

ENCODING AND ORDERING X -CACTUSES

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ABSTRACT. Phylogenetic networks are a generalization of evolutionary or phylogenetic trees that are commonly used to represent the evolution of species which cross with one another. A special type of phylogenetic network is an X -cactus, which is essentially a cactus graph in which all vertices with degree less than three are labelled by at least one element from a set X of species. In this paper, we present a way to *encode* X -cactuses in terms of certain collections of partitions of X that naturally arise from X -cactuses. Using this encoding, we also introduce a partial order on the set of X -cactuses (up to isomorphism), and derive some structural properties of the resulting partially ordered set. This includes an analysis of some properties of its least upper and greatest lower bounds. Our results not only extend some fundamental properties of phylogenetic trees to X -cactuses, but also provides a new approach to solving topical problems in phylogenetic network theory such as deriving consensus networks.

keyword X -cactus, poset, bound, consensus network, supernetwork, phylogenetic network

1. INTRODUCTION

In this paper, we let X denote a finite, non-empty set. An X -tree $\mathcal{T} = (T, \phi)$ is a graph theoretical tree $T = (V, E)$ together with a map $\phi : X \rightarrow V$ whose image includes all vertices in T with degree two or less. In case ϕ is a bijection onto the leaf-set of T , T is called a *phylogenetic tree*. X -trees naturally arise in evolutionary biology where they are commonly used to represent the evolution of a set X of species [14]. A fundamental property of X -trees is that a partial order \leq can be defined on the set $\mathcal{T}(X)$ of X -trees (up to isomorphism) by defining $\mathcal{T} \leq \mathcal{T}'$ for two trees $\mathcal{T}, \mathcal{T}' \in \mathcal{T}(X)$ precisely if a subset of edges in \mathcal{T}' can be contracted so as to obtain \mathcal{T} [14, Section 3.2]. The poset $(\mathcal{T}(X), \leq)$ has several interesting structural properties, some of which have proven useful in developing new insights and methodologies in phylogenetics. For example, lower bounds in $(\mathcal{T}(X), \leq)$ correspond to consensus trees [14, Section 3.6], which are used in phylogenetics to summarise large collections of phylogenetic trees [3].

Recently, there has been a great deal of interest in phylogenetic networks, a generalization of phylogenetic trees that are used to represent the evolution of species which cross with one another, such as plants and viruses [13]. An important class of such networks is the collection of X -cactuses [10] (also known as 1-nested networks [9]), which contains the well-known subclass of *level-1 networks* [8]. A *cactus* is a connected graph N such that any two distinct cycles in N share at most one vertex; an X -cactus

$\mathcal{N} = (N, \phi)$ is a cactus $N = (V, E)$ together with a map $\phi : X \rightarrow V$ whose image includes all vertices in N with degree two or less (e.g., see Figure 1). Note that an X -tree is simply an X -cactus whose underlying graph is a tree. In this paper, we show that by considering edge-contractions for X -cactuses we can obtain a partial order \leq on the set $\mathcal{G}(X)$ of X -cactuses (up to isomorphism) that naturally extends the edge-contraction ordering on X -trees. As well as studying structural properties of the poset $(\mathcal{G}(X), \leq)$ we show that, as with X -trees, we can use the ordering to define consensus networks for X -cactuses, a problem that is of topical interest in the theory of phylogenetic networks.

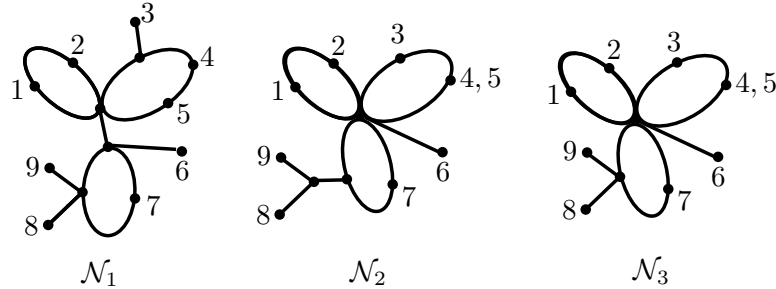


FIGURE 1. Three X -cactuses $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ on the set $X = \{1, \dots, 9\}$. $\mathcal{N}_3 \leq \mathcal{N}_1$ as \mathcal{N}_3 can be obtained from \mathcal{N}_1 by contracting two cut edges and an edge in the top-right cycle. Also, $\mathcal{N}_3 \leq \mathcal{N}_2$ as \mathcal{N}_3 can be obtained from \mathcal{N}_2 by contracting a cut edge.

We now describe the contents of the rest of this paper, at the same time giving an overview of our main results. After presenting some preliminaries in Section 2, in Section 3 we describe a way to *encode* X -cactuses. To help put this statement into context, we first recall that a fundamental property of an X -tree \mathcal{T} is that it is completely determined by the set $\Sigma(\mathcal{T})$ of bipartitions on X that is obtained by removing precisely one edge from \mathcal{T} for each edge in \mathcal{T} . More precisely, the *Splits-Equivalence Theorem for X -trees* states that if Σ is a set of bipartitions of X , then there is an X -tree \mathcal{T} such that $\Sigma = \Sigma(\mathcal{T})$ if and only if Σ satisfies a certain pairwise condition called compatibility, in which case \mathcal{T} is the unique such X -tree up to isomorphism [4] (see also [14, Theorem 3.1.4]).

To obtain our encoding for X -cactuses, we consider the removal of either a cut edge or of all edges in some cycle in an X -cactus. The first removal results in a bipartition of X , just as with X -trees (e.g. removal of the interior cut edge in the cactus \mathcal{N}_1 in Figure 1 gives the bipartition $\{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}\}$). The second removal results in a partition of X whose size is the length of the cycle – clearly, the ordering of the vertices in the cycle is also important, and so we define the concept of a *circular partition* or a partition with a circular ordering, to capture this fact (e.g. removal of the edges in the top right cycle in the cactus \mathcal{N}_2 in Figure 1 gives the circular partition $(\{1, 2, 6, 7, 8, 9\}, \{3\}, \{4, 5\})$). In Theorem 3.3, we show that an X -cactus \mathcal{N} is encoded by its corresponding set $\mathcal{C}(\mathcal{N})$ of circular partitions, and characterise when an arbitrary collection of circular partitions corresponds to a (necessarily unique) X -cactus. As with

X -trees, this characterization is given in terms of a pairwise condition arising from the notion of *strongly compatibility*, a concept that was introduced in [6].

In Section 4, we define an order \leq on the set $\mathcal{G}(X)$ of X -cactuses $\mathcal{G}(X)$ (up to isomorphism). As with X -trees, this is essentially defined via edge contraction, where one network is less than another if it can be obtained by contracting a subset of edges in the first (see e.g. Figure 1). Some care is needed however in case an edge in a 3-cycle is contracted; we define a so-called *triple contraction* to deal with this situation. In Theorem 4.1, we show that $(\mathcal{G}(X), \leq)$ is a poset, and present some of its structural properties. In particular, we show that $\mathcal{G}(X)$ is a graded poset with a unique minimal element (the X -cactus whose underlying graph is a single vertex), and characterize its maximal elements (Theorem 4.1). In Section 5, we show that the poset $(\mathcal{G}(X), \leq)$ can also be given in terms of X -cactus encodings. In particular, we show that $\mathcal{N} \leq \mathcal{N}'$ holds for any two $\mathcal{N}, \mathcal{N}' \in \mathcal{G}(X)$ if and only if $\mathcal{C}(\mathcal{N})$ can be mapped in a special way into $\mathcal{C}(\mathcal{N}')$ (Theorem 5.1).

In Sections 6 and 7, we consider upper and lower bounds in the poset $(\mathcal{G}(X), \leq)$. In general, these bounds have a more complicated behaviour than upper and lower bounds in the poset $(\mathcal{T}(X), \leq)$. Indeed, unlike the poset of X -trees, there may exist non-unique least upper and greatest lower bounds. Even so, we are able to characterize upper and lower bounds in $(\mathcal{G}(X), \leq)$ (Corollary 5.6). In addition, we shed some light on least upper bounds for pairs of X -cactuses (Theorem 6.2), and characterize greatest lower bounds for arbitrary sets of X -cactuses (Theorem 7.2). We expect that this latter result could be a useful starting point for developing methods to find consensus networks for collections of X -cactuses. In Section 8, we conclude by presenting some open problems and new directions.

2. PRELIMINARIES

2.1. Graphs and X -cactuses. All graphs in this paper are *undirected* and *simple*, that is, they contain neither loops nor parallel edges. Let G be a graph. A *leaf* in G is a vertex with degree one and an *internal vertex* of G is a vertex that is not a leaf. A *path* in a graph G is a sequence of distinct vertices v_1, v_2, \dots, v_m such that v_i is adjacent to v_{i+1} for $1 \leq i < m - 1$. If, in addition, v_1 and v_m are adjacent, then the subgraph of G whose vertex set is $\{v_1, \dots, v_m\}$ and whose edge set consists of $\{v_1, v_m\}$ and $\{v_i, v_{i+1}\}$ for $1 \leq i < m - 1$ is a *cycle*. A cycle is called *tiny* if it contains precisely three vertices. A *block* of G is a maximal subgraph of G not containing a cut vertex, and the set of blocks of G is denoted by $\mathbb{B}(G)$. A graph is *trivial* if it contains only one vertex, and *nontrivial* otherwise. Note that the trivial graph is defined as having no blocks.

A *cactus* is a connected graph such that any two distinct cycles in it share at most one vertex. Equivalently, a cactus N is a connected graph in which each edge belongs to at most one cycle so that, in particular, N is a cactus if and only if each edge in N belongs to one and only block in $\mathbb{B}(N)$. Note that each block in a cactus is either a cut edge or a cycle, and that the trivial graph is the only cactus that does not contain any block.

Now, for X a non-empty finite set, an X -cactus, is a pair $\mathcal{N} = (N, \phi)$ where $N = (V, E)$ is a cactus and $\phi : X \rightarrow V$ is a *labelling map*, i.e. a map from X to V such that every vertex of degree at most two in N is contained in its image. To help reduce notational complexity, in case there is little chance for confusion we shall just extend graph theoretical concepts and notations to X -cactuses in the natural way. As mentioned in the introduction, an X -tree is an X -cactus whose underlying graph is a tree. Two X -cactuses $\mathcal{N} = ((V, E), \phi)$ and $\mathcal{N}' = ((V', E'), \phi')$ are *isomorphic* if there exists a bijective map $f : V \rightarrow V'$ such that (i) $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$ and (ii) for all $x \in X$ we have $f(\phi(x)) = \phi'(x)$. The set consisting of all X -cactuses up to isomorphism is denoted by $\mathcal{G}(X)$. An X -cactus $\mathcal{N} = (N, \phi)$ is called *trivial* if N is trivial and it is called *binary* if every internal vertex of N has degree 3. A *phylogenetic X -cactus* is an X -cactus in which the labelling map is a bijection onto its leaves.

We shall use two basic graph operations in this paper which are defined as follows. Given a graph $G = (V, E)$ and a subset $E' \subseteq E$, we let $G - E'$ be the graph with vertex set V and edge set $E - E'$. In case $E' = \{e\}$ we denote $G - E'$ by $G - e$. In addition, if $v \in V$, we let $G - v$ be the graph obtained from G by deleting v and all the edges incident with v . Finally, for a vertex v in G with degree two, the graph $G' = (V', E')$ obtained from G by *suppressing* v has vertex set $V' = V - \{v\}$ and edge set $E' = (E - \{\{u_1, v\}, \{u_2, v\}\}) \cup \{u_1, u_2\}$, where $u_1 \neq u_2$ are adjacent to v in G . Note that suppressing a degree two vertex v in a cactus decreases the number of its edges either by one or by two; the latter occurs if and only if v is contained in a tiny cycle.

2.2. Circular orderings. Given a finite set Y with $m \geq 2$ elements, and a linear order $\theta = (y_1, y_2, \dots, y_m)$ of Y , we let $s(\theta) = (y_2, y_3, \dots, y_m, y_1)$ and $r(\theta) = (y_m, y_{m-1}, \dots, y_2, y_1)$. Two linear orderings θ and θ' are (*circular*) *equivalent*, denoted by $\theta \sim \theta'$, if either $\theta = \theta'$ or there exist $k + 1$ linear orderings $\theta_0 = \theta$, $\theta_k = \theta'$ ($k \geq 1$) such that for $0 \leq i < k$, we have $\theta_{i+1} = r(\theta_i)$ or $\theta_{i+1} = s(\theta_i)$. Using the fact that $s^m(\theta) = \theta = r^2(\theta)$, it is straightforward to check that \sim is an equivalence relation on the set of linear orderings of Y .

A *circular ordering* of Y is an equivalence class of \sim . In particular, if θ is a linear ordering of Y , then $[\theta]$ is the equivalence class consisting of the distinct linear orderings of Y that are equivalent to θ . For example, for $Y = \{1, 2, 3, 4\}$,

$$[(1, 3, 2, 4)] = \{(1, 3, 2, 4), (3, 2, 4, 1), (2, 4, 1, 3), (4, 1, 3, 2), \\ (1, 4, 2, 3), (4, 2, 3, 1), (2, 3, 1, 4), (3, 1, 4, 2)\}.$$

Intuitively, in case $m \geq 3$, a circular ordering of Y is a labelling of the vertices of a regular m -gon by the elements in Y . Indeed, the operations s and r can be seen as the generators of the dihedral group D_m acting on an m -gon (s a rotation and r a reflection), and the equivalence classes of \sim describe the orbits of this action.

2.3. Circular partitions. Recall that a *partition* π of set X with $|X| \geq 2$ is a set consisting of at least two nonempty pairwise disjoint proper subsets of X whose union is equal to X . Each element in π is called a *part* of π and the *size* of π is its number

of parts. For example, $\pi = \{\{1, 3\}, \{2\}, \{5\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4, 5\}$ with four parts; we shall also denote partitions such as π by $13|2|5|4$, where the order of listing the parts does not matter. We refer to a partition as a *split* if it is of size two and a *proper partition* otherwise. The set of partitions of X is denoted by $\Pi(X)$. Two partitions π and π' in $\Pi(X)$ are *compatible* (also known as strongly compatible [7]) if there exists a part $A \in \pi$ and a part $B \in \pi'$ such that $A \cup B = X$, and *incompatible* otherwise. Note that it follows that a split is compatible with itself while a proper partition is incompatible with itself.

A *circular partition* of X is an ordered pair $\sigma = (\pi, \tau)$ where π is a partition of X and τ a circular ordering of the parts in π . We often use $[A_1|A_2|\dots|A_k]$ to denote a circular partition consisting of the partition $\pi = A_1|A_2|\dots|A_k$, $k \geq 2$, of X and the circular ordering $\tau = [(A_1, A_2, \dots, A_k)]$. Note that if $|\pi| = 2$, i.e. π is a split, then there exists only one possible circular ordering of π , and so we shall just call such a circular partition a split. We call a circular partition $[A_1|A_2|\dots|A_k]$ *proper* if $k \geq 3$, and we call a collection of circular partitions *proper* if every partition in the collection is proper. Given a circular partition $\sigma = (\pi, \tau)$ of X we let $\underline{\sigma} = \pi$ and call $\underline{\sigma}$ the partition *induced* by σ . For example, $\sigma_1 = [13|2|5|4]$ and $\sigma_2 = [2|13|5|4]$ are two distinct circular partitions of $\{1, 2, 3, 4, 5\}$ because $[(\{1, 3\}, \{2\}, \{5\}, \{4\})] \neq [(\{2\}, \{1, 3\}, \{5\}, \{4\})]$. Note however that $\underline{\sigma}_1 = \underline{\sigma}_2 = 13|2|4|5$. For a set \mathcal{C} of circular partitions, let $\Pi(\mathcal{C}) := \{\underline{\sigma} : \sigma \in \mathcal{C}\}$ be the set of partitions induced by \mathcal{C} . We shall also let $\mathcal{C}(X)$ denote the set of circular partitions of X , $\mathcal{C}_b(X)$ the set of splits of X , and $\mathcal{C}_p(X)$ the set of proper circular partitions of X .

Given a proper circular partition of X , we can *merge* any two adjacent parts into one part to construct another circular partition of X . For example, merging the two adjacent parts $\{2\}$ and $\{5\}$ in $\sigma_1 = [13|2|5|4]$ results in the circular partition $[13|25|4]$. Given two circular partitions σ and σ' of X we set $\sigma' \preceq \sigma$, if either (i) $\sigma = \sigma'$ or (ii) σ' is proper and σ' can be obtained from σ by applying a (necessarily finite) sequence of merges. Furthermore, let $\sigma' \prec \sigma$ denote $\sigma' \preceq \sigma$ and $\sigma \neq \sigma'$. Note that if σ' is a split and $\sigma' \preceq \sigma$ or $\sigma \preceq \sigma'$, then $\sigma = \sigma'$. Also, if σ' is not a split, $|\underline{\sigma}| = 3$ and $\sigma' \preceq \sigma$ then $\sigma = \sigma'$. Moreover, $\sigma \prec \sigma'$ implies that σ and σ' are incompatible. It is straight-forward to check that \preceq is reflexive, antisymmetric, and transitive, and so $(\mathcal{C}(X), \preceq)$ is a poset.

Two circular partitions σ and σ' of X are *compatible* if $\underline{\sigma}$ and $\underline{\sigma}'$ are compatible, and *incompatible* otherwise. Note that this definition implies that a circular partition σ on X is compatible with itself if and only if $\sigma \in \mathcal{C}_b(X)$. A set \mathcal{C} of circular partitions is compatible if each pair of distinct circular partitions in \mathcal{C} is compatible. Here we use the convention that the emptyset of circular partitions is compatible. Note that the circular orderings of two compatible proper partitions are ‘consistent’ in the following sense.

Lemma 2.1. *Suppose that two proper circular partitions σ_1 and σ_2 of X are compatible. Then there exists a circular partition σ' in $\mathcal{C}(X)$ with $\sigma_1 \preceq \sigma'$ and $\sigma_2 \preceq \sigma'$.*

Proof. Since σ_1 and σ_2 are compatible, there exists a part A_1 in $\underline{\sigma}_1$ and B_1 in $\underline{\sigma}_2$ such that $A_1 \cup B_1 = X$. Without loss of generality, we may assume that $\sigma_1 = [A_1|A_2|\dots|A_t]$ and

$\sigma_2 = [B_k|B_{k-1}|\dots|B_1]$ for some $t, k \geq 3$. Since $A_1 \cup B_1 = X$, it follows that $B_i \subseteq A_1$ for $2 \leq i \leq k$ and $A_i \subseteq B_1$ for $2 \leq j \leq t$.

Now assume first that $A_1 \cap B_1 \neq \emptyset$. Then $\sigma' = [B_k|B_{k-1}|\dots|B_2|B_1 \cap A_1|A_2|\dots|A_t]$ is a proper circular partition of X , and it is straightforward to check that $\sigma_1 \preceq \sigma'$ and $\sigma_2 \preceq \sigma'$. In case $A_1 \cap B_1 = \emptyset$, the same relationships hold for $\sigma' = [B_k|B_{k-1}|\dots|B_2|A_2|\dots|A_t]$. \square

2.4. Tree representations of partitions. For later use, we recall some definitions and results on tree representations of partitions developed in [7]. To this end, we need to generalize the concepts for X -cactuses to semi-labelled X -cactuses. Formally, a *semi-labelled X -cactus* $\mathbf{N} = (N, \psi)$ is an ordered pair where $N = (V, E)$ is a cactus and $\psi : X \rightarrow V$ is a map such that every leaf in N is contained in $\psi(X)$. Note that a semi-labelled cactus $\mathbf{N} = (N, \psi)$ is an X -cactus if and only if every degree two vertex in N is contained in $\psi(X)$. Note that if N is a tree, the pair (N, ψ) is also referred to as a *semi-labelled X -tree*. See Figure 2 for two examples of semi-labelled X -trees.

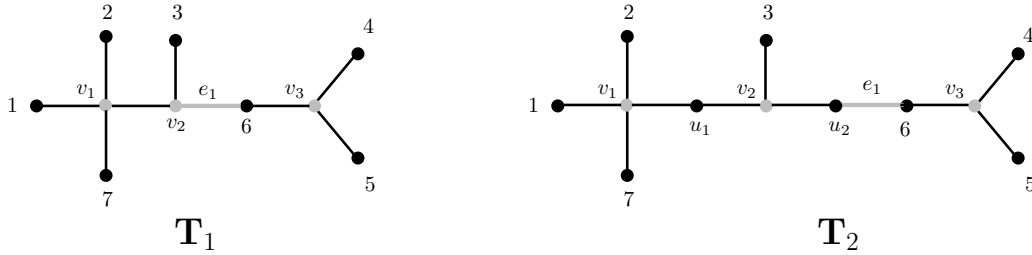


FIGURE 2. Two semi-labelled X -trees \mathbf{T}_1 and \mathbf{T}_2 . For the set $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ with $\pi_1 = 1|2|7|3456$, $\pi_2 = 127|3|456$, $\pi_3 = 12367|5|4$, $\pi_4 = 1237|645$ and $\kappa_j(\pi_i) = v_i$ for $1 \leq i \leq 3$ and $j = 1, 2$ and $\kappa_j(\pi_4) = e_1$ for $j = 1, 2$, the pair (\mathbf{T}_1, κ_1) is a semi-tree representation of Π and the pair (\mathbf{T}_2, κ_2) is the perfect semi-tree representation of Π . In each case $Im(\kappa)$ is indicated in grey. Furthermore, $h(e_1, \kappa_1) = 2$ and $h(e_1, \kappa_2) = 1$.

Given an unlabelled cut vertex $v \in V$ in a semi-labelled X -cactus $\mathbf{N} = (N = (V, E), \phi)$ (i.e. some cut vertex v of N not contained in $\phi(X)$), let $\pi(v)$ be the partition of X induced by $\mathbf{N} - v$. In other words, A is a part in $\pi(v)$ if and only if for all pairs of (not necessarily distinct) vertices $x, x' \in A$, no path between x and x' in N contains v . Similarly, for any cut edge e of N , let $\pi(e)$ be the split of X induced by $N - e$. For example, For example, in Figure 2 the partition induced by $\mathbf{T}_1 - v_2$ is given by $127|3|456$.

We define a *semi-tree representation* of a collection Π of partitions of X to be an ordered pair (\mathbf{T}, κ) consisting of a semi-labelled X -tree $\mathbf{T} = (T = (V, E), \psi)$ and a (necessarily injective) map $\kappa : \Pi \rightarrow (V - \psi(X)) \cup E$ such that for each partition $\pi \in \Pi$, the image $\kappa(\pi)$ is either an unlabelled vertex of degree at least three or an edge, and $\pi = \pi(\kappa(\pi))$ holds. If $\Pi = \emptyset$, then we use the convention that κ is the *empty function*, that is, the image $Im(\kappa)$ of κ is the empty set. Note that if $\Pi \neq \emptyset$ then T must contain

at least one edge. For each edge $e = \{u, v\} \in E$, let $h(e, \kappa)$ be the number of partitions π in Π with $\kappa(\pi) \in \{u, v, e\}$. Note that $0 \leq h(e, \kappa) \leq 3$. The representation (\mathbf{T}, κ) is *perfect* if $h(e, \kappa) = 1$ for all edges e in \mathbf{T} and $\kappa(\pi)$ is an edge in \mathbf{T} if and only if π is a split (see e.g. Figure 2). We use the convention that a semi-tree representation whose underlying X -tree is trivial is perfect.

We now prove a simple but useful extension of [7, Theorem 4] concerning semi-tree representations.

Theorem 2.2. *Suppose that Π is a collection of partitions of X . Then Π is compatible if and only if there exists a semi-tree representation (\mathbf{T}, κ) of Π . Moreover, if Π is compatible, then there exists a (necessarily unique) perfect semi-tree representation (\mathbf{T}, κ) of Π .*

Proof. Clearly if there exists a semi-tree representation of Π , then Π is compatible.

Conversely, suppose Π is compatible. If $\Pi = \emptyset$, then the theorem holds since (\mathbf{T}, κ) is a semi-tree representation for Π , where \mathbf{T} is the semi-labelled trivial X -tree and κ the empty function on Π . So, assume $\Pi \neq \emptyset$.

In [7, Theorem 4] it is proven that, up to isomorphism, there exists a unique semi-tree representation (\mathbf{T}, κ) of Π for which $\mathbf{T} = ((V, E), \phi)$ is an X -tree with $|E| \geq 1$ and $h(e, \kappa) > 0$ holds for all every $e \in E$ (see e.g. Figure 2). We now describe how to obtain a perfect semi-labelled representation $(\mathbf{T}' = ((T', \phi'), \kappa')$ of Π from (\mathbf{T}, κ) which will complete the proof of the theorem. The fact that $(\mathbf{T}' = ((T', \phi'), \kappa')$ is perfect is straight-forward to verify and so we omit this.

The tree \mathbf{T}' is obtained from \mathbf{T} by inserting $h(e, \kappa) - 1$ additional unlabelled degree two vertices into each edge $e \in E$. The labelling map ϕ' is the same as ϕ , i.e., for each $x \in X$, we let $\phi'(x) = \phi(x)$. The map κ' is derived from κ as follows: For $\pi \in \Pi$ with $|\pi| \geq 3$, $\kappa(\pi)$ is a vertex in \mathbf{T} and we set $\kappa'(\pi) = \kappa(\pi)$. For $\pi \in \Pi$ with $|\pi| = 2$, $\kappa(\pi)$ is an edge $e = \{v_1, v_2\} \in E'$. We define $\kappa'(\pi)$ depending on the value $h(e, \kappa)$. If $h(e, \kappa) = 1$, then e is also an edge in E , and we let $\kappa'(\pi) = \kappa(\pi) = e$. If $h(e, \kappa) = 2$, then we subdivide e into two edges e_1 and e_2 , where the indices are chosen in such a way that e_2 is not incident with a vertex in $\kappa(\Pi)$ and we let $\kappa'(\pi) = e_2$. If $h(e, \kappa) = 3$, then we subdivide e into three edges e_1, e_2 , and e_3 , where the indices are chosen in such a way that e_2 is not incident with a vertex in $\kappa(\Pi)$ and we let $\kappa'(\pi) = e_2$. \square

3. ENCODING X -CACTUSES

In this section, we introduce an encoding for X -cactuses that is given in terms of circular partitions of X . We begin by describing a natural way to associate a collection of compatible circular partitions to an X -cactus $\mathcal{N} = (N, \phi)$.

Given an X -cactus (N, ϕ) , define a vertex in N to be *terminal* if it belongs to one and only one block of N . Each terminal vertex in N is of degree two or one, and hence contained in $\phi(X)$. Now suppose N has at least two vertices, and that v is a vertex in a block B of N . Let $(N - B)_v$ be the connected component in $N - E(B)$ that contains

v . Then $(N - B)_v$ contains at least one element in $\phi(X)$ because it contains at least one terminal vertex.

Now, to each block B in N , we associate a circular partition $\tau(B) \in \mathcal{C}(X)$ as follows. Let v_1, \dots, v_k , $k \geq 2$, be a labelling of the vertices in B so that $\{v_i, v_{i+1}\}$ is an edge in the block for each $1 \leq i \leq k$, where the indices are given modulo k . Let V_i be the vertex set of the connected component $(N - B)_{v_i}$ for $1 \leq i \leq k$. Then by the previous paragraph it follows that V_i contains at least one element in $\phi(X)$. Furthermore, it is straightforward to check that $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$, and $V(N) = \bigcup_{1 \leq i \leq k} V_i$. Hence $\tau(B) := [\phi^{-1}(V_1)|\phi^{-1}(V_2)|\dots|\phi^{-1}(V_k)]$ is a circular partition in $\mathcal{C}(X)$. Note that if $k = 2$, then B is a cut edge and $\tau(B)$ is a split. Furthermore, in general $\tau(B)$ is determined by B , but not by the labelling that we chose for its vertices, since any other labelling of this form induces the same circular partition.

Let $\mathcal{C}(\mathcal{N}) = \{\tau(B) : B \in \mathbb{B}(N)\}$. In case \mathcal{N} is the trivial X -cactus (i.e. N is a single vertex), we define $\mathcal{C}(\mathcal{N})$ to be the emptyset. Clearly, $\mathcal{C}(\mathcal{N})$ is the (necessarily disjoint) union of $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b(X) \cap \mathcal{C}(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N}) = \mathcal{C}_p(X) \cap \mathcal{C}(\mathcal{N})$. In particular, $|\mathcal{C}(\mathcal{N})|$ is the number of cycles in N plus the number of cut edges. We now show that the set $\mathcal{C}(\mathcal{N})$ is compatible.

Lemma 3.1. *Suppose that \mathcal{N} is a X -cactus. Then there is a perfect semi-tree representation of $\Pi(\mathcal{C}(\mathcal{N}))$. In particular, $\mathcal{C}(\mathcal{N})$ is a compatible set of circular partitions.*

Proof. Let $\Pi = \Pi(\mathcal{C}(\mathcal{N}))$ and put $\mathcal{N} = (N, \phi)$. If \mathcal{N} is the trivial X -cactus, then $\mathcal{C}(\mathcal{N}) = \emptyset = \Pi(\mathcal{C}(\mathcal{N}))$. The lemma then follows since $\mathcal{C}(\mathcal{N})$ is compatible and (\mathbf{T}, κ) is a perfect semi-tree representation for $\Pi(\mathcal{C}(\mathcal{N}))$, where \mathbf{T} is the semi-labelled trivial X -cactus and κ is the empty function on $\Pi(\mathcal{C}(\mathcal{N}))$.

Now, assume \mathcal{N} is not the trivial X -cactus. We first construct a semi-tree representation $((T, \psi), \kappa)$ of Π (see e.g. Figures 2 and 3). Let $\mathbb{B}_p(N)$ be the set of blocks of N that are cycles, and $\mathbb{B}_b(N)$ be the set of blocks of N that are cut edges. Then $\mathbb{B}(N) = \mathbb{B}_b(N) \amalg \mathbb{B}_p(N)$, and $\mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N})$ are precisely the set of circular partitions induced by the blocks in $\mathbb{B}_b(N)$ and $\mathbb{B}_p(N)$, respectively.

We now construct T : For every block B in $\mathbb{B}_p(N)$, add a new vertex v_B , a new edge $\{v_B, v\}$ for each vertex v in B , and remove all edges in B . Note that the vertex set $V(T)$ of T is the disjoint union of the vertex set $V(N)$ of N and the set $V^*(T) = \bigcup_{B \in \mathbb{B}_p(N)} \{v_B\}$ of new vertices. Furthermore, a vertex v in N is a leaf in T if and only if v is a terminal vertex in N . Finally, the set $E(T) \cap E(N)$ consists of all cut edges in N , one for each of the blocks in $\mathbb{B}_b(N)$.

Let ψ be the labelling map from X to $V(T)$ induced by ϕ , that is, we have $v = \psi(x)$ for vertex v in T and $x \in X$ if and only if $v \in V(N) \subseteq V(T)$ and $v = \phi(x)$. Note that each leaf in T , as a terminal vertex in N , is necessarily contained in $\psi(X)$, and $V^*(T) \cap \psi(X) = \emptyset$.

We now define the map $\kappa : \Pi \rightarrow (V(T) - \psi(X)) \cup E(T)$. If $\pi \in \Pi_p = \Pi(\mathcal{C}_p(\mathcal{N}))$, then there exists a unique circular partition σ in $\mathcal{C}_p(\mathcal{N})$ with $\pi = \underline{\sigma}$ and we let $\kappa(\pi) = v_B$, where B is the unique block B in $\mathbb{B}_p(N)$ with $\tau(B) = \sigma$. Otherwise, $\pi \in \Pi_b = \Pi(\mathcal{C}_b(\mathcal{N}))$

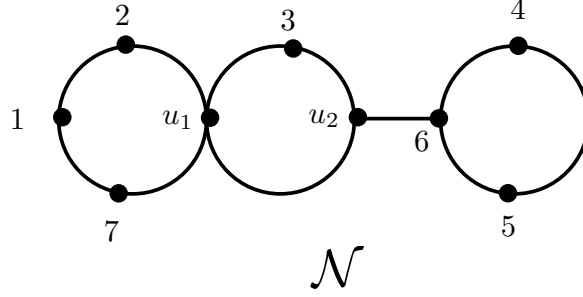


FIGURE 3. Example of an X -cactus \mathcal{N} on $X = \{1, \dots, 7\}$. The cactus contains four blocks: cycles B_1 , B_2 and B_3 (from left the right) and the cut edge B_4 . The set $\mathcal{C}(\mathcal{N})$ contains four circular partitions $\sigma_1 = [1|2|3456|7]$, $\sigma_2 = [127|3|456]$, $\sigma_3 = [4|5|1236]$, and $\sigma_4 = [1237|456]$, where $\sigma_i = \tau(B_i)$ for $1 \leq i \leq 4$. Putting $\sigma_i = (\tau_i, \pi_i)$ for all $1 \leq i \leq 4$, we have $\pi_i = \underline{\sigma}_i$ for all such i , and the perfect semi-tree representation of $\Pi(\mathcal{C}(\mathcal{N})) = \{\underline{\sigma}_i : 1 \leq i \leq 4\}$ is depicted on the right of Figure 2.

and so there exists a unique split σ in $\mathcal{C}_b(\mathcal{N})$ with $\pi = \underline{\sigma}$. In that case, we let $\kappa(\pi) = e_B$ where B is the unique block B in $\mathbb{B}_b(N)$ with $\tau(B) = \sigma$ and e_B is the unique edge in B (which is a cut edge and hence contained in $E(T) \cap E(N)$).

We show that $((T, \psi), \kappa)$ is perfect. Since $\kappa(\pi)$ is an edge if and only if π is a split, we only need to show that $h(e, \kappa) = 1$ for each edge $e = \{u, v\}$ in T . To this end, it suffices to establish that there exists one and only one partition π in Π such that $\kappa(\pi) \in \{u, v, e\}$. This clearly holds if e is contained in $E(T) \cap E(N)$ (i.e., e is a cut edge in N). Indeed, neither u nor v is contained in $V^*(T) = \kappa(\Pi_p)$ and $\kappa(\pi) = e$ holds if and only if $\pi = \tau(B_e)$, where B_e is the block consisting of the cut edge e . The other case is $e \in E(T) - E(N)$. Swapping u and v if necessary, we may assume that u is contained in $V(T) \cap V(N)$, while v is contained in $V^*(T)$. Hence $v = v_B$ for a (necessarily unique) block B in $\mathbb{B}_p(N)$. Then neither u nor e is contained in $\kappa(\Pi)$, and $\kappa(\pi) = v$ if and only if $\pi = \tau(B)$.

The last statement of the lemma follows immediately from Theorem 2.2. \square

We now describe a way to construct an X -cactus from a collection $\mathcal{C} \subseteq \mathcal{C}(X)$ of circular compatible partitions, i.e., the reverse of Lemma 3.1. We first construct a labelled graph $\mathcal{N}(\mathcal{C}) = (N(\mathcal{C}), \phi)$ from \mathcal{C} . In case $\mathcal{C} = \emptyset$, we define $\mathcal{N}(\mathcal{C})$ to be the trivial X -cactus. Otherwise, for $\mathcal{C} \neq \emptyset$ we proceed as follows (see Figure 4):

- (i) Let $(\mathbf{T} = (T, \phi), \kappa)$ be the perfect semi-tree representation of the set $\Pi = \Pi(\mathcal{C})$ of partitions induced by \mathcal{C} that is given by Theorem 2.2.
- (ii) For every vertex v in $\kappa(\Pi(\mathcal{C}_p))$, consider the proper circular partition $\sigma \in \mathcal{C}$ with $\pi(v) = \underline{\sigma}$. Give the vertices adjacent to v a circular ordering $\lambda_v = [(v_1, v_2, \dots, v_m)]$, $m \geq 3$, that is consistent with the ordering of σ , delete v ,

and add the edges $\{v_i, v_{i+1}\}$ for $1 \leq i \leq m$, taking indices modulo m . Denote the resulting graph by $N = N(\mathcal{C})$.

- (iii) Define the labelling map $\phi : X \rightarrow V(N)$ to be the one naturally induced by the map $\phi : X \rightarrow V(T)$.

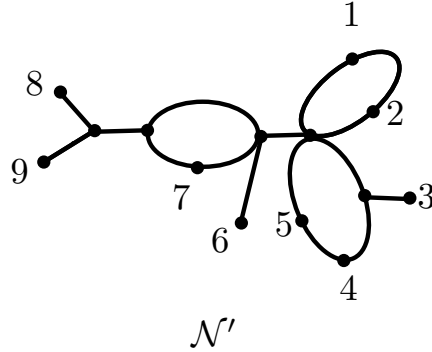


FIGURE 4. The X -cactus $\mathcal{N}' = \mathcal{N}(\mathcal{C})$ on $X = \{1, \dots, 9\}$ for the set $\mathcal{C} = \{\alpha, \beta, \gamma, s_1, s_2, s_3, s_6, s_8, s_9\}$ of circular partitions where $\alpha = [1|2|3456789]$, $\beta = [678912|3|4|5]$, $\gamma = [89|123456|7]$, $s_1 = [12345|6789]$, $s_2 = [89|1234567]$ and $s_x = [x|X - x]$ for $x \in \{3, 6, 8, 9\}$.

Note that in case \mathcal{C} consists solely of circular partitions that are splits, then, since each partition π in $\Pi = \Pi(\mathcal{C})$ is a split, it follows that $\kappa(\pi)$ is necessarily an edge since Step (ii) does not apply, and hence no vertex in $N(\mathcal{C})$ is contained in $\kappa(\Pi)$. In particular, $\mathcal{N} = \mathcal{N}(\mathcal{C})$ is an X -tree, and so \mathcal{N} is an X -cactus with $\mathcal{C}(\mathcal{N}) = \mathcal{C}$. We now show that $\mathcal{N}(\mathcal{C})$ is an X -cactus for an arbitrary collection \mathcal{C} of compatible circular partitions.

Lemma 3.2. *If $\mathcal{C} \subseteq \mathcal{C}(X)$ is a set of compatible circular partitions, then $\mathcal{N}(\mathcal{C})$ is an X -cactus such that $\mathcal{C}(\mathcal{N}(\mathcal{C})) = \mathcal{C}$.*

Proof. The lemma holds in case $\mathcal{C} = \emptyset$ by convention, and if \mathcal{C} contains only splits then it holds by the above remarks. So assume $|\mathcal{C}| > 0$ and that \mathcal{C} contains at least one proper circular partition. Let $\Pi = \Pi(\mathcal{C}) = \{\underline{\sigma} : \sigma \in \mathcal{C}\}$ be the set of partitions on X induced by \mathcal{C} . Since \mathcal{C} is compatible, it follows that Π is also compatible.

Consider the perfect semi-tree representation $(\mathbf{T} = (T, \phi), \kappa)$ of Π given by Theorem 2.2. Then $|E(T)| \geq 1$. Let V_o be the set of unlabelled degree two vertices in T . Let E_1 be the set of edges e in T for which $\kappa(\pi) = e$ holds for some split π in Π .

Let $\mathcal{C}_p = \{\sigma_1, \dots, \sigma_k\}$, $k \geq 1$, be the subset of proper circular partitions in \mathcal{C} . Let $V_1 = \{v_1 = \kappa(\underline{\sigma}_1), \dots, v_k = \kappa(\underline{\sigma}_k)\}$. Because (\mathbf{T}, κ) is a perfect semi-tree representation, for each $v \in V_o$, there exists a vertex $v' \in V_1$ such that v is adjacent to v' . Moreover, each edge e in $E(T) - E_1$ is incident with a vertex v in V_1 .

Let $\mathcal{N}_0 = \mathbf{T}$. For $1 \leq i \leq k$, let \mathcal{N}_i be the graph obtained from \mathcal{N}_{i-1} by performing Step (ii) in the construction of $\mathcal{N}(\mathcal{C})$. For all $1 \leq i \leq k$, let N_i denote the underlying graph of \mathcal{N}_i . Note that, in particular, $N(\mathcal{C}) = N_k$. Since v_i is a cut vertex in N_{i-1} such that each connected component in $N_{i-1} - \{v_i\}$ contains precisely one vertex adjacent with v_i in N_{i-1} , it follows that N_i is a simple connected graph in which two cycles share at most one vertex. Moreover, N_i contains precisely one more cycle, denoted by C_i , than N_{i-1} and, by construction, $\tau(C_i) = \sigma_i$.

Thus $N(\mathcal{C}) = N_k$ is a connected graph in which two cycles share at most one vertex with $\mathcal{C}_p(\mathcal{N}_k) = \mathcal{C}_p$. Since each vertex v in V_o is adjacent to a vertex v_i in T , by construction it follows that v is a vertex in the cycle C_i and hence \mathcal{N}_k does not contain any unlabelled degree two vertices.

Note that each edge added during the process of constructing \mathcal{N}_k from \mathcal{N}_0 belongs to a cycle and each edge e in $E(T) - E_1$ is removed as e is incident with a vertex in V_1 . Thus, the set of cut edges in \mathcal{N}_k is E_1 and these cut edges induce precisely the splits that are contained in Π . It follows that \mathcal{N}_k is an X -cactus with $\mathcal{C}(\mathcal{N}_k) = \mathcal{C}$, which completes the proof of the lemma. \square

We now prove the main result of this section, an analogue of the Splits Equivalence Theorem for X -trees [14, Theorem 3.1.4].

Theorem 3.3. *Let \mathcal{C} be a set of circular partitions of X . Then there is an X -cactus \mathcal{N} such that $\mathcal{C} = \mathcal{C}(\mathcal{N})$ if and only if \mathcal{C} is compatible. Moreover, if such an X -cactus exists, then up to isomorphism, it is unique.*

Proof. We may assume that $\mathcal{C} \neq \emptyset$ since otherwise the theorem holds. We claim that the map which takes any X -cactus $\mathcal{N} \in \mathcal{G}(X)$ to the set $\mathcal{C}(\mathcal{N})$ of circular partitions induces a bijection between $\mathcal{G}(X)$ and the set of compatible sets of circular partitions of X . The theorem then follows immediately from this claim.

By Lemma 3.2, the map in the claim is surjective, and so it suffices to show that it is injective. To this end, suppose that \mathcal{N} and \mathcal{N}' are two X -cactuses with $\mathcal{C}(\mathcal{N}) = \mathcal{C}(\mathcal{N}')$. We show that \mathcal{N} is isomorphic to \mathcal{N}' , which will complete the proof of the claim.

By Lemma 3.1, there is a unique perfect semi-tree representation (\mathbf{T}, κ) of $\Pi(\mathcal{C}(\mathcal{N}))$. Moreover, if we apply Steps (i)–(iii) to (\mathbf{T}, κ) to construct the network $\mathcal{N}(\mathcal{C}(\mathcal{N}))$ it is straightforward to check by considering the construction of (\mathbf{T}, κ) used in the proof of Lemma 3.1 that the network $\mathcal{N}(\mathcal{C}(\mathcal{N}))$ that we obtain is isomorphic to \mathcal{N} . But the same argument holds for \mathcal{N}' , and so it follows that \mathcal{N} is isomorphic to \mathcal{N}' . \square

Remark 3.4. *By Theorem 3.3, the function $d : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathbb{R}_{\geq 0}$ given by $d(\mathcal{N}, \mathcal{N}') = |\mathcal{C}(\mathcal{N}) \Delta \mathcal{C}(\mathcal{N}')|$ for all $\mathcal{N}, \mathcal{N}' \in \mathcal{G}(X)$ is a metric on the set $\mathcal{G}(X)$. This can be regarded as a generalization of the well-known Robinson-Foulds metric (cf. [15, p.25]).*

4. A PARTIAL ORDER ON X -CACTUSES

In this section, we introduce a partial order \leq on the set $\mathcal{G}(X)$ of X -cactuses, and describe some of its basic properties. We begin by defining the following two operations on an X -cactus which are related to local subnetwork operations given in [11].

Suppose that $\mathcal{N} = (N, \phi)$ is a non-trivial X -cactus. First, given any edge $e = \{u, v\} \in E(N)$ that is not contained in a tiny cycle, an *edge contraction (of \mathcal{N} on e)* results in an X -cactus \mathcal{N}' obtained by deleting e , identifying u and v as a new vertex and labelling that vertex by the elements in $\phi^{-1}(u) \cup \phi^{-1}(v)$. Second, given a tiny cycle $C = v_1, v_2, v_3$ in \mathcal{N} , a *triple contraction (of \mathcal{N} on C)* results in an X -cactus \mathcal{N}' obtained from \mathcal{N} by deleting all vertices and edges in C , adding a new vertex v' which we label $\bigcup_{i=1}^3 \phi^{-1}(v_i)$ and replacing each edge $\{v_i, u\}$, $u \in V(N) - \{v_1, v_2, v_3\}$ by a new edge $\{u, v'\}$.

We now introduce the partial order \leq on $\mathcal{G}(X)$. We define a *contraction* on a non-trivial X -cactus in $\mathcal{G}(X)$ to be either an edge contraction or a triple contraction. For two X -cactuses $\mathcal{N}, \mathcal{N}' \in \mathcal{G}(X)$ we then put $\mathcal{N} \leq \mathcal{N}'$ if and only if there is a (possibly empty) sequence of contractions transforming \mathcal{N}' to \mathcal{N} . Furthermore, we put $\mathcal{N} < \mathcal{N}'$ if $\mathcal{N} \leq \mathcal{N}'$ and $\mathcal{N} \neq \mathcal{N}'$.

We now present some basic properties of the ordering \leq on $\mathcal{G}(X)$. For the reader's convenience, we recall two key concepts from poset theory (see e.g. [2, p.4–5]). Suppose (S, \leq) is an arbitrary poset. Then $s' \in S$ is a *coatom* of $s \in S$ (or s *covers* s') if $s' < s$ and there is no element $s'' \in S$ such that $s' < s'' < s$. The poset (S, \leq) is *graded* if it has a *rank function*, that is a map ρ from S into the integers which, for $s, s' \in S$, satisfies (a) if $s' < s$, then $\rho(s') < \rho(s)$, and (b) if s' is a coatom of s , then $\rho(s) = \rho(s') + 1$.

Theorem 4.1. *Assume $|X| \geq 2$. Then the following statements hold:*

- (i) \leq is a partial order on $\mathcal{G}(X)$.
- (ii) For all $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{G}(X)$, \mathcal{N}_1 is a coatom of \mathcal{N}_2 if and only if \mathcal{N}_1 can be obtained from \mathcal{N}_2 by one contraction.
- (iii) The order \leq has a unique minimal element, namely the trivial X -cactus.
- (iv) The maximal elements under \leq are the binary phylogenetic X -cactuses in which every internal vertex is contained in some cycle.
- (v) $\mathcal{G}(X)$ is a graded poset with rank function $\rho : \mathcal{G}(X) \rightarrow \mathbb{Z}$ which assigns to each X -cactus \mathcal{N} its rank $\rho(\mathcal{N})$ given by $\rho(\mathcal{N}) = 0$ in case \mathcal{N} is the trivial X -cactus and, otherwise, by

$$\rho(\mathcal{N}) = \sum_{\sigma \in \mathcal{C}(\mathcal{N})} \chi(\sigma),$$

where $\chi(\sigma) = \max\{1, |\sigma| - 2\}$.

- (vi) If $\mathcal{N} \in \mathcal{G}(X)$, then $0 \leq \rho(\mathcal{N}) \leq 3|X| - 5$. Moreover, $\rho(\mathcal{N}) = 3|X| - 5$ if and only if \mathcal{N} is a binary phylogenetic X -cactus in which every internal vertex is contained in some tiny cycle.

Proof. (i): Suppose $\mathcal{N}, \mathcal{N}', \mathcal{N}'' \in \mathcal{G}(X)$. Then, clearly, $\mathcal{N} \leq \mathcal{N}$ since we can take the empty sequence of contractions. Hence, \leq is reflexive. In addition, since a contraction

applied to a non-trivial X -cactus reduces the number of its edges by at least one, it follows that $\mathcal{N} \leq \mathcal{N}'$ and $\mathcal{N}' \leq \mathcal{N}$ together imply $\mathcal{N} = \mathcal{N}'$. So \leq is antisymmetric. Finally, if $\mathcal{N} \leq \mathcal{N}'$ and $\mathcal{N}' \leq \mathcal{N}''$, then there is a sequence of contractions from \mathcal{N}'' to \mathcal{N}' , and a sequence of contractions from \mathcal{N}' to \mathcal{N} , and therefore a sequence of contractions from \mathcal{N}'' to \mathcal{N} . Thus $\mathcal{N} \leq \mathcal{N}''$, and so \leq is transitive. Thus, \leq is a partial order on $\mathcal{G}(X)$.

(ii): This follows immediately from (i).

(iii): Clearly the trivial X -cactus is a minimal element as no contraction may be applied. Any other X -cactus \mathcal{N} has at least one edge e which either induces a split or is contained in a cycle C . If e induces a split, then an edge contraction of \mathcal{N} on e can be performed. If e is in C , then if C has four or more vertices, an edge contraction of \mathcal{N} on e can be performed. Otherwise C is tiny and a triple contraction of \mathcal{N} on C can be performed. In either case, \mathcal{N} is not minimal.

(iv): Suppose $\mathcal{N} = (N, \phi)$ is a maximal element in $\mathcal{G}(X)$ that is not of the form given in the statement of (iv). If \mathcal{N} has a internal vertex v that is contained in $\phi(X)$ or a leaf v with $|\phi^{-1}(v)| \geq 2$ then, for every $x \in \phi^{-1}(v)$, we add a new vertex w_x which we label by x and the edge $\{v, w_x\}$ and remove all labels of v under ϕ . This results in an X -cactus \mathcal{N}' . The X -cactus \mathcal{N} can then be obtained from \mathcal{N}' by performing a non-empty sequence (possibly of length 1 in case $|\phi^{-1}(v)| = 1$) of edge contractions of \mathcal{N}' on the edges $\{v, w_x\}$ (in any order). So, we may assume that \mathcal{N} is a phylogenetic X -cactus.

Now, if an (unlabelled) internal vertex in \mathcal{N} has degree four or more, then that vertex can be ‘‘popped’’ into two new vertices by inserting an edge e . Since this is clearly the reverse of an edge contraction of the resulting X -cactus on e , we may assume that every internal vertex in \mathcal{N} has degree 3.

Finally, in case there is an internal vertex v in \mathcal{N} of degree three that is not contained in a cycle, then we may replace v by a 3-cycle, i.e. perform the reverse of a triple contraction of \mathcal{N} on v . Statement (iv) now follows.

(v): Since $\mathcal{G}(X)$ is finite, for any $\mathcal{N}, \mathcal{N}' \in \mathcal{G}(X)$ with $\mathcal{N} > \mathcal{N}'$ there is a sequence $\mathcal{N} = \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m = \mathcal{N}'$ of elements in $\mathcal{G}(X)$, such that \mathcal{N}_{i+1} is a coatom of \mathcal{N}_i for all $1 \leq i \leq m - 1$. Hence it suffices to prove that if $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{G}(X)$ are such that \mathcal{N}_1 is a coatom of \mathcal{N}_2 , then $\rho(\mathcal{N}_2) = \rho(\mathcal{N}_1) + 1$.

In view of Statement (ii), \mathcal{N}_1 is obtained from \mathcal{N}_2 by one contraction, and so we shall make a case analysis according to the type of contraction employed. First, suppose that that contraction is a triple contraction of \mathcal{N}_2 on a (necessarily tiny) cycle C . Let σ denote the circular partition induced by C . Then $|\underline{\sigma}| = 3$, $\sigma \notin \mathcal{C}(\mathcal{N}_1)$ and $\mathcal{C}(\mathcal{N}_2) = \mathcal{C}(\mathcal{N}_1) \cup \{\sigma\}$. This implies $\rho(\mathcal{N}_2) = \rho(\mathcal{N}_1) + 1$.

Now, suppose that the contraction is an edge contraction of \mathcal{N} on some edge e . There are two subcases to consider. If e is a cut edge, then let σ denote the split of X induced by e . Then $|\underline{\sigma}| = 2$, $\sigma \notin \mathcal{C}(\mathcal{N}_1)$ and $\mathcal{C}(\mathcal{N}_2) = \mathcal{C}(\mathcal{N}_1) \cup \{\sigma\}$. Thus, $\rho(\mathcal{N}_2) = \rho(\mathcal{N}_1) + 1$ follows again. Otherwise, e is contained in a cycle C of size at least four. Let C' be

the cycle in \mathcal{N}_1 obtained by contracting e . Denote the circular partitions induced by C and C' by σ and σ' , respectively. Then $|\underline{\sigma}| = |\underline{\sigma'}| + 1$ and hence $\chi(\sigma) = \chi(\sigma') + 1$. So $\rho(\mathcal{N}_2) - \rho(\mathcal{N}_1) = \chi(\sigma) - \chi(\sigma') = 1$, which completes the proof of Statement (v).

(vi): Let $n = |X|$. Since a binary phylogenetic X -cactus in which every internal vertex is contained in some tiny cycle has rank $3n - 5$, we need to show that $\rho(\mathcal{N}) \leq 3n - 5$ holds for every X -cactus $\mathcal{N} = (N, \phi)$ in $\mathcal{G}(X)$ and also that the equality holds only if every cycle in \mathcal{N} is tiny. If \mathcal{N} is the trivial X -cactus then $\rho(\mathcal{N}) = 0$ and the stated inequality follows. So assume that \mathcal{N} is not trivial. If $n = 2$ then $\rho(\mathcal{N}) = 1$ as the unique element in $\mathcal{C}(\mathcal{N})$ is the split induced by the sole edge of \mathcal{N} . The stated inequality follows again.

So assume $n \geq 3$. We use induction on n . The base case $n = 3$ is a straightforward consequence of the fact that the maximal element in $\mathcal{G}(X)$ is the X -cactus for which every internal vertex is contained in its unique tiny cycle.

Assume that $n > 3$. Then the stated inequality holds for any X -cactus \mathcal{N}' in $\mathcal{G}(X')$ with $3 \leq |X'| < n$. For the induction step, assume that $\mathcal{N} = (N, \phi)$ is an X -cactus in $\mathcal{G}(X)$ with $|X| = n$. Without loss of generality, we may assume that \mathcal{N} is a maximal element in $\mathcal{G}(X)$ under \leq . By Statement (iv), \mathcal{N} is a binary phylogenetic X -cactus and every internal vertex is contained in some cycle. First we shall show that $\rho(\mathcal{N}) \leq 3n - 5$. Let $x \in X$ and put $X' = X - \{x\}$. Then there exists a leaf v_x in \mathcal{N} such that $x \in \phi^{-1}(v_x)$. Let u_x be a vertex in \mathcal{N} adjacent with v_x . Then $e = \{v_x, u_x\}$ is a cut edge in \mathcal{N} . Let $\sigma_x = x|X'$ be the split induced by e . Note that u_x is an internal vertex of \mathcal{N} and hence contained in some cycle C_x and unlabelled under ϕ because \mathcal{N} is a phylogenetic X -cactus. Denote the circular partition induced by C_x by σ and the two vertices in C_x adjacent to u_x by u_1 and u_2 . Note that u_1 and u_2 must exist as \mathcal{N} is not a simple graph. Note that σ and σ_x are the only two partitions in $\mathcal{C}(\mathcal{N})$ that contain $\{x\}$ as a part. We now have two subcases to consider:

First, suppose C_x contains at least four vertices, i.e., u_1 and u_2 are not adjacent. Deleting the three edges incident with u_x and also the leaf v_x and adding an edge between u_1 and u_2 results in a network \mathcal{N}' on X' , in which u_1 and u_2 are contained in a cycle C' . Let σ' be the circular partition induced by C' . Let $\rho' : \mathcal{G}(X') \rightarrow \mathbb{Z}$ denote the rank function for the graded poset $\mathcal{G}(X')$ which we define analogously to ρ but with ρ replaced by ρ' and χ replaced by χ' . Then we have $\chi'(\sigma') = \chi(\sigma) - 1$. Thus, $\rho(\mathcal{N}) - \rho'(\mathcal{N}') = (\chi(\sigma) - \chi'(\sigma')) + \chi(\sigma_x) = 2$. Together with the induction hypothesis, it follows that $\rho(\mathcal{N}) = \rho'(\mathcal{N}') + 2 \leq (3n - 8) + 2 = 3n - 6$ thereby completing the induction step.

Now suppose that C_x is tiny and contains precisely the three vertices u_x , u_1 , and u_2 . For $i = 1, 2$, let v_i be the vertex adjacent to u_i and not contained in C_x . Denote the split induced by the cut edge $e_i = \{u_i, v_i\}$ by σ_i . Deleting C_x , the leaf v_x and the three edges e , $\{u_1, v_1\}$ and $\{u_2, v_2\}$ and adding an edge between v_1 and v_2 results in an X' -cactus \mathcal{N}' . Note that $\{v_1, v_2\}$ is a cut edge in \mathcal{N}' . Denote the split it induces by σ' . Note that σ' can be obtained by deleting x from either σ_1 or σ_2 . Therefore, we have

$$\rho(\mathcal{N}) - \rho'(\mathcal{N}') = \chi(\sigma) + \chi(\sigma_x) + \chi(\sigma_1) + \chi(\sigma_2) - \chi'(\sigma') = 3$$

since each term in the middle sum equates to 1. By the induction hypothesis, it follows that $\rho(\mathcal{N}) = \rho'(\mathcal{N}') + 3 \leq (3n - 8) + 3 = 3n - 5$. Furthermore, the equality holds only if \mathcal{N}' is an X' -cactus in which every cycle is tiny. By construction, every cycle in \mathcal{N}' is tiny if and only if every cycle in \mathcal{N} is tiny. This completes the proof of the induction step for this subcase too and therefore the proof of Statement (vi). \square

Remark 4.2. *Note that the poset $(\mathcal{T}(X), \leq)$ is pure (i. e. bounded and all maximal chains have the same length), but the poset $(\mathcal{G}(X), \leq)$ does not have this property.*

5. A CHARACTERIZATION OF THE X -CACTUS ORDERING

In this section, we present a characterization of the partial order \leq on $\mathcal{G}(X)$ based on the collection of circular partitions associated to an X -cactus in $\mathcal{G}(X)$. To this end, given two non-empty sets of circular partitions \mathcal{C} and \mathcal{C}' in $\mathcal{C}(X)$, a map $L : \mathcal{C} \rightarrow \mathcal{C}'$ is called a *domination map* if L is injective and $\sigma \preceq L(\sigma)$ holds for each $\sigma \in \mathcal{C}$. We say that \mathcal{C} is *dominated* by \mathcal{C}' , denoted by $\mathcal{C} \trianglelefteq \mathcal{C}'$, if there exists a domination map from \mathcal{C} to \mathcal{C}' . We use the convention that the empty set \emptyset is dominated by any set \mathcal{C} of circular partitions. In this case we also put $\emptyset \trianglelefteq \mathcal{C}$.

The main result of this section can now be stated as follows:

Theorem 5.1. *$\mathcal{N} \leq \mathcal{N}'$ holds for two X -cactuses \mathcal{N} and \mathcal{N}' if and only if $\mathcal{C}(\mathcal{N}) \trianglelefteq \mathcal{C}(\mathcal{N}')$.*

The proof of Theorem 5.1 is presented later on in this section and relies on a number of intermediate results. We start by presenting an observation concerning the poset $(\mathcal{C}(X), \preceq)$.

Lemma 5.2. *Suppose that σ_1, σ_2 and σ'_2 are three circular partitions in $\mathcal{C}(X)$ such that $\sigma_1 \preceq \sigma_2$ and σ_2 is compatible with σ'_2 . Then σ_1 is compatible with σ'_2 . Moreover, if σ'_1 is a circular partition with $\sigma'_1 \preceq \sigma'_2$, then σ_1 is compatible with σ'_1 .*

Proof. Since σ_2 and σ'_2 are compatible, there exists a part X_2 in σ_2 and a part X'_2 in σ'_2 with $X_2 \cup X'_2 = X$. Because $\sigma_1 \preceq \sigma_2$, there exists a part X_1 in σ_1 with $X_2 \subseteq X_1$. This implies $X_1 \cup X'_2 = X$, and hence σ_1 is compatible with σ'_2 . Furthermore, since $\sigma'_1 \preceq \sigma'_2$, there exists a part X'_1 in σ'_1 with $X'_2 \subseteq X'_1$. Therefore $X_1 \cup X'_1 = X$. Hence, σ_1 and σ'_1 are compatible. \square

Lemma 5.3. *Suppose that $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 are three sets of circular partitions in $\mathcal{C}(X)$.*

- (i) *If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$.*
- (ii) *$\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ if and only if $\mathcal{C}_1 \cap \mathcal{C}_b(X) \subseteq \mathcal{C}_2 \cap \mathcal{C}_b(X)$ and $\mathcal{C}_1 \cap \mathcal{C}_p(X) \trianglelefteq \mathcal{C}_2 \cap \mathcal{C}_p(X)$.*
- (iii) *If $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ and $\mathcal{C}_2 \trianglelefteq \mathcal{C}_3$, then $\mathcal{C}_1 \trianglelefteq \mathcal{C}_3$.*

Proof. We assume $\mathcal{C}_1 \neq \emptyset$ as the lemma clearly holds otherwise.

(i) Note that the map $L : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ defined as $L(\sigma) = \sigma$ is a domination map.

(ii) It is straightforward to show that the statement holds if $\mathcal{C}_b(X) \cap \mathcal{C}_1 = \emptyset$ or if $\mathcal{C}_p(X) \cap \mathcal{C}_1 = \emptyset$. So assume that $\mathcal{C}_b(X) \cap \mathcal{C}_1 \neq \emptyset$ and that $\mathcal{C}_p(X) \cap \mathcal{C}_1 \neq \emptyset$. Assume

first that $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ and consider a domination map $L : \mathcal{C}_1 \rightarrow \mathcal{C}_2$. Let $\sigma \in \mathcal{C}_1$. Then $\sigma \in \mathcal{C}_b(X)$ if and only if $L(\sigma)$ is contained in $\mathcal{C}_b(X)$. This implies that the restriction of L to $\mathcal{C}_1 \cap \mathcal{C}_b(X)$ is a domination map from $\mathcal{C}_1 \cap \mathcal{C}_b(X)$ to $\mathcal{C}_2 \cap \mathcal{C}_b(X)$. Thus, we have $\mathcal{C}_1 \cap \mathcal{C}_b(X) \subseteq \mathcal{C}_2 \cap \mathcal{C}_b(X)$. Furthermore, since $\sigma \in \mathcal{C}_p(X)$ if and only if $L(\sigma) \in \mathcal{C}_p(X)$, it follows that $\mathcal{C}_1 \cap \mathcal{C}_p(X) \subseteq \mathcal{C}_2 \cap \mathcal{C}_p(X)$.

Conversely, assume that $\mathcal{C}_1 \cap \mathcal{C}_b(X) \subseteq \mathcal{C}_2 \cap \mathcal{C}_b(X)$ and that $\mathcal{C}_1 \cap \mathcal{C}_p(X) \subseteq \mathcal{C}_2 \cap \mathcal{C}_p(X)$. By Part (i) of the lemma, there exists a domination map L_b from $\mathcal{C}_1 \cap \mathcal{C}_b(X)$ to $\mathcal{C}_2 \cap \mathcal{C}_b(X)$. Let L_p be a domination map from $\mathcal{C}_1 \cap \mathcal{C}_p(X)$ to $\mathcal{C}_2 \cap \mathcal{C}_p(X)$. Now consider the map $L : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ with $L(\sigma) = L_b(\sigma)$ if $\sigma \in \mathcal{C}_1 \cap \mathcal{C}_b(X)$ and $L(\sigma) = L_p(\sigma)$ if $\sigma \in \mathcal{C}_1 \cap \mathcal{C}_p(X)$. Then L is a domination map. Hence, $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$.

(iii) Fix a domination map $L_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and a domination map $L_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_3$. Then the map $L : \mathcal{C}_1 \rightarrow \mathcal{C}_3$ defined by $L(\sigma) = L_2(L_1(\sigma))$ for all $\sigma \in \mathcal{C}_1$ is a domination map. Indeed, L is injective since both L_1 and L_2 are injective, and, $\sigma \preceq L_1(\sigma)$ and $L_1(\sigma) \preceq L_2(L_1(\sigma))$ imply $\sigma \preceq L(\sigma)$. Hence, $\mathcal{C}_1 \trianglelefteq \mathcal{C}_3$. \square

Given any two sets of circular partitions \mathcal{C}_1 and \mathcal{C}_2 in $\mathcal{C}(X)$ with $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$, there could in general be several domination maps from \mathcal{C}_1 to \mathcal{C}_2 . For instance, let $\sigma_1 = [12|3|45|6]$ and $\sigma_2 = [1|2|3|45|6]$, and consider $\mathcal{C}_1 = \{\sigma_1\}$ and $\mathcal{C}_2 = \{\sigma_1, \sigma_2\}$. Then $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ and there are two domination maps from \mathcal{C}_1 to \mathcal{C}_2 : one maps σ_1 to σ_1 , and the other maps σ_1 to σ_2 . However, the following lemma shows that when \mathcal{C}_2 is compatible, the domination map is unique.

Lemma 5.4. *Suppose that \mathcal{C}_1 and \mathcal{C}_2 are two non-empty sets of circular partitions in $\mathcal{C}(X)$ and that \mathcal{C}_2 is compatible. Then for each circular partition $\sigma_1 \in \mathcal{C}_1$, there exists at most one circular partition $\sigma_2 \in \mathcal{C}_2$ with $\sigma_1 \preceq \sigma_2$. Moreover, $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ holds if and only if there exists a unique domination map from \mathcal{C}_1 to \mathcal{C}_2 .*

Proof. Suppose that $\sigma_1 \in \mathcal{C}_1$ and that σ_2 and σ'_2 are two circular partitions in \mathcal{C}_2 with $\sigma_1 \preceq \sigma_2$ and $\sigma_1 \preceq \sigma'_2$. We shall show that $\sigma_2 = \sigma'_2$. This clearly holds if σ_1 is a split because in this case we have $\sigma_2 = \sigma_1 = \sigma'_2$. So, assume for contradiction that σ_1 is proper, that is, $|\underline{\sigma}_1| > 2$, and that $\sigma_2 \neq \sigma'_2$. Since \mathcal{C}_2 is compatible, there exists a part X_2 in σ_2 and X'_2 in σ'_2 such that $X_2 \cup X'_2 = X$. On the other hand, $\sigma_1 \preceq \sigma_2$ implies that there exists a part X_1 in σ_1 with $X_2 \subseteq X_1$. Similarly, since $\sigma_1 \preceq \sigma'_2$ it follows that there exists a part X'_1 in σ_1 with $X'_2 \subseteq X'_1$. However, this implies $X_1 \cup X'_1 = X$, a contradiction to the fact that σ_1 is proper. Thus $\sigma_2 = \sigma'_2$, completing the proof of the first part of the lemma.

To establish the second part of the lemma, it clearly suffices to show that if $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ holds then there must exist a unique domination map from \mathcal{C}_1 to \mathcal{C}_2 . So assume $\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$ and that there exist two distinct domination maps L and L' from \mathcal{C}_1 to \mathcal{C}_2 . Then there exists a circular partition σ in \mathcal{C}_1 with $L(\sigma) \neq L'(\sigma)$. Since L and L' are domination maps we have $\sigma \preceq L(\sigma)$ and $\sigma \preceq L'(\sigma)$, a contradiction to the first part of the lemma. \square

With these results in hand, we now prove Theorem 5.1:

Proof. First assume that \mathcal{N} and \mathcal{N}' are two X -cactuses for which $\mathcal{N} \leq \mathcal{N}'$ holds. Without loss of generality, we may assume that \mathcal{N} is obtained from \mathcal{N}' by one contraction. Furthermore, we may assume that \mathcal{N} is not the trivial X -cactus since in this case the theorem holds in view of Theorem 3.3. By Lemma 5.3(ii) it suffices to show that $\mathcal{C}_b(\mathcal{N}) \subseteq \mathcal{C}_b(\mathcal{N}')$ and $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}')$. To this end, we consider the following possible three cases:

First, assume that \mathcal{N}' is obtained from \mathcal{N} by an edges contraction of \mathcal{N} on a cut edge. Then $\mathcal{C}_b(\mathcal{N}) \subset \mathcal{C}_b(\mathcal{N}')$. Moreover, $\mathcal{C}_p(\mathcal{N}) = \mathcal{C}_p(\mathcal{N}')$, and so $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}')$ in view of Lemma 5.3(i).

Next, assume that \mathcal{N}' is obtained from \mathcal{N} by a triple contraction of \mathcal{N} on a tiny cycle C . Then $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b(\mathcal{N}')$ and $\mathcal{C}_p(\mathcal{N}) \subset \mathcal{C}_p(\mathcal{N}')$, and so $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}')$ by Lemma 5.3(i).

Finally, assume that \mathcal{N}' is obtained from \mathcal{N} by an edge contraction of \mathcal{N} on an edge e in a cycle C that is not tiny. In particular, C is contracted to a cycle C' in \mathcal{N}' with one less edge. Then $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b(\mathcal{N}')$. Denote the circular partition induced by C in \mathcal{N} and C' in \mathcal{N}' by σ and σ' , respectively. Then $\mathcal{C}_p(\mathcal{N}') = (\mathcal{C}_p(\mathcal{N}) - \{\sigma\}) \cup \{\sigma'\}$, from which it is straightforward to verify that $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}')$ by noting that $\sigma' \preceq \sigma$ and using Theorem 3.3 and Lemma 5.4.

Conversely, suppose that \mathcal{N} and \mathcal{N}' are two X -cactuses with $\mathcal{C}(\mathcal{N}) \preceq \mathcal{C}(\mathcal{N}')$. By Lemma 5.3(ii), $\mathcal{C}_b(\mathcal{N}) \subseteq \mathcal{C}_b(\mathcal{N}')$ and $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}')$. If $\mathcal{C}(\mathcal{N}) = \emptyset$ then $\mathcal{N} = \mathcal{N}(\mathcal{C}(\mathcal{N}))$ is the trivial X -cactus and so $\mathcal{N} \leq \mathcal{N}'$ must hold.

So assume that $\mathcal{C}(\mathcal{N}) \neq \emptyset$. Since $\mathcal{C}_p(\mathcal{N}')$ is compatible, by Lemma 5.4 there is a unique domination map from $\mathcal{C}_p(\mathcal{N})$ to $\mathcal{C}_p(\mathcal{N}')$. Let $\mathcal{C}^* \subseteq \mathcal{C}_p(\mathcal{N}')$ be the image of this map. Since a domination map is injective, $|\mathcal{C}^*| = |\mathcal{C}_p(\mathcal{N})|$.

Now, each split σ in $\mathcal{C}_b(\mathcal{N}') - \mathcal{C}_b(\mathcal{N})$ corresponds to a cut edge in \mathcal{N}' that induces σ . Consider the network \mathcal{N}_1 obtained from \mathcal{N}' by performing an edge contraction of \mathcal{N}' on all cut edges that induce some split in $\mathcal{C}_b(\mathcal{N}') - \mathcal{C}_b(\mathcal{N})$. Then $\mathcal{C}_b(\mathcal{N}_1) = \mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N}_1) = \mathcal{C}_p(\mathcal{N}')$.

Next, for each proper partition σ in $\mathcal{C}_p(\mathcal{N}_1) - \mathcal{C}^*$, there is a cycle C_σ in \mathcal{N}_1 that induces σ . Consider the network \mathcal{N}_2 obtained from \mathcal{N}_1 by performing a triple contraction of \mathcal{N}_1 on all cycles in \mathcal{N}_1 that correspond to some proper partition in $\mathcal{C}_p(\mathcal{N}_1) - \mathcal{C}^*$ (i.e., for every $\sigma \in \mathcal{C}_p(\mathcal{N}_1) - \mathcal{C}^*$, first apply a (possibly empty) sequence of edge contractions to covert C_σ into a tiny cycle, and then apply a triple contraction on the resulting tiny cycle). Then $\mathcal{C}_b(\mathcal{N}_2) = \mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N}_2) = \mathcal{C}^*$.

Finally, for each proper partition σ^* in \mathcal{C}^* , there is a cycle C_{σ^*} in \mathcal{N}_2 that induces σ^* . Given such a partition σ^* , let σ be the unique circular partition in $\mathcal{C}_p(\mathcal{N})$ with $\sigma \preceq \sigma^*$. For every C_{σ^*} in \mathcal{N}_2 perform a (possibly empty) series of edge contractions on \mathcal{N}_2 so that in the resulting X -cactus \mathcal{N}_3 there exists a cycle that induces σ . Then $\mathcal{C}_b(\mathcal{N}_3) = \mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N}_3) = \mathcal{C}_p(\mathcal{N})$ because $|\mathcal{C}^*| = |\mathcal{C}_p(\mathcal{N})|$. By Theorem 3.3, \mathcal{N}_3 is isomorphic to \mathcal{N} . Since $\mathcal{N}_3 \leq \mathcal{N}'$ by construction, $\mathcal{N} \leq \mathcal{N}'$ follows. \square

We end this section by giving two consequences of Theorem 5.1. First, recall that, if (S, \leq) and (S', \preceq) are arbitrary posets, then a map $f : S \rightarrow S'$ is an embedding of (S, \leq) into (S', \preceq) if, for all $s, s' \in S$, $s \leq s'$ if and only if $f(s) \preceq f(s')$ [17, p.436]. Now, we have two natural maps $i : \mathcal{T}(X) \rightarrow \mathcal{G}(X)$ and $i' : \mathcal{C}(X) \rightarrow \mathcal{G}(X)$. The first is the inclusion map, and the second is given by taking a circular partition $[A_1 | \dots | A_k]$ of X to the X -cactus which, if $k \geq 3$, is a cycle of length k with each vertex labelled by A_i so that the circular ordering of the A_i 's is preserved and, if $k = 2$, is a cut edge.

Corollary 5.5. *The maps i and i' are both poset embeddings.*

Proof. First consider the map i . Suppose $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{T}(X)$. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ in $\mathcal{T}(X)$ if and only if $\mathcal{C}(\mathcal{T}_1) \subseteq \mathcal{C}(\mathcal{T}_2)$ holds. Since both $\mathcal{C}(\mathcal{T}_1)$ and $\mathcal{C}(\mathcal{T}_2)$ are contained in $\mathcal{C}_b(X)$, by Lemma 5.3(ii), we have $\mathcal{C}(\mathcal{T}_1) \subseteq \mathcal{C}(\mathcal{T}_2)$ if and only if $\mathcal{C}(\mathcal{T}_1) \trianglelefteq \mathcal{C}(\mathcal{T}_2)$. Together with Theorem 5.1, it follows that $\mathcal{T}_1 \leq \mathcal{T}_2$ in $\mathcal{T}(X)$ if and only if $i(\mathcal{T}_1) \leq i(\mathcal{T}_2)$ in $\mathcal{G}(X)$.

Now consider the map i' . Suppose $\sigma_1, \sigma_2 \in \mathcal{C}(X)$. Since $\sigma_1 \preceq \sigma_2$ if and only if $\{\sigma_1\} \trianglelefteq \{\sigma_2\}$, by Theorem 5.1 it follows that $\sigma_1 \preceq \sigma_2$ if and only if $i'(\sigma_1) = \mathcal{N}(\{\sigma_1\}) \leq \mathcal{N}(\{\sigma_2\}) = i'(\sigma_2)$. \square

Finally, we state a useful observation about upper and lower bounds in $(\mathcal{G}(X), \leq)$, which is a straightforward consequence of Lemma 5.3 and Theorem 5.1.

Corollary 5.6. *Let $\mathcal{G} = \{\mathcal{N}_1, \dots, \mathcal{N}_m\} \subseteq \mathcal{G}(X)$, $m \geq 1$, be a collection of X -cactuses. Then the following two statements hold.*

- (i) *An X -cactus \mathcal{N} is an upper bound of \mathcal{G} if and only if $\bigcup_{i=1}^m \mathcal{C}_b(\mathcal{N}_i) \subseteq \mathcal{C}_b(\mathcal{N})$ and, for all $1 \leq i \leq m$, $\mathcal{C}_p(\mathcal{N}_i) \trianglelefteq \mathcal{C}_p(\mathcal{N})$ holds.*
- (ii) *An X -cactus \mathcal{N} is a lower bound of \mathcal{G} if and only if $\mathcal{C}_b(\mathcal{N}) \subseteq \bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i)$ and, for all $1 \leq i \leq m$, $\mathcal{C}_p(\mathcal{N}) \trianglelefteq \mathcal{C}_p(\mathcal{N}_i)$ holds.*

6. UPPER BOUNDS

In Corollary 5.6, we gave a characterization for when a set $\mathcal{G} \subseteq \mathcal{G}(X)$ of X -cactuses has an upper bound in $(\mathcal{G}(X), \leq)$. In this section, we present two further results concerning upper bounds in $(\mathcal{G}(X), \leq)$. Upper bounds are of interest since, if they exist, they can be thought of as ‘‘supernetworks’’ for collections of networks. Note that the behaviour of upper bounds in $(\mathcal{G}(X), \leq)$ is more complicated than it is for the poset $(\mathcal{T}(X), \leq)$ of X -trees. For example, as shown in [14, Theorem 3.3.3], in case a set \mathcal{T} of X -trees has an upper bound in $(\mathcal{T}(X), \leq)$, then it has a unique least upper bound. However, this is not the case for $(\mathcal{G}(X), \leq)$ (e.g. Figure 5).

Our first result gives an insight on the number of circular partitions contained in an upper bound.

Theorem 6.1. *If \mathcal{N} is a least upper bound for a collection $\{\mathcal{N}_1, \dots, \mathcal{N}_m\} \subseteq \mathcal{G}(X)$ of X -cactuses, some $m \geq 1$, then*

$$|\mathcal{C}(\mathcal{N})| \leq \left| \bigcup_{1 \leq i \leq m} \mathcal{C}(\mathcal{N}_i) \right|.$$

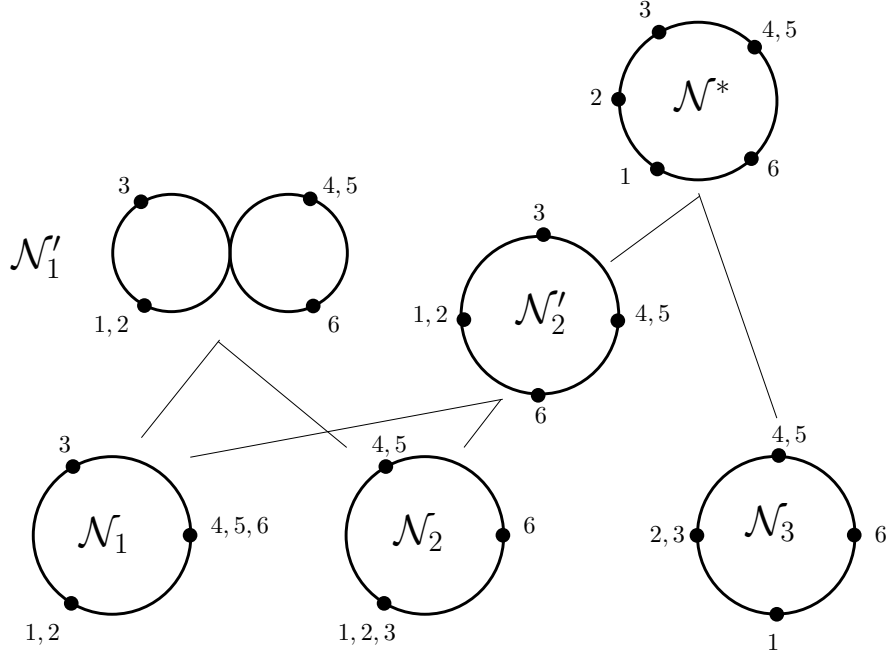


FIGURE 5. The Hasse diagram of six X -cactuses $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}'_1, \mathcal{N}'_2, \mathcal{N}^*$ where $X = \{1, 2, \dots, 6\}$. The set $\{\mathcal{N}_1, \mathcal{N}_2\}$ has two least upper bounds, \mathcal{N}'_1 and \mathcal{N}'_2 , whereas the set $\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$ has a unique least upper bound \mathcal{N}^* . For ease of readability, we have omitted the brackets from the set that labels a vertex.

Proof. Since \mathcal{N} is an upper bound for $\{\mathcal{N}_1, \dots, \mathcal{N}_2\}$, by Corollary 5.6, for each $1 \leq i \leq m$, there is a subset $\mathcal{C}_b^i(\mathcal{N})$ of $\mathcal{C}_b(\mathcal{N})$ so that $\mathcal{C}_b^i(\mathcal{N}) = \mathcal{C}_b(\mathcal{N}_i)$ holds. Furthermore, we can take a minimal subset $\mathcal{C}_p^i(\mathcal{N})$ of $\mathcal{C}_p(\mathcal{N})$ so that $\mathcal{C}_p(\mathcal{N}_i) \leq \mathcal{C}_p^i(\mathcal{N})$ holds, where minimality implies $|\mathcal{C}_p(\mathcal{N}_i)| = |\mathcal{C}_p^i(\mathcal{N})|$. We consider the splits and proper partitions separately.

First, we claim that

$$(1) \quad \mathcal{C}_b(\mathcal{N}) = \bigcup_{1 \leq i \leq m} \mathcal{C}_b^i(\mathcal{N}) = \bigcup_{1 \leq i \leq m} \mathcal{C}_b(\mathcal{N}_i).$$

The right equality holds by definition. To see that the left one holds, note first that by definition $\bigcup_{1 \leq i \leq m} \mathcal{C}_b^i(\mathcal{N}) \subseteq \mathcal{C}_b(\mathcal{N})$. To see that equality holds, assume for contradiction that this is not the case, i.e. that there exists a split σ in $\mathcal{C}_b(\mathcal{N}) - \bigcup_{1 \leq i \leq m} \mathcal{C}_b^i(\mathcal{N})$. Let \mathcal{N}^* be the X -cactus obtained from \mathcal{N} by contracting the edge in \mathcal{N} corresponding to σ . Then by Corollary 5.6, \mathcal{N}^* is an upper bound of $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$ with $\mathcal{N}^* < \mathcal{N}$, a contradiction.

We next claim

$$(2) \quad |\mathcal{C}_p(\mathcal{N})| = \left| \bigcup_{1 \leq i \leq m} \mathcal{C}_p^i(\mathcal{N}) \right| \leq \left| \bigcup_{1 \leq i \leq m} \mathcal{C}_p(\mathcal{N}_i) \right|.$$

The left equality can be shown to hold using a similar argument to the one used to prove Equality (1). To see that the right inequality holds, for each circular partition $\sigma \in \bigcup_{1 \leq i \leq m} \mathcal{C}_p^i(\mathcal{N})$, let $t(\sigma)$ be the smallest index in $\{1, \dots, m\}$ (subject to some ordering) such that σ is contained in $\mathcal{C}_p^{t(\sigma)}(\mathcal{N})$. By construction, there exists a unique circular partition in $\mathcal{C}_p(\mathcal{N}_{t(\sigma)})$, denoted by $f(\sigma)$, such that $\sigma \preceq f(\sigma)$ holds. Since the map $f : \bigcup_{1 \leq i \leq m} \mathcal{C}_p^i(\mathcal{N}) \rightarrow \bigcup_{1 \leq i \leq m} \mathcal{C}_p(\mathcal{N}_i)$ associating each $\sigma \in \bigcup_{1 \leq i \leq m} \mathcal{C}_p^i(\mathcal{N})$ to $f(\sigma) \in \bigcup_{1 \leq i \leq m} \mathcal{C}_p(\mathcal{N}_i)$ is injective, the right inequality follows.

The theorem follows now from Equalities (1) and (2) since $\mathcal{C}(\mathcal{N})$ is a disjoint union of $\mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N})$, and $\bigcup_{1 \leq i \leq m} \mathcal{C}(\mathcal{N}_i)$ is a disjoint union of $\bigcup_{1 \leq i \leq m} \mathcal{C}_b(\mathcal{N}_i)$ and $\bigcup_{1 \leq i \leq m} \mathcal{C}_p(\mathcal{N}_i)$. \square

In our second result, we give an alternative characterization to Corollary 5.6 for when the upper bound for two X -cactuses exists (Theorem 6.2), which gives some more structural insights into determining whether this is the case or not.

Now, for distinct X -cactuses $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{G}(X)$ we define the *incompatibility graph* $\mathbb{I}(\mathcal{N}_1, \mathcal{N}_2) = (W, F)$ to be the graph with vertex set $W = \mathcal{C}(\mathcal{N}_1) \cup \mathcal{C}(\mathcal{N}_2)$ and edge set F consisting of all pairs $\{\sigma_1, \sigma_2\}$ of distinct circular partitions in W such that σ_1 and σ_2 are incompatible. A *resolution* of (W, F) is an injective map $\lambda : F \rightarrow \mathcal{C}(X)$ such that $\lambda(\{\sigma_1, \sigma_2\})$ is an upper bound of σ_1 and σ_2 in $(\mathcal{C}(X), \preceq)$, for every edge $\{\sigma_1, \sigma_2\} \in F$. Such a resolution is called *minimal* if $\lambda(\{\sigma_1, \sigma_2\})$ is a least upper bound of σ_1 and σ_2 in $(\mathcal{C}(X), \preceq)$ for each edge $\{\sigma_1, \sigma_2\}$ in F . Note that if $F = \emptyset$, then we use the convention that the empty function $\lambda : F \rightarrow \mathcal{C}(X)$ with $\lambda(F) = \emptyset$ is the (necessarily unique) minimal resolution of (W, F) .

Theorem 6.2. *Suppose \mathcal{N}_1 and \mathcal{N}_2 are two distinct X -cactuses. Then the following statements are equivalent.*

- (i) \mathcal{N}_1 and \mathcal{N}_2 have an upper bound under \leq ;
- (ii) the incompatibility graph $\mathbb{I}(\mathcal{N}_1, \mathcal{N}_2) = (W, F)$ is a matching (i.e. every vertex has degree 0 or 1), and there exists a resolution λ of (W, F) such that $W_0 \cup \lambda(F)$ is compatible (where $W_0 \subseteq W$ denotes the set of isolated vertices in $\mathbb{I}(\mathcal{N}_1, \mathcal{N}_2)$).

Moreover, if the incompatibility graph (W, F) is a matching and there exists a minimal resolution λ of (W, F) such that $W_0 \cup \lambda(F)$ is compatible, then $\mathcal{N}(W_0 \cup \lambda(F))$ is a least upper bound for \mathcal{N}_1 and \mathcal{N}_2 .

Before proceeding with the proof, to illustrate Theorem 6.2 consider the X -cactuses \mathcal{N}_1 and \mathcal{N}_2 with $X = \{1, 2, \dots, 9\}$ pictured in Figure 1. Then the X -cactus depicted in Figure 4 is a least upper bound for \mathcal{N}_1 and \mathcal{N}_2 since the incompatibility graph $\mathbb{I}(\mathcal{N}_1, \mathcal{N}_2)$ is a matching with sole edge $e = \{\beta, [678912|3|45]\}$ where $\beta = [678912|3|4|5]$ and the map λ assigning e to β is a minimal resolution.

Proof. For simplicity, put $\mathbb{I} = \mathbb{I}(\mathcal{N}_1, \mathcal{N}_2) = (W, F)$.

(i) \Rightarrow (ii): Suppose that \mathcal{N}' is an upper bound of \mathcal{N}_1 and \mathcal{N}_2 under \leq . Note that if $F = \emptyset$, then \mathbb{I} is a matching as every vertex in \mathbb{I} is isolated. Furthermore, the empty

function $\lambda : F \rightarrow \mathcal{C}(X)$ with $\lambda(F) = \emptyset$ is a resolution of \mathbb{I} and $W_0 \cup \lambda(F) = W = \mathcal{C}(\mathcal{N}_1) \cup \mathcal{C}(\mathcal{N}_2)$ is compatible. Hence we may assume that $F \neq \emptyset$.

By Lemma 5.4 and Theorem 5.1, for $i = 1, 2$ there is a unique domination map from $\mathcal{C}(\mathcal{N}_i)$ to $\mathcal{C}(\mathcal{N}')$ which we denote by L_i . We first claim that for each edge $e = \{\sigma_1, \sigma_2\} \in F$, with $\sigma_i \in \mathcal{C}(\mathcal{N}_i)$ for $i = 1, 2$, we have $L_1(\sigma_1) = L_2(\sigma_2)$. Indeed, if this were not the case, then $L_1(\sigma_1)$ and $L_2(\sigma_2)$ would be a pair of compatible circular partitions by Lemma 3.1 (since both are contained in $\mathcal{C}(\mathcal{N}')$). Hence, by the last part of Lemma 5.2 and because $\sigma_i \preceq L_i(\sigma_i)$ for $i = 1, 2$, it follows that σ_1 and σ_2 are compatible, a contradiction as $e \in F$. This proves the claim.

We now show that \mathbb{I} is a matching. Suppose this were not the case. Then there exists a vertex in W with degree two or more. Switching the index if necessary, we may assume that σ_2 is contained in $\mathcal{C}(\mathcal{N}_2)$, and that σ_1 and σ_3 are vertices in $\mathcal{C}(\mathcal{N}_1)$ that are adjacent with σ_2 . By the previous claim it follows that $L_1(\sigma_1) = L_1(\sigma_3)$, a contradiction to the fact that L_1 is injective.

Next, we show that there exists a resolution $\lambda : F \rightarrow \mathcal{C}(X)$ of \mathbb{I} as stated in Statement (ii). For each edge $e = \{\sigma_1, \sigma_2\} \in F$, $\sigma_i \in \mathcal{C}(\mathcal{N}_i)$, $i = 1, 2$, we define $\lambda(e) = L_1(\sigma_1) = L_2(\sigma_2)$ which is clearly well-defined in view of the previous claim. Furthermore, λ is injective because both L_1 and L_2 are injective. Since, by definition, $\lambda(e)$ is an upper bound for σ_i , $i = 1, 2$, it follows that λ is a resolution of \mathbb{I} .

It remains to show that $W_0 \cup \lambda(F)$ is compatible. To see this, note that since $\lambda(e)$ is an upper bound of both σ_1 and σ_2 in $(\mathcal{C}(X), \preceq)$, it follows that $W_0 \cup \lambda(F)$ is a set of circular partitions. Assume that σ and σ' are two distinct circular partitions in $W_0 \cup \lambda(F)$. Note that these two circular partitions are clearly compatible if both of them are contained in W_0 or both in $\lambda(F)$ because $\lambda(F) \subseteq \mathcal{C}(\mathcal{N}')$ and $\mathcal{C}(\mathcal{N}')$ is compatible. Therefore, without loss of generality, we may assume that $\sigma \in W_0 \cap \mathcal{C}(\mathcal{N}_1)$ and $\sigma' = \lambda(e)$ with $e = \{\sigma_1, \sigma_2\} \in F$ for $\sigma_1 \in \mathcal{N}_1$ and $\sigma_2 \in \mathcal{N}_2$. Since $\sigma \preceq L_1(\sigma)$ and noting that $L_1(\sigma)$ and $L_1(\sigma_1) = \sigma'$ are compatible (as they are two distinct circular partitions contained in $\mathcal{C}(\mathcal{N}')$), it follows by the first part of Lemma 5.2 that σ and σ' are compatible.

(ii) \Rightarrow (i): Suppose \mathbb{I} is a matching. We assume $F \neq \emptyset$; the case $F = \emptyset$ can be established in a similar way. Fix a resolution map $\lambda : F \rightarrow \mathcal{C}(X)$ as in Statement (ii). Since $W_0 \cup \lambda(F)$ is compatible, by Theorem 3.3, there exists an X -cactus \mathcal{N}^* such that $\mathcal{C}(\mathcal{N}^*) = W_0 \cup \lambda(F)$. Consider the map $L_\lambda : \mathcal{C}(\mathcal{N}_1) \rightarrow \mathcal{C}(\mathcal{N}^*)$ defined as follows. If $\sigma \in \mathcal{C}(\mathcal{N}_1)$ is an isolated vertex in \mathbb{I} , then $\sigma \in W_0$ and we let $L_\lambda(\sigma) = \sigma$; otherwise there exists a unique circular partition σ_2 in $\mathcal{C}(\mathcal{N}_2)$ such that $\{\sigma, \sigma_2\}$ is an edge in \mathbb{I} because \mathbb{I} is a matching. In this case, we let $L_\lambda(\sigma) = \lambda(\{\sigma, \sigma_2\})$.

We claim that L_λ is a domination map. We first show that L_λ is injective. Assume for contradiction that there exist $\sigma, \sigma_1 \in \mathcal{C}(\mathcal{N}_1)$ such that $L_\lambda(\sigma) = L_\lambda(\sigma_1)$ but $\sigma \neq \sigma_1$. By definition of L_λ and the fact that λ is a resolution and therefore injective, we may assume that $\sigma \in W_0 \cap \mathcal{C}_1(\mathcal{N})$, that $\{\sigma_1, \sigma_2\}$ is an edge in F with $\sigma_1 \in \mathcal{C}(\mathcal{N}_1)$, and that $\sigma_2 \in \mathcal{C}(\mathcal{N}_2)$. Then $\sigma_1 \preceq \lambda(\{\sigma_1, \sigma_2\}) = L_\lambda(\sigma_1) = L_\lambda(\sigma) = \sigma$. Since, by assumption, $\sigma \neq \sigma_1$ it follows that σ and σ_1 are two distinct incompatible circular partitions in

$\mathcal{C}(\mathcal{N}_1)$, a contradiction. Thus, L_λ must be injective. Since $\sigma \preceq L_\lambda(\sigma)$ holds for all σ in $\mathcal{C}(\mathcal{N}_1)$, it follows that L_λ is a domination map, as claimed.

As L_λ is a domination map, $\mathcal{C}(\mathcal{N}_1) \trianglelefteq \mathcal{C}(\mathcal{N}^*)$ and so $\mathcal{N}_1 \leq \mathcal{N}^*$ in view of Theorem 5.1. Using a similar argument, we also have $\mathcal{N}_2 \leq \mathcal{N}^*$. Thus, \mathcal{N}^* is an upper bound of \mathcal{N}_1 and \mathcal{N}_2 under \leq . This completes the proof of the equivalence of Statements (i) and (ii).

To prove the remainder of the theorem, assume that the incompatibility graph $\mathbb{I} = (W, F)$ is a matching and that λ is a minimal resolution of (W, F) such that $W_0 \cup \lambda(F)$ is compatible. Since a minimal resolution is in particular a resolution, our arguments in the previous two paragraphs imply that $\mathcal{N}^* = \mathcal{N}(W_0 \cup \lambda(F))$ is an upper bound for \mathcal{N}_1 and \mathcal{N}_2 . It remains to show that \mathcal{N}^* is a least upper bound.

Assume for contradiction that there exists an X -cactus $\widehat{\mathcal{N}}$ that is an upper bound for \mathcal{N}_1 and \mathcal{N}_2 such that $\widehat{\mathcal{N}} < \mathcal{N}^*$. Without loss of generality we may assume that $\widehat{\mathcal{N}}$ is obtained from \mathcal{N}^* by performing a single contraction. Let $\sigma \in \mathcal{C}(\mathcal{N}^*)$ be the unique circular partition contained in $\mathcal{C}(\mathcal{N}^*)$ but not in $\mathcal{C}(\widehat{\mathcal{N}})$. Since either $\sigma \in W_0$ or $\sigma \in \lambda(F)$ holds, swapping the index of \mathcal{N}_1 and \mathcal{N}_2 if necessary we may assume that $\mathcal{C}(\mathcal{N}_1)$ contains a circular partition σ_0 such that $\sigma_0 \preceq \sigma$. Note that $\mathcal{C}(\mathcal{N}_1) \trianglelefteq \mathcal{C}(\widehat{\mathcal{N}})$ in view of $\mathcal{N}_1 \leq \widehat{\mathcal{N}}$ and Theorem 5.1.

If σ corresponds to a cut edge e of \mathcal{N}^* (that is, $\widehat{\mathcal{N}}$ is obtained from \mathcal{N}^* by an edge contraction of \mathcal{N} on the cut edge e and $\sigma = \tau(e)$) then $\mathcal{C}(\mathcal{N}^*) - \{\sigma\} = \mathcal{C}(\widehat{\mathcal{N}})$. Furthermore, $\sigma_0 = \sigma$ as σ is a split. Thus, $\sigma_0 \in \mathcal{C}_b(\mathcal{N}_1)$ and $\sigma_0 \notin \mathcal{C}_b(\widehat{\mathcal{N}})$, a contradiction in view of $\mathcal{C}(\mathcal{N}_1) \trianglelefteq \mathcal{C}(\widehat{\mathcal{N}})$ and Lemma 5.3(ii).

If σ corresponds to a tiny cycle in \mathcal{N}^* (i.e., $\widehat{\mathcal{N}}$ is obtained from \mathcal{N}^* by a triple contraction of \mathcal{N}^* on this tiny cycle) then $\mathcal{C}(\mathcal{N}^*) - \{\sigma\} = \mathcal{C}(\widehat{\mathcal{N}})$. Furthermore, $\sigma_0 = \sigma$ as $|\sigma| = 3$. Since $\mathcal{N}_1 \leq \widehat{\mathcal{N}} < \mathcal{N}^*$, there exists a circular partition $\widehat{\sigma} \in \mathcal{C}(\widehat{\mathcal{N}})$ and a circular partition $\sigma^* \in \mathcal{C}(\mathcal{N}^*)$ with $\sigma_0 \prec \widehat{\sigma} \preceq \sigma^*$. This implies that $\mathcal{C}(\mathcal{N}^*)$ contains two distinct circular partitions σ and σ^* such that $\sigma_0 \preceq \sigma$ and $\sigma_0 \preceq \sigma^*$; a contradiction in view of Lemma 5.4 and the fact that $\mathcal{C}(\mathcal{N}^*)$ is compatible.

Finally, we consider the case that σ corresponds to a cycle C of at least four edges in \mathcal{N}^* (i.e., $\widehat{\mathcal{N}}$ is obtained from \mathcal{N}^* by an edge contraction of \mathcal{N}^* on an edge of C and $\sigma = \tau(C)$). Let C' be the cycle in $\widehat{\mathcal{N}}$ obtained from C by this edge-contraction and let $\sigma' = \tau(C')$ denote the circular partition corresponding to C' . Then $(\mathcal{C}(\mathcal{N}^*) - \{\sigma\}) \cup \{\sigma'\} = \mathcal{C}(\widehat{\mathcal{N}})$. Now we consider two possible subcases: either $\sigma \in W_0$ or $\sigma \in \lambda(F)$.

Assume first that $\sigma \in W_0$. Then we may further assume that $\sigma \in \mathcal{C}(\mathcal{N}_i)$, some $i \in \{1, 2\}$, $i = 1$, say. Since $\mathcal{N}_1 \leq \widehat{\mathcal{N}} < \mathcal{N}^*$, there exists a circular partition $\widehat{\sigma} \in \mathcal{C}(\widehat{\mathcal{N}})$ and a circular partition $\sigma^* \in \mathcal{C}(\mathcal{N}^*)$ with $\sigma \prec \widehat{\sigma} \preceq \sigma^*$. This implies that $\mathcal{C}(\mathcal{N}^*)$ contains two circular partitions σ and σ^* with $\sigma \prec \sigma^*$, a contradiction to Lemma 5.4 and the fact that $\mathcal{C}(\mathcal{N}^*)$ is compatible.

Finally, assume that $\sigma \in \lambda(F)$. Then there exist $\sigma_1 \in \mathcal{C}(\mathcal{N}_1)$ and $\sigma_2 \in \mathcal{C}(\mathcal{N}_2)$ such that $\{\sigma_1, \sigma_2\} \in F$ and $\lambda(\{\sigma_1, \sigma_2\}) = \sigma$. We claim that $\sigma_1 \preceq \sigma'$, where $\sigma' = \tau(C') \in \mathcal{C}(\widehat{\mathcal{N}})$. Assume for contradiction that this is not the case. Then since $\mathcal{C}(\mathcal{N}_1) \preceq \mathcal{C}(\widehat{\mathcal{N}})$ there exists a circular partition $\sigma'' \in \mathcal{C}(\widehat{\mathcal{N}}) - \{\sigma'\}$ such that $\sigma_1 \preceq \sigma''$. Since $\sigma \notin \mathcal{C}(\widehat{\mathcal{N}})$ and $\sigma'' \in \mathcal{C}(\widehat{\mathcal{N}}) - \{\sigma'\} \subseteq \mathcal{C}(\mathcal{N}^*)$, it follows that σ and σ'' are two distinct circular partitions in the compatible set $\mathcal{C}(\mathcal{N}^*)$ such that $\sigma_1 \preceq \sigma$ and $\sigma_1 \preceq \sigma''$ hold, a contradiction in view of Lemma 5.4. Thus, $\sigma_1 \preceq \sigma'$ as claimed. Since a similar argument also yields $\sigma_2 \preceq \sigma'$ it follows that σ' is an upper bound for σ_1 and σ_2 ; a contradiction to the fact that $\sigma' \prec \sigma$ and the assumption that σ is a least upper bound for σ_1 and σ_2 . This completes the proof of the theorem. \square

7. LOWER BOUNDS

In this section, we investigate properties of greatest lower bounds of subsets in $(\mathcal{G}(X), \preceq)$, which always exists since the trivial X -cactus is a lower bound for any such subset. In particular, as a consequence of the main result of this section (Theorem 7.2), we characterize when a subset of $\mathcal{G}(X)$ has the trivial X -cactus as a greatest lower bound (see Corollary 7.3).

Greatest lower bounds in $\mathcal{G}(X)$ are of interest since they can be considered as “consensus networks”. Indeed, in the poset of X -trees $(\mathcal{T}(X), \preceq)$, the greatest lower bound for any subset \mathcal{T} of X -trees is unique and is known as the *strict-consensus tree* for \mathcal{T} (cf. [14]). However, for arbitrary sets of X -cactuses, greatest lower bounds in $(\mathcal{G}(X), \preceq)$ are not necessarily unique. For example, for $X = \{1, 2, 3, 4\}$ the X -cactuses $\mathcal{N}(\{[14|2|3]\})$ and $\mathcal{N}(\{[1|23|4]\})$ are both greatest lower bounds for the two X -cactuses $\mathcal{N}(\{[1|2|3|4]\})$ and $\mathcal{N}(\{[1|3|2|4]\})$. The main result of this section (Theorem 7.2) gives a characterization for when an X -cactus \mathcal{N} is in the set $glb(\mathcal{N}_1, \dots, \mathcal{N}_m)$ consisting of all of the greatest lower bounds of a subset $\{\mathcal{N}_1, \dots, \mathcal{N}_m\} \subseteq \mathcal{G}(X)$.

To state this result, we introduce some further terminology. Suppose $\mathcal{C}_1, \dots, \mathcal{C}_m$, some $m \geq 1$, are sets of circular partitions of X . An element (μ_1, \dots, μ_m) in the product $\prod_{i=1}^m \mathcal{C}_i$ is said to have a *meet* if its set $glb(\mu_1, \dots, \mu_m)$ of greatest lower bounds under \preceq is non-empty. A subset $\Gamma \subseteq \prod_{i=1}^m \mathcal{C}_i$ is called *feasible* if $\Gamma \neq \emptyset$, for all distinct $(\mu_1, \dots, \mu_m), (\mu'_1, \dots, \mu'_m) \in \Gamma$ we have $\mu_i \neq \mu'_i$, for all $1 \leq i \leq m$, and every element in Γ has a meet. A *meet realisation* of such a subset Γ is a subset \mathcal{C} of $\mathcal{C}(X)$ that consists of precisely one meet for each element in Γ , that is, \mathcal{C} is the minimal subset of $\mathcal{C}(X)$ (under set inclusion) such that $|glb(\mu_1, \dots, \mu_m) \cap \mathcal{C}| = 1$ holds for each element (μ_1, \dots, μ_m) in Γ . Note that this implies $|\mathcal{C}| \leq |\Gamma|$.

To illustrate these concepts, consider the sets $\mathcal{C}_p(\mathcal{N}_1)$ and $\mathcal{C}_p(\mathcal{N}_2)$ of proper circular partitions induced by the networks \mathcal{N}_1 and \mathcal{N}_2 considered in Figure 1. Then $\alpha = [1|2|3456789]$ and $\beta_1 = [678912|3|4|5]$ are two proper circular partitions in $\mathcal{C}_p(\mathcal{N}_1)$ and $\beta_2 = [678912|3|45]$ is a proper circular partition in $\mathcal{C}_p(\mathcal{N}_2)$. Since $\alpha \in \mathcal{C}_p(\mathcal{N}_2)$, it follows that α is a meet for $(\alpha, \alpha) \in \mathcal{C}_p(\mathcal{N}_1) \times \mathcal{C}_p(\mathcal{N}_2)$. Furthermore, β_2 is a meet for $(\beta_1, \beta_2) \in \mathcal{C}_p(\mathcal{N}_1) \times \mathcal{C}_p(\mathcal{N}_2)$. Thus, the set $\Gamma = \{(\alpha, \alpha), (\beta_1, \beta_2)\} \in \mathcal{C}_p(\mathcal{N}_1) \times \mathcal{C}_p(\mathcal{N}_2)$ is

feasible. For $\gamma = [7|89|123456] \in \mathcal{C}(\mathcal{N}_1)$ the set $\{(\gamma, \gamma)\} \cup \Gamma$ is a (maximal) feasible subset of $\mathcal{C}_p(\mathcal{N}_1) \times \mathcal{C}(\mathcal{N}_2)$ and $\mathcal{C}_p(\mathcal{N}_3)$ is a meet realization for it where \mathcal{N}_3 is the X -cactus in Figure 1. In fact, Theorem 7.2 below implies that $\mathcal{N}_3 \in \text{glb}(\mathcal{N}_1, \mathcal{N}_2)$.

Before stating Theorem 7.2, we prove a useful lemma which describes how the splits behave when taking greatest lower bounds in $(\mathcal{G}(X), \leq)$.

Lemma 7.1. *Let $\{\mathcal{N}_1, \dots, \mathcal{N}_m\} \subseteq \mathcal{G}(X)$, some $m \geq 1$. If $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$ then $\mathcal{C}_b(\mathcal{N}) = \bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i)$.*

Proof. Let $\mathcal{C}_b^* = \bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i)$. By Corollary 5.6, we have $\mathcal{C}_b(\mathcal{N}) \subseteq \mathcal{C}_b^*$. We may assume that $\mathcal{C}_b^* \neq \emptyset$ since otherwise the lemma clearly holds.

To see that $\mathcal{C}_b^* \subseteq \mathcal{C}_b(\mathcal{N})$, assume for contradiction that there exists a split σ in $\mathcal{C}_b^* - \mathcal{C}_b(\mathcal{N})$. We claim that $\mathcal{C}(\mathcal{N}) \cup \{\sigma\}$ is compatible. To this end, consider an arbitrary partition σ' in $\mathcal{C}(\mathcal{N})$. We need to show that σ is compatible with σ' . Suppose first that $\sigma' \in \mathcal{C}_b(\mathcal{N})$. Then $\mathcal{C}_b(\mathcal{N}) \cup \{\sigma\} \subseteq \mathcal{C}_b(\mathcal{N}_1)$ implies that σ and σ' are both contained in $\mathcal{C}_b(\mathcal{N}_1)$. Since $\mathcal{C}_b(\mathcal{N}_1)$ is a compatible set of splits it follows that σ and σ' are compatible. Suppose next that $\sigma' \in \mathcal{C}_p(\mathcal{N})$. As $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_1)$ there exists a circular partition σ_1 in $\mathcal{C}_p(\mathcal{N}_1)$ with $\sigma' \preceq \sigma_1$. Together with Lemma 5.2 and the fact that σ and σ_1 are two distinct compatible circular partitions in $\mathcal{C}(\mathcal{N}_1)$, it follows that σ and σ' are compatible, which completes the proof the claim.

Since $\mathcal{C}(\mathcal{N}) \cup \{\sigma\}$ is a compatible set of circular partitions of $\mathcal{C}(X)$, it follows by Theorem 3.3 that there exists an X -cactus \mathcal{N}' with $\mathcal{C}(\mathcal{N}') = \mathcal{C}(\mathcal{N}) \cup \{\sigma\}$. Since $\mathcal{C}(\mathcal{N}) \subset \mathcal{C}(\mathcal{N}')$, by Lemma 5.3(i) we have $\mathcal{C}(\mathcal{N}) \preceq \mathcal{C}(\mathcal{N}')$ and hence $\mathcal{N} < \mathcal{N}'$ in view of Theorem 5.1 because $\mathcal{N} \neq \mathcal{N}'$ as $\mathcal{C}(\mathcal{N}) \subset \mathcal{C}(\mathcal{N}')$. For all $1 \leq i \leq m$, since σ is a split in \mathcal{C}_b^* and $\mathcal{C}_b(\mathcal{N}) \subseteq \mathcal{C}_b^*$, it follows that $\mathcal{C}_b(\mathcal{N}') \subseteq \mathcal{C}_b^* \subseteq \mathcal{C}_b(\mathcal{N}_i)$ and that $\mathcal{C}_p(\mathcal{N}') \preceq \mathcal{C}_p(\mathcal{N}_i)$. By Corollary 5.6, \mathcal{N}' is a lower bound for $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$ with $\mathcal{N} < \mathcal{N}'$; a contradiction to the assumption that $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. \square

We now state and prove the main result of this section.

Theorem 7.2. *Let $\{\mathcal{N}_1, \dots, \mathcal{N}_m\} \subseteq \mathcal{G}(X)$ and let $\mathcal{N} \in \mathcal{G}(X)$. Then $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$ if and only if $\mathcal{C}_b(\mathcal{N}) = \bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i)$ and either (a) $\mathcal{C}_p(\mathcal{N}) = \emptyset$ and $\prod_{i=1}^m \mathcal{C}_p(\mathcal{N}_i)$ contains no feasible subset, or (b) $\mathcal{C}_p(\mathcal{N}) \neq \emptyset$ and $\mathcal{C}_p(\mathcal{N})$ is a meet realization of some maximal feasible subset of $\prod_{i=1}^m \mathcal{C}_p(\mathcal{N}_i)$ (under set inclusion).*

Proof. Let $\mathcal{C}_b^* = \bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i)$ and $\mathcal{C}_p^* = \prod_{i=1}^m \mathcal{C}_p(\mathcal{N}_i)$.

First, suppose $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. Then, by Corollary 5.6(ii), $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_i)$ for $1 \leq i \leq m$ and, by Lemma 7.1, $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b^*$.

To see Statement (a) suppose that $\mathcal{C}_p(\mathcal{N}) = \emptyset$, that is, \mathcal{N} does not contain any cycles. We need to show that \mathcal{C}_p^* contains no feasible subset.

Suppose that this is not the case and that Γ is a feasible subset of \mathcal{C}_p^* . Then $\Gamma \neq \emptyset$, and for every $(\mu_1, \dots, \mu_m) \in \Gamma$ we have that $\text{glb}(\mu_1, \dots, \mu_m) \neq \emptyset$. Now fix an element $(\mu_1, \dots, \mu_m) \in \Gamma \subseteq \mathcal{C}_p^*$ and a partition $\sigma \in \text{glb}(\mu_1, \dots, \mu_m)$. Since $\mu_1 \in \mathcal{C}_p(X)$ and

$\sigma \preceq \mu_1$, it follows that σ must be a proper circular partition, that is, $\sigma \in \mathcal{C}_p(X)$. By an argument similar to the one used in the proof of Lemma 7.1 it follows that $\mathcal{C}' = \mathcal{C}_b^* \cup \{\sigma\}$ is compatible. Therefore by Theorem 3.3, there exists a X -cactus \mathcal{N}' for which $\mathcal{C}(\mathcal{N}') = \mathcal{C}'$ holds. Since $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b^* \subset \mathcal{C}_b^* \cup \{\sigma\} = \mathcal{C}'$ it follows by Lemma 5.3(i) and the assumption that $\mathcal{C}_p(\mathcal{N}) = \emptyset$ that $\mathcal{C}(\mathcal{N}) = \mathcal{C}_b(\mathcal{N}) \preceq \mathcal{C}' = \mathcal{C}(\mathcal{N}')$. Hence, $\mathcal{N} < \mathcal{N}'$ in view of Theorem 5.1.

Next, we claim that \mathcal{N}' is a lower bound for $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$. To this end, consider an arbitrary index $1 \leq i \leq m$. Since $\sigma \in \text{glb}(\mu_1, \dots, \mu_m)$, we have $\sigma \preceq \mu_i$ and hence $\mathcal{C}_p(\mathcal{N}') = \{\sigma\} \preceq \mathcal{C}(\mathcal{N}_i)$. Together with $\mathcal{C}_b(\mathcal{N}') = \mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b^* \subseteq \mathcal{C}(\mathcal{N}_i)$ and Corollary 5.6(ii), it follows that $\mathcal{N}' \leq \mathcal{N}_i$. Thus \mathcal{N}' is a lower bound for $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$; a contradiction since $\mathcal{N} < \mathcal{N}'$ and $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. Thus, \mathcal{C}_p^* cannot contain a feasible subset, which completes the proof of Statement (a).

To see that Statement (b) holds, suppose that $\mathcal{C}_p(\mathcal{N}) \neq \emptyset$. Then $\mathcal{C}_p(\mathcal{N}) = \{\sigma_1, \dots, \sigma_k\}$ for some $k \geq 1$. For each $1 \leq i \leq k$, we construct an m -tuple $(\nu_{i,1}, \dots, \nu_{i,m})$ in \mathcal{C}_p^* by letting $\nu_{i,j}$, $1 \leq j \leq m$, be the circular partition in $\mathcal{C}_p(\mathcal{N}_j)$ with $\sigma_i \preceq \nu_{i,j}$ (which exists because $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_j)$). We claim that $H = \{(\nu_{i,1}, \dots, \nu_{i,m}) : 1 \leq i \leq k\}$ is a feasible subset of \mathcal{C}_p^* and that $\mathcal{C}_p(\mathcal{N})$ is a meet realisation of H .

To see that the claim holds, note first that since $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_j)$ for all $1 \leq j \leq m$, it follows that $\nu_{i,j} \neq \nu_{l,j}$ for every pair $1 \leq i < l \leq k$ because a domination map is injective. To see that $\sigma_i \in \text{glb}(\nu_{i,1}, \dots, \nu_{i,m})$ holds for all $1 \leq i \leq k$, assume for contradiction that there exists some $\widehat{\sigma}_i \in \text{glb}(\nu_{i,1}, \dots, \nu_{i,m})$ with $\sigma_i < \widehat{\sigma}_i$. Employing an argument similar to the one used in the proof of Lemma 7.1 it follows that there exists an X -cactus $\widehat{\mathcal{N}}$ with $\mathcal{C}(\widehat{\mathcal{N}}) = (\mathcal{C}(\mathcal{N}) - \{\sigma_i\}) \cup \{\widehat{\sigma}_i\}$ such that $\mathcal{N} < \widehat{\mathcal{N}}$ and $\widehat{\mathcal{N}}$ is a lower bound for $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$; a contradiction since $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. Thus, H is feasible subset of \mathcal{C}_p^* .

To see that $\mathcal{C}_p(\mathcal{N})$ is a meet realization of H , we need to show that $|\text{glb}(\nu_{i,1}, \dots, \nu_{i,m}) \cap \mathcal{C}_p(\mathcal{N})| = 1$. To this end, note that for each pair i, l with $1 \leq i \leq k$, $1 \leq l \leq m$ and $i \neq l$, we have $\sigma_i \notin \text{glb}(\nu_{l,1}, \dots, \nu_{l,m})$ because otherwise $\sigma_i \preceq \nu_{l,1}$ and $\sigma_i \preceq \nu_{l,1}$ both hold; a contradiction in view of $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_1)$ and Lemma 5.4 because $\nu_{i,l} \neq \nu_{l,1}$. Therefore $|\text{glb}(\nu_{i,1}, \dots, \nu_{i,m}) \cap \mathcal{C}_p(\mathcal{N})| = 1$ and so $\mathcal{C}_p(\mathcal{N})$ is a meet realisation of H . This completes the proof of the claim.

It remains to show that H is a maximal feasible subset of \mathcal{C}_p^* . If not, there exists an element $(\nu_{0,1}, \dots, \nu_{0,m}) \in \mathcal{C}_p^* - H$ such that $H' = H \cup \{(\nu_{0,1}, \dots, \nu_{0,m})\}$ is a feasible subset of \mathcal{C}_p^* . Let $\sigma_0 \in \mathcal{C}_p(X)$ be a circular partition in $\text{glb}(\nu_{0,1}, \dots, \nu_{0,m})$. Then we have $\sigma_0 \notin \mathcal{C}_p(\mathcal{N})$. Indeed, assume for contradiction that $\sigma_0 \in \mathcal{C}_p(\mathcal{N})$. Then if $\sigma_0 = \sigma_i$ for some $1 \leq i \leq k$, then we have $\sigma_i \preceq \nu_{0,1}$ and $\sigma_i \preceq \nu_{0,i}$; a contradiction in view of $\mathcal{C}_p(\mathcal{N}) \preceq \mathcal{C}_p(\mathcal{N}_1)$ and Lemma 5.4 because $\nu_{0,1} \neq \nu_{0,i}$. Thus, $\sigma_0 \notin \mathcal{C}_p(\mathcal{N})$. Furthermore, an argument similar to that in Lemma 7.1 shows that $\mathcal{C}(\mathcal{N}) \cup \{\sigma_0\}$ is compatible. Hence there exists an X -cactus \mathcal{N}' with $\mathcal{C}(\mathcal{N}') = \mathcal{C}(\mathcal{N}) \cup \{\sigma_0\}$ in view of Theorem 3.3. By Lemma 5.3(i) and Theorem 5.1, it follows that $\mathcal{N} < \mathcal{N}'$ because $\sigma_0 \notin \mathcal{C}_p(\mathcal{N})$.

Finally, since $\sigma_0 \in \mathcal{C}_p(X)$ we have $\mathcal{C}_b(\mathcal{N}') = \mathcal{C}_b(\mathcal{N})$ and $\mathcal{C}_p(\mathcal{N}') = \{\sigma_0\} \cup \mathcal{C}_p(\mathcal{N}) = \{\sigma_0, \sigma_1, \dots, \sigma_k\}$. Now for an arbitrary index $1 \leq j \leq m$, consider the map $L_j : \mathcal{C}_p(\mathcal{N}') \rightarrow$

$\mathcal{C}_p(\mathcal{N}_j)$ that maps each σ_i to $\nu_{i,j}$, for all $0 \leq i \leq m$. Noting that $\sigma_i \preceq \nu_{i,j}$ for $0 \leq i \leq k$, and $\nu_{i,j} \neq \nu_{l,j}$ for $0 \leq i < l \leq k$ since H' is a feasible subset of \mathcal{C}_p^* , it follows that L_j is a domination map and hence $\mathcal{C}_p(\mathcal{N}') \trianglelefteq \mathcal{C}_p(\mathcal{N}_j)$. Together with $\mathcal{C}_b(\mathcal{N}') = \mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b^*$, Corollary 5.6 implies that \mathcal{N}' is a lower bound of $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$; a contradiction as $\mathcal{N} < \mathcal{N}'$ and $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. Thus H is a maximal feasible subset of \mathcal{C}_p^* .

We now show that the converse direction in the theorem holds. Assume for contradiction that \mathcal{N} is such that the last statement in the theorem holds, but that $\mathcal{N} \notin \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. We distinguish the cases that $\mathcal{C}_p(\mathcal{N}) = \emptyset$ and that $\mathcal{C}_p(\mathcal{N}) \neq \emptyset$.

First assume $\mathcal{C}_p(\mathcal{N}) = \emptyset$. Then, by Corollary 5.6, \mathcal{N} is a lower bound of $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$. Since $\mathcal{N} \notin \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$, there must exist an X -cactus $\mathcal{N}' \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$ with $\mathcal{N} < \mathcal{N}'$. By Lemma 7.1 and our assumption, we have $\mathcal{C}(\mathcal{N}') = \mathcal{C}_b^* = \mathcal{C}(\mathcal{N})$. Hence $\mathcal{C}_p(\mathcal{N}') \neq \emptyset$ in view of $\mathcal{N} < \mathcal{N}'$. Let σ be a circular partition in $\mathcal{C}_p(\mathcal{N}')$. Since $\mathcal{C}_p(\mathcal{N}') \trianglelefteq \mathcal{C}_p(\mathcal{N}_j)$ and $\mathcal{C}_p(\mathcal{N}_j)$ is compatible, for every $1 \leq j \leq m$, let $\mu_j \in \mathcal{C}_p(\mathcal{N}_j)$ denote the necessarily unique circular partition with $\sigma \preceq \mu_j$. Then $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{C}_p^*$. Furthermore, $\sigma \preceq \mu_i$ for all $1 \leq i \leq m$ and $\text{glb}(\mu_1, \dots, \mu_m) \neq \emptyset$ because σ is a lower bound of $\{\mu_1, \dots, \mu_m\}$ in $(\mathcal{C}(X), \preceq)$. Hence, $\{\mu\}$ is a feasible subset of \mathcal{C}_p^* , a contradiction to Statement (a).

Now, assume $\mathcal{C}_p(\mathcal{N}) \neq \emptyset$ so that $\mathcal{C}_p(\mathcal{N}) = \{\sigma_1, \dots, \sigma_k\}$, some $k \geq 1$. Let H be a maximal feasible subset of \mathcal{C}_p^* such that $\mathcal{C}_p(\mathcal{N})$ is a meet realization of H (which must exist by Statement (b)). Then the elements in H can be enumerated as the m -tuples $(\nu_{i,1}, \dots, \nu_{i,m})$ for $1 \leq i \leq k' = |H|$. For $1 \leq i < l \leq k'$, we have $\text{glb}(\nu_{i,1}, \dots, \nu_{i,m}) \cap \text{glb}(\nu_{l,1}, \dots, \nu_{l,m}) = \emptyset$ in view of Lemma 5.4 and the fact that $\nu_{i,1} \neq \nu_{l,1}$ are two distinct circular partitions contained $\mathcal{C}(\mathcal{N}_1)$ and $\mathcal{C}(\mathcal{N}_1)$ is compatible. As $\mathcal{C}_p(\mathcal{N})$ is a meet realization of H , $|\text{glb}(\mu_1, \dots, \mu_m) \cap \mathcal{C}_p(\mathcal{N})| = 1$, for all $(\mu_1, \dots, \mu_m) \in H$, and so it follows that $k' = k$. Swapping the indices if necessarily, we may assume that $\sigma_i \in \text{glb}(\nu_{i,1}, \dots, \nu_{i,m})$ holds for $1 \leq i \leq k$. Fix an arbitrary index $j \in \{1, \dots, m\}$ and consider the map $L_j : \mathcal{C}(\mathcal{N}) \rightarrow \mathcal{C}(\mathcal{N}_j)$ that maps σ_i to $\nu_{i,j}$ for $1 \leq i \leq k$. Since L_j is injective and $\sigma_i \preceq L_j(\sigma_i) = \nu_{i,j}$ holds for $1 \leq i \leq k$, it follows that L_j is a domination map. Hence $\mathcal{C}_p(\mathcal{N}) \trianglelefteq \mathcal{C}_p(\mathcal{N}_j)$ for $1 \leq j \leq m$. Together with $\mathcal{C}_b(\mathcal{N}) = \mathcal{C}_b^*$ it follows by Corollary 5.6 that \mathcal{N} is a lower bound of $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$.

We conclude the proof of the theorem by showing that $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$. Suppose for contradiction that this is not the case. Then there exists an X -cactus $\mathcal{N}' \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$ with $\mathcal{N} < \mathcal{N}'$. Let $\mathcal{C}'_p = \mathcal{C}_p(\mathcal{N}')$. By Lemma 7.1 and our assumption on \mathcal{N} we obtain $\mathcal{C}_b(\mathcal{N}') = \mathcal{C}_b^* = \mathcal{C}_b(\mathcal{N})$. It follows that $\mathcal{C}_p(\mathcal{N}) \trianglelefteq \mathcal{C}_p(\mathcal{N}')$ and $\mathcal{C}_p(\mathcal{N}) \neq \mathcal{C}_p(\mathcal{N}')$. By Lemma 5.4, there exists a unique domination map $L : \mathcal{C}_p(\mathcal{N}) \rightarrow \mathcal{C}_p(\mathcal{N}')$. We consider the following two subcases.

First, suppose that there exists some $\sigma \in \mathcal{C}_p(\mathcal{N})$ with $\sigma \prec L(\sigma)$, that is, $L(\sigma) \neq \sigma$. Without loss of generality, we may assume that $\sigma = \sigma_1$. By Lemma 5.3(ii) and $\mathcal{C}(\mathcal{N}') \trianglelefteq \mathcal{C}(\mathcal{N}_j)$, there exists, for all $1 \leq j \leq m$, a domination map $L_j : \mathcal{C}_p(\mathcal{N}') \rightarrow \mathcal{C}_p(\mathcal{N}_j)$. By Lemma 5.4 it follows that $L_j(L(\sigma_1)) = \nu_{1,j}$ for all $1 \leq j \leq m$. Hence, $L(\sigma_1) \preceq \nu_{1,j}$ for all such j . Thus, $L(\sigma_1)$ is a lower bound for $(\nu_{1,1}, \dots, \nu_{1,m})$; a contradiction to the fact that $\sigma_1 \in \text{glb}(\sigma_{1,1}, \dots, \sigma_{1,m})$ and $\sigma_1 \prec L(\sigma_1)$.

Finally, suppose that $\sigma_i = L(\sigma_i)$ holds for all $1 \leq i \leq k$. Then we have $\mathcal{C}_p(\mathcal{N}) \subseteq \mathcal{C}_p(\mathcal{N}')$. Since $\mathcal{C}_p(\mathcal{N}) \neq \mathcal{C}_p(\mathcal{N}')$, we may choose a circular partition $\sigma_0 \in \mathcal{C}_p(\mathcal{N}') - \mathcal{C}_p(\mathcal{N})$. Since, for all $1 \leq j \leq m$, $\mathcal{C}_p(\mathcal{N}') \preceq \mathcal{C}_p(\mathcal{N}_j)$ and $\mathcal{C}_p(\mathcal{N}_j)$ is compatible, there exists, by Lemma 5.4, a unique domination map $L'_j : \mathcal{C}_p(\mathcal{N}') \rightarrow \mathcal{C}_p(\mathcal{N}_j)$ from $\mathcal{C}_p(\mathcal{N}')$ to $\mathcal{C}_p(\mathcal{N}_j)$. Put $\nu_{0,j} = L'_j(\sigma_0)$. Then $\nu_0 = (\nu_{0,1}, \dots, \nu_{0,m}) \in \mathcal{C}_p^*$ and $\text{glb}(\nu_{0,1}, \dots, \nu_{0,m}) \neq \emptyset$ because σ_0 is a lower bound of $\{\nu_{0,1}, \dots, \nu_{0,m}\}$ in $(\mathcal{C}(X), \preceq)$. Furthermore, for each pair $1 \leq i \leq k$ and $1 \leq j \leq m$, we have $\nu_{0,j} = L'_j(\sigma_0) \neq L'_j(\sigma_i) = \nu_{i,j}$ as L'_j is a domination map and therefore injective. Hence, $H \cup \{\mu\}$ is a feasible subset of \mathcal{C}_p^* ; a contradiction to the assumption that H is a maximal feasible subset of \mathcal{C}_p^* . This establishes that $\mathcal{N} \in \text{glb}(\mathcal{N}_1, \dots, \mathcal{N}_m)$ and therefore completes the proof of the case $\mathcal{C}_p(\mathcal{N}) \neq \emptyset$. \square

Theorem 7.2 immediately implies

Corollary 7.3. *The trivial X -cactus is the greatest lower bound for a set $\{\mathcal{N}_1, \dots, \mathcal{N}_m\}$ of X -cactuses if and only if $\bigcap_{i=1}^m \mathcal{C}_b(\mathcal{N}_i) = \emptyset$ and none of the subsets in $\prod_{i=1}^m \mathcal{C}_p(\mathcal{N}_i)$ is feasible.*

8. DISCUSSION

In this paper, we have introduced a new poset of X -cactuses and shown that it has several interesting structural properties. We conclude by listing some open problems and possible directions for future research.

- Is it possible to characterize upper bounds for sets of X -cactuses, for example, generalizing Theorem 6.2? Also, it is known that a collection of X -trees has an upper bound in $(\mathcal{T}(X), \leq)$ if and only if every pair of trees in the collection does [14, Theorem 3.3.3]. Is this also true for general collections of X -cactuses in $(\mathcal{G}(X), \leq)$?
- As mentioned above, lower bounds for collections of X -cactuses in $(\mathcal{G}(X), \leq)$ are of interest as they could be used as consensus networks. Bearing this in mind, is it possible to find an efficient algorithm to compute a greatest lower bound for a set of X -cactuses? Our results in Section 7 provide some insights into this problem, however, the computational complexity of this problem remains unresolved.
- It could be of interest to further study structural properties of $(\mathcal{G}(X), \leq)$. For example, what are properties of the Möbius function of this poset? Also, are there alternative ways to define partial orderings of $\mathcal{G}(X)$?
- Can encodings and partial orders be defined for other classes of phylogenetic networks, such as “level- k ” networks or “explicit” networks (see [15, Chapter 10] for a recent overview of phylogenetic networks and definitions of these terms).
- Finally, note that the partially ordered set $(\mathcal{T}(X), \leq)$ of X -trees is intimately related to certain complexes and spaces of phylogenetic trees [1, 16]. It would be of interest to understand how the structure of the so-called order complex of $(\mathcal{G}(X), \leq)$ might be related to phylogenetic network spaces such as those described in, for example, [12] and [5].

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REFERENCES

- [1] Federico Ardila and Caroline J Klivans. The Bergman complex of a matroid and phylogenetic trees. *Journal of Combinatorial Theory, Series B*, 96(1):38–49, 2006.
- [2] Garrett Birkhoff. *Lattice Theory*, volume 25. American Mathematical Soc., 1940.
- [3] David Bryant. A classification of consensus methods for phylogenetics. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 61:163–184, 2003.
- [4] Peter Buneman. The recovery of trees from measures of dissimilarity. *Mathematics in the Archaeological and Historical Sciences*, 1971.
- [5] Satyan Devadoss, Cassandra Durell, and Stefan Forcey. Split network polytopes and network spaces. *arXiv preprint arXiv:1905.11225*, 2019.
- [6] Andreas Dress, Vincent Moulton, and Michael Steel. Trees, taxonomy, and strongly compatible multi-state characters. *Advances in Applied Mathematics*, 19(1):1–30, 1997.
- [7] Andreas Dress, Vincent Moulton, and Taoyang Wu. A topological approach to tree (re-)construction. In A. Dress, K-E. Biebler, Cieslik D., and A. Spillner, editors, *The Maths of Flu*, pages 59–75. Shaker Verlag, Aachen, Germany, 2010.
- [8] Philippe Gambette and Katharina T Huber. On encodings of phylogenetic networks of bounded level. *Journal of Mathematical Biology*, 65(1):157–180, 2012.
- [9] Philippe Gambette, Katharina T Huber, and Guillaume E Scholz. Uprouted phylogenetic networks. *Bulletin of Mathematical Biology*, 79(9):2022–2048, 2017.
- [10] Momoko Hayamizu, Katharina T Huber, Vincent Moulton, and Yukihiro Murakami. Recognizing and realizing cactus metrics. *Information Processing Letters*, 157:105916, 2020.
- [11] Katharina T Huber, Simone Linz, Vincent Moulton, and Taoyang Wu. Spaces of phylogenetic networks from generalized nearest-neighbor interchange operations. *Journal of Mathematical Biology*, 72(3):699–725, 2016.
- [12] Katharina T Huber, Vincent Moulton, and Taoyang Wu. Transforming phylogenetic networks: Moving beyond tree space. *Journal of Theoretical Biology*, 404:30–39, 2016.
- [13] Daniel H Huson, Regula Rupp, and Celine Scornavacca. *Phylogenetic networks: concepts, algorithms and applications*. Cambridge University Press, 2010.
- [14] Charles Semple and Mike A Steel. *Phylogenetics*, volume 24. Oxford University Press, Oxford, 2003.
- [15] Mike Steel. *Phylogeny: Discrete and Random Processes in Evolution*. SIAM, 2016.
- [16] Henryk Trappmann and Günter M Ziegler. Shellability of complexes of trees. *Journal of Combinatorial Theory, Series A*, 82(2):168–178, 1998.
- [17] William T Trotter. Partially ordered sets. *Handbook of Combinatorics*, 1:433–480, 1995.

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