Challenging undergraduate students' mathematical and pedagogical discourses through MathTASK activities

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Mathematics education courses in mathematics undergraduate programs often aim to introduce students to the field of mathematics education research (RME) or/and to prepare them for the profession of mathematics teaching. This aim requires a balance of attention to mathematical content together with attention to RME findings. Such balance is the focus of this chapter in which we propose course activities and an assessment frame for undergraduates' engagement with both mathematical and pedagogical discourses. We draw on the MathTASK[1] program to present the design principles of activities that contextualize the use of mathematics education theory and mathematical content to specific learning situations that may emerge in the classroom. Such activities pose mathematical and pedagogical challenge (MC and PC) to often long-held views about mathematics and its pedagogy. Participants were eight final year undergraduate mathematics students who attended a mathematics education course, in which aforementioned activities were part of their formative and summative assessment. Analysis of student responses through a commognitive lens examined the students' reification of mathematical and pedagogical discourse (RMD and RPD) in response to MC and PC by the end of the course. Here, we sample evidence of how students responded to MC (e.g., in terms of mathematical accuracy) and PC (e.g., in terms of routines, narratives and word use in their engagement with RME) and we explore the relation between RMD and RPD in their responses. Our findings highlight the capacity of MathTASK activities to challenge, and potentially shift, students' often deeply rooted mathematical and pedagogical narratives.

Keywords: mathematical discourse, mathematics education discourse, MathTASK, undergraduate mathematics, mathematical challenge, pedagogical challenge, formative and summative assessment

1. Welcoming Mathematics undergraduates to Mathematics Education

Some mathematics undergraduate programs include in their syllabi also courses on mathematics education. The motivation for such courses is to introduce mathematics students to the field of mathematics education research or/and to prepare

[1] MathTASK: https://www.uea.ac.uk/groups-and-centres/a-z/mathtask

them for mathematics teaching. Very often, these courses familiarize students not only with the new content of the social science of education but also with the new, to them, practices of educational research. Research in mathematics education, however, is a very different enterprise from research in mathematics (Schoenfeld, 2000). For example, in mathematics education, in comparison to mathematics, the perspective is less absolutist and more contextually bounded. There is less attention to error and more focus on the reasons behind the error. Approaches are more relativist on what constitutes knowledge (Nardi, 2015) and evidence is not in the form of proof, but rather more “cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt” (Schoenfeld, 2000, p. 649). Thus, findings are rarely definitive; typically, they are more suggestive.

Such epistemological differences affect the experiences of those who, although familiar with mathematics research and practices, are newcomers to mathematics education. Boaler, Ball and Even (2003) analyze the challenges of mathematics graduates when they embark on postgraduate studies in mathematics education. They describe the epistemological shift these students experience in their transition from systematic enquiry in mathematics to systematic enquiry in mathematics education. To facilitate such transition, Nardi (2015) addresses challenges with such epistemological shifts in the context of a postgraduate program in mathematics education that enrols mathematics graduates by proposing an “activity set designed to facilitate incoming students’ engagement with the mathematics education research literature” (ibid, p. 135) through gradual familiarization with the key journals in the field and through co-engineering with the students steps purposefully designed to develop their skills in identifying, reading, summarising and critically reflecting on literature in our field.

In this chapter, we draw on studies that observe and address such shifts at a postgraduate level (Boaler et al., 2003; Kontorovich & Rouleau, 2018; Nardi, 2015; Rouleau, Kontorovich & Zazkis, 2019) to discuss a course that introduces mathematics education to undergraduate mathematics students. Our work contributes to the broader discussion of the challenges that newcomers face as they enter the field of mathematics education research (Kontorovich & Liljedahl, 2018). Specifically, we propose course activities and an assessment frame for undergraduate engagement with both mathematical and mathematics education discourses. Mathematical discourse is related to the mathematical content seen at school and first year university level, whereas mathematics education discourse is related to theoretical constructs and findings of mathematics education research. The proposed activities aim to challenge often long-held views about mathematics as well as about its learning and teaching (pedagogy).

These activities aim to pose both mathematical and pedagogical challenge (MC and PC) to undergraduate students. Mathematical challenge in these tasks resonates with (Applebaum & Leikin, 2014; Leikin, 2014) and concerns tackling a piece of mathematics – that is familiar from the school mathematics curriculum – in diverse and alternative ways. Pedagogical challenge is seen in relation to how respondents may tackle aforementioned mathematical challenge in the context of a specific classroom situation.

In what follows, we describe the theoretical foundations of the course design and its assessment frame and we exemplify the use of this frame in the context of one assessment item. We then describe the course context, objectives, structure, activities and assessment. Subsequently, we outline – and sample from the findings of – a research study of student responses to certain course activities that posed aforementioned mathematical and pedagogical challenges to the students. We conclude with a broader discussion of the potentialities of such activities in undergraduate programs that introduce mathematics students to mathematics education research.

2. A Mathematics Education course for Mathematics undergraduates: Theoretical foundations

The theoretical perspective of this work is discursive and inspired by the commognitive framework proposed by Sfard (2008) which sees mathematics and mathematics education as distinct discourses. Learning of mathematics and learning about mathematics education research (thereafter RME) are then communication acts within these discourses. We are interested in discursive differences – and potential conflicts – between mathematics and RME and we aim towards a balanced engagement with both.

Specifically, we explore how mathematics undergraduates transform what they know about mathematics from their mathematical studies and about mathematics education research (to which they are introduced during aforementioned courses) into discursive objects that can be used to describe the teaching and learning of mathematics. This transformation is the productive discursive activity of "reification" (Sfard, 2008, p. 118). For example, the reification of the theoretical construct *sociomathematical norms* (Cobb & Yackel, 1996) may be evidenced when the construct is used by an undergraduate to describe a classroom situation in which the teacher and the students customarily negotiate different approaches to solving a mathematical problem. At a "meta-level" (Sfard, 2008, p. 300), we are also interested in how an undergraduate may deploy the theoretical construct of sociomathematical norms in the analysis of a classroom situation as an opportunity to reflect on whether there is value in seeking diverse and alternative solutions to a mathematical problem or whether a lesson must always privilege a single, optimal solution as sanctioned by the teacher.

Our course design is informed by three principles, set out in Nardi (2015), for supporting post-graduate students' (Master's and doctoral levels) introduction to RME: "engaged pedagogy and participation; cultural sensitivity; and, independence, creativity and critical thinking" (p. 140) and by her proposed set of activities for such introduction. In these activities, students are asked to engage with literature from RME and to produce accounts of their readings. In addition, students are asked to produce accounts of instances in "their personal and professional experiences that can be narrated in the language of the theoretical perspective" (ibid, p. 151) featured in those readings. These accounts of students' experiences are called *Data Samples*. Engagement with literature, together with the production of Data Samples, aims to support students with situating readings in their own experiences and their engagement with the discourse of mathematics education research. Nardi's (2015) analysis of student interviews and written responses identifies four milestones regarding students' transition from studies in mathematics to studies in mathematics education: *learning how to identify appropriate mathematics education literature*; *reading increasingly more complex writings in mathematics education*; *coping with the complexity of literate mathematics education discourse*; and, *working towards a contextualized understanding of literate mathematics education discourse* (ibid). The fourth milestone, contextualization of the mathematics education discourse triggered by the Data Samples in (Nardi, 2015), is the inspiration for the course activities that are the focus of this chapter.

Rouleau et al. (2019) also adopt the principles and milestones listed in (Nardi, 2015) to design activities for novice in-service mathematics teachers who study a graduate mathematics education course and "their engagement with scholarly mathematics education literature" (p. 43). In their project, teachers engage with scholarly mathematics education literature in activities such as: reading and critiquing pre-set articles; drawing on their own experiences to comment on these articles; using ideas from the articles to design mathematical activities or problems; and, designing a follow up study to the one reported in the articles they read. The study considers teachers and mathematics education researchers as "members of distinct yet closely related communities" (p. 56) that can mutually benefit from the exchange of experiences and practices. The articles from scholarly mathematics education literature have the potential to "act as boundary objects" (p. 56) between the two communities. Findings highlight the complexity and the challenges of teachers' engagement with this task that invite them to participate in researchers' practices which are different to those of the teachers: making "sense of the theories and terminology that the articles used" (p. 57); acquainting themselves with research methodologies that may challenge previously held views and appreciation for certain research designs (e.g. experimental); or, expecting (and experiencing disappointment when not finding) prescriptive suggestions for overcoming students' mistakes in the research literature. Teachers' challenges with engagement with research are also rooted in conflicts between the role of the teacher who is tempted to intervene and help the student and the role of the researcher who observes the learning process from a distance (Kontorovich & Rouleau, 2018). Rouleau et al.'s (2019) work exposes the challenges that lie in efforts to engage teachers with research literature "ranging from choosing a research article with which to engage; to turning it into an object that has the potential to transfer praxeologically foreign knowledge; and finally, to the development of reading praxes themselves" (p. 58).

Although, the studies we review here (Nardi, 2015; Rouleau et al., 2019) are not about undergraduate students, their relevance to the design of the course and the research study discussed in this chapter is in their focus on engaging newcomers to mathematics education research and the potential challenges such engagement may involve.

Another inspiration for the study we report in this chapter comes from our work with pre- and in- service mathematics teachers in the MathTASK program in which we engage teachers with fictional but realistic classroom situations and ask them to reflect on these situations. We call these activities *mathtasks* (Biza, Nardi & Zachariades, 2007). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students (we discuss a mathtask example in section 3).

Teachers engage with these tasks through reflecting, responding in writing and discussing. At the heart of the MathTASK program is the assumption that, theoretical discussion related to the teaching and learning of mathematics is not productive unless it becomes focused on particular elements of mathematics and its teaching embedded in classroom situations that are likely to occur in actual practice (Speer, 2005). The MathTASK design underlies the course activities we sample in this chapter and which aim to challenge (mathematically: MC; pedagogically: PC) undergraduate students' long held narratives about mathematics and its pedagogy.

The mathematical problem, the students' responses and the teacher's reactions in the mathtask are all inspired by the vast array of issues that typically emerge in the complexity of the mathematics classroom and that prior research has highlighted as seminal (Biza et al., 2007). We see the MC in a mathtask as having three components. One component concerns how the mathematical problem in a mathtask is embedded in school mathematics: the task must be appropriate for students at a certain school level, it should motivate students to complete it and it should develop their mathematical curiosity and interest (Leikin, 2014).

A further component concerns the mathematical problem together with the fictional responses to this problem proposed by the students or/and the teacher in the mathtask scenario. These draw on characteristics of a mathematically challenging task identified by the teachers in the study of Applebaum and Leikin (2014): (1) a problem that requires combination of different mathematical topics; (2) a problem that requires logical reasoning; (3) a problem that has to be solved in different ways; (4) an inquiry based problem; (5) a nonconventional problem; (6) a problem that requires generalization of problem results; (7) proving a new mathematics statement; (8) a problem that requires auxiliary constructions; (9) finding mistakes in solutions; (10) a paradox; (11) a conventional problem that requires knowledge of extracurricular topics; (12) a problem with parameters (p. 399).

A third component concerns the ways in which the mathtask may invite our undergraduate students to see beyond the school mathematical content of the task when they solve the problem and interpret fictional student/teacher responses in the incident and relate its contents to mathematics they may have learned during their university studies.

We see the PC components in the mathtask as being about bringing to the fore and reflecting upon a classroom situation from the epistemological position of mathematics (which our undergraduates typically hold) and from the epistemological position of mathematics education (which our undergraduates are starting to recognize). Thus, mathtasks aim to challenge narratives about mathematics and its pedagogy that are reported in the literature as dominant.

For example, these narratives include:

PC1. absolutist and decontextualized views of mathematics (Schoenfeld, 2000);

PC2. attention to error and less focus on the reasons behind the error (Nardi, 2015);

PC3. seeking evidence in the form of proof (e.g., experimental studies) in definite findings and less attention to research methods that justify valid evidence (Rouleau et al., 2019; Schoenfeld, 2000);

PC4. engagement (or lack of) with RME narratives, word use and routines (Nardi, 2015);

PC5. criticality (or lack of) in the engagement with mathematics education literature (Boaler et al., 2003; Nardi, 2015, Rouleau et al., 2019); and,

PC6. expectations of pedagogical prescription from mathematics education literature (Rouleau et al., 2019).

The undergraduates' responses to the mathtasks are analyzed (for the purposes of course assessment, as we explain in section 4, and for the purposes of research, as we explain in section 5) through a typology of four interrelated characteristics (Biza, Nardi & Zachariades, 2018) that emerged from our prior research with mathematics teachers enrolled on a Master's course in Mathematics Education. That research focused on teachers' engagement with mathematics and RME discourses – particularly in relation to mathematics education theories they had been introduced to during the course.

Our typology is as follows:

Consistency: how consistent is a response in the way it conveys the link between the respondent's stated pedagogical priorities and their intended practice? For example, do respondents who prioritize student participation in class propose a response to a classroom situation that involves such participation of students? Or, does their proposed response involve only telling students the expected answer to a mathematical problem?

Specificity: how contextualized and specific is a response to the teaching situation under consideration? For example, do respondents who write generally about their valuing the use of vivid, visual imagery in mathematics teaching, propose a response to a classroom situation that involves specific examples of such imagery? Or, does the response include only a general or generic statement of their preference?

Reification of pedagogical discourse (RPD): how reified is the pedagogical discourse, the theories and findings from research into the teaching and learning of mathematics – that respondents have become familiar with through the course – in their responses? For example, how productively are terms such as "relational understanding" (Skemp, 1976) or "sociomathematical norms" (Cobb & Yackel, 1996) used in the responses?

Reification of mathematical discourse (RMD): how reified is the mathematical discourse – that respondents are familiar with through prior mathematical studies – in their responses? For example, how productively does prior familiarity with natural, integer, rational and real numbers inform a respondent's discussion about fractions in a primary classroom situation? (Biza & Nardi, 2019, p. 46-47).

In the next section, we see the application of aforementioned theoretical foundations in one mathtask.

3. A mathtask: Students discuss how to solve an algebraic inequality

In Figure 1, we present an example of a mathtask. The context of this mathtask is a Year 12 lesson in which the teacher asks the students to solve an algebraic inequality that involves fractions. Three fictional students, Mary, Ann and Georgia, discuss solutions to the problem. The classroom incident is inspired by the difficulties students face when dealing with algebraic inequalities (e.g., Tsamir & Almog, 2001) and the benefits of overt use of erroneous responses to tasks about inequalities in classroom discussions (Schreiber & Tsamir, 2012). Mary's response involves reversing the fractions (and the inequality) without distinguishing whether the numbers are positive or not. As a result, she misses the point that the inequality does not have a solution for $x \leq 0$. The correct solution to the problem is $0 < x < 2$. Ann and Georgia challenge Mary's choice and trigger an inductive explanation with one example: if $3 > 2$ then $1/3 < 1/2$. Georgia seems convinced by this explanation. Ann however expresses concerns about the explanation and its capacity to result in receiving full marks.

The undergraduates are invited to solve the problem (Q1); to think about the potential aims of giving such a task to students in class (Q2); to reflect on potential issues evidenced in Mary's, Ann's and Georgia's responses (Q3); and, to respond to Mary, Ann and Georgia and to the whole class (Q4).

In a Year 12 lesson, Mr Smith has asked the students to solve the following inequality:

$$"\frac{1}{x} > \frac{1}{2}"$$

Mary, Georgia and Ann are working on the problem while Mr Smith is observing from a distance without intervening:

Mary: *x* is less than 2 [she writes in her notebook: $x<2$]

Ann: Well, how do you know this?

Mary: If you inverse the numbers, the big number becomes small …

Georgia: Mmmm?

Mary: Three is bigger than two [she writes: 3>2], so one over three is less than one over two and the other way round [she writes: $\frac{1}{3} < \frac{1}{2}$].

Georgia: So, if one over x is more than one over 2, then x is less than 2. That's it!

Ann: This sounds too simple to me. I do not feel that this explanation is enough to get full marks.

Questions:

P6.1. Solve the inequality: $\frac{1}{x} > \frac{1}{2}$

P6.2. What is the aim of giving this task to the students?

P6.3. What are the issues in Mary's, Georgia's and Ann's responses?

P6.4. How would you respond to Mary, Georgia, Ann and the whole class?

Figure 1: A mathtask from the course's portfolio of learning outcomes

In terms of MC in the mathtask in Figure 1, undergraduates are invited to solve the problem (Q1) and to identify the issues in students' responses (Q3). From the Applebaum and Leikin (2014) list of (1)-(12), see section 2:

- the problem and the interpretation of fictional student responses require a combination of different mathematical topics (see (1), ibid, p. 399) – e.g., meaning, properties and graphical representations of inequalities and variables;
- the problem and fictional student responses require logical reasoning (see (2), ibid, p. 399) – e.g., why is multiplication with x^2 a correct approach? Or, why does inversing the numbers not necessarily imply inversing the inequality? Or, does Mary's trial of numbers constitute acceptable justification?
- the problem can be solved in different ways (see (3), ibid, p. 399) – e.g. graphical solution, distinguishing cases, multiplying by x^2; and,
- fictional student responses have errors that should be identified, interpreted and acted upon (see (9), ibid, p. 399) – e.g., "[i]f you inverse the numbers, the big number becomes small", Mary says.

In terms of PC in the mathtask in Figure 1, from the list of aforementioned challenges (PC1-PC6), see section 2:

- fictional student responses should be seen in the context of the classroom incident (Year 12 lesson) and the exchanges between Mary, Ann and Georgia (see PC1) – e.g., why does Ann say "[t]his sounds too simple to me. I do not feel that this explanation is enough to get full marks"?;
- errors in fictional student responses should be identified with attention to the reasons behind the error (see PC2) – e.g.: students' intuitive beliefs

about the order of inverse numbers are in conflict with formal properties of numbers; or, students draw on inappropriate analogies between processes applied to equations to processes applied in inequalities; or, students tend to multiply both sides of the inequality with x (Schreiber & Tsamir, 2012; Tsamir & Almog, 2001);

- engagement (or lack of) with RME narratives, word use and routines (see PC4) – e.g., we ask undergraduates to use RME theory and terminology introduced in the sessions in their responses to Q2-Q4;
- critical engagement with mathematics education literature is necessary (see PC5) – e.g., we ask undergraduates to use the literature in their responses to this item (identification of the issues in question Q3 and response to the students in Q4);
- moving beyond prescriptive suggestions from mathematics education literature (see PC6) – e.g., we expect undergraduates to provide a response to the students (see Q4) with the expectation to transform the findings from the literature to pedagogical recommendation and not teaching prescriptions.

We now describe the course context, objectives, structure, activities and assessment.

4. The Course: Context, objectives, structure, activities and assessment

The mathematics education course entitled *The Teaching and Learning of Mathematics* is offered as optional to final year (Year 3) mathematics undergraduate students (BSc in Mathematics) in a research-intensive university in the UK. The aim of the course is to introduce undergraduates to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum (Biza & Nardi, 2020). The learning objectives of the course include: to become familiar with learning theories in mathematics education; to be able to critically appraise research papers in mathematics education; to be able to compose arguments regarding the learning and teaching of mathematics by appraising and synthesizing recent literature; to become familiar with the requirements of teaching mathematics; to become familiar with key findings in research into the use of technology in the learning and teaching of mathematics; and, to practise reading, writing, problem solving and presentation skills with a particular focus on texts of theoretical content, yet embedded in key issues in RME.

Teaching activities, led by Biza, include four hours per week (two for lectures and two for seminars) for a period of twelve weeks. In the lectures, theoretical course content is introduced. In the seminars, undergraduates present and discuss their work that involves preparing presentations of papers they have read, identifying examples from their experience (Data Samples, as per Nardi, 2015), solving problems and reflecting on their solution; and, responding to mathtasks (Biza et al., 2007). Undergraduates are encouraged to upload their contributions in a shared folder before the session. Discussion during the seminars typically draws on their uploaded contributions. In the middle of the course, for the purpose of formative assessment, they are asked to produce a response of about 800 words to a mathtask. Summative assessment is at the end of the course in the format of a portfolio of learning outcomes that involves: questions on mathematics education theory; reflection on undergraduates' own learning experiences in mathematics; solving a mathematical problem and reflecting on the solution; and, responding to mathtasks. Opportunities for verbal and written feedback are interspersed across the seminars. There is also written feedback to formative and summative pieces of writing and a feed-forward session for discussing this feedback once summative assessment is complete.

The mathtask in Figure 1 was in the portfolio of learning outcomes (summative assessment) at the end of the course in a recent academic year. The undergraduates are asked to use mathematics education theory introduced in the course in their preparation of their responses to the task – and their portfolio entries overall:

> In your responses, you are expected to deploy terms that we introduced and used throughout the [course] sessions. You are also expected to refer to a small number (one or two) of research or professional publications in each part […] in addition to the essential publications used in the sessions. (Portfolio guidelines)

Marking criteria are presented in Figure 2 ('arguments and understanding' section adapted from the marking sheet template given to the students). Of those criteria, *consistency*; *specificity*; *use of terms and constructs from mathematics education theory*; and, *use of terms and processes from mathematical theory* are elaboration of the typology of four characteristics (Biza et al., 2018) – consistency, specificity, reification of pedagogical discourse and reification of mathematical discourse – we introduced in section 2.

Portfolio marking criteria:

Arguments and understanding

1. Clarity: How clear, justified and transparent the arguments are.
2. Coherence: How logically connected the arguments are.
3. Consistency: How consistent the arguments are across the text.
4. Specificity: How contextualised and specific the arguments are in the used examples and the discussed situations.
5. Use of terms and constructs from mathematics education theory: How precise and accurate the arguments are in relation to the used mathematics education constructs and terms.
6. Use of terms and processes from mathematical theory: How precise and accurate the arguments are in relation to the used mathematical terms and processes, such as definitions and proofs.

Figure 2: Portfolio of learning outcomes marking criteria

Once the undergraduates' responses to the mathtasks are marked (for the purposes of course assessment), the work of those students who have consented to the use of their work for research purposes, is analyzed through the aforementioned typology of four characteristics (Biza et al., 2018). In what follows, we present findings from this analysis. First, we introduce the participants, the data and the data analysis method.

5. A research study of student responses to a mathtask: Participants, data collection and data analysis method

Of the cohort of thirteen mathematics undergraduates enrolled on the course we described above, eight consented to their work being used for research purposes after the completion of the course assessment. These eight undergraduates are the participants of the study and, at the time of data collection, they were in the third year of a 3-year undergraduate course in Mathematics. For the purposes of this study, we analyzed responses to the mathtask in Figure 1.

As this course aimed primarily to introduce undergraduates to the field of RME, a particular focus of our analysis is on manifestations of undergraduates' engagement with reading, writing, reflecting upon and using the constructs of RME (theory and findings in our field) by the end of the course (PC) and in connection to the mathematical accuracy of their responses (MC). To this purpose, our analysis draws on the typology of the four characteristics (Biza et al. 2018) which we introduced in section 2 and also underpins the marking criteria of the assessment as we described in section 4. Specifically, we aim to identify evidence of reification of the undergraduates' pedagogical discourse (RPD) in tandem with reification of mathematical discourse (RMD). The analysis we sample in what follows naturally weaves in references to the other two characteristics of our typology: specificity and consistency.

6. Analysis of student responses to a mathtask

We start the presentation of the analysis of the undergraduates' responses to the mathtask in Figure 1 by discussing first these responses in terms of reification of mathematical discourse (RMD) as evidenced in how respondents engage with the mathematical challenge (MC) of the problem and the responses of the fictional students (6.1). Then, in the light of the RMD observations, we discuss reification of pedagogical discourse (RPD) as evidenced in how respondents engage with the pedagogical challenge (PC) posed by the situation in the mathtask. Specifically, we explore four themes (6.2-6.5) that emerged from our commognitive analysis of the undergraduates' responses to this mathtask: *engaging with the RME routine of referencing relevant literature (explicitly or implicitly)*; *endorsing the RME narrative of the importance of considering social interactions during mathematical activity*; *ritualized engagement with RME theory and findings*; and, *RME theory as a descriptor of pedagogical prescription*.

6.1. RMD in responses to the mathematical challenge of the mathtask

Of the eight participants, Isaac, Shaun and Tim, agreed with Mary that $x < 2$ is the right response to the problem. They justified their choice by multiplying both sides of the inequality with 2 and x without noticing that x might be a negative number (see Issac's response in Figure 3).

$\frac{1}{x} > \frac{1}{2}$	Multiply both sides by 2
$= \quad \frac{2}{x} > 1$	Multiply both sides by 'x'
$= \quad 2 > x$	Reverse inequality (flip sign and swap sides)
$= \quad x < 2$	

Figure 3: Isaac's response to the problem

The remaining five participants spotted the flaw in Mary's response and solved the problem

- by multiplying both sides with $x^2 > 0$ and then solving the inequality $x(x\text{-}2) < 0$ (Max, Figure 4);
- by distinguishing cases for $x < 0$, $x = 0$ and $x > 0$ and solving the problem in each case (Nicole and Lawrence);
- by saying that the inequality cannot be true if x is not positive and then solving the problem for positive x only (Penny); and,
- by making the graph of the corresponding function and identifying the parts of the graph that satisfy the inequality (Harry).

As for all $x, x^2 > 0$, $\frac{1}{x} > \frac{1}{2} \Rightarrow \frac{x^2}{x} > \frac{x^2}{2} \Rightarrow x > \frac{x^2}{2}$ $[provided\ \ x \neq 0]$

$$\Rightarrow 0 > \frac{x^2}{2} - x \Rightarrow 0 > x^2 - 2x \Rightarrow x^2 - 2x < 0 \Rightarrow x(x-2) < 0$$

Critical Values at $x = 0$ and $x = 2$
Test values of x at the following ranges:

$$x < 0 \Rightarrow x - 2 < -2 < 0.$$

$As\ x < 0$ and $x - 2 < 0, x(x-2) > 0$, making the original inequality a false statement at $x < 0$

$0 < x < 2 \Rightarrow x > 0$ and $x < 2$. $x < 2$ implies that $x - 2 < 0$
As $x > 0$ and $x - 2 < 0, x(x-2) < 0$, making the original inequality true.

$$x > 2 \Rightarrow x > 0\ and\ x - 2 > 0$$

As $x > 0$ and $x - 2 > 0$, $x(x-2) > 0$, making the original inequality false.

In summary, for the original inequality to be true, $0 < x < 2$

Figure 4: Max's response to the problem

We focus now on undergraduates' reflections on the incident in the mathtask as evidenced in their responses to Q2-Q4 in Figure 1. We focus particularly on evidence in their responses of engagement with the RME discourse (theories and findings) they had been introduced to during the course.

6.2. Engaging with the RME routine of referencing relevant literature (explicitly or implicitly)

Unsurprisingly, given the portfolio guidelines and our emphasis in the course sessions, all undergraduates engage to some extent with RME narratives, word use and routines in their responses to the mathtask in Figure 1. Responses draw on theoretical constructs and findings discussed in the course as well as in additional ones found in publications beyond the course resources (PC4). Such engagement is done however at different levels of criticality (PC5).

In some cases, reflections on the incident are well aligned with RME narratives and word use. For example, Nicole writes in her response to the students and the whole class:

> I would also show a graph of y=1/x on the whiteboard and the area where y>1/2, which would reinforce the learning and illustrate the complications of x=0. Tsamir and Almog (2001) found that inequalities were usually solved correctly when graphs were used, with common problems being not rejecting excluded values, and using techniques that apply to equations but not inequalities. (Nicole, Q4)

Nicole proposes the use of a graphical approach as a response to the students and justifies this choice by drawing on relevant literature.

In other cases, the enacted words and narratives are tangentially relevant to the incident under discussion. Harry, for example, in his response to the students, acknowledges the benefits of "constructive conversations" with students and wants to promote more "structure" in student responses through the techniques of problem solving:

> My response to the students would be to first recognise the constructive convers[at]ions they were having with one another to come up with a solution. However, I would then point out to them that their responses have a lack of direction or structure. To address this issue, I would then recommend the students follow Polya's Problem Solving Process. In his book Polya outlines four stages for solving problems. These stages are (Polya, 1957)[2]:
> [… Pólya stages follow]
> In studies such as that by (Griffin & Jitendra, 2009)[3] it found when techniques like Polya's was used this led to an increase in student's problem-solving performance. Therefore, by giving learners this instruction, over time it will become a sociomathematical norm to follow this method of problem solving ensuring that proofs are constructed better in the future." (Harry, Q4)

Although problem solving is at the heart of most mathematical activities, Harry's attempt to connect the situation in the mathtask to Pólya's stages on problem solving is commendable in principle but arguable in its realization. While we see value in his attempt to establish a new sociomathematical norm related to a structured approach to proving and problem solving in the classroom he has been invited to imagine teaching in, his recommendation is related to the specific situation in a tangential, generic manner. It seems to us therefore that Harry's response lacks specificity to the situation in the mathtask – this type of response could be given to many, almost any, classroom situation that involves students talking to each other during problem-solving. We may discern here therefore engagement with the "rituals" (Sfard, 2008, p. 241) of RME discourse: Harry knows he is expected to demonstrate awareness of RME works and does so in a generic manner to fulfil his task-completion obligation. We return to this point in the next sections.

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[2] Pólya, G. (1957). *How to Solve it.* 2 ed. New York: Doubleday Anchor Books.

[3] Griffin, C., & Jitendra, A. (2009). Word problem-solving instruction in inclusive third-grade mathematics classrooms. *The Journal of Educational Research, 102*(3), 187-202.

While Nicole and Harry engage explicitly with RME literature through referencing specific works – a routine in RME discourse that the undergraduates were explicitly encouraged to engage with – other undergraduates did not do so. However, even in those cases, the RME terms used in the course do appear in their responses to the mathtask. Shaun, for example, writes:

> With regards to Mary, it seems that she does have an understanding of how inequalities work when combined with fractions. Although this shows a low level of <u>relational understanding</u> her explanation of her method lacks formal language showing us that she has not yet fully grasped the <u>sociomathematical norms</u> of the class level. She seems to have applied an <u>inductive reasoning</u> to her approach, and although true cannot be relied upon as a <u>formal proof</u>. (Shaun, Q3, our underlining)

Shaun does not see the flaw in Mary's solution and focuses his critique on her justification and whether such justification (inductive reasoning) is considered as acceptable or not in the classroom (sociomathematical norms).

Similarly, Tim, who also did not spot the flaw in Mary's solution, proposes a discussion in the classroom about different "proofs" of the problem, which will be more "deductive" and "convincing":

> I would then ask the other students whether they had any proofs as to why she is right that they prefer to Mary, to see if any of the students would have a proof that is more <u>deductive</u> in style and then ask the class which proof they found <u>more convincing</u> and why. (Tim, Q4, our underlining)

In Shaun's and Tim's responses, we see the enactment of terms (from mathematics and from RME) used in the course as well integrated in their argument, although the relevant literature from which these terms have been drawn is not explicitly referenced. We see such word use as implicit engagement with RME discourse, but we note that engagement with the routine of explicitly referencing the relevant literature is not present. Also, returning to the observation that both Shaun and Tim have not spotted the error in Mary's response, we note that their discussion focuses on the mode of the argument (e.g., inductive vs deductive) and not on the mathematical flaw of the argument. We see this as a missed opportunity to bring in RME literature that proposes potential explanations of the reasons behind such errors and ways to address them.

6.3. Endorsing the RME narrative of the importance of considering social interactions during mathematical activity

In the participants' responses, we observed evidence where the incident in the mathtask was seen beyond its mathematical focus, as an excerpt of student interactions in class. Such responses are attentive to students' learning activity, to the interaction between students or to the norms of the fictional class in the task. We consider such evidence as an indication of the participants' "meta-level learning" (Sfard, 2008, p.300) about a common RME routine: a thoughtful consideration of student contributions in class requires that they are not simply seen as right or wrong (PC1 and PC2).

Nicole, for example, mentions that:

> […] in Vygotsky's (1978)[4] socio-cultural framework the group work enables benefit from the Zone of Proximal Development (ZPD, what they can learn with the support of more knowledgeable others); and from scaffolding (the support they get from others e.g. students, the teacher). (Nicole, Q2)

Later, she returns to this point when she writes "Georgia has understood Mary's response, so has derived some benefit from the ZPD. However she has not noticed Mary's error […]" (Nicole, Q3). Although in Nicole's response the use of ZPD is not precise – it is worded as an approach that can benefit student learning – we can see her attending to the interaction between the students and the potential contribution of this interaction to students' learning. She warrants her support for this type of interaction with her – not-so-precise but appropriately selected – reference to ZPD.

Similarly, Isaac comments on the interaction (and scaffolding) between Georgia and Mary:

> Georgia is hesitant to give her own answer until she hears Mary's explanation for her answer, where she simply agrees. Georgia's agreement with Mary does imply that her concept is expanding and reveals the working of scaffolding between Georgia and Mary within the class. (Isaac, Q3)

Isaac has not spotted the error in Mary's response and his attention is mostly on the justification of why $x < 2$ is the right response and the communication of this justification. In particular, he discusses how Mary tries to persuade Georgia and Ann:

> With Mary's response, observations show that she seems confident in her answer, and is prepared to give answers for how she solved the question. She has a persuading proof, removing doubts the others have (Harel & Sowder, 2007[5], p. 6). Mary's justification and mathematical reasoning does not meet the expected standard for a year 12 class. It would be assumed that in her class, there would be socio-mathematical norms set in which mathematically proving and justifying answers. (Isaac, Q3)

Similarly to Isaac, several students aptly – if not always with precise wording – reference the construct of sociomathematical norms (Cobb & Yackel, 1996) to discuss Ann's concerns whether Mary's solution is enough to receive full marks. Nicole writes in Q3: "Ann appears to have understood Mary's response, but thinks the explanation is too simple. She has considered the sociomathematical norm of 'what counts as an acceptable mathematical explanation' (Cobb and Yackel, 1996, p.178)".

Penny also sees the establishment of sociomathematical norms in the aims of using such a problem in a Year 12 class when she writes that

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[4] Vygotsky, L. S. (1978). *Mind and society: The development of higher mental processes.* Cambridge, MA: Harvard University Press.

[5] Harel, G., & Sowder, L. (2007). Toward Comprehensive Perspectives on the Learning and Teaching of Proof. In F. Lester, (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning, National Council of Teachers of Mathematics*. Greenwich: Information Age Publishing, pp. 805-842.

> [the problem] also aims to tackle a sociomathematical norm explored previously, where students may have the idea that longer, more complicated answers are usually worth more marks, so this task can produce evidence of problems that conflict with this notion and challenge it. (Penny, Q2)

Overall, although some of the participant statements are not precise – e.g., "[Georgia] has derived some benefit from the ZPD" (Nicole) – we see participants' attending to issues of student interaction and the social / sociomathematical norms of the mathematics classroom as endorsement of a common RME narrative: social interactions, not only the mathematical content of such interactions, is a significant and worthy focus of attention when we consider students' contributions in class. The explicit attending to these social interactions in the participants' responses evidences part of what we see as their becoming social scientists, in tandem with being mathematicians: they are endorsing the priorities, foci and methods of the social science of mathematics education while remaining attentive to the mathematical focus of the classroom incident under scrutiny.

6.4. Ritualized engagement with RME theory and findings

We have already seen examples in which engagement with RME narratives and words is relatively inaccurate (e.g. Nicole's reference to ZPD) or not so relevant to the situation (e.g. Harry's generic connection to the problem solving literature). Having in mind that these responses were produced in the context of summative assessment, we acknowledge the undergraduates' understandable effort to appear as knowledgeable and appreciative users of the RME terms introduced in the course (PC4) for the purpose of achieving a higher mark. We detect therefore that they may do so in a ritualized way.

Max uses the constructs of "concept image" and "evoked concept image" (Tall & Vinner, 1981) to describe students' exchanges in the incident. He seems to use "concept image" to describe students' deficiencies:

> Mary's concept image "may cause problems" (Tall and Vinner, 1981) as it does not take into account the cases where x is less than 0 […] Georgia does not have a concept image […] There is "conflict" (Tall and Vinner, 1981) in Ann's concept image, most likely leading to her confusion" (Max, Q3, his quotation marks).

Max's reflection develops around the adequacy or not of Mary, Ann and Georgia's concept image and whether they can see that x might be a negative number. In his response to Q4, he does not attempt a reconstruction of their contributions or address the conflict that may emerge from these contributions. He merely proposes a correct solution to the problem instead. Max's response indicates confidence with the mathematical content (RMD) but also a tendency to focus on what he sees as important: the correctness, or otherwise, of the students' contributions. He takes a largely deficit perspective on these contributions and resorts to the RME literature through a superfluous reference to "concept image" (possibly because he thinks that such a reference may help him gain marks). His alignment with the words, routines and narratives of the RME discourse may therefore be seen as ritualized.

We saw earlier Nicole proposing a visual approach (graphing functions $1/x$ and 1/2 and showing where the former lies above the latter) for her response to the class and grounding this choice in relevant literature. A similar proposition came from Penny: she "would have them draw the graph of $1/x$ to help them visually understand and identify what values x can take in this situation" (Penny, Q4). She justifies this proposition as follows:

> Using a more visual method could also potentially aid those that would be considered Visual Spatial Learners who may struggle to understand problems without a visual representation, as detailed by Rapp[6] (2009) in her paper on the subject. (Penny, Q4).

The reference here to the "Visual Spatial Learners" is one that does not resonate with the focus and principles of the course that explicitly fostered an avoidance of crude characterization of *learners* (as visual, analytic or kinesthetic, for example) and encouraged characterizations of *learning* (and, even more, of learning *in context*). We are aware though that such characterizations proliferate amongst practitioners who find them readily helpful when they plan differentiated activities in their lessons. It is not unlikely that Penny's response may be influenced by recall of uses of such characterizations by, for example, her teachers when she herself was in school.

Yet, Penny's response continues with at least two references directly from those RME works introduced in the course, Ball et al's (2008)[7] Specialized Content Knowledge (SCK) and sociomathematical norms (Cobb et al, 1996).

> This would also fall within Ball, Thames and Phelps' Specialized Content knowledge (Ball, Thames & Phelps, 2008), as without my own understanding of how the problem may be related to graphs, this would not be a viable method. It would also be beneficial to explain that when x is taken to be a non-zero positive, Mary's method would work, but not in all cases, with the above example given, so the students could understand from their own work and thinking where the issues arise. Solving the issue alongside the students could also potentially combat the aforementioned sociomathematical norm that answers must be complicated for high marks. (Penny, Q4)

We see in Penny's response an attempt to bring elements from the literature, some of them directly relevant to the incident (sociomathematical norms) and some a little less directly so (specialized content knowledge). We note however the reflective element in her response when she quotes SCK: for a teacher to be able and willing to offer a confident alternative to solutions proposed by their students, her own SCK needs to be confident. We see merit in Penny's efforts to discern teacher-related issues in a mathtask incident that at face value seems to be largely about learners.

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[6] Rapp, W.H. (2009) Avoiding math taboos: Effective math strategies for Visual Spatial Learners. *Teaching Exceptional Children Plus,* 6(2), 2-12.

[7] Ball. D.L., Thames, M.H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*, 389-407.

6.5. RME theory as a descriptor of pedagogical prescription

We now return to Nicole's response that "group work enables benefit from the Zone of Proximal Development" and that "Georgia has […] derived some benefit from the ZPD". In this statement, ZPD seems enacted not as a tool to explain Georgia's meaning making in her interaction with Mary, but as a didactic approach with potential benefits for learning. Lawrence's response to Q4 evidences a tendency we discerned in the undergraduates' responses to deploy RME theoretical constructs not as explanatory tools but as recommendations for effective teaching practice (PC6):

> Firstly, I would use the teaching triad which introduce by Barbara Jaworski (1994)[8]. According to Management of Learning I have to split the classroom into groups and give them some examples and tasks to check whether the students are familiar with negative numbers and of course negative inequalities. Also, I have to remind them the principle that if we inverse the positive numbers of the inequality then the sign of the inequality changes. For example, if we have two positive numbers 5 > 4 then 1/5 < 1/4. Also, if we multiply an inequality with negative number then the sign of the inequality changed. For instance, let 3 < 6 then if we multiply by -1 both sides then -3 > -6. In addition, when I would finish with these examples, I would encourage the students to participate into a dialogue with the aim to realise if they understand these principles (sensitivity to students). Furthermore, I will split the problem into three cases. First case, when $x < 0$ second case when $0 < x < 2$ and third when $x > 2$. Also, I can use a program like desmos[9] to sketch graphs and I can sketch the graph $y = 1/x$ and $y = 1/2$ and try to find values where $y=1/x$ is above $y=1/2$. Lastly, I would ask the students for any "challenges to engender mathematical thinking and activity" (Potari, D., & Jaworski, B. 2002 p.352-353)[10] with this task and any questions that they appear (Mathematical Challenge)." (Lawrence, Q4)

For Lawrence, Jaworski's Teaching Triad (1994) is not, as its author intended, a lens through which to analyze classroom events; it is instead an alert to three areas of concern that a teacher needs to address: how to manage classroom activity; how to address student needs with sensitivity; and, how to provide precise mathematical support. We discern in this, and other responses of this ilk, a tendency of our newcomers to see RME as an applied field that is able and willing to provide pedagogical prescription. As such, RME theoretical constructs are often construed by our participants not as interpretive instruments but as alerts to what the field prescribes as pedagogically efficient.

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[8] Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry.* London: Falmer Press.

[9] https://www.desmos.com

[10] Potari, D., & Jaworski, B. (2002). Tackling complexity in mathematics teaching development: Using the teaching triad as a tool for reflection and analysis. *Journal of Mathematics Teacher Education, 5*, 351-380.

7. How facing the MC and PC in mathtasks works as a boot-camp experience for newcomers into RME discourse

In this chapter, we present a course that aims to introduce third year undergraduate mathematics students to the field of mathematics education research (RME) by deploying certain course activities and their assessment frame. The course activities are inspired by studies that have identified the epistemological differences between practices in mathematics and mathematics education (Boaler et al. 2003; Kontorovich & Rouleau, 2018; Nardi, 2015; Rouleau et al., 2019; Schoenfeld, 2000) and have addressed these differences in the learning of postgraduate students (Nardi, 2015; Rouleau et al., 2019). Specifically, in this chapter, we focus on one specific type of activity (mathtask) inspired by the principles of the MathTASK program (Biza et al., 2007) that contextualizes the use of RME theory and the mathematical content in specific learning situations. Mathtasks aim to pose both mathematical and pedagogical challenge (MC and PC) to undergraduate students. Undergraduates' responses to such challenges are analyzed for the purposes of course assessment and for the purposes of research through an adaptation of a typology of four interrelated characteristics (Biza et al., 2018): consistency; specificity; reification of pedagogical discourse; and, reification of mathematical discourse. In this chapter, we present findings from the analysis of evidence of reification of mathematical and pedagogical discourses (RMD and RPD, respectively) in the responses of eight undergraduates.

With regard to RMD in response to MC (Applebaum & Leikin, 2014; Leikin, 2014), three undergraduates erroneously multiply both sides of the inequality with x (Schreiber & Tsamir, 2012; Tsamir & Almog, 2001) and conclude with the same incorrect solution, $x < 2$ (as student Mary in the mathtask): they cannot see the error in Mary's response but comment on how Mary warrants her response. The remaining five undergraduates present a range of mathematically valid responses.

With regard to RPD in response to PC (PC1-PC6, see section 2), our analysis highlights that the undergraduates engage with RME literature either explicitly, with the use of theoretical constructs connected to citations of relevant studies, or implicitly, with the use of theoretical constructs without the appropriate citations (PC4). Sometimes this engagement is at different levels of criticality (PC5). The undergraduates do not always realize that an argument in the social science of RME needs to be supported by evidence, either of a first order – namely, data they collected themselves – or of a second order – namely, findings published in peer-reviewed RME outlets (Nardi, 2015).

The literature that the undergraduates choose to reference varies from specific to the topic under discussion to generic and less relevant. We see this as an attempt to gain marks in the course assessment and earn the lecturer's approval: they need to appear as knowledgeable and appreciative users of the terminology the lecturer introduced in the sessions. We see this as ritualized engagement with RME for the purpose of being accepted as a member of the RME community. One of these rituals is referencing the work of eminent members of the community. We see such engagement as a productive, albeit imperfect, path in the epistemological shift from mathematics to mathematics education. More nuanced enculturation can follow.

Furthermore, we observed how the undergraduates attend to social or institutional aspects of the mathematical activity that is contextualized (PC1), goes beyond considering the mathematical correctness of the students' contributions in class and pays attention to group work, student interaction and sociomathematical norms (PC2). In doing so, the undergraduates sometimes conflate theoretical constructs – intended as interpretive tools in the analysis of learning and teaching situations in mathematics – with pedagogical prescriptions (PC6). We see this as a natural step from the prescriptive and normative position that theory may hold in the natural sciences and mathematics to its more interpretive and reflective role in the social sciences. And, again, we see this as a place from which more nuanced enculturation can follow.

RMD is strongly related to RPD. We observed the interface of attending (or not) to certain mathematical issues in the classroom situation presented in the mathtask with the noticing (or not) of certain details of a pedagogical nature. For example, undergraduates who did not spot the mathematical error in the incident tended to focus their attention on how the solution is communicated. Although they reflected on what an acceptable proof would have been – e.g. differences between deductive and inductive proof, persuading others etc. – the opportunity to discuss the mathematics of the problem and to address associated student needs eluded them. This observation illustrates the potency of activities that pose both MC and PC. Discussion of issues related to mathematics as well as to the learning and teaching of mathematics are better situated when MC and PC are seen in synergy. For those who are engaged, or intend to engage with mathematics teaching, mathematical content can be better seen in the context of classroom situations – and pedagogy can be better supported by relevant mathematical content.

We see the potency of the course activities we present in this chapter to welcome mathematics undergraduates into RME in a manner that balances engagement with mathematics and mathematics education discourses productively. Also, we see how the findings from this study can inform us about how undergraduates' epistemological transition from the sciences to the social sciences can be facilitated and how such findings can provide tools for nuanced and targeted formative feedback. Finally, we see this work as contributing to the ongoing endeavor in our field to support the entry of newcomers with diverse backgrounds to mathematics education research (Kontorovich & Liljedahl, 2018).

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