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Scheming and Re-scheming: Secondary Mathematics Teachers' Use and Re-use of Resources

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Abstract
In this article, we examine secondary mathematics teachers' work with resources using the Documentational Approach to Didactics lens. Specifically, we look at the resources and a teacher's scheme of use (aims, rules of actions, operational invariants, and inferences) of these resources across a set of lessons (macro-level analysis) that aim towards students' preparation for the examinations and how this use emerges in a set of three lessons on the same topic (micro-level analysis) as a response to contingent moments. We propose the terms scheming—a teacher's emerging scheme of use related to the same set of resources used for the same aim—and re-scheming, namely, shifts in such scheming. Our analysis of lesson observations and the teacher's reflections on his actions from a post-observation interview demonstrate the interplay between the stable characteristics of the scheme of use and the scheming and re-scheming in individual lessons. We conclude this article with a discussion on the methodological potential of using both macro- and micro-level analyses in the investigation of teachers' use of resources.

Keywords Resources · Schemes · Documentational approach to didactics · *Autograph* · Volume of revolution · Integration

The complexity of mathematics teachers' actions in the preparation of their lessons, as well as in the decisions they make in the classroom, has been reported in research that highlights that classroom interactions are highly contextual and not mere production of teachers' lesson plans, knowledge or beliefs (Biza et al., 2015; Bretscher, 2014). Cyrino (2018) argues that "[teachers] have only limited control in relation to education, the context, the curriculum, being subject to the decisions and rules established by others" (p. 271). Furthermore, different factors and interactions in classrooms impact on teachers' practices. Thus, although teachers may have stable principles underpinning their work, they may also adapt and re-adapt their work

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based on the different factors in their working environment (Adler, 2000; Gueudet & Trouche, 2009).

Factors that come into play in teaching include teachers' and students' roles and approaches, institutional policies (such as curriculum and time pressure), unexpected incidents, and resources (Mishra & Koehler, 2006; Nardi et al., 2012; Turner & Rowland, 2011). This article focuses on the last one, *resources* (e.g., textbooks, on-line materials, etc.), and the role these resources play when teachers plan their lessons and also when they act and interact in their classes. Specifically, we investigate the use of resources in teachers' work at a *macro-level* in a series of lessons with the same overarching aim, and small adaptations of such use at a *micro-level* in a set of consecutive lessons on the same mathematical topic within the same series of lessons.

To do so, we draw on the "Documentational Approach to Didactics" (Gueudet & Trouche, 2009), first to describe the stability of what they term the *scheme of use* of resources employed by one teacher across a series of lessons that aim to prepare upper-secondary students for examinations. We also introduce the terms of *scheming* and *re-scheming* as the dynamic process in which small changes in the use of resources occur when the teacher dealt with *unexpected situations* while he taught the same topic of the volume of revolution to three consecutive lessons. In the next two sections, we draw on the literature to describe how resources are used in this study and how unexpected situations influence teaching actions. Then, in the following section, we introduce the theoretical underpinning of the Documentational Approach to Didactics and our conceptualization of scheming and re-scheming, in order to formulate the research question of the study.

Resources

A *resource* can be anything that informs a mathematics teacher's activity (Trouche et al., 2020). It can be an artifact, some teaching materials or the result of social and cultural interactions that are available for a teacher in her teaching and preparation for teaching (Gueudet & Trouche, 2009). Gueudet and Trouche specify an artifact as "a cultural and social means provided by human activity, offered to mediate another human activity" (p.204). It can be a tool (e.g., a ruler), a mathematical technique (e.g., the quadratic formula), or a piece of software (e.g., *Excel*). In this article, "teaching materials" refer to those designed for mathematics teaching and learning such as school textbooks or mathematics education software (Adler, 2000). "Social and cultural interactions" refer to engagements with the environment, with students, and with colleagues, while results of such interactions that can potentially become a resource for a teacher are, for example, a conversation with or an email sent by a colleague or a student's feedback on an task (Gueudet & Trouche, 2009). Trouche et al. (2019a, 2019b) argue that "resources are essentially social, as they take place, for example, in schools, or via Internet, and often in collectives" (p. 54).

The word "resource" can also be used as "the verb re-source, to source again or differently" (Adler, 2000, p. 207). This implies that teachers interact with resources, manage them, and re-use them in order to achieve their teaching aims (Gueudet, 2017; Trouche et al., 2020a, 2020b), and that, as Adler claims "the effectiveness of resources for mathematical learning lies in their use, that is, in the classroom teaching and learning context" (p. 205). It is noted that different teachers may employ the same resources differently and for different teaching and learning aims (Cohen et al., 2003; González-Martín et al., 2018). Also, the same resources might be employed by the same teacher differently in different classes (Kieran et al., 2012).

Adler argues that "more resources do not necessarily lead to better practice" (p. 206), and that attention should be given to how these resources are employed in the classroom (see also Cohen et al., 2003; Mishra & Koehler, 2006). Hence, what a resource is cannot be separated from its use in the context of the teaching or learning environment. In this article, we examine such use of resources by one teacher by looking closely on how this use is related to his aims to prepare the students for the examinations across lessons and to introduce them to the topic of volume of revolution in a set of three consecutive lessons. Especially in these three related lessons, we focus on how the teacher's use of resources is adapted as a response to unexpected classroom situations, a theme examined in the following subsection.

Dealing with the Unexpected in the Class

When looking at teachers' work with resources in their in-class action, it is important to consider the unexpected moments they face in the classroom. Although teachers can plan their intended classroom actions, they cannot predict their students' responses and contributions (Rowland et al., 2005), and they need to act spontaneously in light of momentary situations (Hole & McEntee, 1999; Lampert & Ball, 1999; Remillard & Heck, 2014). The unpredictable makes decision-making complex and dependent on the individual's instantaneous judgement rather than professionals' advice (Rowland et al., 2015).

Unpredictable instances were named by Rowland and colleagues (Rowland et al., 2005) as *contingent moments* that concern responses to unanticipated or unplanned events, such as responses to unexpected contributions from students or to notable prompt insights from the teacher (Thwaites et al., 2011). Rowland and colleagues (Rowland et al., 2015) recognized three triggers of contingency: the teacher reassessing the lesson plan; students giving an unexpected contribution; and more or fewer pedagogical tools and artifacts being available. Teacher responses to contingent moments include occasions that go under the following codes: *deviation from agenda*; *responding to students' ideas*; *use of opportunities*; *responding to the (un-) availability of tools and resources;* and *teacher insight during instruction* (Thwaites et al., 2011; Turner & Rowland, 2011). The first, third, and fifth codes are triggered by the teacher, while the second code is triggered by students and the fourth by tools and resources (Rowland & Turner, 2017; Rowland et al., 2015).

In response to unexpected students' ideas, a teacher can choose, "to ignore, to acknowledge but put aside, and to acknowledge and incorporate" (Rowland et al., 2015, p. 79). They also assert that one should also distinguish "between the contingent action of the teacher and the contingent potential of the situation" (p. 88). The former refers to the teacher's instantaneous response to a trigger of contingency, while the latter is about teachers' potential development which can result from reflecting on the contingent moments, along with teachers' corresponding actions. This development arises either when a teacher realizes that her/his contingent action can be improved and reflects upon it or when the teacher does not realize such an inadequacy, but a more experienced observer brings it to the teacher's attention for reflection (Rowland et al., 2015).

An unpredictable event is described by Thames and van Zoest (2013) as "[a] pivotal teaching moment" and defined as an instant interruption in the flow of a lesson that requires a teacher to modify instructions, in order to support students' understanding of the mathematical concept. Arafeh et al. (2001) used the term "teachable moments" and defined them as "the set of behaviors within a lesson that indicated students are ripe for, or receptive to, learning because they express confusion, misunderstanding, uncertainty, struggle, or difficulty with a mathematical problem, concept, or procedure" (pp. 2–3).

Technology-related contingent moments were described by Clark-Wilson (2013) as *hiccups*, where the latter are also instances that one cannot predict and, hence, cannot plan for. Although hiccups are similar to the contingent moments suggested by Rowland and colleagues (Rowland et al., 2005), the main difference between the two is the technology factor, as "hiccups" are the result of working with technology in classrooms (Clark-Wilson & Noss, 2015). Another difference is teacher awareness of the hiccups when they occur, which is not always the case with Rowland and colleagues' contingent moments.

For this article, we find "contingent moment" (Turner & Rowland, 2011) a more suitable construct to describe the unexpected, a notion that helps reflect on teachers' practices with all types of resources (including technology), regardless of whether the teacher notices the contingency or not. We argue that these moments have an essential role on how teachers work with resources and how they act within their lessons at a micro-level and at a macro-level more general. In the next section, we describe the theoretical framework we are using to look at such teachers' work with resources: the Documentational Approach to Didactics (Gueudet & Trouche, 2009).

The Documentational Approach to Didactics

The Documentational Approach to Didactics (DAD) analyzes the interaction of teachers with resources, where the latter are seen through a broader view, as we described in the previous section, and can be material resources (e.g., textbooks) or the result of social and cultural interactions (e.g., an email sent by a colleague) (Gueudet, 2019; Gueudet & Trouche, 2009; Trouche et al., 2019a, 2019b). Teachers interact with these resources, and "these interactions play a central role in the teacher's professional activity" (Gueudet et al., 2014, p. 141) and have a great impact on students' learning (Carrillo, 2011). Teachers choose their resources and manage them (Gueudet et al., 2014; Trouche et al., 2020a, 2020b), and their professional experiences influence (and are influenced by) their choices and use of resources. According to Trouche et al., (2019a, 2019b), the DAD considers "resource as process and/or product and [argues] for a focus on both if teachers' activity with resources is to be fully described and understood […, but] a resource is not a benign 'object' with which teachers interact. It itself shapes teachers' work and its effects" (p. vi). Such interaction between resources and teachers' work is studied by the DAD, as we describe below.

According to Gueudet (2019), the "first theoretical source" (p. 20) for the DAD was the *instrumental approach* (Rabardel, 1995). In this, the artifact, a product of the human activity, is distinguished by the instrument developed by a person who uses this artifact. Gueudet adds that the instrument "is a mixed entity, comprising the artefact and a scheme of use" (2019, p. 20). The notion of the *scheme of use* draws on the work of Vergnaud (1998), who defines a *scheme* as "[the] invariant organization of behavior for a certain class of situations" (p. 229). In the scheme, he distinguishes four components: goals, sub-goals, and expectations; rules of action; operational invariants; and possibilities of inferences (p. 229).

In the DAD, as proposed by Gueudet and Trouche (2009), the artifact–instrument dialectic proposed by the instrumental approach is substituted by the resource–document dialectic.[1] Specifically, teachers, while interacting with a set of resources, develop *schemes of use related to these resources*. Still inspired by Vergnaud's definition of "scheme", the scheme of use is an invariant organization of activity for a given class of situations. However, in the DAD, as described in Gueudet (2017), the scheme of use consists of the *aim of the teaching activity* (e.g., to prepare students for examinations or to introduce the volume of revolution that is in the examination requirements[2]); the *rules of action*, which represent teacher actions in using the resources towards the aim(s) (e.g., use examination questions on the volume of revolution for practice); *operational invariants*, which are reasons adopted by a teacher to justify her stable actions in a range of similar situations (e.g., it is useful to use a software and the textbook together to introduce the formula of the volume of revolution); and *inferences* that describe teacher adaptations of activity to the features of a given situation corresponding to the same aim (e.g., if the teacher wants to explain the formula of the volume of revolution, the image from the textbook will be presented first).

A teacher develops a *document* by associating a set of resources with the scheme of use of these resources (Gueudet, 2017). It can be thought of as "the verb *document*: to support something (here, the teacher's professional activity) with documents" (Gueudet & Trouche, 2009, p. 205; *italics in original*). Teachers look for resources, use them for a specific aim, amend them, and share them with other colleagues. This constitutes teachers' *documentation work* that holds an essential part in teachers' professional activity (Gueudet, 2019; Gueudet & Trouche, 2009). For example, a teacher's documentation work towards the teaching of differential equation may include a plan to use a set of resources such as a board, a textbook, or a

[1] An overview of the theoretical and methodological developments of the DAD can be found in Gueudet (2019) and its relation to the long-term tradition of research in the didactics of mathematics can be found in Artigue and Trouche (2021).

[2] We draw on Gueudet (2017) to describe the four parts that consist the scheme of use while we exemplify these parts in the context of our study.

graphing calculator. This plan includes rules of actions of how these resources are used and these actions are justified by the teacher's operational invariants. This plan might be shared with other colleagues or influenced by the work of others or new resources.

The DAD offers lenses for exploring the evolution of a teacher's documents, which, in turn, contribute "to the study of her professional evolution. Naturally, such a study must not be limited to the material aspect of documents, but must also investigate the evolution of usages […] and operational invariants" (Gueudet & Trouche, 2009, p. 211). Following Adler's definition of "the verb re-source," we are interested in the dynamic nature of the use of resources. Especially, we are interested in the potential influence classroom interactions have on the scheme of use, when a teacher uses the same set of resources in series of lessons on the same topic and for the same aims.

Specifically, in this study, we aim to explore the characteristics of one teacher's documents, by investigating his sets of resources and schemes of use across lessons and, in particular, during the teaching of three lessons on the same topic (the volume of revolution). We look at each lesson individually and, then, we look across lessons to identify stable characteristics of the teacher's scheme of use. Then, we return to individual lessons to investigate potential shifts in this scheme of use as a response to contingent moments. To describe such shifts, we introduce the terms of *scheming* and *re-scheming*.

Scheming is a teacher's emerging scheme of use related to the same set of resources used for the same aim. The notion of scheming aims to bring together the stable characteristics of the scheme of use and the emerging characteristics of teacher's use of resources in certain lessons. Re-scheming, a term preliminarily proposed in Kayali and Biza (2018), refers to shifts in the scheming as a response to unexpected situations. Scheming and re-scheming might be momentary and temporary, with the potential to gain stability and become part of the scheme of use that describes the stable organization for facing a given class of situations.

The role of contingent moments in the interplay between scheming and scheme of use is the focus of this article in which we report on our investigation the following research question: *How does a teacher's scheming shift in response to contingent moments that s/he faces in lessons on the same topic when using the same set of resources*? We sought a response to this question from one teacher's actions in the observed lessons and his reflections in a post-observation interview, which we describe in the next section.

Method

This article reports outcomes from a Ph.D. study conducted in the UK by the first author (Kayali, 2020), which looks at four upper-secondary (students aged 16–18 years) mathematics teachers' documentation work, specifically their schemes of use. Here, we focus on one teacher, George (pseudonym). Similar to the other participants, George was initially selected for the doctoral study because he mentioned that he used a wide range of resources, including computer-based ones.

Indeed, in the lessons we observed, George integrated a wide range of resources including paper- and technology-based ones. Such a range offered us the opportunity to observe the complexity of teaching when a wide range of resources is employed and to examine teachers' ways of navigating across them. We selected him for this article for two reasons: he welcomed and integrated students' contributions in the flow of his lessons, an approach that often-generated contingent moments; and, with him, we had the opportunity to observe consequent lessons on the same topic—findings from the analysis of three consequent lessons on the volume of revolution are reported in this article.

Data were collected in three stages: a pre-observation interview (not used in this article), lesson observations, and a post-observation interview. Also, we collected the teaching materials he used in his teaching (e.g., school or personal website, textbook, past exam papers, etc.). The interviews were audio-recorded. George's lesson observations were video-recorded while field notes were kept, and single pictures were taken of the work done on the board. The field note protocol was on the used resources, how these resources were used, who was using them, and how they were used in teacher–student interaction. The study was approved by the Research Ethics Committee of our institution and George and his students consented to the anonymized use of their data for the purposes of this research.

The study employed qualitative analysis based on an interpretative research methodology (Stake, 2010). To analyze the data, we watched lesson videos, together with field notes and photographs, and constructed *factual narrative accounts* summarizing each observed lesson with attention on the use of resources, teacher actions, and reactions to student responses. The factual narrative accounts were used for the identification of instances of when the teacher made a decision that shaped classroom interactions (such decisions were triggered by the teacher's or students' actions), as well as opportunities that emerged from classroom interactions which were not taken up by the teacher (Kayali, 2020; Thames & van Zoest, 2013). Such decisions reveal the teacher's approach and purpose (Kayali, 2020). Contingent moments, where a teacher was faced with situations he had not anticipated, were also included in these instances. Post-observation interview protocols were based on these instances and aimed to invite the teacher to interpret and reflect on his choices and decisions during lessons.

Then, for each lesson, we analyzed classroom observations in the light of the post-observation interviews in relation to teacher's documents, namely the resource used and his scheme of use (aims of the teaching activity, rules of actions, operational invariants, and inferences). We outlined the outcomes of this analysis by drawing teachers' *scheme of use tables* for each lesson (micro-level analysis), similar to the ones used by Gueudet (2017) in her analysis of university teachers' work. While, in that study, such tables were based on 1-h interview with lecturers in which they shared the resources they used with the researcher, in our work, the construction of the tables drew on classroom observations for each lesson, as well as on data from interview and the used resources. The analysis of individual lessons captured the emerging characteristics of George's scheming.

Then, the scheme of use tables for each lesson was deployed to identify the common parts of George's used resources and the scheme of use of these resources

Table 1 Resources used by George across the eleven lessons

Artifact and teaching materials
• Interactive whiteboard
• Board
• Curriculum of Year 13
• Past-examination questions and mark schemes
• Calculators
• Autograph
• Notebooks
• School policy documents
• Textbook: Wiseman and Searle (2005), Advanced Maths for AQA series
Results of social and cultural interactions
• Past teaching experience
• Students' contributions

across lessons (macro-level analysis). The common part of resources used by George and his scheme of use across lessons is outlined in Table 1 and Table 2, respectively. The macro-level analysis drew on the micro-level analysis of individual lessons and was aimed towards the identification of the stable characteristics of George's documentation work across lessons. Lastly, consequent lessons on the same topic were analyzed further in the frame of the overarching scheme of use and in relation to contingent moments observed in George's teaching. In this article, we present episodes where contingency was triggered—regardless of whether the teacher noticed and acted upon the potential of the contingent moment or not—with attention to how such contingency influenced (or not) the teacher's work in the subsequent lessons on the same topic.

In the next section, we present George and his documentation work across lessons (macro-level analysis). Then, we focus on his teaching in a trio of consecutive lessons on the volume of revolution. In relation to these three lessons, we first present his scheming with reference to his overarching scheme of use across lessons and then we discuss two episodes in which contingent moments took place in the first lesson and then impacted (or not) on his work in subsequent lessons. Each episode was selected to demonstrate George's scheming and re-scheming in lessons that used the same set of resources and had the same aim. In the presentation of the episodes, we use excerpts from factual narrative accounts that described the context and quotations from the observations and the interviews. We conclude each episode with the findings of our analysis in relation to George's scheming with reference to contingency codes, marked relevantly in italics (e.g., *responding to students' ideas*).

George and His Documentation Work Towards Students' Preparation for the Exams

George is a mathematics teacher with 15 years of secondary teaching experience, mostly in upper-secondary education (students aged 16–18 years), and he was the head of mathematics at his school at the time of the data collection. He holds

Table 2 George's scheme of use of resources across eleven lessons (macro-level analysis)

Aims	Rules of action (ways of using the resources)	Operational invariants (reason of using the resources in this way)
A1. To prepare students for the exams A2. To introduce a mathematical topic that is in the examination requirements	R1. Introduce a new mathematical topic with examples on the board R2. Connect mathematical ideas (e.g. by connecting a new mathematical topic to students' previous knowledge or to other areas of mathematics) R3. Use *Autograph* (and the interactive whiteboard) to orchestrate the need of a new definition, to invite conjectures, to explain a formula or to visualize figures R4. Use calculators and notebooks for quick calculations and reference R5. Invite students' contributions R6. Respond to students' contributions R7. Use exam questions for practice and use mark schemes to demonstrate exam expectations R8. Allow some time for students to solve textbook exercises independently or in small groups R9. Navigate across available resources	O1. *Autograph* supports students' learning by visualizing mathematical ideas, triggering the need or explaining the purpose of a new definition, demonstrating a range of examples, and making the lesson more interesting ("fun") O2. The use of exam-style questions supports students' preparation for the examinations O3. Connecting new ideas with students' previous knowledge is essential for student learning

a Bachelor of Science in Mathematics and Postgraduate Certificate in Education (PGCE) in secondary mathematics teaching. The school he worked at provided interactive whiteboards, iPads for students' use, and *Autograph* software (Autograph, 2020). *Autograph* offers two-dimensional (2D) or three-dimensional (3D) environments and allows the user to see multiple representations (e.g., graphical, tabular or algebraic) of the same mathematical idea (Jones et al., 2010).

We observed 11 lessons from George all with Year 13 students (aged 17–18 years) who were working towards the A-level (Advanced-level) examination in mathematics—a school-leaving and university admission qualification. Before presenting George's use of resources in specific lessons (micro-level), we first describe a summary of his documentation work across lessons (macro-level). Such work emerged from the analysis of the observed lessons, as well as from the follow-up interview.

The resources that are common in George's teaching of the different lessons we observed are summarized in Table 1. We have grouped them into artifacts and teaching materials and into results of social and cultural interactions. The former group includes paper- and technology-based resources: interactive whiteboard, board, curriculum of Year 13, past-examination questions and mark schemes, calculators, notebooks, and *Autograph*. School policy and textbooks were also used by George, which we considered part of his set of resources. For the observed lessons, George used a textbook (Wiseman & Searle, 2005), which is part of a series of books (Advanced Maths for AQA) for student preparation for the A-level examination.

In relation to resources as a result of George's social and cultural interactions, we have included his past teaching experience and students' contributions. For his past teaching experience, we sought evidence when he explicitly explained his choices in the lesson. For example, when George was asked in the interview why he had chosen the *sine* function to draw a 3D shape on Autograph, he said, "partly from using that in previous lessons. So, knowing that is going to give an interesting shape, and from playing around with sine graphs and things like that in previous lessons." (I[3])

Examples of George using students' contributions as a resource were frequently noticed. When explaining a mathematical idea, he invited responses to specific questions and built his explanations by following the students' answers. For instance, he asked students to estimate the formula for a given trigonometric graph and used their responses to introduce general formulae or had them experiment with equations of their choice and then drew on the outcomes of their experimentation to introduce an iteration method for solving equations (Kayali, 2020). Overall, he was open to use students' contributions at different stages of the lesson and was confident working with examples chosen on the spot, which occasionally generated contingent moments which he mostly dealt with confidently. We discuss examples of these moments in the episodes that follow.

Table 2 summarizes George's scheme of use of the resources listed in Table 1 with his aims across the eleven lessons we observed (first column), the rules of actions (second column), and the operational invariants (third column).

[3] We use the following coding for the quotations: I for interview or O for observation, O is followed by Lx, where x (1–11) is the number of the lesson (L) that was observed.

Preparing students for the exams was an explicit aim of George's teaching throughout the eleven-lesson observations (A1); this was reflected in the use of resources such as past-examination papers and mark schemes (R7). He justified the use of the latter as a response to student requests and a good way to prepare them for the examination (O2): "once you are ready, you just practice past papers, because they are the best way to get you the most experience of examination-style questions" (I). In several lessons, we also observed him trying to address connections between mathematical ideas (R2). When, for example, he was asked about his choice of functions that he drew on *Autograph*, he responded: "what I'm also doing there, I am also reinforcing or going back over, making sure that they know about their transformations, so I'm kind of teaching two topics at once, so although we are doing this volume of revolution, I am also reminding them of what they do when they do their transformations because I know they are going to get asked about that one" (I).

In the interview and across the observed lessons, George expressed confidence when using *Autograph* with a range of mathematical topics he aimed to introduce in his lessons (A2), as he acknowledged: "knowing that the software was there and it was easy to use, and so I can just show the graph and we can go from there" (I). For example, he employed the software to orchestrate the need of a new definition, to invite conjectures, or to explain a formula (R3): "with *Autograph* I know that I can just go there and five minutes before the lesson, I can go right I need this, this, this and this and it will be ready" (I).

In the lessons, we observed George's navigating across different resources, for example, by adding notes on an image projected on the whiteboard from *Autograph* or the textbook; by drawing on the board; and by experimenting on the software with problems seen in the exam questions (R9). Such navigation across resources was not always planned in advance, as we exemplify in the next section. From the interviews, as well as lesson observations, George expressed his appreciation of the software (O1), emphasis on examination papers (O2), and valuing connecting ideas to students' previous knowledge (O3). We consider these operational invariants of his scheme of use of resources (see Table 2).

He chose a more specific teaching aim for each lesson, based on the topic from the examination requirements he was planning to teach that day (A2). His rules of actions always started by introducing mathematical ideas with examples on the board (R1), inviting students' contributions (R5) and giving them some time to solve textbook exercises independently or in small groups (R8). His rules of actions also included connecting mathematical ideas, for example, by connecting a new mathematical topic to students' previous knowledge (R2); using the software to orchestrate the need of a new definition, to invite conjectures, to explain a formula, or to visualize figures (R3); using calculators and notebooks for quick calculations and reference (R4); responding to students' contributions (R6); and navigating across the available resources (R9).

The inference component of the scheme of use has not been included in the macro-level documentation work summarized in Table 2. Although there were opportunities in which we observed George adapting his activity to specific features of a given situation corresponding to the same aim—for example, in the post-observation interview, George expressed that if one activity did not work well, he would amend it based on student feedback—we recorded these inferences in the context of specific lessons (micro-level), as we elaborate in the next section, but not at the macro-level.

We look now at George's documentation work more closely in specific lessons with the aim to identify shifts in his scheming. For the purpose of this article, we have chosen a discussion of three lessons on volume of revolution from the set of eleven of his that we observed. As mentioned earlier in the method section, this choice was due to having observed a series of three lessons on the same topic in which we were able to seek evidence of George's scheming and re-scheming his use of resources in lessons with the same aim, as we explain in the next section.

George's Scheming and Re-scheming in the Introduction to the Volume of Revolution

Each lesson on the volume of revolution lasted for 50 min. The first and third of these lessons were taught to one group (G1) and the second to another group (G2). Both were groups of Year 13 students. In the first and the second lesson, George introduced the idea of volume of revolution to the corresponding groups, while in the third lesson, he reviewed the same idea with G1 students. Over the three lessons, in addition to the resources used across the eleven lessons (see Table 1), George also used formula sheets, personal website, school website and formulae cards. The formulae cards were on display next to the board all the time, but shown to the students only in the second and third lessons, while the formula sheets were printed and handed out to students and used in the first and second lessons. Overall, the resources stayed almost the same across the three lessons.

We discuss here George's scheme of use of these three lessons with reference to his macro-level documentation work summarized in Table 2. His general aim, across his teaching, was to prepare students for the examinations (A1), while the specific aim of these lessons is to introduce students to the volume of revolution, which is one of the required exam topics (A2 in the context of the three lessons we micro-analyze). His rules of actions included those we had seen across lessons (see Table 2), as adapted to the context of these three specific lessons.

For example, the rule of action of introducing the volume of revolution with examples on the board (R1), in connection with students' previous knowledge (R2) and with the use of Autograph (R3), was materialized through specific actions, such as reminding the students the formula used to find the area of a circle; showing a previously drawn graph in 3D on *Autograph* (e.g., the rotation of $y = x(3 - x)$ around the x-axis); explaining that his aim was to find the area underneath the graph and that he would use the trapezium rule which the students had learnt about in previous lessons and Simpson's rule on the software; explaining that Simpson's rule is more accurate than the trapezium rule in this case; rotating the shaded area around the x-axis; showing the students different positions and rotations of the shaded area that looked like a "pointy sphere," a "Pacman," or a "smarty" as he described them to the students; and showing another example prepared previously for the graph of $y = \sqrt{x}$.

His operational invariants include general invariants, such as George's appreciation of Autograph (O1) and his value on connections of new ideas to students' previous knowledge (O3) adapted to the specific aim of these lessons of introducing the volume of revolution. Specifically, George acknowledged in the interview that the software helped students visualize the volume of revolution in 3D; with it, he could "make weird shapes and have a bit of fun" (I); the use of familiar graph on the software helped to "reinforce previous knowledge" (I); his use of pre-prepared graphs helped in focusing on the new topic without overloading the students with information:

> At that point, I am then thinking about: what do I want them to learn today? ... What that lesson is all about I want them to learn that I don't want to get too bogged down into what's going on here, because that's too many things for them to understand. So, yeah, I wouldn't start off with something completely new. (I)

Regarding the operational invariant on the benefits of using past-examination questions (O2), he raised in the interview that such use was in response to students' needs and to give them some practice for the examination. Specifically, George mentioned that students felt that not all textbook questions were examination-style questions, and some were even "harder" than the latter. As a result, he chose to use past-examination questions for practice on every topic he taught:

> I think the teaching has influenced what we have done. So, in the past, we used to use the textbooks and, at the end of every chapter, there are revision questions, and we made the students do those. But, then the students said they didn't like them, because they were a little bit harder than they needed to be and they were a different style from exam questions. So, then we said OK then we'll get some exam questions and put the exam questions that fit that chapter, so that's where the end of chapter exam questions came from. So, it was the students who kind of determined that and then the use of past papers. I think all schools do it once you are ready, you just practice past papers because they are the best way to get you the most experience of exam-style questions. (I)

In the first lesson, we observed tensions in the explanation of the formula for the volume of revolution that was not planned in advance. Also, we saw George dealing with an unexpected question from a student. We consider these as contingent moments with potential for of scheming and re-scheming that may have resulted in adaptations in the scheme of use. In the next two subsections, we focus on a pair of episodes initiated by these two contingent moments triggered in the first lesson and their impact on the subsequent two lessons.

Episode 1: Introducing the Formula of the Volume of Revolution (Autograph vs Textbook)

The episode we discuss here started in the first lesson with G1 and continued in lesson two with G2 and lesson three with G1 again and regards George's explanation of

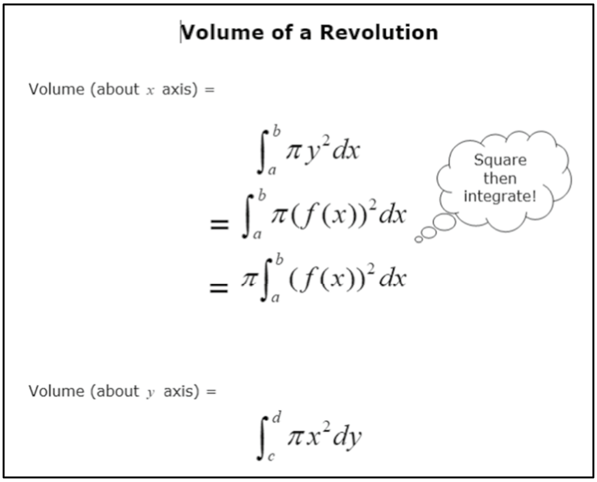


Fig. 1 Formula sheet for volume of revolution, prepared by George

the formula of the volume of revolution to his students (Fig. 1). George had already demonstrated some examples of graph rotation in 3D on Autograph (e.g., the rotation of $y = x(3 - x)$ around the x-axis), he had reminded the students of the formula used to find the area of the circle, and other methods of calculating the area and the next step was to explain how the volume of revolution is calculated and why we use the formulae in Fig. 1.

For this purpose, he offered the pre-prepared graph for $y = \sqrt{x}$ on 3D *Autograph* and showed the students the rotation of a specific area between this graph and the x-axis, around the x-axis (Fig. 2). He then rotated the same graph around the x-axis

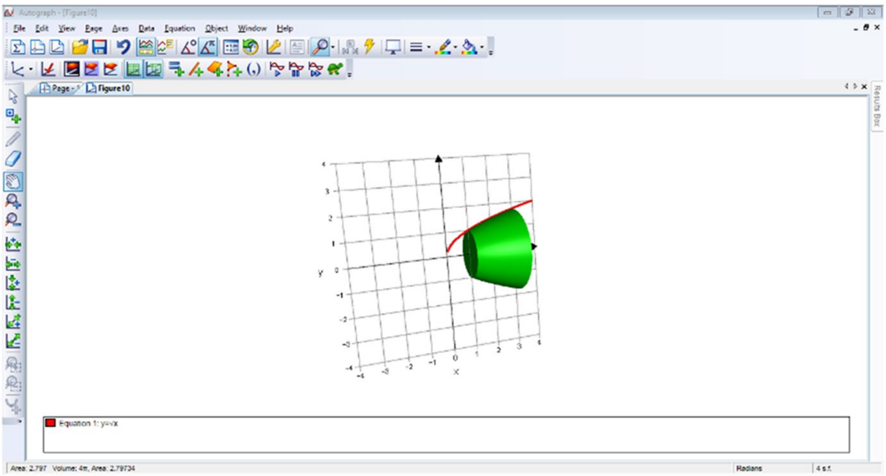


Fig. 2 The rotation of the area between $y = \sqrt{x}$ and the x-axis, around the x-axis (reproduction of George's work)

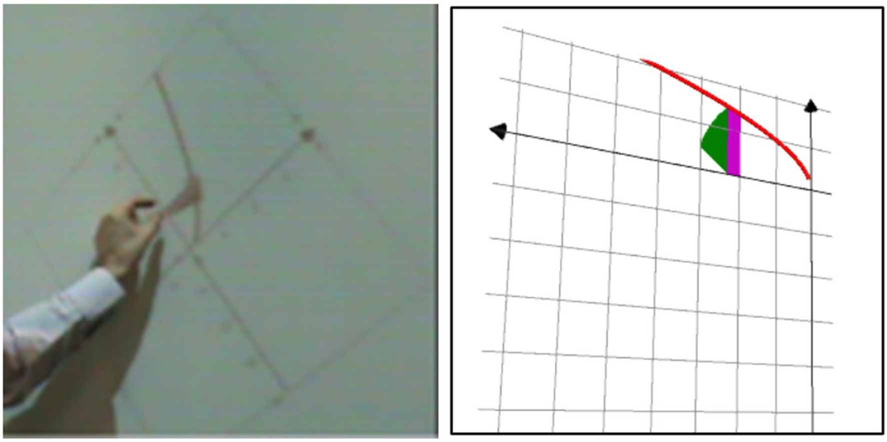


Fig. 3 George's unsuccessful attempt to show a thin slice of the shaded area on Autograph (actual attempt to the left, reconstruction to the right)

with a small angle this time in order to show a sector of the shape resulted from the rotation, what he called a "thin slice" (O, L4). He started dragging the end of the arc further from the axes, changing the boundaries to make the shaded area smaller and then bigger. He again dragged a point to show a thin sector of the shaded area (Fig. 3) by trying different commands in *Autograph*. It seemed that George was aiming to demonstrate tiny disks of the 3D shapes and the area of each can be calculated with the formula of the circle (πr^2). The integral would be the total of these tiny shapes that would then explain the formula in Fig. 1. However, it seemed that the actions he took with the *Autograph* produced only a slice of the disk (Fig. 3) that was not enough to employ the formula of the circle. George was not satisfied with the demonstration. He again tried to explain how the formula could be concluded by pointing to what he already graphed on Autograph, highlighting that the formula would include integrating πr^2 between the limits given in the question:

> So, what I'd like to imagine is actually imagine that was a slither, a really, really thin slice, it's like a disk and so if it was infinitely thin then the area would be, sorry the volume would be that multiplied by, I guess the height, OK? So, the radius is whatever really $\sqrt{x}$ is, so the y value. The height there or the width is the difference between the two really small *xs* [the x coordinates]… Now the whole point of integration, I think you guys learnt kind of that integration is the opposite of differentiation. So, what integration really is, is when we integrate something we're integrating to get the area, all of these really, really thin strips, like the trapeziums. But what we do with it, what the integration does is it makes those strips infinitely thin and then just adds them all up, OK? So, what we get here is we've got lots of infinitely thin strips and, or disks and each disk has got an area of that. So, where we get to is we get to this equation here… And, hopefully what you can see in there is that the volume of this thing is. Let me move it out again, not it's not working, OK. (O, L4)

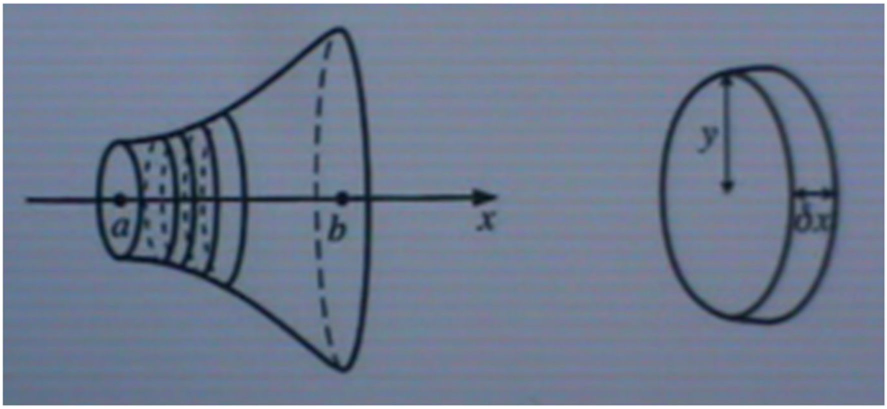
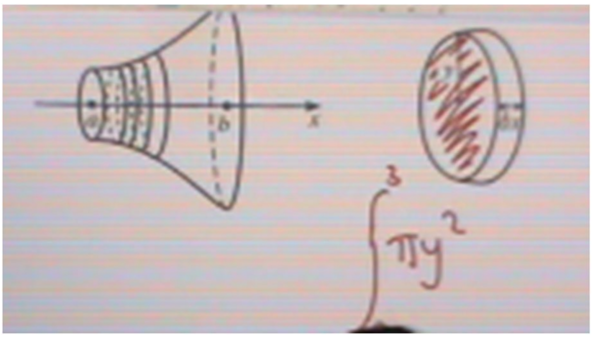


Fig. 4 Diagram illustrating the formula of volume of revolution in the textbook by Wiseman and Searle (2005, p. 108)

Then, George invited his students to practice some textbook questions (Wiseman & Searle, 2005). This is when he noticed a figure in the textbook (Fig. 4, p. 108) illustrating the formula for volume of revolution. As an immediate reaction, he used this image to conclude the explanation of the formula he had initiated earlier with *Autograph*.

In the second lesson with G2, George used the software to visualize the rotation of the graphs, but did not attempt to use it to introduce the formula for volume of revolution, as he had in first lesson. He used only the textbook diagram (Fig. 4) for that purpose, which he had scanned from the textbook and then displayed on the interactive whiteboard. He explained that the formula of the volume of revolution could be viewed as "lots of slices" being added up using integration. He asked the students to consider one "slice" and think of its area which was the area of a circle (πr^2). He demonstrated that instead of r, it is y in this case (Fig. 5). Then, he used the equation he entered on Autograph to find the volume of revolution

Fig. 5 Textbook diagram used by George on the interactive whiteboard



and to demonstrate to the students the limits of the integration for the example of $y = x(3 - x)$ and the calculation of the volume $V = \int_0^3 \pi y^2 dx$.

In the third lesson, again with G1, he continued working on the volume of revolution topic by quickly showing the graph of the function $y = x(x - 3)$ and rotations using the software, in a way similar to the way he followed in the first and second lessons. However, when he returned to the formula, he drew on the textbook diagram to explain one more time how it was derived.

We discussed his shift from *Autograph* to the textbook in his effort to explain the formula and he commented that:

> I think that textbook image shows it cut up a little bit easier. Again, the idea of using *Autograph* here is to give them that kind of hook, that interesting thing, so they can see exactly what we have going on here. After that, it's then more getting into the nuts and bolts of it, what's the maths behind this, what do they have to do and this idea of that being in slices to show them the little bits and to convince them that, actually, when they are doing that, it's the 'what is it?' it's the volume is pi times r squared. See, there's your, well, not *r* but *y* squared isn't it and that *y* is the radius of all these little strips, so to make the connection between area of a circle and then volume of this revolution, so at that point the textbook diagram was better than anything I could do on *Autograph*. (I)

If we revisit episode one with attention to George's scheming across the three lessons, we see that "use *Autograph* to explain a formula" (R3, Table 2), in our case the formula of volume of revolution around the *x*-axis, was observed as a rule of action in the first lesson. This is in agreement with what we have seen of George across the observations. However, this rule of action was shifted in the following lessons. The unexpected visualization on *Autograph* and his challenge with controlling the graph generated a contingent moment when he realized that *Autograph* (or his use of it) could not demonstrate the thin disk of the shape he was aiming for (*responding to the (un)availability of tools and resources*). He did not manage to adjust the demonstration on the spot.

The textbook diagram (see Fig. 4) that appeared spontaneously in the first lesson—another contingent moment for George—shifted his actions. Initially, it would appear as a momentary action applicable only in the context of the first lesson, as what Rowland and colleagues (Rowland et al., 2015) call "contingent action". However, following George in the other two lessons with the same aim, we observed him using the textbook diagram to explain the formula of volume of revolution. So, this diagram became part of his repertoire of resources and the use of the diagram became a rule of action in lessons two and three.

In the interview, when we asked George about the choice of the textbook in these lessons, he commented that the textbook diagram "shows it cut up a little bit easier […] and was better than anything I could do on *Autograph*" (I). So, George's initial inference from the first lesson is that he found the textbook diagram more helpful in explaining the formula of the volume of revolution. He added that he found it useful to have "both *Autograph* and the textbook":

> The *Autograph* bit shows it well in 3D, it shows the whole rotating business, but it doesn't show those disks and so where the idea of the formula comes from. […] I think it is useful to have both. It would be brilliant, I guess, to be able to overlay that image on here, so you could really make that connection, but yeah it's something I've not done yet and, at this point, it is then showing actually we can make all these weird different shapes and have a bit of fun with it. (I)

This leads us to attribute one more inference to his scheme of use: it is useful to use both software and textbook for the introduction of the volume of revolution and its formula. These inferences became operational invariants for lessons two and three. George expressed that he would follow this combination in future lessons. So, the micro-level analysis provided us with evidence of a shift in the use of resources for the introduction of the volume of revolution—this is what we consider an evidence of re-scheming. Furthermore, George's reflection in the interview indicates that such re-scheming gained stability in his document for the introduction to the volume of revolution and is expected to become part of his work in future lessons.

Episode 2: the Impact of Student Questions

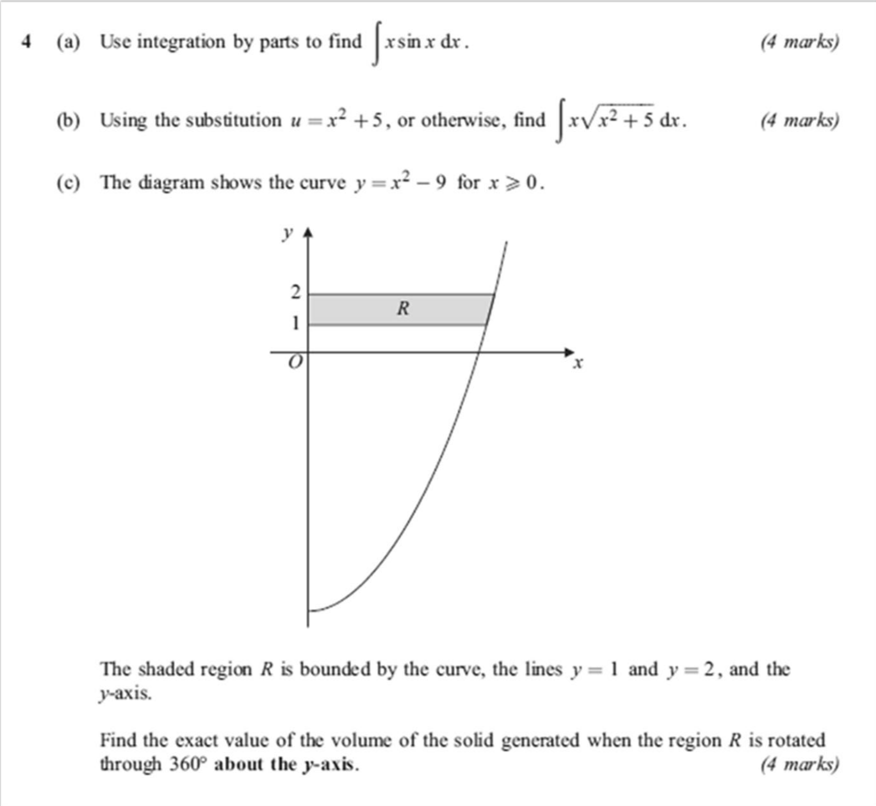
In the first lesson, after the introduction of the formula, George proceeded with past-examination questions. He displayed the school website on the interactive whiteboard where such questions are archived, and chose the question in Fig. 6 to solve on the board.

In part (c) of the question in Fig. 6, the curve $y = x^2 - 9$ was given. George commented that the students needed to find x^2 for this question and wrote $x^2 = y + 9$. So, $V = \ldots$. At this point, a student (student A) asked how he could find the volume of revolution for the shaded area shown in Fig. 6 when the rotation was around the x-axis:

> The graph on the board, if you rotated that around $y = 0$ to get like a doughnut shape… you could do the whole thing and then minus the bottom. So, you could do rotation between zero and one and then between zero and two and take them away from each other, but will it work? I don't know…(O, L4).

George reacted by saying "you could… cylinder," but student A objected the "cylinder" idea because "the edge was slanted." Then, the following exchanges took place:

> George: You confused two types of questions; I think so. This rotates vertically, whereas you're asking me about rotating around the x-axis. Let me see if there is one like that …
> Student A: I don't think there will be because there is a gap in the middle.
> George: It is quite easy to do because you could do your volume of the whole thing take away the cylinder. And the cylinder would be quite easy because it's just what's the volume of the cylinder? Pi r squared h. So, your r here is constant throughout, you're literally just gonna work that out. Your h is whatever



4 (a) Use integration by parts to find $\int x \sin x \, dx$. *(4 marks)*

(b) Using the substitution $u = x^2 + 5$, or otherwise, find $\int x\sqrt{x^2+5}\, dx$. *(4 marks)*

(c) The diagram shows the curve $y = x^2 - 9$ for $x \geqslant 0$.

The shaded region R is bounded by the curve, the lines $y = 1$ and $y = 2$, and the y-axis.

Find the exact value of the volume of the solid generated when the region R is rotated through 360° **about the y-axis.** *(4 marks)*

Fig. 6 A past-examination question chosen by George to solve on the board

the length of the cylinder is. So, you do the big volume, the whole thing, take away the volume of the cylinder,
Student B: He's saying you're wrong, sir!
Student A: You're assuming it's a cylinder when it's not… it's slanted, it's not exact… (O, L7)

George then went on the school website looking at past-examination questions for a similar question to what student A had asked about, but he could not find one. He only found what he described as "classic questions." Then, he returned to *Autograph* with the graph of $y = x(x-3)$, which earlier in the lesson he had rotated around the x-axis (Fig. 7). But this time, he rotated the part of the graph within the interval [0,3] around a line parallel to the x-axis ($y = 1$) and got the shape in Fig. 8. The 3D shape produced on *Autograph* was not what student A asked about:

Student A: This is bigger in the middle; I think it's just confusing.
George: Yes, it is confusing.

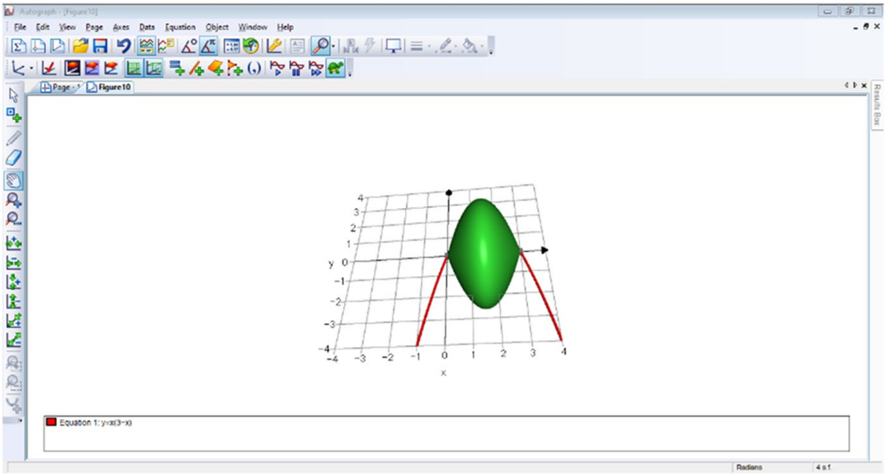


Fig. 7 Rotation of $y = x(3 - x)$ around the x-axis

Then, George went back to the original rotation he produced for $y = x(x - 3)$ around the x-axis (Fig. 7).

Student B: Delete it all, and start doing it again.
George: I'm not sure if there is a way of making a hole in the middle. So, it doesn't work, don't even ask… [Student A] for your one, for that one …
Student A: It's fine, I don't need to know, it's just this… no one will get a question like that [in the exams].

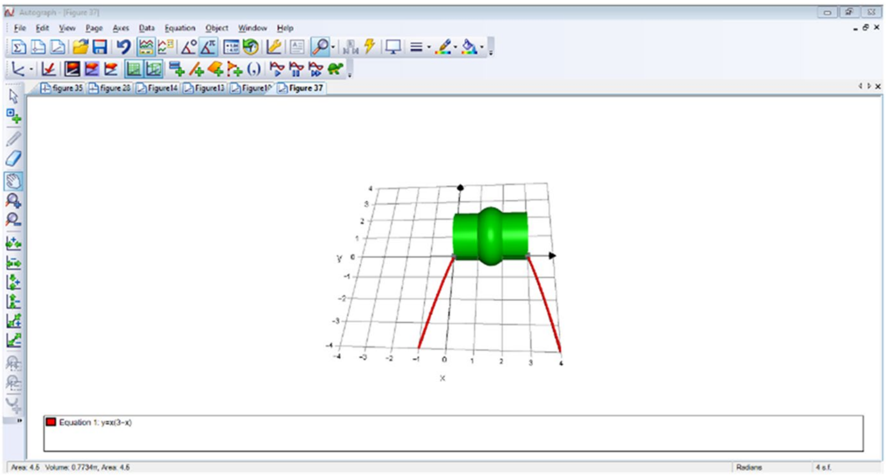


Fig. 8 The shape George created on Autograph in an attempt to answer student A's question (reproduction of George's work)

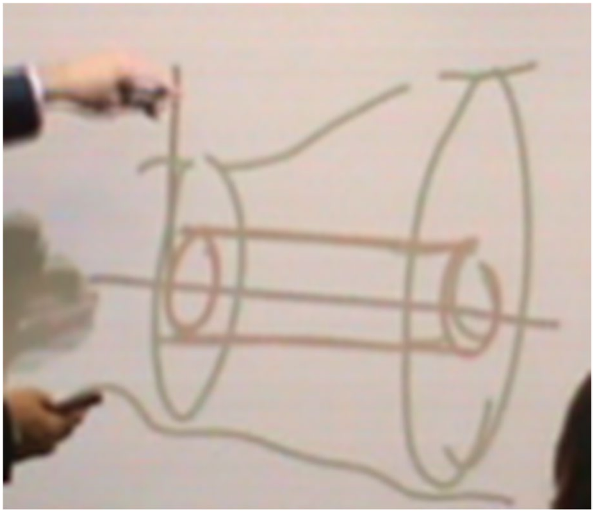


Fig. 9 A shape sketched by George during his second lesson on volume of revolution

> George: You won't get a question like that [in the exams], because you need to know that, end of it… What we're gonna do now, find your *y* squared or your *x* squared in this case, plug it in and work it out. (O, L4)

In lesson 2, with G2, George returned to the question asked by students in the previous lesson, with G1. He mentioned that one student asked about how to find the volume of revolution of a shaded area that leaves a gap when it is rotated around the *x*-axis. But, instead of using the image in Fig. 6 that had motivated student A's question, he used a different shape that he drew quickly on the board (see Fig. 9). A student voluntarily said "cylinder." The teacher responded "Exactly, exactly, exactly, you do your volume of revolution as normal, take away that cylinder…" (O, L9). Later, in the third lesson, George did not return to student A's question and he did not mention the example in Fig. 9 either.

If we revisit episode two with attention to George's scheming across the three lessons, we see that "respond to students' contributions" (R6, Table 2) was observed as a rule of action in the first lesson. George attempted a response to student A's question on the board without success, while student A kept challenging George's explanation. This was a contingent moment (*responding to students' ideas*) that he tried to address on the spot. He interrupted the flow of the lesson (*deviation from agenda*) and took the opportunity to discuss the alternative problem that was proposed by student A (*use of opportunities*).

He consulted the school website to find a similar problem in past-examination questions (R7, Table 2), again without success (*responding to the (un)availability of tools and resources*). Then, he attempted an example in *Autograph* (R3 and R9, Table 2) and encouraged students' contributions to help with the graph (R5, Table 2). However, the commands he used on the software did not give the shape student A had asked about (*responding to the (un)availability of tools and resources*). This led

George to say that the question student A had asked was not required for the examinations and reassured students who were worried (R7, Table 2).

Initially, it would appear that the opportunity of the contingent moment, what Rowland and colleagues (Rowland et al., 2015) call "contingent potential," was missed in the first lesson and was not transformed to a contingent action. George, in consistency with his overarching aim to prepare the students for the examinations, marginalized the problem as one that is not required for the examinations. However, this contingent moment followed George's actions in the next lesson, where we can see how student A's question from the first lesson was perceived by George and then how his interpretation of this question became a resource for the following lesson. His revised version of the problem was closer to what is asked in examinations. So, the micro-level analysis provided us with evidence of a shift in the use of resources on tasks on the volume of revolution.

Similar to episode one, we consider this observation as an instance of re-scheming in which the question from student A was transformed by George to a new exam-type task. However, in episode two as opposed to episode one, we do not have enough evidence to claim that this re-scheming gained stability and became part of George's document. We can claim, though, that his overarching aim of preparing students for the examinations impacted on his reaction to student A and his subsequent choice of task in the second lesson.

Scheming and Re-scheming Across Lessons and the Teacher's Scheme of Use of Resources

In this article, we use the Documentational Approach to Didactics (Gueudet, 2017, 2019; Gueudet & Trouche, 2009) to identify one teacher's (George) scheme of use of resources at a macro-level in a series of lessons with the same overarching aim of preparing the students for the examinations, and shifts in such use at a micro-level in a set of consecutive lessons on the same mathematical topic, the volume of revolution, within the same series of lessons. Especially for the latter, we studied such shifts in relation to contingent moments the teacher faced in his class.

To describe such shifts, and inspired by Adler's (2000) view of "the verb re-source, to source again or differently" (p. 207) that captures the dynamic nature of the use of resources, we introduced the terms *scheming* and *re-scheming*. Scheming captures the emerging characteristics of a teacher's scheme of use. In practice, scheming in individual lessons or series of lessons on the same topic is used for the identification of stable characteristics of the scheme of use across lessons with the same aim, and, vice versa, the scheme of use provides the frame that explains teacher's actions in individual lessons. Thus, scheming brings together the macro-level view of the scheme of use and the micro-level of certain lessons.

Re-scheming, on the other hand, reflects the shifts in the scheming as a response to unexpected situations. Scheming and re-scheming might be momentary and temporary with the potential to gain stability, while the scheme of use describes the stable organization for facing a given class of situations (Gueudet, 2017). Thus, re-scheming may involve shifts in a scheme of use for a class of similar situations in the future. It can be a result of an inference teachers reach in one lesson and decide to act on by changing their scheme of use for the next lessons on the same or similar topic.

For example, George's lessons aimed to prepare students for the examination, and he introduced mathematical topics that are in the requirements of these examinations. Across lessons, he used *Autograph* to orchestrate a need of a new definition to explain formulae or to visualize figures. However, in the three lessons on volume of revolution, George re-schemed his second lesson based on his experience in the first lesson. In episode 1, he tried to use *Autograph* to explain the formula of volume of revolution. However, he felt that his students could not see the link between the software demonstration and the formula of volume of revolution.

Contingency was evident during the unsuccessful use of *Autograph* (unavailability of tools). Having spotted the textbook diagram, he then responded to the availability of resources (i.e., the textbook diagram) and offered an alternative demonstration. Due to software image not working as he expected, George decided not to use it for that purpose in his second and third lessons, and to offer an image from the textbook instead. Through the lens of the DAD (Gueudet, 2017), we spotted the inference George had reached based on his teaching experience, and this inference led him to avoid using *Autograph* to explain the formula for volume of revolution in the consequent lessons.

It seems that the inference, in this case, became an operational invariant later, with the teacher believing that relying on the textbook image to explain the formula was better than "anything" he could use on *Autograph*. Hence, episode 1 shows how George's use of this software was re-thought and built with and around the other resources in the lesson. Another example is related to George's use of student contributions as a resource—a result of social and cultural interaction that is available both for teaching and preparation for teaching (Gueudet & Trouche, 2009). Such an approach has the potential of generation of contingent moments, in response to students' ideas.

In episode 2, a student asked a question that George had not anticipated. This question, although initially neglected as not appropriate for the examinations, was adapted to a version that may appear in the examinations and became a resource for George in the second lesson, with a different group. This intentionally or unintentionally modified question was the outcome of George's interaction with the students that eventually became a resource for his teaching, at least for the next lesson. Such a modification of the question might also be the influence of the overarching aim of preparing students for examinations, and giving an indication that the scheme of use at a macro-level and the scheming at a micro-level are interrelated.

The findings from the analysis of George's data (and other data in Kayali, 2020) demonstrate the potential of using DAD together with observations of teachers' work in the class. The interplay between micro- and macro-analysis gives us the opportunity to access shifts at a micro-level and the rationale behind these shifts from one lesson to another (Kayali & Biza, 2018) and inform teachers' work with resources across lessons at a macro-level and, vice versa, see the actions in specific lessons in the frame of the overarching work with resources across lessons.

Scheming and re-scheming precisely describe the emergence and the manifestation of the scheme of use.

This work showed scheming and re-scheming as active learning experiences, offering researchers windows on teachers' knowledge in action and their professional development. We argue that the episodes discussed in this article provide rich narratives that can be used for professional development purposes. Rowland et al. (2015) refer to "the contingent potential of the situation," where teachers' potential development can result from reflecting on the contingent moments and on teachers' corresponding actions. We see this potential in various episodes, not only with the participating teachers reflecting on their own work, but also with other teachers/trainee teachers reflecting on situations derived from these episodes in professional development courses.

Overall, the Documentational Approach to Didactics, together with observations of teachers' work in class, has the potential to capture the complexity of teachers' work with resources and the variability of such work. Especially in the current circumstances, in which teachers have been asked for flexibility between in-class and on-line teaching, and for adaptation of existing resources to new approaches to teaching, looking at how teachers' scheming is of additional interest.

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author declares no competing interests.

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