# Planar Phylogenetic Networks

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**Abstract**—A phylogenetic network is a rooted, directed acyclic graph, with a single root, whose sinks correspond to a set of species. Such networks are commonly used to represent the evolution of species that have undergone reticulate evolution. Recently, there has been great interest in developing the theory behind and algorithms for constructing such networks. One issue with phylogenetic networks is that, unlike rooted evolutionary trees, they can be highly non-planar (i.e. it is not possible to draw them in the plane without some arcs crossing). This can make them difficult to visualise and interpret. In this paper we investigate properties of planar phylogenetic networks and algorithms for deciding whether or not phylogenetic networks have certain special planarity properties. In particular, as well as defining some natural subclasses of planar phylogenetic networks and working out their interrelationships, we characterize when the well-known level-$k$ networks are planar as well as showing that planar regular networks are closely related to pyramids. Our results make use of the highly developed field of planar digraphs, and we believe that the link between phylogenetic networks and this field should prove useful in future for developing new approaches to both construct and visualise phylogenetic networks.

**Index Terms**—Phylogenetic network, planar digraph, upward planar, level-$k$ network, pyramid, regular network



## 1 Introduction

Phylogenetic networks are a generalization of evolutionary trees that are commonly used to study species whose evolution have involved reticulate processes such as recombination, hybridization and lateral gene transfer. These networks have been used to analyze the evolution of organisms such as viruses, plants, bacteria, as well as animals (see e.g. [1]). In general, phylogenetic networks come in two main types: Those that aim to represent evolutionary patterns in the data, and those that try to represent the actual evolutionary history (sometimes called *implicit* and *explicit* networks, respectively; see e.g. [2]). Note that for the special case of evolutionary trees, implicit and explicit networks can be thought of as corresponding to unrooted and rooted trees, respectively. More generally, implicit networks tend to be undirected graphs whereas explicit ones are directed.

Although the concept of a phylogenetic network has been considered in the literature for quite some time, the first algorithmic approaches to produce such networks mainly focused on generating implicit networks. In particular, one of the archetypal types of implicit networks are the so-called *split networks* that were first introduced in [3]. For example, in the left panel of Figure 1 we present an example of a split network generated for collection of HIV virus sequences described in [4, Chapter 21] that was generated by the Splitstree program [5] using the NeighborNet algorithm [6]. In general, split networks are isometric subgraphs of hypercubes [3], and which means that they can be potentially highly non-planar (i.e. it is not possible to draw them in the plane without some edges crossing one another). Even so, split networks have proven to be very popular, one of the main reasons probably being that algorithms such as the NeighborNet algorithm are designed to generate split networks that *are planar* which greatly facilitates their visualization (see e.g. [7], [8], [9]) and subsequent biological interpretation.

More recently, the focus of the phylogenetic networks literature has shifted more towards explicit networks. Essentially, an explicit phylogenetic network (which from now on we will just call a phylogenetic network) is a directed acyclic graph with a single source or root, whose sinks (which are usually the leaves of the network) correspond to the set of species. For example, in the right panel of Figure 1 we present a network generated using the Trilonet program [10]. There are several types of phylogenetic networks (see e.g. [11, Chapter 10] for a recent review), and various programs are

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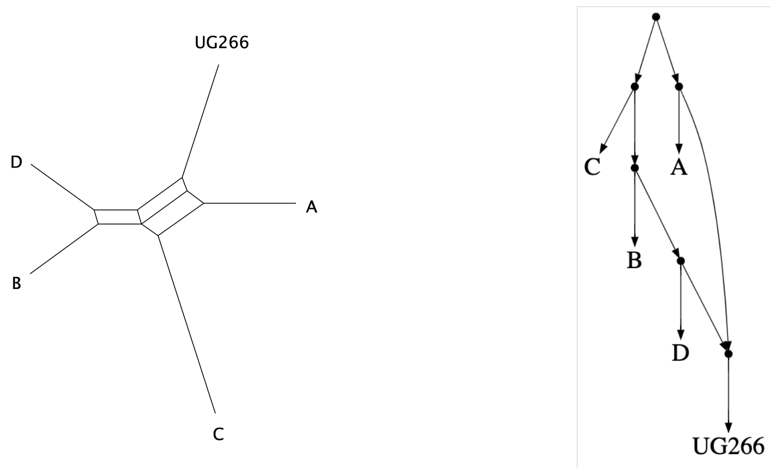


Fig. 1: Examples of implicit and explicit phylogenetic networks on a set of HIV viruses. The labels A–D correspond to viral subtypes, whereas UG266 corrsponds to a virus that is an A/D-recombinant. Left: A split network generated by Neighbornet; Right: a level-1 network generated by Trilonet. Note that directions in the right network make it somewhat clearer that UG266 has arisen as an A/D-recombinant.

available for their generation (e.g. [12], [13], [14] and [15] for a recent comparison of some of these programs). Even so, although some of these types of networks (such as so-called level-1 networks) are always planar, just as with split networks this is not true in general. Thus, it is interesting to study planarity of phylogenetic networks, with the view to developing new approaches and algorithms for generating phylogenetic networks that can be more readily visualized.

In this paper we shall present some new concepts and results concerning planar phylogenetic networks, which we now briefly summarize. To help prove our results we shall use the highly developed theory of planar directed graphs and posets, some of which we shall review in the next section. In Section 3, we then consider planar phylogenetic networks and introduce and compare three subclasses of such networks: upward planar, tip planar, and outer planar networks. Upward and outer planar phylogenetic networks are special examples planar diagrahs with the same name, whereas tip planar networks are networks where the source and sinks (i.e., the root and leaves) must all lie in the so-called outer face of some planar drawing of the network. We present a characterisation of tip planar networks in Theorem 3.1, and use this result to show that the four types of planar networks form a hierarchy (Theorem 3.2). As a consequence we also show that there are polynomial time algorithms for deciding whether or not a phylogenetic network is contained in each one of these classes or not (Theorem 3.3).

In Sections 4 and 5 we investigate planarity for two special classes of phlogenetic networks: level-$k$ ($k \geq 0$) [16] and regular networks [17]. Loosely speaking, the level of a network is measure of how far the network is from being a tree, where level-0 networks are trees. For level-$k$ networks we show that level-1, -2, -3 networks are always outer, tip, and upward planar, respectively, and that level-4 networks are not necessarily planar (Theorem 4.1). We also characterise when level-2 networks are tip planar (Theorem 4.2). A regular network can be thought of as being a Hasse diagram of some collection of clusters of a set of species. We show that a regular network is tip planar if and only if the underlying set of clusters forms a *pyramid* [18] (Theorem 5.1). We conclude in Section 6 with a discussion and some open problems.

## 2 PRELIMINARIES

In this paper we shall assume that $X$ is a finite set with $|X| \geq 2$. The set $X$ usually corresponds to a collection of extant species.

### 2.1 Posets and cluster systems

A set $Y$ with a partial order $\preceq$ (i.e., a binary reflexive, antisymmetric, and transitive relation) is called a partially ordered set or *poset*, and denoted by $(Y, \preceq)$. A poset $(Y, \preceq)$ is a *lattice* if for each pair of elements $(y_1, y_2)$ in $Y$, the set $Y$ contains both a join $y_1 \vee y_2$ (i.e., a least upper bound) and a meet $y_1 \wedge y_2$ (i.e., a greatest lower bound) of $y_1$ and $y_2$. Note that in case either a join or a meet exists then it is necessarily unique. The partial order $\preceq$ is referred to as a *linear order* if in addition, for each pair of distinct elements $y_1$ and $y_2$ in $Y$, we have either $y_1 \preceq y_2$ or $y_2 \preceq y_1$ (but not both). Clearly, a linear order is necessarily a lattice. Given a poset $P$, we let $P^*$ be $P$ if it has the bottom element (i.e. a least element), and $P$ with an extra bottom element $\hat{0}$ added otherwise.

The *Hasse diagram* $H(P)$ of a poset $P = (Y, \preceq)$ is the digraph with vertex set $Y$ and so that $(y_1, y_2)$ is a directed edge from $y_1$ to $y_2$ in $H(P)$ if $y_1$ *covers* $y_2$, that is, $y_2 \preceq y_1$ and there is no element $y_3 \in Y - \{y_1, y_2\}$ with $y_2 \preceq y_3 \preceq y_1$. Note that the direction of the edges in our definition of the Hasse diagram is chosen for the convenience later on, and is the opposite to the way commonly used in the poset literature. However, this difference will be immaterial for the results presented here. It is well-known in poset theory that each partial order $\preceq$ on $Y$ has a *linear extension*, that is, a linear order $\preceq_1$ on $Y$ such that $y_1 \preceq y_2$ in $Y$ implies that $y_1 \preceq_1 y_2$. Following [19], the (Dushnik and Miller) *dimension* of a poset $(Y, \preceq)$ is the smallest cardinal number $m$ such that $\preceq$ is the intersection of $m$ linear extensions $\preceq_1, \ldots, \preceq_m$ of $\preceq$, that is, $y_1 \preceq y_2$ if and only if $y_1 \preceq_i y_2$ for each $1 \leq i \leq m$.

A *cluster system $\mathcal{C}$ (on $X$)* is a collection of non-empty subsets of $X$ that contains $X$ and the subset $\{x\}$ for each $x \in X$, where each subset of $X$ in the system is referred to as a *cluster*. Furthermore, $\mathcal{C}$ is *closed under intersection* if for each pair of sets $A$ and $B$ in $\mathcal{C}$, we have either $A \cap B = \emptyset$ or $A \cap B \in \mathcal{C}$. Note that any cluster system $\mathcal{C}$ is a poset $P(\mathcal{C}) = (\mathcal{C}, \subseteq)$ under the ordering induced by taking set-inclusion; we let $P^*(\mathcal{C})$ be the poset obtained from $P(\mathcal{C})$ by adding the emptyset $\emptyset$ as the bottom element. We now make a simple observation that we will use later on. We include a proof for completeness.

***Lemma 2.1.*** Suppose that $\mathcal{C}$ is a cluster system on $X$ that is closed under intersection. Then the poset $P^*(\mathcal{C})$ is a lattice.

*Proof:* Let $A$ and $B$ be two distinct elements in $\mathcal{C}$. Let $A \wedge B$ be $\emptyset$ if $A$ and $B$ are disjoint, and $A \cap B$ otherwise. Since $\mathcal{C}$ is closed under intersection, it follows that $A \wedge B$ is contained in $P^*(\mathcal{C})$ and it is in fact the meet of $A$ and $B$.

Next, let $u(A, B)$ be the collection of subsets $C$ in $\mathcal{C}$ such that $A \subseteq C$ and $B \subseteq C$. Then $u(A, B)$ is non-empty as it contains $X$. Now, for $C_1$ and $C_2$ in $u(A, B)$, we have $A \cup B \subseteq C_1 \cap C_2$, and hence $C_1 \cap C_2 \neq \emptyset$. Therefore, $C_1 \cap C_2 \in u(A, B)$. In other words, $u(A, B)$ has a minimum element $C_0$ such that $C_0 \subseteq C$ for all $C \in u(A, B)$. Clearly, $C_0$ is the join of $A$ and $B$ in $P(\mathcal{C})$, and hence also their join in $P^*(\mathcal{C})$. Hence $P^*(\mathcal{C})$ is a lattice. □

## 2.2 Planar graphs

We shall assume that all sets and graphs are finite, and that all graphs are simple, that is, they contain neither loops nor parallel edges. For integers $m, n \geq 1$, let $K_m$ be the complete graph with $m$ vertices, and $K_{m,n}$ be the complete bipartite graph with $m$ vertices in one set and $n$ vertices in the other set.

We now recall some standard definitions for planar graphs and graph drawing, largely following [20] and [21]. A *drawing* $\Gamma = \Gamma(G)$ of a graph $G$ maps each vertex $v$ to a distinct point $\Gamma(v)$ of the Euclidean plane (endowed with the $x$ and $y$ coordinates) and each edge $(u, v)$ to an *arc* $\Gamma(u, v)$, that is, a simple open Jordan curve with endpoints $\Gamma(u)$ and $\Gamma(v)$. For clarity, the points $\Gamma(v)$ corresponding to a vertex $v$ in $G$ will be referred to as a node in the drawing, to differentiate them from the points inside of the arcs. Such a drawing is called a *straight-line drawing* if each arc in the drawing is a straight line segment between its endpoints. A drawing is *planar* is no two distinct arcs intersect except, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing. Note that such a drawing partitions the plane into connected regions called *faces*. There is precisely one unbounded face of $\Gamma$, referred to as the *outer-face* of $\Gamma$. If a planar graph has a drawing in which every vertex is contained in the outer-face, then the graph is called *outer planar*.

Given two graphs $G$ and $G'$, we say $G'$ is a *minor* of $G$ if $G'$ can be obtained from $G$ by repeated deletion (of vertices and edges) and contractions. Furthermore, we say $H$ is a subdivision of $G$ if $H$ is obtained from $G$ by replacing the (directed) edge of $G$ with (directed) paths between their ends (so that none of these paths has an inner vertex on another path or in $G$). The well-known Kuratowski theorem states that a graph $G$ is planar if and only if none of the subgraph of $G$ is a subdivision of either $K_5$ or $K_{3,3}$; equivalently, the Wagner theorems states that $G$ is planar if and only if $G$ contains neither $K_5$ nor $K_{3,3}$ is a minor (see, e.g. [20, Theorem 4.4.6] for a proof of both theorems). Moreover, $G$ is outer planar if and only if no subgraph of $G$ is a subdivision of either $K_4$ or $K_{2,3}$; see, e.g. [22, p.117].

Note that if $G$ is a digraph, then each edge $(u, v)$ will be mapped to a directed arc from $\Gamma(u)$ to $\Gamma(v)$.

Clearly, a digraph $G$ is planar if and only if its underlying graph $U(G)$ (e.g. the one obtained from $G$ by forgetting the directions of edges) is planar. Furthermore, a planar drawing of a digraph is called *upward* if each arc is a curve of monotonically nondecreasing in the $y$-coordinate. As noted in [23], a digraph $G$ is *upward planar* if it admits a straight-line upward planar drawing. Another useful result concerning upward-planar graphs is as follows. An *st-digraph* is an acyclic digraph with a single source, denoted by $s$, and a single sink, denoted by $t$, and the arc $(s,t)$. Then $G$ is upward-planar if and only if it is a spanning subgraph of a planar $st$-digraph, that is, directed edges can be added to $G$ such that the resulting digraph is a planar $st$-digraph [23, Theorem 1].

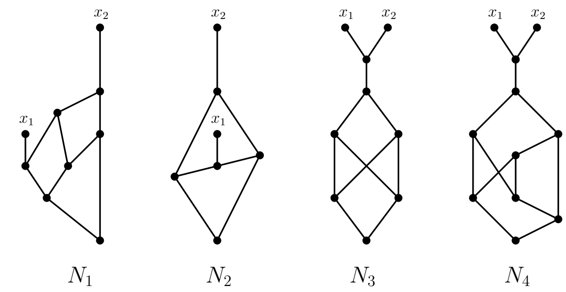


Fig. 2: Examples of phylogenetic networks on $X = \{x_1, x_2\}$ that are (a) tip planar but not outer planar, (b) upward planar but not tip planar, (c) planar but not upward planar, (d) not planar. The direction of each edge is pointed upward, and omitted here for simplicity; a convention used for all directed graphs in this paper depicted after this one.

## 3 Planar phylogenetic networks

In the following we shall mainly use the definitions for phylogenetic networks presented in [11, Section 10]. A *phylogenetic network on $X$*, or simply a phylogenetic network when the underlying set $X$ is clear from the context, is an acyclic directed graph $N = (V, A)$ with $X \subseteq V$, a single source $\rho_N$ (the root), $X$ the set of sinks in $N$, and no vertices with indegree 1 and outdegree 1. In particular, each edge is directed away from the root of $N$. Two phylogenetic networks $N = (V, A)$ and $N' = (V', A')$ on $X$ are *isomorphic* if there exists a bijective map $f : V \to V'$ such that $(u, v) \in A$ if and only if $(f(u), f(v)) \in A'$, and for all $x \in X$, $f(x) = x$.

A phylogenetic network on $X$ is called *planar, outer planar or upward planar* if as a digraph it satisfies each of these properties. Furthermore, we defined such a network to be *tip planar*[1] if admits a planar drawing in which the set $\{\rho_N\} \cup X$ is contained in the outer-face (note that this is analogous to *outer-labelled split networks* [7], where the elements in $X$ must all lie in the outer face of the network). Clearly, if a phylogenetic network is outer planar, then it is tip planar. Also, note that a phylogenetic network $N$ is outer planar (respectively tip planar or planar) if and only if its underlying graph $U(N)$ is outer planar (respectively tip planar or planar). In Fig. 2 we present some examples of these concepts; we shall prove that the networks in this figure have the stated properties in Theorem 3.2 and Theorem 4.1 below.

1. We were unable to find an agreed name for a vertex in an arbitrary digraph that is either a source or a sink, so for phylogenetic networks we call such a vertex a *tip* of the network.

We now present a result relating tip planar and upward planar phylogenetic networks on $X$. Given a phylogenetic network $N = (V, E)$ on $X$, we let $N^*$ be the digraph with vertex set $V \cup \{t\}$ for some $t \notin V$ which will be referred to as the *sink vertex*, and directed edge set $A \cup \{(x,t) : x \in X\}$. We call $N^*$ the *completion* of $N$.

***Theorem 3.1.*** Suppose that $N$ is a phylogenetic network on $X$. Then $N$ is tip planar if and only if the completion $N^*$ of $N$ is upward planar.

*Proof:* Suppose first that $N^*$ is upward planar. Fix an upward drawing $\Gamma^*$ of $N^*$. Consider the drawing $\Gamma$ of $N$ obtained from $\Gamma(N^*)$ by removing the sink $\Gamma^*(t)$ and the arcs incident with it. Then $\Gamma$ is a planar drawing of $N$ such that the set $\{\rho_N\} \cup X$ is contained in the outer-face of $\Gamma$. So $N$ is tip planar.

Conversely, suppose that $N$ is tip planar. Fix a planar drawing $\Gamma$ of $N$ in the plane such that the set $\{\rho_N\} \cup X$ is contained in the outer-face. Without loss of generality, we may assume that the $y$-coordinate of each point in the drawing $\Gamma$ is strictly positive. Take a point $w$ in the interior of the outer-face in the plane whose $y$-coordinate is negative. Join this point to each element $x \in X$ with a curve $\gamma_{w,x}$ contained in the outer-face such that $\gamma_{w,x} \cap \gamma_{w,x'} = \{w\}$ for all $x, x' \in X$. Then add in an arc between the $\rho_N$ and $w$ which does not intersect any existing arcs, except possibly in $w$ or $\rho_N$. This gives a planar drawing $\Gamma'$ of the digraph

$$N' = (V \cup \{t\}, A \cup \{(x,t) : x \in X\} \cup \{(\rho_N, t)\})$$

in which $t$ is mapped to $w$ under $\Gamma'$. So $N'$ is a planar $st$-digraph (with source $s = \rho_N$ and sink

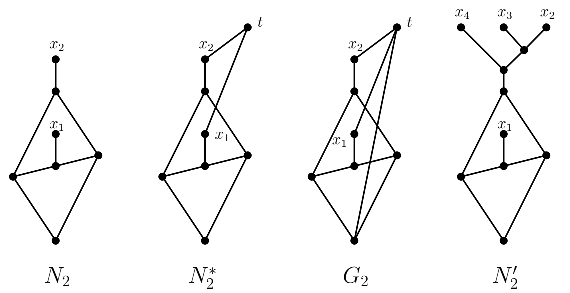


Fig. 3: Four graphs in the proof of Theorem 3.2.

$t$). It follows that $N^*$ is upward planar by [23, Theorem 1] that we stated above. Finally, as $N$ is a subgraph of $N^*$, it follows that $N$ is also upward planar. □

Let $\mathcal{P}_p(X)$, $\mathcal{P}_u(X)$, $\mathcal{P}_t(X)$, $\mathcal{P}_o(X)$ denote the classes of planar, upward planar, tip planar, and outer planar phylogenetic networks on $X$, respectively. We now show how these classes are related to one another.

***Theorem 3.2.*** For any $X$, $|X| \geq 2$, we have

$$\mathcal{P}_o(X) \subsetneq \mathcal{P}_t(X) \subsetneq \mathcal{P}_u(X) \subsetneq \mathcal{P}_p(X).$$

*Proof:* By definition we have $\mathcal{P}_o(X) \subseteq \mathcal{P}_t(X)$ and $\mathcal{P}_u(X) \subseteq \mathcal{P}_p(X)$. Furthermore, $\mathcal{P}_t(X) \subseteq \mathcal{P}_u(X)$ follows from Theorem 3.1.

We now show that all of the set inclusions in the statement of the theorem are strict, for the cases $Y = \{x_1, x_2\}$ and $X = \{x_1, x_2, \dots, x_t\}$, $n \geq 3$, which will complete the proof of the theorem.

First, consider the network $N_2$ on $Y$ in Fig. 2 which is clearly contained in $\mathcal{P}_u(Y)$. We claim that $N_2 \notin \mathcal{P}_t(Y)$. Indeed, if this were not the case, then $N_2$ is tip planar, and hence its completion $N_2^*$ is upward planar in view of Theorem 3.1. Therefore, $N_2^*$ is a spanning subgraph of an $st$-digraph denoted by $\widetilde{N}_2^*$. This implies that the graph $G_2$ in Fig. 3 is a minor of $U(\widetilde{N}_2^*)$. However, $G_2$ contains $K_{3,3}$ as one of its minors, and hence not planar, a contradiction. Now, let $T_n = T(x_2, \dots, x_n)$ be the binary tree on $\{x_2, \dots, x_n\}$ given by taking the Hasse diagram of the cluster system that contains all clusters of the form $\{x_2, \dots, x_k\}$, $3 \leq k \leq n$, and construct the phylogenetic network $N_2'$ on $X$ by replacing $x_2$ with the tree $T(x_2, \dots, x_n)$ (see Fig. 3 for an example with $n = 4$. Since both $N_2$ and $T(x_2, \dots, x_n)$ are upward planar, it is straightforward to construct an upward planar drawing for $N_2'$. A similar to the proof that $N_2$ is not tip planar then implies that $N_2' \in \mathcal{P}_u(X) \setminus \mathcal{P}_t(X)$.

Next, consider the network $N_1$ on $Y$ in Fig. 2 which is clearly contained in $\mathcal{P}_t(Y)$. Furthermore, $N_1$ is not outer planar because $U(N_1)$ contains a subdivision of $K_{2,3}$. Now we construct the phylogenetic network $N_1'$ on $X$ by replacing $x_2$ with the tree $T_n$ as in the last paragraph. Since $T_n$ is outer planar and $N_1$ is tip planar, we start with a planar drawing of $N_1$ whose outer-face contains the root and leaves of $N_1$ and then naturally extend it to a planar drawing of $N_1'$ whose outer-face contains the root and leaves of $N_1'$. It follows that $N_1'$ is tip planar. Furthermore, $N_1'$ is not outer planar because $U(N_1')$ again contains a subdivision of $K_{2,3}$. Hence $N_1' \in \mathcal{P}_t(X) \setminus \mathcal{P}_o(X)$.

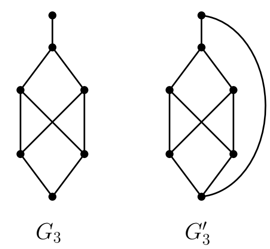


Fig. 4: Two graphs in the proof of Theorem 3.2.

Finally, we claim that $N_3$ in Fig. 2, which is easily seen to be planar, is not in $\mathcal{P}_u(Y)$. If not, then the directed graph $G_2$ (see Fig. 4) obtained from $N_3$ by removing the sinks $x_1, x_2$ and the directed edges incident with them is also upward planar. That is, $G_3$ is a spanning subgraph of an $st$-digraph $\widetilde{G}_3$ and hence $G_3'$ in Fig. 4 is a minor of $U(\widetilde{G}_3)$. However, $G_3'$ contains $K_{3,3}$ as its minor, a contradiction. We now construct a phylogenetic network $N_3'$ on $X$ by replacing $x_2$ with the tree $T_n$ as above. Since $T_n$ is planar and it can be drawn in an arbitrary small neighborhood of $x_2$, it follows that $N_3'$ is also planar. An argument similar to the proof of that $N_3$ is not upward planar now implies $N_3' \in \mathcal{P}_p(X) \setminus \mathcal{P}_u(X)$. □

We now consider some algorithmic aspects of planar phylogenetic networks. Our main result will follow more-or-less directly from some results in the literature concerning planar digraphs which we now briefly review. Suppose that $G$ is a digraph with $m$ vertices. Then it can be checked in $O(m)$ time if $G$ is planar/outer planar, and a straight-line-planar/outer-planar drawing for $G$ can be produced in $O(m)$ time in case it is (see [24]/ [25], [26]). Similarly, there are also $O(m)$ algorithms for checking upward planarity and producing an upward planar drawing for $G$ in case $G$ has a single source (see [27], and also [23, Theorem 15]), although the decision problem for deciding whether or not a digraph is upward planar is NP-complete in general [28].

***Theorem 3.3.*** Suppose $N$ is a phylogenetic network on $X$ with $m$ vertices. Then it can be checked in $O(m)$ time whether or not $N$ is planar/upward planar/outer planar/tip planar, and a planar drawing with the corresponding property can be produced in $O(m)$ time if $N$ satisfies any of these properties. In particular, if the number of vertices in $N$ is bounded by some polynomial $p$ in $|X|$, then we can check if $N$ satisfies any of these planarity properties and if so produce a corresponding planar drawing in $O(p(|X|))$-time.

*Proof:* The results concerning planar, upper planar, or outer planar follow from the results mentioned before the statement of the theorem.

To check whether $N$ is tip planar, we can construct its completion $N^*$ which contains $m+1$ vertices in $O(m)$ time. Since it can be checked in $O(m)$ time whether or not $N^*$ is upper planar, by Theorem 3.1 we can check whether $N$ is tip planar in $O(m)$ time. Furthermore, when $N$ is tip planar, a tip planar drawing of $N$ can be obtained from an upper planar drawing of $N^*$ in $O(m)$ time using the procedure in the proof of Theorem 3.1. □

Using Theorem 3.3, we can now say something more about checking planarity for some well-known classes of phylogenetic networks. We begin by recalling some definitions (see e.g. [11, Chapter 10]). A vertex in a phylogenetic network is either a *tree vertex* (i.e., a vertex of indegree 1 or the root vertex) or a *reticulation vertex* (i.e., a vertex of indegree 2 or more). Furthermore, a phylogenetic network is called *binary* if the root has outdegree 2, the leaves have indegree 1, and all of the all other vertices have indegree 1 and outdegree 2 or indegree 2 and outdegree 1.

Now, a *tree-child network* is a phylogenetic network for which every non-leaf vertex has a child that is a tree vertex (note a vertex $v$ in a phylogenetic network $N$ is a child of a vertex $u$ in $N$ if $(u, v)$ is an arc in $N$). A vertex $v$ is said to be *visible* if there is a leaf $x_v \in X$ so that every directed path from the root to $x_v$ passes through $v$. A *reticulation-visible network* is a phylogenetic network for which each reticulation vertex is visible. Since a phylogenetic network is tree-child if and only if each vertex is visible, it follows that a tree-child network is necessarily reticulation-visible. In particular, the corollary below concerning reticulation-visible networks also holds for tree-child networks. Finally, a phylogenetic network is *normal* if it is tree-child and has it does not contain any arc $(u, v)$ such that there is another directed path from $u$ to $v$ (these are know as *redundant* arcs).

It is known that a binary reticulate-visible network on $X$ has $O(|X|)$ vertices (see e.g. [11, Proposition 10.11]), and that a normal network on $X$ has $O(|X|^2)$ vertices (see [11, Theorem 10.11], and the discussion directly after). Hence by Theorem 3.3 we immediately see that the following holds:

***Corollary 3.4.*** Suppose that $N$ is a phylogenetic network on $X$. If $N$ is binary and reticulate-visible or normal, then it can be checked in $O(|X|)$ or $O(|X|^2)$ time, respectively, if $N$ is planar, upper planar, outer planar or tip planar and in case $N$ satisfies any of these properties, a planar drawing can be produced with the corresponding property in the same time.

**Remark:** There are non-binary, tip planar, tree-child networks on a set $X$ that do not have a polynomial number of vertices in $|X|$ (see e.g. [11, Fig. 10.5 (ii)]).

## 4 LEVEL-$k$ NETWORKS

In this section, we shall consider planarity of level-$k$ networks [16], which are defined as follows. A biconnected component in a graph is a maximal subgraph that does not contain any cut vertex (i.e. a vertex whose removal disconnects the graph). A phylogenetic network is *level-$k$*, $k \geq 0$, if it is binary and it can be converted into a tree by deleting at most $k$ directed edges from each biconnected component [11, p.247]. Clearly, a level-0 phylogenetic network is necessary a tree, and a binary phylogenetic network $N$ is level-$k$ if and only if $U(N)$ can be converted into a tree by deleting at most $k$ edges from each biconnected component of $U(N)$. We let $\mathcal{L}_k(X)$ denote the set of level-$k$ phylogenetic networks on $X$ for $k \geq 0$, so that $\mathcal{L}_i(X) \subseteq \mathcal{L}_k(X)$ clearly holds for all $0 \leq i \leq k$.

To prove the first main result of this section we will use the following observation [29]: A connected single-source digraph is upward planar if and only if all of its biconnected components are upward planar (see also [23, Lemma 4]). Since each phylogenetic network $N$ on $X$ is a connected and single-source, it follows that $N$ is upward planar if and only if all the biconnected components of $N$ are upward planar.

***Theorem 4.1.*** The following statements hold:

(i) All level-1 networks are outer-planar, that is, $\mathcal{L}_1(X) \subseteq \mathcal{P}_o(X)$.
(ii) All level-2 networks are upward planar, but some level-2 networks are not tip planar, that is, $\mathcal{L}_2(X) \subseteq \mathcal{P}_u(X)$ and $\mathcal{L}_2(X) \setminus \mathcal{P}_t(X) \neq \emptyset$.
(iii) All level-3 networks are planar, but some level-3 network are not upward planar, that is $\mathcal{L}_3(X) \subseteq \mathcal{P}_p(X)$ and $\mathcal{L}_3(X) \setminus \mathcal{P}_u(X) \neq \emptyset$.
(iv) Some level-4 networks are non-planar, that is, $\mathcal{L}_4(X) \setminus \mathcal{P}_p(X) \neq \emptyset$.

*Proof:* (i) Assume that $N$ is a level-1 network on $X$ and consider its underlying undirected graph $U(N)$. Since each non-trivial biconnected component of $U(N)$ is a cycle, it follows that none of the subgraphs of $U(N)$ is a subdivision of either $K_4$ or $K_{2,3}$, and hence $U(N)$ is outer planar. Thus $N$ is also outer planar.

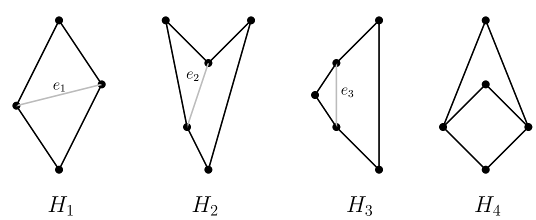


Fig. 5: The four digraphs for level-2 networks in the proof of Theorem 4.1; modified from [30, Fig. 5]. The three edges highlighted in grey and labelled as $e_1, e_2, e_3$ are referred in the proof of Theorem 4.1(ii).

(ii) Assume that $N$ is a level-2 network on $X$. To show that $N$ is upward planar, it is sufficient to establish the claim that each biconnected component of $N$ is upward planar. To this end, consider an arbitrary biconnected component $H$ of $N$. Without loss of generality we may further assume that $H$ contains more than one cycle as otherwise the claim clearly holds. Using the results in [30, Section 4] concerning the generators of level-2 networks, it follows that $H$ is a subdivision of one of the four digraphs in Fig. 5. Since each of these four digraphs in Fig. 5 is upward planar, it follows that $H$ is also upward planar, as required.

Finally, note that the network $N_2'$ constructed in the proof of Theorem 3.2 is clearly contained in $\mathcal{L}_2(X) \setminus \mathcal{P}_t(X)$.

(iii) First, we shall show that $\mathcal{L}_3(X) \subseteq \mathcal{P}_p(X)$. If not, consider a level-3 phylogenetic network $N$ on $X$ such that $N$ is not planar. Then $U(N)$ contains either $K_{3,3}$ or $K_5$ as its minor. We shall only consider the case where $U(N)$ contains $K_{3,3}$ as a minor; the proof for $K_5$ a minor is similar. In this case, there exists a biconnected component $B$ of $U(N)$ such that $B$ contains $K_{3,3}$ as a minor. This implies that we need to delete at least 4 edges from $B$ to convert it into a tree, in contradiction to the assumption that $N$ is level-3.

Finally note that for the network $N_3'$ constructed in the proof of Theorem 3.2 we clearly have $N_3' \in \mathcal{L}_3(X) \setminus \mathcal{P}_u(X)$.

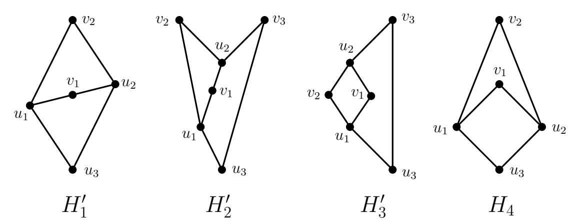


Fig. 6: The four digraphs in Theorem 4.2.

(iv) This follows since the network $N_4$ on $\{x_1, x_2\}$ in Fig 2 is level-4 but not planar as $K_{3,3}$ is a minor of $U(N_4)$. Now construct a phylogenetic network $N_4'$ on $X$ by replacing $x_1$ with any binary rooted phylogenetic tree on $\{x_1, x_3, .., x_n\}$. Then it is straight-forward to see that $N_4' \in \mathcal{L}_4(X) \setminus \mathcal{P}_p(X)$. □

In the last theorem we showed that all level-1 networks are outer planar. In the second main result of this section, we show that for a level-2 network outer planarity of the network is equivalent to it being tip planar, and characterize those level-2 networks that have this property.

***Theorem 4.2.*** Suppose that $N$ is a level-2 phylogenetic network on $X$. Then the following statements are equivalent:

(i) $N$ is tip planar.
(ii) $N$ is outer planar.
(iii) $N$ does not contain any subdivision of any of the four digraphs in Fig. 6.
(iv) $U(N)$ does not contain a subdivision of $K_{2,3}$.

*Proof:* Since $(ii) \Rightarrow (i)$ clearly holds, it suffices to show that $(i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii)$.

$(i) \Rightarrow (iii)$: First assume that $N$ contains a subdivision of $H_1'$ in Fig. 6. Since $N$ is a binary network, it follows that there exists two elements $x_1$ and $x_2$ in $X$ such that for $i = 1, 2$, there exists a directed path from $v_i$ to $x_i$ which does not contain any directed edge in $H_1'$. Now an argument similar to the proof of Theorem 4.1(ii) (see also Fig. 3) shows that $N$ is not tip planar, a contradiction.

The other three cases can be established in a similar manner.

$(iii) \Rightarrow (iv)$: Assume that $U(N)$ contains a subdivision of $K_{2,3}$. We need to show that $N$ contains a subdivision of one of the four digraphs in Fig. 6. Let $H$ be a biconnected component in $N$ such that $U(H)$ contains a subdivision of $K_{2,3}$, which must clearly exist. Then $H$ contains a subdivision of one of the four digraphs $H_i$ in Fig. 5. We assume that $H$ contains a subdivision of $H_2$; the other cases can be established in a similar manner.

We shall show that $H$ contains a subdivision of $H_2'$. To this end, label the vertices and edges in $H_2$ as in Fig. 7(A). Since $U(H)$ contains a subdivision of $K_{2,3}$, there exists two vertices $u, v$ in $U(H)$ such that there are three internally disjoint paths between them and each of these three paths contains at least two edges. Noting that $H_2'$ is obtained from $H_2$ by subdividing the edge $e_2 = \{u_1, u_2\}$ with the path $u_1, v_1, u_2$, it is now sufficient to establish the following claim:

**Claim:** Under the above assumptions, $\{u, v\} = \{u_1, u_2\}$.

To establish the claim we consider the following subcases based on the size of intersection $\{u, v\} \cap V(H_2)$.

Case (a): Neither $u$ nor $v$ is contained in $V(H_2)$. Then both $u$ and $v$ are contained in edges of $H_2$ and are added in the subdivision step when obtaining $H$ from $H_2$. We have two subcases to consider. The first one assumes that both $u$ and $v$ are contained in the same edge. We assume that both of them are contained in $e_1$; the other cases are similar (see Fig. 7(B) for an illustration). Then at most one path between $u$ and $v$ in $U(H)$ does not contain the vertex $u_2$, and hence $U(H)$ does not contain a subdivision of $K_{2,3}$, a contradiction. The second subcase assumes that $u$ and $v$ are contained in two different edges. We assume $v$ is contained in $e_1$ and $u$ is contained in $e_4$; the other cases are similar (see Fig. 7(C) for an illustration). Then each path between $u$ and $v$ in $U(H)$ contains either $u_1$ or $u_2$, and hence $U(H)$ does not contain a subdivision of $K_{2,3}$, a contradiction.

Case (b): Precisely one of the vertices between $u$ and $v$ is contained in $V(H_2)$. Without loss of generality, we assume $u \in V(H_2)$ and $v \notin V(H_2)$. Then there are 30 combinations to consider: five choices for $u$ and six choices of $v$. Here we consider the combination $u = u_3$ and $v$ is contained in $e_1$ (see Fig. 7(D) for an illustration); the other combinations can be proved similarly. For this combination, each path between $u$ and $v$ in $U(H)$ contains either $u_1$ or $u_2$, and hence $U(H)$ does not contain a subdivision of $K_{2,3}$, a contradiction.

Case (c): Both vertices are contained in $V(H_2)$, that is, $\{u, v\} \subseteq V(H_2)$. Since $\{u, v\} \neq \{u_1, u_2\}$, then an argument similar to the proof of Case (a) shows that $U(H)$ does not contain a subdivision of $K_{2,3}$, a contradiction. This completes the proof of the claim, and hence also the implication $(iii) \Rightarrow (iv)$.

$(iv) \Rightarrow (ii)$: Since $N$ is outer planar if and only if $U(N)$ does not contain a subdivision of $K_{2,3}$ or $K_4$, it remains to show that $N$ does not contain a subdivision of $K_4$. If this were not the case, then there must exist a biconnected component $B$ of $U(N)$ such that $B$ contains a subdivision of $K_4$. This implies that we need to delete at least 3 edges from $B$ to convert it into a tree, in contradiction to the assumption that $N$ is level-2. □

**Remark:** Note that the condition that $N$ is level-2 is optimal in Theorem 4.2 in the following sense: The network $N_1$ in Fig. 2 is level-3 and tip planar, but not outer planar.

## 5 Regular networks

In this section, we shall consider planarity of regular phylogenetic networks [17] that are defined as follows. First, note that given any cluster system $\mathcal{C}$ on $X$, we can regard the Hasse diagram $H(\mathcal{C})$ as a phylogenetic network on $X$, where the root is the vertex corresponding to the cluster $X$, and each element $x \in X$ corresponds to the singleton set $\{x\}$. Conversely, given a phylogenetic network $N = (V, A)$ on $X$, consider the partial ordering $\preceq_N$ on $V$ (or $\preceq$ for short) given by taking $u \preceq_N v$ for $u, v \in V$ if there exists a directed path from $u$ to $v$ (which includes the special case that $u = v$). The *cluster induced* by a vertex $v$ in $N$, denoted by $\mathcal{C}(v)$, is the set of elements $x \in X$ such that $v \preceq x$ (so $u \preceq_N v$, $u, v \in V$ if and only if $\mathcal{C}(v) \subseteq \mathcal{C}(u)$), and the cluster system $\mathcal{C}(N)$ *induced by* $N$ consists of all clusters induced by the vertices in $N$. A phylogenetic network $N$ on $X$ is *regular* if $N$ is isomorphic to $H(\mathcal{C}(N))$ [11, p.252].

Interestingly, there is a special type of planar, regular phylogenetic network on $X$ that first appeared sometime ago in the literature [18] (using different terminology). Following [31], a cluster system $\mathcal{C}$ of $X$ is a *prepyramid* if it can be represented as a family of intervals, that is, there exists a bijective mapping $f$ from $X$ to $[n] := \{1, \ldots, n\}$ such that $f(A)$ is an interval of $\{1, \ldots, n\}$ for each cluster $A$ in $\mathcal{C}$, and a *pyramid* if it is both prepyramid and closed under intersection [18]. Here

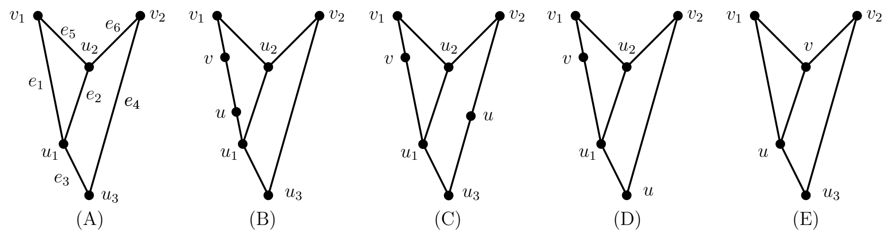


Fig. 7: Digraphs for level-2 networks in the proof of Theorem 4.2.

we use the convention that singleton sets are also intervals. We define a *(pre)pyramid network* to be a phylogenetic network on $X$ of the form $H(\mathcal{C})$ for $\mathcal{C}$ a (pre)pyramid on $X$. In [18] pyramid networks were implicitly defined and shown to be planar. An example of such a network also appeared recently in [32, Fig 2.].

In this section, we shall show that tip planar regular networks and pyramidal networks are in fact one and the same thing. To prove this, we shall use the following three results rephrased in our terminology: (F1) the poset induced by a planar digraph with one source and one sink is a lattice [23, Theorem 5], (F2) a lattice is upward planar if and only if it has dimension at most 2 [33, Proposition 5.2], and (F3) if $\mathcal{C}$ is a prepyramid on $X$, then the poset $(\mathcal{C}, \subseteq)$ has dimension at most 2 [19, Theorem 3.6.1].

***Theorem 5.1.*** Suppose that $N$ is a regular phylogenetic network on $X$. Then $N$ is tip planar if and only if $N$ is pyramidal.

*Proof:* Let $N$ be a regular network on $X$. Denote its cluster system by $\mathcal{C} = \mathcal{C}(N)$ so that, since $N$ is regular, $N \cong H(P(\mathcal{C}))$. Clearly the isomorphism between $N$ and $H(P(\mathcal{C}))$ can be extended to give a digraph isomorphism between the completion $N^*$ of $N$ and the Hasse diagram $H(P^*(\mathcal{C}))$, by mapping the sink vertex $t$ in $N^*$ to the vertex in $H(P^*(\mathcal{C}))$ associated with the bottom element.

For the 'if' direction, suppose that $N$ is pyramidal, so that $\mathcal{C}$ is a pyramid. Then by Lemmas 2.1 and the fact (F3) above, $P^*(\mathcal{C})$ is a lattice with dimension at most 2. It follows that $H(P^*(\mathcal{C}))$ is upward planar by the fact (F2) above. Hence, as $N^*$ is isomorphic to $H(P^*(\mathcal{C}))$, it follows that $N^*$ is upward planar, and so $N$ is tip planar by Theorem 3.1.

Now, for the 'only if' direction, suppose that $N$ is tip planar. By Theorem 3.1 $N^*$ is upward planar. As $N^*$ is isomorphic to $H(P^*(\mathcal{C}))$, by the fact (F1) above it follows that $P^*(\mathcal{C})$ is a lattice. Therefore, $\mathcal{C}$ is closed under intersection, that is, for any two vertices $u$ and $v$ in $N$ with $\mathcal{C}(u) \cap \mathcal{C}(v) \neq \emptyset$, there exists a vertex $w$ in $N$ such that $\mathcal{C}(u) \cap \mathcal{C}(v) = \mathcal{C}(w)$.

Fix an upward planar drawing $\Gamma^*$ of $N^*$, and let $\Gamma$ be the tip planar drawing of $N$ induced by $\Gamma^*$, that is, $\Gamma$ is obtained from $\Gamma^*$ by removing the node corresponding to the sink vertex in $N^*$ and all arcs incident with it. Consider the ordering of the elements in $X$ obtained from the drawing $\Gamma$ of $N$ by following the outer-face of $\Gamma$ in a clockwise direction starting from the node associated with the root of $N$. This gives a bijective map $f$ from $X$ to $\{1, \ldots, n\}$, where $n = |X|$. For simplicity, we may further assume that the elements in $X$ are enumerated in a way such that $f(x_i) = i$ holds for all $1 \leq i \leq n$. We claim that for each cluster $A$ in $\mathcal{C}$, $f(A)$ is an interval of $\{1, \ldots, n\}$ which, as $\mathcal{C}$ is closed under intersection, will complete the proof of the theorem.

To establish the claim, consider an arbitrary vertex $v$ in $N$. Without loss of generality, we suppose that $a < b$ are two numbers in $\{1, \ldots, n\}$ with $a < b - 1$ and both $x_a \preceq v$ and $x_b \preceq v$ hold for $x_a := f^{-1}(a)$ and $x_b := f^{-1}(b)$, as otherwise the claim clearly holds. It suffices to show that for each $c$ with $a < c < b$ we have $x_c \preceq v$ for the leaf $x_c := f^{-1}(c)$. This clearly holds if $v$ is the root of $N$, and hence we may assume that $v$ is not the root for the remainder of the proof of the claim.

Fix two directed paths $H_a$ and $H_b$ in $N^*$ between $v$ and $x_a$ and $x_b$, respectively. Furthermore, let $e_a = (x_a, t)$ and $e_b = (x_b, t)$ denote the two directed edges connecting $x_a$ and $x_b$ to the sink element $t$, respectively. Without loss of generality, we may further assume that $v$ is the only common vertex contained in both $H_a$ and $H_b$, as otherwise we can replace $v$ by the lowest vertex in $H_a$ that is also contained in $H_b$.

Let $L_a = \Gamma^*(H_a)$ and $L_b = \Gamma^*(H_b)$ be the curves in the drawing $\Gamma^*$ corresponding to $H_a$ and $H_b$, respectively. Then the curve $L = L_a \cup L_b \cup \Gamma^*(e_a) \cup \Gamma^*(e_b)$ forms a closed Jordan curve in the

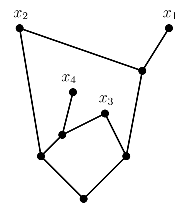


Fig. 8: A regular network that is a prepyramid network and upward planar, but not tip planar.

plane. By the Jordan Curve Theorem (see e.g. [20, Theorem 4.1.1]), it follows that $L$ partitions the plane into precisely two connected components, one containing the node corresponding to root $\rho$, and the other containing the node corresponding to vertex $x_c$.

Now, fix a path $H_c$ in $N^*$ from the root $\rho$ to $x_c$. Then, as $L$ partitions the plane into precisely two connected components, $\Gamma^*(H_c)$ must intersect at least one node in $L$. That is, there exists a vertex $u$ in $N^*$ such that $\Gamma^*(u) \in L \cap \Gamma^*(H_c)$. Since $\Gamma^*$ is an upward planar drawing of $N^*$, we can further assume $u \notin \{x_a, x_b, t\}$. Since $v \preceq u \preceq x_c$, it follows that $x_c \in \mathcal{C}(v)$, which completes the proof of the claim. □

Note that there are regular networks on $X$ can have $2^{|X|} - 1$ vertices (see, e.g. [11, Section 10.3.4]), but by Theorem 5.1, the number of vertices in any tip planar regular network is less than or equal to $\binom{|X|}{2}$, the number of intervals of $\{1, 2, \ldots, |X|\}$. Thus by Theorem 3.3 we have:

***Corollary 5.2.*** If $N$ is a regular phylogenetic network on $X$, then we can check in $O(|X|^2)$ if $N$ is tip planar.

**Remark:** Consider the phylogenetic network $N$ on $X = \{x_1, x_2, x_3, x_4\}$ in Fig. 8. Then since

$$\mathcal{C}(N)=\{\{x_1,x_2,x_3,x_4\},\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_1,x_2\},\{x_3,x_4\},\{x_1\},\{x_2\},\{x_3\},\{x_4\}\},$$

it is straight-forward to see that $N$ is a regular and that $\mathcal{C}(N)$ is a prepyramid. Moreover, $N$ is *not* a pyramid network because the intersection $\{x_1, x_2, x_3\} \cap \{x_2, x_3, x_4\} = \{x_2, x_3\}$ is not contained $\mathcal{C}(N)$. Thus, this gives an example of a prepyramid $\mathcal{C}$ such that the phylogenetic network $H(\mathcal{C})$ on $X$ is upward planar (in view of Fig. 8), but not tip planar (interestingly, in this example the poset $P(\mathcal{C}(N))$ has dimension 2).

## 6 Discussion

In this paper, we have considered the concept of planar phylogenetic networks, and shown that the theory and algorithms coming from the area of planar digraphs can be very useful for handling and understanding these structures.

There remain several open questions. For example:

- Given a planar phylogenetic network on $X$, let $\widetilde{N}$ be the digraph obtained from $N$ by adding an additional arc $(x, \rho)$ for each element $x$ in $X$ if $(\rho, x)$ is not already an arc in $N$. If $N$ is tip planar, then it is straightforward to see that $\widetilde{N}$ is planar, but does the converse hold?
- Given a normal network $N$, we define $Comp(N)$ to be the network which is obtained from $N$ by collapsing any arc $(u, v)$ in $N$, where $u$ has outdegree 1 and $v$ has indegree 1. Note that that $Comp(N)$ must be regular (see e.g. [11, p.253]). Is $N$ is tip planar if and only $Comp(N)$ is tip planar?
- Is there a prepyramid cluster system $\mathcal{C}$ such that $H(\mathcal{C})$ is not upward planar?
- Although there are some general algorithms for drawing planar, upper planar, tip planar and outer planar networks (Theorem 3.3), are there more specific algorithms for drawing special types of planar phylogenetic networks such as tree-child networks?

More generally, the phylogenetic networks that we have considered in this paper did not have arc weights. It would be interesting to understand how arc weights might affect our results, especially when we want to make a straight-line drawing where the length of an arc is proportional to its weights. A special case that could be considered first are temporal phylogenetic networks which incorporate a natural vertical time axis which provides timings for past evolutionary events (see e.g. [11, Chapter 10.3.3]). This concept appears to be related to upward planarity, and it would be interesting to further investigate this relationship. Moreover, in general as the theory of phylogenetic networks continues to grow it could be worthwhile to develop new algorithms to produce planar phylogenetic networks from biological data.

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