define $\pi_1(X) = \pi\big(\varphi^{-1}(X)\big)$ where $X \in \mathcal{A}_1$. Thus,

$$\begin{aligned}
\pi_1(\varnothing) &= \pi\big(\varphi^{-1}(\varnothing)\big) = 0 \\
\pi_1\big(\{[0, \frac{1}{2}]\}\big) &= \pi\big(\varphi^{-1}(\{[0, \frac{1}{2}]\})\big) = \frac{1}{2} \\
\pi_1\big(\{[\frac{1}{2}, 1]\}\big) &= \pi\big(\varphi^{-1}(\{[\frac{1}{2}, 1]\})\big) = \frac{1}{2} \\
\pi_1\big(\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}\big) &= \pi\big(\varphi^{-1}(\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\})\big) = 1.
\end{aligned}$$

Thus, we get a probability space $J_1 = (J_1, \mathcal{A}_1, \pi_1)$. From the above calculation we notice that all the sets in $\mathcal{A}_1$ have positive measure except $\varnothing$. Hence, π_1 has full support.

For $S, T \in J_1$, we define $W_1(S,T) = W'(x,y)$ for any $x \in S, y \in T$. Now, $\varphi : J \to J_1$ and define $W_1 : J_1 \times J_1 \to \mathbb{R}$. Let us consider any point $x \in [0,1]$, say $x = \frac{1}{4}$, then $\varphi(x) = [0, \frac{1}{2}]$. Then

$$W_1^{\varphi}\big(\frac{1}{4}, \frac{1}{4}\big) = W_1\big([0, \frac{1}{2}], [0, \frac{1}{2}]\big) = 1 = W'\big(\frac{1}{4}, \frac{1}{4}\big),$$

where W_1^{φ} is the *pullback* of W_1, i.e $[0,1] \times [0,1] \xrightarrow{\varphi\times\varphi} J_1 \times J_1 \xrightarrow{W_1} \mathbb{R}$ then we get $W_1^{\varphi} : J \times J \to \mathbb{R}$. Thus, we have $W_1^{\varphi} = W' = W$ almost everywhere. Now, we have a twin-free graphon $W_1 : J_1 \times J_1 \to \mathbb{R}$ and we want to see that is this twin-free graphon pure? To show that W_1 is pure we need to show (J_1, r_{W_1}) is a complete separable metric space and π_1 has full support.

We already knew that π_1 has full support. We only need to prove that (J_1, r_{W_1}) is complete and separable metric space. Since our metric space (J_1, r_{W_1}) is finite, then it is compact (every open cover of (J_1, r_{W_1}) has a finite subcover). Then, the metric space (J_1, r_{W_1}) is complete. Moreover, the metric space (J_1, r_{W_1}) is separable because it is compact. Therefore, the twin-free graphon W_1 is a pure graphon which is weakly isomorphic to W.

4

k-Uniform Hypergraphs and k-Uniform Hypergraphons

This chapter explores a notion of k-uniform hypergraphon from different perspectives such as Elek and Szegedy in [8] and Zhao in [22]. We give a new definition of k-uniform hypergraphon on arbitrary measure space. Furthermore, we show that every k-uniform hypergraphon is weakly isomorphic to a twin-free separable k-uniform hypergraphon by following the work of Borge, Chayes and Lovász with graphons [6].

4.1 History/Known results

According to Lovász's book [13], a hypergraph is a pair $H = (V, E)$ where V is a finite set and $E \subseteq V^k$ for some $k \geqslant 2$. We consider exclusively k-uniform hypergraphs where $H = (V, E)$ is a k-uniform hypergraph if and only if $E \subseteq V^k$ is symmetric in the sense that

$$(x_1, \dots, x_k) \in E \iff (x_{\sigma(1)}, \dots, x_{\sigma(k)}) \in E$$

for every permutation σ of $\{1, \dots, k\} = [k]$. In particular, 2-uniform hypergraphs are equivalent to simple graphs. That means graphs are a special case of hypergraphs.

Suppose we have two k-uniform hypergraphs F and H, and a map $\varphi : V(F) \to V(H)$. If $e = \{v_1, v_2, \dots, v_k\} \in E(F)$, then we define $\varphi(e) = \{\varphi(v_1), \varphi(v_2), \dots, \varphi(v_k)\} \in V(H)^k$. We say that φ is a k-uniform hypergraph homomorphism if $\varphi(e) \in E(H)$ for every $e \in E(F)$. The number of homomorphisms from F to H is denoted $\hom(F, H)$. The homomorphism density of F in H

$$t(F, H) = \frac{\hom(F, H)}{|V(H)|^{|V(F)|}},$$

denotes the probability that a random map of $V(F)$ into $V(H)$ is a homomorphism [8, p. 1735].

Zhao in [22] states that a sequence of k-uniform hypergraphs $H_1, H_2, \dots$ is called *convergent* if the sequence $t(F, H_1), t(F, H_2), \dots$ converges for every k-uniform hypergraph F. From this definition, it would seem that we could define a hypergraphon as a function $W : [0,1]^k \to [0,1]$ which is the limit of a sequence of hypergraphs. However, as Zhao explains this definition does not work. We give Zhao's definition below.

Elek and Szegedy built up the theory of *limit hypergraphs* from scratch using an *ultraproduct*, and they defined a *k-uniform hypergraphon* as a limit object of convergent sequences of k-uniform hypergraphs. For more details see [8]. Below we discuss the relation between two definitions of k-uniform hypergraphon, the first by Elek and Szegedy and the second by Zhao.

First of all, let us introduce some notation. We will then describe a k-uniform hypergraphon. For a set A, define $r(A)$ to be the set of all nonempty subsets of A, and $r_<(A)$ to be the set of all nonempty proper subsets of A.

If X is a set and R is a finite set, we define X^R to be $|R|$ copies of X with each copy indexed by an element of R.

Definition 4.1. The symmetric group S_k acts naturally on a power set $P([k])$. Suppose that $c \subseteq P([k])$ is closed under the action of S_k. This induces a bijection $X^c \to X^c$ for any set X. We say that a function $f : X^c \to Y$ is *symmetric* if $f(x) = f(\sigma(x))$ for all $\sigma \in S_k$ and all $x \in X^c$.

Definition 4.2. [8] A k-uniform ES-hypergraphon is a symmetric, under the action of S_k, measurable function $\mathcal{H} : [0,1]^{r([k])} \to \{0,1\}$.

Definition 4.3. [22] A k-uniform $\mathcal{Z}$-hypergraphon is a symmetric, under the action of S_k, measurable function $\mathcal{H} : [0,1]^{r_<([k])} \to [0,1]$

We notice a difference between Elek and Szegedy's Definition 4.2 and Zhao's Definition 4.3 of k-uniform hypergraphons. Therefore, our purpose here is to explore the relationship between those definitions.

Definition 4.4. We say that a ES-hypergraphon $\mathcal{H}$ is *well-behaved* if for every $\vec{v} \in [0,1]^{r_<([k])}$ there exactly one $x \in [0,1]$ such that $\mathcal{H}(\vec{v}, x) = 1$.

Proposition 4.5. From every k-uniform $\mathcal{Z}$-hypergraphon we can produce a k-uniform ES-hypergraphon.

Proof. Notice that for a finite set A,

$$r(A) = P(A) \backslash \{\varnothing\} = r_<(A) \cup \{A\}$$

Hence $[0,1]^{r([k])} = [0,1]^{r_<([k])} \times [0,1]$.

- Subsets of $[0,1]^{r([k])}$ can be represented as functions $[0,1]^{r([k])}$ to $\{0,1\}$ by identifying every subset $B \subseteq [0,1]^{r([k])}$ by the characteristic function χ_B, where

$$\chi_B(\vec{x}) = \begin{cases} 1 & \text{if} \quad \vec{x} \in B, \\ 0 & \quad \text{otherwise}, \end{cases} \tag{4.1.1}$$

  for $\vec{x} \in [0,1]^{r(k)}$.

- Every function $f : [0,1]^{r_<([k])} \to [0,1]$ can be understood as a subset of $[0,1]^{r_<([k])} \times [0,1]$ by identifying f with its graph $\{(\vec{v}, f(\vec{v})) : \vec{v} \in [0,1]^{r_<([k])}\}$. Hence every k-uniform $\mathcal{Z}$-hypergraphon is a k-uniform ES-hypergraphon.

$\square$

Proposition 4.6. If $\mathcal{H}$ is a well-behaved ES-hypergraphon, then we can produce a $\mathcal{Z}$-hypergraphon.

We prove this proposition by reversing the argument of the Proposition 4.5 proof.

Proof. Suppose $\mathcal{H}$ is a well-behaved ES-hypergraphon $\mathcal{H} : [0,1]^{r([k])} \to \{0,1\}$ such that for every $\vec{v} \in [0,1]^{r_<([k])}$ there exactly one $x \in [0,1]$ such that $\mathcal{H}(\vec{v}, x) = 1$. Identify $\mathcal{H}$ with the subset of $[0,1]^{([k])}$ given by

$$\{\vec{x} \in [0,1]^{r([k])} : \mathcal{H}(\vec{x}) = 1\}.$$

Then, we identify this with a subset L of $[0,1]^{r_<([k])} \times [0,1]$. Since $\mathcal{H}$ is well-behaved, for each $\vec{v} \in [0,1]^{r_<([k])}$ there is a unique $x \in [0,1]$ with $(\vec{v}, x) \in L$. Define a $\mathcal{Z}$-hypergraphon $\mathcal{H}_{\mathcal{Z}}(\vec{v}) = x$. □

Note that if $\mathcal{H}$ is not well-behaved then we have $\mathcal{H}(\vec{v}, x) = \mathcal{H}(\vec{v}, y)$ for some $\vec{v} \in [0,1]^{r_<([k])}$ and $x \neq y \in [0,1]$. In this case, we can identify the function $[0,1]^{r([k])} \to \{0,1\}$ with a subset of $[0,1]^{r_<([k])} \times [0,1]$ but not with a unique map $[0,1]^{r_<([k])} \to [0,1]$.

Proposition 4.7. [8] Suppose $\mathcal{H}$ is an ES-hypergraphon. Then we can define a $\mathcal{Z}$-hypergraphon $\mathcal{H}_{\mathcal{Z}}$ by setting

$$\mathcal{H}_{\mathcal{Z}}(\underline{x}) = \int_0^1 \mathcal{H}(\underline{x}, x)dx$$

for $\underline{x} \in [0,1]^{r_<([k])}$.

4.1.1 A hypergraphon from Zhao's point of view

Here we give a brief explanation about the definition of hypergraphon by Zhao [22] with more details and examples.

For a set A, $r(A, m) = [A]^{\leqslant m} \backslash \varnothing$, which is the set of all nonempty subsets of A with size at most m . Any permutation σ on a set A induces a permutation on $r(A, m)$.

Example 4.8. Let $A = \{1, 2, 3\} = [3]$ and $m = 2$. Then,

$$r_<([3]) = \Big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\Big\}$$

Suppose $\sigma : A \to A$ is the permutation that maps $1 \mapsto 3$, $2 \mapsto 2$, and $3 \mapsto 1$. Then σ induces the permutation on $r(A, m)$ given by

$$\begin{aligned} \{1\} &\mapsto \{3\}, & \{1,2\} &\mapsto \{2,3\}, \\ \{2\} &\mapsto \{2\}, & \{1,3\} &\mapsto \{1,3\}, \\ \{3\} &\mapsto \{1\}, & \{2,3\} &\mapsto \{1,2\}. \end{aligned}$$

A function $\mathcal{H} : [0, 1]^{r([k],m)} \to [0, 1]$ is symmetric if it is invariant under any permutation of $r([k], m)$ induced by permutations of k.

Example 4.9. We work out what it means exactly for a function $\mathcal{H}$: $[0, 1]^{r([3],2)} \to [0, 1]$ to be symmetric . Now we have to consider all possible permutations of $\{1, 2, 3\}$ which are given by

$$S_3 = \big\{I, (12), (13), (23), (123), (132)\big\}.$$

We view our function $\mathcal{H}$ as $\mathcal{H} : [0, 1]^6 \to [0, 1]$ where the copies of $[0, 1]$ are indexed by the elements of $r_<([3])$.

In Zhao's paper [22] the notation for a typical element of $[0, 1]^6$ reflects this: we write $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ for a typical element of $[0, 1]^6 = [0, 1]^{r_<([3])}$. So, a symmetric $\mathcal{H}$ should satisfy:

$$\begin{aligned} &\mathcal{H}(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = \mathcal{H}(x_1, x_3, x_2, x_{13}, x_{12}, x_{23}) \\ = &\mathcal{H}(x_3, x_2, x_1, x_{23}, x_{13}, x_{12}) = \mathcal{H}(x_2, x_1, x_3, x_{12}, x_{23}, x_{13}) \\ = &\mathcal{H}(x_2, x_3, x_1, x_{23}, x_{12}, x_{13}) = \mathcal{H}(x_3, x_1, x_2, x_{13}, x_{23}, x_{12}) \end{aligned}$$

We note that being symmetric does not imply being invariant under all permutations of $r_<([k])$. For example, it is not necessary to have

$$\mathcal{H}(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = \mathcal{H}(x_{13}, x_2, x_3, x_{12}, x_1, x_{23})$$

4.2 k-uniform hypergraphons from our point of view

After we discussed the relationship between Elek and Szegedy's definition and Zhao's definition of k-uniform hypergraphon in the previous sections, we would like to introduce our definition. Rather than working only with the space $[0, 1]$, we will work with an arbitrary measure space. This will enable us to construct twin-free hypergraphons. We can see in Example 3.9 why changing the space is sometimes necessarily. From now on we fix $k \geqslant 2$ and we let $l = 2^k - 2$. Hence l is the number of proper nonempty subsets of $[k]$.

Definition 4.10. Consider an arbitrary measure space $(J, \mathcal{A}, \pi)$. We say that a k-uniform hypergraphon is a map $\mathcal{H} : J^l \to [0, 1]$ which is symmetric under the action of S_k and is a bounded measurable function with respect to $(J^l, \overline{\mathcal{A}^l}, \pi^l)$.

Suppose we have a sequence $\underline{x} = (x_1, x_2, \ldots, x_r)$. We denote

$$\underline{x}(\hat{j}) = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r)$$

the sequence obtained by removing the j^{th} term from $\underline{x}$. Furthermore, we denote

$$\underline{x}^y(\check{j}) = (x_1, x_2, \ldots, x_{j-1}, y, x_j, \ldots, x_r)$$

the sequence obtained by inserting y before the j^{th} term of $\underline{x}$. When we integrate over multiple terms, we will use the notation $d(\underline{x})$ to denote $dx_1 dx_2 \ldots dx_r$; similarly for $d(\underline{x}(\hat{j}))$ and $d(\underline{x}(\check{j}))$.

We identify $\underline{x} \in J^l$ as $\underline{x} = (x_1, x_2, \dots, x_k, x_{12}, x_{13}, \dots, x_{23\dots k})$ where the subscripts correspond to the nonempty proper subsets of $[k]$. For $1 \leqslant i \leqslant k-1$, set

$$\varrho(i) = \left[\sum_{j=1}^{i-1} \binom{k}{j}\right] + 1$$

which is the coordinate where $x_{1\dots i}$ lies. Then $\underline{x}(\widehat{\varrho(i)})$ denotes the sequence $\underline{x} = (x_1, x_2, \dots, x_k, x_{12}, \dots, x_{23\dots k})$ with the term $x_{1\dots i}$ removed.

Now we are going to define some notions that are related to how "nice" a k-uniform hypergraphon and the underlying measure space are.

Let $\mathcal{H}$ be a k-uniform hypergraphon on an arbitrary measure space $(J, \mathcal{A}, \pi)$. Then $\mathcal{H}$ is *strong* if it is measurable with respect to the σ-algebra $\mathcal{A}^l$. The completion of $\mathcal{H}$, denoted by $\overline{\mathcal{H}}$, is the same function of the k-uniform hypergraphon $\mathcal{H}$ but considered with respect to the completion of $\mathcal{A}$, which is $\overline{\mathcal{A}}$. If a k-uniform hypergraphon equal to its completion, then it is *complete*. We say that the k-uniform hypergraphon $\mathcal{H}$ is *separable* if the measure space $(J, \mathcal{A}, \pi)$ is separable.

If S is a set and $\underline{x} = (x_i : i \in S) \in J^S$, then if $T \subseteq S$, we write $\underline{x}_T = (x_i : i \in T) \in J^T$. In particular, we will consider the case where $F = (V, E)$ is a k-uniform hypergraph and $S = r(V(F), k-1)$ and $T = r(A, k-1)$ where $A \in E(F)$.

The next definitions are analogues of the definitions given in [6].

Definition 4.11. Let $F = (V, E)$ be a k-uniform hypergraph and let $\mathcal{H}$ be a k-uniform hypergraphon. We define the homomorphism density

$$t(F, \mathcal{H}) = \int\limits_{J^{r(F,k-1)}} \prod_{A \in E} \mathcal{H}(\underline{x}_{r<(A)}) d\underline{x}.$$

For example, let $F_6^{(3)} = \{123, 124, \dots, 156, 234, \dots, 256, 345, \dots, 356, 456\}$ be the complete 3-uniform hypergraph on 6 vertices and $\mathcal{H}$ a 3-uniform hypergraphon. Then,

$$t(F_6^{(3)}, \mathcal{H}) = \int\limits_{J^{21}} \mathcal{H}(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})\mathcal{H}(x_1, x_2, x_4, x_{12}, x_{14}, x_{24}) \dots$$
$$\dots \mathcal{H}(x_3, x_5, x_6, x_{35}, x_{36}, x_{56})\mathcal{H}(x_4, x_5, x_6, x_{45}, x_{46}, x_{56}) \quad d_1 d_2 \dots d_{56}.$$

We can obtain a new k-uniform hypergraphon by applying a "pull-back" using a measure preserving function. Let $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ be two measure spaces. Suppose that $\mathcal{H}'$ is a k-uniform hypergraphon on $(J', \mathcal{A}', \pi')$, and $\varphi : (J, \mathcal{A}, \pi) \to (J', \mathcal{A}', \pi')$ is a measure preserving function. If $\underline{x} = (x_1, x_2, \dots, x_{23\dots k}) \in J^l$, we define

$$\varphi(\underline{x}) = \big(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{23\dots k})\big) \in (J')^l.$$

Then we define the *pull-back* $\mathcal{H} = (\mathcal{H}')^{\varphi}$ to be the k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$ given by

$$(\mathcal{H}')^{\varphi}(\underline{x}) = \mathcal{H}'\big(\varphi(\underline{x})\big)$$

for all $\underline{x} \in J^l$.

We say that two k-uniform hypergraphons $\mathcal{H}$ and $\mathcal{H}'$ on $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ are isomorphic mod 0 if there exists a measure preserving map $\varphi : J \to J'$ such that φ is an isomorphism mod 0 and $(\mathcal{H}')^{\varphi} = \mathcal{H}$ almost everywhere in J^l.

Definition 4.12. Let $\mathcal{H}$ and $\mathcal{K}$ be two k-uniform hypergraphons on the measure spaces $(J, \mathcal{A}, \pi)$ and $(\Lambda, \mathcal{B}, \nu)$. If $\varphi : J \to \Lambda$ is a measure preserving function from $\overline{\mathcal{A}}$ into $\mathcal{B}$ such that $\mathcal{H} = \mathcal{K}^{\varphi}$ almost everywhere, then φ is a *weak isomorphism* from $\mathcal{H}$ to $\mathcal{K}$.

Definition 4.13. Two k-uniform hypergraphons $\mathcal{H}$ and $\mathcal{H}'$ are *weakly isomorphic* if we have another k-uniform hypergraphon $\mathcal{K}$ and weak isomorphism functions from $\mathcal{H}$ and $\mathcal{H}'$ into $\mathcal{K}$.

Now we are going to generalize the definition of a twins of a graphon given in [6].

Definition 4.14. Let $\mathcal{H}$ be a k-uniform hypergraphon on a measure space $(J, \mathcal{A}, \pi)$. For $\rho \in S_l$ and $\underline{z} \in J^l$, we define $\underline{z}^\rho$ to be z with the entries permuted by ρ. We say that $x, x' \in J$ are twins if

$$\mathcal{H}(x, y_2, \ldots, y_{23\ldots k})^\rho = \mathcal{H}(x', y_2, \ldots, y_{23\ldots k})^\rho$$

for almost all $\underline{y} = (y_2, y_3, \ldots, y_{23\ldots k}) \in J^{l-1}$, and all $\rho \in S_l$.

We call the k-uniform hypergraphon $\mathcal{H}$ twin-free if no two points in J are twins in $\mathcal{H}$. Furthermore, we say that the k-uniform hypergraphon $\mathcal{H}$ is almost twin-free if there exists a null set N of J such that no two points in $J \backslash N$ are twins.

We are going to investigate how can we convert a k-uniform hypergraphon into a twin-free separable k-uniform hypergraphon.

4.3 Reduction of k-uniform hypergraphons

Analogously to [6, Theorem 3.2] we want to show that every k-uniform hypergraphon is weakly isomorphic to a twin-free separable k-uniform hypergraphon. To show this we are going to manipulate the corresponding σ-algebra and modify the k-uniform hypergraphon. However, we need to recall Theorem 2.32.

Suppose that $(J_i, \mathcal{A}_i, \pi_i)$, where $i = 1, \ldots, l$, are finite measure spaces. Then there exists a unique measure $\pi_1 \otimes \cdots \otimes \pi_l$ on the product space $(J_1 \times \cdots \times J_l, \mathcal{A}_1 \times \cdots \times \mathcal{A}_l)$ with the property that

$$(\pi_1 \otimes \cdots \otimes \pi_l)(A_1 \times \cdots \times A_l) = \pi_1(A_1) \ldots \pi_l(A_l)$$

for all $A_1 \times \cdots \times A_l \in \mathcal{A}_i$.

The following lemma allows us to change a σ-algebra to a countably generated one.

Lemma 4.15. Suppose $(J, \mathcal{A})$ is a measurable space, and let

$$\mathcal{F} : J^m \to \mathbb{R}$$

be a measurable function with respect to $\mathcal{A}^m$. Then there exist a countably generated σ-algebra $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{F}$ is measurable with respect to $\mathcal{B}^m$.

The proof of this lemma is similar to the graphon case in [6] by using the monotone class theorem for functions, Theorem 2.28.

Proof. Let $\mathcal{G}$ be the set of all bounded functions f which are measurable with respect to $\mathcal{A}^m$ such that Lemma 4.15 is true. The set $\mathcal{G}$ is a vector space since it is closed under linear combinations, so that $af_1 + bf_2 \in \mathcal{G}$ where $f_1, f_2 \in \mathcal{G}$ and $a, b \in \mathbb{R}$, and it contains the multiplicative identity 1. Furthermore, $\mathcal{G}$ is closed under bounded increasing limits, meaning that for any bounded sequence from $\mathcal{G}$ with $\mathcal{F}_n \leqslant \mathcal{F}_{n+1}$ for all n, we have $\lim\limits_{n\to\infty} \mathcal{F}_n \in \mathcal{G}$. By the monotone class theorem 2.28, $\mathcal{G}$ contains all bounded functions that are measurable with respect to the σ-algebra generated by $\mathcal{A}^m$. □

Next we explore how to construct a new k-uniform hypergraphon from a given k-uniform hypergraphon using the idea of a push-forward, as we explain.

Suppose we have two measure spaces, say $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$, and let $\varphi : J \to J'$ be a measure preserving map. We say that a k-uniform hypergraphon $\mathcal{H}_\varphi$ on $(J', \mathcal{H}', \pi')$ is the push-forward of $\mathcal{H}$ by φ if

$$\int\limits_{(A')^l} \mathcal{H}_\varphi(\underline{x}') \quad d\pi'(\underline{x}') = \int\limits_{\varphi^{-1}((A')^l)} \mathcal{H}(\underline{x}) \quad d\pi(\underline{x}) \tag{4.3.1}$$

holds for all $A' \in \mathcal{A}'$.

In order to prove next lemma we need to state a theorem of Radon-Nikodym [18].

Theorem 4.16 (Radon-Nikodym Theorem)**.** Let $(J, \mathcal{A}, \pi)$ be a σ-finite measure space and μ a σ-finite measure defined on the space $(J, \mathcal{A})$ that is absolutely continuous with respect to π. Then there exists a nonnegative π-measurable function $\mathcal{H}$ such that

$$\mu(A) = \int_A \mathcal{H} d\pi$$

for all $A \in \mathcal{A}$. The function $\mathcal{H}$ is unique in the sense that if h is any nonnegative π-measurable function on J that also has this property, then $h = \mathcal{H}$ almost everywhere with respect to π. The function $\mathcal{H}$ is called the Radon-Nikodym derivative of μ and is denoted by $\mathcal{H} = \frac{d\mu}{d\pi}$.

The following lemma says that the push-forward $\mathcal{H}_\varphi$ is well defined and it gives the relation between the push-forward and the pull back $(\mathcal{H}_\varphi)^\varphi$.

Lemma 4.17. Suppose that we have two measure spaces $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ and a measure preserving map $\varphi : J \to J'$. Let $\mathcal{H}$ be a given k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$. Then

(i) There is a function $\mathcal{H}_\varphi : (J')^l \to \mathbb{R}$ which is measurable with respect to $(\mathcal{A}')^l$ and the measure π' which satisfies $\mathcal{H}_\varphi = d\mu/d(\pi')$. In particular, μ is the measure on $(\mathcal{A}')^l$ defined by

$$\mu(A'_1 \times \cdots \times A'_l) = \int_{A'_1 \times \cdots \times A'_l} \mathcal{H}_\varphi d\pi'$$

for $A'_1, \ldots, A'_l \in \mathcal{A}'$. The function $\mathcal{H}_\varphi$ is unique up to changes on a set of π'- measure zero in $(J')^l$.

(ii) If $\mathcal{A}_\varphi = \{\varphi^{-1}(A) : A \in \mathcal{A}'\}$, then $(\mathcal{H}_\varphi)^\varphi = E(\mathcal{H}|\mathcal{A}_\varphi^l)$ almost everywhere.

(iii) If φ is actually a measure preserving embedding of $(J, \mathcal{A}, \pi)$ into $(J', \mathcal{A}', \pi')$, then $(\mathcal{H}_\varphi)^\varphi = \mathcal{H}$ almost everywhere.

Proof. (i) Let us define a measure μ on $(\mathcal{A}')^l$ by

$$\mu(A_1' \times \cdots \times A_l') = \int\limits_{\varphi^{-1}(A_1') \times \cdots \times \varphi^{-1}(A_l')} \mathcal{H} d\pi$$

for $A_1', \ldots, A_l' \in \mathcal{A}$. Since $\mathcal{H}$ takes values in $[0, 1]$ we have

$$\mu(A_1' \times \cdots \times A_l') \leqslant \pi(\varphi^{-1}(A_1') \times \cdots \times \varphi^{-1}(A_l')).$$

Since φ is a measure preserving map from J to J', we have

$$\mu(A_1' \times \cdots \times A_l') \leqslant \pi'(A_1' \times \cdots \times A_l')$$

for all $A_1' \times \cdots \times A_l' \in \mathcal{A}'$. That means μ is absolutely continuous with respect to π' and hence the Radon-Nikodym derivative of μ with respect to π' is well defined and is equal to $\mathcal{H}_\varphi$, i.e.

$$\mathcal{H}_\varphi = \frac{d\mu}{d\pi'}.$$

The uniqueness of $\mathcal{H}_\varphi$ follows from the Radon-Nikodym theorem.

(ii) Assume that $A_1, \ldots, A_l \in \mathcal{A}_\varphi$ where $A_1 = \varphi^{-1}(A_1'), \ldots, A_l = \varphi^{-1}(A_l')$ for some $A_1', \ldots, A_l' \in \mathcal{A}'$. Since φ is measure preserving, and $\mathcal{H}_\varphi : (J')^l \to \mathbb{R}$ we show that $\mathcal{H} = (\mathcal{H}_\varphi)^\varphi$ almost everywhere. It suffices to show that

$$\int_{A_1\times\cdots\times A_l} \mathcal{H}(\underline{x}) \quad d\pi(\underline{x}) = \int_{A'_1\times\cdots\times A'_l} \mathcal{H}_\varphi(\underline{x'}) \quad d\pi'(\underline{x'})$$

$$= \int_{A_1\times\cdots\times A_l} \mathcal{H}_\varphi(\varphi(\underline{x})) \quad d\pi(\underline{x}) = \int_{A_1\times\cdots\times A_l} (\mathcal{H}_\varphi)^\varphi(\underline{x}) \quad d\pi(\underline{x})$$

Therefore, $(\mathcal{H}_\varphi)^\varphi = E(\mathcal{H}|\mathcal{A}^l_\varphi)$ almost everywhere.

(iii) Now suppose that φ is embedding. Then φ is an isomorphism between $(J, \mathcal{A}, \pi)$ and a subspace of $(J', \mathcal{A}', \pi')$. For any $A \in \mathcal{A}$ we get $A' \in \mathcal{A}'$ such that $A' \cap \varphi(J) = \varphi(A)$. However, $\varphi^{-1}(A') = \varphi^{-1}(\varphi(A)) = A$ which means $A \in \mathcal{A}_\varphi$. Hence, $\mathcal{A} = \mathcal{A}_\varphi \implies \mathcal{H} = (\mathcal{H}_\varphi)^\varphi$ almost everywhere. □

Definition 4.18. Suppose that $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ are measure spaces, and $\varphi : J \to J'$ is a measure preserving map. Let $\mathcal{H}$ and $\mathcal{H}_\varphi$ be k-uniform hypergraphons on $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$ respectively. We say that φ is an embedding of $\mathcal{H}$ into $\mathcal{H}_\varphi$ if φ is embedding of $(J, \mathcal{A}, \pi)$ into $(J', \mathcal{A}', \pi')$ and $(\mathcal{H}_\varphi)^\varphi = \mathcal{H}$ almost everywhere.

Now by using the construction of the push-forward we can define quotients of k-uniform hypergraphons.

Definition 4.19. Let $\mathcal{H}$ be a k-uniform hypergraphon on a measure space $(J, \mathcal{A}, \pi)$. Let $\mathcal{P}$ be an arbitrary partition of J into disjoint sets. For $x \in J$, we let $[x]$ be the class in $\mathcal{P}$ which contains x. Then we define a k-uniform hypergraphon

$$\mathcal{H}/\mathcal{P} = (J/\mathcal{P}, \mathcal{A}/\mathcal{P}, \pi/\mathcal{P})$$

and a measure preserving map $\varphi : J \to J/\mathcal{P}$ as follows:

- the points in $J/\mathcal{P}$ are the classes of the partition $\mathcal{P}$.
- φ is the map $x \mapsto [x]$.
- $\mathcal{A}/\mathcal{P}$ is the σ-algebra consisting of the sets $A' \subseteq J/\mathcal{P}$ such that $\varphi^{-1}(A') \in \mathcal{A}$.

- $(\pi/\mathcal{P})(A') = \pi(\varphi^{-1}(A'))$.

Then φ is measure preserving. We define $\mathcal{H}/\mathcal{P} = \mathcal{H}_\varphi$ as in (4.3.1).

Analogously to the graphon case in [6] we are going to express several lemmas that describe how we can reduce every k-uniform hypergraphon to a twin-free separable k-uniform hypergraphon.

Lemma 4.20. Suppose $\mathcal{H}$ is a k-uniform hypergraphon on a measure space $(J, \mathcal{A}, \pi)$. Then we can obtain a strong k-uniform hypergraphon by changing the value of $\mathcal{H}$ on a set of measure zero with respect to π.

Proof. Let $\mathcal{H}$ be a k-uniform hypergraphon. We want to find a strong k-uniform hypergraphon $\mathcal{H}'$ equivalent to $\mathcal{H}$. Define $\mathcal{H}' = E(\mathcal{H}|\mathcal{A}^l)$. That means $\mathcal{H}'$ is measurable with respect to $\mathcal{A}^l$. Then it is sufficient to show that $\mathcal{H} = \mathcal{H}'$ almost everywhere. For all $A \in \mathcal{A}$, define the measurable functions

$$X_A = \int_A \mathcal{H} d\pi, \quad g_A = E(1_A|\mathcal{A}), \quad Y_A = \int_A \mathcal{H} g_A d\pi.$$

Thus,

$$\begin{aligned}
\int_{A^l} \mathcal{H} \quad d\pi_1 \times \cdots \times d\pi_l &= \int X_A 1_{A^{l-1}} \quad d\pi_1 \times \cdots \times d\pi_{l-1} \\
&= \int X_A g_{A^{l-1}} \quad d\pi_1 \times \cdots \times d\pi_{l-1} \\
&= \int X_A g_A g_{A^{l-2}} \quad d\pi_1 \times \cdots \times d\pi_{l-1} \\
&= \int Y_A g_{A^{l-2}} 1_A \quad d\pi_1 \times \cdots \times d\pi_{l-1} \\
&= \int Y_A g_{A^{l-2}} g_A \quad d\pi_1 \times \cdots \times d\pi_{l-1} \\
&= \int \mathcal{H} g_A g_{A^{l-2}} g_A \quad d\pi_1 \times \cdots \times d\pi_l \\
&= \int \mathcal{H} g_{A^l} \quad d\pi_1 \times \cdots \times d\pi_l
\end{aligned}$$

$$= \int \mathcal{H}' g_{A^l} \quad d\pi_1 \times \cdots \times d\pi_l$$
$$= \int \mathcal{H}' 1_{A^l} \quad d\pi_1 \times \cdots \times d\pi_l$$
$$= \int_{A^l} \mathcal{H}' \quad d\pi_1 \times \cdots \times d\pi_l$$

for all $A^l \in \mathcal{A}^l$.

Similarly for all sets S in $\overline{\mathcal{A}^l}$ we have that

$$\int_S \mathcal{H}' \quad d\pi = \int_S \mathcal{H} \quad d\pi.$$

Therefore, $\mathcal{H} = \mathcal{H}'$ almost everywhere. □

Lemma 4.21. If $\mathcal{H}$ is a k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$, then there exists a countably generated σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{H}$ is measurable with respect to $(\mathcal{A}')^l$.

Proof. It is enough to use Lemma 4.15. □

Definition 4.22. Let $\mathcal{H}$ be a k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$. The σ-algebra $\mathcal{A}$ of subsets of J induces a partition $\mathcal{P}_{[\mathcal{A}]}$ of J by using the relation of equivalence, $x_1 \sim x_2$ if and only if for every $A \in \mathcal{A}$ either $x_1, x_2 \in A$ or $x_1, x_2 \notin A$.

Lemma 4.23. Let $\mathcal{H}$ be a k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$. Then, the k-uniform hypergraphon $\mathcal{H}/\mathcal{P}_{[\mathcal{A}]}$ is separating. If $\mathcal{H}$ is countably generated, then so is $\mathcal{H}/\mathcal{P}_{[\mathcal{A}]}$.

Proof. Let $\mathcal{B} = \mathcal{A}/\mathcal{P}_{[\mathcal{A}]}$ be the σ-algebra of $J/\mathcal{P}_{[\mathcal{A}]}$. Then by construction, $\mathcal{B}$ is separating so $\mathcal{H}/\mathcal{P}_{[\mathcal{A}]}$ is separating. Moreover, if $\mathcal{H}$ is countably generated then its σ-algebra $\mathcal{A}$ is countably generated.

Now, if we identify elements in the same class of the partition $\mathcal{P}_{[\mathcal{A}]}$, then $\mathcal{B}$ isomorphic to $\mathcal{A}$. Thus, we can see that $\mathcal{H}/\mathcal{P}_{[\mathcal{A}]}$ is countably generated. □

Lemma 4.24. If $\mathcal{H}$ is a separating k-uniform hypergraphon on a measure space with a countable basis, then $\overline{\mathcal{H}}$ can be embedded in a separable k-uniform hypergraphon.

Proof. Let $\mathcal{H}$ be a separating k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$. Suppose that $\mathcal{A}$ is generated by a countable set C, then C is a basis for the completion of $(J, \mathcal{A}, \pi)$. Thus, we have an embedding map φ from the completion of $(J, \mathcal{A}, \pi)$ to a separable measure space $(J', \mathcal{A}', \pi')$. If we take $\mathcal{H}'$ to be the push-forward of $\mathcal{H}$, then $\mathcal{H}' = \mathcal{H}_\varphi$. By Lemma 4.17 we get that $(\mathcal{H}')^\varphi = (\mathcal{H}_\varphi)^\varphi = \mathcal{H}$ almost everywhere. That means φ is an embedding of $\overline{\mathcal{H}}$ into the separable k-uniform hypergraphon $\mathcal{H}'$. □

Lemma 4.25. Suppose that $\mathcal{H}$ is a k-uniform hypergraphon, and let $\mathcal{P}$ be the partition into the twin-classes of $\mathcal{H}$. Then $\mathcal{H}/\mathcal{P}$ is twin-free. If $\mathcal{H}$ is separable, then $\mathcal{H}/\mathcal{P}$ is separable as well. Moreover, the projection $\mathcal{H} \to \mathcal{H}/\mathcal{P}$ is a weak isomorphism.

Proof. Let $\mathcal{H}$ be a k-uniform hypergraphon on $(J, \mathcal{A}, \pi)$. By using Lemma 4.21, we can choose a countably generated σ-algebra $\mathcal{A}_1$ instead of $\mathcal{A}$. Note that the relation of being twins remains the same. Since $\mathcal{H}$ is measurable with respect to $(\mathcal{A}_1)^l$, then x and x' are twins with respect to $\mathcal{H}$ if and only if they are twins with respect to the hypergraphon $\mathcal{H}_1$ obtained by replacing $\mathcal{A}$ with $\mathcal{A}_1$.

Now define $\mathcal{D}$ to be the sub-σ-algebra of $\mathcal{A}$ consisting of sets D such that if $x_1, x_2 \in J^l$ are twins for $\mathcal{H}$ then D does not separate x_1 and x_2. Define $\mathcal{H}' = E(\mathcal{H}|\mathcal{D}^l)$. Then we are going to show that $\mathcal{H} = \mathcal{H}'$ almost everywhere.

Claim: $\mathcal{H} = \mathcal{H}'$ almost everywhere.

Similarly to Lemma 4.20 above, we show that

$$\int_{A_1 \times \cdots \times A_l} \mathcal{H} \quad d\pi = \int_{A_1 \times \cdots \times A_l} \mathcal{H}' \quad d\pi$$

for all $A_1, \ldots, A_l \in \mathcal{A}$.

Recall that $\mathcal{H}' = E(\mathcal{H}|\mathcal{D}^l)$, then we are going to see that $\mathcal{H}'$ on the measure space $(J, \mathcal{D}, \pi)$ is a k-uniform hypergraphon which is weakly isomorphic to $\mathcal{H}$.

Let J' be the set of equivalence classes with respect to the relation of being twins on J, and for $x \in J$ let $\varphi(x)$ be the equivalence class containing x. Let $(\mathcal{A}')^l = \{\varphi(X) : X \in \mathcal{D}^l\}$, and for $X \in (\mathcal{A}')^l$ we define $(\pi')^l(X) = \pi^l(\varphi^{-1}(X))$. Then, we have a measure space $\big((J')^l, (\mathcal{A}')^l, (\pi')^l\big)$.

If $(S_1, S_2, \ldots, S_l) \in (J')^l$, then we define

$$\widetilde{\mathcal{H}}(S_1, S_2, \ldots, S_l) = \mathcal{H}'(x_1, x_2, \ldots, x_l) = \mathcal{H}(x_1, x_2, \ldots, x_l)$$

for any $x_i \in S_i$ where $1 \leqslant i \leqslant l$.

If $\varphi : J^l \to (J')^l$ and $\widetilde{\mathcal{H}} : (J')^l \to [0,1]$, we can see that $(\widetilde{\mathcal{H}})^{\varphi} : J^l \to [0,1]$. Thus, $(\widetilde{\mathcal{H}})^{\varphi} = \mathcal{H}' = \mathcal{H}$ almost everywhere.

Now define N to be the set of points $x \in J$ for which

$$\big\{y_1, \ldots, y_{23\ldots k} \in J : \mathcal{H}'(y_1, \ldots, x, \ldots, y_{23\ldots k}) \neq \mathcal{H}(y_1, \ldots, x, \ldots, y_{23\ldots k})\big\}$$

has positive measure. Thus, N is a null set, and $x, x' \in J \backslash N$ are twins in $\mathcal{H}$ if and only if they are twins in $\mathcal{H}'$. Therefore, we obtained a k-uniform hypergraphon $\mathcal{H}/\mathcal{P}$ from $\mathcal{H}'$ which is twin-free.

Now, we need to prove that if $\mathcal{H}$ is separable, then $\mathcal{H}/\mathcal{P}$ is separable too. Consider $\mathcal{B}$ to be a countable set generating $\mathcal{A}$, closed under finite intersections. For $A \in \mathcal{A}$, $x \in J$, let

$$\lambda_x(A) = \int_A \mathcal{H}(y_1, \ldots, x, \ldots, y_{23\ldots k}) d\pi(y_1) \ldots d\pi(y_{23\ldots k})$$

Because $\mathcal{H}$ is a bounded measurable function with respect to $\mathcal{A}^l$, then the function $A \mapsto \lambda_x(A)$ is a finite measure for all $x \in J$, and the function $x \mapsto \lambda_x(A)$ is a measurable function with respect to $\mathcal{A}$ on J for each $A \in \mathcal{A}$.

Then equivalently we can say that x and x' are twins if and only if $\lambda_x(A) = \lambda_{x'}(A)$ for all $A \in \mathcal{A}$, and since each measure $\lambda_x(\cdot)$ is uniquely determined by sets in $\mathcal{B}$, x and x' are twins if and only if $\lambda_x(B) = \lambda_{x'}(B)$ for all $B \in \mathcal{B}$.

For every $B \in \mathcal{B}$ and $r \in \mathbb{Q}$, let us have the sets

$$S_{B,r} = \{x \in J : \lambda_x(B) \geqslant r\}.$$

These are countably many, $S_{B,r} \in \mathcal{D}$.

Suppose that x and x' are not twins. Then $B \in \mathcal{B}$ such that $\lambda_x(B) \neq \lambda_{x'}(B)$. Now, assume that $\lambda_x(B) > \lambda_{x'}(B)$, then for any $r \in \mathbb{Q}$ with $\lambda_x(B) > r > \lambda_{x'}(B)$, we get that $x \in S_{B,r}$ but $x' \notin S_{B,r}$. That means the countable family of sets $S_{B,r}$ separates x and x'. □

Theorem 4.26. Every k-uniform hypergraphon has a weak isomorphism into a twin-free strong separable k-uniform hypergraphon.

Proof. Let $\mathcal{H}$ be a k-uniform hypergraphon. From Lemma 4.20, we can change the value of $\mathcal{H}$ and obtain a strong k-uniform hypergraphon $\mathcal{H}_1$. Then the identity map on $(J, \mathcal{A}, \pi)$ will be a measurable weak isomorphism $\varphi_1 : \mathcal{H} \to \mathcal{H}_1$. From Lemmas 4.21,4.23, and 4.24, the completion of $\mathcal{H}_1$ has an embedding map $\varphi_2 : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_2$ is a separable k-uniform hypergraphon. Finally, by using Lemma 4.25 we get a twin-free separable k-uniform hypergraphon $\mathcal{H}'$ such that $\varphi_3 : \mathcal{H}_2 \to \mathcal{H}'$ is a weak isomorphism. Because $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$ is a measurable map between $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$, there is a weak isomorphism from $\mathcal{H}$ into $\mathcal{H}'$. □

Conjecture 4.3.1. *Let us consider two k-uniform hypergraphons, $\mathcal{H}$ and*

$\mathcal{H}'$ on measure spaces $(J, \mathcal{A}, \pi)$ and $(J', \mathcal{A}', \pi')$, which satisfy

$$t(F, \mathcal{H}) = t(F, \mathcal{H}')$$

for all k-uniform hypergraphs F. By Lemmas 4.20 to 4.25 and Theorem 4.26, we can find twin-free separable k-uniform hypergraphons $\mathcal{W}$ and $\mathcal{W}'$ on $(\Omega, \mathcal{B}, \lambda)$ and $(\Omega', \mathcal{B}', \lambda')$ and weak isomorphisms $\varphi : \mathcal{H} \to \mathcal{W}$ and $\varphi' : \mathcal{H}' \to \mathcal{W}'$. Then $t(F, \mathcal{W}) = t(F, \mathcal{W}')$ holds for every k-uniform hyperhraph F if and only if $\mathcal{W}$ is isomorphic mod 0 to $\mathcal{W}'$, i.e. $\mathcal{W} \cong \mathcal{W}'$.

4.4 Neighbourhood distance in a k-uniform hypergraphon

Recall that a k-uniform hypergraphon is a symmetric function $\mathcal{H} : J^l \to \mathbb{R}$ where $l = 2^k - 2$ which is measurable with respect to a given measure space $(J, \mathcal{A}, \pi)$.

Definition 4.27. For each $1 \leqslant s \leqslant k-1$ we define a distance r_s as follows. Let $x, y \in J$. Then

$$r_s(x, y) = \int_{J^{l-1}} \left| \mathcal{H}(\underline{x}^x(\overbrace{\varrho(s)})) - \mathcal{H}(\underline{x}^y(\overbrace{\varrho(s)})) \right| \quad d\pi(\underline{x}).$$

Example 4.28. Let $k = 3$ then $1 \leqslant s \leqslant 2$. Thus,

$$r_1(x, y) = \int_{J^5} \Big| \mathcal{H}(x, x_2, x_3, x_{12}, x_{13}, x_{23}) - \mathcal{H}(y, x_2, x_3, x_{12}, x_{13}, x_{23}) \Big| \quad d\pi(x_2)d\pi(x_3)d\pi(x_{12})d\pi(x_{23})d\pi(x_{23})$$

$$r_2(x, y) = \int_{J^5} \Big| \mathcal{H}(x_1, x_2, x_3, x, x_{13}, x_{23}) - \mathcal{H}(x_1, x_2, x_3, y, x_{13}, x_{23}) \Big| \quad d\pi(x_1)d\pi(x_2)d\pi(x_3)d\pi(x_{13})d\pi(x_{23}).$$

Then x and y are twins if and only if $r_1(x,y) = 0 = r_2(x,y)$.

We note that the distance r_s is not necessarily a metric on J, but it is easily seen that r_s is a pseudometric.

Definition 4.29. Let $x, y \in J$. We define the neighbourhood distance

$$r_{\mathcal{H}}(x,y) = \max_{1 \leqslant s \leqslant k-1} \{r_s(x,y)\}.$$

Lemma 4.30. Let $x, y \in J$, then

(i) x and y are twins if and only if $r_{\mathcal{H}}(x,y) = 0$.

(ii) $r_{\mathcal{H}}(x,y)$ is a metric if and only if $\mathcal{H}$ is twin-free.

Proof. *(i)* ($\Longrightarrow$): Assume that x and y are twins. Then by Definition 4.14

$$\mathcal{H}\big(x, \underline{z}\big)^{\rho} = \mathcal{H}\big(y, \underline{z}\big)^{\rho}$$

for almost all $\underline{z} \in J^{l-1}$, and all $\rho \in S_l$. Then,

$$r_s(x,y) = \int\limits_{J^{l-1}} \left| \mathcal{H}(\underline{z}^x(\overset{\frown}{\varrho(s)})) - \mathcal{H}(\underline{z}^y(\overset{\frown}{\varrho(s)})) \right| \quad d\pi(\underline{z}) = 0$$

for all s. By definition 4.29 we have that $r_{\mathcal{H}}(x,y) = 0$.

($\Longleftarrow$): Assume that $r_{\mathcal{H}}(x,y) = 0$. That implies $\max_{1 \leqslant s \leqslant k-1} \{r_s(x,y)\} = 0$. That means $r_s(x,y) = 0$ for all $1 \leqslant s \leqslant k-1$.

Suppose that $\rho \in S_l$ and $\underline{z} \in J^{l-1}$. Let s be the position of x and y in $(x, \underline{z})^{\rho}$ and $(y, \underline{z})^{\rho}$. Then $r_s(x,y) = 0$ implies that $\mathcal{H}(x, \underline{z})^{\rho} - \mathcal{H}(y, \underline{z})^{\rho} = 0$ for almost all $\underline{z}$. Hence, x and y are twins.

(ii) ($\Longrightarrow$): Assume that $r_{\mathcal{H}}$ is a metric. If $x, y \in J$, then

$$r_{\mathcal{H}}(x,y) > 0 \iff x \neq y.$$

Hence, $\mathcal{H}$ is twin-free.

($\Longleftarrow$): Suppose $\mathcal{H}$ is twin-free. Then each pseudometric r_s is a metric. Now, we show that in this case $r_{\mathcal{H}}$ is a metric.

Since $\mathcal{H}$ is twin-free, then there are no twins x and y in J with $x \neq y$. Then, $r_{\mathcal{H}}(x,y) > 0$ for every $x \notin y$ in J. Furthermore,

$$r_{\mathcal{H}}(x,y) = \max_{1 \leqslant s \leqslant k-1} \{r_s(x,y)\} = \max_{1 \leqslant s \leqslant k-1} \{r_s(y,x)\} = r_{\mathcal{H}}(y,x).$$

Now, we are going to show that $r_{\mathcal{H}}(x,z) \leqslant r_{\mathcal{H}}(x,y) + r_{\mathcal{H}}(y,z)$ for all $x, y, z \in J$. Let us suppose that

$$r_{\mathcal{H}}(x,y) = r_s(x,y),$$
$$r_{\mathcal{H}}(y,z) = r_i(y,z),$$
$$r_{\mathcal{H}}(x,z) = r_j(x,z),$$

for some $1 \leqslant s, i, j \leqslant k-1$. Then,

$$\begin{aligned} r_{\mathcal{H}}(x,z) &= r_j(x,z) \\ &\leqslant r_j(x,y) + r_j(y,z) \\ &\leqslant r_s(x,y) + r_i(y,z) \\ &= r_{\mathcal{H}}(x,y) + r_{\mathcal{H}}(y,z). \end{aligned}$$

Thus, $r_{\mathcal{H}}$ satisfies the metric conditions, hence it is a metric. □

The advantage of having $r_{\mathcal{H}}$ defined is that it is a way of measuring the distance which we can use to define purity.

5

The Purification of k-uniform Mixed Hypergraphons

5.1 Motivation and a new definition of k-uniform hypergraphon

In this chapter, we come up with a new generalization of a k-uniform hypergraphon which is called a k-uniform mixed hypergraphon. Moreover, we are going to define the notion of a pure k-uniform mixed hypergraphon and then show that for every twin-free separable k-uniform mixed hypergraphon, there is a pure k-uniform mixed hypergraphon isomorphic, up to a null set, to it.

The following example shows that the analogous result cannot be applied with the definition of k-uniform hypergraphon that we adopted in Definition 4.10.

Example 5.1. Let $\mathcal{H} : [0,1]^6 \to [0,1]$ be the hypergraphon defined by

$$\mathcal{H}(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = x_1 x_2 x_3$$

If $x, y \in [0,1]$, then

$$r_1(x,y) = 0 \iff x = y,$$
$$r_2(x,y) = 0 \quad \text{for all } x \text{ and } y$$

so that $r(x,y) > 0$ for all $x \neq y$ and according to the current definition $\mathcal{H}$ is twin-free. This would cause problems when we look at the purification because $r_2(x,y)$ definitely isn't a metric!

We get round this with another generalization of a hypergraphon and another definition of twins.

Definition 5.2. Suppose that $(J_i, \mathscr{A}_i, \pi_i)$ are measure spaces where $1 \leqslant i \leqslant$

$k-1$. A k-uniform mixed hypergraphon is a function

$$\mathscr{H} : J_1^{\binom{k}{1}} \times J_2^{\binom{k}{2}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}} \to [0,1]$$

which is symmetric in the sense of Definition 4.1. We say that $\mathscr{H}$ is a k-uniform mixed hypergraphon with respect to $J_1, \ldots, J_{k-1}$.

Definition 5.3. Fix $1 \leqslant i \leqslant k-1$. We say that $x, y \in J_i$ are twins for i if

$$\mathscr{H}\left(\underline{x}^x(\widetilde{\varrho(i)})\right) = \mathscr{H}\left(\underline{x}^y(\widetilde{\varrho(i)})\right)$$

for almost all $\underline{x} \in J_1^{\binom{k}{1}} \times \cdots \times J_i^{\binom{k}{i}-1} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$. We say that $\mathscr{H}$ is a *twin-free k-uniform mixed hypergraphon* if there is no $1 \leqslant i \leqslant k-1$ such that there are $x \neq y$ twins for J_i.

With this new definition, Example 5.1 is no longer twin-free. However, with Definition 5.2 we can remove the twins for $i = 2$ but keep J_1 the same. Namely set $J_1 = [0,1]$ and $J_2 = \{\bullet\}$ and define

$$\widetilde{\mathscr{H}} : J_1 \times J_1 \times J_1 \times J_2 \times J_2 \times J_2 \to [0,1],$$
$$\widetilde{\mathscr{H}} : (x_1, x_2, x_3, \bullet, \bullet, \bullet) \mapsto x_1 x_2 x_3.$$

A k-uniform mixed hypergraphon $\mathscr{H}$ with respect to $J_1, J_2, \ldots, J_{k-1}$ is *strong* if it is measurable with respect to the σ-algebra $\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_{k-1}$. For a k-uniform mixed hypergraphon $\mathscr{H}$, we define its completion $\overline{\mathscr{H}}$ as being the same function of $\mathscr{H}$ but considered with respect to the completion of σ-algebra $\mathscr{A}$. We say that a k-uniform mixed hypergraphon is complete if it is equal to its completion.

A k-uniform mixed hypergraphon is *separable* if the measure space $J_1 \times \cdots \times J_{k-1}$ is separable.

Suppose that $(J_i, \mathscr{A}_i, \pi_i)$ and $(J_i', \mathscr{A}_i', \pi_i')$ are measure spaces where

$i = 1, \ldots, k-1$, and let $\mathscr{H}'$ be a k-uniform mixed hypergraphon with respect to $J'_1, J'_2, \ldots, J'_{k-1}$. Suppose that $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{k-1})$ is a tuple of measure preserving maps where $\varphi_i : J_i \to J'_i$. Then we define $\varphi : J_1^{\binom{k}{1}} \times J_2^{\binom{k}{2}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}} \to {J'_1}^{\binom{k}{1}} \times {J'_2}^{\binom{k}{2}} \times \cdots \times {J'_{k-1}}^{\binom{k}{k-1}}$ as follows; if $\underline{x} = (x_1, x_2, \ldots, x_{23\ldots k}) \in J_1^{\binom{k}{1}} \times J_2^{\binom{k}{2}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$, then

$$\varphi(\underline{x}) = \big(\varphi_1(x_1), \varphi_1(x_2), \ldots, \varphi_{k-1}(x_{23\ldots k})\big)$$

We define the *pull-back* $(\mathscr{H}')^{\varphi}$ to be the k-uniform mixed hypergraphon on $J_1, J_2, \ldots, J_{k-1}$ defined by

$$(\mathscr{H}')^{\varphi}(\underline{x}) = \mathscr{H}'\big(\varphi(\underline{x})\big)$$

for all $\underline{x} \in J_1^{\binom{k}{1}} \times J_2^{\binom{k}{2}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$.

Suppose that $\mathscr{H}$ is a k-uniform mixed hypergraphon on $J_1, J_2, \ldots, J_{k-1}$. If for all i the function φ_i is measure preserving from $\overline{\mathscr{A}_i}$ to $\mathscr{A}'_i$ and $\mathscr{H} = (\mathscr{H}')^{\varphi}$ almost everywhere, then we call φ a *weak isomorphism* from $\mathscr{H}$ to $\mathscr{H}'$.

We say that two k-uniform mixed hypergraphons $\mathscr{H}$ and $\mathscr{H}'$ are *weakly isomorphic* if there is another k-uniform mixed hypergraphon $\mathscr{H}''$ and weak isomorphisms from $\mathscr{H}$ and $\mathscr{H}'$ into $\mathscr{H}''$.

5.2 Reduction of the k-uniform mixed hypergraphon

The main goal of this section is to explore how can we adapt a k-uniform mixed hypergraphon into a twin-free separable k-uniform mixed hypergraphon. We use the same measure theory notions and results from section 4.3.

In the next lemma we modify σ-algebra to a countably generated one.

Lemma 5.4. Let $(J_1, \mathscr{A}_1), \ldots, (J_t, \mathscr{A}_t)$ be measurable spaces, and suppose

$$\mathscr{L} : J_1 \times \cdots \times J_t \to \mathbb{R}$$

is a function which is measurable with respect to $\mathscr{A}_1 \times \cdots \times \mathscr{A}_t$. Then, there exist countably generated σ-algebras $\mathscr{B}_1 \subseteq \mathscr{A}_1, \ldots, \mathscr{B}_t \subseteq \mathscr{A}_t$ such that $\mathscr{L}$ is measurable with respect to $\mathscr{B}_1 \times \cdots \times \mathscr{B}_t$.

Proof. The proof of this lemma is similar to the proof of Lemma 4.15. □

Now, we investigate how can we come by a new k-uniform mixed hypergraphon from a given k-uniform mixed hypergraphon by applying the idea of a push-forward as follows.

Let $(J_i, \mathscr{A}_i, \pi_i)$ and $(J'_i, \mathscr{A}'_i, \pi'_i)$ be measure spaces where $i = 1, \ldots, k-1$, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{k-1})$ be a tuple of measure preserving maps where $\varphi_i : J_i \to J'_i$. Suppose that $\mathscr{H}$ is a k-uniform mixed hypergraphon on $J_1, J_2, \ldots, J_{k-1}$. A k-uniform mixed hypergraphon $\mathscr{H}_\varphi$ on $J'_1, J'_2, \ldots, J'_{k-1}$ is said to be the *push-forward* of $\mathcal{H}$ by φ if

$$\int_c \mathscr{H}_\varphi(\underline{x}') \quad d\pi'(\underline{x}') = \int_{\varphi^{-1}(c)} \mathscr{H}(\underline{x}) \quad d\pi(\underline{x}) \tag{5.2.1}$$

for all $c \in \mathscr{A}'$ where $\mathscr{A}' = (\mathscr{A}'_1)^{\binom{k}{1}} \times \cdots \times (\mathscr{A}'_{k-1})^{\binom{k}{k-1}}$.

As in Lemma 4.17 we are going to show that the push-forward $\mathscr{H}_\varphi$ is well defined and that there is a relation between $\mathscr{H}_\varphi$ and the pull back $(\mathscr{H}_\varphi)^\varphi$.

Lemma 5.5. Let $(J_i, \mathscr{A}_i, \pi_i)$ and $(\Omega_i, \mathscr{B}_i, \mu_i)$ be measure spaces where $i = 1, \ldots, k-1$, and let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{k-1})$ be a tuple of measure preserving maps where $\varphi_i : J_i \to \Omega_i$. Suppose that $\mathscr{H}$ is a k-uniform mixed hypergraphon with respect to $J_1, J_2, \ldots, J_{k-1}$. Then

(i) There is a bounded, symmetric function

$$\mathscr{H}_\varphi : \Omega_1^{\binom{k}{1}} \times \Omega_2^{\binom{k}{2}} \times \cdots \times \Omega_{k-1}^{\binom{k}{k-1}} \to \mathbb{R}$$

which is measurable with respect to $\mathscr{B} = \mathscr{B}_1^{\binom{k}{1}} \times \cdots \times \mathscr{B}_{k-1}^{\binom{k}{k-1}}$ and the measure μ which satisfies $\mathscr{H}_\varphi = \frac{d\nu}{d\mu}$. In particular, ν is a measure on $\mathscr{B}$ defined by

$$\nu(c) = \int\limits_c \mathscr{H}_\varphi d\mu$$

for all $c \in \mathscr{B}$. Thus $\mathscr{H}_\varphi$ is a unique up to changes on a set of μ-measure zero in $\Omega_1^{\binom{k}{1}} \times \Omega_2^{\binom{k}{2}} \times \cdots \times \Omega_{k-1}^{\binom{k}{k-1}}$.

(ii) If $\mathscr{A}_\varphi = \{\varphi^{-1}(B) : B \in \mathscr{B}\}$, then

$$(\mathscr{H}_\varphi)^\varphi = E\big(\mathscr{H}\big|(\mathscr{A}_1^{\binom{k}{1}})_{\varphi_1} \times \cdots \times (\mathscr{A}_{k-1}^{\binom{k}{k-1}})_{\varphi_{k-1}}\big)$$

almost everywhere.

(iii) If φ is a measure preserving embedding of $(J_i, \mathscr{A}_i, \pi_i)$ into $(\Omega_i, \mathscr{B}_i, \mu_i)$, then $(\mathscr{H}_\varphi)^\varphi = \mathscr{H}$ almost everywhere.

Proof. (i) Define a measure ν on $\mathscr{B} = \mathscr{B}_1^{\binom{k}{1}} \times \cdots \times \mathscr{B}_{k-1}^{\binom{k}{k-1}}$ by

$$\nu(c) = \int\limits_{\varphi^{-1}(c)} \mathscr{H}(x_1, \ldots, x_k, \ldots, x_{23\ldots k}) \quad d\pi_1(x_1) \ldots d\pi_{k-1}(x_{23\ldots k})$$

for $c \in \mathscr{B}$. Because $\mathscr{H}$ takes values in $[0,1]$, we have

$$\nu(c) \leqslant \pi_1 \times \cdots \times \pi_{k-1}(\varphi^{-1}(c)).$$

Since $\varphi_i : J_i \to \Omega_i$ is measure preserving map, then $\nu(c) \leqslant \mu(c)$ for all $c \in \mathscr{B}$. Thus the measure ν is absolutely continuous with respect to μ, denoted as $\nu(c) << \mu(c)$. Hence, the Radon-Nikodym derivative of ν with respect to μ is well defined and is equal to $\mathscr{H}_\varphi$.

Note that by the Radon-Nikodym Theorem 4.16, $\mathscr{H}_\varphi$ is unique.

(ii) Assume that $e \in \mathscr{A}_\varphi$ where $e = \varphi^{-1}(c)$ for some $c \in \mathscr{B}$. From the function

$$\mathscr{H}_\varphi : \Omega_1^{\binom{k}{1}} \times \Omega_2^{\binom{k}{2}} \times \cdots \times \Omega_{k-1}^{\binom{k}{k-1}} \to \mathbb{R}$$

where φ is a tuple of measure preserving maps, and the pull-back of $\mathscr{H}_\varphi$, i.e. $(\mathscr{H}_\varphi)^\varphi$, we show that

$$\int_e \mathscr{H}(\underline{x}) \quad d\pi(\underline{x}) = \int_c \mathscr{H}_\varphi(\underline{x}') \quad d\mu(\underline{x}')$$

$$= \int_e \mathscr{H}_\varphi(\varphi(\underline{x})) \quad d\pi(\underline{x}) = \int_e (\mathscr{H}_\varphi)^\varphi(\underline{x}) \quad d\pi(\underline{x})$$

Thus,

$$(\mathscr{H}_\varphi)^\varphi = E\big(\mathscr{H} \big| (\mathscr{A}_1^{\binom{k}{1}})_{\varphi_1} \times \cdots \times (\mathscr{A}_{k-1}^{\binom{k}{k-1}})_{\varphi_{k-1}}\big)$$

almost everywhere.

(iii) Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{k-1})$ be an isomorphism between $(J_i, \mathscr{A}_i, \pi_i)$ and a subspace of $(\Omega_i, \mathscr{B}_i, \mu_i)$. For any $e \in \mathscr{A}$, we have $c \in \mathscr{B}$ such that the intersection of c and $\varphi(J)$ is $\varphi(e)$ where $J = J_1^{\binom{k}{1}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$. However, $\varphi^{-1}(c) = \varphi^{-1}\big(\varphi(e)\big) = e$, thus $e \in \mathscr{A}_\varphi$. Hence, $\mathscr{A} = \mathscr{A}_\varphi$ which implies that $\mathscr{H} = (\mathscr{H}_\varphi)^\varphi$ almost everywhere. $\square$

As a result, the function φ is said to be an embedding of $\mathscr{H}$ into $\mathscr{H}_\varphi$ if it satisfies the following conditions:

- φ is an embedding of $(J_i, \mathscr{A}_i, \pi_i)$ into $(\Omega_i, \mathscr{B}_i, \mu_i)$,
- $\mathscr{H} = (\mathscr{H}_\varphi)^\varphi$ almost everywhere.

Now, we can define quotients of k-uniform mixed hypergraphons.

Definition 5.6. Let $\mathscr{H}$ be a k-uniform mixed hypergraphon with respect to measure spaces $J_1, J_2, \ldots, J_{k-1}$. For each i, let $\mathscr{P}_i$ be an arbitrary partition of J_i into disjoint sets. For $x \in J_i$, we let $[x]$ be the class in $\mathscr{P}_i$ which contains x. Let $\mathscr{P} = \mathscr{P}_1 \times \cdots \times \mathscr{P}_{k-1}$, and we define a k-uniform mixed hypergraphon

$$\mathscr{H}/\mathscr{P} : \Big(J_1/\mathscr{P}_1\Big)^{\binom{k}{1}} \times \cdots \times \Big(J_{k-1}/\mathscr{P}_{k-1}\Big)^{\binom{k}{k-1}} \longrightarrow [0,1]$$

where $(J_i/\mathscr{P}_i, \mathscr{A}_i/\mathscr{P}_i, \pi_i/\mathscr{P}_i)$ is the measure space defined as follows:

- the points in $J_i/\mathscr{P}_i$ are the classes of the partition $\mathscr{P}_i$.
- φ_i is the measure preserving map $x \mapsto [x]$.
- $\mathscr{A}_i/\mathscr{P}_i$ is the σ-algebra consisting of the sets $A' \subseteq J_i/\mathscr{P}_i$ such that $\varphi_i^{-1}(A') \in \mathscr{A}_i$.
- $(\pi_i/\mathscr{P}_i)(A') = \pi_i(\varphi_i^{-1}(A'))$.

So φ is measure preserving, and we define $\mathscr{H}/\mathscr{P} = \mathscr{H}_\varphi$ as in (5.2.1).

In a similar way to section 4.3, we show that every k-uniform mixed hypergraphon is weakly isomorphic to a twin-free separable k-uniform mixed hypergraphon.

Theorem 5.7. Let $\mathscr{H}$ be a k-uniform mixed hypergraphon with respect to $J_1, J_2, \ldots, J_{k-1}$ where each J_i stands for a measure space $(J_i, \mathscr{A}_i, \pi_i)$. Let $J = J_1^{\binom{k}{1}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$, similarly for $\mathscr{A}$ and π. Then

(i) We can define a strong k-uniform mixed hypergraphon by changing the value of $\mathscr{H}$ on a set of measure zero with respect to π.

(ii) There exists a countably generated σ-algebra $\mathscr{B} \subseteq \mathscr{A}$ such that $\mathscr{H}$ is measurable with respect to $\mathscr{B}$ where $\mathscr{B} = \mathscr{B}_1^{\binom{k}{1}} \times \cdots \times \mathscr{B}_{k-1}^{\binom{k}{k-1}}$.

(iii) The σ-algebra $\mathscr{A}/\mathscr{P}_{[\mathscr{A}]}$ is separating. If $\mathscr{A}$ is countably generated, then so is $\mathscr{A}/\mathscr{P}_{[\mathscr{A}]}$.

(iv) If $\mathscr{H}$ is a separating k-uniform mixed hypergraphon on a measure space with a countable basis, then $\overline{\mathscr{H}}$ can be embedded into a separable k-uniform mixed hypergraphon.

(v) Suppose that $\mathscr{H}$ is a k-uniform mixed hypergraphon, and let $\mathscr{P}$ be the partition into the twin-classes of $\mathscr{H}$. Then $\mathscr{H}/\mathscr{P}$ is twin-free. If $\mathscr{H}$ is separable, then $\mathscr{H}/\mathscr{P}$ is separable as well. Moreover, the projection $\mathscr{H} \to \mathscr{H}/\mathscr{P}$ is a weak isomorphism.

Proof. Here we sketch our proof since this theorem is analogous to the Lemmas from 4.20 to 4.25.

We explain how we can find a strong k-uniform mixed hypergraphon $\mathscr{H}'$ which is equal to $\mathscr{H}$ almost everywhere. We define $\mathscr{H}' = E(\mathscr{H}|\mathscr{A})$ the conditional expectation with respect to $\mathscr{A}$. Clearly $\mathscr{H}'$ is measurable with respect to $\mathscr{A}$ and in a similar way to Lemma 4.20 we have

$$\int\limits_e \mathscr{H} \quad d\pi_1 \dots d\pi_{k-1} = \int\limits_e \mathscr{H}' \quad d\pi_1 \dots d\pi_{k-1}$$

for all $e \in \mathscr{A}$. This implies $\mathscr{H} = \mathscr{H}'$ almost everywhere. This proves (i). Next, if $\mathscr{H}$ is a k-uniform mixed hypergraphon with respect to $J_1, J_2, \dots, J_{k-1}$, then by Lemma 5.4 we have (ii).

Consider the partition $\mathscr{P}_{[\mathscr{A}]}$ of J induced by the σ-algebra $\mathscr{A}$. If we identify elements in the same class of $\mathscr{P}_{[\mathscr{A}]}$, we obtain the σ-algebra $\mathscr{A}/\mathscr{P}_{[\mathscr{A}]}$. By Lemma 2.22, the σ-algebras $\mathscr{A}$ and $\mathscr{A}/\mathscr{P}_{[\mathscr{A}]}$ are isomorphic. Then clearly if $\mathscr{A}$ is countably generated so is $\mathscr{A}/\mathscr{P}_{[\mathscr{A}]}$, which proves (iii).

For (iv), suppose that the k-uniform mixed hypergraphon $\mathscr{H}$ is separating with respect to $J_1, J_2, \dots, J_{k-1}$. Let σ-algebra $\mathscr{A}$ be generated by a countable set $\mathscr{S} = \{\mathscr{S}_1, \mathscr{S}_2, \dots, \mathscr{S}_{k-1}\}$. That means $\mathscr{S}$ is a basis for

$(\overline{\mathscr{A}}_i, \overline{\pi}_i)$ where $i = 1, 2, \ldots, k-1$. Thus we have an embedding functions φ_i from the completion of $(J_i, \mathscr{A}_i, \pi_i)$ to a separable measure space $(\Omega_i, \mathscr{B}_i, \mu_i)$ for each i. If we let $\mathscr{K}$ to be the push-forward of $\mathscr{H}$, then $\mathscr{K} = \mathscr{H}_\varphi$. By using part (iii) of Lemma 5.5, the pull-back of $\mathscr{K}$ equals to the pull-back of $\mathscr{H}_\varphi$ which is equal to $\mathscr{H}$ almost everywhere. Hence, φ is an embedding of $\overline{\mathscr{H}}$ into $\mathscr{K}$.

To show (v), suppose that $\mathscr{H}$ is a k-uniform mixed hypergraphon with respect to $J_1, J_2, \ldots, J_{k-1}$. Then, by (ii) there exists a countably generated σ-algebra $\mathscr{B} \subseteq \mathscr{A}$ such that $\mathscr{H}$ is measurable with respect to $\mathscr{B}$ where $\mathscr{B} = \mathscr{B}_1^{\binom{k}{1}} \times \cdots \times \mathscr{B}_{k-1}^{\binom{k}{k-1}}$. That means any two points in $\mathscr{B} \subseteq \mathscr{A}$ are twins with respect to $\mathscr{H}$ if and only if they are twins with respect to the k-uniform mixed hypergraphon $\mathscr{H}_1$ obtained by replacing $\mathscr{A}$ with $\mathscr{B}$.

Define $\mathscr{H}' = E(\mathscr{H}|\mathscr{D} = \mathscr{D}_1^{\binom{k}{1}} \times \cdots \times \mathscr{D}_{k-1}^{\binom{k}{k-1}})$ where $\mathscr{D} \subseteq \mathscr{A}$ that consisting of those sets in $\mathscr{A}$ that do not separate any twin points. By (i), we can see $\mathscr{H}$ is equal to $\mathscr{H}'$ almost everywhere, and $\mathscr{H}'$ with respect to $J_1, J_2, \ldots, J_{k-1}$ and $\mathscr{D}$ is weakly isomorphic to $\mathscr{H}$.

Now, let us consider a null set $N_i \subseteq J_i$. Then for any two points in $J_i \backslash N_i$ are twins in $\mathscr{H}$ if and only if they are twins in $\mathscr{H}'$. If we identify indistinguishable elements in partition $\mathscr{P}$, then we can get a k-uniform mixed hypergraphon $\mathscr{H}/\mathscr{P}$ from $\mathscr{H}'$. Thus $\mathscr{H}/\mathscr{P}$ is twin-free.

Finally, in the same way to Lemma 4.25 we can see that if the k-uniform mixed hypergraphon $\mathscr{H}$ is separable then $\mathscr{H}/\mathscr{P}$ is separable.

□

The proof of the main theorem in this section follows immediately.

Theorem 5.8. Every k-uniform mixed hypergraphon admits a weak isomorphism into a twin-free separable k-uniform mixed hypergraphon.

5.3 The purification of twin-free separable k-uniform mixed hypergraphons

Let $\mathscr{H}$ be a twin-free separable k-uniform mixed hypergraphon. Fix $1 \leqslant i \leqslant k-1$. For each $z \in J_i$ we have a function (section) $\mathscr{H}_z^i : T_i \to [0,1]$ defined by

$$\mathscr{H}_z^i(\underline{x}) = \mathscr{H}(\underline{x}^z \, \breve{\underline{\varrho}(i)})$$

for $\underline{x} \in T_i$ where $T_i = J_1^{\binom{k}{1}} \times \cdots \times J_i^{\binom{k}{i}-1} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$.

We see that $\mathscr{H}_z^i : T_i \to [0,1]$ is measurable function since $\mathscr{H}$ is measurable, and we see that

$$\int_{J_i} \left(\int_{T_i} \mathscr{H}_z^i \quad d(\underline{y}(\widehat{\underline{\varrho}(i)})) \right) dx = \int_{J_1^{\binom{k}{1}} \times J_2^{\binom{k}{2}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}} \mathscr{H} \quad d(\underline{y}) \quad < \infty \qquad (5.3.1)$$

for each i. That means $\mathscr{H}_z^i \in L^1(T_i)$.

Now, let us define $\varphi_{\mathscr{H}^i} : J_i \to L^1(T_i)$ by $\varphi_{\mathscr{H}^i}(z) = \mathscr{H}_z^i$, which is a measurable function.

For $A \subseteq L^1(T_i)$, we define

$$\pi_{\mathscr{H}^i}(A) = \pi_i(\varphi_{\mathscr{H}^i}^{-1}(A))$$

which is the measure on $L^1(T_i)$ induced by the measure π_i on J_i.

Now let us define

$$J_{\mathscr{H}^i} = \left\{ f \in L^1(T_i) : \text{for every open set } U \text{ that contains } f, \pi_{\mathscr{H}^i}(U) > 0 \right\}$$

This is the support of $\pi_{\mathscr{H}^i}$. Thus, $J_{\mathscr{H}^i}$ is a subset of $L^1(T_i)$.

Proposition 5.9. For each i, $J_{\mathscr{H}^i}$ is a separable Banach space, and the measure $\pi_{\mathscr{H}^i}$ has full support on $J_{\mathscr{H}^i}$.

To prove this proposition we need to state the following theorem.

Theorem 5.10. Let (Ψ, μ) be a measure space. The metric space $L^1(\Psi)$ is separable if and only if the measure μ is separable.

Proof of Proposition *5.9.* First of all, we want to show that $J_{\mathscr{H}^i}$ is closed. We are going to show that $J_{\mathscr{H}^i}$ is closed by proving that $L^1(T_i)\backslash J_{\mathscr{H}^i}$ is open.

Let $f \in L^1(T_i)\backslash J_{\mathscr{H}^i}$. There is a neighbourhood U_f of f such that $\pi_{\mathscr{H}^i}(U_f) = 0$. Then for every $g \in U_f$, U_f itself is a neighbourhood of g such that $\pi_{\mathscr{H}^i}(U_f) = 0$ and therefore $g \notin J_{\mathscr{H}^i}$. This shows $U_f \subseteq L^1(T_i)\backslash J_{\mathscr{H}^i}$. Hence, $L^1(T_i)\backslash J_{\mathscr{H}^i}$ is open set and so $J_{\mathscr{H}^i}$ is closed set.

Now we want to show that $J_{\mathscr{H}^i}$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $J_{\mathscr{H}^i}$. That means this Cauchy sequence is also in $L^1(T_i)$. Thus, f is a limit point of $\{f_n\}$ in $L^1(T_i)$. However, since $J_{\mathscr{H}^i}$ is closed then it contains all of its limit points. Thus, $f \in J_{\mathscr{H}^i}$, so it is complete.

For separability of $J_{\mathscr{H}^i}$ by using Theorem 5.10 we see that $(J_{\mathscr{H}^i}, \pi_{\mathscr{H}^i})$ is separable if $\pi_{\mathscr{H}^i}$ is separable, which follows from our assumption. Therefore, $J_{\mathscr{H}^i}$ is a complete separable metric space for each i.

Now, we want to show that for each i, $\pi_{\mathscr{H}^i}$ has full support on $J_{\mathscr{H}^i}$. Assume that we have an open subset U of $J_{\mathscr{H}^i}$ with $\pi_{\mathscr{H}^i}(U) = 0$. Then, $U = D \cap J_{\mathscr{H}^i}$ for some open subset D of $L^1(T_i)$. That means $\pi_{\mathscr{H}^i}(D) = \pi_{\mathscr{H}^i}(U) = 0$. Hence, D is subset of $L^1(T_i)\backslash J_{\mathscr{H}^i}$ and $U = D \cap J_{\mathscr{H}^i} = \varnothing$. $\square$

Definition 5.11. Let $y_1, y_2 \in J_i$. Then we define

$$\begin{aligned} r_{\mathscr{H}^i}(y_1, y_2) = \Bigg| \int_{T_i} & \mathscr{H}^i_{y_1}(x_1, x_2, \ldots, x_{23\ldots k-1}) d\pi_1(x_1) \ldots d\pi_{k-1}(x_{23\ldots k-1}) \\ & - \int_{T_i} \mathscr{H}^i_{y_2}(x_1, x_2, \ldots, x_{23\ldots k-1}) d\pi_1(x_1) \ldots d\pi_{k-1}(x_{23\ldots k-1}) \Bigg| \\ & = \|\mathscr{H}^i_{y_1} - \mathscr{H}^i_{y_2}\|_{L^1} \end{aligned}$$

Lemma 5.12. The function $r_{\mathscr{H}^i}$ is a metric.

Proof. We have assumed that J_i does not contain any twin points of $\mathscr{H}$. Thus, $r_{\mathscr{H}^i}(y_1, y_2) > 0$ for every $y_1 \neq y_2$ in J_i . Furthermore, for $y_1, y_2 \in J_i$ we see that

$$r_{\mathscr{H}^i}(y_1, y_2) = r_{\mathscr{H}^i}(y_2, y_1)$$

Moreover, in the same way as in Lemma 4.30 the triangle inequality holds

$$r_{\mathscr{H}^i}(y_1, y_2) \leqslant r_{\mathscr{H}^i}(y_1, z) + r_{\mathscr{H}^i}(z, y_2)$$

for all $z \in J_i$. Hence, $r_{\mathscr{H}^i}$ is a metric. □

Now we are going to state the definition of a pure k-uniform mixed hypergraphon.

Definition 5.13. We say that the twin-free k-uniform mixed hypergraphon $\mathscr{H}$ is *pure* if, for all i,

- The metric space $(J_i, r_{\mathscr{H}^i})$ is complete and separable,
- $\pi_{\mathscr{H}^i}$ has full support with respect to $r_{\mathscr{H}^i}$, i.e. for every $x \in J_i$ and $\epsilon > 0$,
$$\int_{\{y: r_{\mathscr{H}^i}(x,y) < \epsilon\}} d\pi_{\mathscr{H}^i}(x) > 0$$

Lemma 5.14. A k-uniform mixed hypergraphon $\mathscr{H}$ is pure if and only if $\varphi_{\mathscr{H}^i} : J_i \to J_{\mathscr{H}^i}$ is a bijection, for all i.

Proof. ($\Longrightarrow$): Assume that $\mathscr{H}$ is a pure k-uniform mixed hypergraphon. That means $(J_i, r_{\mathscr{H}^i})$ is complete and separable metric space and $\pi_{\mathscr{H}^i}$ has full support for all i. Then, J_i is twin-free since the metric $r_{\mathscr{H}^i}(y_1, y_2) > 0$ iff $y_1 \neq y_2$ for all $y_1, y_2 \in J_i$. That means $\varphi_{\mathscr{H}^i} : J_i \to J_{\mathscr{H}^i}$ is injective. Now, let us consider a set $S = \{x \in J_i : \varphi^{-1}_{\mathscr{H}^i}(x) = \varnothing\}$. Then the measure of S is zero, hence $\varphi_{\mathscr{H}^i}$ is bijective for all i.

($\Longleftarrow$): Assume that $\varphi_{\mathscr{H}^i} : J_i \to J_{\mathscr{H}^i}$ is a bijection for all i. Since $\varphi_{\mathscr{H}^i}$ is measurable and $\pi_{\mathscr{H}^i}(A) = \pi_i(\varphi^{-1}_{\mathscr{H}^i}(A))$ for each $A \subseteq L^1(T_i)$, then $\varphi_{\mathscr{H}^i}$ is measure preserving bijection. We know that $J_{\mathscr{H}^i}$ is a subset of $L^1(T_i)$ so $J_{\mathscr{H}^i}$ is a complete separable metric space and $\pi_{\mathscr{H}^i}$ has full support on $J_{\mathscr{H}^i}$. Therefore, $\mathscr{H}$ is a pure k-uniform mixed hypergraphon. □

Theorem 5.15. Every twin-free separable k-uniform mixed hypergraphon is weakly isomorphic to a pure k-uniform mixed hypergraphon.

Proof. Let $\mathscr{H}$ be a twin-free separable k-uniform mixed hypergraphon. That means there is no $1 \leqslant i \leqslant k-1$ such that $x \neq y$ with x and y are twins for J_i. For each $x \in J_i$, we have a function $\mathscr{H}^i_x : T_i \to [0,1]$ where $T_i = J_1^{\binom{k}{1}} \times \cdots \times J_i^{\binom{k}{i}-1} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$. From (5.3.1) we see that $\mathscr{H}^i_x$ is measurable since $\mathscr{H}$ is measurable. Then, $\mathscr{H}^i_x \in L^1(T_i)$.

Now, let $\varphi_{\mathscr{H}^i} : J_i \to L^1(T_i)$ be defined by $\varphi_{\mathscr{H}^i}(x) = \mathscr{H}^i_x$, which is measurable. Let $A \subseteq L^1(T_i)$, then

$$\pi_{\mathscr{H}^i}(A) = \pi_i(\varphi^{-1}_{\mathscr{H}^i}(A))$$

defines a measure on $L^1(T_i)$.

Recall that

$$J_{\mathscr{H}^i} = \{f \in L^1(T_i) : \text{for every open set } U \text{ that contains } f, \pi_{\mathscr{H}^i}(U) > 0\}$$

is the support of $\pi_{\mathscr{H}^i}$. Then, $J_{\mathscr{H}^i} \subseteq L^1(T_i)$. From the Proposition 5.9 we can see that $J_{\mathscr{H}^i}$ is a separable Banach space, and $\pi_{\mathscr{H}^i}$ has full support on $J_{\mathscr{H}^i}$.

Suppose that Ω_i is the set of elements in J_i for which $\mathscr{H}^i_x \in J_{\mathscr{H}^i}$, and let

$\Omega_{\mathscr{H}^i} = \{\mathscr{H}_x^i : x \in \Omega_i\}$. We define $\psi_i : \Omega_i \to \Omega_{\mathscr{H}^i}$ by $x \mapsto \mathscr{H}_x^i$ which is bijective for all i since $\mathscr{H}$ is twin-free.

Claim. $\pi_i(J_i \backslash \Omega_i) = 0$

Proof: Suppose that $g \in L^1(T_i) \backslash J_{\mathscr{H}_x^i}$. That means there is a neighbourhood U_g of g such that $U_g \cap J_{\mathscr{H}^i} = \varnothing$, and $\pi_{\mathscr{H}^i}(U_g) = 0$. Thus, $U_g \subseteq L^1(T_i) \backslash J_{\mathscr{H}^i}$, and $\pi_i(\{x \in J_i : \mathscr{H}_x^i \in U_g\}) = 0$.

Let $U = \bigcup_{g \notin J_{\mathscr{H}^i}} U_g$. Since $L^1(T_i)$ is separable, then U equals the union of some countable subfamily $\{U_{g_s} : s \in \mathbb{N}\}$, and so $\pi_i\big(\{x \in J_i : \mathscr{H}_x^i \in U\}\big) = 0$ by countable additive of π_i. We know that for each $x \in J_i$, the function $\mathscr{H}_x^i \in L^1(T_i)$. Since $J_i \backslash \Omega_i \subseteq U$, then we see that $\pi_i(J_i \backslash \Omega_i) = 0$. ■

The function ψ_i and the measure π_i induce a measure ν_i on $J_{\mathscr{H}^i}$ given by $\nu_i = \pi_i \circ \psi_i^{-1}$. Equipped with this measure, $J_{\mathscr{H}^i}$ is a complete separable metric space and every open set has a positive measure. Since ψ_i is a bijection, we see that $\pi_i(J_i \backslash \Omega_i) = 0$ implies that $\nu_i(J_{\mathscr{H}^i} \backslash \Omega_{\mathscr{H}^i}) = 0$.

Now, we define a k-uniform mixed hypergraphon

$$\widetilde{\mathscr{H}} : \Omega_{\mathscr{H}^1}^{\binom{k}{1}} \times \Omega_{\mathscr{H}^2}^{\binom{k}{2}} \times \cdots \times \Omega_{\mathscr{H}^{k-1}}^{\binom{k}{k-1}} \to [0,1]$$

as follows. Set $\mathcal{I} = \big\{A \subset \{1, 2, \ldots, k\} : 1 \leqslant |A| \leqslant k-1\big\}$. Let

$$\mathbf{f} = (f_1, f_2, \ldots, f_k, f_{12}, \ldots, f_{k-1k}, \ldots, f_{12\ldots k-1}, \ldots, f_{23\ldots k}) \in \Omega_{\mathscr{H}^1}^{\binom{k}{1}} \times \Omega_{\mathscr{H}^2}^{\binom{k}{2}} \times \cdots \times \Omega_{\mathscr{H}^{k-1}}^{\binom{k}{k-1}}$$

For each $I \in \mathcal{I}$ there exists $x_I \in J_i$, where $i = |I|$, such that

$$f_I = \mathscr{H}_{x_I}^i = \varphi_{\mathscr{H}^i}(x_I).$$

Set $\varphi_{\mathscr{H}} = \big(\varphi_{\mathscr{H}^1}, \varphi_{\mathscr{H}^2}, \ldots, \varphi_{\mathscr{H}^{k-1}}\big)$ where $\varphi_{\mathscr{H}^i} : J_i \to L^1(T_i)$. Define $\widetilde{\mathscr{H}}$ to be the pullback of $\mathscr{H}$ by $\varphi_{\mathscr{H}}$, that is

$$\widetilde{\mathscr{H}}(\mathbf{f}) = \widetilde{\mathscr{H}}\big(\varphi_{\mathscr{H}}(\underline{x})\big) = (\widetilde{\mathscr{H}})^{\varphi_{\mathscr{H}}}(\underline{x}) = \mathscr{H}(\underline{x}) \tag{5.3.2}$$

where $\underline{x} = (x_1, x_2, \dots, x_k, x_{12}, \dots, x_{23\dots k})$.

For instance, if $k = 2$ then $\widetilde{\mathscr{H}}(f_1, f_2)$ is defined as follows; if $f_1, f_2 \in \Omega_{\mathscr{H}^1}$ then $f_1 = \mathscr{H}^1_{x_1}$ and $f_2 = \mathscr{H}^1_{x_2}$ for some $x_1, x_2 \in J_1$ and we set

$$\widetilde{\mathscr{H}}(f_1, f_2) = \widetilde{\mathscr{H}}\big(\varphi_{\mathscr{H}^1}(x_1), \varphi_{\mathscr{H}^1}(x_2)\big) = \widetilde{\mathscr{H}}^{\varphi_{\mathscr{H}^1}}(x_1, x_2) = \mathscr{H}(x_1, x_2)$$

We claim that $\widetilde{\mathscr{H}}$ is pure. We shall actually prove that $r_{\widetilde{\mathscr{H}}^i}$ agrees with the L^1 norm on $\Omega_{\widetilde{\mathscr{H}}^i}$. Let us recall an important notation.

Suppose we have a sequence $\underline{x} = (x_1, x_2, \dots, x_r)$. We denote

$$\underline{x}^y(\check{j}) = (x_1, x_2, \dots, x_{j-1}, y, x_j, \dots, x_r)$$

the sequence obtained by inserting y before the j^{th} term of $\underline{x}$. Then if we have $\underline{x}_i \in T_i$, we define $\underline{x}_i^y\big(\widetilde{\varrho(i)}\big)$ to be the sequence obtained by putting y before the $\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{i-1} + 1$ term of $\underline{x}$.

Now, for each $f \in \Omega_{\mathscr{H}^i}$, we have a function (section) $\widetilde{\mathscr{H}^i_f} : \widetilde{T}_i \to [0,1]$ where $\widetilde{T}_i = \Omega_{\mathscr{H}^1}^{\binom{k}{1}} \times \dots \times \Omega_{\mathscr{H}^i}^{\binom{k}{i}-1} \times \dots \times \Omega_{\mathscr{H}^{k-1}}^{\binom{k}{k-1}}$ defined as follows. Let $\underline{h}_i \in \widetilde{T}_i$. Then

$$\widetilde{\mathscr{H}^i_f}(\underline{h}_i) = \widetilde{\mathscr{H}}\big(\underline{h}_i^f\big(\widetilde{\varrho(i)}\big)\big).$$

Now, for any $f, g \in \Omega_{\mathscr{H}^i} \cong \Omega_i$ there are x and y such that $f = \mathscr{H}^i_x$ and $g = \mathscr{H}^i_y$. By Definition 5.11 we have that

$$\begin{aligned}
r_{\widetilde{\mathscr{H}}^i}(f,g) &= \int\limits_{\widetilde{T}_i} \left| \widetilde{\mathscr{H}}^i_f(\underline{h}_i) - \widetilde{\mathscr{H}}^i_g(\underline{h}_i) \right| \quad d\nu(\underline{h}_i) \\
&= \int\limits_{\widetilde{T}_i} \left| \widetilde{\mathscr{H}}\left(\underline{h}_i^f\left(\widetilde{\varrho(i)}\right)\right) - \widetilde{\mathscr{H}}\left(\underline{h}_i^g\left(\widetilde{\varrho(i)}\right)\right) \right| \quad d\nu(\underline{h}_i) \\
&= \int\limits_{T_i} \left| \mathscr{H}\left(\underline{z}_i^x\left(\widetilde{\varrho(i)}\right)\right) - \mathscr{H}\left(\underline{z}_i^y\left(\widetilde{\varrho(i)}\right)\right) \right| \quad d\pi(\underline{z}_i) \\
&= \int\limits_{T_i} \left| \mathscr{H}^i_x(\underline{z}_i) - \mathscr{H}^i_y(\underline{z}_i) \right| \quad d\pi(\underline{z}_i) \\
&= r_{\mathscr{H}^i}(x,y)
\end{aligned}$$

Hence,

$$\begin{aligned}
r_{\mathscr{H}^i}(x,y) &= \int\limits_{T_i} \left| \mathscr{H}^i_x(\underline{z}_i) - \mathscr{H}^i_y(\underline{z}_i) \right| \quad d\pi(\underline{z}_i) \\
&= \int\limits_{T_i} \left| f(\underline{z}_i) - g(\underline{z}_i) \right| \quad d\pi(\underline{z}_i) \\
&= ||f-g||_{L^1}
\end{aligned}$$

Since we have shown that $r_{\widetilde{\mathscr{H}}^i}$ agrees with the norm L^1, we conclude that $\widetilde{\mathscr{H}}$ satisfies the conditions of purity in Definition 5.13. Hence, $\widetilde{\mathscr{H}}$ is pure.

To conclude, from (5.3.2) above we can see that the twin-free separable k-uniform mixed hypergraphon $\mathscr{H}$ and the pure k-uniform mixed hypergraphon $\widetilde{\mathscr{H}}$ are weakly isomorphic. □

Recall that a standard measure space is the measure space that is the completion of a Borel space. By the completion we mean adding all subsets of sets of measure zero to the σ-algebra. If we have a twin-free k-uniform mixed hypergraphon on a standard measure space, we call it a standard twin-free k-uniform mixed hypergraphon.

In the next theorem we are going to show that any two weakly isomorphic standard twin-free separable k-uniform mixed hypergraphons are isomorphic up to a null set. However, we need to define what it means for two k-uniform mixed hypergraphons to be isomorphic.

Definition 5.16. Let $\mathscr{H}$ and $\mathscr{H}'$ be two k-uniform mixed hypergraphons with respect to $J_1, J_2, \ldots, J_{k-1}$ and $J'_1, J'_2, \ldots, J'_{k-1}$. Then we say that $\mathscr{H}$ and $\mathscr{H}'$ are isomorphic up to a null set if there are invertible measure preserving functions $\varphi_i : J_i \to J'_i$ such that

$$\mathscr{H}'\big(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_{k-1}(x_{23\ldots k})\big) = \mathscr{H}(x_1, x_2, \ldots, x_{23\ldots k})$$

almost everywhere.

Theorem 5.17. If two standard twin-free separable k-uniform mixed hypergraphons are weakly isomorphic, then they are isomorphic up to a null set.

Proof. Assume that $\mathscr{D}$ and $\mathscr{G}$ are two weakly isomorphic twin-free separable k-uniform mixed hypergraphons on standard measure spaces $(\Omega_i, \mathscr{B}_i, \mu_i)$ and $(\Lambda_i, \mathscr{C}_i, \tau_i)$ where $1 \leqslant i \leqslant k-1$. We shall show that then there is a third k-uniform mixed hypergraphon $\mathscr{H}$ on $(J_i, \mathscr{A}_i, \pi_i)$ such that there are weak isomorphisms $\varphi_{\mathscr{D}^i} : \Omega_i \to J_i$ and $\varphi_{\mathscr{G}^i} : \Lambda_i \to J_i$ making $\mathscr{D}, \mathscr{G}$, and $\mathscr{H}$ are weakly isomorphic to each other.

Recall that $\varphi_{\mathscr{D}} = \big(\varphi_{\mathscr{D}^1}, \varphi_{\mathscr{D}^2}, \ldots, \varphi_{\mathscr{D}^{k-1}}\big)$ is a sequence of measure preserving maps. Then, we define

$$\varphi_{\mathscr{D}} : \Omega_1^{\binom{k}{1}} \times \cdots \times \Omega_{k-1}^{\binom{k}{k-1}} \to J_1^{\binom{k}{1}} \times \cdots \times J_{k-1}^{\binom{k}{k-1}}$$

as follows; let $\underline{g} = (g_1, g_2, \ldots, g_{23\ldots k}) \in \Omega_1^{\binom{k}{1}} \times \cdots \times \Omega_{k-1}^{\binom{k}{k-1}}$, then

$$\varphi_{\mathscr{D}}(\underline{g}) = \big(\varphi_{\mathscr{D}^1}(g_1), \varphi_{\mathscr{D}^2}(g_2), \ldots, \varphi_{\mathscr{D}^{k-1}}(g_{23\ldots k})\big)$$

Now, we can define a pullback $\mathscr{D} = \mathscr{H}^{\varphi_{\mathscr{D}}}$ as

$$\mathscr{H}^{\varphi_{\mathscr{D}}}(\underline{g}) = \mathscr{H}(\varphi_{\mathscr{D}}(\underline{g}))$$

For all i, the function $\varphi_{\mathscr{D}^i}$ is measure preserving from $\overline{\mathscr{B}_i}$ to $\mathscr{A}_i$ and $\mathscr{D} = \mathscr{H}^{\varphi_{\mathscr{D}}}$ and hence $\varphi_{\mathscr{D}^i}$ is a weak isomorphism from $\mathscr{D}$ to $\mathscr{H}$.

By a similar argument to that of the proof of the Claim in Theorem 5.15, we can see that for each i, the measure μ_i has full support on Ω_i.

Since $\mathscr{D}$ is a twin-free separable k-uniform mixed hypergraphon, the function $\varphi_{\mathscr{D}^i}$ is injective. Furthermore, the set $\{x \in J_i : \varphi^{-1}_{\mathscr{D}^i}(x) = \varnothing\}$ has measure zero. That means $\varphi_{\mathscr{D}^i} : \Omega_i \to J_i$ is bijective (up to a null set).

By similar arguments as above, we have that $\varphi_{\mathscr{G}^i} : \Lambda_i \to J_i$ is bijective. Therefore, $\varphi_{\mathscr{D}^i}$ and $\varphi_{\mathscr{G}^i}$ are isomorphisms between $\mathscr{D}$, $\mathscr{G}$ and $\mathscr{H}$. □

Theorem 5.18. Every twin-free separable k-uniform mixed hypergraphon is isomorphic, up to a null set, to a pure k-uniform mixed hypergraphon.

Proof. Let $\mathscr{H}$ be a twin-free separable k-uniform mixed hypergraphon with respect to $J_1, J_2, \ldots, J_{k-1}$. From Theorem 5.15 there exists a pure k-uniform mixed hypergraphon $\widetilde{\mathscr{H}}$ which is weakly isomorphic to $\mathscr{H}$. By Theorem 5.17, we see that $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ are isomorphic up to a null set. □

5.4 Future work

In the theory of graphons, Lovász and Szegedy in [17] defined an *automorphism* for any graphon. They made the following definition:

Definition 5.19. An automorphism of a given graphon W on J is an invertible measure preserving function $\varphi : J \to J$ such that

$$W(x^\varphi, y^\varphi) = W(x, y)$$

for almost all $x, y \in J$.

However, if this definition is used, we may have weakly isomorphic graphons which have very different automorphism groups.

Example 5.20. Suppose that W_1 and W_2 are graphons on J_1 and J_2. Let $J_1 = [0,1]$, and define $W_1(x,y) = \frac{1}{2}$ for all $x, y \in J_1$. Let $J_2 = \{X\}$. Define $W_2(x,x) = \frac{1}{2}$.

Let F be a graph. Then

$$\begin{aligned} t(F, W_1) &= \int\limits_{[0,1]^V} \prod_{(i,j)\in E} W_1(x_i, x_j) \prod_{s\in V} dx_s \\ &= \int\limits_{[0,1]^V} \prod_{(i,j)\in E} \frac{1}{2} \quad \prod_{s\in V} dx_s \\ &= \int\limits_{[0,1]^V} \prod_{(i,j)\in E} W_2(x_i, x_j) \prod_{s\in V} dx_s \\ &= t(F, W_2) \end{aligned}$$

Therefore, $t(F, W_1) = t(F, W_2)$ for all simple graph F. Hence, W_1 and W_2 are weakly isomorphic.

Lovász and Szegedy motivated the definition of automorphism so that it only held for pure graphons. They made the following definition:

Definition 5.21. [17] Let W be a pure graphon on J. A measure preserving bijection $\varphi : J \to J$ is called an automorphism of W if for every $x \in J$, we have $W(x^\varphi, y^\varphi) = W(x,y)$ for almost all $y \in J$.

When it comes to pure graphons, the second definition is stronger than the first. It means that one cannot interchange two arbitrary points in J and obtain an automorphism.

Lovász stated in [17] an important theorem.

Theorem 5.22. The automorphism group of a pure graphon is compact.

This theorem is a result of the following fact which is proved in [17] Lemma 11.

Fact : The automorphisms of a pure graphon W on J form a closed subgroup of the isometry group of $(\bar{J}, \bar{r}_W)$

where $\bar{J}$ is the completion of J and $\bar{r}_W$ is called the *similarity metric* (2-neighborhood metric). That is

$$\bar{r}_W(x,y) = r_{W \circ W}(x,y) = \int_J \left| \int_J (W(x,u) - W(y,u))W(u,z)du \right| dz$$

We write dz instead of $d\pi(z)$, where π is the probability measure of the graphon.

Now, in the theory of k-uniform mixed hypergraphon we may ask several questions.

(1) How do we define the similarity distance for a pure k-uniform mixed hypergraphon?

(2) What is the definition of the automorphism group of a k-uniform mixed hypergraphon?

If we answer those questions, then we may have the opportunity to prove the following conjecture.

Conjecture 5.4.1. *The automorphism group of a pure k-uniform mixed hypergraphon is compact.*

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