TOPOLOGICAL FINITENESS PROPERTIES OF MONOIDS PART 1: FOUNDATIONS

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ABSTRACT. We initiate the study of higher dimensional topological finiteness properties of monoids. This is done by developing the theory of monoids acting on CW complexes. For this we establish the foundations of M-equivariant homotopy theory where M is a discrete monoid. For projective M-CW complexes we prove several fundamental results such as the homotopy extension and lifting property, which we use to prove the M-equivariant Whitehead theorems. We define a left equivariant classifying space as a contractible projective M-CW complex. We prove that such a space is unique up to M-homotopy equivalence and give a canonical model for such a space via the nerve of the right Cayley graph category of the monoid. The topological finiteness conditions left-F_n and left geometric dimension are then defined for monoids in terms of existence of a left equivariant classifying space satisfying appropriate finiteness properties. We also introduce the bilateral notion of M-equivariant classifying space, proving uniqueness and giving a canonical model via the nerve of the two-sided Cayley graph category, and we define the associated finiteness properties bi-F_n and geometric dimension. We explore the connections between all of the these topological finiteness properties and several well-studied homological finiteness properties of monoids which are important in the theory of string rewriting systems, including FP_n, cohomological dimension, and Hochschild cohomological dimension. We also introduce a theory of M-equivariant collapsing schemes which gives new results giving sufficient conditions for a monoid to be of type F_∞ (or bi-F_∞). We identify some families of monoids to which these theorems apply, and in particular provide topological proofs of results of Anick, Squier and Kobayashi that monoids which admit presentations by complete rewriting systems are left- right- and bi-FP_∞. This is the first in a series of three papers proving that all one-relator monoids are of type FP_∞, settling a question of Kobayashi from 2000.

1. INTRODUCTION

The study of the higher dimensional finiteness properties of groups was initiated fifty years ago by C. T. C. Wall [Wal65] and Serre [Ser71]. An Eilenberg–MacLane complex $K(G,1)$ for a discrete group G, also called a classifying space, is an aspherical CW complex with fundamental group G. Such a space can always be constructed for any group G (e.g. via the bar construction) and it is unique up to homotopy equivalence. While useful for theoretical purposes, this canonical $K(G,1)$-complex is very big and is often not useful for practical purposes, specifically if one wants to compute the homology of the group. It is therefore natural to seek a 'small' $K(G,1)$ for a given group by imposing various finiteness conditions on the space. Two of the most natural and well-studied such conditions are the topological finiteness property F_n and the geometric dimension $\mathrm{gd}(G)$ of the group.

Date: September 29, 2021.

2010 *Mathematics Subject Classification.* 20M50, 20M05, 20J05, 57M07, 20F10, 20F65.

Key words and phrases. CW complex, monoid, equivariant homotopy theory, homological finiteness property FP_n, cohomological dimension, Hoschild cohomology, rewriting system, collapsing scheme, discrete Morse theory.

This work was supported by the EPSRC grant EP/N033353/1 'Special inverse monoids: subgroups, structure, geometry, rewriting systems and the word problem'. The second author was supported in part by United States-Israel Binational Science Foundation #2012080 and NSA MSP #H98230-16-1-0047.

Property F_n was introduced by C. T. C. Wall in [Wal65]. A group G is said to be of type F_n if it has an Eilenberg–MacLane complex $K(G,1)$ with finite n-skeleton. It is easily verified that a group is finitely generated if and only if it is of type F_1 and is finitely presented if and only if it is of type F_2. Thus property F_n generalises the two fundamental finiteness properties of being finitely generated, or finitely presented, to higher dimensions. The geometric dimension of G, denoted $\mathrm{gd}(G)$, is the smallest non-negative integer n such that there exists an n-dimensional $K(G,1)$ complex. If no such n exists, then we set $\mathrm{gd}(G) = \infty$. For more general background on higher dimensional finiteness properties of groups we refer the reader to the books [Bro94, Chapter 8], [Geo08, Chapters 6-9], or the survey article [Bro10].

Each of these topological finiteness properties has a natural counterpart in homological algebra given in terms of the existence of projective resolutions of $\mathbb{Z}G$-modules. The analogue of F_n in this context is the homological finiteness property FP_n, while geometric dimension corresponds to the cohomological dimension of the group. Recall that a group G is said to be of type FP_n (for a positive integer n) if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for $i \leq n$. We say that G is of type FP_∞ if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ with P_i finitely generated for all i. The property FP_n was introduced for groups by Bieri in [Bie76] and since then has received a great deal of attention in the literature; see [BB97, BH01, Bro87, BW07, FMWZ13, GS06]. For groups, F_n and FP_n are equivalent for $n = 0, 1$, while important results of Bestvina and Brady [BB97] show that FP_2 is definitely weaker than F_2. For higher n there are no further differences, in that a group G is of type F_n $(2 \leq n \leq \infty)$ if and only if it is finitely presented and of type FP_n.

The cohomological dimension of a group G, denoted $\mathrm{cd}(G)$, is the smallest non-negative integer n such that there exists a projective resolution $P = (P_i)_{i \geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ of length $\leq n$, i.e., satisfying $P_i = 0$ for $i > n$. (Or, if no such n exists, then we set $\mathrm{cd}(G) = \infty$.) The geometric dimension of a group provides an upper bound for the cohomological dimension. It is easily seen that $\mathrm{gd}(G) = \mathrm{cd}(G) = 0$ if and only if G is trivial. It follows from important results of Stallings [Sta68] and Swan [Swa69] that $\mathrm{gd}(G) = \mathrm{cd}(G) = 1$ if and only if G is non-trivial free group. Eilenberg and Ganea [EG57] proved that for $n \geq 3$ the cohomological and the geometric dimensions of a group are the same. The famous Eilenberg–Ganea problem asks whether this also holds in dimension two.

Working in the more general context of monoids, and projective resolutions of left $\mathbb{Z}M$-modules, gives the notion of left-FP_n, and left cohomological dimension, of a monoid M. There is an obvious dual notion of monoids of type right-FP_n, working with right $\mathbb{Z}M$-modules. Working instead with bimodules resolutions of the $(\mathbb{Z}M, \mathbb{Z}M)$-bimodule $\mathbb{Z}M$ one obtains the notion *bi*-FP_n introduced in [KO01]. Property bi-FP_n is of interest from the point of view of Hochschild cohomology, which is the standard notion of cohomology for rings; [Hoc45], [Wei94, Chapter 9], or [Mit72]. For monoids all these notions of FP_n are known to be different, while for groups they are all equivalent; see [Coh92, Pri06]. Similarly there is a dual notion of the right cohomological dimension of a monoid which again is in general not equal to the left cohomological dimension; see [GP98]. The two-sided notion is the Hochschild cohomological dimension [Mit72].

In monoid and semigroup theory the property FP_n arises naturally in the study of string rewriting systems. The history of rewriting systems in monoids and groups has roots in fundamental work of Dehn and Thue. A central topic in this area is the study of complete rewriting systems and in methods for computing normal forms. A finite complete rewriting system is a finite presentation for a monoid of a particular form (both confluent and Noetherian) which in particular gives a solution of the word problem for the monoid; see [BO93]. It is therefore of considerable interest to develop an understanding of which monoids are presentable by such rewriting systems. Many important classes of groups are known to be presentable by finite complete rewriting systems, including surface groups, Coxeter groups, and many closed three-manifold groups. Rewriting systems continue to receive a lot of attention in the literature; see [Cho06, CS09, DDM09, GS08, HS99]. The connection between complete rewriting systems and homological finiteness properties is given by a result of Anick [Ani86] (see also [Bro92]) which shows that a monoid that admits such a presentation must be of type left- and right-FP_∞; the special case of FP_3 was also handled by Squier [Squ87]. More generally Kobayashi [Kob05] proved that any such monoid is of type bi-FP_∞. A range of other interesting related homotopical and homological finiteness properties have been studied in relation to monoids defined by compete rewriting systems; see [GM12, PO04, PO05, SOK94]. Results on cohomology, and cohomological dimension, of monoids include [AR67, GP98, Nic69, Nov98, Nun95]. The cohomological dimension of left regular bands was recently considered in [MSS15b] and [MSS15a] where connections with the Leray number of simplicial complexes [KM05] and the homology of cell complexes was obtained.

It is often easier to establish the topological finiteness properties F_n for a group than the homological finiteness properties FP_n, especially if there is a suitable geometry or topological space available on which the group acts cocompactly. The desired homological finiteness properties can then be derived by the above-mentioned result for groups, that F_n (for $n \geq 2$) is equivalent to being finitely presented and of type FP_n. In contrast, no corresponding theory of F_n for monoids currently exists. Similarly, there is currently no analogue of geometric dimension of monoids in the literature. The study of homological finiteness properties of monoids should greatly profit from the development of a corresponding theory of topological finiteness properties of monoids. The central aim of the present article is to lay the foundations of such a theory and show how it can be applied to prove results about homological properties of monoids using topology. This article is the first of four papers (the other three being [GS18, GS19, GS20]) in which we prove several new results about homological properties of monoids using the topological approach introduced here.

When attempting to develop a theory of topological finiteness properties of monoids a number of stumbling blocks exist that those familiar with the corresponding theory for groups might not think of as an issue at first sight. We shall now outline some of the key difficulties in developing such a theory for monoids, and explain how the new approaches that we introduce and develop in this paper overcome these obstacles.

The first key issue that arises is that that there is no fundamental monoid of a space and monoid actions by covering transformations and so the classical notion of a classifying space of a monoid is not well behaved and has no applications to homological properties. In more detail, for a theory of topological finiteness properties of monoids to be useful in the study of homological finiteness properties the definitions need to be made in such a way that left-F_n implies left-FP_n, and that the left geometric dimension provides an upper bound for the left cohomological dimension. The fundamental question that needs to be addressed when developing this theory is to determine the correct analogue of the $K(G,1)$-complex in the theory for monoids? There is a natural notion of classifying space $|BM|$ of a monoid M. This is obtained by viewing M as a one-point category, letting BM denote the nerve of this category, and setting $|BM|$ as the geometric realisation of the nerve; see Section 5 for full details of this construction. For a group G this space $|BG|$ is a $K(G,1)$-complex, it is the canonical complex for G. Since $K(G,1)$s are unique up to homotopy equivalence the finiteness conditions F_n and cohomological dimension can all be defined in terms of existence of CW complexes homotopy equivalent to $|BG|$ satisfying the appropriate finiteness property. Indeed in group theory it is a common approach in the study of these topological finiteness properties to begin with the space $|BG|$

and then seek transformations on the space which preserve the homotopy equivalence class, but make the space smaller. It could be regarded as natural therefore to try define and study topological finiteness properties of a monoid M in terms of the space $|BM|$. We note that there is an extensive literature on the study of classifying spaces $|BM|$ of monoids and related topics; see for instance [Fie84, Hur89, KT76, LN01, McD79, MS76, Nun95].

It turns out, however, that using $|BM|$ to define topological finiteness properties of monoids is not the right approach in the sense that it would lead to a definition of F_n for monoids which does not imply left- or right-FP_n, and there are similar issues for the corresponding definition of geometric dimension. Indeed, by applying results of MacDuff [McD79] it is possible to show that there are examples of monoids which are not of type left-FP_1 even though $|BM|$ is contractible. For example, if M is an infinite left zero semigroup (a semigroup with multiplication $xy = x$ for all elements x and y) with an identity adjoined then by [McD79, Lemma 5] the space $|BM|$ is contractible while it is straightforward to show that M does not even satisfy the property left-FP_1 (this also follows from Theorem 6.13 below). This shows that one should not define property F_n for monoids using the space $|BM|$. Similar comments apply to attempts to define geometric dimension–if one tries to define geometric dimension using $|BM|$ then if M is any monoid with a left zero element but no right zero element, the left $\mathrm{cd}(M)$ would not equal zero (by Proposition 6.28) while the geometric dimension would be zero.

This issue in fact arose in work of Brown [Bro92] when he introduced the theory of collapsing schemes. In that paper Brown shows that if a monoid M admits a presentation by a finite complete rewriting system, then $|BM|$ has the homotopy type of a CW complex with only finitely many cells in each dimension. When M is a group this automatically implies that the group is of type FP_∞. Brown goes on to comment

> *"We would like, more generally, to construct a 'small' resolution of this type for any monoid M with a good set of normal forms, not just for groups. I do not know any way to formally deduce such a resolution from the existence of the homotopy equivalence for $|BM|$ above".*

As the comments above show, just knowing about the homotopy equivalence class of $|BM|$ will never suffice in order to deduce that the monoid is left- (or right-) FP_∞.

Since the space $|BM|$ cannot be used to give useful definitions of F_n, or geometic dimension, for monoids a different approach is needed. The results we prove in this paper will show that, in fact, the correct framework for studying topological finiteness properties of monoids is to develop an M-equivariant homotopy theory and use this to define an appropriate notion of a left equivariant classifying space of a monoid. The definitions of left-F_n and left geometric dimension of a monoid may then naturally be given in terms of the existence of a left equivariant classifying space for the monoid satisfying appropriate finiteness properties.

To achieve this, we need to resolve the problem of finding the correct notion of a monoid action on a CW complex to get a theory for monoids that can successfully be used in applications. It is not initially obvious what the correct approach for monoids should be. This can be seen, for example, by the fact that Bieri and Renz in their work on higher Sigma invariants (see [BR88] and Renz's PhD thesis, page 30) give a notion of a free M-CW complex for a monoid M that requires the monoid to act by injective cellular maps and to have the stabilizer of each point be trivial. But if a monoid M has elements m, m' with $mm' = m'$ and $m \neq 1$, then it can never act on a cell complex with trivial stabilizers. This happens in particular if the monoid M has non-identity idempotents. Hence their definition of free M-CW complex only applies to a very restricted class of monoids. So to develop a theory that works for arbitrary monoids, as we do in this paper, one must be careful with the notion of a free action. We note that, in the special case of monoids that embed naturally in their group of fractions (which is what happens in the theory of Σ-invariants) one can, in fact, reformulate their higher topological invariants in terms of the topological finiteness properties we introduce in this paper of the monoid associated to a character, or valuation.

The discussion above shows that past topological approaches to studying homological properties of monoids either do not work, like trying to use the classifying space $|BM|$, or they do work but only for very particular and restricted classes of monoids, like the free M-CW complex definition from the work of Bieri and Renz discussed above. Given this, to develop the correct theory of topological finiteness properties of monoids in this paper we need to give a new definition of a monoid action on a CW complex that works for arbitrary monoids, and that also can be applied to prove results about homological properties of monoids. We now highlight some examples of the problems that we resolve in this paper in order to obtain the correct notion of M-CW complex, and the corresponding theory of topological finiteness properties of monoids, that we introduce here.

- Monoids do not act by permutations and so when a monoid acts on a CW complex one needs to decide what kinds of maps should be allowed. We will see in this paper that the correct notion of an M-CW complex gives actions that do not allow cells to be mapped to cells of a different dimension, but do allow two different cells to be mapped to the same cell. So for example, the two endpoints of an edge may get collapsed by a monoid element, but the edge cannot be collapsed to a point.
- There are actions of monoids that are projective (in the categorical sense) but not free, and these can be used to build projective resolutions and we need to encompass these in our theory. We will show in Section 6 that there are examples where it is possible to find an M-finite equivariant classifying space for M which is projective but where no M-finite free equivariant classifying space exists. This will be relevant when considering geometric dimension. Also, for our applications in later papers [GS18, GS19] it is necessary to use projective M-CW complexes that are not free, so this is an essential part of the theory.
- If H is a subgroup of a group G then $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module and G is a free H-set. This does not usually happen with monoids and so base change must be handled carefully (see Section 3).
- We need to prove equivariant versions of the standard Whitehead theorems for projective M-CW complexes because we cannot exploit covering space theory as is typically done in group theory. We need this to get the uniqueness of equivariant classifying spaces up to equivariant homotopy that we prove in Section 6.
- There are added technical issues when dealing with the bimodule version of F_n, since we need to act on disconnected CW complexes on both the left and right; see Section 7. These issues do not arise for groups, since for groups the properties left- right- and bi-F_n all coincide.

By resolving the issues listed above, in this paper we introduce for this first time a theory of topological finiteness properties of monoids that can successfully applied to prove results about homological properties of monoids. Specifically, in this paper we shall define a left equivariant classifying space as a contractible projective M-CW complex (see Section 2) and then use this to define in a natural way the corresponding finiteness properties left-F_n and left geometric dimension. Their are obvious dual notions of right-F_n and right geometric dimension, and we also develop the two-sided analogue of this theory with corresponding finiteness properties bi-F_n and geometric dimension.

Applying the results and theory we introduce in this paper we have been able to prove several new results about homological properties of monoids which had previously been out of reach using only algebraic means. The success we have found in applying these methods, both in this paper and also in the papers [GS18, GS19, GS20], shows that the new definitions and methods we introduce in this article are the right ones. We now summarise some of the main applications that have been obtained, so far, of the results from this paper.

One major application of the methods we introduce in this paper is the solution [GS18, GS19] to Kobayashi's problem [Kob00] on homological finiteness properties of one-relator monoids, that is, monoids defined by presentations of the form $\langle A \mid u = v \rangle$. It is a classical result of Magnus (see [LS01, Chapter 4, Section 5]) that one-relator groups have decidable word problem. In contrast, it is a longstanding open problem whether all one-relator monoids have decidable word problem. Closely related to this is the open problem of whether every one-relator monoid admits a presentation by a finite complete rewriting system. A positive answer to this question would show decidability of the word problem for one-relator monoids. Given this, and since by Anick [Ani86] any monoid admitting such a presentation must be of type left- and right-FP_∞, it is natural to ask whether all one-relator monoids are of type left- and right-FP_∞. This question was posed by Kobayashi in [Kob00] where he proved that one-relator monoids are of type FP_3. Additional motivation for this question comes from Lyndon's identity theorem [Lyn50] which gives resolutions for one-relator groups that imply, among other things, that all one-relator groups are of type FP_∞. In [GS18, GS19] we use the topological foundations from this paper to prove that all one-relator monoids are of type left- and right-F_∞ (and hence also FP_∞), answering positively Kobayashi's question. We also apply our results to classify the one-relator monoids of cohomological dimension at most 2, and to describe the relation module of a torsion-free one-relator monoid presentation. These results are an analogue of Lyndon's identity theorem for one-relator monoids.

In [GS18] we apply the topological methods developed in this paper to prove results about the homological finiteness properties of special monoids, that is, monoids defined by finite presentations of the form $\langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle$. We prove that if M is a special monoid with group of units G then if G is of type FP_n with $1 \leq n \leq \infty$, then M is of type left- and right-FP_n; and moreover that $\operatorname{cd} G \leq \operatorname{cd} M \leq \max\{2, \operatorname{cd} G\}$. As a corollary we obtain that all special one-relator monoids are of type left- and right-FP_∞, which is an important case in our Lyndon's identity theorem for arbitrary one-relator monoids described above.

Also in [GS18] we use our topological methods to show that a Bass-Serre theory á la graphs of spaces can be done equivariantly to construct equivariant classifying spaces for certain amalgamated free products and HNN extensions of monoids. We then apply this to prove new results in [GS18] about the closure properties of left-F_n and bi-F_n for: amalgamated free products of monoids (simplifying and vastly improving on some results from [CO98]), HNN-like extensions in the sense of Otto and Pride [PO04] (in particular generalising [PO04, Theorem 1] to higher dimensions), and HNN extensions of the sort considered in [How63].

As explained above, Brown's original topological proof of the Anick–Groves–Squier theorem for groups using $|BG|$ cannot work for monoids using $|BM|$. One application of the theory of topological finiteness properties of monoids introduced in this paper is given in this paper in Sections 8-11 where we use our methods to introduce an M-equivariant collapsing scheme theory for monoids and then apply this to give a topological proof of the Anick–Groves–Squier theorem for monoids. In more detail, in Sections 8-11 we introduce the notion of an M-equivariant collapsing scheme for a monoid and prove several theorems which, when combined with the foundational results from earlier in the paper, lead to Theorem 10.2, and Corollary 10.3, which give new results giving sufficient conditions for a monoid to be of type F_∞ (or bi-F_∞) via the notion of a guarded collapsing scheme. These results are in part applications of the theory we develop earlier in the paper, although several additional new definitions and proofs are also needed, including the notions of M-equivariant collapsing scheme, and guarded collapsing scheme, both of which we introduce in this paper. Not only do these results give topological proofs of both the Anick–Groves–Squier Theorem [Ani86] in the one-sided case and a theorem of Kobayashi [Kob05] in the two-sided case (see Corollaries 11.2 and 11.3) for monoids defined by finite complete rewriting systems, but our theorem also applies to other families of monoids including e.g. monoids that admit a factorability structure in the sense of [HO14] (see Corollary 11.4).

The paper is structured as follows. We begin in Section 2 by introducing our definitions of free and projective M-CW complexes, and proving some foundational results including the M-equivariant Whitehead theorems and cellular approximation theorem. In Section 3 some useful base change theorems are established, and then in Sections 4 and 5 we give canonical models of M-equivariant (respectively two-sided M-equivariant) classifying spaces of a monoid, by taking the geometric realisations of the right Cayley graph category (respectively two-sided Cayley graph category) of the monoid. In Section 6 we will define left-equivariant, and dually right-equivariant, classifying spaces for a monoid M. We prove that such spaces always exist, and that they are unique up to M-homotopy equivalence. Using the notion of left-equivariant classifying space we define property left-F_n for monoids. We prove several fundamental results about this property, including results showing its relationship with property left-FP_n, its connection with the properties of being finitely generated and finitely presented, and results relating the property holding in M to it holding in certain maximal subgroups of M. We also introduce the notion of left geometric dimension of a monoid in this section. We show that it is possible to find an M-finite equivariant classifying space for M which is projective when no M-finite free equivariant classifying space exists, justifying our choice to work with projective M-CW complexes. The geometric dimension is proved to provide an upper bound for the cohomological dimension, and we characterize monoids with left geometric dimension equal to zero. In Section 7 we introduce the bilateral notion of a classifying space in order to introduce the stronger property of bi-F_n. We prove results for bi-F_n analogous to those previously established for left- and right-F_n. In particular we show that bi-F_n implies bi-FP_n which is of interest from the point of view of Hochschild cohomology. We also define the geometric dimension as the minimum dimension of a bi-equivariant classifying space and show how it relates to the Hochschild dimension. Finally, as already explained above, in Sections 8-10 we introduce the notion of an M-equivariant collapsing scheme for a monoid and prove several theorems which, when combined with the foundational results from earlier in the paper, give new results giving sufficient conditions for a monoid to be of type F_∞ (or bi-F_∞) via the notion of a guarded collapsing scheme. In Section 11, we show that these results give topological proofs of both Anick–Groves–Squier Theorem [Ani86], Kobayashi's theorem [Kob05], and can also be applied to monoids that admit a factorability structure in the sense of [HO14].

2. Projective M-CW complexes and M-homotopy theory

2.1. **CW complexes.** For background on CW complexes, homotopy theory, and algebraic topology for group theory, we refer the reader to [Geo08] and [May99]. Throughout B^n will denote the closed unit ball in $\mathbb{R}^n$, S^{n-1} the $(n-1)$-sphere which is the boundary ∂B^n of the n-ball, and $I = [0,1]$ the unit interval. We use e^n to denote an open n-cell, homeomorphic to the open n ball $\mathring{B}^n = B^n - \partial B^n$, ∂e denotes the boundary of e and $\bar{e} = cl(e)$ the closure of e, respectively. We identify $I^r = I^r \times 0 \subset I^{r+1}$.

A *CW complex* is a space X which is a union of subspaces X_n such that, inductively, X_0 is a discrete set of points, and X_n is obtained from X_{n-1} by attaching balls B^n along *attaching maps* $j\colon S^{n-1} \to X_{n-1}$. The resulting maps $B^n \to X_n$ are called the *characteristic maps*. So X_n is the quotient space obtained from $X_{n-1} \cup (J_n \times B^n)$ by identifying (j, x) with $j(x)$ for $x \in S^{n-1}$, where J_n is the discrete set of such attaching maps. Thus X_n is obtained as a pushout of spaces:

$$\begin{array}{ccc} J_n \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ J_n \times B^n & \longrightarrow & X_n. \end{array}$$

The topology of X should be that of the inductive limit $X = \varinjlim X_n$

A CW complex X is then equal as a set to the disjoint union of (open) cells $X = \bigcup_\alpha e_\alpha$ where the e_α are the images of $\mathring{B}^n$ under the characteristic maps. Indeed, an alternative way of defining CW complex, which shall be useful for us to use later on, is as follows. A CW complex is a Hausdorff space K along with a family $\{e_\alpha\}$ of open cells of various dimensions such that, letting

$$K^j = \bigcup \{e_\alpha \colon \dim e_\alpha \leq j\},$$

the following conditions are satisfied

(CW1) $K = \bigcup_\alpha e_\alpha$ and $e_\alpha \cap e_\beta = \varnothing$ for $\alpha \neq \beta$.

(CW2) For each cell e_α there is a map $\varphi_\alpha \colon B^n \to K$ (called the characteristic map) where B^n is a topological ball of dimension $n = \dim e_\alpha$, such that

(a) $\varphi_\alpha|_{\mathring{B}^n}$ is a homeomorphism onto e_α;

(b) $\varphi_\alpha(\partial B^n) \subset K^{n-1}$.

(CW3) Each $\overline{e_{\alpha_0}}$ is contained in a union of finitely many e_α.

(CW4) A set $A \subset K$ is closed in K if and only if $A \cap \overline{e}_\alpha$ is closed in $\overline{e}_\alpha$ for all e_α.

Note that each characteristic map $\varphi \colon B^n \to K$ gives rise to a characteristic map $\varphi' \colon I^n \to K$ be setting $\varphi' = \varphi h$ for some homeomorphism $h \colon I^n \to B^n$. So we can restrict our attention to characteristic maps with domain I^n when convenient. If $\varphi \colon B^n \to K$ is a characteristic map for a cell e then $\varphi|_{\partial B^n}$ is called an attaching map for e. A *subcomplex* is a subset $L \subset K$ with a subfamily $\{e_\beta\}$ of cells such that $L = \bigcup e_\beta$ and every $\overline{e_\beta}$ is contained in L. If L is a subcomplex of K we write $L < K$ and call (K, L) a CW pair. If e is a cell of K which does not lie in (and hence does not meet) L we write $e \in K - L$. An isomorphism between CW complexes is a homeomorphism that maps cells to cells.

Let M be a monoid. We shall define notions of free and projective M-CW complexes and then use these to study topological finiteness properties of M. The notion of a free M-CW complex is a special case of a free C-CW complex for a category C considered by Davis and Lück in [DL98] and so the cellular approximation theorem, HELP Theorem and Whitehead Theorem in this case can be deduced from their results. The HELP Theorem and Whitehead Theorem for projective M-CW complexes can be extracted with some work from the more general results of Farjoun [DFZ86] on diagrams of spaces but to keep things elementary and self-contained we present them here.

2.2. **The category of M-sets.** A left M-set consists of a set X and a mapping $M \times X \to X$ written $(m, x) \mapsto mx$ called a left action, such that $1x = x$ and $m(nx) = (mn)x$ for all $m, n \in M$ and $x \in X$. Right M-sets are defined dually, they are the same thing as left M^{op}-sets. A *bi-M-set* is an $M \times M^{op}$-set. There is a category of M-sets and M-equivariant mappings, where $f \colon X \to Y$ is M-equivariant if $f(mx) = mf(x)$ for all $x \in X$, $m \in M$.

A (left) M-set X is said to be *free* on a set A if there is a mapping $\iota\colon A \to X$ such that for any mapping $f\colon A \to Y$ with Y an M-set, there is a unique M-equivariant map $F\colon X \to Y$ such that

$$\begin{array}{ccc} A & \xrightarrow{\iota} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} \\ & & Y \end{array}$$

commutes. The mapping ι is necessarily injective. If X is an M-set and $A \subseteq X$, then A is a free basis for X if and only if each element of X can be uniquely expressed as ma with $m \in M$ and $a \in A$.

The free left M-set on A exists and can be realised as the set $M \times A$ with action $m(m', a) = (mm', a)$ and ι is the map $a \mapsto (1, a)$. Note that if G is a group, then a left G-set X is free if and only if G acts freely on X, that is, each element of X has trivial stabilizer. In this case, any set of orbit representatives is a basis.

An M-set P is *projective* if any M-equivariant surjective mapping $f\colon X \to P$ has an M-equivariant section $s\colon P \to X$ with $f \circ s = 1_P$. Free M-sets are projective and an M-set is projective if and only if it is a retract of a free one.

Each projective M-set P is isomorphic to an M-set of the form $\coprod_{a\in A} Me_a$ (disjoint union, which is the coproduct in the category of M-sets) with $e_a \in E(M)$. Here $E(M)$ denotes the set of idempotents of the monoid M. In particular, projective G-sets are the same thing as free G-sets for a group G. (See [Kna72] for more details.)

2.3. **Equivariant CW complexes.** A *left M-space* is a topological space X with a continuous left action $M \times X \to X$ where M has the discrete topology. A right M-space is the same thing as an M^{op}-space and a *bi-M-space* is an $M \times M^{op}$-space. Each M-set can be viewed as a discrete M-space. Note that colimits in the category of M-spaces are formed by taking colimits in the category of spaces and observing that the result has a natural M-action.

Let us define a (projective) *M-cell* of dimension n to be an M-space of the form $Me \times B^n$ where $e \in E(M)$ and B^n has the trivial action; if $e = 1$, we call it a *free M-cell*. We will define a projective M-CW complex in an inductive fashion by imitating the usual definition of a CW complex but by attaching M-cells $Me \times B^n$ via M-equivariant maps from $Me \times S^{n-1}$ to the $(n-1)$-skeleton.

Formally, a *projective (left) relative M-CW complex* is a pair (X, A) of M-spaces such that $X = \varinjlim X_n$ with $i_n\colon X_n \to X_{n+1}$ inclusions, $X_{-1} = A$, $X_0 = P_0 \cup A$ with P_0 a projective M-set and where X_n is obtained as a pushout of M-spaces

$$\begin{array}{ccc} P_n \times S^{n-1} & \longrightarrow & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ P_n \times B^n & \longrightarrow & X_n \end{array} \tag{2.1}$$

with P_n a projective M-set and B^n having a trivial M-action for $n \geq 1$. As usual, X_n is called the *n-skeleton* of X and if $X_n = X$ and $P_n \neq \emptyset$, then X is said to have *dimension n*. Notice that since P_n is isomorphic to a coproduct of M-sets of the form Me with $e \in E(M)$, we are indeed attaching M-cells at each step. If $A = \emptyset$, we call X a *projective M-CW complex*. Note that a projective M-CW complex is a CW complex and the M-action is cellular (in fact, takes n-cells to n-cells). We can define projective right M-CW complexes and projective bi-M-CW complexes by replacing M with M^{op} and $M \times M^{op}$, respectively. We say that X is a *free M-CW complex* if each P_n is a free M-set. If G is a group, a CW complex with a G-action is a free G-CW complex if and only if G acts freely, cellularly, taking cells to cells, and the setwise stabilizer of each cell is trivial [Geo08, Appendix of Section 4.1].

More generally we define an M-CW complex in the same way as above except that the P_i are allowed to be arbitrary M-sets. Most of the theory developed below is only valid in the projective setting, but there will be a few occasions (e.g. when we discuss M-simplicial sets) where it will be useful for us to be able to refer to M-CW complexes in general. For future reference we should note here that, just as for the theory of G-CW complexes, there is an alternative way of defining M-CW complex in terms of monoids acting on CW complexes. This follows the same lines as that of groups, see for example [Geo08, Section 3.2 and page 110] or [May96]. Let Y be a left M-space where M is a monoid and $Y = \bigcup_\alpha e_\alpha$ is a CW complex with characteristic maps $\varphi_\alpha \colon B^n \to Y$. We say that Y is a *rigid left M-CW complex* if it is:

- Cellular and dimension preserving: For every e_α and $m \in M$ there exists an e_β such that $me_\alpha = e_\beta$ and $\dim(e_\beta) = \dim(e_\alpha)$; and

- Rigid on cells: If $me_\alpha = e_\beta$ then $m\varphi_\alpha(k') = \varphi_\beta(k')$ for all $k' \in B^n - \partial B^n$.

If the action of M on the set of n-cells is free (respectively projective) then we call Y a free (respectively projective) rigid left M-CW complex. The inductive process described above for building (projective, free) left M-CW complexes is easily seen to give rise to rigid (projective, free) left M-CW complexes, in the above sense. Conversely every rigid (projective, free) left M-CW complex arises in this way. In other words, the two definitions are equivalent. For an explanation of this in the case of G-CW complexes see, for example, [Geo08, page 110]. The proof for monoids is analogous and is omitted. Similar comments apply for rigid right M-CW complexes and rigid bi-M-CW complexes.

We say that a projective M-CW complex X is of *M-finite type* if P_n is a finitely generated projective M-set for each n and we say that X is *M-finite* if it is finite dimensional and of M-finite type (i.e., X is constructed from finitely many M-cells).

Notice that if $m \in M$, then mX is a subcomplex of X for all $m \in M$ with n-skeleton mX_n. Indeed, $mX_0 = mP_0$ is a discrete set of points and mX_n is obtained from mX_{n-1} via the pushout diagam

$$\begin{array}{ccc} mP_n \times S^{n-1} & \longrightarrow & mX_{n-1} \\ \big\downarrow & & \big\downarrow \\ mP_n \times B^n & \longrightarrow & mX_n. \end{array}$$

A *projective M-CW subcomplex* of X is an M-invariant subcomplex $A \subseteq X$ which is a union of M-cells of X. In other words, each P_n (as above) can be written $P_n = P_n' \coprod P_n''$ with the images of the $P_n' \times B^n$ giving the cells of A. Notice that if A is a projective M-CW subcomplex of X, then (X, A) can be viewed as a projective relative M-CW complex in a natural way. Also note that a cell of X belongs to A if and only if each of its translates do.

A projective $\{1\}$-CW complex is the same thing as a CW complex and $\{1\}$-finite type ($\{1\}$-finite) is the same thing as finite type (finite).

If $e \in E(M)$ is an idempotent and $m \in Me$, then left multiplication by m induces an isomorphism $H_n(\{e\} \times B^n, \{e\} \times S^{n-1}) \to H_n(\{m\} \times B^n, \{m\} \times S^{n-1})$ (since it induces a homeomorphism $\{e\} \times B^n/\{e\} \times S^{n-1} \to \{m\} \times B^n/\{m\} \times S^{n-1}$) and so if we choose an orientation for the n-cell $\{e\} \times B^n$, then we can give $\{m\} \times B^n$ the orientation induced by this isomorphism. If $m \in M$ and $m' \in Me$, then the isomorphism

$$H_n(\{e\} \times B^n, \{e\} \times S^{n-1}) \to H_n(\{mm'\} \times B^n, \{mm'\} \times S^{n-1})$$

induced by mm' is the composition of the isomorphism

$$H_n(\{e\} \times B^n, \{e\} \times S^{n-1}) \to H_n(\{m'\} \times B^n, \{m'\} \times S^{n-1})$$

induced by m' and the isomorphism

$$H_n(\{m'\} \times B^n, \{m'\} \times S^{n-1}) \to H_n(\{mm'\} \times B^n, \{mm'\} \times S^{n-1})$$

induced by m and so the action of m preserves orientation. We conclude that the degree n component of the cellular chain complex for X is isomorphic to $\mathbb{Z}P_n$ as a $\mathbb{Z}M$-module and hence is projective (since $\mathbb{Z}\left[\coprod_{a\in A} Me_a\right] \cong \bigoplus_{a\in A} \mathbb{Z}Me_a$ and $\mathbb{Z}M \cong \mathbb{Z}Me \oplus \mathbb{Z}M(1-e)$ for any idempotent $e \in E(M)$).

If X is a projective M-CW complex then so is $Y = M \times I$ where I is given the trivial action. If we retain the above notation, then $Y_0 = X_0 \times \partial I \cong X_0 \coprod X_0$. The n-cells for $n \geq 1$ are obtained from attaching $P_n \times B^n \times \partial I \cong (P_n \coprod P_n) \times B^n$ and $P_{n-1} \times B^{n-1} \times I$. Notice that $X \times \partial I$ is a projective M-CW subcomplex of $X \times I$.

If X, Y are M-spaces, then an *M-homotopy* between M-equivariant continuous maps $f, g\colon X \to Y$ is an M-equivariant mapping $H\colon X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for $x \in X$ where I is viewed as having the trivial M-action. We write $f \simeq_M g$ in this case. We say that X, Y are *M-homotopy equivalent*, written $X \simeq_M Y$, if there are M-equivariant continuous mappings (called *M-homotopy equivalences*) $f\colon X \to Y$ and $g\colon Y \to X$ with $gf \simeq_M 1_X$ and $fg \simeq_M 1_Y$. We write $[X, Y]_M$ for the set of M-homotopy classes of M-equivariant continuous mappings $X \to Y$.

Lemma 2.1. *Let X, Y be projective M-CW complexes and A a projective M-CW subcomplex of X. Let $f\colon A \to Y$ be a continuous M-equivariant cellular map. Then the pushout $X \coprod_A Y$ is a projective M-CW complex.*

Proof. It is a standard result that $X \coprod_A Y$ is a CW complex whose n-cells are the n-cells of Y together with the n-cells of X not belonging to A. In more detail, let $q\colon X \to X \coprod_A Y$ be the canonical mapping and view Y as a subspace of the adjunction space. Then the attaching map of a cell coming from Y is the original attaching map, whereas the attaching map of a cell of X not belonging to A is the composition of q with its original attaching mapping. It follows from the definition of a projective M-CW subcomplex and the construction that $X \coprod_A Y$ is a projective M-CW complex. Here it is important that a translate by M of a cell from $X \setminus A$ is a cell of $X \setminus A$. □

A *free M-CW subcomplex* of a free M-CW complex X is an M-invariant subcomplex $A \subseteq X$ which is a union of M-cells of X.

The proof of Lemma 2.1 yields the following.

Lemma 2.2. *Let X, Y be free M-CW complexes and A a free M-CW subcomplex of X. Let $f\colon A \to Y$ be a continuous M-equivariant cellular map. Then the pushout $X \coprod_A Y$ is a free M-CW complex.*

A continuous mapping $f\colon X \to Y$ of spaces is an *n-equivalence* if

$$f_*\colon \pi_q(X, x) \to \pi_q(Y, f(x))$$

is a bijection for $0 \leq q < n$ and a surjection for $q = n$ where $\pi_0(Z, z) = \pi_0(Z)$ (viewed as a pointed set with base point the component of z). It is a *weak equivalence* if it is an n-equivalence for all n, i.e., f_* is a bijection for all $q \geq 0$. We will consider a weak equivalence as an ∞-equivalence. We shall see later that an M-equivariant weak equivalence of projective M-CW complexes is an M-homotopy equivalence.

Let $\mathrm{Top}(X,Y)$ denote the set of continuous maps $X \to Y$ for spaces X, Y and $\mathrm{Top}_M(X,Y)$ denote the set of continuous M-equivariant maps $X \to Y$ between M-spaces X, Y.

Proposition 2.3. *Let X be a space with a trivial M-action, $e \in E(M)$ and Y an M-space. Then there is a bijection between $\mathrm{Top}_M(Me \times X, Y)$ and $\mathrm{Top}(X, eY)$. The bijection sends $f\colon Me \times X \to Y$ to $\overline{f}\colon X \to eY$ given by $\overline{f}(x) = f(e,x)$ and $g\colon X \to eY$ to $\widehat{g}\colon Me \times X \to Y$ given by $\widehat{g}(m,x) = mg(x)$.*

Proof. If $x \in X$, then $\overline{f}(x) = f(e,x) = f(e(e,x)) = ef(e,x) \in eY$. Clearly, $\overline{f}$ is continuous. As $\widehat{g}$ is the composition of $1_{Me} \times g$ with the action map, it follows that $\widehat{g}$ is continuous. We show that the two constructions are mutually inverse. First we check that

$$\widehat{\overline{f}}(m,x) = m\overline{f}(x) = mf(e,x) = f(m(e,x)) = f(me,x) = f(m,x)$$

for $m \in Me$ and $x \in X$. Next we compute that

$$\overline{\widehat{g}}(x) = \widehat{g}(e,x) = eg(x) = g(x)$$

since $g(x) \in eY$. This completes the proof. □

Proposition 2.3 is the key tool to transform statements about projective M-CW complexes into statement about CW complexes. We shall also need the following lemma relating equivariant n-equivalences and n-equivalences.

Lemma 2.4. *Let Y, Z be M-spaces and let $k\colon Y \to Z$ be an M-equivariant n-equivalence with $0 \leq n \leq \infty$. Let $e \in E(M)$ and $k' = k|_{eY}\colon eY \to eZ$. Then k' is an n-equivalence.*

Proof. First note that $k(ey) = ek(y)$ and so $k|_{eY}$ does indeed have image contained in eZ. Let $y \in eY$ and $q \geq 0$. Let $\alpha\colon eY \to Y$ and $\beta\colon eZ \to Z$ be the inclusions. Then note that the action of e gives retractions $Y \to eY$ and $Z \to eZ$. Hence we have a commutative diagram

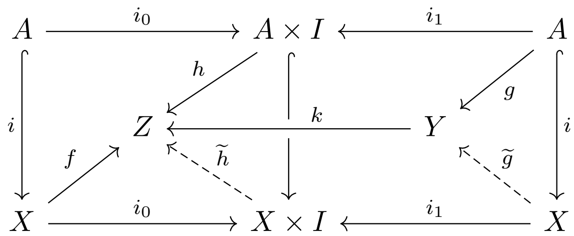
$$\begin{array}{ccc} \pi_q(Y,y) & \xrightarrow{k_*} & \pi_q(Z,k(y)) \\ {\scriptstyle\alpha_*}\uparrow\downarrow{\scriptstyle e_*} & & {\scriptstyle\beta_*}\uparrow\downarrow{\scriptstyle e_*} \\ \pi_q(eY,y) & \xrightarrow{k'_*} & \pi_q(eZ,k(y)) \end{array}$$

with $e_*\alpha_*$ and $e_*\beta_*$ identities. Therefore, if k_* is surjective, then k'_* is surjective and if k_* is injective, then k'_* is injective. The lemma follows. □

2.4. **Whitehead's theorem.** With Lemma 2.4 in hand, we can prove an M-equivariant version of HELP (homotopy extension and lifting property) [May99, Page 75], which underlies most of the usual homotopy theoretic results about CW complexes. If X is a space, then $i_j\colon X \to X \times I$, for $j = 0, 1$, is defined by $i_j(x) = (x, j)$.

Theorem 2.5 (HELP). *Let (X, A) be a projective relative M-CW complex of dimension at most $n \in \mathbb{N} \cup \{\infty\}$ and $k\colon Y \to Z$ an M-equivariant n-equivalence of M-spaces. Then given M-equivariant continuous mappings $f\colon X \to Z$, $g\colon A \to Y$ and $h\colon A \times I \to Z$ such that $kg = hi_1$ and $fi = hi_0$ (where $i\colon A \to X$ is the inclusion), there exist M-equivariant continuous mappings*

$\widetilde{g}$ and $\widetilde{h}$ making the diagram

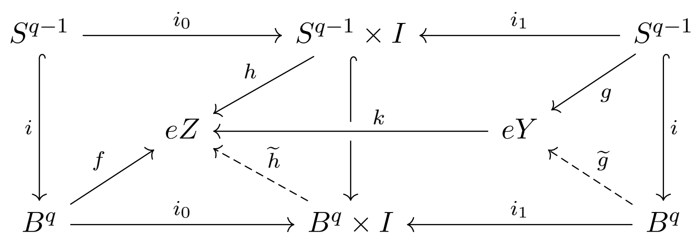


commute.

Proof. Proceeding by induction on the skeleta and adjoining an M-cell at a time, it suffices to handle the case that

$$(X, A) = (Me \times B^q, Me \times S^{q-1})$$

with $0 \leq q \leq n$. By Proposition 2.3 it suffices to find continuous mappings $\widetilde{g}$ and $\widetilde{h}$ making the diagram



commute where we have retained the notation of Proposition 2.3. The mapping $k\colon eY \to eZ$ is an n-equivalence by Lemma 2.4 and so we can apply the usual HELP theorem [May99, Page 75] for CW complexes to deduce the existence of $\widetilde{g}$ and $\widetilde{h}$. This completes the proof. $\square$

As a consequence we may deduce the M-equivariant Whitehead theorems.

Theorem 2.6 (Whitehead)**.** *If X is a projective M-CW complex and $k\colon Y \to Z$ is an M-equivariant n-equivalence of M-spaces, then the induced mapping $k_*\colon [X, Y]_M \to [X, Z]_M$ is a bijection if* $\dim X < n$ *or $n = \infty$ and a surjection if* $\dim X = n < \infty$.

Proof. For surjectivity we apply Theorem 2.5 to the pair $(X, \emptyset)$. If $f\colon X \to Z$, then $\widetilde{g}\colon X \to Y$ satisfies $kg \simeq_M f$. For injectivity, we apply Theorem 2.5 to the pair $(X \times I, X \times \partial I)$ and note that $X \times I$ has dimension one larger than X. Suppose that $p, q\colon X \to Y$ are such that $kp \simeq_M kq$ via a homotopy $f\colon X \times I \to Z$. Put $g = p \coprod q\colon X \times \partial I \to Y$ and define $h\colon X \times \partial I \times I \to Z$ by $h(x, s, t) = f(x, s)$. Then $\widetilde{g}\colon X \times I \to Y$ is a homotopy between p and q. $\square$

Corollary 2.7 (Whitehead)**.** *If $k\colon Y \to Z$ is an M-equivariant weak equivalence (n-equivalence) between projective M-CW complexes (of dimension less than n), then k is an M-homotopy equivalence.*

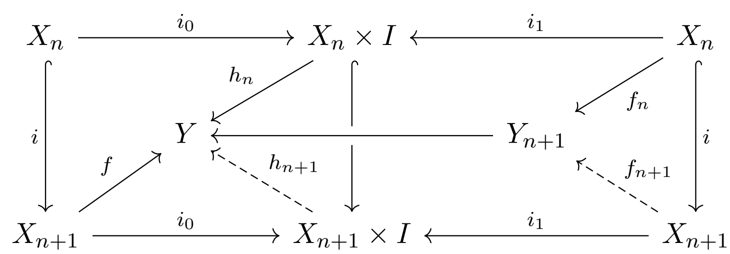
Proof. Under either hypothesis, $k_*\colon [Z, Y]_M \to [Z, Z]_M$ is a bijection by Theorem 2.6 and so $kg \simeq_M 1_Z$ for some M-equivariant $g\colon Z \to Y$. Then $kgk \simeq_M k$ and hence, since $k_*\colon [Y, Y] \to [Y, Z]$ is a bijection by Theorem 2.6, we have that $gk \simeq_M 1_Y$. This completes the proof. $\square$

2.5. **Cellular approximation.** Our next goal is to show that every M-equivariant continuous mapping of projective M-CW complexes is M-homotopy equivalent to a cellular one. We shall need the well-known fact that if Y is a CW complex, then the inclusion $Y_n \hookrightarrow Y$ is an n-equivalence for all $n \geq 0$ [May99, Page 76].

Theorem 2.8 (Cellular approximation)**.** *Let $f\colon X \to Y$ be a continuous M-equivariant mapping with X a projective M-CW complex and Y a CW complex with a continuous action of M by cellular mappings. Then f is M-homotopic to a continuous M-equivariant cellular mapping. Any two cellular approximations are homotopy equivalent via a cellular M-homotopy.*

Proof. We prove only the first statement. The second is proved using a relative version of the first whose statement and proof we omit. Note that Y_n is M-equivariant for all $n \geq 0$ because M acts by cellular mappings. We construct by induction M-equivariant continuous mappings $f_n\colon X_n \to Y_n$ such that $f|_{X_n} \simeq_M f_n|_{X_n}$ via an M-homotopy h_n and f_n, h_n extend f_{n-1}, h_{n-1}, respectively (where we take $X_{-1} = \emptyset$). We have, without loss of generality, $X_0 = \coprod_{a\in A} Me_a$. Since e_aY is a subcomplex of Y with 0-skeleton e_aY_0 and $f(e_a) \in e_aY$, we can find a path p_a in e_aY from $f(e_a)$ to an element $y_a \in e_aY_0$. Define $f_0(me_a) = my_a$ and $h_0(me_a, t) = mp_a(t)$, cf. Proposition 2.3.

Assume now that f_n, h_n have been defined. Since the inclusion $Y_{n+1} \to Y$ is an M-equivariant $(n+1)$-equivalence, Theorem 2.5 gives a commutative diagram



thereby establishing the inductive step. We obtain our desired cellular mapping and M-homotopy by taking the colimit of the f_n and h_n. □

3. Base change

If A is a right M-set and B is a left M-set, then $A \otimes_M B$ is the quotient of $A \times B$ by the least equivalence relation $\sim$ such that $(am, b) \sim (a, mb)$ for all $a \in A$, $b \in B$ and $m \in M$. We write $a \otimes b$ for the class of (a, b) and note that the mapping $(a, b) \mapsto a \otimes b$ is universal for mappings $f\colon A \times B \to X$ with X a set and $f(am, b) = f(a, mb)$. If M happens to be a group, then M acts on $A \times B$ via $m(a, b) = (am^{-1}, mb)$ and $A \otimes_M B$ is just the set of orbits of this action. The tensor product $A \otimes_M ()$ preserves all colimits because it is a left adjoint to the functor $X \mapsto X^A$.

If B is a left M-set there is a natural preorder relation $\leq$ on B where $x \leq y$ if and only if $Mx \subseteq My$. Let $\approx$ denote the symmetric-transitive closure of $\leq$. That is, $x \approx y$ if there is a sequence $z_1, z_2, \ldots, z_n$ of elements of B such that for each $0 \leq i \leq n-1$ either $z_i \leq z_{i+1}$ or $z_i \geq z_{i+1}$. This is clearly an equivalence relation and we call the $\approx$-classes of B the *weak orbits* of the M-set. This corresponds to the notion of the weakly connected components of a directed graph. If B is a right M-set then we use B/M to denote the set of weak orbits of the M-set. Dually, if B is a left M-set we use $M\backslash B$ to denote the set of weak orbits. Note that if 1 denotes the trivial right M-set and B is a left M-set, then we have $M\backslash B = 1 \otimes_M B$.

Let M, N be monoids. An *M-N-biset* is an $M \times N^{op}$-set. If A is an M-N-biset and B is a left N-set, then the equivalence relation defining $A \otimes_N B$ is left M-invariant and so $A \otimes_N B$ is a left M-set with action $m(a \otimes b) = ma \otimes b$.

Proposition 3.1. *Let A be an M-N-biset that is (finitely generated) projective as an M-set and let B be a (finitely generated) projective N-set. Then $A \otimes_N B$ is a (finitely generated) projective M-set.*

Proof. As B is a (finite) coproduct of N-sets Ne with $e \in E(N)$, it suffices to handle the case $B = Ne$. Then $A \otimes_N Ne \cong Ae$ via $a \otimes n \mapsto an$ with inverse $a \mapsto a \otimes e$ for $a \in Ae$. Now define $r\colon A \to Ae$ by $r(a) = ae$. Then r is an M-equivariant retraction. So Ae is a retract of a (finitely generated) projective and hence is a (finitely generated) projective. $\square$

If X is a left M-space and A is a right M-set, then $A \otimes_M X$ is a topological space with the quotient topology. Again the functor $A \otimes_M ()$ preserves all colimits. In fact, $A \otimes_M X$ is the coequalizer in the diagram

$$\coprod_{A\times M} X \rightrightarrows \coprod_{A} X \to A \otimes_M X$$

where the top map sends x in the (a, m)-component to mx in the a-component and the bottom map sends x in the (a, m)-component to x in the am-component.

Corollary 3.2. *If A is an M-N-biset that is projective as an M-set and X is a projective N-CW complex, then $A \otimes_N X$ is a projective M-CW complex. If A is in addition finitely generated as an M-set and X is of N-finite type, then $A \otimes_N X$ is of M-finite type. Moreover,* $\dim A \otimes_N X = \dim X$.

Proof. Since $A \otimes_N ()$ preserves colimits, $A \otimes_N X = \varinjlim A \otimes_N X_n$. Moreover, putting $X_{-1} = \emptyset$, we have that if X_n is obtained as per the pushout square (2.1), then $A \otimes_N X_n$ is obtained from the pushout square

$$\begin{array}{ccc} (A \otimes_N P_n) \times S^{n-1} & \longrightarrow & A \otimes_N X_{n-1} \\ \big\downarrow & & \big\downarrow \\ (A \otimes_N P_n) \times B^n & \longrightarrow & A \otimes_N X_n \end{array}$$

by preservation of colimits and the observation that if C is a trivial left N-set and B is a left N-set, then $A \otimes_N (B \times C) \cong (A \otimes_N B) \times C$ via $a \otimes (b, c) \mapsto (a \otimes b, c)$. The result now follows from Proposition 3.1. $\square$

By considering the special case where M is trivial and A is a singleton, and observing that a projective M-set P is finitely generated if and only if $M\backslash P$ is finite, we obtain the following corollary.

Corollary 3.3. *Let X be a projective M-CW complex. Then $M\backslash X$ is a CW complex. Moreover, X is M-finite (of M-finite type) if and only if $M\backslash X$ is finite (of finite type).*

The following observation will be used many times.

Proposition 3.4. *Let X be a locally path connected N-space and A an M-N-biset. Then $\pi_0(X)$ is an N-set and $\pi_0(A \otimes_N X) \cong A \otimes_N \pi_0(X)$.*

Proof. Note that the functor $X \mapsto \pi_0(X)$ is left adjoint to the inclusion of the category of N-sets into the category of locally path connected M-spaces and hence it preserves all colimits. The result now follows from the description of tensor products as coequalizers of coproducts. $\square$

The advantage of working with M-homotopies is that they behave well under base change.

Proposition 3.5. *Let A be an M-N-biset and let X, X' be N-homotopy equivalent N-spaces. Then $A \otimes_N X$ is M-homotopy equivalent to $A \otimes_N X'$. In particular, if Y, Z are M-spaces and $Y \simeq_M Z$, then $M\backslash Y \simeq M\backslash Z$.*

Proof. It suffices to prove that if Y, Z are N-spaces and $f, g\colon Y \to Z$ are N-homotopic N-equivariant maps, then

$$A \otimes_N f, A \otimes_N g\colon A \otimes_N Y \to A \otimes_N Z$$

are M-homotopic. This follows immediately from the identification of $A \otimes_N (Y \times I)$ with $(A \otimes_N Y) \times I$. For if $H\colon Y \times I \to Z$ is an N-homotopy between f and g, then $A \otimes_N H$ provides the M-homotopy between $A \otimes_N f$ and $A \otimes_N g$. □

The following base change lemma, and its dual, is convenient for dealing with bisets.

Lemma 3.6. *Let A be an $M \times M^{op}$-set and consider the right $M \times M^{op}$-set M with the right action $m(m_L, m_R) = mm_L$. Then A/M is a left M-set and there is an M-equivariant isomorphism $A/M \to M \otimes_{M \times M^{op}} A$.*

Proof. Clearly, $A/M = A \otimes_M 1$ is a left M-set. Write $[a]$ for the class of a in A/M. Define $f\colon A/M \to M \otimes_{M \times M^{op}} A$ by $f([a]) = 1 \otimes a$. This is well defined and M-equivariant because if $a \in A$ and $m \in M$, then $1 \otimes am = 1 \otimes (1, m)a = 1(1, m) \otimes a = 1 \otimes a$ and $1 \otimes ma = 1 \otimes (m, 1)a = 1(m, 1) \otimes a = m \otimes a$. Define $G\colon M \times A \to A/M$ by $G(m, a) = [ma]$. If $m, m_L, m_R \in M$, then $G(m(m_L, m_R), a) = G(mm_L, a) = [mm_La]$ and $G(m, (m_L, m_R)a) = [mm_Lam_R] = [mm_La]$. Therefore, G induces a well defined mapping $g\colon M \otimes_{M \times M^{op}} A \to A/M$. Then we check that $gf([a]) = g(1 \otimes a) = [a]$ and $fg(m \otimes a) = f([ma]) = 1 \otimes ma = 1 \otimes (m, 1)a = 1(m, 1) \otimes a = m \otimes a$. Thus f and g are inverse isomorphisms. □

The following basic result will be used later.

Proposition 3.7. *Let G be a group. Then $G \times G$ is a $(G \times G^{op})$-G-biset that is free as a right G-set on cardinality of G generators under the right action $(g, g')h = (gh, h^{-1}g')$.*

Proof. It is easy to check that the right action of G is indeed an action commuting with the left action of $G \times G^{op}$. Moreover, the right action of G is free and two elements (g_1, g_2) and (g_1', g_2') are in the same right G-orbit if and only if $g_1g_2 = g_1'g_2'$. This completes the proof. □

Corollary 3.8. *Let M be a monoid and X a projective $M \times M^{op}$-CW complex.*

(1) X/M is a projective M-CW complex and $M\backslash X$ is a projective M^{op}-CW complex.
(2) If X is of $M \times M^{op}$-finite type, then X/M is of M-finite type and dually for $M\backslash X$.
(3) $\dim X/M = \dim X = \dim M\backslash X$.
(4) If X, Y are $M \times M^{op}$-homotopic projective $M \times M^{op}$-CW complexes, then X/M and Y/M (respectively, $M\backslash X$ and $M\backslash Y$) are M-homotopic projective M-CW complexes (respectively, M^{op}-homotopic projective M^{op}-CW complexes).

Proof. The first three items follow from Corollary 3.2 and Lemma 3.6 (and their duals). The final statement follows from Lemma 3.6 and Proposition 3.5. □

We shall frequently use without comment that if A is an M-N-biset and B is an N-set, then $\mathbb{Z}[A \otimes_N B] \cong \mathbb{Z}A \otimes_{\mathbb{Z}N} \mathbb{Z}B$ as left $\mathbb{Z}M$-modules. Indeed, there are natural isomorphisms of abelian groups

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}M}(\mathbb{Z}A \otimes_{\mathbb{Z}N} \mathbb{Z}B, V) &\cong \mathrm{Hom}_{\mathbb{Z}N}(\mathbb{Z}B, \mathrm{Hom}_{ZM}(\mathbb{Z}A, V)) \\
&\cong \mathrm{Hom}_N(B, \mathrm{Hom}_M(A, V)) \\
&\cong \mathrm{Hom}_M(A \otimes_N B, V) \\
&\cong \mathrm{Hom}_{\mathbb{Z}M}(\mathbb{Z}[A \otimes_N B], V)
\end{aligned}$$

for a $\mathbb{Z}M$-module V and so we can apply Yoneda's Lemma.

4. Simplicial sets

An important source of examples of rigid M-CW complexes will come from simplicial sets which admit suitable monoid actions. In this section we introduce the notion of a rigid M-simplicial set, and we show how these give rise to rigid M-CW complexes via the geometric realisation functor. For further background on simplicial sets we refer the reader to [Wei94, Chapter 8] or [May67].

Let Δ denote the *simplicial category*. It has objects all the finite linearly ordered sets $[n] = \{0, 1, \ldots, n-1\}$ $(n \geq 0)$ and morphisms given by (non-strictly) order-preserving maps. A *simplicial set* X is then a functor $X \colon \Delta^{op} \to \boldsymbol{Set}$ from Δ^{op} to the category of sets. For each n, the image of $[n]$ under X is denoted X_n and is called the *set of n-simplicies* of the simplicial set. Any simplicial set X may be defined combinatorially as a collection of sets X_n $(n \geq 0)$ and functions $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$ $(0 \leq i \leq n)$ satisfying

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad (i < j) \\ s_i s_j &= s_{j+1} s_i \quad (i \leq j) \\ d_i s_j &= \begin{cases} 1 & i = j,\ j+1 \\ s_{j-1} d_i & i < j \\ s_j d_{i-1} & i > j+1. \end{cases} \end{aligned}$$

Here the d_i are called the *face maps* and the s_i are called the *degeneracy maps*. We say that an n-simplex $x \in X_n$ is *degenerate* if it is the image of some degeneracy map.

A *simplicial morphism* $f \colon X \to Y$ between simplicial sets is a natural transformation between the corresponding functors, i.e., a sequence of functions $f_n \colon X_n \to Y_n$ for each $n \geq 0$ such that $f_{n-1} d_i = d_i f_n$ and $f_{n+1} s_j = s_j f_n$. There is a functor $|\cdot| \colon \mathbf{SSet} \to \mathcal{CG}$, called the *geometric realization functor*, from the category $\mathbf{SSet}$ of simplicial sets and the category $\mathcal{CG}$ of compactly-generated Hausdorff topological spaces. Let $K = \bigcup_{i \geq 0} K_i$ be a simplicial set with degeneracy and face maps d_i, s_i. The *geometric realisation* $|K|$ of K is the CW complex constructed from K in the following way. Let

$$\Delta_n = \left\{ (t_0, \ldots, t_n) : 0 \leq t_i \leq 1, \sum t_i = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

denote the standard topological n-simplex. Define

$$\begin{array}{cccc} \delta_i \colon & \Delta_{n-1} & \to & \Delta_n \\ & (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}), \end{array}$$

and

$$\begin{array}{cccc} \sigma_i \colon & \Delta_{n+1} & \to & \Delta_n \\ & (t_0, \ldots, t_{n+1}) & \mapsto & (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n-1}). \end{array}$$

Then

$$|K| = \left(\bigsqcup_{n \geq 0} K_n \times \Delta_n \right) /\!\sim$$

where $\sim$ is the equivalence relation generated by

$$(x, \delta_i(u)) \sim (d_i(x), u), \quad (x, \sigma_i(v)) \sim (s_i(x), v).$$

We give

$$\left(\bigsqcup_{0 \leq n \leq q} K_n \times \Delta_n \right) /\!\sim$$

the quotient topology for all q and take the inductive limit of the resulting topologies. The geometric realisation $|K|$ is a CW complex whose cells are in natural bijective correspondence with the non-degenerate simplicies of K. To see this, write

$$\overline{K} = \bigsqcup_{n\geq 0} K_n \times \Delta_n.$$

Then a point $(k, x) \in \overline{K}$ is called *non-degenerate* if k is a non-degenerate simplex and x is an interior point. The following is [Mil57, Lemma 3].

Lemma 4.1. *Each point $(k, x) \in \overline{K}$ is $\sim$-equivalent to a unique non-degenerate point.*

In each case, the point in question is determined by the maps δ_i, d_i, σ_i and s_i (see [Mil57] for details). This lemma is the key to proving that $|K|$ is a CW complex: we take as n-cells of $|K|$ the images of the non-degenerate n-simplices of $\overline{K}$, and the above lemma shows that the interiors of these cells partition $|K|$. The remaining properties of a CW complex are then easily verified. The following lemma shows that geometric realisation defines a functor from **SSet** to $\mathcal{CG}$.

The next result is [Mil57, Lemma 4].

Lemma 4.2. *If $K = \bigcup K_i$ and $L = \bigcup L_i$ are simplicial sets and $f\colon K \to L$ is a simplicial morphism then $\overline{f}$ given by*

$$\overline{f}_n\colon K_n \times \Delta_n \to L_n \times \Delta_n, \quad (x, u) \mapsto (f(x), u)$$

is continuous, and induces a well-defined continuous map

$$|f|\colon |K| \to |L|, \quad (x, u)/\!\sim \;\mapsto\; (f(x), u)/\!\sim$$

of the corresponding geometric realizations, which is cellular.

A left M-simplicial set is a simplicial set equipped with a left action of M by simplicial morphisms. In order to construct rigid M-CW complexes we shall need the following special kind of M-simplicial set.

Definition 4.3 (Rigid M-simplicial set)**.** Let $K = \bigcup_{i\geq 0} K_i$ be a simplicial set with degeneracy and face maps d_i, s_i, and let M be a monoid. We call K a *rigid left M-simplicial set* if K comes equipped with an action of $M \times K \to K$ such that

- M is acting by simplicial morphisms, i.e., M maps n-simplicies to n-simplicies, and commutes with d_i and s_i;
- M preserves non-degeneracy, i.e., for every non-degenerate n-simplex x and every $m \in M$ the n-simplex mx is also non-degenerate.

A *rigid right M-simplicial set* is defined dually, and a *rigid bi-M-simplicial set* is simultaneously both a left and a right M-simplicial set, with commuting actions. A bi-M-simplicial set is the same thing as a left $M \times M^{op}$-simplicial set. Note that it follows from the condition that M acts by simplicial morphisms that, under the action of M, degenerate n-simplicies are sent to degenerate n-simplicies. The geometric realisation construction defines a functor from the category of left M-simplicial sets (with M-equivariant simplicial morphisms) to the category of left M-spaces. In particular, this functor associates with each rigid left M-simplicial set a rigid M-CW complex. Corresponding statements hold for both rigid right and bi-M-simplicial sets.

Lemma 4.4. *For any rigid left M-simplicial set $K = \bigcup_{i\geq 0} K_i$ the geometric realisation $|K|$ is a rigid left M-CW complex with respect to the induced action given by*

$$m \cdot [(x, u)/\!\sim] = (m \cdot x, u)/\!\sim.$$

Proof. It follows from Lemma 4.2 that the action is continuous. By the definition of rigid left M-simplicial set the M-action maps non-degenerate simplices to non-degenerate simplices, and the cells of $|K|$ are in natural bijective correspondence with the non-degenerate simplicies of K. It follows that the action of M on $|K|$ sends n-cells to n-cells. The action is rigid by definition. Thus $|K|$ is a rigid M-CW complex. □

There are obvious right- and bi-M-simplicial set analogues of Lemma 4.4 obtained by replacing M by M^{op} and $M \times M^{op}$, respectively.

5. Standard constructions of projective M-CW complexes

In this section we shall give a fundamental method that, for any monoid M, allows us to construct in a canonical way free left-, right- and bi-M-CW complexes. These constructions will be important when we go on to discuss M-equivariant classifying spaces later on in the article. Each of the constructions in this section is a special case of the general notion of the nerve of a category.

To any (small) category C we can associate a simplicial set $N(C)$ called the *nerve* of the category. For each $k \geq 0$ we let $N(C)_k$ denote the set of all sequences $(f_1, \ldots, f_k)$ composable arrows

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} A_k \tag{5.1}$$

where we allow objects to repeat in these sequences. The objects of C are the 0-simplices. The face map $d_i\colon N(C)_k \to N(C)_{k-1}$ omits A_i, so it carries the above sequence to

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_{i+1}\circ f_i} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_k} A_k$$

while the degeneracy map $s_i\colon N(C)_k \to N(C)_{k+1}$ carries it to

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} A_i \xrightarrow{\mathrm{id}_{A_i}} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_k} A_k$$

The *classifying space* of a (small) category C is the geometric realisation $|N(C)|$ of the nerve $N(C)$ of C.

The nerve is a functor from **Cat** (the category of small categories) to **SSet** (the category of simplicial sets, with simplicial morphisms) given by applying the functor to the diagram (5.1). From this it follows that a functor between small categories C and D induces a map of simplicial sets $N(C) \to N(D)$, which in turn induces a continous map between the classifying spaces $|N(C)| \to |N(D)|$. Also, a natural transformation between two functors between C and D induces a homotopy between the induced maps on the classifying spaces. In particular, equivalent categories have homotopy equivalent classifying spaces. Any functor which is left or right adjoint induces a homotopy equivalence of nerves. Consequently, $|N(C)|$ is contractible if C admits an initial or final object. (For a proof of this see [Sri96, Corollary 3.7].)

It is obvious from the nerve construction that the nerve of a category which is not connected is the disjoint union of the nerves of the connected components of the category. Thus, if every component of C admits an initial or final object, then each of the components of $|N(C)|$ will be contractible.

It is well know that the geometric realisations of the nerve of a category C and its reversal C^{op} are homeomorphic.

5.1. **The classifying space $|BM|$ of a monoid M.** In the general context above, given a monoid M we can construct a category with a single object, one arrow for every $m \in M$, and composition given by multiplication. The *classifying space* $|BM|$ of the monoid M is then the geometric realisation of the nerve of the category corresponding to M^{op} (the reversal is for the technical reason of avoiding reversals in the face maps). In more detail, the nerve of this category is the simplicial set BM with n-simplices: $\sigma = (m_1, m_2, ..., m_n)$, n-tuples of elements of M. The face maps are given by

$$d_i\sigma = \begin{cases} (m_2, \dots, m_n) & i = 0 \\ (m_1, \dots, m_{i-1}, m_i m_{i+1}, m_{i+2}, \dots, m_n) & 0 < i < n \\ (m_1, \dots, m_{n-1}) & i = n, \end{cases}$$

and the degeneracy maps are given by

$$s_i\sigma = (m_1, \dots, m_i, 1, m_{i+1}, \dots, m_n) \quad (0 \leq i \leq n).$$

The geometric realisation $|BM|$ is called the *classifying space* of M. Then $|BM|$ is a CW complex with one n-cell for every non-degenerate n-simplex of BM, i.e., for every n-tuple all of whose entries are different from 1. As mentioned in the introduction, classifying spaces of monoids have received some attention in the literature.

5.2. **Right Cayley graph category.** Let $\Gamma_r(M)$ denote the right Cayley graph category for M, which has

- Objects: M;
- Arrows: $(x, m)\colon x \to xm$; and
- Composition of arrows: $(xm, n) \circ (x, m) = (x, mn)$.

The identity at x is $(x, 1)$. This composition underlies our use of M^{op} in defining BM.

Let $\overrightarrow{EM}$ be the nerve of the category $\Gamma_r(M)$. The n-simplies of $\overrightarrow{EM}$ may be written using the notation $m(m_1, m_2, ..., m_n) = m\tau$ where $\tau = (m_1, m_2, ..., m_n)$ is an n-simplex of BM. Here $m(m_1, m_2, ..., m_n)$ denotes the n-tuple of composable arrows in the category $\Gamma_r(M)$ where we start at m and the follow the path labelled by $m_1, m_2, ..., m_n$.

The face maps in $\overrightarrow{EM}$ are given by

$$d_i(m(m_1, m_2, ..., m_n)) = \begin{cases} mm_1(m_2, ..., m_n) & i = 0 \\ m(m_1, m_2, ..., m_i m_{i+1}, ..., m_n) & 0 < i < n \\ m(m_1, m_2, ..., m_{n-1}) & i = n \end{cases}$$

and the degeneracy maps are given by

$$s_i\sigma = m(m_1, \dots, m_i, 1, m_{i+1}, \dots, m_n) \quad (0 \leq i \leq n).$$

where $\sigma = m(m_1, ..., m_n)$.

Let $|\overrightarrow{EM}|$ denote the geometric realisation of $\overrightarrow{EM}$. So $|\overrightarrow{EM}|$ is a CW complex with one n-cell for every non-degenerate n-simplex of $\overrightarrow{EM}$, that is, for every $m(m_1, m_2, \dots, m_n)$ with $m_j \neq 1$ for $1 \leq j \leq n$. As a consequence, by an n-cell of $\overrightarrow{EM}$ we shall mean a non-degenerate n-simplex.

Consider the right Cayley graph category $\Gamma_r(M)$. For each $m \in M$ there is precisely one morphism $(1, m)$ from 1 to m. Since the category has an initial object we conclude that the geometric realisation of its nerve $|\overrightarrow{EM}|$ is contractible.

Applying the nerve functor to the projection functor from the category $\Gamma_r(M)$ to the one-point category M^{op}, which identifies all the vertices of $\Gamma_r(M)$ to a point, gives a simplicial morphism $\pi\colon \overrightarrow{EM} \to BM$ between the corresponding nerves, which maps

$$m(m_1, m_2, ..., m_n) \mapsto (m_1, m_2, ..., m_n).$$

Observe that, for each n, the projection π maps the set of n-cells of $\overrightarrow{EM}$ onto the set of n-cells BM. If we then apply the geometric realisation functor we obtain a projection $\pi\colon |\overrightarrow{EM}| \to |BM|$ (we abuse notation slightly by using the same notation π to denote this map).

The monoid M acts by left multiplication on the category $\Gamma_r(M)$. By functoriality of the nerve, it follows that M acts on the left of $\overrightarrow{EM}_n$ by simplicial morphisms via

$$s \cdot m(m_1, m_2, ..., m_n) = sm(m_1, m_2, ..., m_n).$$

Under this action $\overrightarrow{EM}_n$ is a free left M-set with basis BM_n. Also, if we restrict to the n-cells (i.e., non-degenerate simplices), then we obtain a free left M-set with basis the set of n-cells of BM. It is an easy consequence of the definitions that this is an action by simplicial morphisms and that it preserves non-degeneracy in the sense that $s \cdot m\sigma$ is an n-cell if and only if $m\sigma$ is an n-cell for all $s \in M$ and $m\sigma \in \overrightarrow{EM}$. Therefore $\overrightarrow{EM}$ is a rigid left M-simplicial set. Combining these observations with Lemma 4.4 we conclude that $|\overrightarrow{EM}|$ is a free left M-CW complex which is contractible.

Dually, we use $\overleftarrow{EM}$ to denote the nerve of the left Cayley graph category $\Gamma_l(M)$. The simplicial set $\overleftarrow{EM}$ satisfies all the obvious dual statements to those above for $\overrightarrow{EM}$. In particular M acts freely via right multiplication action on $\overleftarrow{EM}$ by simplicial morphisms, and $|\overleftarrow{EM}|$ is a free right M-CW complex which is contractible.

5.3. **Two-sided Cayley graph category.** Let $\overleftrightarrow{\Gamma(M)}$ denote the two-sided Cayley graph category for M, which has

- Objects: $M \times M$;
- Arrows: $M \times M \times M$ where $(m_L, m, m_R)\colon (m_L, mm_R) \to (m_L m, m_R)$; and
- Composition of arrows: $(n_L, n, n_R) \circ (m_L, m, m_R) = (m_L, mn, n_R)$ where $(m_L m, m_R) = (n_L, nn_R)$. Equivalently this is the same as the composition $(m_L m, n, n_R) \circ (m_L, m, nn_R) = (m_L, mn, n_R)$ and corresponds to the path

$$(m_L, mnn_R) \to (m_L m, nn_R) \to (m_L mn, n_R).$$

This is in fact the kernel category of the identity map, in the sense of Rhodes and Tilson [RT89]. There is a natural $M \times M^{op}$ action of the category $\overleftrightarrow{\Gamma(M)}$.

Let $\overleftrightarrow{EM}$ be the nerve of the category $\overleftrightarrow{\Gamma(M)}$. The simplicial set $\overleftrightarrow{EM}$ parallels the two-sided geometric bar construction of J. P. May; see [May72, May75]. The n-simplies of $\overleftrightarrow{EM}$ may be written using the notation $m(m_1, m_2, ..., m_n)s = m\tau s$ where $\tau = (m_1, m_2, ..., m_n)$ is an n-simplex of BM.

Here $m(m_1, m_2, ..., m_n)s$ denotes the n-tuple of composable morphisms in the category $\overleftrightarrow{\Gamma(M)}$ where we start at $(m, m_1 m_2 \dots m_n s)$ and follow the path

$$(m, m_1 m_2 m_3 \dots m_n s) \to (mm_1, m_2 m_3 \dots m_n s) \to \dots$$

$$\dots (mm_1 m_2, m_3 \dots m_n s) \to \dots (mm_1 m_2 \dots m_n, s)$$

labelled by the edges

$$(m, m_1, m_2 m_3 \dots m_n s), (mm_1, m_2, m_3 \dots m_n s), \dots,$$

$$(mm_1 m_2 \dots m_{n-2}, m_{n-1}, m_n s), (mm_1 m_2 \dots m_{n-1}, m_n, s)$$

and finish at $(mm_1m_2\dots m_n, s)$. The face maps in the nerve $\overleftrightarrow{EM}$ are given by

$$d_i(m(m_1, m_2, ..., m_n)s) = \begin{cases} mm_1(m_2, ..., m_n)s & i = 0 \\ m(m_1, m_2, ..., m_im_{i+1}, ..., m_n)s & 0 < i < n \\ m(m_1, m_2, ..., m_{n-1})m_ns & i = n \end{cases}$$

and the degeneracy maps are given by

$$s_i\sigma = m(m_1, \dots, m_i, 1, m_{i+1}, \dots, m_n)s \quad (0 \le i \le n),$$

where $\sigma = m(m_1, ..., m_n)s$.

Let $|\overleftrightarrow{EM}|$ denote the geometric realisation of $\overleftrightarrow{EM}$. So $|\overleftrightarrow{EM}|$ is a CW complex with one n-cell for every non-degenerate n-simplex of $\overleftrightarrow{EM}$. Observe that (m_L, m_R) and (m'_L, m'_R) are in the same component of the two-sided Cayley graph category $\overleftrightarrow{\Gamma}(M)$ if and only if $m_Lm_R = m'_Lm'_R$. Moreover, for each $m \in M$ the vertex $(1, m)$ is initial in its component. It follows from these observations that $\pi_0(|\overleftrightarrow{EM}|) \cong M$ as an $M \times M^{op}$-set, and each component of $|\overleftrightarrow{EM}|$ is contractible. There is a natural projection from $\overleftrightarrow{\Gamma}(M)$ to the one-point category M^{op} mapping (m_L, m, m_R) to its middle component m. Applying the nerve functor to this projection gives a simplicial morphism $\pi\colon \overleftrightarrow{EM} \to BM$ given by

$$m(m_1, m_2, ..., m_n)s \mapsto (m_1, m_2, ..., m_n).$$

As in the one-sided case, this projection sends n-cells to n-cells and induces a map $\pi\colon |\overleftrightarrow{EM}| \to |BM|$ between the corresponding geometric realisations.

The monoid M has a natural two-sided action on $\overleftrightarrow{EM}_n$ via

$$x \cdot [m(m_1, m_2, ..., m_n)s] \cdot y = xm(m_1, m_2, ..., m_n)sy.$$

Under this action $\overleftrightarrow{EM}$ is a free rigid bi-M-simplicial set. Combining these observations with Lemma 4.4 we conclude that $|\overleftrightarrow{EM}|$ is a free bi-M-CW complex such that $\pi_0(|\overleftrightarrow{EM}|) \cong M$ as an $M \times M^{op}$-set and each component of $|\overleftrightarrow{EM}|$ is contractible.

6. One-sided classifying spaces and finiteness properties

We will define left and right equivariant classifying spaces for a monoid M. Two-sided equivariant classifying spaces will be defined in the next section. As we shall see, the examples discussed in Section 5 will serve as the standard models of such spaces.

We say that a projective M-CW complex X is a *(left) equivariant classifying space* for M if it is contractible. A right equivariant classifying space for M will be a left equivariant classifying space for M^{op}. Notice that the augmented cellular chain complex of an equivariant classifying space for M provides a projective resolution of the trivial (left) $\mathbb{Z}M$-module $\mathbb{Z}$.

Example 6.1. The bicyclic monoid is the monoid B with presentation $\langle a, b \mid ab = 1\rangle$. It is not hard to see that each element of B is uniquely represented by a word of the form b^ia^j where $i, j \ge 0$. Figure 1 shows an equivariant classifying space for B. The 1-skeleton is the Cayley graph of B and there is a 2-cell glued in for each path labelled ab. This example is a special case of far more general results about equivariant classifying spaces of special monoids which will appear in a future paper of the current authors [GS18].

Our first goal is to show that any two equivariant classifying spaces for M are M-homotopy equivalent.

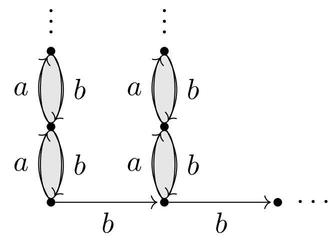
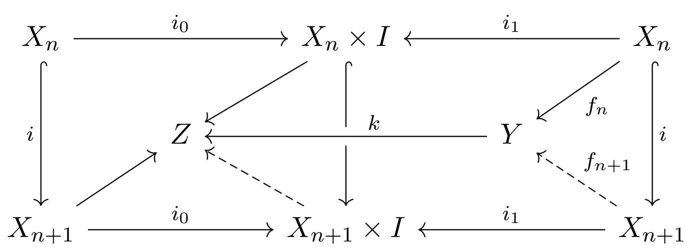


FIGURE 1. An equivariant classifying space for the bicyclic monoid

Lemma 6.2. *Let X be an equivariant classifying space for M and let Y be a contractible M-space. Then there exists a continuous M-equivariant mapping $f\colon X \to Y$.*

Proof. The proof constructs by induction M-equivariant continuous mappings $f_n\colon X_n \to Y$ with f_n extending f_{n-1}. To define f_0, observe that $X_0 = \coprod_{a\in A} Me_a$ (without loss of generality) and so, by Proposition 2.3, $\mathbf{Top}_M(X_0, Y) \cong \prod_{a\in A} e_aY \neq \emptyset$ and so we can define f_0. Assume that $f_n\colon X_n \to Y$ has been defined. Let Z be the one-point space with the trivial M-action and let $k\colon Y \to Z$ be the unique M-equivariant map. Then k is a weak equivalence. So by Theorem 2.5 we can construct a commutative diagram



with f_{n+1} M-equivariant. Now take f to be the colimit of the f_n. □

Theorem 6.3. *Let X, Y be equivariant classifying spaces for M. Then X and Y are M-homotopy equivalent by a cellular M-homotopy equivalence.*

Proof. By Corollary 2.7 and Theorem 2.8 it suffices to construct an M-equivariant continuous mapping $f\colon X \to Y$. But Lemma 6.2 does just that. □

We now give an elementary proof that contractible free M-CW complexes exist and hence there are equivariant classifying spaces for M. A more canonical construction, using simplicial sets, was given in the previous section.

Lemma 6.4. *Let M be a monoid.*

(1) If X_0 is a non-empty projective (free) M-set, then there is a connected projective (free) M-graph X with vertex set X_0.
(2) If X is a connected projective (free) M-CW complex such that $\pi_q(X) = 0$ for $0 \leq q < n$, then there exists a projective M-CW complex Y containing X as a projective M-CW subcomplex and such that $Y_n = X_n$ and $\pi_q(Y) = 0$ for $0 \leq q \leq n$.
(3) If X is a connected projective (free) M-CW complex such that $\pi_q(X)$ is trivial for $0 \leq q < n$, then there exists a contractible projective (free) M-CW complex Y containing X as a projective M-CW subcomplex and such that $Y_n = X_n$.

Proof. For the first item, fix $x_0 \in X_0$. The edge set of X will be in bijection with $M \times X_0$ with the edge corresponding to (m, x) connecting mx_0 to mx. Then X is a projective (free) M-graph and each vertex x is connected to x_0 via the edge $(1, x)$.

For the second item, we show that we can add free M-cells of dimension $n+1$ to X to obtain a new projective M-CW complex Y with $\pi_n(Y) = 0$. If $\pi_n(X) = 0$, then take $Y = X$. So assume that $\pi_n(X)$ is non-trivial. Fix a base point $x_0 \in X_0$ and let $f_a\colon S^n \to X$, $a \in A$, be mappings whose based homotopy classes form a set of generators for $\pi_n(X, x_0)$. As $X_n \to X$ is an n-equivalence, we may assume without loss of generality that $f_a\colon S^n \to X_n$. Suppose that X is constructed from pushouts as per (2.1). Note that $M \times A$, where A has the trivial action, is a free M-set. Let us define Y by putting $Y_k = X_k$ for $0 \leq k \leq n$, defining Y_{n+1} to be the pushout

$$\begin{array}{ccc} (P_{n+1} \times S^n) \coprod (M \times A \times S^n) & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ (P_{n+1} \times B^{n+1}) \coprod (M \times A \times B^{n+1}) & \longrightarrow & Y_{n+1}, \end{array}$$

where the top map is the union of the attaching map for X with the mapping $(m, a, x) \mapsto mf_a(x)$ (cf. Proposition 2.3), and putting $Y_k = X_k \cup Y_{n+1}$ for $k > n+1$. Then Y is a projective M-CW complex containing X as a projective M-CW subcomplex and with $Y_n = X_n$. Moreover, since $X_n = Y_n \to Y$ is an n-equivalence, it follows that the based homotopy classes of the $f_a\colon S^n \to Y$ generate $\pi_n(Y, x_0)$. By construction the f_a can be extended to $B^{n+1} \to Y$ and so their classes are trivial in $\pi_n(Y, x_0)$. Thus $\pi_n(Y) = 0$. Also, because $X_n = Y_n \to Y$ is an n-equivalence, we have that $\pi_q(Y) = 0$ for $0 \leq q < n$.

The final item follows from Whitehead's theorem, iteration of the second item and that $Y_n \to Y$ is an n-equivalence for all $n \geq 0$. □

Corollary 6.5. *Let M be a monoid. Then there exists a contractible free M-CW complex.*

Proof. Put $X_0 = M$. Then by Lemma 6.4 we can find a connected free M-graph X with vertex set X_0. Now applying the third item of Lemma 6.4 we can find a contractible free M-CW complex with 1-skeleton X. □

Example 6.6. It follows from the definitions and results in Section 5 that the geometric realisation $|\overrightarrow{EM}|$ of the nerve of the right Cayley graph category of M is a left equivariant classifying space for M.

Corollary 6.7. *If X, Y are equivariant classifying spaces for M, then $M\backslash X$ and $M\backslash Y$ are homotopy equivalent. In particular, $M\backslash X \simeq |BM|$. Therefore, if G is a group and X is an equivariant classifying space for G, then $G\backslash X$ is an Eilenberg-Mac Lane space for G. Conversely, the universal cover of any Eilenberg-Mac Lane space for G is an equivariant classifying space for G.*

Proof. The first statement follows from Theorem 6.3 and Proposition 3.5. The second statement follows from the first as $|\overrightarrow{EM}|$ is an equivariant classifying space for M. The group statements then follow from the previous statements and classical covering space theory. □

If M and N are monoids, then $E(M \times N) = E(M) \times E(N)$ and $(M \times N)(e, f) = Me \times Nf$. It follows that if P is a (finitely generated) projective M-set and Q a (finitely generated) projective N-set, then $P \times Q$ is a (finitely generated projective) $M \times N$-set.

Proposition 6.8. *Let M, N be monoids and let X, Y be equivariant classifying spaces for M, N, respectively. Then $X \times Y$ is an $M \times N$-equivariant classifying space, which is of $M \times N$-finite type whenever X is of M-finite type and Y is of N-finite type.*

Proof. Assume that X is obtained via attaching projective M-cells $P_n \times B^n$ and that Y is obtained by attaching projective N-cells $Q_n \times B^n$. Then the n-cells of $X \times Y$ are obtained by adjoining $\coprod_{i=0}^{n} P_i \times Q_{n-i} \times B^n$ and hence $X \times Y$ is an $M \times N$-projective CW complex which is of $M \times N$-finite type whenever X is of M-finite type and Y is of N-finite type. As $X \times Y$ is contractible, we deduce that it is an $M \times N$-equivariant classifying space. □

6.1. **Monoids of type left-F_n.** A key definition for this paper is the following. A monoid M is of type *left*-F_n if there is an equivariant classifying space X for M such that X_n is M-finite, i.e., such that $M\backslash X$ has finite n-skeleton. We say that M is of type *left*-F_∞ if M has an equivariant classifying space X that is of M-finite type, i.e., $M\backslash X$ is of finite type. The monoid M is defined to have type *right*-F_n if M^{op} is of type left-F_n for $0 \leq n \leq \infty$. The following proposition contains some basic facts.

Proposition 6.9. *Let M be a monoid.*

1. *A group is of type left-F_n if and only if it is of type F_n in the usual sense for any $0 \leq n \leq \infty$.*
2. *For $0 \leq n \leq \infty$, if M is of type left-F_n, then it is of type left-FP_n.*
3. *If M is of type left-F_∞, then it is of type left-F_n for all $n \geq 0$.*

Proof. The first item follows from Corollary 6.7 and Corollary 3.3. The second is immediate using that the augmented cellular chain complex of an equivariant classifying space X gives a projective $\mathbb{Z}M$-resolution of the trivial $\mathbb{Z}M$-module since if X is built up from pushouts as per (2.1), then the n^{th}-chain module is isomorphic to $\mathbb{Z}P_n$. The final item is trivial. □

Note that, trivially, if M is a finite monoid then $|\overrightarrow{EM}|$ has finitely many cells in each dimension and thus M is of type left-F_∞.

Sometimes it will be convenient to use the following reformulation of the property left-F_n.

Proposition 6.10. *Let M be a monoid. The following are equivalent for $0 \leq n < \infty$.*

1. *M is of type left-F_n*
2. *There is a connected M-finite projective M-CW complex X of dimension at most n with $\pi_q(X) = 0$ for $0 \leq q < n$.*

Proof. If Y is an equivariant classifying space for M such that Y_n is M-finite, then since $Y_n \to Y$ is an n-equivalence, we deduce that $X = Y_n$ is as required for the second item. Conversely, if X is as in the second item, we can construct by Lemma 6.4 an equivariant classifying space Y for M with $Y_n = X$. Thus M is of type left-F_n. □

Recall that the fundamental group of $|BM|$ is isomorphic to the universal group (or maximal group image, or group completion) $U(M)$ of M i.e., the group with generators M and relations the multiplication table of M (cf. [GZ67]).

Corollary 6.11. *Let M be a monoid. If M is of type left-F_1, then $U(M)$ is finitely generated. If M is of type left-F_2, then $U(M)$ is finitely presented.*

Proof. By Corollary 6.7, $|BM|$ in the first case is homotopy equivalent to a CW complex with finite 1-skeleton and in the second case to a CW complex with finite 2-skeleton by Corollary 3.3. Thus $U(M) \cong \pi_1(BM)$ has the desired properties in both cases. □

Recall that an inverse monoid is a monoid M with the property that for every $m \in M$ there is a unique element $m' \in M$ such that $mm'm = m$ and $m'mm' = m'$. For more on inverse monoids, and other basic concepts from semigroup theory we refer the reader to [How95].

Corollary 6.12. *Let M be a monoid such that $|BM|$ is an Eilenberg-Mac Lane space (e.g., if M is cancellative with a left or right Ore condition or if M is an inverse monoid) and suppose that $0 \leq n \leq \infty$. If M is of type left-F_n, then $U(M)$ is of type F_n.*

Proof. If X is an equivariant classifying space for M, then $M\backslash X$ is homotopy equivalent to $|BM|$ by Corollary 6.7 and hence is an Eilenberg-Mac Lane space for $U(M)$. The result now follows from Corollary 3.3. □

Since, as already mentioned above, D. McDuff [McD79] has shown that every path-connected space has the weak homotopy type of the classifying space of some monoid, not every monoid $|BM|$ is an Eilenberg-Mac Lane space. So not every monoid satisfies the hypotheses of Corollary 6.12. The fact that if M is cancellative with a left or right Ore condition then $|BM|$ is an Eilenberg-Mac Lane space is well known. If M is an inverse monoid then $|BM|$ may also be shown to be an Eilenberg-Mac Lane space. Both of these results can easily be proved by appealing to Quillen's theorem A, see [Wei13, Chapter 4], and should be considered folklore.

The converse of Corollary 6.12 does not hold. For example, the free inverse monoid on one generator is not of type left-F_2 while its maximal group image $\mathbb{Z}$ is F_∞ (this proof of the fact that the free inverse monoid on one generator is not left-F_2 will appear in [GS18]).

For groups, being of type F_1 is equivalent to finite generation. For monoids, the condition of being left-F_1 is considerably weaker. Recall that if M is a monoid and $A \subseteq M$, then the (right) *Cayley digraph* $\Gamma(M, A)$ of M with respect to A is the graph with vertex set M and with edges in bijection with $M \times A$ where the directed edge (arc) corresponding to (m, a) starts at m and ends at ma. Notice that $\Gamma(M, A)$ is a free M-CW graph and is M-finite if and only if A is finite.

Theorem 6.13. *Let M be a monoid. The following are equivalent.*

1. *M is of type left-F_1.*
2. *M is of type left-FP_1.*
3. *There is a finite subset $A \subseteq M$ such that $\Gamma(M, A)$ is connected as an undirected graph.*

In particular, any finitely generated monoid is of type left-F_1.

Proof. Item (1) implies (2) by Proposition 6.9, whereas (2) implies (3) by a result due to Kobayashi [Kob07]. For completeness, let us sketch the proof that (2) implies (3). Let $\varepsilon\colon \mathbb{Z}M \to \mathbb{Z}$ be the augmentation map; the ideal $I = \ker \varepsilon$ is called the augmentation ideal. If M is of type left-FP_1, then I must be finitely generated because the augmentation map gives a partial free resolution. But I is generated by all elements of the form $m - 1$ with $m \in M$. Hence there is a finite subset $A \subseteq M$ such that the elements $a - 1$ with $a \in A$ generate I. Consider the Cayley digraph $\Gamma(M, A)$. Then M acts cellularly on $\Gamma(M, A)$ and hence acts on $\pi_0(\Gamma(M, A))$. There is a surjective $\mathbb{Z}M$-module homomorphism $\eta\colon \mathbb{Z}M \to \mathbb{Z}\pi_0(\Gamma(M, A))$ mapping $m \in M$ to the connected component of the vertex m of $\Gamma(M, A)$. Moreover, the augmentation ε factors through η. Thus to show that $\Gamma(M, A)$ is connected, it suffices to show that $I = \ker \eta$. By construction $\ker \eta \subseteq I$. But if $a \in A$, then a and 1 are in the same connected component of $\Gamma(M, A)$ and thus $a - 1 \in \ker \eta$. Since the $a - 1$ with $a \in A$ generate I, we deduce that $I \subseteq \ker \eta$ and hence $\Gamma(M, A)$ is connected.

Finally, (3) implies (1) by Proposition 6.10 as $\Gamma(M, A)$ is an M-finite connected free M-CW complex of dimension at most 1. □

We next show that a finitely presented monoid is of type left-F_2. In fact, we shall see later that finitely presented monoids are of type bi-F_2, which implies left-F_2, but the proof of this case is instructive.

Theorem 6.14. *Let M be a finitely presented monoid. Then M is of type left-*F_2

Proof. Suppose that M is generated by a finite set A with defining relations $u_1 = v_1, \ldots, u_n = v_n$. Let us construct a 2-dimensional, M-finite, free M-CW complex X with 1-skeleton the Cayley graph $\Gamma(M, A)$ by attaching a free M-cell $M \times B^2$ for each relation. Let p_i, q_i be the paths from 1 to m_i labelled by u_i and v_i, respectively, where m_i is the image of u_i (and v_i) in M. Then we glue in a disk d_i with boundary path $p_i q_i^{-1}$ and glue in $M \times B^2$ using Proposition 2.3 (so $\{m\} \times B^2$ is sent to md_i). Then X is an M-finite connected free M-CW complex of dimension at most 2. See Figure 1 for this construction for the bicyclic monoid. By Proposition 6.10, it suffices to prove that X is simply connected.

A digraph is said to be *rooted* if there is a vertex v so that there is a directed path from v to any other vertex. For instance, $\Gamma(M, A)$ is rooted at 1. It is well known that a rooted digraph admits a spanning tree, called a *directed spanning tree*, such that the geodesic from the root to any vertex is directed. Let T be a directed spanning tree for $\Gamma(M, A)$ rooted at 1 (this is the same thing as a prefix-closed set of normal forms for M so, for instance, shortlex normal forms would do). Let $e = m \xrightarrow{a} ma$ be a directed edge not belonging to T. Then the corresponding generator of $\pi_1(X, 1)$ is of the form peq^{-1} where p and q are directed paths from 1 to m and ma, respectively. Let u be the label of p and v be the label of q. Then $ua = v$ in M. Thus it suffices to prove that if $x, y \in A^*$ are words which are equal in M to an element m', then the loop ℓ labelled xy^{-1} at 1, corresponding to the pair of parallel paths 1 to m' labelled by x and y, is null homotopic. By induction on the length of a derivation from x to y, we may assume that $x = wu_iw'$ and $y = wv_iw'$ for some $i = 1, \ldots, n$. Let m_0 be the image of w in M. Then $m_0 d_i$ is a 2-cell with boundary path the loop at m_0 labeled by $u_i v_i^{-1}$. It follows that ℓ is null homotopic. This completes the proof. □

The converse of Theorem 6.14 is not true, e.g., the monoid $(\mathbb{R}, \cdot)$ is of type left-F_2 (by Corollary 6.23) but is not even finitely generated. It is natural to ask whether there is a nice characterisation, analogous to Theorem 6.13(3), for left-F_2 in terms of the right Cayley graph together with the left action of M. We would guess that M is of type left-F_2 if and only if it has a finite subset $A \subseteq M$ such that $\Gamma(M, A)$ is connected, and finitely many free M-2-cells can be adjoined to make a simply connected 2-complex.

It is well known that, for finitely presented groups, the properties F_n and FP_n are equivalent for $3 \leq n \leq \infty$. We now provide the analogue in our context. Here we replace finitely presented by left-F_2.

Theorem 6.15. *Let M be a monoid of type left-F_2. Then M is of type left-F_n if and only if M is of type left-FP_n for $0 \leq n \leq \infty$.*

Proof. We prove that if there is a connected M-finite projective M-CW complex X of dimension at most n with $\pi_q(X) = 0$ for $0 \leq q < n$ with $n \geq 2$ and M is of type left-FP_{n+1}, then there is a connected M-finite projective M-CW complex Y of dimension at most $n+1$ with $Y_n = X$ and $\pi_q(Y) = 0$ for all $0 \leq q < n+1$. This will imply the theorem by Proposition 6.10, Proposition 6.9 and induction.

Since X is simply connected, $H_q(X) = 0$ for $1 \leq q < n$ and $H_n(X) \cong \pi_n(X)$ by the Hurewicz theorem. Therefore, the augmented cellular chain complex of X gives a partial projective resolution of $\mathbb{Z}$ of length at most n, which is finitely generated in each degree. Therefore, since M is of type left-FP_{n+1}, it follows that $H_n(X) = \ker d_n \colon C_n(X) \to C_{n-1}(X)$ is finitely generated as a left $\mathbb{Z}M$-module. Choose representatives of $f_a \colon S^n \to X$, with $a \in A$, of a finite set of elements of $\pi_n(X)$ that map to a finite $\mathbb{Z}M$-module generating set of $H_n(X)$ under the Hurewicz isomorphism. Then form Y by adjoining $M \times A \times B^{n+1}$ to X via the attaching map $M \times A \times S^n \to X_n$ given by $(m, a, x) \mapsto mf_a(x)$. Then Y is an M-finite projective M-CW complex of dimension $n+1$ with $Y_n = X$. Since the inclusion of $X = Y_n$ into Y is an n-equivalence, we deduce that $\pi_q(Y) = 0$ for $1 \le q < n$ and that the inclusion $X \to Y$ is surjective on π_n. But since the Hurewicz map in degree n is natural and is an isomorphism for both X and Y, we deduce that the inclusion $X \to Y$ induces a surjection $H_n(X) \to H_n(Y)$. But, by construction, the images of the $\mathbb{Z}M$-module generators of $H_n(X)$ are trivial in $H_n(Y)$ (since they represent trivial elements of $\pi_n(Y)$). We deduce that $H_n(Y) = 0$ and hence, by the Hurewicz theorem, $\pi_n(Y) = 0$. This completes the induction. □

Notice that Theorem 6.15 implies that M is of type left-F_∞ if and only if M is of type left-F_n for all $n \ge 0$.

Proposition 6.16. *If M is of type left-F_n with $n \ge 1$, then M has a free contractible M-CW complex X such that X_n is M-finite.*

Proof. This is clearly true for $n = 1$ by Theorem 6.13. Note that Lemma 6.4 and the construction in the proof of Theorem 6.15 show that if Y is a simply connected M-finite free M-CW complex of dimension at most 2 and M is of type left-FP_n, then one can build a contractible free M-CW complex X with $X_2 = Y$ such that X_n is M-finite. Thus it remains to prove that if there is a simply connected M-finite projective M-CW complex Y of dimension 2, then there is a simply connected M-finite free M-CW complex X of dimension 2.

Note that $Y_0 = \coprod_{a \in A} Me_a$ with A a finite set. Define $X_0 = M \times A$. Identifying Me_a with $Me_a \times \{a\}$, we may view Y_0 as an M-subset of X_0. Using this identification, we can define X_1 to consists of Y_1 (the edges of Y) along with some new edges. We glue in an edge from (m, a) to (me_a, a) for each $m \in M$ and $a \in A$; that is we glue in a free M-cell $M \times A \times B^1$ where the attaching map takes $(m, a, 0)$ to (m, a) and $(m, a, 1)$ to (me_a, a). Notice that all vertices of $X_0 \setminus Y_0$ are connected to a vertex of Y_0 in X_1 and so X_1 is connected as Y_1 was connected.

To define X_2, first we keep all the two-cells from Y_2. Notice that if T is a spanning tree for Y, then a spanning tree T' for X can be obtained by adding to T all the edges $(m, a) \longrightarrow (me_a, a)$ with $m \notin Me_a$ (all vertices of $X_0 \setminus Y_0$ have degree one). Thus the only edges in $X_1 \setminus Y_1$ that do not belong to T' are the loop edges $(m, a) \longrightarrow (me_a, a)$ for $m \in Me_a$ that we have added. So if we attach $M \times A \times B^2$ to X_1 by the attaching map $M \times A \times S^1 \to X_1$ mapping $\{m\} \times \{a\} \times S^1$ to the loop edge $(me_a, a) \longrightarrow (me_a, a)$ from $X_1 \setminus Y_1$, then we obtain a simply connected free M-CW complex X which is M-finite. This completes the proof. □

In light of Proposition 6.16, one might wonder why we bother allowing projective M-CW complexes rather than just free ones. The reason is because projective M-CW complexes are often easier to construct and, as we are about to see, sometimes it is possible to find an M-finite equivariant classifying space for M which is projective when no M-finite free equivariant classifying space exists. This will be relevant when considering geometric dimension.

Let M be a non-trivial monoid with a right zero element z. Then $Mz = \{z\}$ is a one-element set with the trivial action. Since z is idempotent, it follows that the trivial M-set is projective but not free. Therefore, the one-point space with the trivial M-action is an M-finite equivariant classifying space for M, which is not free. We will show that if M has a finite number of right zeroes (e.g., if M has a zero element), then there is no finite free resolution of the trivial module which is finitely generated in each degree. In this case, every free equivariant classifying space for M of M-finite type will be infinite dimensional.

A finitely generated projective module P over a ring R is said to be *stably free* if there are finite rank free modules F, F' such that $P \oplus F' \cong F$. The following lemma is well known, but we include a proof for completeness.

Lemma 6.17. *Let P be a finitely generated projective (left) module over a ring R. Then P has a finite free resolution, finitely generated in each degree, if and only if P is stably free.*

Proof. Suppose that P is stably free, say $P \oplus F' \cong F$ with F, F' finite rank free R-modules. Then the exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow P \longrightarrow 0$$

provides a finite free resolution of P that is finitely generated in each degree.

Conversely, suppose that

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow P \longrightarrow 0$$

is a free resolution with F_i finitely generated for all $0 \leq i \leq n$. We also have a projective resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow P \longrightarrow 0$$

with $P_0 = P$ and $P_i = 0$ for $1 \leq i \leq n$ because P is projective. By the generalized Schanuel's lemma, we have that

$$P_0 \oplus F_1 \oplus P_2 \oplus F_3 \oplus \cdots \cong F_0 \oplus P_1 \oplus F_2 \oplus P_3 \oplus \cdots$$

and hence

$$P \oplus F_1 \oplus F_3 \oplus \cdots \cong F_0 \oplus F_2 \oplus \cdots$$

and so P is stably free. □

So we are interested in showing that the trivial module for $\mathbb{Z}M$ is not stably free if M is a non-trivial monoid with finitely many right zeroes (and at least one).

Recall that a ring R is said to have the *Invariant Basis Number property* (IBN) if whenever $R^n \cong R^m$ as R-modules, one has $m = n$ (where m, n are integers); in this definition it does not matter if one uses left or right modules [Lam99].

Our first goal is to show that if M is a monoid with zero z, then the contracted monoid ring $\mathbb{Z}M/\mathbb{Z}z$ has IBN. This result is due to Pace Nielsen, whom we thank for allowing us to reproduce it. It is equivalent to show that if M is a monoid and I is a proper ideal of M, then $\mathbb{Z}M/\mathbb{Z}I$ has IBN. The proof makes use of the Hattori-Stallings trace (see [Wei13, Chapter 2]).

Let $\sim$ be the least equivalence relation on a monoid M such that $mn \sim nm$ for all $m, n \in M$; this relation, often called *conjugacy*, has been studied by a number of authors.

Lemma 6.18. *Let M be a monoid and $e \in M$ an idempotent. Suppose that e is conjugate to an element of an ideal I. Then $e \in I$.*

Proof. Suppose that e is conjugate to $m \in I$. Then we can find elements $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$ with $e = x_1y_1$, $y_ix_i = x_{i+1}y_{i+1}$ and $y_nx_n = m$. Then $e = e^{n+1} = (x_1y_1)^{n+1} = x_1(y_1x_1)^ny_1 = x_1(x_2y_2)^ny_1 = x_1x_2(y_2x_2)^{n-1}y_2y_1 = \cdots = x_1 \cdots x_n(y_nx_n)y_ny_{n-1} \cdots y_1 \in I$ as $y_nx_n = m \in I$. □

If R is a ring, then $[R, R]$ denotes the additive subgroup generated by all commutators $ab - ba$ with $a, b \in R$. The abelian group $R/[R, R]$ is called the *Hattori-Stallings trace* of R; this is also the 0-Hochschild homology group of R. Cohn proved that if $1 + [R, R]$ has infinite order in $R/[R, R]$, then R has IBN (see [Lam99, Exercise 1.5]). The point is that if A is an $m \times n$ matrix over R and B is an $n \times m$-matrix over R with $AB = I_m$ and $BA = I_n$, then $m + [R, R] = T(AB) = T(BA) = n + [R, R]$ where, for a square matrix C over R, we define $T(C)$ to be the class of the sum of the diagonal entries of C in $R/[R, R]$. The following proposition is an elaboration of Pace Nielsen's argument (see [Nie]).

Proposition 6.19. *Let M be a monoid and I a (possibly empty) ideal. Let $R = \mathbb{Z}M/\mathbb{Z}I$. Then $R/[R,R]$ is a free abelian group on the conjugacy classes of M that do not intersect I. More precisely, if T is a transversal to the conjugacy classes of M not intersecting I, then the elements of the form $m + [R,R]$ with $m \in T$ form a basis for $R/[R,R]$.*

Proof. We view R as having basis $M \setminus I$ subject to the relations of the multiplication table of M and that the elements of I are 0. Let A be the free abelian group on the conjugacy classes of M that do not intersect I. Write $[m]$ for the conjugacy class of $m \in M$. Define an abelian group homomorphism $f\colon A \to R/[R,R]$ by $f([m]) = m + [R,R]$. This is well defined because $xy + [R,R] = yx + [R,R]$ for $x, y \in M$ with $xy, yx \notin I$. To see that f is surjective, note that if $m \in M \setminus I$ with $[m] \cap I \neq \emptyset$, then $m + [R,R] = [R,R]$. This follows because if $m = x_1y_1$, $y_ix_i = x_{i+1}y_{i+1}$, for $i = 1, \dots, n-1$, and $y_nx_n \in I$, then $[R,R] = y_nx_n + [R,R] = x_ny_n + [R,R] = \cdots = y_1x_1 + [R,R] = m + [R,R]$.

Let us define $g\colon R \to A$ on $m \in M \setminus I$ by

$$g(m) = \begin{cases} [m], & \text{if } [m] \cap I = \emptyset \\ 0, & \text{else.} \end{cases}$$

Then if $a, b \in R$ with, say,

$$a = \sum_{m \in M \setminus I} c_m m, \ b = \sum_{n \in M \setminus I} d_n n$$

then we have that

$$ab - ba = \sum_{m,n \in M \setminus I} c_m d_n (mn - nm).$$

Since $mn \sim nm$, either both map to 0 under g or both map to $[mn] = [nm]$. Therefore, $ab - ba \in \ker g$ and so g induces a homomorphism $g'\colon R/[R,R] \to A$. Clearly, if $[m] \cap I = \emptyset$, then $gf([m]) = g'(m + [R,R]) = g(m) = [m]$. It follows that f is injective and hence an isomorphism. The result follows. □

As a consequence we deduce the result of Nielsen.

Corollary 6.20. *Let M be a monoid and I a proper ideal (possibly empty). Then $\mathbb{Z}M/\mathbb{Z}I$ has IBN. In particular, contracted monoid rings have IBN.*

Proof. Put $R = \mathbb{Z}M/\mathbb{Z}I$. If I is a proper ideal, then 1 is not conjugate to any element of I by Lemma 6.18. It follows from Proposition 6.19 that $1 + [R,R]$ has infinite order in $R/[R,R]$ and hence R has IBN. □

Theorem 6.21. *Let M be a non-trivial monoid with finitely many right zeroes (and at least one). Then the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ is projective but not stably free and hence does not have a finite free resolution that is finitely generated in each degree.*

Proof. Let I be the set of right zero elements of M and fix $z \in I$. Observe that I is a proper two-sided ideal. Note that $z, 1-z$ form a complete set of orthogonal idempotents of $\mathbb{Z}M$ and so $\mathbb{Z}M \cong \mathbb{Z}Mz \oplus \mathbb{Z}M(1-z)$ and hence $\mathbb{Z}Mz \cong \mathbb{Z}$ is projective. Suppose that $\mathbb{Z}$ is stably free, that is, $\mathbb{Z} \oplus F' \cong F$ with F, F' free $\mathbb{Z}M$-modules of rank r, r', respectively.

There is a exact functor from $\mathbb{Z}M$-modules to $z\mathbb{Z}Mz$-modules given by $V \mapsto zV$. Note that $z\mathbb{Z}Mz = \mathbb{Z}z \cong \mathbb{Z}$ as a ring. Also, $z\mathbb{Z}M = \mathbb{Z}I$ is a free abelian group (equals $z\mathbb{Z}Mz$-module) of rank $|I| < \infty$. Therefore,

$$\mathbb{Z}^{|I|r} \cong (\mathbb{Z}I)^r \cong zF \cong \mathbb{Z} \oplus zF' \cong \mathbb{Z} \oplus (ZI)^{r'} \cong \mathbb{Z}^{1+|I|r'}$$

as $\mathbb{Z}$-modules and hence $r|I| = 1 + r'|I|$ as $\mathbb{Z}$ has IBN. But putting $R = \mathbb{Z}M/\mathbb{Z}I$ and observing that $\mathbb{Z}/\mathbb{Z}I \cdot \mathbb{Z} = 0$, we have that

$$R^r \cong R \otimes_{\mathbb{Z}M} F \cong (R \otimes_{\mathbb{Z}M} \mathbb{Z}) \oplus (R \otimes_{\mathbb{Z}M} F') \cong R^{r'}$$

and hence $r = r'$ as R has IBN by Corollary 6.20. This contradiction completes the proof. □

There is, of course, a dual result for left zeroes. In particular, if M is non-trivial monoid with a zero element, then $\mathbb{Z}$ is not stably free as either a left or right $\mathbb{Z}M$-module. Thus, if M is a non-trivial monoid with zero, it has no M-finite free left or right equivariant classifying space but it has an M-finite projective one. This justifies considering projective M-CW complexes.

If L is a left ideal of M containing an identity e, then $L = Me = eMe$. Note that $\varphi\colon M \to Me$ given by $\varphi(m) = me$ is a surjective monoid homomorphism in this case since $\varphi(1) = e$ and $\varphi(mn) = mne = mene = \varphi(m)\varphi(n)$ as $ne \in Me = eMe$. Also note that the left M-set structure on Me given by inflation along φ corresponds to the left M-set structure on Me induced by left multiplication because if $m \in M$ and $n \in Me$, then $n = en$ and so $mn = men = \varphi(m)n$. Notice that if $f \in E(Me)$, then $f \in E(M)$ and $Mef = Mf$ is projective as both an Me-set and an M-set. Thus each (finitely generated) projective Me-set is a (finitely generated) projective M-set via inflation along φ. Note that if A is a left M-set, then $eM \otimes_M A \cong eA$ via $em \otimes a \mapsto ema$.

Proposition 6.22. *Suppose that M and $e \in E(M)$ with $Me = eMe$ and $0 \leq n \leq \infty$. If Me is of type left-F_n, then so is M. The converse holds if eM is a finitely generated projective left eMe-set.*

Proof. If X is a projective Me-CW complex constructed via pushouts as in (2.1) (but with M replaced by Me), then each P_n is a projective M-set via inflation along φ and so X is a projective M-CW complex. Moreover, if X is of Me-finite type (respectively, Me-finite), then it is of M-finite type (respectively, M-finite). Thus if Me is of type left-F_n, then so is M.

Suppose that X is an equivariant classifying space for M and eM is a finitely generated projective left eMe-set. Then $eM \otimes_M X \cong eX$. Now eX is a projective eMe-CW complex and if X_n is M-finite, then $(eX)_n = eX_n$ is eMe-finite by Corollary 3.2. Moreover, since eX is a retract of X as a CW complex and X is contractible, it follows that eX is contractible. Thus eX is an equivariant classifying space for eMe. The result follows. □

Our first corollary is that having a right zero guarantees the property left-F_∞ (which can be viewed as a defect of the one-sided theory).

Corollary 6.23. *If M contains a right zero, then M is of type left-F_∞. Hence any monoid with a zero is both of type left- and right-F_∞.*

Proof. If e is a right zero, then $Me = \{e\} = eMe$ and $\{e\}$ is of type left-F_∞. Thus M is of type left-F_∞ by Proposition 6.22. □

Recall that two elements m and n in a monoid M are said to be $\mathscr{L}$-related if and only if they generate the same principal left ideal, i.e., if $Mm = Mn$. Clearly $\mathscr{L}$ is an equivalence relation on M.

Corollary 6.24. *Suppose that M is a monoid and $e \in E(M)$ with eM a two-sided minimal ideal of M and $0 \leq n \leq \infty$. Note that $G_e = eMe$ is the maximal subgroup at e. If G_e is of type-F_n, then M is of type left-F_n. If eM contains finitely many $\mathscr{L}$-classes, then the converse holds.*

Proof. Note that $Me = eMe = G_e$ and so the first statement is immediate from Proposition 6.22. For the converse, it follows from Green's lemma [How95, Lemma 2.2.1] that eM is free left G_e-set and that the orbits are the $\mathscr{L}$-classes of eM. Thus the second statement follows from Proposition 6.22. □

Corollary 6.24 implies that if M is a monoid with a minimal ideal that is a group G, then M is of type left-F_n if and only if G is of type F_n and dually for right-F_n.

The following is a slight extension of the fact that a finite index subgroup of a group of type F_n is also of type F_n; see [Bro94, Chapter VIII, Proposition 5.1].

Proposition 6.25. *Let M be a monoid and N a submonoid such that M is a finitely generated projective left N-set. If M is of type left-F_n, then N is of type left-F_n, as well.*

Proof. Observe that each finitely generated free left M-set is a finitely generated projective N-set. Hence each finitely generated projective M-set, being a retract of a finitely generated free left M-set, is a finitely generated projective N-set. Thus any equivariant classifying space for M is also an equivariant classifying space for N. □

An immediate consequence of Proposition 6.8 is the following.

Proposition 6.26. *Let M, N be monoids of type left-F_n. Then $M \times N$ is of type left-F_n.*

6.2. **Left geometric dimension.** Let us define the *left geometric dimension* of M to be the minimum dimension of a left equivariant classifying space for M. The right geometric dimension is, of course, defined dually. Clearly, the geometric dimension is an upper bound on the cohomological dimension $\operatorname{cd} M$ of M. Recall that the (left) *cohomological dimension* of M is the projective dimension of the trivial module $\mathbb{Z}$, that is, the shortest length of a projective resolution of $\mathbb{Z}$. As mentioned in the introduction, for groups of cohomological dimension different than 2, it is known that geometric dimension coincides with cohomological dimension, but the general case is open.

Theorem 6.27. *Let M be a monoid. Then M has an equivariant classifying space of dimension $\max\{\operatorname{cd} M, 3\}$.*

Proof. If M has infinite cohomological dimension, then this is just the assertion that M has an equivariant classiyfing space. So assume that $\operatorname{cd} M < \infty$. Put $n = \max\{\operatorname{cd} M, 3\}$. Let Y be an equivariant classifying space for M. As the inclusion $Y_{n-1} \to Y$ is an $(n-1)$-equivalence, we deduce that $\pi_q(Y_{n-1})$ is trivial for $0 \leq q < n-1$. Also, as the augmented cellular chain complex of Y_{n-1} provides a partial projective resolution of the trivial module of length $n-1$ and $\operatorname{cd} M \leq n$, it follows that $\ker d_{n-1} = H_{n-1}(Y_{n-1})$ is a projective $\mathbb{Z}M$-module. By the Eilenberg swindle, there is a free $\mathbb{Z}M$-module F such that $H_{n-1}(Y_{n-1}) \oplus F \cong F$. Suppose that F is free on a set A. Fix a basepoint $y_0 \in Y_{n-1}$. We glue a wedge of $(n-1)$-spheres, in bijection with A, into Y_{n-1} at y_0 as well as freely gluing in its translates. That is we form a new projective M-CW complex Z with $Z_{n-2} = Y_{n-2}$ and where $Z = Z_{n-1}$ consists of the $(n-1)$-cells from Y_{n-1} and $M \times A \times B^{n-1}$ where the attaching map $M \times A \times S^{n-2}$ is given by $(m, a, x) \mapsto my_0$.

Notice that $C_{n-1}(Z) \cong C_{n-1}(Y_{n-1}) \oplus F$ as a $\mathbb{Z}M$-module and that the boundary map is zero on the F-summand since the boundary of each of the new $(n-1)$-cells that we have glued in is a point and $n \geq 3$. Therefore, $H_{n-1}(Z) = \ker d_{n-1} = H_{n-1}(Y_{n-1}) \oplus F \cong F$. As the inclusion $Y_{n-2} = Z_{n-2} \to Z$ is an $(n-2)$-equivalence, we deduce that $\pi_q(Z)$ is trivial for $0 \leq q \leq n-2$. In particular, Z is simply connected as $n \geq 3$. By the Hurewicz theorem, $\pi_{n-1}(Z, y_0) \cong H_{n-1}(Z)$. Choose mappings $f_a \colon S^{n-1} \to Z$, for $a \in A$, whose images under the Hurewicz mapping from a $\mathbb{Z}M$-module basis for $H_{n-1}(Z) \cong F$. Then form X by attaching $M \times A \times B^n$ to $Z = Z_{n-1}$ via the mapping

$$M \times A \times S^{n-1} \to Z$$

sending (m, a, x) to $mf_a(x)$.

Note that X is an n-dimensional CW complex with $X_{n-1} = Z_{n-1}$ and hence the inclusion $Z = X_{n-1} \to X$ is an $(n-1)$-equivalence. Therefore, $\pi_q(X) = 0 = H_q(X)$ for $0 \leq q \leq n-2$. Also $\pi_{n-1}(X, y_0) \cong H_{n-1}(X)$ via the Hurewicz isomorphism. Moreover, as the inclusion $Z = X_{n-1} \to X$ is an $(n-1)$-equivalence, we deduce that the inclusion induces a surjective homomorphism $\pi_{n-1}(Z, y_0) \to \pi_{n-1}(X, y_0)$ and hence a surjective homomorphism $H_{n-1}(Z) \to H_{n-1}(X)$. As the $\mathbb{Z}M$-module generators of $H_{n-1}(Z)$ have trivial images in $H_{n-1}(X)$ by construction and the Hurewicz map, we deduce that $H_{n-1}(X) = 0$.

Recall that $C_n(X) = H_n(X_n, X_{n-1})$. By standard cellular homology $H_n(X_n, X_{n-1})$ is a free $\mathbb{Z}M$-module on the images of the generator of the relative homology of (B^n, S^{n-1}) under the characteristic mappings

$$h_a \colon (\{1\} \times \{a\} \times B^n, \{1\} \times \{a\} \times S^{n-1}) \to (X_n, X_{n-1})$$

and the boundary map $\partial_n \colon H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1})$ sends the class corresponding to $a \in A$ to the image of the generator of S^{n-1} under the map on homology induced by the attaching map $f_a \colon S^{n-1} \to X_{n-1}$. Hence a free basis of $H_n(X_n, X_{n-1})$ is sent by ∂_n bijectively to a free basis for $H_{n-1}(X_{n-1})$ and so ∂_n is an isomorphism. The long exact sequence for reduced homology and the fact that an $(n-1)$-dimensional CW complex has trivial homology in degree n provides an exact sequence

$$0 = H_n(X_{n-1}) \longrightarrow H_n(X_n) \longrightarrow H_n(X_n, X_{n-1}) \xrightarrow{\partial_n} H_{n-1}(X_{n-1})$$

and so $H_n(X) = H_n(X_n) \cong \ker \partial_n = 0$. As X is a simply connected n-dimensional CW complex with $H_q(X) = 0$ for $0 \leq q \leq n$, we deduce that X is contractible by the Hurewicz and Whitehead theorems. Therefore, X is an n-dimensional equivariant classifying space for M, completing the proof. $\square$

We end this section by observing that monoids of left cohomological dimenison 0 are precisely the monoids of left geometric dimension 0. The following result generalises [GP98, Lemma 1 and Theorem 1].

Proposition 6.28. *Let M be a monoid. Then the following are equivalent.*

(1) M has a right zero element.
(2) M has left cohomological dimension 0.
(3) M has left geometric dimension 0.

Proof. If M has a right zero z, then $Mz = \{z\}$ is a projective M-set and hence the one point space is an equivariant classifying space for M. Thus M has left geometric dimension zero. If M has left geometric dimension zero, then it has left cohomological dimension zero. If M has left cohomological dimension zero, then $\mathbb{Z}$ is a projective $\mathbb{Z}M$-module and so the augmentation mapping $\varepsilon \colon \mathbb{Z}M \to \mathbb{Z}$ splits. Let P be the image of the splitting, so that $\mathbb{Z}M = P \oplus Q$. As P is a retract of $\mathbb{Z}M$ and each endomorphism of $\mathbb{Z}M$ is induced by a right multiplication, we have that $\mathbb{Z} \cong P = \mathbb{Z}Me$ for some idempotent $e \in \mathbb{Z}M$ with $\varepsilon(e) = 1$. Then since $me = e$ for all $m \in M$ and e has finite support X, we must have that M permutes X under left multiplication. Let G be the quotient of M that identifies two elements if they act the same on X. Then G is a finite group and $\mathbb{Z}$ must be a projective $\mathbb{Z}G$-module. Therefore, G is trivial. But this means that if $x \in X$, then $mx = x$ for all $m \in M$ and so M has a right zero. $\square$

We do not know whether left geometric dimension equals left cohomological dimension for monoids of cohomological dimension one or two, although the former is true for groups by the Stallings-Swan theorem.

7. Bi-equivariant classifying spaces

Let M be a monoid. We now introduce the bilateral notion of a classifying space in order to introduce a stronger property, bi-F_n. It will turn out that bi-F_n implies both left-F_n and right-F_n, but is strictly stronger. Moreover, bi-F_n implies bi-FP_n which is of interest from the point of view of Hochschild cohomology, which is the standard notion of cohomology for rings. Many of the results are similar to the previous section, but the proofs are more complicated.

First recall that M is an $M \times M^{op}$-set via the action $(m_L, m_R)m = m_L m m_R$. We say that a projective $M \times M^{op}$-CW complex X is a *bi-equivariant classifying space for M* if $\pi_0(X) \cong M$ as an $M \times M^{op}$-set and each component of X is contractible; equivalently, X has an $M \times M^{op}$-equivariant homotopy equivalence to the discrete $M \times M^{op}$-set M.

We can augment the cellular chain complex of X via the canonical surjection $\varepsilon\colon C_0(X) \to H_0(X) \cong \mathbb{Z}\pi_0(X) \cong \mathbb{Z}M$. The fact that each component of X is contractible guarantees that this is a resolution, which will be a projective bimodule resolution of $\mathbb{Z}M$ and hence suitable for computing Hochschild cohomology. We begin by establishing the uniqueness up to $M \times M^{op}$-homotopy equivalence of bi-equivariant classifying spaces.

Lemma 7.1. *Let X be a bi-equivariant classifying space for M and let Y be a locally path connected $M \times M^{op}$-space with contractible connected components. Suppose that $g\colon \pi_0(X) \to \pi_0(Y)$ is an $M \times M^{op}$-equivariant mapping. Then there exists a continuous $M \times M^{op}$-equivariant mapping $f\colon X \to Y$ such that the mapping $f_*\colon \pi_0(X) \to \pi_0(Y)$ induced by f is g.*

Proof. Let $r\colon X \to \pi_0(X)$ and $k\colon Y \to \pi_0(Y)$ be the projections to the set of connected components. Then k and r are continuous $M \times M^{op}$-equivariant maps where $\pi_0(X)$ and $\pi_0(Y)$ carry the discrete topology. Our goal will be to construct an $M \times M^{op}$-equivariant continuous mapping $f\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle k} \\ \pi_0(X) & \xrightarrow[g]{} & \pi_0(Y) \end{array}$$

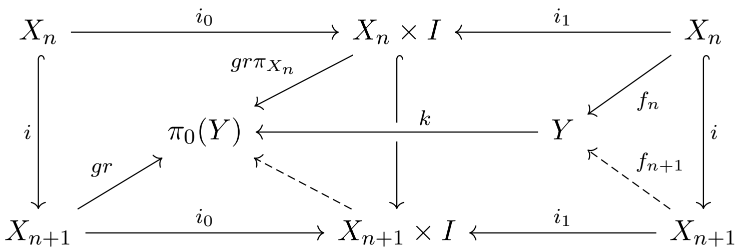
commutes. We construct, by induction, $M \times M^{op}$-equivariant continuous mappings $f_n\colon X_n \to Y$ such that

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle k} \\ \pi_0(X) & \xrightarrow[g]{} & \pi_0(Y) \end{array} \tag{7.1}$$

commutes and f_n extends f_{n-1}.

To define f_0, observe that $X_0 = \coprod_{a \in A} Me_a \times e'_a M$. Choose $y_a \in Y$ with $k(y_a) = g(r(e_a, e'_a))$. Then $k(e_a y e'_a) = e_a k(y_a) e'_a = e_a g(r(e_a, e'_a)) e'_a = g(r(e_a, e'_a))$ and so replacing y_a by $e_a y e'_a$, we may assume without loss of generality that $y_a \in e_a Y e'_a$. Then by Proposition 2.3 there is an $M \times M^{op}$-equivariant mapping $X_0 \to Y$ given by $(me_a, e'_a m') \mapsto m e_a y_a e'_a m'$ for $a \in A$ and $m \in M$. By construction, the diagram (7.1) commutes.

Assume now that f_n has been defined. The map $k\colon Y \to \pi_0(Y)$ is $M \times M^{op}$-equivariant and a weak equivalence (where $\pi_0(Y)$ has the discrete topology) because Y has contractible connected components. So by Theorem 2.5 we can construct a commutative diagram



where f_{n+1} is $M \times M^{op}$-equivariant and π_{X_n} is the projection. Note that $kf_{n+1} \simeq gr$ and hence, since $\pi_0(Y)$ is a discrete space, we conclude that $kf_{n+1} = gr$. Now take f to be the colimit of the f_n. This completes the proof. □

Theorem 7.2. *Let X, Y be bi-equivariant classifying spaces for M. Then X and Y are $M \times M^{op}$-homotopy equivalent by a cellular $M \times M^{op}$-homotopy equivalence.*

Proof. As $\pi_0(X) \cong M \cong \pi_0(Y)$ as $M \times M^{op}$-sets, there is an $M \times M^{op}$-equivariant isomorphism $g\colon \pi_0(X) \to \pi_0(Y)$. Then by Lemma 7.1, there is an $M \times M^{op}$-equivariant continuous mapping $f\colon X \to Y$ inducing g on connected components. It follows that f is a weak equivalence as X and Y both have contractible connected components. The result now follows from Corollary 2.7 and Theorem 2.8. □

Next we prove in an elementary fashion that bi-equivariant classifying spaces for M exist. A more canonical construction, using simplicial sets, was described earlier.

Lemma 7.3. *Let M be a monoid.*

(1) If X is a projective (free) $M \times M^{op}$-CW complex such that $\pi_0(X) \cong M$ and $\pi_q(X, x) = 0$ for all $1 \le q < n$ and $x \in X$, then there exists a projective $M \times M^{op}$-CW complex Y containing X as a projective $M \times M^{op}$-CW subcomplex and such that $Y_n = X_n$ and $\pi_q(Y, y) = 0$ for all $y \in Y$ and $1 \le q \le n$.

(2) If X is a projective (free) $M \times M^{op}$-CW complex such that $\pi_0(X) \cong M$ and $\pi_q(X, x) = 0$ for all $1 \le q < n$ and $x \in X$, then there exists a projective (free) $M \times M^{op}$-CW complex Y with contractible connected components containing X as a projective $M \times M^{op}$-CW subcomplex and such that $Y_n = X_n$.

Proof. This is a minor adaptation of the proof of Lemma 6.4 that we leave to the reader. □

Corollary 7.4. *Let M be a monoid. Then there exists a free $M \times M^{op}$-CW complex X with $\pi_0(X) \cong M$ and each connected component of X contractible.*

Proof. By Lemma 7.3 it suffices to construct a free $M \times M^{op}$-graph X with $\pi_0(X) \cong M$. We take $X_0 = M \times M$ and we take an edge set in bijection with $M \times M \times M$. The edge (m_L, m_R, m) will connect (m_L, mm_R) to $(m_L m, m_R)$. Then X is a free $M \times M^{op}$-graph. Notice that if (m_1, m_2) is connected by an edge to (m_1', m_2'), then $m_1 m_2 = m_1' m_2'$. On the other hand, $(1, m_2, m_1)$ is an edge from $(1, m_1 m_2)$ to (m_1, m_2) and hence there is a bijection $\pi_0(X) \to M$ sending the component of (m_1, m_2) to $m_1 m_2$ and this mapping is an $M \times M^{op}$-equivariant bijection. □

Example 7.5. It follows from the definitions and results in Section 5 that the geometric realisation $|\overleftrightarrow{EM}|$ of the nerve of the two-sided Cayley graph category of M is a bi-equivariant classifying space for M.

Corollary 7.6. *If X is a bi-equivariant classifying space for M, then $M \backslash X / M \simeq |BM|$.*

Proof. We have $M\backslash|\overleftrightarrow{EM}|/M \cong |BM|$. The result now follows from Theorem 7.2 and Proposition 3.5. □

Another important definition for this paper is the following. A monoid M is of type *bi*-F_n if there is a bi-equivariant classifying space X for M such that X_n is $M\times M^{op}$-finite, i.e., $M\backslash X/M$ has finite n-skeleton. We say that M is of type *bi*-F_∞ if M has a bi-equivariant classifying space X that is of $M\times M^{op}$-finite type, i.e., $M\backslash X/M$ is of finite type. Clearly by making use of the canonical two-sided classifying space $|\overleftrightarrow{EM}|$ we can immediately conclude that any finite monoid is of type bi-F_∞.

Recall that a monoid M is said to be of type *bi*-FP_n if there is a partial resolution of the $(\mathbb{Z}M,\mathbb{Z}M)$-bimodule $\mathbb{Z}M$

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z}M \to 0$$

where $A_0, A_1, \ldots, A_n$ are finitely generated projective $(\mathbb{Z}M,\mathbb{Z}M)$-bimodules. Monoids of type bi-FP_n were studied by Kobayashi and Otto in [KO01]. We note that this differs from the definition of bi-FP_n considered in [AH03], which is called *weak bi*-FP_n by Pride in [Pri06] where it is shown to be equivalent to being simultaneously of type left- and right-FP_n. In this paper by bi-FP_n we shall always mean bi-FP_n in the above sense of Kobayashi and Otto. The property bi-FP_n is of interest because of its connections with the study of Hochschild cohomology [Wei94, Chapter 9]. Kobayashi investigated bi-FP_n in [Kob05, Kob07, Kob10] proving, in particular, that any monoid which admits a presentation by a finite complete rewriting system is of type bi-FP_n. This has applications for the computation of Hochschild cohomology. We shall recover this theorem of Kobayashi below in Section 11 as an application of our results on equivariant discrete Morse theory and collapsing schemes. See also [Pas08] for further related results on bi-FP_n.

The following result relates bi-F_n with bi-FP_n.

Proposition 7.7. *Let M be a monoid.*

1. *For $0 \le n \le \infty$, if M is of type bi-F_n, then it is of type bi-FP_n.*
2. *If M is of type bi-F_∞, then it is of type bi-F_n for all $n \ge 0$.*
3. *If M is of type bi-F_n for $0 \le n \le \infty$, then M is of type left-F_n and type right-F_n.*
4. *For $0 \le n \le \infty$, a group is of type bi-F_n if and only if it is of type F_n.*

Proof. The first item follows using that the cellular chain complex of a bi-equivariant classying space X can be augmented, as discussed earlier, to give a bimodule resolution of $\mathbb{Z}M$ and that if X is built up from pushouts as per (2.1) (with $M\times M^{op}$ in place of M), then the n^{th}-chain module is isomorphic to $\mathbb{Z}P_n$ as a bimodule and hence is projective. The second item is trivial.

For the third item, one verifies that if $\overleftrightarrow{EM}$ is the two-sided bar construction, then $\overleftarrow{EM} \cong \overleftrightarrow{EM}/M$ where $\overleftarrow{EM}$ is the left bar construction. Suppose now that X is a bi-equivariant classifying space for M such that X_n is $M\times M^{op}$-finite. Then $X \simeq_{M\times M^{op}} \overleftrightarrow{EM}$ by Theorem 7.2. Therefore, $X/M \simeq_M |\overleftrightarrow{EM}|/M = |\overleftarrow{EM}|$ and X/M is a projective M-CW complex with $(X/M)_n = X_n/M$ being M-finite by Corollary 3.8. Thus if M is of type bi-F_n for $0 \le n \le \infty$, then M is of type left-F_n and dually right-F_n.

If G is a group of type bi-F_n, then it is of type F_n by the previous item and Proposition 6.9. Conversely, suppose that X is a free G-CW complex with G-finite n-skeleton. Then using the right G-set structure on $G\times G$ from Proposition 3.7 we have that $Y = (G\times G)\otimes_G X$ is a projective $G\times G^{op}$-CW complex by Proposition 3.2 such that Y_n is $G\times G^{op}$-finite. Moreover, $\pi_0(Y) \cong (G\times G)\otimes_G \pi_0(X)$ by Proposition 3.4. But $\pi_0(X)$ is the trivial G-set and $(G\times G)\otimes_G 1 \cong G$ as a $G\times G^{op}$-set via $(g,h)\otimes 1 \mapsto gh$. Finally, since G is a free right G-set of on G-generators by Proposition 3.7, it follows that as a topological space $Y = \coprod_G X$ and hence each component of Y is contractible. This completes the proof. $\square$

The proof of Proposition 7.7 establishes the following proposition.

Proposition 7.8. *If X is a bi-equivariant classifying space for M, then X/M is an equivariant classifying space for M.*

Sometimes it will be convenient to use the following reformulation of the property bi-F_n.

Proposition 7.9. *Let M be a monoid. The following are equivalent for $0 \leq n < \infty$.*

(1) M is of type bi-F_n

(2) There is an $M \times M^{op}$-finite projective $M \times M^{op}$-CW complex X of dimension at most n with $\pi_0(X) \cong M$ and $\pi_q(X, x) = 0$ for $1 \leq q < n$ and $x \in X$.

Proof. This is entirely analogous to the proof of Proposition 6.10. $\square$

If M is a monoid and $A \subseteq M$, then the *two-sided Cayley digraph* $\overleftrightarrow{\Gamma(M, A)}$ is the digraph with vertex set $M \times M$ and with edges in bijection with elements of $M \times M \times A$. The directed edge (m_L, m_R, a) goes from (m_L, am_R) to $(m_L a, m_R)$ and we draw it as

$$(m_L, am_R) \xrightarrow{a} (m_L a, m_R).$$

Note that $\overleftrightarrow{\Gamma(M, A)}$ is a free $M \times M^{op}$-graph and is $M \times M^{op}$-finite if and only if A is finite. Also note that if (m_1, m_2) is connected to (m_1', m_2') by an edge, then $m_1 m_2 = m_1' m_2'$. Hence multiplication of the coordinates of a vertex induces a surjective $M \times M^{op}$-equivariant mapping $\pi_0(\overleftrightarrow{\Gamma(M, A)}) \to M$. If A is a generating set for M, then the mapping is an isomorphism because if $m_1, m_2 \in M$ and $u \in A^*$ is a word representing m_1, then there is a directed path labelled by u from $(1, m)$ to (m_1, m_2). Namely, if $u = a_1 \cdots a_k$ with $a_i \in A$, then the path labelled by u from $(1, m)$ to (m_1, m_2) is

$$(1, m) \xrightarrow{a_1} (a_1, a_2 \cdots a_k m_2) \xrightarrow{a_2} (a_1 a_2, a_3 \cdots a_k m_2) \xrightarrow{a_3} \cdots \xrightarrow{a_k} (m_1, m_2). \tag{7.2}$$

A monoid M is said to be *dominated* by a subset A if whenever $f, g\colon M \to N$ are monoid homomorphisms with $f|_A = g|_A$, one has $f = g$. In other words, the inclusion $\langle A \rangle \hookrightarrow M$ is an epimorphism (in the category theory sense). Of course, a generating set of M dominates M. Note that if A is a subset of an inverse monoid M (e.g., a group), then A dominates M if and only if A generates M as an inverse monoid. Hence M is finitely generated if and only if M is finitely dominated. Kobayashi gives an example of an infinitely generated monoid that is finitely dominated. See [Kob07] for details.

Theorem 7.10. *The following are equivalent for a monoid M.*

(1) M is of type bi-F_1.

(2) M is of type bi-FP_1.

(3) There is a finite subset $A \subseteq M$ such that the natural mapping $\pi_0(\overleftrightarrow{\Gamma(M, A)}) \to M$ is an isomorphism.

(4) There is a finite subset $A \subseteq M$ that dominates M.

In particular, any finitely generated monoid is of type bi-F_1.

Proof. The equivalence of (2) and (4) was established by Kobayashi [Kob07] using Isbel's zig-zag lemma (actually, the equivalence of (3) and (4) is also direct from Isbel's zig-zag lemma).

Assume that (3) holds. Then $\overleftrightarrow{\Gamma(M, A)}$ is $M \times M^{op}$-finite and so M is of type bi-F_1 by Proposition 7.9. Proposition 7.7 shows that (1) implies (2).

Assume that M is of type bi-FP_1. Then we have a partial free resolution

$$\mathbb{Z}M \otimes \mathbb{Z}M \xrightarrow{\mu} \mathbb{Z}M \longrightarrow 0$$

of finite type where μ is induced by the multiplication in $\mathbb{Z}M$. Since M is of type bi-FP_1, $\ker \mu$ is finitely generated. It is well known that $\ker \mu$ is generated as a $\mathbb{Z}M \otimes \mathbb{Z}M^{op}$-module by the elements $m \otimes 1 - 1 \otimes m$ with $m \in M$. Indeed, if $\sum c_i m_i \otimes n_i$ is in $\ker \mu$, then $\sum c_i m_i n_i = 0$ and so

$$\begin{aligned}\sum c_i m_i \otimes n_i &= \sum c_i m_i \otimes n_i - \sum c_i (1 \otimes m_i n_i)\\ &= \sum c_i (m_i \otimes n_i - 1 \otimes m_i n_i)\\ &= \sum c_i (m_i \otimes 1 - 1 \otimes m_i) n_i.\end{aligned}$$

Hence there is a finite subset $A \subseteq M$ such that $\ker \mu$ is generated by the elements $a \otimes 1 - 1 \otimes a$ with $a \in A$. We claim that the natural surjective mapping $\pi_0(\overleftrightarrow{\Gamma(M,A)}) \to M$ is an isomorphism.

Identifying $\mathbb{Z}[M \times M^{op}]$ with $\mathbb{Z}M \otimes \mathbb{Z}M^{op}$ as rings and $\mathbb{Z}[M \times M]$ with $\mathbb{Z}M \otimes \mathbb{Z}M$ as bimodules, we have a bimodule homomorphism

$$\lambda\colon \mathbb{Z}[M \times M] \to \mathbb{Z}\pi_0(\overleftrightarrow{\Gamma(M,A)})$$

sending (m_L, m_R) to its connected component in $\overleftrightarrow{\Gamma(M,A)}$ and μ factors as λ followed by the natural mapping

$$\mathbb{Z}\pi_0(\overleftrightarrow{\Gamma(M,A)}) \to \mathbb{Z}M.$$

Clearly, $\ker \lambda \subseteq \ker \mu$ and so to prove the result it suffices to show that $\ker \mu \subseteq \ker \lambda$. Now $\ker \mu$ is generated by the elements $(1,a) - (a,1)$ with $a \in A$ under our identifications. But $(1,1,a)$ is an edge from $(1,a)$ to $(a,1)$. Thus $(1,a) - (a,1) \in \ker \lambda$ for all $a \in A$. This establishes that (2) implies (3), thereby completing the proof. □

If G is a group, then it follows from Theorem 7.10 that $G \cup \{0\}$ is of type bi-F_1 if and only if G is finitely generated. Indeed, $G \cup \{0\}$ is an inverse monoid and hence finitely dominated if and only if finitely generated. But $G \cup \{0\}$ is finitely generated if and only if G is finitely generated. On the other hand, $G \cup \{0\}$ is both of type left- and right-F_∞ for any group G by Corollary 6.23. Thus bi-F_n is a much stronger notion.

Remark 7.11. It can be shown that if M is a monoid and M^0 is the result of adjoining a 0 to M, then if M^0 is of type bi-F_n, then M is of type bi-F_n. The idea is that if X is a bi-equivariant classifying space for M^0, then the union Y of components of X corresponding to elements of M is a bi-equivariant classifying space for M and Y_n will be $M \times M^{op}$-finite if X_n is $M^0 \times (M^0)^{op}$-finite. More generally, if T is a submonoid of M such that $M \setminus T$ is an ideal, then M being of type bi-F_n implies T is also of type bi-F_n.

Next we show that finitely presented monoids are of type bi-F_2. The proof is similar to the proof of Theorem 6.14, which is in fact a consequence.

Theorem 7.12. *Let M be a finitely presented monoid. Then M is of type bi-*F_2.

Proof. Suppose that M is generated by a finite set A with defining relations $u_1 = v_1, \ldots, u_n = v_n$. We construct an $M \times M^{op}$-finite 2-dimensional free $M \times M^{op}$-CW complex X with 1-skeleton the two-sided Cayley graph $\overleftrightarrow{\Gamma(M,A)}$ by attaching an $M \times M^{op}$-cell $M \times M \times B^2$ for each relation. Suppose that u_i and v_i map to m_i in M. Let p_i, q_i be the paths from $(1, m_i)$ to $(m_i, 1)$ labelled by u_i and v_i, respectively, cf. (7.2). Then we glue in a disk d_i with boundary path $p_i q_i^{-1}$ and glue in $M \times M \times B^2$ using Proposition 2.3 (so $\{(m_L, m_R)\} \times B^2$ is sent to

$m_Ld_im_R$). Then X is a free $M \times M^{op}$-CW complex of dimension at most 2 that is $M \times M^{op}$-finite and $\pi_0(X) \cong M$. By Proposition 6.10, it suffices to prove that each connected component of X is simply connected.

The connected component $X(m)$ of X corresponding to $m \in M$ is a digraph rooted at $(1, m)$ by (7.2). Let T_m be a directed spanning tree for $X(m)$ rooted at $(1, m)$. Let $e = (n_1, an_2) \xrightarrow{a} (n_1a, n_2)$ be a directed edge of $X(m)$ not belonging to T_m. Then the corresponding generator of $\pi_1(X(m), (1, m))$ is of the form peq^{-1} where p and q are directed paths from $(1, m)$ to (n_1, an_2) and (n_1a, n_2), respectively. Let u be the label of p and v be the label of q. Then $ua = v$ in M. Thus it suffices to prove that if $x, y \in A^*$ are words which are equal in M to m' labelling respective paths from $(1, m)$ to (m', m'') with $m'm'' = m$, then the corresponding loop ℓ labelled xy^{-1} at $(1, m)$ is null homotopic.

By induction on the length of a derivation from x to y, we may assume that $x = wu_iw'$ and $y = wv_iw'$ for some $i = 1, \ldots, n$. Then the path labelled by w starting at $(1, m)$ ends at $(w, m_iw'm'')$ where we recall that m_i is the image of u_i, v_i in M. Then $wd_iw'm''$ is a 2-cell bounded by parallel paths from $(w, m_iw'm'')$ to $(wm_i, w'm'')$ labeled by u_i and v_i, respectively. It follows that the paths labelled by x and y from $(1, m)$ to (m', m'') are homotopic relative to endpoints and hence ℓ is null homotopic. This completes the proof that $X(m)$ is simply connected. □

Remark 7.13. We currently do not know the precise relationship between bi-F_2 and finitely presentability for monoids. Specifically we have the question: Is there a finitely generated bi-F_2 monoid that is not finitely presented? Even for inverse monoids this question remains open.

We next observe that finitely generated free monoids are bi-F_∞.

Proposition 7.14. *Let A be a finite set. Then the free monoid A^* is of type bi-*F_∞.

Proof. Each connected component of $\overleftrightarrow{\Gamma(M, A)}$ is a tree and hence contractible. Thus $\overleftrightarrow{\Gamma(M, A)}$ is an $A^* \times (A^*)^{op}$-finite bi-equivariant classifying space for A^*. □

Theorem 6.15 has an analogue for bi-F_n and bi-FP_n with essentially the same proof, which we omit.

Theorem 7.15. *Let M be a monoid of type bi-*F_2*. Then M is of type bi-*F_n *if and only if M is of type bi-*FP_n *for $0 \leq n \leq \infty$.*

Observe that Theorem 7.15 implies that M is of type bi-F_∞ if and only if M is of type bi-F_n for all $n \geq 0$. The analogue of Proposition 6.16 in our setting again admits a very similar proof that we omit.

Proposition 7.16. *If M is of type bi-*F_n *with $n \geq 1$, then M has a free $M \times M^{op}$-CW complex X that is a bi-equivariant classifying space for M where X_n is $M \times M^{op}$-finite.*

Proposition 6.26 also has a two-sided analogue.

Proposition 7.17. *If M, N are of type bi-*F_n*, then $M \times N$ is of type bi-*F_n.

Let us turn to some inheritance properties for bi-F_n. If I is an ideal of M containing an identity e, then e is a central idempotent and $MeM = Me = eM = eMe$. Indeed, $em = (em)e = e(me) = me$ as $em, me \in I$. If $f, f' \in E(M)$, then $fe, f'e \in E(eMe)$ and $e(Mf \times f'M)e = eMefe \times f'eMe$ as an eMe-eMe-biset and hence is finitely generated projective. Thus if P is a (finitely generated) projective $M \times M^{op}$-set, then ePe is a (finitely generated) projective $eMe \times eMe^{op}$-set.

Proposition 7.18. *Let M be a monoid and $0 \leq n \leq \infty$ and $e \in E(M)$ be a central idempotent. If M is of type bi-*F_n*, then so is eMe.*

Proof. Let X be a bi-equivariant classifying space of M such that X_n is $M \times M^{op}$-finite. Suppose that X is obtained via pushouts as per (2.1) (but with $M \times M^{op}$ in place of M). Then each eP_ke is a projective $eMe \times eMe^{op}$-set and is finitely generated whenever P_k was finitely generated by the observation preceding the proposition. Thus eXe is a projective $eMe \times eMe^{op}$-CW complex and $(eXe)_n = eX_ne$ is $eMe \times eMe^{op}$-finite. Also, since $eXe \cong (eM \times Me) \otimes_{M \times M^{op}} X$, we deduce that $\pi_0(eXe) \cong e\pi_0(X)e \cong eMe$ by Proposition 3.4. If $X(m)$ is the component of X corresponding to $m \in eMe$, then $eX(m)e$ is the component of eXe corresponding to m in eXe and is a retract of $X(m)$. But $X(m)$ is contractible and hence $eX(m)e$ is contractible. This shows that eXe is a bi-equivariant classifying space for eMe, completing the proof. □

Two monoids M and N are *Morita equivalent* if the categories of left M-sets and left N-sets are equivalent. It is known that this is the case if and only if there is an idempotent $e \in E(M)$ such that $xy = 1$ for some $x, y \in M$ with $ey = y$ and $eMe \cong N$ [Kna72]. It follows easily that if M and N are Morita equivalent, then so are M^{op} and N^{op}. Note that if e is as above, then the functor $A \mapsto eA \cong eM \otimes_M A$ from M-sets to N-sets (identifying N with eMe) is an equivalence of categories with inverse $B \mapsto Me \otimes_N B$. This uses that $Me \otimes_{eMe} eM \cong M$ as $M \times M^{op}$-sets via the multiplication map (the inverse bijection takes $m \in M$ to $mxe \otimes y$) and $eM \otimes_M Me \cong eMe$ as $eMe \times (eMe)^{op}$-sets (via the multiplication with inverse $eme \mapsto em \otimes e$). It follows that if P is a (finitely generated) projective M-set, then eP is a (finitely generated) projective N-set (as being projective is categorical and a projective is finitely generated if and only if it is a coproduct of finitely many indecomposable projectives, which is also categorical). In particular, eM is a finitely generated projective N-set.

Proposition 7.19. *Let M and N be Morita equivalent monoids and $0 \leq n \leq \infty$.*

(1) M is of type left-F_n *if and only if N is of type left-*F_n*.*
(2) M is of type right-F_n *if and only if N is of type right-*F_n*.*
(3) M is of type bi-F_n *if and only if N is of type bi-*F_n*.*

Proof. By symmetry it suffice to prove the implications from left to right. We may assume without loss of generality that $N = eMe$ where $1 = xy$ with $ey = y$. Notice that $1 = xy = xey$ and so replacing x by xe, we may assume that $xe = x$. To prove (1), suppose that X is an equivariant classifying space for M such that X_n is M-finite. Then $eM \otimes_M X \cong eX$ is a projective N-CW complex by Proposition 3.2 such that $(eX)_n = eX_n$ is N-finite. But eX is a retract of X and hence contractible. We deduce that N is of type left-F_n.

The proof of (2) is dual. To prove (3), observe that $(e,e)(M \times M^{op})(e,e) = eMe \times (eMe)^{op}$ and that we have $(x,y)(y,x) = (1,1)$ and $(e,e)(y,x) = (y,x)$ in $M \times M^{op}$ because $xy = e$, $ey = y$ and $xe = x$. Thus $M \times M^{op}$ is Morita equivalent to $N \times N^{op}$ and $eM \times Me$ is a finitely generated projective $N \times N^{op}$-set. Suppose that X is a bi-equivariant classifying space for M such that X_n is $M \times M^{op}$-finite. Then $(eM \times Me) \otimes_{M \times M^{op}} X \cong eXe$ is a projective $N \times N^{op}$-CW complex such that $(eXe)_n = eX_ne$ is $N \times N^{op}$-finite by Corollary 3.2. Also, $\pi_0(eXe) \cong e\pi(X)e \cong N$ by Proposition 3.4. Moreover, if $m \in eMe$ and $X(m)$ is the component of X corresponding to m, then the component of eXe corresponding to m is $eX(m)e$, which is a retract of $X(m)$ and hence contractible. Thus eXe is a bi-equivariant classifying space of N. The result follows. □

There are examples of Morita equivalent monoids that are not isomorphic; see [Kna72].

We define the *geometric dimension* of M to be the minimum dimension of a bi-equivariant classifying space for M. The *Hochschild cohomological dimension* of M, which we write $\dim M$, is the length of a shortest projective resolution of $\mathbb{Z}M$ as a $\mathbb{Z}[M \times M^{op}]$-module. Of course, the Hochschild cohomological dimension bounds both the left and right cohomological dimension and the geometric dimension bounds the Hochschild cohomological dimension. Also the geometric dimension bounds both the left and right geometric dimensions because if X is a bi-equivariant classifying space for M of dimension n, then X/M is a classifying space of dimension n.

The following theorem has an essentially identical proof to Theorem 6.27.

Theorem 7.20. *Let M be a monoid. Then M has a bi-equivariant classifying space of dimension* $\max\{\dim M, 3\}$.

Free monoids have a forest for a bi-equivariant classifying space and hence have geometric dimension 1. It is well known (see e.g. [Mit72]) that they have Hochschild cohomological dimension 1.

It is known that a monoid has Hochschild cohomological dimension 0 if and only if it is a finite regular aperiodic monoid with sandwich matrices invertible over $\mathbb{Z}$ (see [Che84]). For instance, any finite aperiodic inverse monoid has Hochschild cohomogical dimension 0. A non-trivial monoid of Hochschild cohomological dimension 0 does not have geometric dimension 0 because M would have to be a projective M-biset. So $M \cong Me \times fM$, with $e, f \in E(M)$, via an equivariant map φ sending (e, f) to 1 (as M being finite aperiodic implies that 1 is the unique generator of M as a two-sided ideal). But then $f = \varphi(e, f)f = \varphi(e, f) = 1$ and similarly $e = 1$ and so M is trivial. Thus non-trivial monoids of Hochschild cohomological dimension 0 do not have geometric dimension 0.

8. Brown's theory of collapsing schemes

The theory of collapsing schemes was introduced by Brown in [Bro92]. Since then it has become an important and often-used tool for proving that certain groups are of type F_∞. The first place the idea appears is in a paper of Brown and Geoghegan [BG84] where they had a cell complex with one vertex and infinitely many cells in each positive dimension, and they showed how it could be collapsed to a quotient complex with only two cells in each positive dimension. Brown went on to develop this idea further in [Bro92] formalising it in his theory of collapsing schemes, and applying it to give a topological proof that groups which admit presentations by finite complete rewriting systems are of type F_∞ (see Section 11 below for the definition of complete rewriting system). Brown's theory of collapsing schemes was later rediscovered under the name of discrete Morse theory [For95, For02], an important area in algebraic combinatorialists. Chari [Cha00] formulated discrete Morse theory combinatorially via Morse matchings, which turn out to be the same thing as collapsing schemes.

The basic idea of collapsing schemes for groups is a follows. Suppose we are given a finitely presented group G and we would like to prove it is of type F_∞. Then we can first begin with the big $K(G, 1)$ complex $|BG|$ with infinitely many n-cells for each n. Then in certain situations it is possible to show how one can collapse away all but finitely many cells in each dimension resulting in a $K(G, 1)$ much smaller than the one we started with. The collapse is carried out using a so-called *collapsing scheme* associated with the simplicial set BG. It turns out that any group which is presentable by a finite complete rewriting system admits a collapsing scheme that, using this process, can be used to prove the group is of type F_∞; see [Bro92, page 147].

As mentioned in the introduction above, Brown in fact develops this theory for monoids in general, and applies the theory of collapsing schemes to show that if M admits a presentation by a finite complete rewriting system then its classifying space $|BM|$ has the homotopy type of a CW complex with only finitely many cells in each dimension. However, as discussed in detail in the introduction to this article, this information about the space $|BM|$ is not enough on its own to imply that the monoid M is of type left-FP_∞.

Motivated by this, in this section we shall develop the theory of M-equivariant collapsing schemes. We shall prove that if an M-simplicial set admits an M-equivariant collapsing scheme of finite type then the monoid is of type left-F_∞. We then prove that if M admits a finite complete rewriting system then $\overrightarrow{EM}$ admits an M-equivariant collapsing scheme of finite type, thus giving a topological proof that such monoids are of type left-F_∞. To do this, we shall identify conditions under which a collapsing scheme for BM can be lifted to give an M-equivariant collapsing scheme for $\overrightarrow{EM}$. These conditions will hold in a number of different situations, including when M admits a presentation by a finite complete rewriting system and when M is a, so-called, factorable monoid [HO14]. We also develop the two-sided theory. As a consequence we also obtain a topological proof of the fact that such a monoid is of type bi-F_∞, recovering a theorem of Kobayashi [Kob05].

8.1. **Collapsing schemes.** Let $K = \bigcup_{i \geq 0} K_i$ be a simplicial set and let $X = |K|$ be its geometric realisation. We identify the cells of X with the non-degenerate simplices of K. A collapsing scheme for K consists of the following data:

- A partition of the cells of X into three classes, $\mathfrak{E}$, $\mathfrak{C}$, $\mathfrak{R}$, called the *essential*, *collapsible* and *redundant* cells, respectively, where the collapsible cells all have dimension at least one.
- Mappings c and i which associate with each redundant n-cell τ a collapsible $(n+1)$-cell $c(\tau)$, and a number $i(\tau)$, such that $\tau = d_{i(\tau)}(c(\tau))$.

Let $\sigma = c(\tau)$. If τ' is a redundant n-cell such that $\tau' = d_j\sigma$ for some $j \neq i(\tau)$ then we call τ' an immediate predecessor of τ and write $\tau' \prec \tau$. Furthermore, the conditions for a collapsing scheme are satisfied, which means:

(C1) for all n, the mapping c defines a bijection between $\mathfrak{R}_n$ (the redundant n-cells) and $\mathfrak{C}_{n+1}$ (the collapsible $(n+1)$-cells).

(C2) there is no infinite descending chain $\tau \succ \tau' \succ \tau'' \succ \cdots$ of redundant n-cells.

Condition (C2) clearly implies that there is a unique integer i such that $\tau = d_i(c(\tau))$ (otherwise we would have $\tau \succ \tau$, leading to an infinite descending chain). It also follows from (C2) that, by Königs lemma, there cannot be arbitrarily long descending chains $\tau_0 \succ \cdots \succ \tau_k$. This is a key fact in the proof of [Bro92, Proposition 1] since it gives rise to the notion of 'height':

Definition 8.1 (Height)**.** The *height* of a redundant cell τ, written height(τ), is the maximum length of a descending chain $\tau = \tau_0 \succ \tau_1 \succ \cdots \succ \tau_k$.

We say that a collapsing scheme is of *finite type* if it has finitely many essential cells of each dimension.

In the construction of the 'small' CW complex in the proof of [Bro92, Proposition 1] the redundant n-cells are adjoined in order of their heights, guaranteeing in the proof that the adjunction of τ and $c(\tau)$ is an elementary expansion. This is the the key idea in the proof which is that each pair $(\tau, c(\tau))$ of redundant and corresponding collapsible cells may be adjoined without changing the homotopy type, and so in the end it is only the essential cells that matter. More precisely, Brown proves that if K be a simplical set with a collapsing scheme, then its geometric realisation $X = |K|$ admits a canonical quotient CW complex Y, whose cells are in 1–1 correspondence with the essential cells of X. This notion of height in Brown's theory relates to the values taken by the discrete Morse function in Forman's theory (see [For02, page 10]). A discrete Morse function gives one a way to build the simplicial complex by attaching the simplices in the order prescribed by the function, i.e., adding first the simplices which are assigned the smallest values. Brown's essential cells are called 'critical' in Forman's theory.

9. M-equivariant collapsing schemes

In this section we develop the theory of M-equivariant collapsing schemes, or equivalently, of M-equivariant discrete Morse theory. Results on G-equivariant discrete Morse theory, for G a group, may be found in [Fre09].

Let $K = \bigcup_{i \geq 0} K_i$ be a simplicial set with degeneracy and face operators d_i, s_i, and equipped with a collapsing scheme (E, R, C, c, i). Here E, R and C partition the cells (which are in bijective correspondence with the non-degenerate simplices) of K.

Let M be a monoid acting on the simplicial set K with the following conditions satisfied:

(A1) The action of M maps n-simplicies to n-simplicies, and commutes with d_i and s_i, that is, M is acting by simplicial morphisms.

(A2) For every n-simplex σ and $m \in M$, σ is a cell (i.e. is a non-degenerate simplex) if and only if $m\sigma$ is a cell, in which case $\sigma \in E$ (respectively R, C) if and only if $m\sigma \in E$ (respectively R, C).

(A3) If $(\sigma, \tau) \in R_n \times C_{n+1}$ is a matched redundant-collapsible pair (i.e. $\tau = c(\sigma)$) then so is the pair $m(\sigma, \tau) = (m\sigma, m\tau) \in R_n \times C_{n+1}$, i.e., $c(m\sigma) = mc(\sigma)$ for $\sigma \in R_n$.

(A4) There is a subset $B \subseteq E \cup R \cup C$ such that for all n the set of n-cells is a free left M-set with basis B_n (where B_n is the set of n-cells in B). Let $E^B = E \cap B$, $R^B = R \cap B$ and $C^B = C \cap B$. Then E_n is a free left M-set with basis E^B_n, and similarly for R_n and C_n.

(A5) For every matched pair $(\sigma, \tau) \in R \times C$, $\sigma \in R^B$ if and only if $\tau \in C^B$. In particular, for every matched pair (σ, τ) there is a unique pair $(\sigma', \tau') \in R^B \times C^B$ and $m \in M$ such that $(\sigma, \tau) = m(\sigma', \tau')$; namely, if $\sigma = m\sigma'$ with $\sigma' \in R^B$ and $\tau' = c(\sigma')$, then $m\tau' = mc(\sigma') = c(m\sigma') = c(\sigma) = \tau$.

(A6) For every redundant cell τ and every $m \in M$

$$\text{height}(\tau) = \text{height}(m\tau),$$

with height defined as in Definition 8.1 above.

These conditions imply that K is a rigid free left M-simplicial set and hence by Lemma 4.4 the action of M on K induces an action of M on the geometric realisation $|K|$ by continuous maps making $|K|$ into a free left M-CW complex. When the above axioms hold, we call (E, R, C, c, i) an *M-equivariant collapsing scheme* for the rigid free left M-simplicial set K. Dually, given a rigid free right M-simplicial set K with a collapsing scheme satisfying the above axioms for K as an M^{op}-simplicial set we call (E, R, C, c, i) an M-equivariant collapsing scheme for K. If K is a bi-M-simplicial set we say (E, R, C, c, i) an M-equivariant collapsing scheme if the axioms are satisfied for K as a left $M \times M^{op}$-simplicial set.

Our aim is to prove a result about the M-homotopy type of $|K|$ when K has an M-equivariant collapsing scheme. Before doing this we first make some observations about mapping cylinders and the notion of elementary collapse.

9.1. **Mapping cylinders and elementary collapse.** If X is a subspace of a space Y then $D \colon Y \to X$ is a *strong deformation retraction* if there is a map $F \colon Y \times I \to Y$ such that, with $\mathrm{F}_t \colon Y \to Y$ defined by $\mathrm{F}_t(y) = F(y, t)$, we have (i) $\mathrm{F}_0 = 1_Y$, (ii) $\mathrm{F}_t(x) = x$ for all $(x, t) \in X \times I$, and (iii) $\mathrm{F}_1(y) = D(y)$ for all $y \in Y$. If $D \colon X \to Y$ is a strong deformation retraction then D is a homotopy equivalence, a homotopy inverse of which is the inclusion $i \colon X \hookrightarrow Y$.

Definition 9.1 (Mapping cylinder)**.** Let $f \colon X \to Y$ be a cellular map between CW complexes. The *mapping cylinder* M_f is defined to be the adjunction complex $Y \coprod_{f_0} (X \times I)$ where $f_0 \colon X \times$

$\{0\}$ is the map $(x,0) \mapsto f(x)$. Let $i_1\colon X \to X \times I, x \mapsto (x,1)$ and let $i_0\colon X \to X \times I, x \mapsto (x,0)$. Let p be the projection $p\colon X \times I \to X, (x,i) \mapsto x$. Also set $i = k \circ i_1$, with k as below. Thus we have

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle p}\uparrow \;\downarrow{\scriptstyle i_1}\;\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle j} \\ X \times I & \xrightarrow[\quad k \quad]{} & M_f \end{array}$$

The map from $X \times I$ to Y taking (x,t) to $f(x)$, and the identity map on Y, together induce a retraction $r\colon M_f \to Y$.

The next proposition is [Geo08, Proposition 4.1.2].

Proposition 9.2. *The map r is a homotopy inverse for j, so r is a homotopy equivalence. Indeed there is a strong deformation retraction $D\colon M_f \times I \to M_f$ of M_f onto Y such that $D_1 = r$.*

The following result is the M-equivariant analogue of [Geo08, Proposition 4.1.2]. Recall that if X is a projective (resp. free) M-CW complex then $Y = M \times I$ is a projective (resp. free) M-CW complex, where I is given the trivial action.

Lemma 9.3. *Let $f\colon X \to Y$ be a continuous M-equivariant cellular map between free (projective) M-CW complexes X and Y. Let M_f be the pushout of*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle j} \\ X \times I & \xrightarrow[\quad k \quad]{} & M_f \end{array}$$

where $i_0\colon X \to X \times I, x \mapsto (x,0)$. Then:

- *(i) M_f is a free (projective) M-CW complex; and*
- *(ii) there is an M-equivariant strong deformation retraction $r\colon M_f \to Y$. In particular M_f and Y are M-homotopy equivalent.*

Proof. It follows from Lemma 2.2 that M_f has the structure of a free (projective) M-CW complex, proving part (i). For part (ii) first note that the map from $X \times I$ to Y taking (x,t) to $f(x)$, and the identity map on Y, together induce a retraction $r\colon M_f \to Y$. It follows from Proposition 9.2 that r is a homotopy equivalence with homotopy inverse j. By Corollary 2.7 to show that r is an M-homotopy equivalence it suffices to verify that r is an M-equivariant map between the sets M_f and Y. But M-equivariance of r follows from the definitions of r and M_f, the definition of the action of M on M_f which in turn is determined by the actions of M on $X \times I$ and on Y, together with the assumption that $f\colon X \to Y$ is M-equivariant. $\square$

The fundamental idea of collapsing schemes, and discrete Morse theory, is that of a collapse. The following definition may be found in [Coh73, page 14] and [FS05, Section 2]. We use the same notation as in [Coh73]. In particular $\approx$ denotes homeomorphism of spaces.

Definition 9.4 (Elementary collapse)**.** If (K,L) is a CW pair then K collapses to L by an elementary collapse, denoted $K \searrow^{e} L$, if and only if:

1. (1) $K = L \cup e^{n-1} \cup e^n$ where e^n and e^{n-1} are open cells of dimension n and $n-1$ respectively, which are not in L, and
2. (2) there exists a ball pair $(B^n, B^{n-1}) \approx (I^n, I^{n-1})$ and a map $\varphi\colon B^n \to K$ such that
   - (a) φ is a characteristic map for e^n

(b) $\varphi|_{B^{n-1}}$ is a characteristic map for e^{n-1}
(c) $\varphi(P^{n-1}) \subseteq L^{n-1}$ where $P^{n-1} = cl(\partial B^n - B^{n-1})$.

Note that in this statement P^{n-1} is an $(n-1)$-ball (i.e. is homeomorphic to I^{n-1}). We say that K collapses to L, writing $K \searrow L$, if L may be obtained from K by a sequence of elementary collapses. We also say that K is an *elementary expansion* of L. An elementary collapse gives a way of modifying a CW complex K, by removing the pair $\{e^{n-1}, e^n\}$, without changing the homotopy type of the space. We can write down a homotopy which describes such an elementary collapse $K \searrow^e L$ as follows. Let (K, L) be a CW pair such that $K \searrow^e L$. Set $\varphi_0 = \varphi|_{p^{n-1}}$ in the above definition. Then

$$\varphi_0 \colon (P^{n-1}, \partial P^{n-1}) \to (L^{n-1}, L^{n-2}),$$

(using the identification $P^{n-1} \approx I^{n-1}$) and

$$(K, L) \approx (L \coprod_{\varphi_0} B^n, L).$$

The following is [Coh73, (4.1)].

Lemma 9.5. *If $K \searrow^e L$ then there is a cellular strong deformation retraction $D \colon K \to L$.*

Indeed, let $K = L \cup e^{n-1} \cup e^n$. There is a map $\varphi_0 \colon I^{n-1} \approx P^{n-1} \to L^{n-1}$ such that

$$(K, L) \approx (L \coprod_{\varphi_0} B^n, L).$$

But $L \coprod_{\varphi_0} B^n$ is the mapping cylinder M_{φ_0} of $\varphi_0 \colon I^{n-1} \to L^{n-1}$. Thus by Proposition 9.2 there is a strong deformation retraction

$$D \colon K \approx L \coprod_{\varphi_0} I^n \to L$$

such that $D(\overline{e^n}) = \varphi_0(I^{n-1}) \subset L^{n-1}$. The map D is given by the map r in Definition 9.1.

We may now state and prove the main result of this section.

Theorem 9.6. *Let K be a rigid free left M-simplicial set with M-equivariant collapsing scheme (E, R, C, c, i) (that is, the conditions (A1)–(A6) are satisfied). Then, with the above notation, there is a free left M-CW complex Y with $Y \simeq_M |K|$ and such that the cells of Y are in bijective correspondence with E, and under this bijective correspondence Y_n is a free left M-set with basis E_n^B for all n.*

Proof. Let X be the geometric realisation $|K|$ of the simplicial set K. By axiom (A1) we have that K is a left M-simplical set, and it follows that $X = |K|$ has the structure of a left M-CW complex where the M-action is given by Lemma 4.4. In fact, by assumptions (A2)-(A6), X is a free M-CW complex where, for each n, the set $E_n^B \cup R_n^B \cup C_n^B$ is a basis for the n-cells.

Write X as an increasing sequence of subcomplexes

$$X_0 \subseteq X_0^+ \subseteq X_1 \subseteq X_1^+ \subseteq \ldots$$

where, X_0 consists of the essential vertices, X_n^+ is obtained from X_n by adjoining the redundant n-cells and collapsible $(n+1)$-cells, and X_{n+1} is obtained from X_n^+ by adjoining the essential $(n+1)$-cells. We write X_n^+ as a countable union

$$X_n = X_n^0 \subseteq X_n^1 \subseteq X_n^2 \subseteq \ldots$$

with $X_n^+ = \coprod_{i \geq 0} X_n^i$ where X_n^{j+1} is constructed from X_n^j by adjoining $(\tau, c(\tau))$ for every redundant n-cell τ of height j. From assumptions (A1)-(A6), for every n and j, each of X_n^+, X_n and X_n^j is a free M-CW subcomplex of X.

As argued in the proof of [Bro92, Proposition 1], for every redundant n-cell τ of height j the adjunction of $(\tau, c(\tau))$ is an elementary expansion. In this way X_n^{j+1} can be obtained from X_n^j by a countable sequence of simultaneous elementary expansions. The same idea, together with Lemma 9.3, can be used to obtain an M-homotopy equivalence between X_n^j and X_n^{j+1}. The details are as follows.

Recall that X_n^{j+1} is obtained from X_n^j by adjoining $(\tau, c(\tau))$ for every redundant n-cell τ of height j. It follows from the axioms (A1)-(A6) that this set of pairs $(\tau, c(\tau))$ is a free M-set with basis $\{(\tau, c(\tau)) \in R_n^B \times C_{n+1}^B : \text{height}(\tau) = j\}$. Let $(\tau, c(\tau)) \in R_n^B \times C_{n+1}^B$ with $\text{height}(\tau) = j$, and let $m \in M$. From the assumptions (A1)-(A6) it follows that

$$m \cdot (\tau, c(\tau)) = (m\tau, c(m\tau)), \quad \text{and} \quad \text{height}(m\tau) = \text{height}(\tau) = j.$$

The pair

$$(X_n^{j+1}, X_n^{j+1} - \{m\tau, c(m\tau)\})$$

satisfies the conditions of an elementary collapse. Indeed (as argued in the proof of [Bro92, Proposition 1]) every face of $c(m\tau)$ other than $m\tau$ is either (i) a redundant cell of height less than j, (ii) is essential (so has height 0), or (iii) is collapsible or degenerate. It follows that the adjunction of $m\tau$ and $c(m\tau)$ is an elementary expansion. This is true for every pair $(m\tau, c(m\tau))$ where $(\tau, c(\tau)) \in R_n^B \times C_{n+1}^B$ with $\text{height}(\tau) = j$ and $m \in M$. Now X_n^{j+1} is obtained from X_n^j by adjoining all such pairs $(m\tau, c(m\tau))$. Let $A = \{(\tau, c(\tau)) \in R_n^B \times C_{n+1}^B : \text{height}(\tau) = j\}$ and let $M \times A$ denote the free left M-set with basis $\{(1, a) : a \in A\}$ and action $m(n, a) = (mn, a)$. Then $(M \times A) \times I^{n+1}$ is a disjoint union of the free M-cells $(M \times \{a\}) \times I^{n+1}$ with $a \in A$. The characteristic maps for the collapsible $n+1$ cells of height j combine to give an M-equivariant map $\varphi\colon (M \times A) \times I^{n+1} \to X_n^{j+1}$ such that

- (E1) φ restricted to the $(m, a) \times I^{n+1}$, $(m \in M, a \in A)$, gives characteristic maps for each of the collapsible $n+1$ cells of height j;
- (E2) φ restricted to the $(m, a) \times I^n$, $(m \in M, a \in A)$, gives characteristic maps for each $\tau \in R_n$ such that $c(\tau)$ is a collapsible $n+1$ cell of height j;
- (E3) $\varphi((M \times A) \times P^n) \subseteq (X_n^{j+1})^{\leq n}$ (where $(X_n^{j+1})^{\leq n}$ is the subcomplex of X_n^{j+1} of cells of dimension $\leq n$) where $P^n = cl(\partial I^{n+1} - I^n)$.

Set $\varphi_0 = \varphi|_{(M \times A) \times P^n}$. Then

$$\varphi_0\colon (M \times A) \times P^n \to (X_n^{j+1})^{\leq n}$$

is a continuous M-equivariant cellular map between free M-CW complexes $(M \times A) \times P^n$ and $(X_n^{j+1})^{\leq n}$. It follows that X_n^{j+1} is M-equivariantly isomorphic to

$$(X_n^{j+1} - \{(\tau, c(\tau)) \in R_n \times C_{n+1} \colon \text{height}(\tau) = j\}) \coprod_{\varphi_0} ((M \times A) \times I^{n+1}). \tag{9.1}$$

But since $P^n = cl(\partial I^{n+1} - I^n) \approx I^n$ we conclude that (9.1) is just the mapping cylinder of φ_0. Thus we can apply Lemma 9.3 to obtain a strong deformation retraction $r_j\colon X_n^{j+1} \to X_n^j$ which is also an M-homotopy equivalence. It follows that there is a retraction $r_n\colon X_n^+ \to X_n$ which is an M-equivariant homotopy equivalence.

We build the space Y such that $|K| \simeq_M Y$ inductively. First set $Y_0 = X_0$. Now suppose that we have an M-homotopy equivalence $\pi^{n-1}\colon X_{n-1}^+ \to Y_{n-1}$ is given. Define Y_n to be the the M-CW complex $Y_{n-1} \cup (M \times E_n^B)$ where $(M \times E_n^B)$ is a collection of free M-cells indexed by E_n^B. These free M-cells are attached to Y_n by composing with π^{n-1} the attaching maps for the essential n-cells of X. This makes sense because X_{n-1}^+ contains the $(n-1)$-skeleton of X. Extend π^{n-1} to an M-homotopy equivalence $\widehat{\pi^{n-1}}\colon X_n \to Y_n$ in the obvious way. This is possible since X_n is obtained from X_{n-1}^+ by adjoining the essential n-cells. Composing r_n with $\widehat{\pi^{n-1}}$ then gives an M-homotopy equivalence $X_n^+ \to Y_n$. Passing to the union gives the M-homotopy equivalence $X \simeq_M Y$ stated in the theorem. □

There is an obvious dual result for right simplicial sets which follows from the above result by replacing M by M^{op}. We also have the two-sided version.

Theorem 9.7. *Let K be a rigid free bi-M-simplicial set with M-equivariant collapsing scheme (E, R, C, c, i). Then, with the above notation, there is a free bi-M-CW complex Y with $Y \simeq_M |K|$ and such that the cells of Y are in bijective correspondence with E, and under this bijective correspondence Y_n is a free bi-M-set with basis E_n^B for all n.*

Proof. This follows by applying Theorem 9.6 to the rigid free left $M \times M^{op}$-simplicial set K. □

10. Guarded collapsing schemes

In this section we introduce the idea of a *left guarded collapsing scheme.* We shall prove that whenever BM admits a left guarded collapsing scheme then $\overrightarrow{EM}$ will admit an M-equivariant collapsing scheme whose M-orbits of cells are in bijective correspondence with the essential cells of the given collapsing scheme for BM. Applying Theorem 9.6 it will then follow that when BM admits a left guarded collapsing scheme of finite type then the monoid M is of type left-F_∞. Analogous results will hold for right guarded and right-F_∞, and (two-sided) guarded and bi-F_∞. In later sections we shall give some examples of monoids which admit guarded collapsing schemes of finite type, including monoids with finite complete presentations (rewriting systems), and factorable monoids in the sense of [HO14].

Definition 10.1 (Guarded collapsing schemes)**.** Let $K = \bigcup_{i \geq 0} K_i$ be a simplicial set and let X be its geometric realisation. We identify the cells of X with the non-degenerate simplices of K, and suppose that K admits a collapsing scheme (E, C, R, c, i). We say that this collapsing scheme is

- *left guarded* if for every redundant n-cell τ we have $i(\tau) \neq 0$;
- *right guarded* if for every redundant n-cell τ we have $i(\tau) \neq n+1$;
- *guarded* if it is both left and right guarded.

In other words, the collapsing scheme is guarded provided the function i never takes either of its two possible extreme allowable values. The aim of this section is to prove the following result.

Theorem 10.2. *Let M be a monoid and suppose that BM admits a collapsing scheme (E, C, R, c, i).*

(a) If (E, C, R, c, i) is left guarded , then there is an M-equivariant collapsing scheme $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ for the free left M-simplicial set $\overrightarrow{EM}$ such that, for each n, the set of essential n-cells $\mathfrak{E}_n$ is a free left M-set of rank $|E_n|$.

(b) If (E, C, R, c, i) is right guarded , then there is an M-equivariant collapsing scheme $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ for the free right M-simplicial set $\overleftarrow{EM}$ such that, for each n, the set of essential n-cells $\mathfrak{E}_n$ is a free right M-set of rank $|E_n|$.

(c) If (E, C, R, c, i) is guarded , then there is an M-equivariant collapsing scheme $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ for the free bi-M-simplicial set $\overleftrightarrow{EM}$ such that, for each n, the set of essential n-cells $\mathfrak{E}_n$ is a free $M \times M^{op}$-set of rank $|E_n|$.

Corollary 10.3. *Let M be a monoid and let $|BM|$ be its classifying space.*

(a) If BM admits a left guarded collapsing scheme of finite type then M is of type left F_∞ *(and therefore also is of type left-*FP_∞*).*

(b) If BM admits a right guarded collapsing scheme of finite type then M is of type right-F_∞ *(and therefore also is of type right-*FP_∞*).*

(c) If BM admits a guarded collapsing scheme of finite type then M is of type bi-F_∞ *(and therefore also is of type bi-*FP_∞*).*

Proof. This follows directly from Theorem 9.6 and its dual, and Theorems 9.7 and 10.2. □

We shall give some examples of monoids to which this corollary applies in the next section.

The rest of this section will be devoted to the proof of Theorem 10.2. Clearly part (b) of the theorem is dual to part (a). The proofs of parts (a) and (c) are very similar, the only difference being that the stronger guarded condiion is needed for (c), while only left guarded is needed for (a). We will begin by giving full details of the proof of Theorem 10.2(a). Then afterwards we will explain the few modifications in the proof necessary to obtain the two-sided proof for (c), in particular highlighting the place where the guarded condition is needed.

10.1. **Proof of Theorem 10.2(a).** Let (E, C, R, c, i) be a left guarded collapsing scheme for BM. We can now 'lift' this collapsing scheme to a collapsing scheme $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ for the simplicial set $\overrightarrow{EM}$ in the following natural way. First observe that

$$\begin{aligned} & m(m_1, m_2, ..., m_n) \text{ is an } n\text{-cell of } EM \\ \Leftrightarrow\ & m_i \neq 1 \text{ for all } 1 \leq i \leq n \\ \Leftrightarrow\ & (m_1, m_2, ..., m_n) \text{ is an } n\text{-cell of } BM. \end{aligned}$$

Define an n-cell $m(m_1, m_2, ..., m_n)$ of $\overrightarrow{EM}$ to be essential (respectively redundant, collapsible respectively) if and only if $(m_1, m_2, ..., m_n)$ is essential (respectively redundant, collapsible respectively) in the collapsing scheme (E, R, C, c, i). This defines the partition $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C})$ of the n-cells of $\overrightarrow{EM}$ for each n. We call these sets the essential, redundant and collapsible cells, respectively, of $\overrightarrow{EM}$. For the mappings κ and ι, given $m\tau \in \overrightarrow{EM}$ where $\tau = (m_1, ..., m_n)$ is a redundant n-cell of BM we define

$$\iota(m\tau) = i(\tau) \quad (10.1)$$

$$\kappa(m\tau) = m(c(\tau)). \quad (10.2)$$

We claim that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is an M-equivariant collapsing scheme for the free left M-simplicial set $\overrightarrow{EM}$ such that, for each n, the set of essential n-cells $\mathfrak{E}_n$ is a free left M-set of rank $|E_n|$. Once proved this will complete the proof of Theorem 10.2(a).

We begin by proving that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is a collapsing scheme for $\overrightarrow{EM}$, and then we will verify that all of the conditions (A1) to (A6) are satisfied.

10.1.1. *Proving that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is a collapsing scheme.*

Proposition 10.4. *With the above definitions, $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is a collapsing scheme for the simplicial set $\overrightarrow{EM}$.*

We have already observed that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C})$ partitions the cells of $\overrightarrow{EM}$. To complete the proof of the proposition we need to verify that the conditions (C1) and (C2) in the definition of collapsing scheme are both satisfied. For part of the proof we will find it useful to recast the ideas in terms of certain bipartite digraphs. The idea of viewing collapsing schemes in terms of matchings in bipartite graphs is a natural one and has been used in the literature; see for example [Cha00]. Let us now introduce the terminology and basic observations about digraphs that we shall need.

A *directed graph* D consists of: a set of edges ED, a set of vertices VD and functions α and β from ED to VD. For $e \in E$ we call $\alpha(e)$ and $\beta(e)$ the initial and terminal vertices of the directed edge e. A *directed path* of length n in D is a sequence of edges $e_1 e_2 \ldots e_n$ such that $\beta(e_i) = \alpha(e_{i+1})$ for each directed edge. Note that edges in paths are allowed to repeat, and vertices can also repeat in the sense that $\beta(e_i) = \beta(e_j)$ is possible for distinct i and j (in graph theory literature what we call a path here is often called a walk). By an *infinite directed path* we mean a path $(e_i)_{i \in \mathbb{N}}$. Note that an infinite directed path need not contain infinitely many distinct edges. For example, if a digraph contains a directed circuit $e_1 e_2 e_3$ then $e_1 e_2 e_3 e_1 e_2 e_3 \ldots$ would be an infinite directed path. A *bipartite digraph* D with bipartition $VD_1 \cup VD_2$ has vertex set $VD = VD_1 \cup VD_2$ where VD_1 and VD_2 are disjoint, such that for every $e \in ED$ we have either $\alpha(e) \in VD_1$ and $\beta(e) \in VD_2$, or $\alpha(e) \in VD_2$ and $\beta(e) \in VD_1$ (i.e., there are no directed edges between vertices in the same part of the bipartition). A *homomorphism* $\varphi\colon D \to D'$ between digraphs is a map $\varphi\colon (VD \cup ED) \to (VD' \cup ED')$ which maps vertices to vertices, edges to edges, and satisfies $\alpha(\varphi(e)) = \varphi(\alpha(e))$ and $\beta(\varphi(e)) = \varphi(\beta(e))$. If $p = e_1 e_2 \ldots e_n$ is a path of length n in D and $\varphi\colon D \to D'$ is a digraph homomorphism then we define $\varphi(p) = \varphi(e_1)\varphi(e_2)\ldots\varphi(e_n)$ which is a path of length n in D'. Note that in general a homomorphism is allowed to map two distinct vertices (resp. edges) of D to the same vertex (resp. edge) of D'. Since digraph homomorphisms map paths to paths, we have the following basic observation.

Lemma 10.5. *Let $\varphi\colon D \to D'$ be a homomorphism of directed graphs. If D has an infinite directed path than D' has an infinite directed path.*

For each $n \in \mathbb{N}$ let $\Gamma^{(n)}(BM)$ be the directed bipartite graph defined as follows. The vertex set $V\Gamma^{(n)}(BM) = \mathcal{C}_n \cup \mathcal{C}_{n+1}$ where $\mathcal{C}_i$ denotes the set of i-cells BM. The directed edges EV of V are of two types:

(DE1) A directed edge $\tau \longrightarrow d_j(\tau)$ (with initial vertex τ and terminal vertex $d_j(\tau)$) for every collapsible $\tau \in \mathcal{C}_{n+1}$ and $j \in \{0, \ldots, n+1\}$ such that $d_j(\tau)$ is a redundant n-cell (i.e., is a redundant non-degenerate n-simplex) and either $c(d_j(\tau)) \neq \tau$ or $c(d_j(\tau)) = \tau$ but $j \neq i(d_j(\tau))$;

(DE2) A directed edge $\sigma \longrightarrow c(\sigma)$ (with initial vertex σ and terminal vertex $c(\sigma)$) for every redundant $\sigma \in \mathcal{C}_n$.

We sometimes refer to these two types of directed arcs as the "down-arcs" and the "up-arcs" (respectively) in the bipartite graph. Note that condition (C2) in the collapsing scheme definition is equivalent to saying that the digraph $D(n, n+1)$ does not contain any infinite directed path, and in particular does not contain any directed cycles. Let $\Gamma^{(n)}(\overrightarrow{EM})$ be the corresponding directed bipartite graph defined in the same way using $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$, with vertex set the n and $n+1$ cells of $\overrightarrow{EM}$ and directed edges determined by the maps κ and ι. To simplify notation, let us fix $n \in \mathbb{N}$ and set $\Gamma(BM) = \Gamma^{(n)}(BM)$ and $\Gamma(EM) = \Gamma^{(n)}(EM)$.

Lemma 10.6. *Let $\pi\colon V\Gamma(\overrightarrow{EM}) \cup E\Gamma(\overrightarrow{EM}) \to V\Gamma(BM) \cup E\Gamma(BM)$ be defined on vertices by:*

$$m(m_1, \ldots, m_k) \mapsto (m_1, \ldots, m_k) \ (k \in \{n, n+1\})$$

and defined on edges (DE1) and (DE2) by $\pi(x \to y) = \pi(x) \to \pi(y)$. Then π is a digraph homomorphism.

Proof. We need to prove, for each directed edge $x \to y$ from $E\Gamma(\overrightarrow{EM})$, that $\pi(x) \to \pi(y)$ is a directed edge in $E\Gamma(BM)$. There are two cases that need to be considered depending on arc type. The two arc types depend on whether the arc is going downwards from level $n+1$ to level n (arc type 1) or upwards from level n to level $n+1$ (arc type 2).

Case: Arc type 1. Let $m(m_1, \ldots, m_{n+1})$ be a collapsible $n+1$ cell in $\overrightarrow{EM}$ and let $j \in \{0, \ldots, n+1\}$ and suppose that

$$m(m_1, \ldots, m_{n+1}) \longrightarrow d_j(m(m_1, \ldots, m_{n+1}))$$

is an arc in $\Gamma(\overrightarrow{EM})$. This means that $d_j(m(m_1, \ldots, m_{n+1}))$ is a redundant n-cell in $\overrightarrow{EM}$ and if $\kappa(d_j(m(m_1, \ldots, m_{n+1}))) = m(m_1, \ldots, m_{n+1})$, then $j \neq \iota(d_j(m(m_1, \ldots, m_{n+1})))$. Note that $j = 0$ or $j = n+1$ is possible here. We claim that

$$\pi(m(m_1, \ldots, m_{n+1})) \longrightarrow \pi(d_j(m(m_1, \ldots, m_{n+1})))$$

in $\Gamma(BM)$. Indeed, we saw above in Section 5 that the projection $\pi\colon \overrightarrow{EM} \to BM$ is a simplicial morphism with

$$\pi(d_j(m(m_1, \ldots, m_{n+1}))) = d_j(m_1, \ldots, m_{n+1}) = d_j(\pi(m(m_1, \ldots, m_{n+1}))).$$

Therefore

$$\begin{aligned} &\pi(m(m_1, \ldots, m_{n+1})) \\ &= (m_1, \ldots, m_{n+1}) \to d_j(m_1, \ldots, m_{n+1}) \\ &= \pi(d_j(m(m_1, \ldots, m_{n+1}))), \end{aligned}$$

in $\Gamma(BM)$, since if $c(d_j(m_1, \ldots, m_{n+1})) = (m_1, \ldots, m_{n+1})$, then

$$\kappa(d_j(m(m_1, \ldots, m_{n+1}))) = \kappa(md_j(m_1, \ldots, m_{n+1})) = mc(d_j(m_1, \ldots, m_{n+1})) = m(m_1, \ldots, m_{n+1})$$

and hence by definition of ι we have

$$j \neq \iota(d_j(m(m_1, \ldots, m_{n+1}))) = i(d_j(m_1, \ldots, m_{n+1})).$$

Case: Arc type 2. These are the arcs that go up from level n to level $n+1$. A typical such arc arises as follows. Let $m\sigma \in \mathcal{C}_n$ be a redundant cell in $\overrightarrow{EM}$ where σ is a redundant cell in BM and $m \in M$. Then $\kappa(m\sigma) \in \mathcal{C}_{n+1}$ is collapsible and

$$d_{\iota(m\sigma)}(\kappa(m\sigma)) = m\sigma.$$

A typical type 2 arc has the form

$$m\sigma \longrightarrow \kappa(m\sigma) = m(c(\sigma)),$$

by definition of κ. Applying π to this arc gives

$$\pi(m\sigma) = \sigma \longrightarrow c(\sigma) = \pi(m(c(\sigma))),$$

which is a type 2 arc in $\Gamma(BM)$ completing the proof of the lemma. □

Proof of Proposition 10.4. We must check that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ satisfies the two collapsing scheme conditions (C1) and (C2).

Verifying collapsing scheme condition (C1). We must prove that the map κ defines a bijection from the redundant n-cells to the collapsible $n+1$-cells.

The map is injective since $\kappa(m\tau) = \kappa(m'\sigma)$ implies $mc(\tau) = m'c(\sigma)$ so $m = m'$ and $\tau = \sigma$ since c is injective by assumption. Also, given an arbitrary collapsible $n+1$ cell $m\sigma$ there exists $\sigma = c(\tau)$ where τ is a redundant n-cell an so $m\sigma = \kappa(m\tau)$.

Moreover, for every redundant n-cell $m\tau$, we have

$$\begin{aligned} d_{\iota(m\tau)}(\kappa(m\tau)) &= d_{i(\tau)}(mc(\tau)) \text{ (by definition)} \\ &= md_{i(\tau)}(c(\tau)) \text{ (since } i(\tau) \neq 0) \\ &= m\tau, \end{aligned}$$

and this concludes the proof that collapsing scheme condition (C1) holds.

Note that it is in the second line of the above sequence of equations that we appeal to our assumption that (E, C, R, c, i) is left guarded (which implies $i(\tau) \neq 0$). In fact, this will be the only place in the proof of Theorem 10.2(a) that the left guarded assumption is used.

Verifying collapsing scheme condition (C2). To see that collapsing scheme condition (C2) holds let $\Gamma(BM)$ and $\Gamma(\overrightarrow{EM})$ be the level $(n, n+1)$ bipartite digraphs of BM and $\overrightarrow{EM}$, respectively, defined above. By Lemma 10.6 the mapping π defines a digraph homomorphism from $\Gamma(\overrightarrow{EM})$ to $\Gamma(BM)$. If $\Gamma(\overrightarrow{EM})$ contained an infinite directed path then by Lemma 10.5 the image of this path would be an infinite directed path in $\Gamma(BM)$ which is impossible since (E, R, C, c, i) is a collapsing scheme. Therefore $\Gamma(\overrightarrow{EM})$ contains no infinite path and thus $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ satisfies condition (C2).

This completes the proof of Proposition 10.4. □

Remark 10.7. It follows from Proposition 10.4 that every down arc in $\Gamma(\overrightarrow{EM})$ (that is, every arc of type (DE1)) is of the form $\tau \to d_j(\tau)$ where $\tau \in \mathfrak{C}_{n+1}$, $d_j(\tau) \in \mathfrak{R}_n$, and $\kappa(d_j(\tau)) \neq \tau$.

10.1.2. *Proving $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is a left M-equivariant collapsing scheme.* To prove that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is a left M-equivariant collapsing scheme for $\overrightarrow{EM}$ we need to verify that conditions (A1)–(A6) hold. In Section 5 we proved that $\overrightarrow{EM}$ is a rigid free left M-simplicial set. In addition to this, from the definitions an n-cell $m(m_1, m_2, ..., m_n)$ of $\overrightarrow{EM}$ essential (redundant, collapsible respectively) if and only if $(m_1, m_2, ..., m_n)$ is essential (redundant, collapsible respectively) in the collapsing scheme (E, R, C, c, i) of BM. These facts prove that (A1) and (A2) both hold:

- (A1) The action of M on $\overrightarrow{EM}$ maps n-simplicies to n-simplicies, and commutes with d_i and s_i, that is, M is acting by simplicial morphisms on $\overrightarrow{EM}$.
- (A2) For every n-simplex σ and $m \in M$, σ is a cell (i.e. is a non-degenerate simplex) if and only if $m\sigma$ is a cell, in which case $\sigma \in \mathfrak{E}$ (respectively $\mathfrak{R}$, $\mathfrak{C}$) if and only if $m\sigma \in \mathfrak{E}$ (respectively $\mathfrak{R}$, $\mathfrak{C}$).

The next axiom we need to verify is:

- (A3) If $(\sigma, \tau) \in \mathfrak{R}_n \times \mathfrak{C}_{n+1}$ is a matched redundant-collapsible pair (i.e. $\tau = c(\sigma)$) then so is the pair $m(\sigma, \tau) = (m\sigma, m\tau) \in \mathfrak{R}_n \times \mathfrak{C}_{n+1}$.

We shall prove a little more than this. We consider the bipartite digraph $\Gamma(\overrightarrow{EM})$ between levels n and $n+1$ defined above. We want to prove that M acts on this digraph, that is, that arcs are sent to arcs under the action. We note that the action of M on the vertices preserves the bipartition. As above, there are two types of directed arcs that need to be considered, the up-arcs and the down-arcs.

First consider the down-arcs. By Remark 10.7, a typical down-arc has the form

$$m\sigma \xrightarrow{d_j} d_j(m\sigma)$$

where $\kappa(d_j(m\sigma)) \neq m\sigma$. Let $n \in M$ be arbitrary. Then we have

$$nm\sigma \xrightarrow{d_j} d_j(nm\sigma) = nd_j(m\sigma),$$

by definition of d_j. This is a down-arc because if $\kappa(d_j(nm\sigma)) = nm\sigma$, then from $nmd_j(\sigma) = d_j(nm\sigma)$ we deduce that $nmc(d_j(\sigma)) = nm\sigma$ and so $c(d_j(\sigma)) = \sigma$, whence $\kappa(d_j(m\sigma)) = \kappa(md_j(\sigma)) = mc(d_j(\sigma)) = m\sigma$, which is a contradiction.

Now consider up-arcs. A typical up-arc has the form $m\sigma \to \kappa(m\sigma)$. Let $n \in M$. Then

$$nm\sigma \to \kappa(nm\sigma) = nm(c(\sigma)) = n\kappa(m\sigma),$$

which is an up-arc as required.

This covers all types of arcs in $\Gamma(\overrightarrow{EM})$ and we conclude that M acts on $\Gamma(\overrightarrow{EM})$ by digraph endomorphisms. This fact together with (A2) then implies property (A3), since the up-arcs in this bipartite graph are precisely the matched redundant-collapsible pairs. Next consider

(A4) There is a subset $\mathfrak{B} \subseteq \mathfrak{E} \cup \mathfrak{R} \cup \mathfrak{C}$ such that for all n the set of n-cells is a free left M-set with basis $\mathfrak{B}_n$ (the n-cells in $\mathfrak{B}$). Let $\mathfrak{E}^{\mathfrak{B}} = \mathfrak{E} \cap \mathfrak{B}$, $\mathfrak{R}^{\mathfrak{B}} = \mathfrak{R} \cap \mathfrak{B}$ and $\mathfrak{C}^{\mathfrak{B}} = \mathfrak{C} \cap \mathfrak{B}$. Then $\mathfrak{E}_n$ is a free left M-set with basis $\mathfrak{E}_n^{\mathfrak{B}}$, and similarly for $\mathfrak{R}_n$ and $\mathfrak{C}_n$.

We saw in Section 5 that the set of n-cells of $\overrightarrow{EM}$ is a free left M-set with basis the set of n-cells

$$\mathfrak{B} = \{(m_1, \ldots, m_n) : \quad m_i \neq 1 \text{ for all } i\}$$

of BM. The last clause of (A4) then follows from (A2). Now we shall prove

(A5) For every matched pair $(\sigma, \tau) \in \mathfrak{R} \times \mathfrak{C}$, $\sigma \in \mathfrak{R}^{\mathfrak{B}}$ if and only if $\tau \in \mathfrak{C}^{\mathfrak{B}}$. In particular, for every matched pair (σ, τ) there is a unique pair $(\sigma', \tau') \in \mathfrak{R}^{\mathfrak{B}} \times \mathfrak{C}^{\mathfrak{B}}$ and $m \in M$ such that $(\sigma, \tau) = m(\sigma', \tau')$.

The matched pairs are the up-arcs in the graph $\Gamma(\overrightarrow{EM})$. A typical up-arc has the form

$$m(m_1, \ldots, m_n) \to \kappa(m(m_1, \ldots, m_n) = mc(m_1, \ldots, m_n).$$

So this pair is

$$m \cdot ((m_1, \ldots, m_n) \to c(m_1, \ldots, m_n))$$

where

$$(m_1, \ldots, m_n) \to c(m_1, \ldots, m_n)$$

is a uniquely determined matched basis pair. Also, if (σ, τ) is a matched pair then $\sigma = (m_1, \ldots, m_n)$ belongs to the basis if and only if $\kappa(m_1, \ldots, m_n) = c(m_1, \ldots, m_n)$ belongs to the basis, completing the proof of (A5). Finally we turn our attention to showing axiom

(A6) For every redundant cell τ and every $m \in M$

$$\text{height}(\tau) = \text{height}(m\tau)$$

where height is taken with respect to the collapsing scheme $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$.

The following lemma will be useful.

Lemma 10.8 (Path lifting property). *Define a mapping* $\pi\colon V\Gamma(\overrightarrow{EM})\cup E\Gamma(\overrightarrow{EM}) \to V\Gamma(BM)\cup E\Gamma(BM)$ *defined on vertices by:*

$$m(m_1,\dots,m_k) \mapsto (m_1,\dots,m_k) \ (k \in \{n, n+1\})$$

and defined on edges (DE1) and (DE2) by $\pi(x \to y) = \pi(x) \to \pi(y)$. *Let* μ *be a redundant* n*-cell in* $\overrightarrow{EM}$. *Then for each path* p *in* $\Gamma(BM)$, *with initial vertex* $\pi(\mu)$ *there is a path* $\hat{p}$ *in* $\Gamma(\overrightarrow{EM})$, *with initial vertex* μ, *such that* $\pi(\hat{p}) = p$.

Proof. We shall establish two claims, from which the lemma will be obvious by induction on path length. First we claim that if $y = m\tau$ is a redundant n-cell of $\overrightarrow{EM}$ (with $m \in M$ and τ and n-cell of BM) and $\sigma \in V\Gamma(BM)$ is a vertex such that there is a directed edge $\tau = \pi(y) \to \sigma$, then there is a vertex $z \in V\Gamma(\overrightarrow{EM})$ such that $y \to z$ is a directed edge in $E\Gamma(\overrightarrow{EM})$, and $\pi(y \to z) = \pi(y) \to \sigma$. Indeed, set $z = \kappa(y)$. Then $y \to z$ is a directed edge in $E\Gamma(\overrightarrow{EM})$ by definition and $\pi(z) = \pi(\kappa(y)) = \pi(\kappa(m\tau)) = \pi(m(c(\tau))) = c(\tau) = \sigma$.

Next we claim that if $x \to y$ is an up-arc of $E\Gamma(\overrightarrow{EM})$ as in (DE2) and σ is a vertex in $V\Gamma(BM)$ such that $\pi(y) \to \sigma$ is a directed edge in $E\Gamma(BM)$, then there exists a vertex $z \in V\Gamma(\overrightarrow{EM})$ such that $y \to z$ is a directed edge in $E\Gamma(\overrightarrow{EM})$, and $\pi(y \to z) = \pi(y) \to \sigma$. Write $x = m\tau$ where $m \in M$ and τ is a redundant n-cell of BM. Then $x \to y$ is equal to $m\tau \longrightarrow \kappa(m\tau) = mc(\tau)$. In $\Gamma(BM)$ we have the path $\pi(x) \to \pi(y) \to \sigma$ which equals $\tau \to c(\tau) \to \sigma$. Therefore $c(\tau) \to \sigma$ is an arc in $\Gamma(BM)$ of type 1. Therefore, by Remark 10.7 applied to the graph $\Gamma(BM)$, it follows that σ is a redundant n-cell with $\sigma = d_j(c(\tau))$ for some $j \in \{0,\dots,n+1\}$ and $\sigma \neq \tau$ (using that c is a bijection). Set $z = d_j(y)$. We need to show that $\pi(z) = \sigma$ and that $y \to z$ is a directed edge in $\Gamma(\overrightarrow{EM})$.

For the first claim, since π is a simplicial morphism we have

$$\pi(z) = \pi(d_j(y)) = d_j(\pi(y)) = d_j(c(\tau)) = \sigma.$$

For the second claim, to verify that $y \to z$ is a directed edge in $\Gamma(\overrightarrow{EM})$ we just need to show that z is a redundant cell and $y \neq \kappa(z)$. From the definitions it follows that under the mapping π any vertex in the preimage of a redundant cell is a redundant cell. Thus, since σ is redundant and $\pi(z) = \sigma$ it follows that z is redundant. Finally, if $y = \kappa(z)$, then $z = x$ because κ is injective. Therefore, $\sigma = \pi(z) = \pi(x) = \tau$, which is a contradiction.

We can now construct the path $\hat{p}$ by induction on the length of p, where the inductive step uses the first claim if the lift of the previous portion ends at a redundant cell and uses the second claim if it ends at a collapsible cell. □

Axiom (A6) is then easily seen to be a consequence of the following lemma.

Lemma 10.9. *Let* τ *be a redundant cell in the collapsing scheme* $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$. *Write* $\tau = m\sigma$ *where* σ *is a redundant cell in* BM. *Let* $\mathrm{height}_{\overrightarrow{EM}}(m\sigma)$ *denote the height of* $m\sigma$ *with respect to the collapsing scheme* $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$, *and let* $\mathrm{height}_{BM}(\sigma)$ *denote the height of* σ *with respect to the collapsing scheme* (E, R, C, c, i). *Then* $\mathrm{height}_{\overrightarrow{EM}}(m\sigma) = \mathrm{height}_{BM}(\sigma)$.

Proof. Let

$$m\sigma = \tau = \tau_0 \succ \tau_1 \succ \cdots \succ \tau_k$$

be a descending chain of redundant n-cells from $\mathfrak{R}$. It follows that there is a directed path

$$p = \tau_0 \to \kappa(\tau_0) \to \cdots \to \kappa(\tau_{k-1}) \to \tau_k$$

in $\Gamma(\overrightarrow{EM})$. Since π is a digraph homomorphism it follows that $\pi(p)$ is a directed path in $\Gamma(BM)$ and hence

$$\sigma = \pi(\tau_0) \succ \pi(\tau_1) \succ \cdots \succ \pi(\tau_k)$$

is a descending chain of redundant n-cells in R. This proves that $\text{height}_{\overrightarrow{EM}}(m\sigma) \leq \text{height}_{BM}(\sigma)$.

For the converse, let

$$\sigma = \sigma_0 \succ \sigma_1 \succ \cdots \succ \sigma_k$$

be a descending chain of redundant n-cells from R. Then there is a directed path

$$q = \sigma_0 \to c(\sigma_0) \to \cdots \to c(\sigma_{k-1}) \to \sigma_k$$

in $\Gamma(BM)$. By Lemma 10.8 we can lift q to a path $\hat{q}$ in $\Gamma(\overrightarrow{EM})$ with initial vertex $m\sigma$ and such that $\pi(\hat{q}) = q$. But then the redundant cells in the path $\hat{q}$ form a decending chain of length k starting at $m\sigma$, proving that $\text{height}_{\overrightarrow{EM}}(m\sigma) \geq \text{height}_{BM}(\sigma)$. □

This completes the proof of Theorem 10.2(a) and its dual Theorem 10.2(b).

10.2. **Proof of Theorem 10.2(c).** We shall explain how the above proof of Theorem 10.2(a) is modified to prove the two-sided analogue Theorem 10.2(c).

Let (E, C, R, c, i) be a guarded collapsing scheme. Define an n-cell $m(m_1, m_2, ..., m_n)u$ of $\overleftrightarrow{EM}$ to be essential (respectively redundant, collapsible) if and only if $(m_1, m_2, ..., m_n)$ is essential (respectively redundant, collapsible) in the collapsing scheme (E, R, C, c, i). This defines $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C})$.

For the mappings κ and ι, given $m\tau u \in \overleftrightarrow{EM}$ where $\tau = (m_1, ..., m_n)$ is an n-cell of BM we define

$$\iota(m\tau u) = i(\tau) \tag{10.3}$$
$$\kappa(m\tau u) = m(c(\tau))u. \tag{10.4}$$

With these definitions we claim that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is an M-equivariant collapsing scheme for the free bi-M-simplicial set $\overleftrightarrow{EM}$ such that for each n the set of essential n-cells $\mathfrak{E}_n$ is a free bi-M-set of rank $|E_n|$.

10.2.1. *Proving that* $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ *is a collapsing scheme.*

Proposition 10.10. *With the above definitions,* $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ *is a collapsing scheme for the simplicial set* $\overleftrightarrow{EM}$*.*

The proof is analogous to the proof of Proposition 10.4. As in the proof of that proposition, we have a digraph homomorphism $\pi\colon V\Gamma(\overleftrightarrow{EM}) \cup E\Gamma(\overleftrightarrow{EM}) \to V\Gamma(BM) \to E\Gamma(BM)$ given by

$$m(m_1, \ldots, m_k)n \to (m_1, \ldots, m_k),$$
$$\pi(x \to y) = \pi(x) \to \pi(y).$$

To verify collapsing scheme condition (C1) it suffices to observe that for every redundant n-cell $m\tau$ we have

$$\begin{aligned} d_{\iota(m\tau n)}(\kappa(m\tau n)) &= d_{i(\tau)}(mc(\tau)n) \text{ (by definition)} \\ &= md_{i(\tau)}(c(\tau))n \text{ (since } i(\tau) \neq 0 \text{ and } i(\tau) \neq n+1) \\ &= m\tau n. \end{aligned}$$

Here the second line appeals to the fact that the original collapsing scheme is guarded. Collapsing scheme condition (C2) holds again by applying Lemma 10.5 and the fact that π is a digraph homomorphism.

10.2.2. *Proving* $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ *is an* M*-equivariant collapsing scheme.* We have already seen in Section 5 that $\overleftrightarrow{EM}$ is a bi-M-simplicial set. We need to show that $(\mathfrak{E}, \mathfrak{R}, \mathfrak{C}, \kappa, \iota)$ is an M-equivariant collapsing scheme for $\overleftrightarrow{EM}$. By definition, for this we need to verify that axioms (A1)-(A6) are satisfied for $\overleftrightarrow{EM}$ as a left $M \times M^{op}$-simplicial set.

In Section 5 we saw that $\overleftrightarrow{EM}$ is a rigid free left $M \times M^{op}$-simplicial set. Together with the definition of the action of $M \times M^{op}$ on $\overleftrightarrow{EM}$ and the definition of $\mathfrak{E}$, $\mathfrak{R}$ and $\mathfrak{C}$, axioms (A1) and (A2) both then follow. Axioms (A3)-(A6) are then proved exactly as above just with $M \times M^{op}$ in place of M in the proof. This then completes the proof of Theorem 10.2(c).

11. Monoids admitting guarded collapsing schemes

In this section we give examples of classes of monoids to which the above theory of equivariant collapsing schemes applies. In particular, this will allow us to use the theory developed above to give a topological proof of the fact that monoids which admit finite complete presentations are of type bi-F_∞.

Let M be a monoid defined by a finite presentation $\langle A \mid R \rangle$ with generators A and defining relations $R \subseteq A^* \times A^*$. Thus, M is isomorphic to $A^*/\overset{*}{\leftrightarrow}_R$ where $\overset{*}{\leftrightarrow}_R$ is the smallest congruence on A^* containing R. We view $\langle A \mid R \rangle$ as a string rewriting system, writing $l \to r$ for the pair $(l, r) \in R$. We define a binary relation $\to$ on A^*, called a *single-step reduction*, in the following way:

$$u \to v \;\Leftrightarrow\; u \equiv w_1 l w_2 \text{ and } v \equiv w_1 r w_2$$

for some $(l, r) \in R$ and $w_1, w_2 \in X^*$. A word is called *irreducible* if no single-step reduction rule may be applied to it. The transitive and reflexive closure of $\to_R$ is denoted by $\xrightarrow{*}_R$.

This rewriting system is called *noetherian* if there are no infinite descending chains

$$w_1 \to_R w_2 \to_R w_3 \to_R \cdots \to_R w_n \to_R \cdots .$$

It is called *confluent* if whenever we have $u \xrightarrow{*}_R v$ and $u \xrightarrow{*}_R v'$ there is a word $w \in X^*$ such that $v \xrightarrow{*}_R w$ and $v' \xrightarrow{*}_R w$. If R is simultaneously noetherian and confluent we say that R is *complete*. The presentation $\langle A \mid R \rangle$ is called complete if the rewriting system R is complete.

It is well-known (see for example [BO93]) that if $\langle A \mid R \rangle$ is a finite complete presentation then each $\overset{*}{\leftrightarrow}_R$-class of A^* contains exactly one irreducible element. So the set of irreducible elements give a set of normal forms for the elements of the monoid M. In particular, if a monoid admits a presentation by a finite complete rewriting system then the word problem for the monoid is decidable.

In [Bro92, page 145] a method is given for constructing a collapsing scheme on BM for any monoid M that is given by a finite complete rewriting system. It is easily observed from [Bro92, page 145] that in the given collapsing scheme (E, R, C, c, i) the function i never takes either of its two extreme allowable values, that is, the collapsing scheme for BM given in [Bro92] is guarded in the sense of Definition 10.1. Also, as Brown observes (see [Bro92, page 147]), it follows easily from his definition that there are only finitely many essential cells in each dimension. Thus we have:

Proposition 11.1. *Let* M *be a monoid. If* M *admits a presentation by a finite complete rewriting system then* BM *admits a guarded collapsing scheme of finite type.*

It follows that the theory of M-equivariant collapsing schemes developed in the previous section applies to monoids with complete presentations, giving:

Corollary 11.2. *Let M be a monoid that admits a presentation by a finite complete rewriting system. Then M is of type left-F_∞, right-F_∞ and bi-F_∞.*

Proof. By Proposition 11.1 the simplicial set BM admits a guarded collapsing scheme of finite type. Then the result follows from Corollary 10.3. □

We obtain the following result of Kobayashi as a special case.

Corollary 11.3 ([Kob05])**.** *Let M be a monoid that admits a presentation by a finite complete rewriting system. Then M is of type bi-FP_∞.*

Proof. Follows from Proposition 7.7 and Corollary 11.2. □

More recently the class of, so-called, factorable monoids was introduced in work of Hess and Ozornova [HO14]. Since it is quite technical we shall not give the definition of factorable monoid here, we refer the reader to [HO14] to the definition, and we shall use the same notation as there. In their work they show that a number of interesting classes of monoids admit factorability structures. In some cases (e.g. Garside groups) the rewriting systems associated with factorability structures are finite and complete, and so in these cases the monoids are bi-F_∞. On the other hand, in [HO14, Appendix] they give an example of a factorable monoid where the associated rewriting system is not noetherian and thus not complete (although it is not discussed there whether the monoid admits a presentation by some other finite complete presentation). So, there are some examples where factorability structures may be seen to exist, even when the given presentation is not complete. In [HO14, Section 9] the authors construct a matching on the reduced, inhomogeneous bar complex of a factorable monoid. As they say in their paper, the matching they construct is very similar to the construction used by Brown giving a collapsing scheme for monoids defined by finite complete rewriting systems [Bro92, page 145]. Details of the matching they construct for factorable monoids may be found on pages 27 and 28 of [HO14]. It is immediate from the definition of the mapping μ on page 28 that their construction defines a guarded collapsing scheme for the simplicial set BM where M is any factorable monoid $(M, \mathcal{E}, \eta)$. If the generating set $\mathcal{E}$ for the monoid is finite, then the number of essential cells in each dimension is finite, and so we have a guarded collapsing scheme of finite type, giving:

Corollary 11.4. *Let M be a monoid. If M admits a factorability structure $(M, \mathcal{E}, \eta)$ with finite generating set $\mathcal{E}$ then BM admits a guarded collapsing scheme of finite type. In particular M is of type left-F_∞, right-F_∞ and bi-F_∞.*

References

[AH03] J. M. Alonso & S. M. Hermiller. 'Homological finite derivation type'. *Internat. J. Algebra Comput.*, 13, no. 3 (2003), pp. 341–359. DOI: 10.1142/S0218196703001407.

[Ani86] D. J. Anick. 'On the homology of associative algebras'. *Trans. Amer. Math. Soc.*, 296, no. 2 (1986), pp. 641–659. DOI: 10.2307/2000383.

[AR67] W. W. Adams & M. A. Rieffel. 'Adjoint functors and derived functors with an application to the cohomology of semigroups'. *J. Algebra*, 7 (1967), pp. 25–34. DOI: 10.1016/0021-8693(67)90065-8.

[BB97] M. Bestvina & N. Brady. 'Morse theory and finiteness properties of groups'. *Invent. Math.*, 129, no. 3 (1997), pp. 445–470. DOI: 10.1007/s002220050168.

[BG84] K. S. Brown & R. Geoghegan. 'An infinite-dimensional torsion-free FP_∞ group'. *Invent. Math.*, 77, no. 2 (1984), pp. 367–381. DOI: 10.1007/BF01388451.

[BH01] R. Bieri & J. Harlander. 'On the FP_3-conjecture for metabelian groups'. *J. London Math. Soc. (2)*, 64, no. 3 (2001), pp. 595–610.

[Bie76] R. Bieri. *Homological dimension of discrete groups.* Mathematics Department, Queen Mary College, London, 1976. Queen Mary College Mathematics Notes.

[BO93] R. V. Book & F. Otto. *String-rewriting systems*. Texts and Monographs in Computer Science. Springer-Verlag, New York, 1993.

[BR88] R. Bieri & B. Renz. 'Valuations on free resolutions and higher geometric invariants of groups'. *Comment. Math. Helv.*, 63, no. 3 (1988), pp. 464–497. DOI: 10.1007/BF02566775.

[Bro87] K. S. Brown. 'Finiteness properties of groups'. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, vol. 44, pp. 45–75, 1987. DOI: 10.1016/0022-4049(87)90015-6.

[Bro92] K. S. Brown. 'The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem'. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, vol. 23 of *Math. Sci. Res. Inst. Publ.*, pp. 137–163. Springer, New York, 1992. DOI: 10.1007/978-1-4613-9730-4-6.

[Bro94] K. S. Brown. *Cohomology of groups*, vol. 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[Bro10] K. S. Brown. 'Lectures on the cohomology of groups'. In *Cohomology of groups and algebraic K-theory*, vol. 12 of *Adv. Lect. Math. (ALM)*, pp. 131–166. Int. Press, Somerville, MA, 2010.

[BW07] K.-U. Bux & K. Wortman. 'Finiteness properties of arithmetic groups over function fields'. *Invent. Math.*, 167, no. 2 (2007), pp. 355–378.

[Cha00] M. K. Chari. 'On discrete Morse functions and combinatorial decompositions'. *Discrete Math.*, 217, no. 1-3 (2000), pp. 101–113. Formal power series and algebraic combinatorics (Vienna, 1997). DOI: 10.1016/S0012-365X(99)00258-7.

[Che84] C. C.-a. Cheng. 'Separable semigroup algebras'. *J. Pure Appl. Algebra*, 33, no. 2 (1984), pp. 151–158. DOI: 10.1016/0022-4049(84)90003-3.

[Cho06] F. Chouraqui. 'Rewriting systems in alternating knot groups'. *Internat. J. Algebra Comput.*, 16, no. 4 (2006), pp. 749–769.

[CO98] R. Cremanns & F. Otto. 'FP_∞ is undecidable for finitely presented monoids with word problems decidable in polynomial time'. *Mathematische Schriften Kassel 11/98*, Universitat Kassel, September (1998) .

[Coh73] M. M. Cohen. *A course in simple-homotopy theory*. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 10.

[Coh92] D. E. Cohen. 'A monoid which is right FP_∞ but not left FP_1'. *Bull. London Math. Soc.*, 24, no. 4 (1992), pp. 340–342.

[CS09] J. Cassaigne & P. V. Silva. 'Infinite periodic points of endomorphisms over special confluent rewriting systems'. *Ann. Inst. Fourier*, 59, no. 2 (2009), pp. 769–810.

[DDM09] V. Diekert, A. Duncan, & A. Miasnikov. 'Geodesic rewriting systems and pregroups'. *To appear in Combinatorial and Geometric Group Theory, Dortmund and Carleton Conferences. Series: Trends in Mathematics Bogopolski, O.; Bumagin, I.; Kharlampovich, O.; Ventura, E. (Eds.) 2009.*, arXiv:0906.2223v1, 2009.

[DFZ86] E. Dror Farjoun & A. Zabrodsky. 'Homotopy equivalence between diagrams of spaces'. *J. Pure Appl. Algebra*, 41, no. 2-3 (1986), pp. 169–182. DOI: 10.1016/0022-4049(86)90109-X.

[DL98] J. F. Davis & W. Lück. 'Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory'. *K-Theory*, 15, no. 3 (1998), pp. 201–252. DOI: 10.1023/A:1007784106877.

[EG57] S. Eilenberg & T. Ganea. 'On the Lusternik-Schnirelmann category of abstract groups'. *Ann. of Math. (2)*, 65 (1957), pp. 517–518. DOI: 10.2307/1970062.

[Fie84] Z. Fiedorowicz. 'Classifying spaces of topological monoids and categories'. *Amer. J. Math.*, 106, no. 2 (1984), pp. 301–350. DOI: 10.2307/2374307.

[FMWZ13] M. G. Fluch, M. Marschler, S. Witzel, & M. C. B. Zaremsky. 'The Brin-Thompson groups sV are of type F_∞'. *Pacific J. Math.*, 266, no. 2 (2013), pp. 283–295. DOI: 10.2140/pjm.2013.266.283.

[For95] R. Forman. 'A discrete Morse theory for cell complexes'. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pp. 112–125. Int. Press, Cambridge, MA, 1995.

[For02] R. Forman. 'A user's guide to discrete Morse theory'. *Séminaire Lotharingien de Combinatoire*, B48c (2002), pp. 1–35. URL: `http://www.emis.ams.org/journals/SLC/wpapers/s48forman.pdf`.

[Fre09] R. Freij. 'Equivariant discrete Morse theory'. *Discrete Math.*, 309, no. 12 (2009), pp. 3821–3829. DOI: 10.1016/j.disc.2008.10.029.

[FS05] D. Farley & L. Sabalka. 'Discrete Morse theory and graph braid groups'. *Algebr. Geom. Topol.*, 5 (2005), pp. 1075–1109. DOI: 10.2140/agt.2005.5.1075.

[Geo08] R. Geoghegan. *Topological methods in group theory*, vol. 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008. DOI: 10.1007/978-0-387-74614-2.

[GM12] Y. Guiraud & P. Malbos. 'Higher-dimensional normalisation strategies for acyclicity'. *Adv. Math.*, 231, no. 3-4 (2012), pp. 2294–2351. DOI: 10.1016/j.aim.2012.05.010.

[GP98] V. S. Guba & S. J. Pride. 'On the left and right cohomological dimension of monoids'. *Bull. London Math. Soc.*, 30, no. 4 (1998), pp. 391–396. DOI: 10.1112/S0024609398004676.

[GS06] V. S. Guba & M. V. Sapir. 'Diagram groups and directed 2-complexes: homotopy and homology'. *J. Pure Appl. Algebra*, 205, no. 1 (2006), pp. 1–47. DOI: 10.1016/j.jpaa.2005.06.012.

[GS08] O. Goodman & M. Shapiro. 'On a generalization of Dehn's algorithm'. *Internat. J. Algebra Comput.*, 18, no. 7 (2008), pp. 1137–1177.

[GS18] R. D. Gray & B. Steinberg. 'Topological finiteness properties of monoids. Part 2: special monoids, one-relator monoids, amalgamated free products, and HNN extensions'. *arXiv:1805.03413*, 2018.

[GS19] R. D. Gray & B. Steinberg. 'A Lyndon's identity theorem for one-relator monoids'. *arXiv:1910.09914*, 2019.

[GS20] R. D. Gray & B. Steinberg. 'Free inverse monoids are not FP_2'. *C. R. Math. Acad. Sci. Paris*, (to appear), *arXiv:2002.07690*, 2020.

[GZ67] P. Gabriel & M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[HO14] A. Hess & V. Ozornova. 'Factorability, string rewriting and discrete morse theory'. *arXiv:1412.3025*, 2014.

[Hoc45] G. Hochschild. 'On the cohomology groups of an associative algebra'. *Ann. of Math. (2)*, 46 (1945), pp. 58–67. DOI: 10.2307/1969145.

[How63] J. M. Howie. 'Embedding theorems for semigroups'. *Quart. J. Math. Oxford Ser. (2)*, 14 (1963), pp. 254–258. DOI: 10.1093/qmath/14.1.254.

[How95] J. M. Howie. *Fundamentals of semigroup theory*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1995. L.M.S. Monographs, No. 7.

[HS99] S. Hermiller & M. Shapiro. 'Rewriting systems and geometric three-manifolds'. *Geom. Dedicata*, 76, no. 2 (1999), pp. 211–228.

[Hur89] C. M. Hurwitz. 'On the homotopy theory of monoids'. *J. Austral. Math. Soc. Ser. A*, 47, no. 2 (1989), pp. 171–185.

[KM05] G. Kalai & R. Meshulam. 'A topological colorful Helly theorem'. *Adv. Math.*, 191, no. 2 (2005), pp. 305–311. DOI: 10.1016/j.aim.2004.03.009.

[Kna72] U. Knauer. 'Projectivity of acts and Morita equivalence of monoids'. *Semigroup Forum*, 3, no. 4 (1971/1972), pp. 359–370.

[KO01] Y. Kobayashi & F. Otto. 'On homotopical and homological finiteness conditions for finitely presented monoids'. *Internat. J. Algebra Comput.*, 11, no. 3 (2001), pp. 391–403. DOI: 10.1142/S0218196701000577.

[Kob00] Y. Kobayashi. 'Every one-relation monoid has finite derivation type'. In *Proceedings of the Third Symposium on Algebra, Languages and Computation (Osaka, 1999)*, pp. 16–20. Shimane Univ., Matsue, 2000.

[Kob05] Y. Kobayashi. 'Gröbner bases of associative algebras and the Hochschild cohomology'. *Trans. Amer. Math. Soc.*, 357, no. 3 (2005), pp. 1095–1124 (electronic). DOI: 10.1090/S0002-9947-04-03556-1.

[Kob07] Y. Kobayashi. 'The homological finiteness property FP_1 and finite generation of monoids'. *Internat. J. Algebra Comput.*, 17, no. 3 (2007), pp. 593–605. DOI: 10.1142/S0218196707003743.

[Kob10] Y. Kobayashi. 'The homological finiteness properties left-, right-, and bi-FP_n of monoids'. *Comm. Algebra*, 38, no. 11 (2010), pp. 3975–3986. DOI: 10.1080/00927872.2010.507562.

[KT76] D. M. Kan & W. P. Thurston. 'Every connected space has the homology of a $K(\pi, 1)$'. *Topology*, 15, no. 3 (1976), pp. 253–258.

[Lam99] T. Y. Lam. *Lectures on modules and rings*, vol. 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. DOI: 10.1007/978-1-4612-0525-8.

[LN01] I. J. Leary & B. E. A. Nucinkis. 'Every CW-complex is a classifying space for proper bundles'. *Topology*, 40, no. 3 (2001), pp. 539–550. DOI: 10.1016/S0040-9383(99)00073-7.

[LS01] R. C. Lyndon & P. E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition. DOI: 10.1007/978-3-642-61896-3.

[Lyn50] R. C. Lyndon. 'Cohomology theory of groups with a single defining relation'. *Ann. of Math. (2)*, 52 (1950), pp. 650–665. URL: `https://doi.org/10.2307/1969440`.

[May67] J. P. May. *Simplicial objects in algebraic topology*. Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[May72] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.

[May75] J. P. May. 'Classifying spaces and fibrations'. *Mem. Amer. Math. Soc.*, 1, no. 1, 155 (1975), pp. xiii+98.

[May96] J. P. May. *Equivariant homotopy and cohomology theory*, vol. 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. DOI: 10.1090/cbms/091.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[McD79] D. McDuff. 'On the classifying spaces of discrete monoids'. *Topology*, 18, no. 4 (1979), pp. 313–320. DOI: 10.1016/0040-9383(79)90022-3.

[Mil57] J. Milnor. 'The geometric realization of a semi-simplicial complex'. *Ann. of Math. (2)*, 65 (1957), pp. 357–362.

[Mit72] B. Mitchell. 'Rings with several objects'. *Advances in Math.*, 8 (1972), pp. 1–161. DOI: 10.1016/0001-8708(72)90002-3.

[MS76] D. McDuff & G. Segal. 'Homology fibrations and the "group-completion" theorem'. *Invent. Math.*, 31, no. 3 (1975/76), pp. 279–284.

[MSS15a] S. Margolis, F. Saliola, & B. Steinberg. 'Cell complexes, poset topology and the representation theory of algebras arising in algebraic combinatorics and discrete geometry'. *ArXiv e-prints*, 2015. arXiv: 1508.05446.

[MSS15b] S. Margolis, F. Saliola, & B. Steinberg. 'Combinatorial topology and the global dimension of algebras arising in combinatorics'. *J. Eur. Math. Soc. (JEMS)*, 17, no. 12 (2015), pp. 3037–3080. DOI: 10.4171/JEMS/579.

[Nic69] W. R. Nico. 'On the cohomology of finite semigroups'. *J. Algebra*, 11 (1969), pp. 598–612. DOI: 10.1016/0021-8693(69)90093-3.

[Nie] P. Nielsen. 'Can the trivial module be stably free for a monoid ring?'. MathOverflow. URL:https://mathoverflow.net/q/248371 (version: 2016-08-26). URL: `https://mathoverflow.net/q/248371`.

[Nov98] B. V. Novikov. 'Semigroups of cohomological dimension one'. *J. Algebra*, 204, no. 2 (1998), pp. 386–393. DOI: 10.1006/jabr.1997.7363.

[Nun95] M. Nunes. 'Cohomological results in monoid and category theory via classifying spaces'. *J. Pure Appl. Algebra*, 101, no. 2 (1995), pp. 213–244. DOI: 10.1016/0022-4049(94)00056-O.

[Pas08] E. Pasku. 'On some homotopical and homological properties of monoid presentations'. *Semigroup Forum*, 76, no. 3 (2008), pp. 427–468. DOI: 10.1007/s00233-007-9037-1.

[PO04] S. J. Pride & F. Otto. 'For rewriting systems the topological finiteness conditions FDT and FHT are not equivalent'. *J. London Math. Soc. (2)*, 69, no. 2 (2004), pp. 363–382. DOI: 10.1112/S0024610703004903.

[PO05] S. J. Pride & F. Otto. 'On higher order homological finiteness of rewriting systems'. *J. Pure Appl. Algebra*, 200, no. 1-2 (2005), pp. 149–161.

[Pri06] S. J. Pride. 'Homological finiteness conditions for groups, monoids, and algebras'. *Comm. Algebra*, 34, no. 10 (2006), pp. 3525–3536. DOI: 10.1080/00927870600796110.

[RT89] J. Rhodes & B. Tilson. 'The kernel of monoid morphisms'. *J. Pure Appl. Algebra*, 62, no. 3 (1989), pp. 227–268. DOI: 10.1016/0022-4049(89)90137-0.

[Ser71] J.-P. Serre. 'Cohomologie des groupes discrets'. (1971), pp. 77–169. Ann. of Math. Studies, No. 70.

[SOK94] C. C. Squier, F. Otto, & Y. Kobayashi. 'A finiteness condition for rewriting systems'. *Theoret. Comput. Sci.*, 131, no. 2 (1994), pp. 271–294. DOI: 10.1016/0304-3975(94)90175-9.

[Squ87] C. C. Squier. 'Word problems and a homological finiteness condition for monoids'. *J. Pure Appl. Algebra*, 49, no. 1-2 (1987), pp. 201–217. DOI: 10.1016/0022-4049(87)90129-0.

[Sri96] V. Srinivas. *Algebraic K-theory*, vol. 90 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996. DOI: 10.1007/978-0-8176-4739-1.

[Sta68] J. R. Stallings. 'On torsion-free groups with infinitely many ends'. *Ann. of Math. (2)*, 88 (1968), pp. 312–334. DOI: 10.2307/1970577.

[Swa69] R. G. Swan. 'Groups of cohomological dimension one'. *J. Algebra*, 12 (1969), pp. 585–610. DOI: 10.1016/0021-8693(69)90030-1.

[Wal65] C. T. C. Wall. 'Finiteness conditions for CW-complexes'. *Ann. of Math. (2)*, 81 (1965), pp. 56–69.

[Wei94] C. A. Weibel. *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. DOI: 10.1017/CBO9781139644136.

[Wei13] C. A. Weibel. *The K-book*, vol. 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

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