# FREE INVERSE MONOIDS ARE NOT $\mathbf{FP_2}$

ROBERT D. GRAY [1] and BENJAMIN STEINBERG[2]

Abstract. We give a topological proof that a free inverse monoid on one or more generators is neither of type left-$\mathrm{FP}_2$ nor right-$\mathrm{FP}_2$. This strengthens a classical result of Schein that such monoids are not finitely presented as monoids.

Given how easy it is to prove that a group $G$ is finitely presented as a group if and only if it is finitely presented as a monoid, it is rather surprising that the same result does not hold for inverse monoids. Indeed it is a classical result of Schein [13] that free inverse monoids on a non-empty set of generators are not finitely presented as monoids.

Our goal in this paper is to prove the following stronger result about free inverse monoids.

**Theorem 1.** *A free inverse monoid on one or more generators is neither of type left-*$\mathrm{FP}_2$ *nor right-*$\mathrm{FP}_2$*.*

The free inverse monoid is an object of central importance in inverse semigroup theory. Recall that an *inverse monoid* is a monoid $S$ with the property that for every element $s \in S$ there is a unique $t \in S$ such that $sts = s$ and $tst = t$. The element $t$ is called the *inverse* of $s$ and is usually denoted $t = s^{-1}$. Since every group clearly satisfies this property, inverse monoids form a class of structures that lies between groups and arbitrary monoids. As explained in [7], inverse monoids arise naturally in mathematics when studying partial symmetries of structures. Inverse monoids form a variety of algebras, in the sense of universal algebra, and as a consequence it follows that free inverse monoids exist; see [4, Exercise 1.1.20]. Free inverse monoids were studied in detail in classical work of Munn [8] and Scheiblich [12]. As we will explain in more detail below, it follows from that work that the word problem is decidable for free inverse monoids. For a general introduction to the theory of inverse monoids, including proofs of the basic facts about inverse monoids mentioned above, we refer the reader to [5, Chapter 5] and [7].

Recall that a monoid $M$ is said to be of type *left*-$\mathrm{FP}_n$ if there is a projective resolution $P = (P_i)_{i\geq 0}$ of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ such that $P_i$ is finitely generated for $i \leq n$. There is a dual notion of *right*-$\mathrm{FP}_n$, and we say a monoid is of type $\mathrm{FP}_n$ if it is both of type left- and right-$\mathrm{FP}_n$. It is well known (see e.g. [9]) that every finitely presented monoid is of type left- and right-$\mathrm{FP}_2$. Hence an immediate corollary of Theorem 1 is Schein's theorem [13] that free inverse monoids on a non-empty set of generators are not finitely presented.

**Corollary 2.** *Free inverse monoids on one or more generators are not finitely presented.*

Since inverse monoids are isomorphic to their duals, it suffices to show that $M$ is not of type left-$\mathrm{FP}_2$, which henceforth shall be called simply $\mathrm{FP}_2$. Pride [11] showed that the class of monoids of type $\mathrm{FP}_2$ is closed under taking retracts. Since the free monogenic inverse monoid

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[1]School of Mathematics, University of East Anglia, Norwich NR4 7TJ, England. Email Robert.D.Gray@uea.ac.uk.

[2] Department of Mathematics, City College of New York, Convent Avenue at 138th Street, New York, New York 10031, USA. Email bsteinberg@ccny.cuny.edu.

$M$ is a retract of any free inverse monoid on a non-empty set of generators, it suffices to prove that $M$ is not of type $\mathrm{FP}_2$. Here, an inverse monoid is called *monogenic* if it is generated by a single element.

**Theorem 3.** *The free monogenic inverse monoid is not of type* $\mathrm{FP}_2$.

Before proving this result we briefly review some facts about free inverse monoids and the representation of their elements via Munn trees. For a full account of this theory we refer the reader to [7, Chapter 6]. Let $X$ be a non-empty set and let $X^{-1}$ be a set disjoint from $X$ and in bijective correspondence with $X$ via $x \mapsto x^{-1}$. The *free inverse monoid* $\mathrm{FIM}(X)$ is defined to be $Y^*/\rho$ where $Y = X \cup X^{-1}$ and $\rho$ is the congruence generated by the set

$$\{(ww^{-1}w, w) : w \in Y^*\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in Y^*\}.$$

Recall that a congruence $\eta$ on a monoid $S$ is an equivalence relation on $S$ that is compatible with multiplication in the sense that $(s,t), (s',t') \in \eta$ implies $(ss', tt') \in \eta$ for all $s, s', t, t' \in S$. Also, for any relation $\sigma$ on $S$, the congruence generated by $\sigma$ is the intersection of all congruences on $S$ containing $\sigma$, that is, it is the smallest congruence on $S$ containing $\sigma$.

For each word $u \in Y^*$ we associate a tree $\mathrm{MT}(u)$, called the *Munn tree*, of $u$ where $u$ is obtained by tracing the word $u$ in the Cayley graph $\Gamma(\mathrm{FG}(X))$ of the free group $\mathrm{FG}(X)$ with respect to the generating set $X$. So $\mathrm{MT}(u)$ is a finite birooted subtree of $\Gamma(\mathrm{FG}(X))$ with initial vertex (also called in-vertex) 1 and terminal vertex (also called out-vertex) the reduced form $\mathrm{red}(u)$ of the word $u$ in the free group. Here we use 1 to denote the empty word which is the identity element of $\mathrm{FG}(x)$. Munn's solution to the word problem in $\mathrm{FIM}(X)$ says that $u = v$ in $\mathrm{FIM}(X)$ if and only if $\mathrm{MT}(u) = \mathrm{MT}(v)$ as birooted trees. For a detailed explanation of free inverse monoids and the theory of Munn trees we refer the reader to [4, Chapter 2] and also [10, Chapter VIII, Section 3]. In this paper we will only be concerned with the special case of the free monogenic inverse monoid, that is, the inverse monoid $\mathrm{FIM}(X)$ with $|X| = 1$. This monoid is considered in detail in [10, Chapter IX, Section 1] where several different constructions of this monoid are exhibited. For the convenience of the reader, we will give full details below of the theory of Munn trees, and how it can be used to solve the word problem, in the particular case of the free monogenic inverse monoid.

So, let us now turn our attention to the special case of the free monogenic inverse monoid and the proof of Theorem 3. For the remainder of this article, let $M$ denote the free monogenic inverse monoid. Let $x$ be the free generator of $M$ and to simplify notation let $y$ denote its inverse $y = x^{-1}$. Given two words $w_1, w_2 \in \{x,y\}^*$ we shall write $w_1 \equiv w_2$ to denote that $w_1$ and $w_2$ are equal as words in the free monoid $\{x,y\}^*$.

Following [10, Chapter VIII] we shall now explain how to determine when two words $w_1, w_2 \in \{x,y\}^*$ are equal in the free monogenic inverse monoid $M$. In the usual way we identify the elements of the free group $\mathrm{FG}(x)$ with the freely reduced words over $\{x, x^{-1}\}$ so $\mathrm{FG}(x) = \{x^i : i \in \mathbb{Z}\}$. For any word $w \in \{x,y\}^*$ we use $\mathrm{red}(w)$ to denote the reduced word obtained by freely reducing the word $w$ in the free group $\mathrm{FG}(x)$. For example $\mathrm{red}(xyxyxx) = \mathrm{red}(xx^{-1}xx^{-1}xx) = x^2$ while $\mathrm{red}(xyyy) = \mathrm{red}(xx^{-1}x^{-1}x^{-1}) = x^{-2}$. Also, for any word $w \in \{x,y\}^*$ we use $\mathrm{pref}(w)$ to denote the set of all prefixes of the word $w$ in $\{x,y\}^*$. Here $u \in \{x,y\}^*$ is a *prefix* of $w \in \{x,y\}^*$ if $w \equiv uv$ for some word $v \in \{x,y\}^*$. Furthermore, for each $w \in \{x,y\}^*$ we define

$$P_w = \{\mathrm{red}(u) : u \text{ is a prefix of } w\} \subseteq \mathrm{FG}(x).$$

For example if $w = xyyxxxy$ then

$$\begin{aligned} P_w &= \{1, \mathrm{red}(x), \mathrm{red}(xy), \mathrm{red}(xyy), \mathrm{red}(xyyx), \mathrm{red}(xyyxx), \mathrm{red}(xyyxxx), \mathrm{red}(xyyxxxy)\} \\ &= \{1, x, 1, x^{-1}, 1, x, x^2, x\} \\ &= \{x^i : -1 \le i \le 2\}. \end{aligned}$$

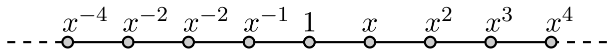
Note that in this example $P_w$ is a prefix closed subset of the free group $\mathrm{FG}(x)$, that is, it is a set of the form $\{x^i : m \le i \le n\}$ for some $m \le 0$ and $n \ge 0$. In fact, it follows from the definition that for any word $w \in \{x, y\}^*$ the set $P_w$ is a prefix closed subset of $\mathrm{FG}(x)$. Also note that by definition $\mathrm{red}(w) \in P_w$ for every $w \in \{x, y\}^*$. Thus using these definitions we see that for each word $w \in \{x, y\}^*$ we can associate a pair $(P_w, \mathrm{red}(w))$ where $P_w$ is a prefix closed subset of $\mathrm{FG}(x)$ and $\mathrm{red}(w) \in P_w$. The following result shows that two words give rise to the same pair if and only if they represent the same element of the free monogenic inverse monoid $M$. We refer the reader to [10, Construction VIII 1.2 and Theorem VIII 1.5] for a proof of this result.

**Lemma 4.** *Let $w_1, w_2 \in \{x, y\}^*$. Then $w_1 = w_2$ in the free monogenic inverse monoid $M$ if and only if $(P_{w_1}, \mathrm{red}(w_1)) = (P_{w_2}, \mathrm{red}(w_2))$.*

In fact, the proof of [10, Theorem VIII 1.5] shows that the map $w \mapsto (P_w, \mathrm{red}(w))$ defines a surjection from $\{x, y\}^*$ to the set of all pairs $(P, t)$ where $P$ is a finite prefix closed subset of $\mathrm{FG}(x)$ and $t \in P$. Hence this map defines a bijection between elements of the free monogenic inverse monoid $M$ and the set of all such pairs $(P, t)$.

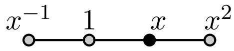
When doing computations in the free monogenic inverse monoid $M$ rather than computing the pairs $(P_{w_1}, w_1)$ and $(P_{w_2}, w_2)$ each time we want to see whether two words $w_1$ and $w_2$ are equal in $M$, it is usually easier to think in terms of Munn trees, as we now explain. For any word $w \in \{x, y\}^*$ the Munn tree $\mathrm{MT}(w)$ of $w$ is defined to be the subgraph of the Cayley graph of $\mathrm{FG}(x)$ (with respect to the generating set $\{x, x^{-1}\}$) induced on the set $P_w$. The Munn tree $\mathrm{MT}(w)$ also comes with two distinguished vertices, the initial vertex (also called in-vertex) which in this formulation of Munn trees we will always set to be the vertex 1, and the terminal vertex (also called out-vertex) which we set to be $\mathrm{red}(w)$. Hence the Munn $\mathrm{MT}(w)$ of a word $w$ is a finite connected induced subgraph of the Cayley graph of the free group $\mathrm{FG}(x)$ containing the vertex 1, with initial vertex 1 and terminal vertex $\mathrm{red}(w)$. In terms of Munn trees, Lemma 4 says that two words $w_1$ and $w_2$ are equal in $M$ if their Munn trees are equal, meaning that their Munn trees have equal vertex sets $P_{w_1} = P_{w_2}$, and have equal terminal vertices $\mathrm{red}(w_1) = \mathrm{red}(w_2)$.

The intuition behind the Munn tree $\mathrm{MT}(w)$ of a word $w \in \{x, y\}^*$ is that we start at the vertex 1 in the Cayley graph of $\mathrm{FG}(x)$ and we follow the walk in this Cayley graph labelled by the word $w$. If we visualise the Cayley graph of $\mathrm{FG}(x)$ drawn in the plane as follows




then to compute $\mathrm{MT}(w)$ where $w \in \{x, y\}^*$ we start at 1, then we read the word $w$ one letter at a time from left to right. When we read an $x$ we take one step to the right in the Cayley graph (corresponding to right multiplication by $x$) and whenever we read a $y$ we take one step left in the Cayley graph (corresponding to right multiplication by $x^{-1} = y$). As we trace out this walk, we keep a record of the set of all the vertices that were visited during the walk, this is the set $P_w$, and we keep a record of the final vertex of the walk, this is the terminal vertex which is equal to $\mathrm{red}(w)$.

For example if $w = xyyxxxy$, which is the word we considered in an earlier example above when we defined $P_w$, then the Munn tree is obtained by starting at 1 in the Cayley graph of $\mathrm{FG}(x)$, taking one step right (reading $x$), then two steps left (reading $yy$), then three steps right (reading $xxx$), followed finally by one last step left (reading the final letter $y$ of the word $w$). Tracing out this walk in the Cayley graph of $\mathrm{FG}(x)$ and recording the terminal vertex $\mathrm{red}(w) = x$ we see that the Munn tree of this word $w = xyyxxxy$ is




where the initial vertex of this Munn tree is 1 and the terminal vertex of this Munn tree is $x$ which is coloured in black. Note that this Munn tree does indeed have vertex set $P_w$ and terminal vertex $\mathrm{red}(w)$. Comparing this with the calculation we made above of $P_w$ for this word $w = xyyxxxy$ one can see that the computation of the set $P_w$ is exactly the same as recording the vertices visited by the walk labelled by the word $w$.

Thinking in terms of Munn trees gives a useful way of checking whether two words are equal in the free monogenic inverse monoid $M$. For example, continuing with the above example, if we want to prove that the equality $w_1 = w_2$ holds in $M$ where $w_1 \equiv xyyxxxy$ and $w_2 \equiv xxyyyxxxy$, then we can either compute the sets $P_{w_1}$ and $P_{w_2}$ and verify that $(P_{w_1}, \mathrm{red}(w_1)) = (P_{w_2}, \mathrm{red}(w_2))$, or equivalently we can compute the Munn tree for the word $w_2$ just as we did for the word $w_1$ above and observe that both Munn trees have the same vertex set, and the same terminal vertex. This is true since if we start at 1 and read the word $w_2$ it says take two steps right in the Cayley graph of $\mathrm{FG}(x)$ (reading $xx$), then three steps left (reading $yyy$), then three steps right (reading $xxx$), then finally take one step left (reading the last letter $y$ of $w_2$). The set of vertices visited by this walk is $\{x^{-1}, 1, x, x^2\}$ and the terminal vertex of the walk is $x$. Hence we obtain the same Munn tree as we computed for the word $w_1$ above. This proves that $xyyxxxy = xxyyyxxxy$ in the free monogenic inverse monoid $M$. Throughout this article we will use this method to prove equalities between words in the free monogenic inverse monoid $M$.

Recall that if $S$ is a monoid and $A \subseteq S$, then the (right) *Cayley digraph* $\Gamma(S, A)$ of $S$ with respect to $A$ is the graph with vertex set $S$ and with edges in bijection with $S \times A$ where the directed edge (arc) corresponding to $(s, a)$ starts at $s$ and ends at $sa$. Let $\Gamma$ be the Cayley digraph of the free monogenic inverse monoid $M$ with respect to the generating set $\{x, y\}$ where $y = x^{-1}$. Then $M$ acts on the left of $\Gamma$ by cellular mappings. The augmented cellular chain complex of $\Gamma$ gives a partial resolution of the trivial module

$$C_1(\Gamma) \xrightarrow{d_1} C_0(\Gamma) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Moreover, since the vertices of $\Gamma$ form a free $M$-set on 1 generator (the vertex 1) and the edges form a free $M$-set on 2 generators (the arrows $1 \xrightarrow{x} x$ and $1 \xrightarrow{y} y$), this is, in fact, a partial free resolution which is finitely generated in each degree. Therefore, if $M$ is of type $\mathrm{FP}_2$, we must have that $\ker d_1 = H_1(\Gamma)$ is finitely generated as a $\mathbb{Z}M$-module (by [1, Proposition VIII.4.3]). So our goal now is to show that $H_1(\Gamma)$ is not finitely generated as a $\mathbb{Z}M$-module. We remark that $H_1(\Gamma)$ is isomorphic as a $\mathbb{Z}M$-module to the relation module of $M$ in the sense of Ivanov [6]; see [3, Section 6].

If $p$ is a path in $\Gamma$, there is a corresponding element $\overline{p}$ of $C_1(\Gamma)$ which is the weighted sum of the edges traversed by $p$, where an edge receives a weight of $n - k$ if it is traversed $n$ times in the forward direction and $k$ times in the reverse direction.

If $T$ is a spanning tree for $\Gamma$ (and we will choose a particular one shortly), then $H_1(\Gamma)$ is a free abelian group with a basis in bijection with the directed edges of $\Gamma \setminus T$. If $v, w$ are vertices, then $[v, w]$ will denote the geodesic in $T$ from $v$ to $w$. The basis element $b_e$ of $H_1(\Gamma)$ corresponding to a directed edge $e$ of $\Gamma \setminus T$ is $\overline{[1, \iota(e)]e[1, \tau(e)]^{-1}}$ where $\iota, \tau$ denote the initial and terminal vertex functions, respectively. If $p$ is a closed path in $\Gamma$, then the homology class of $\overline{p}$ is the weighted sum of the basis elements $b_e$ where the weight of $b_e$ is $n - k$ with $n$ the number of traversals of $e$ by $p$ in the forward direction and $k$ the number of traversals in the reverse direction.

We now use the theory of Munn trees for $M$ described above to identify a prefix-closed set of normal form words for the elements of the free monogenic inverse monoid $M$, which we will use to define our spanning tree. Recall from above that these Munn trees are finite connected prefix closed subgraphs of the Cayley graph of $\mathrm{FG}(x)$. To obtain normal forms for these Munn trees the idea is that, starting at 1, we first sweep to the right in the Munn-tree as far as possible, then to the left as far as possible, and then, if necessary, back to the right. This leads to the set of normal form words in the following lemma where we end up with two families of normal form words which correspond to whether or not the Munn tree contains negative powers of $x$. See Figure 1 for one example of each kind of normal form word. We note that the normal form we give in this lemma closely relates to a normal form for elements of free inverse monoids using a left-right-left sweep defined by Gluskin in [2].


Figure 1. The Munn tree on the left has normal form $x^5y^3$, while the Munn tree on the right has normal form $x^4y^6x^3$. In each example, the in-vertex is the identity 1 of the free group $\mathrm{FG}(x)$ and the out-vertex is coloured in black.

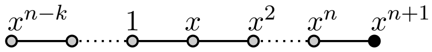



Figure 2. The Munn tree used in the proof of Lemma 6 part (1) to show that $x^ny^kx^{k+1} = x^{n+1}y^{k+1}x^{k+1}$ in $M$ for $k > n \geq 0$. The in-vertex is 1 and the out-vertex is $x^{n+1}$ which is coloured in black.

**Lemma 5.** *The set of elements of the forms $x^ny^k$ with $0 \leq k \leq n$ and $x^ny^kx^j$ with $0 \leq n < k$ and $0 \leq j \leq k$ constitute a prefix-closed set of normal forms for $M$.*

*Proof.* This normal form is essentially the dual of the normal form established in [10, IX.1.5 Proposition]. For the convenience of the reader we provide a proof here. From the results on Munn trees and the free monogenic inverse monoid above we see that there are two kinds of Munn trees: those whose vertex set contains negative powers of $x$, and those whose vertex set does not. We consider each case separately.

First consider a Munn tree which does not contain negative powers of $x$. So this Munn tree has vertex set $\{x^i : 0 \leq i \leq n\}$ for some $n$ and terminal vertex in this set which we can write as $x^{n-k}$ where $0 \leq k \leq n$. We can read this Munn tree starting at 1, then reading right until the largest power $x^n$ in the vertex set, and then reading left and stopping when we reach the terminal vertex $x^{n-k}$. This reading of the Munn tree gives the word $x^ny^k$ with $0 \leq k \leq n$. Clearly distinct choices of $n$ and $k$ give distinct Munn trees.

On the other hand, given a Munn tree which does contain negative powers of $x$ we can read the Munn tree by starting at 1, reading right and stopping at the rightmost vertex $x^n$, then reading left to the leftmost vertex $x^{n-k}$ with $k > n$, and then finally reading right up until the terminal vertex $x^{n-k+j}$ where $0 \leq j \leq k$. So this is the Munn tree with vertex set $\{x^i : n-k \leq i \leq n\}$ and terminal vertex $x^{n-k+j}$. This reading of this Munn tree gives the word $x^ny^kx^j$ with $0 \leq n < k$ and $0 \leq j \leq k$. Clearly, distinct choices of the parameters $n, k$ and $j$ give rise to distinct Munn trees.

This completes the proof that these words constitute a set of normal forms for the free monogenic inverse monoid $M$. Finally, it is immediate from the definition that any prefix of one of these normal form words is again one of these normal form words. □

Since the set of normal forms in Lemma 5 is prefix-closed it defines a spanning tree of the Cayley graph $\Gamma$ of $M$. Let $T$ be the spanning tree of $\Gamma$ corresponding to the set of normal forms in Lemma 5. Hence for any edge $w_1 \xrightarrow{z} w_2$ from $\Gamma$, where $w_1$ and $w_2$ are both normal form words and $z \in \{x, y\}$, this edge belongs to the spanning tree $T$ if and only if $w_2 \equiv w_1z$ in the free monoid $\{x, y\}^*$. Note that $[1, x^ny^k]$ consists of $n$ $x$-edges followed by $k$ $y$-edges for $0 \leq k \leq n$ and $[1, x^ny^kx^j]$ consists of $n$ $x$-edges, followed by $k$ $y$-edges, followed by $j$ $x$-edges for $0 \leq n < k$ and $0 \leq j \leq k$. Notice that $T$ is a directed spanning tree rooted at 1.

A directed edge of $\Gamma$ is called a *transition edge* if its initial and terminal vertices are in different strongly connected components of $\Gamma$. Edges of $T$ will be called *tree edges*. Here, we say that two vertices $u$ and $v$ of the Cayley graph $\Gamma$ belong to the same strongly connected component of $\Gamma$ if and only if there is a directed path from $u$ to $v$, and also a directed path from $v$ to $u$.

**Lemma 6.** *The following equalities hold in $M$.*

(1) $x^ny^kx^{k+1} = x^{n+1}y^{k+1}x^{k+1}$ *for* $k > n \geq 0$.

(2) $yx^ny^k = x^{n-1}y^nx^{n-k}$ *for* $n \geq 1$ *and* $0 \leq k \leq n$.

*(3)* $yx^ny^k = x^{n-1}y^k$ *if* $0 < n < k$.

*Proof.* (1) Let $k, n \in \mathbb{Z}$ with $k > n \geq 0$, and set $w_1 = x^ny^kx^{k+1}$ and $w_2 = x^{n+1}y^{k+1}x^{k+1}$. The equality $w_1 = w_2$ holds in $M$ since the Munn trees of both of these words have vertex set $\{x^i : n-k \leq i \leq n+1\}$ and terminal vertex $x^{n+1}$. Indeed, consider the Munn tree with vertex set $\{x^i : n-k \leq i \leq n+1\}$ and terminal vertex $x^{n+1}$. This Munn tree is illustrated in Figure 2. The word $w_1 = x^ny^kx^{k+1}$ is obtained by reading this Munn tree starting at 1, then taking $n$ steps to the right, then $k > n$ steps left, and finally $k+1$ steps right, ending on the vertex $x^{n+1}$. This walk clearly visits every vertex in the Munn tree. On the other hand, the word $w_2 = x^{n+1}y^{k+1}x^{k+1}$ is obtained by reading the same Munn tree starting at 1, then taking $n+1$ steps right, then $k+1$ steps left, and then $k+1$ steps right ending on the vertex $x^{n+1}$. Again this walk clearly visits every vertex of the Munn tree. This completes the proof that $x^ny^kx^{k+1} = x^{n+1}y^{k+1}x^{k+1}$ in $M$ for $k > n \geq 0$.

(2) This equality holds in $M$ since for $n \geq 1$ and $0 \leq k \leq n$ the Munn trees of the words $yx^ny^k$ and $x^{n-1}y^nx^{n-k}$ are the same. Specifically both of these words have the Munn tree with vertex set $\{x^i : -1 \leq i \leq n-1\}$ and terminal vertex $x^{n-1-k}$.

(3) This equality holds since for $0 < n < k$ the words $yx^ny^k$ and $x^{n-1}y^k$ both have Munn tree with vertex set $\{x^i : n-1-k \leq i \leq n-1\}$ and terminal vertex $x^{n-1-k}$. □

The following lemma describes the right action of the generators $\{x, y\}$ on the normal form words from Lemma 5. These computations describe all the edges of the Cayley graph $\Gamma$.

**Lemma 7.** *The right multiplicative action of the generators* $\{x, y\}$ *on the normal form words* $x^ny^k$ *with* $0 \leq k \leq n$ *is given by*

$$(x^ny^k)x = \begin{cases} x^ny^{k-1} & \text{if } k > 0 \\ x^{n+1} & \text{if } k = 0 \end{cases}, \quad \text{and} \quad (x^ny^k)y = x^ny^{k+1}.$$

*The action on the normal form words* $x^ny^kx^j$ *with* $0 \leq n < k$ *and* $0 \leq j \leq k$ *is given by*

$$(x^ny^kx^j)x = \begin{cases} x^ny^kx^{j+1} & \text{if } j < k \\ x^{n+1}y^{k+1}x^{k+1} & \text{if } j = k \end{cases}, \quad \text{and} \quad (x^ny^kx^j)y = \begin{cases} x^ny^kx^{j-1} & \text{if } j > 0 \\ x^ny^{k+1} & \text{if } j = 0. \end{cases}$$

*Proof.* First we consider the action on normal form words $x^ny^k$ with $0 \leq k \leq n$. For right multiplication by $x$, if $k > 0$ then $x^ny^kx = x^ny^{k-1}$ in the free monogenic inverse monoid $M$ where $x^ny^{k-1}$ is a normal form word. The equality $x^ny^kx = x^ny^{k-1}$ holds in $M$ since both these words have the same Munn tree with vertex set $\{1, x, \ldots, x^n\}$ and terminal vertex $x^{n-k+1}$. On the other hand, if $k = 0$ and $(x^ny^k)x = x^nx = x^{n+1}$ which is a normal form word. For right multiplication by $y$ we have $(x^ny^k)y = x^ny^{k+1}$ which is already a normal form word.

Now consider the action on the normal forms words $x^ny^kx^j$ where $0 \leq n < k$ and $0 \leq j \leq k$. For right multiplication by $x$, if $j < k$ then $(x^ny^kx^j)x = x^ny^kx^{j+1}$ which is a normal form word. If $j = k$ then $(x^ny^kx^j)x = x^ny^kx^{k+1}$ where $x^ny^kx^{k+1}$ is not a normal form word. It follows from Lemma 6(1) that $x^ny^kx^{k+1} = x^{n+1}y^{k+1}x^{k+1}$ where $x^{n+1}y^{k+1}x^{k+1}$ is a normal form word. Hence we have shown that $(x^ny^kx^j)x = x^{n+1}y^{k+1}x^{k+1}$ in the case that $j = k$. For right multiplication by $y$, if $j > 0$ then $(x^ny^kx^j)y = x^ny^kx^{j-1}$ where $x^ny^kx^{j-1}$ is a normal form word. The equality of words holds in $M$ since they both have Munn tree with vertex set $\{x^i : n-k \leq i \leq n\}$ and terminal vertex $x^{n-k+j}$. Finally, if $j = 0$ then $(x^ny^kx^j)y = x^ny^ky = x^ny^{k+1}$ which is a normal form word. Since all cases have now been considered, this completes the proof. □

Now we describe which edges of $\Gamma$ are in $T$.

**Proposition 8.** *The following edges belong to* $T$*:*

*(1)* $x^n \xrightarrow{x} x^{n+1}$ *with* $n \geq 0$.
*(2)* $x^ny^k \xrightarrow{y} x^ny^{k+1}$ *with* $n \geq 0$ *and* $k \geq 0$.
*(3)* $x^ny^kx^j \xrightarrow{x} x^ny^kx^{j+1}$ *with* $0 \leq n < k$ *and* $0 \leq j < k$.

*All remaining edges do not belong to $T$.*

*Proof.* For any edge $w_1 \xrightarrow{z} w_2$ from $\Gamma$, where $w_1$ and $w_2$ are both normal form words and $z \in \{x, y\}$, by definition, this edge belongs to $T$ if and only if $w_2 \equiv w_1 z$ in the free monoid $\{x, y\}^*$. We can now apply Lemma 7 to identify all of these edges. There are two cases.

First suppose that $w_1 \equiv x^n y^k$ where $0 \leq k \leq n$. Then by the first part of Lemma 7, the word $w_1 z$ is a normal form if and only if either $z = y$, or $z = x$ and $k = 0$. This gives the edges $x^n \xrightarrow{x} x^{n+1}$ with $n \geq 0$, and $x^n y^k \xrightarrow{y} x^n y^{k+1}$ with $0 \leq k \leq n$.

Now suppose that $w_1 \equiv x^n y^k x^j$ with $0 \leq n < k$ and $0 \leq j \leq k$. Then by the second part of Lemma 7, the word $w_1 z$ is a normal form if and only if either $z = x$ and $j < k$, or $z = y$ and $j = 0$. This gives the edges $x^n y^k x^j \xrightarrow{x} x^n y^k x^{j+1}$ with $0 \leq n < k$ and $0 \leq j < k$, and $x^n y^k \xrightarrow{y} x^n y^{k+1}$ with $0 \leq n < k$. This covers all cases, and hence completes the proof of the lemma. □

Next we consider the edges of $\Gamma$ that do not belong to $T$. We begin with non-transition edges. It follows from the definitions that two elements $m$ and $n$ of $M$ belong to the same strongly connected component of $\Gamma$ if and only if $mM = nM$, that is, $m$ and $n$ generate the same principal right ideal in $M$. Such elements are said to be $\mathscr{R}$-related. Necessary and sufficient conditions for two elements of $M$ to be $\mathscr{R}$-related are given in [4, Theorem 2.1.15] and [10, VIII 3.9 Proposition]. See in particular the proof of part (ii) of [10, VIII 3.9 Proposition]. Using the conventions of the present article, these results say that two elements of $M$ are $\mathscr{R}$-related if and only if their Munn trees have the same vertex sets (but the terminal vertices of the Munn trees need not be the same). Combining these observations and results gives the following lemma.

**Lemma 9.** *Two normal form words $w_1$ and $w_2$ belong to the same strongly connected component of $\Gamma$ if and only if $P_{w_1} = P_{w_2}$, that is, their Munn trees have the same vertex sets.*

We can applying this lemma to prove the following result.

**Proposition 10.** *An edge of $\Gamma \setminus T$ belongs to a strongly connected component if and only if it is of one of the following two forms:*

*(1) $x^n y^k \xrightarrow{x} x^n y^{k-1}$ with $0 < k \leq n$;*

*(2) $x^n y^k x^j \xrightarrow{y} x^n y^k x^{j-1}$ with $0 \leq n < k$ and $0 < j \leq k$.*

*Moreover, if $e$ is as in (1), then*

$$b_e = \overline{(x^n y^{k-1} \xrightarrow{y} x^n y^k)(x^n y^k \xrightarrow{x} x^n y^{k-1})}$$

*and if $e$ is as in (2), then*

$$b_e = \overline{(x^n y^k x^{j-1} \xrightarrow{x} x^n y^k x^j)(x^n y^k x^j \xrightarrow{y} x^n y^k x^{j-1})}.$$

*Proof.* Consider an edge $w_1 \xrightarrow{z} w_2$ of $\Gamma$, where $w_1$ and $w_2$ are normal form words and $z \in \{x, y\}$. Further suppose that this edge does not belong to $T$ and that $w_1$ and $w_2$ belong to the same same strongly connected component of $\Gamma$, which by Lemma 9 means that the Munn trees of $w_1$ and $w_2$ have equal vertex sets. We now apply Lemma 7 to identify all edges satisfying these conditions. There are two cases to consider.

First suppose that $w_1 \equiv x^n y^k$ where $0 \leq k \leq n$. In this case we cannot have $z = y$ since then the edge $w_1 \xrightarrow{z} w_2$ would be equal to $x^n y^k \xrightarrow{y} x^n y^{k+1}$ which belongs to $T$. Hence we have $z = x$. Then we must have $k > 0$ since if $k = 0$ then the edge $w_1 \xrightarrow{z} w_2$ would be equal to $x^n \xrightarrow{x} x^{n+1}$ which belongs to $T$. So in this case we obtain the set of edges $x^n y^k \xrightarrow{x} x^n y^{k-1}$ with $0 < k \leq n$ none of which belong to $T$ by Proposition 8. Furthermore all these edges do connect vertices in the same strongly connected component since when $0 < k \leq n$ the Munn trees of $x^n y^k$ and $x^n y^{k-1}$ both have vertex set $\{1, x, \ldots, x^n\}$.

Now suppose that $w_1 \equiv x^n y^k x^j$ with $0 \leq n < k$ and $0 \leq j \leq k$. If $z = x$ then by the second part of Lemma 7 we cannot have $j = k$ since the Munn trees of $x^n y^k x^j$ and of $x^{n+1} y^{k+1} x^{k+1}$ have different vertex sets. But then $j < k$ and the edge $w_1 \xrightarrow{z} w_2$ is equal to $x^n y^k x^j \xrightarrow{x} x^n y^k x^{j+1}$ which belongs to $T$. This is a contradiction. Hence in this case there are no edges satisfying the conditions and with $z = x$. On the other hand, if $z = y$ then by the second part of Lemma 7 we cannot have $j = 0$ since the edge $x^n y^k \xrightarrow{y} x^n y^{k+1}$ belongs to $T$. Hence $j > 0$, so in this case we obtain the set of edges $x^n y^k x^j \xrightarrow{y} x^n y^k x^{j-1}$ with $0 \le n < k$ and $0 < j \le k$. By Proposition 8 none of these edges belong to $T$. Furthermore all these edges do connect vertices in the same strongly connected component since when $0 \le n < k$ and $0 < j \le k$. the Munn trees of $x^n y^k x^j$ and $x^n y^k x^{j-1}$ both have vertex set $\{x^i : n - k \le i \le n\}$.

This completes the proof that an edge of $\Gamma \setminus T$ belongs to a strongly connected component if and only if it is of one of the two forms (1) and (2) given in the statement of the proposition.

For the first of the final two statements of the proposition, if $e$ is an edge as in (1) then by definition

$$b_e = \overline{[1, \iota(e)]e[1, \tau(e)]^{-1}} = \overline{[1, x^n y^k]e[1, x^n y^{k-1}]^{-1}}$$

where $0 < k \le n$. Note that $x^n y^k$ is a normal form word, and all prefixes of this word are normal form words, hence distinct prefixes of this word represent distinct elements of $M$. Since the paths $[1, x^n y^{k-1}]$ and $[1, x^n y^k]$ have a common initial segment $[1, x^n y^{k-1}]$, it follows that all of the edges in this common initial segment cancel each other out in when computing the weighted sum $b_e$ leaving only

$$b_e = \overline{(x^n y^{k-1} \xrightarrow{y} x^n y^k)(x^n y^k \xrightarrow{x} x^n y^{k-1})}.$$

For the final claim in the proposition, let $e$ be an edge as in (2). Then by definition

$$b_e = \overline{[1, \iota(e)]e[1, \tau(e)]^{-1}} = \overline{[1, x^n y^k x^j]e[1, x^n y^k x^{j-1}]^{-1}}$$

where $0 \le n < k$ and $0 < j \le k$. Since $[1, x^n y^k x^{j-1}]$ is a common initial segment of both $[1, x^n y^k x^j]$ and $[1, x^n y^k x^{j-1}]$ it follows that in the weighted sum $b_e$ the edges in this common initial segment cancel out leaving just two edges contributing to the sum, giving

$$b_e = \overline{(x^n y^k x^{j-1} \xrightarrow{x} x^n y^k x^j)(x^n y^k x^j \xrightarrow{y} x^n y^k x^{j-1})}.$$

□

The next result shows that there is only one type of transition edge not belonging to $T$.

**Proposition 11.** *The transition edges of $\Gamma$ not belonging to $T$ are of the form $x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}$ with $0 \le n < k$. The corresponding basis element of $H_1(\Gamma)$ is*

$$\overline{[x^n, x^n y^k x^k]} + (x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}) - \overline{[x^n, x^{n+1} y^{k+1} x^{k+1}]}.$$

*Proof.* Let $w_1 \xrightarrow{z} w_2$ be a transition edge not belonging to $T$, where $w_1$ and $w_2$ are normal form words and $z \in \{x, y\}$. There are two cases depending on the form of the word $w_1$.

First suppose that $w_1 \equiv x^n y^k$ where $0 \le k \le n$. By the first part of Lemma 7 and Proposition 8, since $w_1 \xrightarrow{z} w_2$ does not belong to $T$ the only possibility is that it is the edge $x^n y^k \xrightarrow{x} x^n y^{k-1}$ and $k > 0$. But this is not a transition edge by Proposition 10(1). This proves that we cannot have $w_1 \equiv x^n y^k$ where $0 \le k \le n$.

Now suppose that $w_1 \equiv x^n y^k x^j$ with $0 \le n < k$ and $0 \le j \le k$. By the second part of Lemma 7, Proposition 8, and Proposition 10(2), the only possibility for the edge $w_1 \xrightarrow{z} w_2$ is that it is equal to $x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}$ with $0 \le n < k$. It also follows from Proposition 8, and Proposition 10(2) that these are all transition edges and they do not belong to $T$. Since all cases for $w_1$ have been considered, this completes the proof that these are exactly the transition edges not belonging to $T$.

To complete the proof of the proposition, let $e$ be the edge $x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}$ for some $0 \le n < k$. Then by definition

$$b_e = \overline{[1, \iota(e)]e[1, \tau(e)]^{-1}} = \overline{[1, x^n y^k x^k]e[1, x^{n+1} y^{k+1} x^{k+1}]^{-1}}.$$

Since

$$[1, x^n y^k x^k] = [1, x^n][x^n, x^n y^k x^k] \text{ and } [1, x^{n+1} y^{k+1} x^{k+1}] = [1, x^n][x^n, x^{n+1} y^{k+1} x^{k+1}]$$

have common initial segment $[1, x^n]$, all of the edges in this common segment cancel each other out in the weighted sum $b_e$, which gives

$$b_e = \overline{[x^n, x^n y^k x^k]} + (x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}) - \overline{[x^n, x^{n+1} y^{k+1} x^{k+1}]}.$$

□

Our next goal is to assign a number, called the *depth*, to the basis element $b_e$ of $H_1(\Gamma)$ corresponding to a directed edge $e$ of $\Gamma \setminus T$. If $e$ belongs to a strongly connected component of $\Gamma$, then we set $b_e$ to have depth zero. If $e$ is as in Proposition 11, then we set $b_e$ to have depth $k$ (which is greater than 0). Also, for any directed edge $e$ of $\Gamma \setminus T$, by the depth of the edge $e$ we mean the depth of the corresponding basis element $b_e$. Let $W_k$ be the subgroup of $H_1(\Gamma)$ generated by the $b_e$ of depth at most $k$. Then we have a strictly increasing chain of subgroups

$$W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots$$

with $\bigcup_{k \geq 0} W_k = H_1(\Gamma)$. Our goal is to show that each $W_k$ with $k \geq 0$ is a $\mathbb{Z}M$-submodule. Since a finitely generated module cannot be written as the union of a strictly increasing chain of submodules, this will prove that $H_1(\Gamma)$ is not a finitely generated $\mathbb{Z}M$-module and hence $M$ is not of type $\mathrm{FP}_2$.

**Proposition 12.** *The subgroup $W_0$ is a $\mathbb{Z}M$-submodule of $H_1(\Gamma)$.*

*Proof.* By Proposition 10, if $b_e$ has depth zero then $b_e = \overline{p}$ where $p$ is a directed cycle of length 2. But any translate of a closed directed path is a closed directed path and hence contained in a strongly connected component of $\Gamma$. Since every edge of a strongly connected component either belongs to the tree $T$ or has depth zero, we see that $W_0$ is indeed a $\mathbb{Z}M$-submodule. □

We now extend this to all values of $k$.

**Proposition 13.** *For all $k \geq 0$, $W_k$ is a $\mathbb{Z}M$-submodule of $H_1(\Gamma)$.*

*Proof.* Proposition 12 handles the case $k = 0$. By the definition of $W_k$, to complete the proof of the proposition it suffices to show that for all $k \geq 1$, if $e$ is an edge of the form $x^n y^k x^k \xrightarrow{x} x^{n+1} y^{k+1} x^{k+1}$ with $0 \leq n < k$ and $z \in \{x, y\}$, then $zb_e \in W_k$. By Proposition 11, this means we need to show that $ze$ and edges of $z[x^n, x^n y^k x^k]$, $z[x^n, x^{n+1} y^{k+1} x^{k+1}]$ are of depth at most $k$ or tree edges.

Let us start with $z = y$. In what follows, $x^{-1}$ should be interpreted as $y$; this situation arises when $n = 0$. We consider first $y[x^n, x^n y^k x^k]$. Note that

$$[x^n, x^n y^k x^k] = [x^n, x^n y^n][x^n y^n, x^n y^k][x^n y^k, x^n y^k x^k].$$

By Lemma 6, we have $yx^n = x^{n-1} y^n x^n$ and $yx^n y^n = x^{n-1} y^n$, which belong to the same strongly connected component. Thus each edge of $y[x^n, x^n y^n]$ is either a tree edge or an edge of depth zero. On the other hand, $y[x^n y^n, x^n y^k]$ is a string of $k - n$ $y$-edges from $x^{n-1} y^n$ to $yx^n y^k = x^{n-1} y^k$ (by Lemma 6) and these are all tree edges. Finally, $y[x^n y^k, x^n y^k x^k]$ is a string of $k$ $x$-edges from $x^{n-1} y^k$ to $x^{n-1} y^k x^k$. Since $k > n > n - 1$, these are again tree edges.

Next, we consider $y[x^n, x^{n+1} y^{k+1} x^{k+1}]$. Write

$$[x^n, x^{n+1} y^{k+1} x^{k+1}] = [x^n, x^{n+1}][x^{n+1}, x^{n+1} y^{n+1}][x^{n+1} y^{n+1}, x^{n+1} y^{k+1}] \cdot [x^{n+1} y^{k+1}, x^{n+1} y^{k+1} x^{k+1}]$$

As $yx^n = x^{n-1} y^n x^n$ and $yx^{n+1} = x^n y^{n+1} x^{n+1}$, by Lemma 6, we see that $y[x^n, x^{n+1}] = x^{n-1} y^n x^n \xrightarrow{x} x^n y^{n+1} x^{n+1}$ is an edge of depth $n < k$ (or a tree edge if $n = 0$ by Proposition 8(3)). Note that $x^n y^{n+1} x^{n+1}$ and $x^n y^{n+1}$ are in the same strongly connected component of $\Gamma$ by Lemma 9, since they both have Munn trees with vertex set $\{x^i : -1 \leq i \leq n\}$. Since $yx^{n+1} = x^n y^{n+1} x^{n+1}$ and $yx^{n+1} y^{n+1} = x^n y^{n+1}$ (see Lemma 6) belong to the same strongly connected component, we have that $y[x^{n+1}, x^{n+1}y^{n+1}]$ consists of tree edges and edges of depth zero. Next, we have that the translate $y[x^{n+1}y^{n+1}, x^{n+1}y^{k+1}]$ is a string of $k-n$ $y$-edges from $yx^{n+1}y^{n+1} = x^n y^{n+1}$ to $yx^{n+1}y^{k+1} = x^n y^{k+1}$, and all these edges are tree edges. Finally, $y[x^{n+1}y^{k+1}, x^{n+1}y^{k+1}x^{k+1}]$ is a string of $k+1$ $x$-edges from $yx^{n+1}y^{k+1} = x^n y^{k+1}$ to $yx^{n+1}y^{k+1}x^{k+1} = x^n y^{k+1}x^{k+1}$ by Lemma 6. These are again tree edges.

The translate $ye$ is $x^{n-1}y^k x^k \xrightarrow{x} x^n y^{k+1}x^{k+1}$, which is an edge of depth $k$, using that $n-1<k$, $yx^n y^k x^k = x^{n-1}y^k x^k$ and $yx^{n+1}y^{k+1}x^{k+1} = x^n y^{k+1}x^{k+1}$ by Lemma 6, unless $n=0$, in which case it is a tree edge. This completes the argument that $yb_e \in W_k$.

So we next turn to $z = x$. There are two cases, $k > n+1$ and $k = n+1$.

Assume first that $k > n+1$. Then $x[x^n, x^n y^k x^k] = [x^{n+1}, x^{n+1}y^k x^k]$ and $x[x^n, x^{n+1}y^{k+1}x^{k+1}] = [x^{n+1}, x^{n+2}y^{k+1}x^{k+1}]$ consist of tree edges and $xe = x^{n+1}y^k x^k \xrightarrow{x} x^{n+2}y^{k+1}x^{k+1}$ is an edge of depth $k$. Thus, in this case, $xb_e \in W_k$.

Finally, suppose that $k = n+1$. Then $xx^n y^k x^k = x^{n+1}y^{n+1}x^{n+1} = x^{n+1}$. Therefore, $x[x^n, x^n y^k x^k]$ is a directed path from $x^{n+1}$ to $x^{n+1}$ and hence uses only tree edges and edges of depth zero as it is contained in a strongly connected component. Observe that $xx^{n+1}y^{k+1}x^{k+1} = x^{n+2}y^{n+2}x^{n+2} = x^{n+2}$. Writing $[x^n, x^{n+1}y^{k+1}x^{k+1}] = [x^n, x^{n+1}][x^{n+1}, x^{n+1}y^{k+1}x^{k+1}]$, we see that $x[x^n, x^{n+1}y^{k+1}x^{k+1}]$ is the concatenation of the tree edge $x^{n+1} \xrightarrow{x} x^{n+2}$ with a directed path from $x^{n+2}$ to itself and the latter path uses only tree edges and edges of depth zero as it is contained in a strongly connected component. Also, we have that $xe = x^{n+1} \xrightarrow{x} x^{n+2}$ is a tree edge. We conclude that $xb_e \in W_k$ in this case as well. This completes the proof that $W_k$ is a $\mathbb{Z}M$-submodule of $H_1(\Gamma)$. □

Proposition 13 completes the proof of Theorem 3 in light of the discussion preceding Proposition 12.

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