Homomorphisms between Specht modules of KLR algebras

A thesis submitted to the School of Mathematics at the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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March 2020

Abstract

THIS thesis is concerned with the representation theory of the symmetric groups and related algebras, in particular the combinatorics underlying the representations of the Khovanov-Lauda-Rouquier (KLR) algebras. These algebras are of particular interest since they possess cyclotomic quotients which were shown by Brundan and Kleshchev to be isomorphic to the Ariki-Koike algebras. The Ariki-Koike algebras generalise Iwahori-Hecke algebras of the symmetric group, and so in turn generalise the symmetric groups themselves. Via this isomorphism, we are able to utilise the grading of the KLR algebras in the setting of the Ariki-Koike algebras, and thus study graded Specht modules.

Specht modules of the KLR algebras admit a definition which lends them well to diagrammatic combinatorics. We shall first develop an arsenal of combinatorial lemmas related to the manipulation of braid diagrams. Then, we will use these to demonstrate the existence of explicit homomorphisms between Specht modules of certain KLR algebras, related to moving particular shapes between the multipartitions associated to these Specht modules. We shall begin by considering moving single nodes between bipartitions, but eventually consider moving multiple large connected shapes of nodes between components of multipartitions in higher levels.

We will then use the obtained homomorphisms to investigate the homomorphism spaces between Specht modules that lie in core blocks of level 2 KLR algebras whose base tuples consists entirely of zeroes. We will completely describe the dominated homomorphism spaces between Specht modules in these blocks. In particular, when the quantum characteristic is not 2 and the multicharge entries are distinct, we will completely describe all homomorphism spaces between Specht modules in these blocks. We will also give a conjecture about replacing the base tuple with any arbitrary base tuple.

Contents

Acknowledgements

FIRST of all, I would of course like to thank my supervisor, Sinéad Lyle, for her support and guidance, for introducing me to an enjoyable area of research, and for being that one person within a fifty mile radius who actually has some sort of idea about the mathematics in this thesis.

I would also like to thank Matt Fayers for his KLR algebra-focused GAP packages. These have been immensely useful in almost all areas of this thesis.

It goes without saying that I would like to thank all of the UEA maths PhD students who I have had the pleasure of meeting during my time here. In particular, thanks go to:

- Mike, for the many games of cribbage.
- Andy, for the abundance of Christmas merriment and general mirth.
- Carl-Fredrik, for teaching me more about semigroups than I ever needed to know, a keen love of words, and for proofreading this whole thing!
- Mark, for being a korfball comrade.
- OG Ben, for the funky tunes, geocaching excitement and all the games of squash we played.
- BBQ Ben, for yet more thrilling games of squash, and always being up for a pub quiz, even though there never was an Ariana Grande round...

More generally, I thank everyone who has made my time at UEA enjoyable. In particular, anyone that I have worked with in the maths department or for outreach, my friends Ben, Adam and Aryan for some entertaining lunches and Skype for Business conversations, and of course the entirety of the UEA Korfball Club.

Finally, I give thanks to my family for not demanding too strongly that I get a 'proper job', and to Amy (who may have been a little more contemptuous towards my state of employment), who has been a constant companion throughout.

Introduction

THE representation theory of the symmetric group $\mathfrak{S}_n$ over the complex numbers can be traced back to the work of Young [You00], Frobenius [Fro03] and Specht [Spe35], whose ideas are still present today. Somewhat more recently, James progressed the topic by working over an arbitrary field, not just the complex numbers, to construct the irreducible modules as quotients of *Specht modules*. A key reference that we will use and a great introduction to James's work on the symmetric group is given in the following reference [Jam78a].

In [DJ86], Dipper and James introduced the Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, which acts as a generalisation of the symmetric group. As such, results in the representation theory of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ can be used to recover corresponding results from the representation theory of $\mathfrak{S}_n$. Subsequently, Murphy [Mur92, Mur95] obtained a basis for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ which gave a new approach to studying the representations of the Iwahori-Hecke algebra. The Murphy basis is an example of a cellular basis, as defined later by Graham and Lehrer [GL96] and we have that the Specht modules here arise as the cell modules.

Generalising further, in [AK94] Ariki and Koike introduced the Ariki-Koike algebra $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. A cellular basis constructed by Dipper, James and Mathas [DJM98] gave the representation theory of the Ariki-Koike algebra a similar framework to that of the Iwahori-Hecke algebra. Again, since the Ariki-Koike algebra is a generalisation of the Iwahori-Hecke algebra, many of the results for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ have corresponding results for $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$, and we have Specht modules arising as cell modules.

In [BK09], Brundan and Kleshchev showed that the Ariki-Koike algebras are isomorphic to certain $\mathbb{Z}$-graded algebras defined by Khovanov and Lauda [KL09] [KL11] and Rouquier [Rou08]. This isomorphism gives a non-trivial $\mathbb{Z}$-grading on $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. In fact we can additionally grade the Specht modules and thus study graded representation theory.

In Chapter 1, we shall define the algebras that we are to work with, along with giving an overview of any background material that we will need in order to study their representation theory. This will include both the algebraic setup that we require along with combinatorial definitions such as braid diagrams for Specht modules of KLR algebras, partitions, tableaux and abacuses. We shall also give a more in-depth definition of the Specht modules for KLR algebras, since these will be the main objects of our study.

Once the necessary background is established, in Chapter 2 we present various ways in which we can manipulate braid diagrams associated to KLR algebras and their Specht modules. When making our way through calculations whilst trying to show the existence of explicit homomorphisms within our KLR algebras, we will often encounter similar looking patterns of strings within our braid diagrams. Thus we aim to establish general methods for dealing with such crossings, as this will be useful for cutting down the amount of work needed in our calculations.

In Chapter 3, we use the processes outlined in the previous chapter in order to prove the existence of explicit homomorphisms between certain Specht modules for KLR algebras where the corresponding pair of multipartitions differ by moving nodes. We will first use our approach in order to find homomorphisms and explicitly state the image of the generator, which arise between one-node Carter-Payne pairs as studied by Lyle and Mathas [LM14]. We will then build upon our techniques so that in Theorem 3.14 and its corollaries we may exhibit new homomorphisms that arise when considering two multipartitions which differ by the moving of a large connected set of nodes. We follow this with a conjecture concerning improving the strictness of the hypotheses of the theorem. We conclude the chapter with a brief discussion about homomorphisms that occur when we do not have the *diagonal residue condition*, by means of exploring some examples.

Finally, in Chapter 4, we consider some recent work by Fayers [Fay06, Fay07] concerning the *core blocks* of the Ariki-Koike and KLR algebras and discuss some relevant ideas using the combinatorics of the abacus in the KLR setting, based on personal communication with Sinéad Lyle. We shall show that within these core blocks, the homomorphisms constructed in the previous chapter arise, and Theorem 4.26 will allow us to describe the entire set of *dominated* homomorphism spaces within certain classes of core blocks. Due to a result of Speyer [FS16], we are able to state a condition for when this set coincides with the entire set of homomorphism spaces in Theorem 4.27. We finish by discussing the possibilities of extending the results of this chapter to all core blocks, exploring some new examples with links to those at the end of the previous chapter, and ultimately stating a relevant conjecture.

Chapter 1

Background

In this chapter we will state the necessary background information related to the algebras that we will work with, and the relevant combinatorial ideas that we will need. In particular, we will detail the definition of a Specht module for a KLR algebra and describe the combinatorics involved. Some more general details related to the background, e.g. the definition of an algebra defined by generators and relations, can be found in [EGH+11].

1.1 The symmetric group

Fix $n \geq 1$ and let $\mathfrak{S}_n$ be the symmetric group of degree n. For $1 \leq i < n$ let s_i be the transposition $(i, i+1)$. Then $\mathfrak{S}_n$ is generated by the elements $s_1, \dots, s_{n-1}$ subject to the relations:

$$\begin{aligned} s_i^2 &= 1, && 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, && 1 \leq i < j-1 \leq n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && 1 \leq i \leq n-2. \end{aligned}$$

For $w \in \mathfrak{S}_n$, write $w = s_{i_1} \cdots s_{i_k}$ for some $k \geq 0$. If k is minimal then we say that $s_{i_1} \cdots s_{i_k}$ is a *reduced expression* for w and we say that w has *length* k and write $l(w) = k$. The identity element 1 has length 0. Given $w \in \mathfrak{S}_n$, we define the *signature* of w to be $\operatorname{sgn}(w) \coloneqq (-1)^{l(w)}$.

Given $w, w' \in \mathfrak{S}_n$, then we say that w is greater than w' in the *Bruhat order* (and write $w \succeq w'$) if there is a reduced expression for w such that w' can be obtained as a subexpression of this reduced expression. If $w \succeq w'$ then we also write $w' \preceq w$ and say w' is smaller than w in the Bruhat order.

1.2 The Iwahori-Hecke algebra

Definition 1.1. Let $\mathbb{F}$ be a field and let q be an arbitrary non-zero element of $\mathbb{F}$. The *Iwahori-Hecke algebra* $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is the unital associative $\mathbb{F}$-algebra with generators $T_1, \dots, T_{n-1}$ and relations:

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, && 1 \le i \le n-1, \\ T_iT_j &= T_jT_i, && 1 \le i < j-1 \le n-2, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && 1 \le i \le n-2. \end{aligned}$$

For brevity, we may write $\mathcal{H}$ for $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. When $q = 1$, the first relation becomes $T_i^2 = 1$, and so in this case $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is isomorphic to the group algebra $\mathbb{F}\mathfrak{S}_n$.

Definition 1.2. Define $e \in \{2, 3, 4, \dots\}$ to be the *quantum characteristic* of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, that is, the smallest integer e such that

$$1 + q + q^2 + \ldots + q^{e-1} = 0.$$

If no such integer exists, let $e = \infty$. Note that if $q = 1$ (as in the case of $\mathfrak{S}_n$) then $e = \operatorname{char} \mathbb{F}$.

Suppose $w \in \mathfrak{S}_n$ and let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression for w. We define

$$T_w = T_{i_1} \cdots T_{i_k}.$$

If w is the identity element of $\mathfrak{S}_n$ then we identify T_w with the identity element of $\mathbb{F}$. By Matsumoto's Theorem for reduced expressions [Mat64], we have that T_w is independent of the choice of reduced expression for w and hence is well-defined.

The following result details how we perform right multiplication in $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.

Proposition 1.3. *[Mat99, Lemma 1.1.2] Let $w \in \mathfrak{S}_n$, then*

$$T_wT_{s_i} = \begin{cases} T_{ws_i}, & \text{if } l(ws_i) > l(w), \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws_i) < l(w). \end{cases}$$

Example 1.4. Let $w = (1,2,3) = (2,3)(1,2)$ and consider $s_2 = (2,3)$. Then $ws_2 = (1,3) = (2,3)(1,2)(2,3)$, so $l(ws_2) = 3 > 2 = l(w)$ hence

$$T_{(1,2,3)}T_{(2,3)} = T_{(1,3)}.$$

If instead we consider $s_1 = (1,2)$, then $ws_1 = (2,3)$, so $l(ws_1) = 1 < 2 = l(w)$ hence

$$T_{(1,2,3)}T_{(1,2)} = qT_{(2,3)} + (q-1)T_{(1,2,3)}.$$

$\diamondsuit$

By Lemma 1.3 we see that the elements T_w for $w \in \mathfrak{S}_n$ span $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. It can also be shown that they are linearly independent to obtain the following theorem.

Theorem 1.5. *[Mat99, Theorem 1.1.3] The Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ is free as an $\mathbb{F}$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$.*

1.3 The Ariki-Koike algebra

Definition 1.6. Let $\mathbb{F}$ be a field, let q be a non-zero element of $\mathbb{F}$, let $l \geq 1$ and let $Q = (Q_1, \dots, Q_l) \in \mathbb{F}^l$ with $Q_i \neq 0$ for $0 \leq i \leq l$. The *Ariki-Koike algebra* $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ is the unital associative $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$ and relations:

$$\begin{aligned}
(T_0 - Q_1)(T_0 - Q_2)\dots(T_0 - Q_l) &= 1, & \\
(T_i - q)(T_i + 1) &= 0, & 1 \leq i \leq n-1, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, & \\
T_iT_j &= T_jT_i, & 0 \leq i < j-1 \leq n-2, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, & 1 \leq i \leq n-2.
\end{aligned}$$

We may write just $\mathcal{H}_{\mathbb{F},q,Q}$ for $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. Note in the case when $l = 1$, the first relation becomes $T_0 = 1 + Q_1 \in \mathbb{F}$, thus the third relation is trivially satisfied, and so we recover the presentation for the Iwahori-Hecke algebra.

We define the *quantum characteristic* e of the Ariki-Koike algebra identically to that of the Iwahori-Hecke algebra. Similarly, for $w \in \mathfrak{S}_n$ we set $T_w = T_{i_1} \cdots T_{i_k}$ where $s_{i_1} \cdots s_{i_k}$ is a reduced expressions for w, and we have the same multiplication formula as given by Proposition 1.3.

We say Q is *q-connected* if for each i, $Q_i = q^{a_i}$ for some $a_i \in \mathbb{Z}$. In [DM02], Dipper and Mathas prove that any Ariki-Koike algebra is Morita equivalent to a direct sum of tensor products of smaller Ariki-Koike algebras, each of which has q-connected parameters. Thus we may assume that we are always working with a Ariki-Koike algebra where each Q_i is an integral power of q.

We call l the *level* of $\mathcal{H}_{\mathbb{F},q,Q}$. Given $e \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ set $I = \mathbb{Z}/e\mathbb{Z}$ (which we identify with $\{0, 1, \ldots, e-1\}$) unless $e = \infty$, in which case set $I = \mathbb{Z}$. $\mathfrak{S}_n$ acts on the left on elements of I^n by place permutations. We call an l-tuple $\kappa = (\kappa_1, \ldots, \kappa_l) \in I^l$ an *e-multicharge of level l*. If we consider a particular choice of $\mathcal{H}_{\mathbb{F},q,Q}$, by assumption there exists $\kappa = (\kappa_1, \ldots, \kappa_l) \in I^l$ such that $Q_i = q^{\kappa_i}$ for every $i \in \{1, \ldots, l\}$. We call this particular κ the *e-multicharge* of level l of $\mathcal{H}_{\mathbb{F},q,Q}$.

1.4 Lie-theoretic setup

Let $e \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and let I be defined as in Section 1.3. Let Γ_e be the quiver with vertex set I and an arrow from i to $i-1$ for $i \in I$. If some $i, j \in I$ are not connected by an edge in Γ_e then we write $i \not\!\!-\, j$. Some examples are shown below.

$$\begin{tikzcd} 0 \arrow[d, shift left] \\ 1 \arrow[u, shift left] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dr] & \\ 1 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 3 \arrow[d] \\ 1 \arrow[u] & 2 \arrow[l] \end{tikzcd}$$

Γ_2 Γ_3 Γ_4

$$\cdots \longleftarrow -1 \longleftarrow 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots$$

Γ_∞

The *Cartan matrix* $(a_{i,j})_{i,j\in I}$ is given by:

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } j \neq i, i \pm 1, \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j, \\ -2 & \text{if } i \leftrightarrows j. \end{cases}$$

As in [Kac90], let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of the Cartan matrix, giving us simple roots α_i and fundamental dominant weights Λ_i for $i \in I$, along with a bilinear form $(\ ,\ )$ satisfying $(\alpha_i, \alpha_j) = a_{i,j}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$ for $i, j \in I$. Define $P^+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i$, the *positive weight lattice* and define $Q^+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$, the *positive root lattice*. For $\alpha = \sum_{i\in I} c_i\alpha_i \in Q^+$, the *height* of α is defined to be $\sum_{i\in I} c_i$.

Consider $\kappa = (\kappa_1, \dots, \kappa_l)$, an e-multicharge of level l, as defined in Section 1.3. We define the corresponding dominant weight to be $\Lambda_\kappa := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_l}$.

1.5 Graded algebras

Definition 1.7. An $\mathbb{F}$-algebra A is $\mathbb{Z}$*-graded* if for every $i \in \mathbb{Z}$ there exists a vector space A_i such that

(i) $A_iA_j \subseteq A_{i+j}$ for every $i, j \in \mathbb{Z}$, and

(ii) there is a direct sum decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ as vector spaces.

We will usually refer to $\mathbb{Z}$-graded algebras simply as *graded algebras*.

Example 1.8. (i) Any algebra A is graded by taking $A_0 = A, A_i = 0$ for $i \neq 0$.

(ii) Let $A = \mathbb{F}[x]$, the polynomial ring with coefficients in $\mathbb{F}$. Let

$$A_i = \begin{cases} \langle x^i \rangle_{\mathbb{F}} & \text{if } i \geq 0, \\ 0 & \text{if } i < 0. \end{cases}$$

Then this defines a grading on A.

$\diamondsuit$

Definition 1.9. An element $a \in A$ is *homogeneous of degree* i if $a \in A_i$. We write $\deg(a) = i$.

Definition 1.10. Let A be a graded $\mathbb{F}$-algebra and let M be an A-module. We say M is $\mathbb{Z}$*-graded* if for every $i \in \mathbb{Z}$ there exists a vector space M_i such that

(i) $M_i A_j \subseteq M_{i+j}$ for every $i, j \in \mathbb{Z}$, and

(ii) there is a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as a vector space.

We will refer to $\mathbb{Z}$-graded modules simply as *graded modules*.

Definition 1.11. Let A be a graded algebra and let M and N be graded A-modules. A homomorphism of A-modules $\varphi\colon M \to N$ is called *homogeneous of degree* d if $\varphi(M_i) \subseteq N_{i+d}$ for every $i \in \mathbb{Z}$.

Definition 1.12. If M is a graded module and $k \in \mathbb{Z}$, then we define $M\langle k\rangle$ to be the module isomorphic to M but whose grading is shifted by k. That is, we have $M\langle k\rangle_d = M_{d-k}$.

1.6 KLR algebras

Suppose we have fixed an $e \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and defined I as in Section 1.3. As in Section 1.4 we have a Cartan matrix $(a_{i,j})_{i,j \in I}$, simple roots α_i and the positive root lattice Q^+.

Definition 1.13. Let $\mathbb{F}$ be a field and suppose $\alpha \in Q^+$ has height n. Define I^α to be the set

$$I^\alpha = \{i \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha\}.$$

Then define $\mathscr{H}_\alpha$ to be the unital associative $\mathbb{F}$-algebra with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

and relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i}), \tag{1.1}$$

$$\sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) = 1, \tag{1.2}$$

$$y_r = e(\mathbf{i})y_r, \tag{1.3}$$

$$\psi_r e(\mathbf{i}) = e(s_r\mathbf{i})\psi_r, \tag{1.4}$$

$$y_r y_s = y_s y_r, \tag{1.5}$$

$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1, \tag{1.6}$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \text{if } |r-s| > 1, \tag{1.7}$$

$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r \psi_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \tag{1.8}$$

$$y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \tag{1.9}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_r \to i_{r+1}, \\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \leftrightarrows i_{r+1}, \\ e(\mathbf{i}), & \text{otherwise,} \end{cases} \tag{1.10}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} \to i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1} + y_r - 2y_{r+1} + y_{r+2})e(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftrightarrows i_{r+1}, \\ (\psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i}), & \text{otherwise,} \end{cases} \tag{1.11}$$

for all $\mathbf{i}, \mathbf{j} \in I^\alpha$ and all admissible r and s.

The *affine Khovanov-Lauda-Rouquier algebra* (or *quiver Hecke algebra*) $\mathscr{H}_n$ is defined to be $\bigoplus_\alpha \mathscr{H}_\alpha$, the sum being over all $\alpha \in Q^+$ of height n.

Once again recall the Lie-theoretic notation of Section 1.4. Given $\alpha \in Q^+$ and

κ, an e-multicharge of level l, we obtain the dominant weight Λ_κ. We define $\mathscr{H}_\alpha^{\Lambda_\kappa}$ to be the quotient of $\mathscr{H}_\alpha$ by the relations

$$y_1^{(\Lambda_\kappa,\alpha_{i_1})}e(\mathbf{i}) = 0 \tag{1.12}$$

for $\mathbf{i} \in I^\alpha$.

Definition 1.14. The *cyclotomic Khovanov-Lauda-Rouquier algebra* $\mathscr{H}_n^{\Lambda_\kappa}$ is defined to be $\bigoplus_\alpha \mathscr{H}_\alpha^{\Lambda_\kappa}$, the sum being over all $\alpha \in Q^+$ of height n.

Note that we will frequently abbreviate Khovanov-Lauda-Rouquier to "KLR". We remark that in the case of $\mathscr{H}_n^{\Lambda_\kappa}$, due to [LM07] or [Bru08, Theorem 1] the blocks are given by the algebras $\mathscr{H}_\alpha^{\Lambda_\kappa}$.

Proposition 1.15. *[BK09, Corollary 1] There is a unique $\mathbb{Z}$-grading on $\mathscr{H}_\alpha^{\Lambda_\kappa}$ such that*

$$\deg(e(\mathbf{i})) = 0, \qquad \deg(y_r) = 2, \qquad \deg(\psi_r e(\mathbf{i})) = -a_{i_r,i_{r+1}},$$

for each admissible r and $\mathbf{i} \in I^\alpha$.

Now we state the theorem which motivates our use of the KLR algebras.

Theorem 1.16. *[BK09, Main Theorem] Suppose the Ariki-Koike algebra $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ has e-multicharge κ. Then $\mathscr{H}_n^{\Lambda_\kappa} \cong \mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$.*

Theorem 1.16 implies that the Ariki-Koike algebras are (non-trivially) $\mathbb{Z}$-graded. As special cases, we have that if $l = 1$ then $\mathscr{H}_n^{\Lambda_\kappa} \cong \mathbb{F}\mathfrak{S}_n$ when $q = 1$ and $\mathscr{H}_n^{\Lambda_\kappa} \cong \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ otherwise.

We will perform multiple calculations within KLR algebras. We will make liberal use of the commuting relation (1.7) without reference. Also, for ease when writing elements, we use the following shorthand notation.

For $a \leq b$,

$$\Psi_a{\uparrow}^b := \psi_a\psi_{a+1}\cdots\psi_b.$$

If $a > b$ then $\Psi_a{\uparrow}^b := 1$. For $a \le b \le c$,

$$(\Psi_b{\uparrow}^c){\downarrow}_a := \Psi_b{\uparrow}^c \, \Psi_{b-1}{\uparrow}^{c-1} \cdots \Psi_a{\uparrow}^{a+c-b} \, .$$

Similarly, for $a \le b$,

$$\Psi^b{\downarrow}_a := \psi_b \psi_{b-1} \cdots \psi_a.$$

If $a > b$ then $\Psi^b{\downarrow}_a := 1$. For $a \le b \le c$,

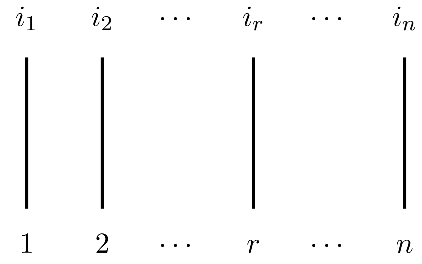
$$(\Psi^c{\downarrow}_b){\uparrow}^a := \Psi^c{\downarrow}_b \, \Psi^{c-1}{\downarrow}_{b-1} \cdots \Psi^{c-b+a}{\downarrow}_a \, .$$

1.7 Braid diagrams

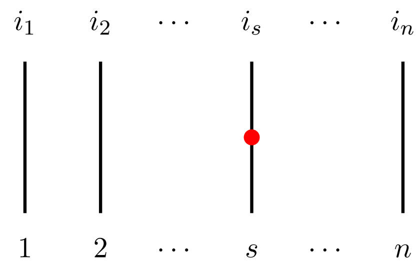
When working with elements of KLR algebras we will prefer to work diagrammatically, and so we associate a braid diagram to each element of $\mathscr{H}_n^{\Lambda_\kappa}$ as in [KL09]. We define an *n-braid diagram* to be a graph whose vertices are labelled by $\{1', 2', \ldots, n', 1, 2, \ldots, n\}$, with every vertex in $\{1', 2', \ldots, n'\}$ connected to a unique vertex in $\{1, 2, \ldots, n\}$, and each edge labelled by an element of I and donning a finite number of dots. If we consider the braid diagram B as a map from $\{1', 2', \ldots, n'\}$ to $\{1, 2, \ldots, n\}$, we obtain a permutation $\pi_{\mathsf{B}} \in \mathfrak{S}_n$.

Since braid diagrams are graphs, we can draw them in the plane, and we do not distinguish between over and under crossings. We place the vertices $\{1', 2', \ldots, n'\}$ at the top of the diagram and the vertices $\{1, 2, \ldots, n\}$ at the bottom, with both sets ordered from left to right in the natural way. We explicitly indicate the labels of the vertices $\{1, \ldots, n\}$ at the bottom of the diagram whilst at the top we label the vertices with $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$. For $r \in \{1, \ldots, n\}$, we call the edge connected to the vertex r the *r-string of residue i*, where i is the component of $\mathbf{i}$ labelling the other vertex of the edge.

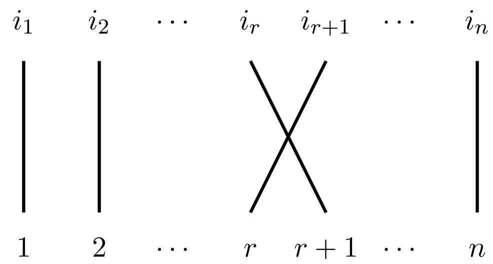
If $\mathbf{i} = (i_1, \ldots, i_n)$, then the braid diagram representing $e(\mathbf{i})$ is

$e(\mathbf{i}) =$  .

Now for $r \in \{1, 2, \ldots, n-1\}$, ψ_r acts on a braid diagram by crossing the r-string with the $(r+1)$-string at the bottom of the diagram. For $s \in \{1, 2, \ldots, n\}$, y_s acts on the braid diagram by adding a dot to the bottom of the s-string. For $\mathbf{j} \in I^n$, $e(\mathbf{j})$ acts as the identity on a diagram if $\mathbf{j} = \pi_{\mathsf{B}}^{-1}\mathbf{i}$, otherwise it acts as zero. The element $e(\mathbf{i})y_s$ can be drawn as

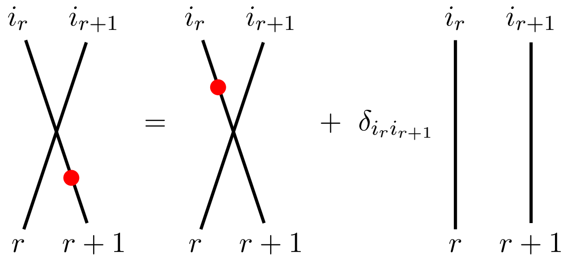
$e(\mathbf{i})y_s =$  ,

whilst $e(\mathbf{i})\psi_r$ can be drawn as

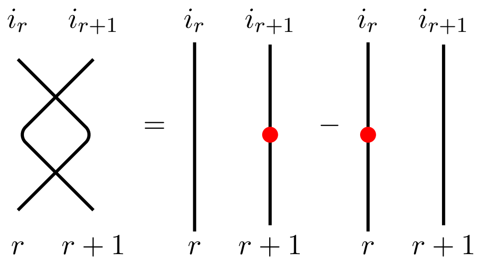
$e(\mathbf{i})\psi_r =$  .

We can multiply diagrams by concatenation from top to bottom and so we have braid diagram versions of the relations (1.1)-(1.12). For example, if we use relation

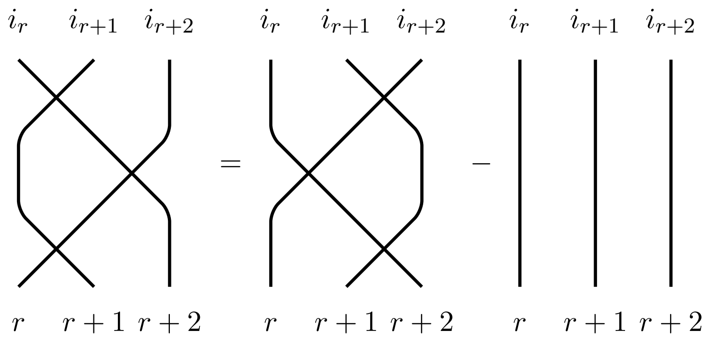
(1.8), part of the diagram will undergo the diagrammatic relation:



Suppose that $i_r \to i_{r+1}$, then when using relation (1.10), part of the diagram will undergo the following:



If we suppose that $i_r = i_{r+2} \to i_{r+1}$, then when using relation (1.11), part of the diagram will undergo the following:



Often we will not draw entire diagrams for elements since there will be many strings which do not undergo any crossings or involve dots and so are of no importance to the calculation in question, just as in the few diagrams directly above.

For $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{r_1} \cdots s_{i_k}$ and define

$$\psi_w := \psi_{r_1} \cdots \psi_{r_k}.$$

Note that ψ_w may depend on the choice of reduced expression for w. We can obtain an associated braid diagram for ψ_w. Proposition 1.18 will be useful in determining when an expression for a permutation is reduced. Let us first define the set $N(w)$ for $w \in \mathfrak{S}_n$;

$$N(w) := \{(i,j) \in \mathfrak{S}_n \mid 1 \leq i < j \leq n \text{ and } iw > jw\},$$

and then state an associated result:

Proposition 1.17. *[Mat98] Suppose that $w \in \mathfrak{S}_n$. Then $l(w) = \#N(w)$.*

Now we can prove the following result that will be of use to us when dealing with reduced expressions.

Proposition 1.18. *A permutation $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ is reduced if and only if in the corresponding braid diagram B, no two strings cross twice.*

Proof. Suppose that two strings cross twice. Then remove the corresponding crossings from the diagram to obtain a new diagram B' such that the length of $\pi_{\mathsf{B}'}$ is two less than that of π_{B}. Hence we have an expression for w that is shorter than that we started with so that $w = s_{i_1} \cdots s_{i_k}$ was not reduced.

Conversely, assume that no two strings cross twice. Then for $i < j$, string i crosses string j if and only if $(i,j) \in N(w)$. Hence $\#N(w) = k$ and so by Proposition 1.17 we have the desired result. $\square$

1.8 Partitions and tableaux

A *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $|\lambda| := \sum_{i \geq 1} \lambda_i = n$. Since $n < \infty$, there is a k such that $\lambda_i = 0$ for $i > k$ and we may write $\lambda = (\lambda_1, \ldots, \lambda_k)$. We write $\varnothing$ for the empty composition $(0, 0, \ldots)$. If a composition has repeated parts we group them together with an index. For example,

$$(5, 5, 3, 1, 1, 1, 0, 0, \ldots) = (5, 5, 3, 1, 1, 1) = (5^2, 3, 1^3).$$

A *partition* of n is a composition of n satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$.

An *l-multicomposition* of n is an ordered l-tuple of compositions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)})$ such that $|\lambda| := |\lambda^{(1)}| + \cdots |\lambda^{(l)}| = n$. If each $\lambda^{(i)}$ is a partition, we say that λ is an *l-multipartition* of n. We may often refer to a 2-multipartition as a *bipartition*.

If λ is a partition, we define the *conjugate partition* λ' to be the partition with ith part

$$\lambda'_i = \#\{j \geq 1 \mid \lambda_j \geq i\}.$$

If λ is an l-multipartition, then the *conjugate multipartition* is defined as

$$\lambda' = (\lambda^{(l)'}, \dots, \lambda^{(1)'}).$$

Given l-multicompositions λ and μ, we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{k=1}^{m-1} |\lambda^{(k)}| + \sum_{i=1}^{s} \lambda_i^{(m)} \geq \sum_{k=1}^{m-1} |\mu^{(k)}| + \sum_{i=1}^{s} \mu_i^{(m)}$$

for all $1 \leq m \leq l$ and $s \geq 1$. If $\lambda \trianglerighteq \mu$ then we also write $\mu \trianglelefteq \lambda$.

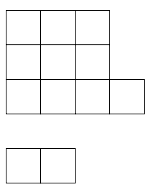
If λ is an l-multicomposition, we define the *diagram* of λ to be

$$[\lambda] = \{(r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, l\} \mid c \leq \lambda_r^{(m)}\}.$$

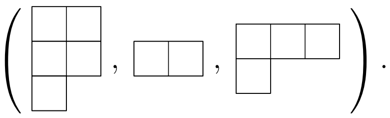
The elements of $[\lambda]$ are called *nodes*. Note that if λ is a composition then we can consider λ as a 1-multicomposition and in this case we identify $[\lambda]$ with

$$\{(r, c) \in \mathbb{N} \times \mathbb{N} \mid c \leq \lambda_r\}.$$

Given a composition we may draw its diagram in the plane, drawing each node as a box, with the r coordinate increasing down the page and the c coordinate increasing from left to right. For example, the diagram of $(3, 3, 4, 0, 2)$ is drawn as follows.



Similarly we can draw the diagram of an l-multicomposition as an l-tuple of the diagrams of its component compositions. For example, the diagram of $((2,2,1),(2),(3,1))$ is drawn as



We say a node (r,c,m) lies *above* (r',c',m') if either $m < m'$ or ($m = m'$ and $r < r'$). Similarly, we say a node (r,c,m) lies *below* (r',c',m') if either $m > m'$ or ($m = m'$ and $r > r'$).

For an l-multipartition λ, we say that an element $B \in \mathbb{N} \times \mathbb{N} \times \{1,\ldots l\}$ is an *addable* node if $B \notin [\lambda]$ and $[\lambda] \cup \{B\}$ is the diagram of a multipartition. We say that a node $A \in [\lambda]$ is *removable* if $[\lambda] \setminus A$ is also the the diagram of a multipartition.

Given a partition λ, let $\lambda_{\hat{k}} := (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \lambda_{k+2}, \ldots)$, i.e. $\left[\lambda_{\hat{k}}\right]$ is $[\lambda]$ with the rightmost node on the kth row removed.

Definition 1.19. Given λ, an l-multipartition of n, a λ-*tableau* is a bijection $\mathfrak{t}\colon [\lambda] \to \{1,\ldots,n\}$. We can represent a λ-tableau $\mathfrak{t}$ by drawing $[\lambda]$ and then filling in the box at position (r,c,m) with its image under $\mathfrak{t}$.

We say a λ-tableau $\mathfrak{t}$ is *row standard* if its entries increase along the rows of each component of its diagram, and we say $\mathfrak{t}$ is *standard* if, in addition to being row standard, its entries increase down the columns of each component of its diagram. We write $\mathrm{Std}(\lambda)$ for the set of standard λ-tableaux.

Example 1.20. Let $\lambda = ((3,2),(1^2))$. Then

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\,,\ \begin{array}{|c|}\hline 6\\\hline 7\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\,,\ \begin{array}{|c|}\hline 7\\\hline 6\\\hline\end{array}\right)\ \text{and}\ \left(\begin{array}{|c|c|c|}\hline 2&7&3\\\hline 1&6\\\cline{1-2}\end{array}\,,\ \begin{array}{|c|}\hline 5\\\hline 4\\\hline\end{array}\right)$$

are examples of λ-tableau, the first of which is standard, the second of which is row standard but not standard, and the third of which is neither. $\diamondsuit$

The symmetric group acts naturally on λ-tableaux on the right by permuting the entries. In Example 1.20 above, the permutation $(2,3,5,4)(6,7)$ sends the first tableau to the second.

Definition 1.21. Let λ be a l-multipartition of n. The *initial tableau* $\mathfrak{t}^\lambda$ is defined to be the tableau obtained by writing the numbers $1, \ldots, n$ in order from left to right, going down the rows of each successive component of λ. Given a λ-tableau $\mathfrak{t}$ we define the permutation $d(\mathfrak{t}) \in \mathfrak{S}_n$ by $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$.

Example 1.22. Let $\lambda = ((2,2,1),(2),(3,1))$, a 3-multipartition of 11. Then we have

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}\ , \ \begin{array}{|c|c|} \hline 6 & 7 \\ \hline \end{array}\ , \ \begin{array}{|c|c|c|} \hline 8 & 9 & 10 \\ \hline 11 \\ \cline{1-1} \end{array} \right).$$

If

$$\mathfrak{t} = \left(\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}\ , \ \begin{array}{|c|c|} \hline 6 & 7 \\ \hline \end{array}\ , \ \begin{array}{|c|c|c|} \hline 8 & 10 & 11 \\ \hline 9 \\ \cline{1-1} \end{array} \right),$$

then $d(\mathfrak{t}) = (2,4,5,3)(9,10,11)$. $\diamondsuit$

Recall the Bruhat order $\succeq$ on $\mathfrak{S}_n$ as defined in Section 1.1. Given a multipartition λ we define a dominance order on the set of λ-tableaux: for λ-tableaux $\mathfrak{t}$ and $\mathfrak{s}$ we have

$$\mathfrak{t} \trianglerighteq \mathfrak{s} \quad \text{if and only if} \quad d(\mathfrak{t}) \succeq d(\mathfrak{s}).$$

If $\mathfrak{t} \trianglerighteq \mathfrak{s}$ then we also write $\mathfrak{s} \trianglelefteq \mathfrak{t}$.

1.9 Residues and degrees

Suppose $e \in \{2,3,4,\ldots\} \cup \{\infty\}$ and that we have an e-multicharge $\kappa = (\kappa_1, \ldots, \kappa_l)$ as defined in Section 1.3. Given a node $A = (r,c,m)$ define its *residue* $\operatorname{res}A$ to be

$$\operatorname{res}A = \kappa_m + c - r \mod e.$$

Note that in the theory of $\mathfrak{S}_n$ and $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ we have $l = 1$, and the choice of multicharge has no consequence on the structure of the algebra, thus the residue of a node A is usually defined just as

$$\operatorname{res}A = c - r \mod e.$$

We say A is an *i-node* if $\mathrm{res}A = i$. Define the *residue diagram* of λ to be the diagram formed by filling in the box of $[\lambda]$ at node A with $\mathrm{res}A$. If λ is a multipartition of n and $f \in \mathbb{F}$, let $c_f(\lambda)$ be the number of nodes in $[\lambda]$ of residue f. We define the *residue content* of λ to be

$$\mathrm{cont}(\lambda) = \begin{cases} (c_0(\lambda), c_1(\lambda), \dots, c_{e-1}(\lambda)) & \text{if } e \in \{2,3,4,\dots\}, \\ (\dots, c_{-2}(\lambda), c_{-1}(\lambda), c_0(\lambda), c_1(\lambda), c_2(\lambda), \dots) & \text{if } e = \infty. \end{cases}$$

Recall the simple roots α_i constructed in Section 1.4. If we are in this setting we may also define the residue content to be

$$\sum_{A \in [\lambda]} \alpha_{\mathrm{res}A} \in Q^+.$$

Given a λ-tableau $\mathfrak{t}$, where λ is a multipartition of n, define the *residue sequence* of $\mathfrak{t}$ to be $\mathbf{i}(\mathfrak{t}) = (i_1, \dots, i_n)$ where i_k is the residue of the node whose image is k under $\mathfrak{t}$. In particular, we define $\mathbf{i}^\lambda := \mathbf{i}(\mathfrak{t}^\lambda)$. We write $\mathrm{res}_\nu(a)$ for the residue of the node containing a in $\mathfrak{t}^\nu$.

Example 1.23. Let $\lambda = ((2,2,1),(2),(3,1))$, $e = 3$ and $\kappa = (0,2,1)$. Then we have the residue diagram

$$\left(\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|} \hline 2 & 0 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|} \hline 1 & 2 & 0 \\ \hline 0 \\ \cline{1-1} \end{array} \right)$$

and we have that $\mathbf{i}^\lambda = (0,1,2,0,1,2,0,1,2,0,0)$. ◊

Suppose λ is an l-multipartition of n and that A is an i-node of λ. Define

$$d_A(\lambda) := \#\left\{ \begin{array}{c} \text{addable } i\text{-nodes of } \lambda \\ \text{below } A \end{array} \right\} - \#\left\{ \begin{array}{c} \text{removable } i\text{-nodes of } \lambda \\ \text{below } A \end{array} \right\},$$

and

$$d^A(\lambda) := \#\left\{ \begin{array}{c} \text{addable } i\text{-nodes of } \lambda \\ \text{above } A \end{array} \right\} - \#\left\{ \begin{array}{c} \text{removable } i\text{-nodes of } \lambda \\ \text{above } A \end{array} \right\}.$$

We define the *degree* and *codegree* of $\mathfrak{t} \in \mathrm{Std}(\lambda)$ recursively. We set $\deg(\mathfrak{t}) = 0 = \operatorname{codeg}(\mathfrak{t})$ when $\mathfrak{t}$ is the $\varnothing$-tableau. For $\mathfrak{t} \in \mathrm{Std}(\lambda)$ with $|\lambda| \neq 0$, set

$$\deg(\mathfrak{t}) := d_A(\lambda) + \deg(\mathfrak{t}_{<n}) \qquad \text{and} \qquad \operatorname{codeg}(\mathfrak{t}) := d^A(\lambda) + \operatorname{codeg}(\mathfrak{t}_{<n}),$$

where $A = \mathfrak{t}^{-1}(n)$ and $\mathfrak{t}_{<n}$ is the tableau obtained by removing A from $\mathfrak{t}$.

Example 1.24. Let $\lambda = ((2,1),(1^2))$, $e = 3$, $\kappa = (2,1)$ and consider $\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline\end{array}\right)$. We have the corresponding residue diagram $\left(\begin{array}{|c|c|}\hline 2 & 0 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}\right)$. Then, taking A to be the node $\mathfrak{t}^{-1}(5) = (2,1,2)$, we get $d_A(\lambda) = 0 - 0 = 0$, $d^A(\lambda) = 1 - 1 = 0$, and $\mathfrak{t}_{<5} = \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array}\right)$. Continuing in this manner, we find that $\deg(\mathfrak{t}) = 2$ and $\operatorname{codeg}(\mathfrak{t}) = 0$. $\diamondsuit$

1.10 Abacuses

Another way to represent multipartitions is to use abacus configurations. Suppose λ is a partition and that we have fixed $a \in \mathbb{Z}$. For every $j \geq 1$ we define the *β-number* β_j to be

$$\beta_j := \lambda_j - j + a$$

and we define the set of β-numbers associated to λ with respect to a to be

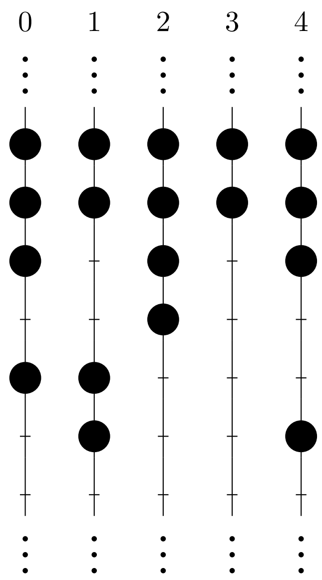
$$\beta_a(\lambda) = \{\beta_j \mid j \geq 1\}.$$

Now suppose we have an abacus whose runners extend infinitely and are indexed from left to right by the elements of I and whose possible bead positions are labelled with the elements of $\mathbb{Z}$ from left to right and then top to bottom, with position 0 appearing on runner 0. Then the abacus configuration associated to λ with respect to a is the abacus configuration with a bead placed at position β_j for every $j \geq 1$.

Example 1.25. Suppose $e = 5$, $a = 3$ and $\lambda = (12, 10, 6^2, 4, 2, 1)$. Then we have

$$\beta_a(\lambda) = \{14, 11, 6, 5, 2, -1, -3, -5, -6, -7, \ldots\}$$

and the abacus configuration is



$\diamondsuit$

Given an l-multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ and $\mathfrak{a} = (a_1, a_2, \ldots, a_l) \in \mathbb{Z}^l$, we define the abacus configuration associated to λ with respect to $\mathfrak{a}$ to be the l-tuple of abacuses where the ith abacus corresponds to the β-numbers $\beta_{a_i}(\lambda^{(i)})$.

Note that if $\kappa = (\kappa_1, \ldots, \kappa_l)$ is a multicharge and $a_i \equiv \kappa_i$ for $i \in \{1, \ldots, l\}$ then each bead corresponds to the end of a row of the diagram of λ (or to a row of length 0), and by the definition of the β-numbers the node at the end of the row (if it exists) has residue i if and only if the corresponding bead is on runner i of the abacus. Thus if we increase any β-number by one, this is equivalent to moving a bead from runner j to runner $j+1 \mod e$ which is equivalent to adding a node of residue $j+1$ to the diagram of λ. Similarly decreasing a β-number by one is equivalent to moving a bead from runner j to runner $j-1 \mod e$ which is equivalent to removing a node of residue j from the diagram of λ.

Example 1.26. Recall the setup in Example 1.25. We see, for example, that increasing β_1 from 14 to 15 corresponds to adding a node of residue 0, whilst decreasing β_5 from 2 to 1 corresponds to removing a node of residue 2. The corresponding residue diagram is shown below with the addable node in red and the removable node in blue.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & \boxed{0} \\
\hline
2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
\cline{1-10}
1 & 2 & 3 & 4 & 0 & 1 \\
\cline{1-6}
0 & 1 & 2 & 3 & 4 & 0 \\
\cline{1-6}
4 & 0 & 1 & \boxed{2} \\
\cline{1-4}
3 & 4 \\
\cline{1-2}
2 \\
\cline{1-1}
\end{array}$$

◊

1.11 Specht modules for $\mathscr{H}_n^{\Lambda_\kappa}$

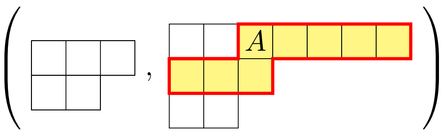
We will now define Specht modules for $\mathscr{H}_n^{\Lambda_\kappa}$. By Theorem 1.16 we have that each KLR algebra is isomorphic to some Ariki-Koike algebra. There are well established ways of defining Specht modules of Ariki-Koike algebras as cell modules corresponding to cellular bases (see [GL96] for the first such way, or [DJM98]), and in a similar vein Hu and Mathas construct an explicit homogeneous cellular basis for $\mathscr{H}_n^{\Lambda_\kappa}$ in [HM10]. However, we will define the Specht modules for $\mathscr{H}_n^{\Lambda_\kappa}$ in such a way that gives us a different insight into the representation theory of the Ariki-Koike algebras.

By Proposition 1.15 we know that $\mathscr{H}_n^{\Lambda_\kappa}$ is a graded algebra, but it is not directly obvious that certain classes of modules should admit a grading. Nevertheless, graded Specht modules for $\mathscr{H}_n^{\Lambda_\kappa}$ were first exhibited in [BKW11], however they are defined in such a way that computations rely on repeatedly using Theorem 1.16. Instead, following the method used in [KMR12], we may present the Specht modules in terms of a single homogeneous generator and relations, allowing us to consider a different approach to the Specht modules as opposed to using cellular bases. Note that this grading can of course be transferred to $\mathcal{H}_{\mathbb{F},q,Q}$ (and hence also $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ and $\mathbb{F}\mathfrak{S}_n$) via Theorem 1.16.

Fix $e \in \{2, 3, 4, \dots\} \cup \{\infty\}$ and let κ be an e-multicharge of level l. Let λ be an l-multipartition of n, and let $A = (a, b, m) \in [\lambda]$. We say A is a (row) *Garnir node* of λ if $(a + 1, b, m) \in [\lambda]$. The (row) *Garnir belt* $\mathbf{B}^A$ is defined to be the set of nodes

$$\mathbf{B}^A = \{(a, c, m) \in [\lambda] \mid c \geq b\} \cup \{(a + 1, c, m) \in [\lambda] \mid c \leq b\}.$$

Example 1.27. Let $\lambda = ((3, 2), (7, 3, 2))$ and $A = (1, 3, 2)$. Then the Garnir belt $\mathbf{B}^A$ is shown highlighted below.



$\Diamond$

We define a λ-tableau called the (row) *Garnir tableau* $\mathtt{G}^A$ by taking the initial tableau $\mathfrak{t}^\lambda$ and rewriting the entries within $\mathbf{B}^A$ so that they increase from bottom left to top right.

Example 1.28. Continuing Example 1.27, we have the tableau

$$\mathfrak{t}^\lambda = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \; , \; \begin{array}{ccccccc} 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & & & & \\ 16 & 17 & & & & & \end{array} \right)$$

and

$$\mathtt{G}^A = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \; , \; \begin{array}{ccccccc} 6 & 7 & 11 & 12 & 13 & 14 & 15 \\ 8 & 9 & 10 & & & & \\ 16 & 17 & & & & & \end{array} \right).$$

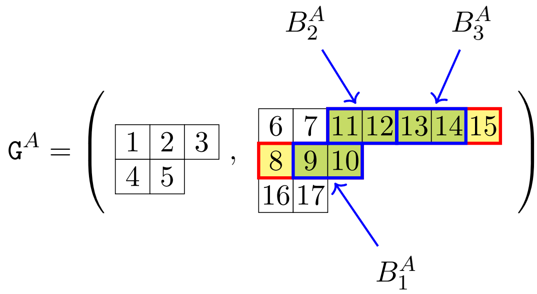
$\Diamond$

We define a (row) *brick* to be a set of e successive nodes

$$\{(c, d, m), (c, d + 1, m), \ldots, (c, d + e - 1, m)\} \subseteq \mathbf{B}^A$$

such that $\operatorname{res}(c, d, m) = \operatorname{res} A$. So $\mathbf{B}^A$ is a disjoint union of bricks together with less than e nodes not in a brick at the end of row a and less than e nodes not in a brick at the beginning of row $a + 1$. Let $f = f^A$ be the number of bricks in row a and $g = g^A$ be the number of bricks in row $a + 1$. Set $k = k^A = f^A + g^A$, i.e. k is the total number of bricks in $\mathbf{B}^A$. Then we label the bricks $B_1^A, B_2^A, \ldots, B_k^A$ from bottom left to top right.

Example 1.29. Continuing Example 1.28, if we suppose $e = 2$ then the bricks are illustrated below.



$\diamondsuit$

If $k > 0$ let $d = d^A$ be the smallest entry in B_1^A. In Example 1.29, we see that $d = 9$. For each $r \in \{1, \dots, k-1\}$, we define a *brick transposition*

$$w_r^A := \prod_{x=d+re-e}^{d+re-1} (x, x+e)$$

which swaps the bricks B_r^A and B_{r+1}^A. These elements are the Coxeter generators for a symmetric group:

$$\mathfrak{S}^A := \langle w_1^A, w_2^A, \dots, w_{k-1}^A \rangle \cong \mathfrak{S}_k.$$

If $k = 0$ then we set $\mathfrak{S}^A$ to be the trivial group.

Define Gar^A to be the set of all row standard λ-tableaux which are obtained from $\mathtt{G}^A$ by brick permutations, i.e. by acting on $\mathtt{G}^A$ by $\mathfrak{S}^A$. Note that every tableau in Gar^A is standard except for $\mathtt{G}^A$, and $\mathtt{G}^A$ is the minimal element of Gar^A with respect to the Bruhat order.

Let $\mathtt{T}^A$ be the λ-tableau obtained from $\mathtt{G}^A$ by reordering the bricks so that their entries increase along row a and then along row $a+1$.

Example 1.30. Continuining Example 1.29, we have

$$\mathtt{T}^A = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}\ , \ \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 7 & 9 & 10 & 11 & 12 & 15 \\ \hline 8 & 13 & 14 \\ \cline{1-3} 16 & 17 \\ \cline{1-2} \end{array} \right).$$

$\Diamond$

Define $\mathscr{D}^A$ to be the set of minimal length right coset representatives of $\mathfrak{S}_f \times \mathfrak{S}_g$ in $\mathfrak{S}^A \cong \mathfrak{S}_k$. Let $\mathbf{i}^\mathbf{A} = \mathbf{i}(\mathtt{T}^A)$ be the residue sequence of $\mathtt{T}^A$. Recall that for $w \in \mathfrak{S}_n$, we fix a reduced expression $w = s_{r_1} \cdots s_{i_k}$ and define $\psi_w := \psi_{r_1} \cdots \psi_{r_k}$. Note that the ψ_w may depend on the choice of reduced expression for w. Now if $u \in \mathscr{D}^A$, choose a reduced expression $u = w^A_{r_1} \cdots w^A_{r_t}$ for d and define

$$\tau^A_u := e(\mathbf{i}^A)(\psi_{w^A_{r_1}} + 1) \cdots (\psi_{w^A_{r_t}} + 1).$$

By [KMR12, Theorem 5.11], τ^A_u does not rely on the choice of reduced expression for u, nor the choice of reduced expression for each $\psi_{w^A_{r_i}}$. Let $\mathfrak{t}$ be a λ-tableau, then after fixing a choice of reduced expression for $d(\mathfrak{t})$ we may define $\psi^\mathfrak{t} := \psi_{d(\mathfrak{t})}$.

Definition 1.31. Suppose λ is an l-multipartition of n and A is a Garnir node of $[\lambda]$. The (row) *Garnir element* is defined as

$$g^A := \sum_{u \in \mathscr{D}^A} \psi^{\mathtt{T}^A} \tau^A_u.$$

Example 1.32. We continue Example 1.30 and compute the corresponding Garnir element. We have $f = 2$, $g = 1$, and so $\mathscr{D}^A = \{1, s_2, s_2 s_1\}$. The brick transpositions are

$$\begin{aligned} w^A_1 &= (9, 11)(10, 12), \\ w^A_2 &= (11, 13)(12, 14), \end{aligned}$$

and so we get

$$\begin{aligned} \tau^A_1 &= e(\mathbf{i}^A), \\ \tau^A_{s_2} &= e(\mathbf{i}^A)(\psi_{w^A_2} + 1) && = e(\mathbf{i}^A)(\psi_{(11,13)(12,14)} + 1), \\ \tau^A_{s_2 s_1} &= e(\mathbf{i}^A)(\psi_{w^A_2} + 1)(\psi_{w^A_1} + 1) && = e(\mathbf{i}^A)(\psi_{(11,13)(12,14)} + 1)(\psi_{(9,11)(10,12)} + 1). \end{aligned}$$

Now

$$d(\mathtt{T}^A) = (12,13)(13,14)(14,15)(11,12)(10,11)(9,10)(8,9),$$

hence

$$\psi^{\mathtt{T}^A} = \psi_{(12,13)}\psi_{(13,14)}\psi_{(14,15)}\psi_{(11,12)}\psi_{(10,11)}\psi_{(9,10)}\psi_{(8,9)}.$$

Putting these together, we get

$$\begin{aligned}
g^A &= \psi^{\mathtt{T}^A} e(\mathbf{i}^A)(1 + (\psi_{(11,13)(12,14)} + 1) + (\psi_{(11,13)(12,14)} + 1)(\psi_{(9,11)(10,12)} + 1)) \\
&= \psi^{\mathtt{T}^A} e(\mathbf{i}^A)(3 + 2\psi_{(11,13)(12,14)} + \psi_{(9,11)(10,12)} + \psi_{(11,13)(12,14)}\psi_{(9,11)(10,12)}) \\
&= \psi^{\mathtt{T}^A} e(\mathbf{i}^A)(3 + 2\psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)} + \psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)} \\
&\qquad + \psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)}\psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)}).
\end{aligned}$$

Note that since $\mathtt{T}^A = \mathfrak{t}^\lambda d(\mathtt{T}^A)$, using relation (1.4) we see that $\psi^{\mathtt{T}^A} e(\mathbf{i}^A) = \psi_{d(\mathtt{T}^A)} e(\mathbf{i}^A) = e(\mathbf{i}^\lambda)\psi_{d(\mathtt{T}^A)}$, and so

$$\begin{aligned}
g^A &= e(\mathbf{i}^\lambda)\psi^{\mathtt{T}^A}(3 + 2\psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)} + \psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)} \\
&\qquad + \psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)}\psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)}). \\
&= e(\mathbf{i}^\lambda)\psi_{(12,13)}\psi_{(13,14)}\psi_{(14,15)}\psi_{(11,12)}\psi_{(10,11)}\psi_{(9,10)}\psi_{(8,9)}(3 \\
&\qquad + 2\psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)} + \psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)} \\
&\qquad + \psi_{(12,13)}\psi_{(11,12)}\psi_{(13,14)}\psi_{(12,13)}\psi_{(10,11)}\psi_{(9,10)}\psi_{(11,12)}\psi_{(10,11)}).
\end{aligned}$$

$\diamondsuit$

Now we can define the Specht modules of $\mathscr{H}_n^{\Lambda_\kappa}$, or rather, in light of Theorem 1.16, give an alternative presentation for the Specht modules of the Ariki-Koike algebras.

Definition 1.33. Suppose λ is an l-multipartition of n. The *Specht module* S^λ of $\mathscr{H}_n^{\Lambda_\kappa}$ is the $\mathscr{H}_n^{\Lambda_\kappa}$-module generated by the homogeneous element $v^{\mathfrak{t}^\lambda}$ of degree $\deg(\mathfrak{t}^\lambda)$ subject to the relations:

(i) $v^{\mathfrak{t}^\lambda} e(\mathbf{i}^\lambda) = v^{\mathfrak{t}^\lambda}$;

(ii) $v^{\mathfrak{t}^\lambda} y_r = 0$, for all $r \in \{1, \ldots, n\}$;

(iii) $v^{\mathfrak{t}^\lambda}\psi_r = 0$, for all $r \in \{1, \dots, n-1\}$ such that r and $r+1$ are in the same row of $\mathfrak{t}^\lambda$;

(iv) $v^{\mathfrak{t}^\lambda} g^A = 0$, for every Garnir node A of $[\lambda]$.

We refer to the relations in (iii) as *row relations*, and those in (iv) as *Garnir relations*.

Recall that for any λ-tableau $\mathfrak{t}$, we have a corresponding element $\psi^{\mathfrak{t}}$, which depends on a choice of reduced expression for $d(\mathfrak{t})$. For any λ-tableau $\mathfrak{t}$ we define $v^{\mathfrak{t}} := v^{\mathfrak{t}^\lambda}\psi^{\mathfrak{t}}$. The next two results do not depend on the choice of reduced expression.

Proposition 1.34. *[KMR12, Proposition 5.14] Suppose λ is a multipartition of n and that $\mathfrak{t}$ is a standard λ-tableau. Then $\deg(v^{\mathfrak{t}}) = \deg(\mathfrak{t})$.*

Proposition 1.35. *[KMR12, Corollary 6.24] Suppose λ is a multipartition of n. Then*

$$\{v^{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

is a homogeneous basis for S^λ over $\mathbb{F}$.

We state two results which are useful for calculations involving the Specht modules.

Proposition 1.36. *[BKW11, Corollary 4.6 & Proposition 4.7] Let λ be a multipartition of n, $\mathfrak{t}$ be a standard λ-tableau, and $d(\mathfrak{t}) = s_{i_1} \cdots s_{i_t}$ be a reduced expression. Then*

$$v^{\mathfrak{t}^\lambda}\psi_{i_1}\cdots\psi_{i_t} = v^{\mathfrak{t}} + \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mathfrak{s}\lhd\mathfrak{t}}} a_{\mathfrak{s}} v^{\mathfrak{s}}$$

for some $a_{\mathfrak{s}} \in \mathbb{F}$. Furthermore if $a_{\mathfrak{s}} \neq 0$ then $\mathbf{i}(\mathfrak{s}) = \mathbf{i}(\mathfrak{t})$.

Proposition 1.37. *[BKW11, Lemma 4.8] Let λ be a multipartition of n, $\mathfrak{t}$ be a standard λ-tableau, and $r \in \{1, \dots, n\}$. Then*

$$v^{\mathfrak{t}} y_r = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mathfrak{s}\lhd\mathfrak{t}}} a_{\mathfrak{s}} v^{\mathfrak{s}}$$

for some $a_\mathfrak{s} \in \mathbb{F}$*. Furthermore if* $a_\mathfrak{s} \neq 0$ *then* $\mathbf{i}(\mathfrak{s}) = \mathbf{i}(\mathfrak{t})$*.*

Note that Proposition 1.36 does depend on the reduced expression of $d(\mathfrak{t})$. The following example illustrates this.

Example 1.38. Let $\lambda = ((3,1),(2,1))$, $e = 4$, $\kappa = (0,2)$. Consider $\mathfrak{t} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 7 \\ \cline{1-1} \end{array} \right)$, so then $d(\mathfrak{t}) = (3,5)$. Now both $(3,4)(4,5)(3,4)$ and $(4,5)(3,4)(4,5)$ are reduced expressions for $d(\mathfrak{t})$, and so suppose we fix $s_3 s_4 s_3 = (3,4)(4,5)(3,4)$ to be our preferred reduced expression. Then

$$v^{\mathfrak{t}^\lambda}\psi_3\psi_4\psi_3 = v^{\mathfrak{t}^\lambda}\psi_{d(\mathfrak{t})} = v^{\mathfrak{t}^\lambda}\psi^{\mathfrak{t}} = v^{\mathfrak{t}}.$$

But now also

$$\begin{aligned} v^{\mathfrak{t}^\lambda}\psi_4\psi_3\psi_4 &= v^{\mathfrak{t}^\lambda}e(\mathbf{i}^\lambda)\psi_4\psi_3\psi_4 \\ &= v^{\mathfrak{t}^\lambda}\psi_4\psi_3\psi_4 e(\mathbf{i}(\mathfrak{t})) \end{aligned}$$

and we have $e(\mathbf{i}^\lambda) = (0,1,2,3,2,3,1) = e(\mathbf{i}(\mathfrak{t}))$, $i_3 = i_5 \leftarrow i_4$, so by relation (1.11) we get

$$\begin{aligned} v^{\mathfrak{t}^\lambda}\psi_4\psi_3\psi_4 &= v^{\mathfrak{t}^\lambda}\psi_4\psi_3\psi_4 e(\mathbf{i}(\mathfrak{t})) \\ &= v^{\mathfrak{t}^\lambda}(\psi_3\psi_4\psi_3 + 1)e(\mathbf{i}(\mathfrak{t})) \\ &= v^{\mathfrak{t}} + v^{\mathfrak{t}^\lambda}. \end{aligned}$$

$\diamondsuit$

Recall that any KLR algebra $\mathscr{H}_n^{\Lambda_\kappa}$ is isomorphic to an Ariki-Koike algebra $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ and that in the latter we have a cellular structure so that the Specht modules S^λ arise as cell modules. From the theory of cellular algebras, each module S^λ has an attached bilinear form, and from this we can obtain every simple $\mathcal{H}_{\mathbb{F},q,Q}$-module as the quotient of an S^λ by the radical associated to this form. Let us write D^λ for the simple module we obtain. Then we have the following:

Theorem 1.39. *[DJM98, Theorem 3.30]*

$$\{D^\lambda \mid \lambda \textit{ is an multipartition of } n \textit{ such that } D^\lambda \neq 0\}$$

is a complete set of non-isomorphic irreducible $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$*-modules over* $\mathbb{F}$.

Using the isomorphism discussed above, we can view these irreducible modules in the KLR setting also.

We can in fact say exactly when $D^\lambda \neq 0$. Given a multipartition λ and $i \in I$ (where I is defined as in 1.3), define the *i-signature* of λ by assessing the addable and removable nodes of λ in turn from the highest to the lowest and writing A for each addable i-node and R for each removable i-node. Then repeatedly remove any adjacent pairs RA until none are left. The remaining removable i-nodes in the sequence are called *normal* i-nodes, and the highest of these (i.e. that which lies above the others, if it exists) is called the *good* i-node. We say that λ is *Kleshchev* if λ is the empty multipartition, or if there is a good node x of λ (of any residue) such that removing x from λ still gives a Kleshchev multipartition. Then the following holds.

Theorem 1.40. *[Ari01, Theorem 4.3]* $D^\lambda \neq 0$ *if and only if* λ *is a Kleshchev multipartition.*

Example 1.41. Let $e = 2$, $\kappa = (0, 0)$ and $\lambda = (\varnothing, (2, 1))$. Then

$$[\lambda] = \left(\varnothing \,,\, \begin{array}{|c|c|} \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \quad \text{has residue diagram} \quad \left(\varnothing \,,\, \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

and has reduced 1-signature RR thus the highest 1-node is a good node. Then

$$\left(\varnothing \,,\, \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \right) \quad \text{has residue diagram} \quad \left(\varnothing \,,\, \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array} \right)$$

and has reduced 1-signature AR thus the 1-node is good. Then

$$\left(\varnothing \,,\, \begin{array}{|c|} \hline \\ \hline \end{array} \right) \quad \text{has residue diagram} \quad \left(\varnothing \,,\, \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right)$$

and has reduced 0-signature AR so the 0-node is good. Since by removing it we have the empty multipartition this means that λ is Kleshchev. On the other hand, if we consider $\mu = ((2,1), \varnothing)$, then we can similarly reduce the situation to considering the multipartition $(1, \varnothing)$, but

$$\left(\boxed{}\ , \varnothing \right) \quad \text{has residue diagram} \quad \left(\boxed{0}\ , \varnothing \right)$$

with 0-signature RA, so the reduced 0-signature is empty and there are no good nodes that we can remove. So μ is not Kleshchev. $\diamondsuit$

Now we use the grading to define graded decomposition numbers. Since the Specht modules S^λ are graded, we have that the quotients D^λ are also graded. Recall Definition 1.11.

Definition 1.42. We define the *graded decomposition number* of D^μ as a composition factor of S^λ to be

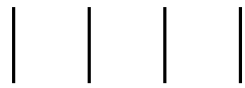
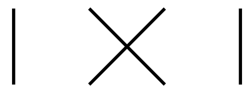
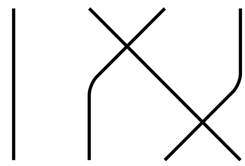
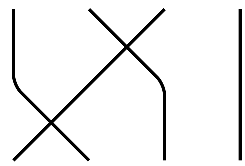
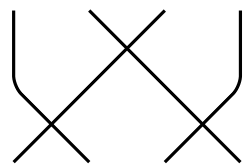
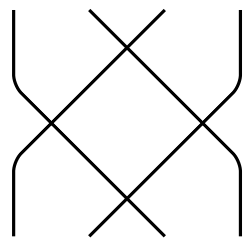
$$d_{\lambda\mu}(v) = [S^\lambda \colon D^\mu]_v \coloneqq \sum_{k \in \mathbb{Z}} [S^\lambda \colon D^\mu\langle k \rangle] v^k$$

where v is an indeterminate over $\mathbb{Z}$ and $[S^\lambda \colon D^\mu\langle k \rangle]$ is the number of times $D^\mu\langle k \rangle$ appears as a composition factor of S^λ.

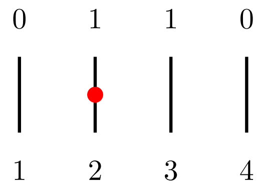
It is a long standing open problem to determine the decomposition numbers, even just in the symmetric group case.

In Section 1.7 we have seen how we can use braid diagrams to work with elements of KLR algebras. In particular we can work with Specht modules using these braid diagrams. We can represent the generator of a Specht module S^λ as the braid diagram corresponding to $e(\mathbf{i}^\lambda)$ and then apply KLR generators of the form y_s and ψ_r as dots and crossings respectively.

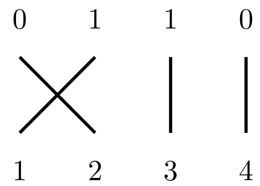
Example 1.43. Let λ be the multipartition $((2), (1^2))$, let $e = 3$ and suppose $\kappa = (0, 1)$. Then the residue sequence of t^λ is $(0, 1, 1, 0)$ and the standard tableaux, corresponding basis elements of S^λ and braid diagrams are shown below.

Standard tableaux	Basis element	Braid diagram
$\left(\,\boxed{1\,\vert\,2}\;,\;\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}$	0 1 1 0  1 2 3 4
$\left(\,\boxed{1\,\vert\,3}\;,\;\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}\psi_2$	0 1 1 0  1 2 3 4
$\left(\,\boxed{1\,\vert\,4}\;,\;\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}\psi_2\psi_3$	0 1 1 0  1 2 3 4
$\left(\,\boxed{2\,\vert\,3}\;,\;\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}\psi_2\psi_1$	0 1 1 0  1 2 3 4
$\left(\,\boxed{2\,\vert\,4}\;,\;\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}\psi_2\psi_1\psi_3$	0 1 1 0  1 2 3 4
$\left(\,\boxed{3\,\vert\,4}\;,\;\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}\,\right)$	$v^{\mathfrak{t}^\lambda}\psi_2\psi_1\psi_3\psi_2$	0 1 1 0  1 2 3 4

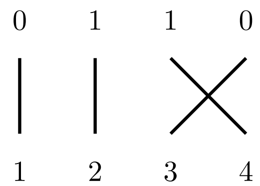
We also know that $v^{\mathfrak{t}^\lambda} y_s = 0$ for all $s \in \{1, \dots, 4\}$, so for example we have:

$v^{\mathfrak{t}^\lambda} y_2 =$  $= 0.$

Similarly, since 1 and 2 belong to the same row of $\mathfrak{t}^\lambda$, we know that

$v^{\mathfrak{t}^\lambda} \psi_1 =$  $= 0.$

We have one Garnir node, that containing 3 in $\mathfrak{t}^\lambda$, and since $e = 3$ we have no row bricks and thus we find that the corresponding Garnir element is just $g = \psi_3$. So we have:

$v^{\mathfrak{t}^\lambda} \psi_3 =$  $= 0.$

◊

We would like to note that, when performing calculations in examples such as in the above and in later sections when computing homomorphisms, we have been incredibly reliant on the KLR algebra-focused GAP packages provided by Matt Fayers.

Chapter 2

Manipulating braid diagrams

OUR first goal is to prepare ourselves with the necessary tools in order to prove our main theorem. Thus, in this chapter we will present numerous ways of dealing with elements of KLR algebras through a series of lemmas. When manipulating the braid diagrams corresponding to elements of KLR algebras we frequently encounter the same 'patterns' of strings. With these lemmas, we aim to eliminate the need to constantly re-explain how one performs the basic relations from the presentation of $\mathscr{H}_n^{\Lambda_\kappa}$ on large sections of diagrams. Many of the lemmas act as natural extensions of the KLR relations; they may appear convoluted when written algebraically, but are much easier to comprehend when viewed with respect to the corresponding diagrams.

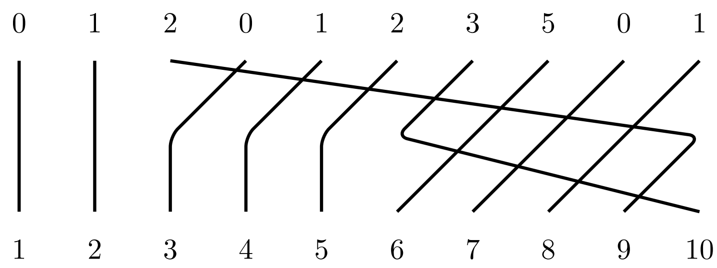
2.1 Motivation

Recall the notational shortcuts that we defined at the end of Section 1.6. The following examples will help motivate how such lemmas can speed up our computations.

Example 2.1. Let $e = 6$, $\kappa = (0, 0)$, $n = 10$ and consider the associated algebra $\mathscr{H}_n^{\Lambda_\kappa}$. Let $\lambda = ((2), (4, 4))$ and $\mu = ((3), (4, 3))$. Suppose that we are trying to show that there is a homomorphism from S^λ to S^μ, given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}}$, where $\mathfrak{s}$ is the tableau

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 10 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & \\ \cline{1-3} \end{array} \right).$$

We wish to show that the Garnir element $\Psi_6 \uparrow^9$ for S^λ kills $v^{\mathfrak{s}}$, that is, that $v^{\mathfrak{t}^\mu} \Psi_3 \uparrow^9 \Psi_6 \uparrow^9 = 0$. Writing the left hand side using the braid diagram combinatorics seen at the end of Section 1.11 we have:



We shall perform just the first step in showing that this is zero. If we only apply the relations for $\mathscr{H}_n^{\Lambda_\kappa}$ as written our calculations take an inordinate amount of time:

$$\begin{aligned}\Psi_3{\uparrow}^9\ \Psi_6{\uparrow}^9 &= \Psi_3{\uparrow}^5\ \psi_6\psi_7\psi_6\Psi_8{\uparrow}^9\ \Psi_7{\uparrow}^9\\ &= \Psi_3{\uparrow}^5\ \psi_7\psi_6\psi_7\Psi_8{\uparrow}^9\ \Psi_7{\uparrow}^9\end{aligned}$$

by relation (1.11) as $2 \neq 5$,

$$\begin{aligned}&= \psi_7\Psi_3{\uparrow}^6\ \psi_7\psi_8\psi_7\psi_9\Psi_8{\uparrow}^9\\ &= \psi_7\Psi_3{\uparrow}^6\ \psi_8\psi_7\psi_8\psi_9\Psi_8{\uparrow}^9\end{aligned}$$

by relation (1.11) as $2 \neq 0$,

$$\begin{aligned}&= \Psi_7{\uparrow}^8\ \Psi_3{\uparrow}^7\ \psi_8\psi_9\psi_8\psi_9\\ &= \Psi_7{\uparrow}^8\ \Psi_3{\uparrow}^7\ \psi_9\psi_8\psi_9\psi_9\end{aligned}$$

again by relation (1.11) as $2 \neq 1$,

$$= \Psi_7{\uparrow}^9\ \Psi_3{\uparrow}^9\ \psi_9.$$

Ideally, instead of all this we wish to just be able to say something in the spirit of: “We have $\Psi_3{\uparrow}^9\ \Psi_6{\uparrow}^9 = \Psi_3{\uparrow}^9\ \Psi_6{\uparrow}^8\ \psi_9$ and then as $2 \neq 5, 0, 1$ this is equal to $\Psi_7{\uparrow}^9\ \Psi_3{\uparrow}^9\ \psi_9$.”

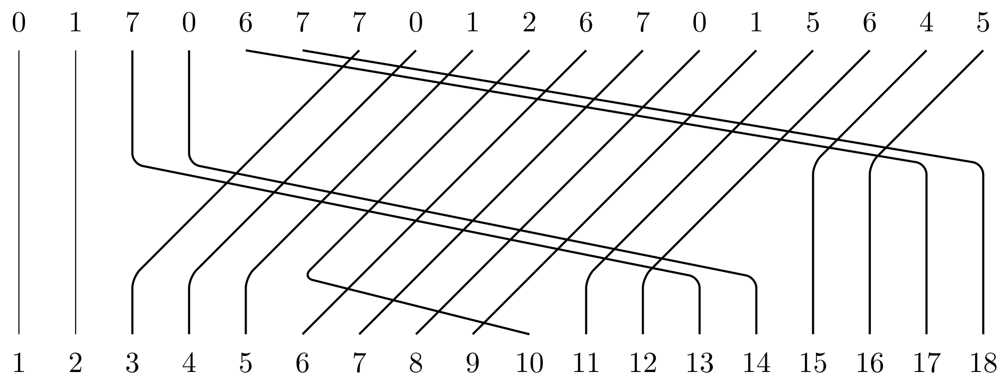
$\lozenge$

The second of the two examples exhibits how things can be much worse than in the first example, greatly increasing our need for some lemmas which speed up the combinatorics.

Example 2.2. Suppose that $e = 8$, $\kappa = (0, 7)$, $n = 18$ and consider the associated algebra $\mathscr{H}_n^{\Lambda_\kappa}$. Let $\lambda = ((2), (4^4))$ and $\mu = ((2^3), (4^2, 2^2))$. Suppose that we are trying to show that there is a homomorphism from S^λ to S^μ, given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}}$ where $\mathfrak{s}$ is the tableau

$$\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 13 & 14 \\ \hline 17 & 18 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|c|c|} \hline 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & 10 \\ \hline 11 & 12 & \multicolumn{2}{c}{} \\ \cline{1-2} 15 & 16 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \right).$$

We wish to show that the Garnir element $\Psi_6 \uparrow^9$ for S^λ kills $v^{\mathfrak{s}}$, i.e. that $v^{\mathfrak{t}^\mu} \left(\Psi_6 \uparrow^{17}\right) \downarrow_5 \left(\Psi_4 \uparrow^{13}\right) \downarrow_3 \Psi_6 \uparrow^9 = 0$. Then writing the left hand side as a braid diagram we have:



and it is immediately clear that using individual applications of the braid relation here as in Example 2.1 will be largely inefficient, when really all we wish to do is notice that the string of residue 2 can be 'pulled over the other crossings' so as to arrive at the top of the diagram (giving us the Garnir relation $\Psi_{10} \uparrow^{13}$ for S^μ at the top).

$\Diamond$

2.2 Lemmas

The first lemma we prove acts as a generalisation of relation (1.11). It streamlines the process of pulling two strings over each other, when the residues are such that there is no need to add extra terms as in (1.11).

Lemma 2.3. *Suppose we have the crossings* $\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1} \cdot \psi_{x+g+1}$ *with residues* l*,* m*, and* $r_1, \ldots, r_g$ *as shown in Figure 2.1. Suppose also that one of the following occurs:*

(i) $l \not- m$*,*

(ii) $r_i \not- m$ *for every* $i \in \{1, \ldots, g\}$*,*

(iii) $l \neq r_i$ *for every* $i \in \{1, \ldots, g\}$*.*

Then

$$\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1} \cdot \psi_{x+g+1} = \psi_{x+1} \cdot \left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1},$$

the right hand side of the equality being shown in Figure 2.2.

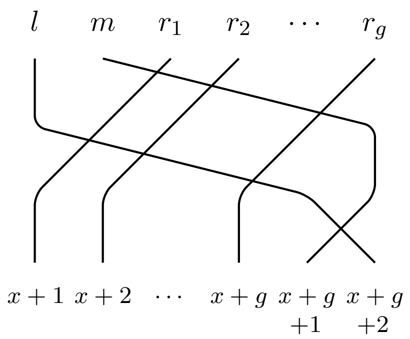


Figure 2.1: Crossings at the start of Lemma 2.3.

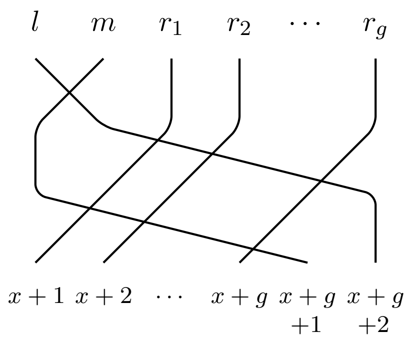


Figure 2.2: Crossings at the end of Lemma 2.3.

Proof. Using the braid relation (1.11) we have

$$\begin{aligned}
&\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1} \cdot\ \psi_{x+g+1} \\
&\qquad\qquad = \left(\Psi^{x+2}\!\downarrow_{x+1}\right)\!\uparrow^{x+g} \cdot \psi_{x+g+1}\psi_{x+g}\psi_{x+g+1} \\
&\qquad\qquad = \left(\Psi^{x+2}\!\downarrow_{x+1}\right)\!\uparrow^{x+g} \cdot \psi_{x+g}\psi_{x+g+1}\psi_{x+g} \\
&\qquad\qquad = \left(\Psi^{x+2}\!\downarrow_{x+1}\right)\!\uparrow^{x+g-1} \cdot \psi_{x+g}\psi_{x+g-1}\psi_{x+g}\psi_{x+g+1}\psi_{x+g},
\end{aligned}$$

then by continually applying the braid relation again

$$= \psi_{x+1} \cdot \left(\Psi^{x+2}{\downarrow}_{x+1}\right){\uparrow}^{x+g+1}$$
$$= \psi_{x+1} \cdot \left(\Psi_{x+2}{\uparrow}^{x+g+1}\right){\downarrow}_{x+1}\ .$$

□

The next lemma extends Lemma 2.3 by considering when we can pull a string over multiple other strings instead of just one when the residues are sufficiently spread apart.

Lemma 2.4. *Suppose we have the crossings*

$$\Psi_{x+f+1}{\uparrow}^{x+f+g} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+1}$$

with residues $l_1, \ldots, l_f$, m, and $r_1, \ldots, r_g$ as shown in Figure 2.3. Suppose that one of the following occurs:

(i) $l_i \not\to m$ for every $i \in \{1, \ldots, f\}$,

(ii) $r_i \not\to m$ for every $i \in \{1, \ldots, g\}$, or

(iii) $l_i \neq r_j$ for every $i \in \{1, \ldots, f\}$ and $j \in \{1, \ldots, g\}$.

Then

$$\Psi_{x+f+1}{\uparrow}^{x+f+g} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+1} = \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+1} \cdot \Psi_{x+1}{\uparrow}^{x+g},$$

the right hand side of the equality being shown in Figure 2.4.

Proof. We have

$$\begin{aligned}
&\Psi_{x+f+1}{\uparrow}^{x+f+g} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+1} \\
&\qquad = \left(\Psi_{x+f+1}{\uparrow}^{x+f+g}\right){\downarrow}_{x+f} \cdot \psi_{x+f+g} \cdot \left(\Psi_{x+f-1}{\uparrow}^{x+f+g-1}\right){\downarrow}_{x+1} \\
&\qquad = \psi_{x+f} \cdot \left(\Psi_{x+f+1}{\uparrow}^{x+f+g}\right){\downarrow}_{x+f} \cdot \left(\Psi_{x+f-1}{\uparrow}^{x+f+g-1}\right){\downarrow}_{x+1}
\end{aligned}$$

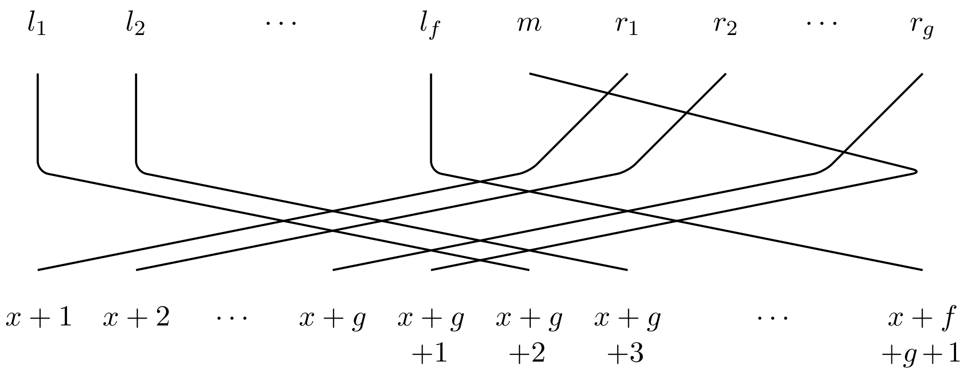


Figure 2.3: Crossings at the start of Lemma 2.4.

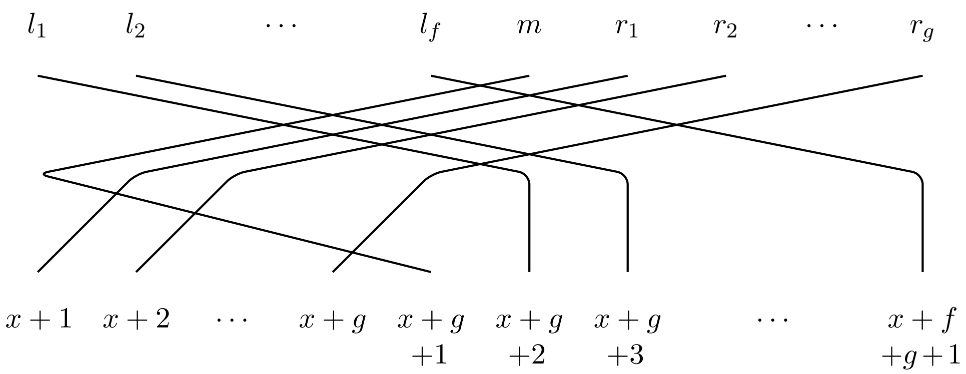


Figure 2.4: Crossings at the end of Lemma 2.4.

by Lemma 2.3 (since one of (i), (ii), or (iii) occurs)

$$
\begin{aligned}
&= \Psi_{x+f}{\uparrow}^{x+f+g} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g-1}\right){\downarrow}_{x+f-1} \cdot \psi_{x+f+g-1} \\
&\qquad\qquad\qquad\qquad \cdot \left(\Psi_{x+f-2}{\uparrow}^{x+f+g-2}\right){\downarrow}_{x+1} \\
&= \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+1} \cdot \Psi_{x+1}{\uparrow}^{x+g}
\end{aligned}
$$

by repeatedly applying Lemma 2.3.

□

The following lemma is an extension of Lemma 2.4 and deals with pulling multiple strings over each other where the residues are sufficiently spread so that no extra terms are created due to relation (1.11).

Lemma 2.5. *Suppose we have the crossings*

$$\left(\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+f+1}\cdot\left(\Psi_{x+f}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+1} \tag{2.1}$$

with residues $l_1,\ldots,l_f$, $m_1,\ldots,m_h$ *and* $r_1,\ldots,r_g$ *as shown in Figure 2.5. Suppose also that one of the following occurs:*

(i) $l_i \not- m_j$ *for every* $i\in\{1,\ldots,f\}, j\in\{1,\ldots,h\}$,

(ii) $r_i \not- m_j$ *for every* $i\in\{1,\ldots,g\}, j\in\{1,\ldots,h\}$,

(iii) $l_i \neq r_j$ *for every* $i\in\{1,\ldots,f\}$ *and* $j\in\{1,\ldots,g\}$.

Then (2.1) is equal to:

$$\left(\Psi_{x+f}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+h}\uparrow^{x+g+h-1}\right)\downarrow_{x+1}.$$

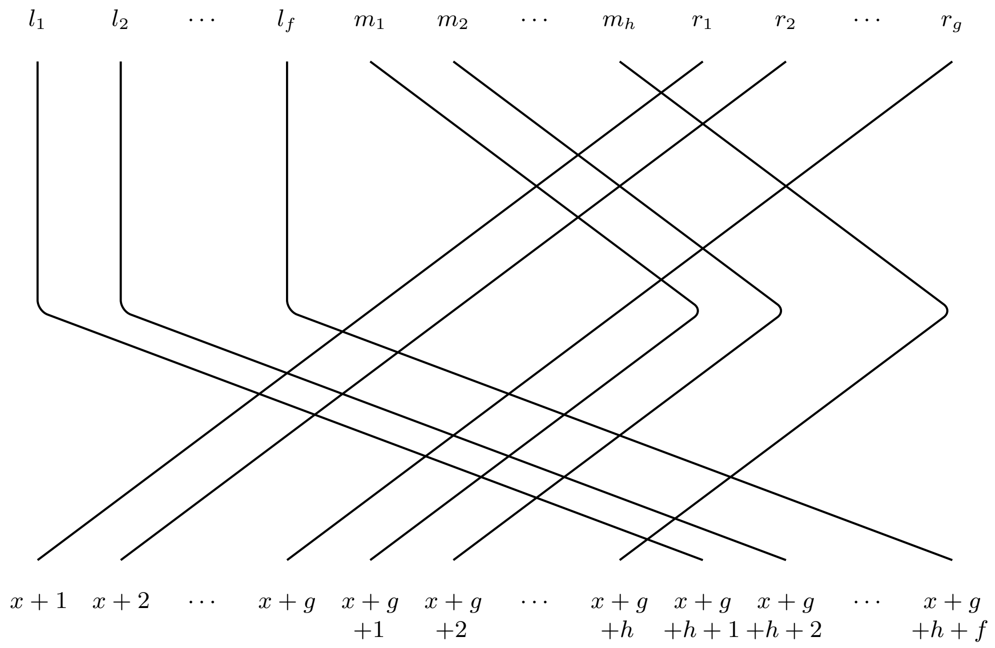


Figure 2.5: Crossings at the start of Lemma 2.5.

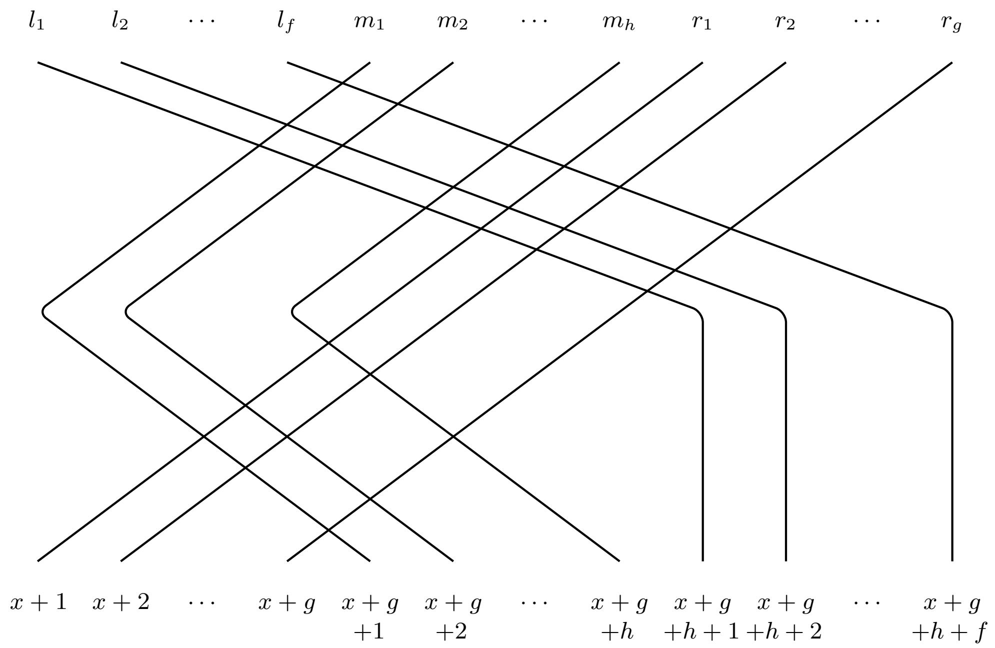


Figure 2.6: Crossings at the end of Lemma 2.5.

Proof. We will prove that for $0 \leq k \leq h-1$ we have that (2.1) is equal to

$$\begin{aligned}\left(\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+f+k+2}\cdot\left(\Psi_{x+f}\uparrow^{x+f+k-1}\right)\downarrow_{x+1}\\ \cdot\,\Psi_{x+f+k+1}\uparrow^{x+f+g+k}\cdot\left(\Psi_{x+f+k}\uparrow^{x+f+g+k}\right)\downarrow_{x+k+1}\\ \cdot\left(\Psi_{x+k}\uparrow^{x+g+k-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+g+k+1}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+g+k+2}\,.\end{aligned}$$

We can see that if $k=0$ we recover (2.1). So now by induction, suppose $\gamma \in \{0, 1, \ldots, h-2\}$ and consider

$$\begin{aligned}\left(\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+f+\gamma+2}\cdot\left(\Psi_{x+f}\uparrow^{x+f+\gamma-1}\right)\downarrow_{x+1}\\ \cdot\,\Psi_{x+f+\gamma+1}\uparrow^{x+f+g+\gamma}\cdot\left(\Psi_{x+f+\gamma}\uparrow^{x+f+g+\gamma}\right)\downarrow_{x+\gamma+1}\\ \cdot\left(\Psi_{x+\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+g+\gamma+1}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+g+\gamma+2}\end{aligned} \tag{2.2}$$

which is shown in Figure 2.7. Applying Lemma 2.4 to the second line (since one

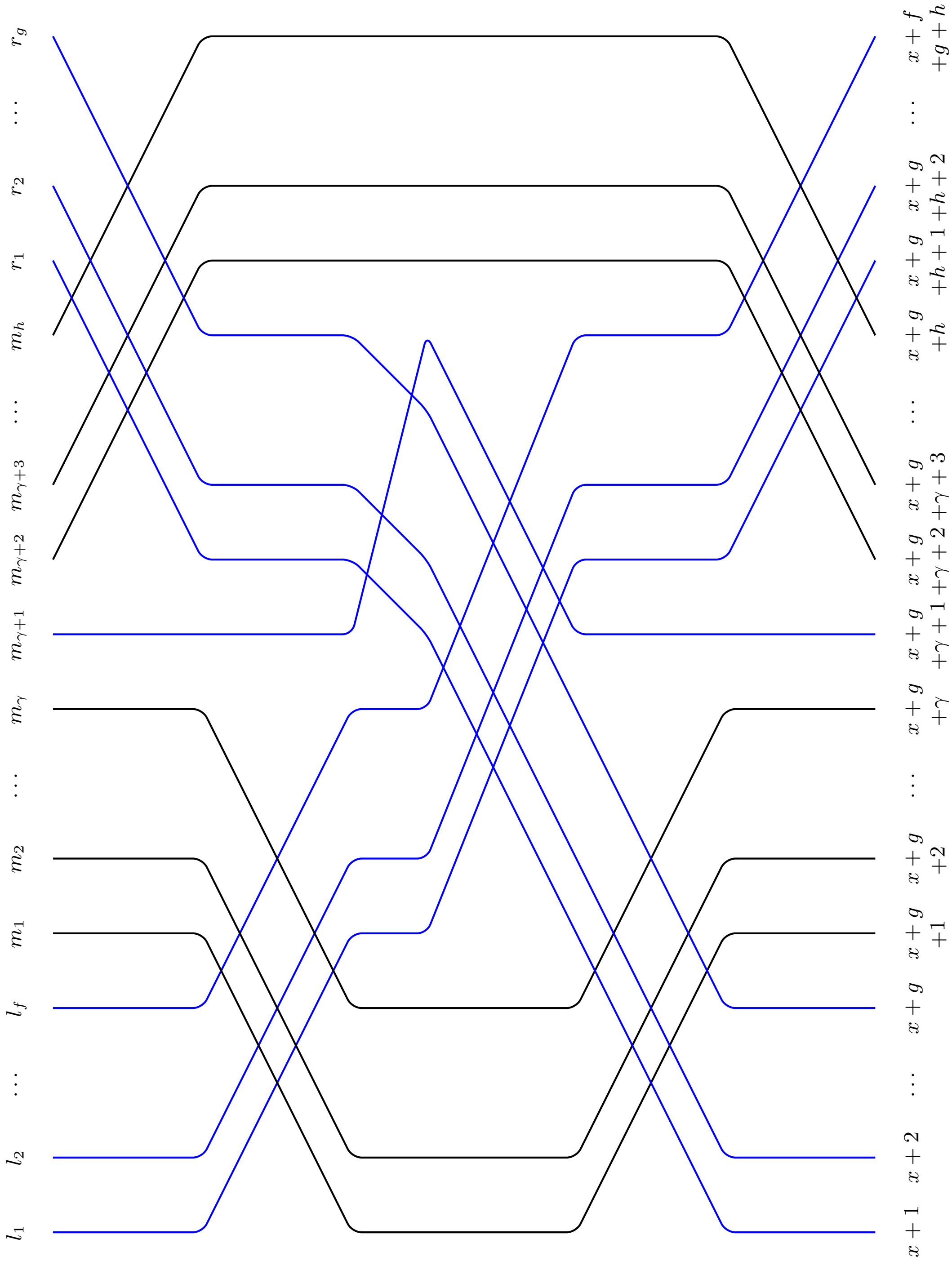


Figure 2.7: Crossings from (2.2). The strings involved in the application of Lemma 2.4 are coloured blue.

of (i), (ii) or (iii) holds), gives us that (2.2) is equal to

$$\begin{aligned}
&\left(\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+f+\gamma+2}\cdot\left(\Psi_{x+f}\uparrow^{x+f+\gamma-1}\right)\downarrow_{x+1}\\
&\quad\cdot\left(\Psi_{x+f+\gamma}\uparrow^{x+f+g+\gamma}\right)\downarrow_{x+\gamma+1}\cdot\Psi_{x+\gamma+1}\uparrow^{x+g+\gamma}\\
&\quad\cdot\left(\Psi_{x+\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+g+\gamma+1}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+g+\gamma+2}\\
&=\left(\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+f+\gamma+3}\cdot\left(\Psi_{x+f}\uparrow^{x+f+\gamma}\right)\downarrow_{x+1}\\
&\quad\cdot\Psi_{x+f+\gamma+2}\uparrow^{x+f+g+\gamma+1}\cdot\left(\Psi_{x+f+\gamma+1}\uparrow^{x+f+g+\gamma+1}\right)\downarrow_{x+\gamma+2}\\
&\quad\cdot\left(\Psi_{x+\gamma+1}\uparrow^{x+g+\gamma}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+g+\gamma+2}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+g+\gamma+3},
\end{aligned}$$

proving the inductive step.

So setting $k=h-1$ we have that (2.1) is equal to:

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+h-2}\right)\downarrow_{x+1}\cdot\Psi_{x+f+h}\uparrow^{x+f+g+h-1}\\
&\quad\cdot\left(\Psi_{x+f+h-1}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+h}\cdot\left(\Psi_{x+h-1}\uparrow^{x+g+h-2}\right)\downarrow_{x+1}\\
&=\left(\Psi_{x+f}\uparrow^{x+f+g+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+h}\uparrow^{x+g+h-1}\right)\downarrow_{x+1}
\end{aligned}$$

by applying Lemma 2.4 to the second line, which is what we want. □

In many cases when we will wish to use Lemma 2.5, we will want to pull the strings of residues $m_1,\ldots,m_h$ over other strings of which only a part will give the crossings in the setup of the lemma. Hence our final improvement in this circumstance is to introduce these extraneous crossings to the setup.

Corollary 2.6. *Suppose we have the crossings:*

$$\left(\Psi_{x+f+k+h}\uparrow^{x+f+k+h+g-1}\right)\downarrow_{x+k+f+1}\cdot\left(\Psi_{x+f}\uparrow^{x+f+k+h+g+t-1}\right)\downarrow_{x+1}\tag{2.3}$$

with residues $p_1,\ldots,p_k$, $l_1,\ldots,l_f$, $m_1,\ldots,m_h$, $r_1,\ldots,r_g$ and $q_1,\ldots,q_t$ as shown in Figure 2.8. Suppose also that one of the following occurs:

(i) $l_i \not\rightarrow m_j$ for every $i\in\{1,\ldots,f\}, j\in\{1,\ldots,h\}$,

(ii) $r_i \not\rightarrow m_j$ for every $i\in\{1,\ldots,g\}, j\in\{1,\ldots,h\}$,

(iii) $l_i \neq r_j$ *for every* $i \in \{1, \ldots, f\}$ *and* $j \in \{1, \ldots, g\}$.

Then (2.3) is equal to:

$$\left(\Psi_{x+f} \uparrow^{x+f+k+h+g+t-1}\right)\downarrow_{x+1} \cdot \left(\Psi_{x+k+h} \uparrow^{x+k+h+g-1}\right)\downarrow_{x+k+1} \cdot$$

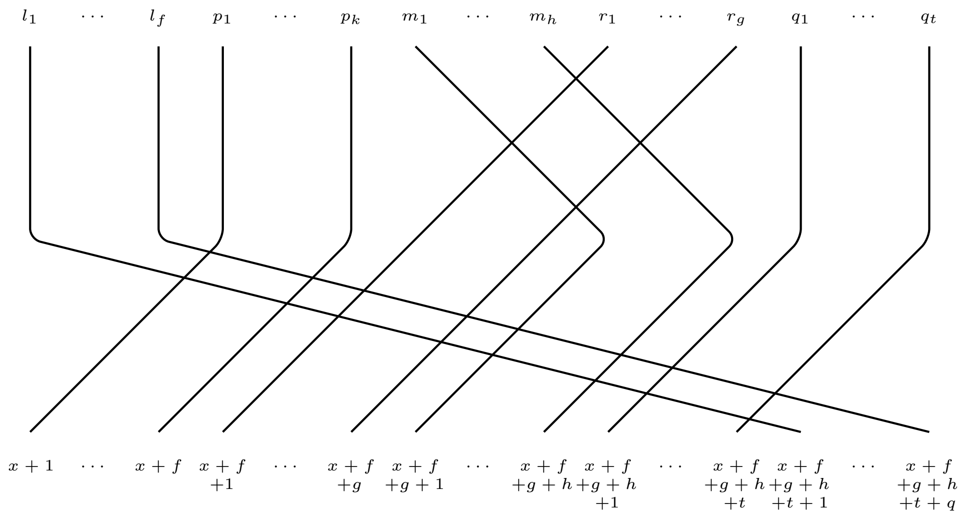


Figure 2.8: Crossings at the start of Corollary 2.6.

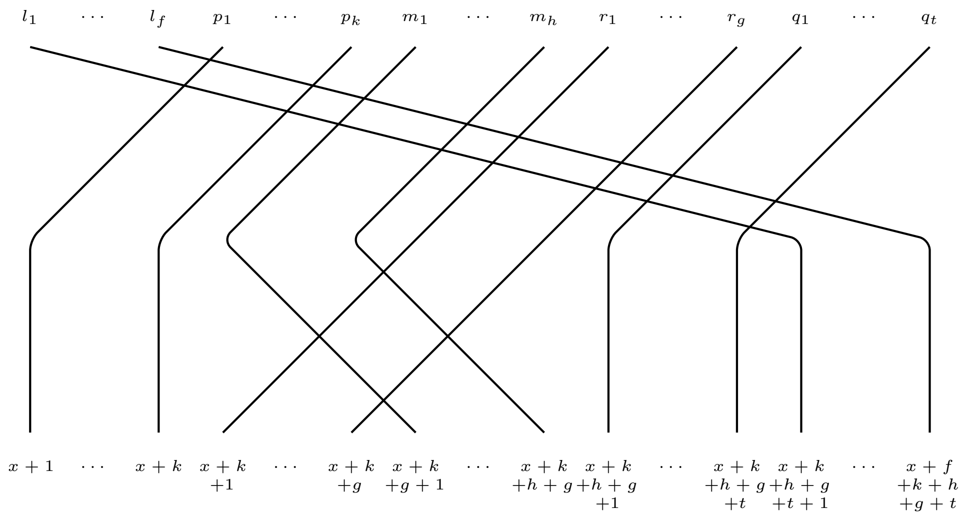


Figure 2.9: Crossings at the end of Corollary 2.6.

Proof.

$$
\begin{aligned}
&\left(\Psi_{x+f+k+h}{\uparrow}^{x+k+f+h+g-1}\right){\downarrow}_{x+k+f+1} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+k+h+g+t-1}\right){\downarrow}_{x+1} \\
= \;& \left(\Psi_{x+f}{\uparrow}^{x+f+k-1}\right){\downarrow}_{x+1} \cdot \left(\Psi_{x+f+k+h}{\uparrow}^{x+k+f+h+g-1}\right){\downarrow}_{x+k+f+1} \\
&\qquad \cdot \left(\Psi_{x+f+k}{\uparrow}^{x+f+k+h+g-1}\right){\downarrow}_{x+k+1} \\
&\qquad\quad \cdot \left(\Psi_{x+f+k+h+g}{\uparrow}^{x+f+k+h+g+t-1}\right){\downarrow}_{x+k+h+g+1} \\
= \;& \left(\Psi_{x+f}{\uparrow}^{x+f+k-1}\right){\downarrow}_{x+1} \cdot \left(\Psi_{x+f+k}{\uparrow}^{x+f+k+h+g-1}\right){\downarrow}_{x+k+1} \\
&\qquad \cdot \left(\Psi_{x+k+h}{\uparrow}^{x+k+h+g-1}\right){\downarrow}_{x+k+1} \\
&\qquad \cdot \left(\Psi_{x+f+k+h+g}{\uparrow}^{x+f+k+h+g+t-1}\right){\downarrow}_{x+k+h+g+1}
\end{aligned}
$$

by Lemma 5,

$$
= \left(\Psi_{x+f}{\uparrow}^{x+f+k+h+g+t-1}\right){\downarrow}_{x+1} \cdot \left(\Psi_{x+k+h}{\uparrow}^{x+k+h+g-1}\right){\downarrow}_{x+k+1} .
$$

□

Now we consider what happens if we have the setup of Lemma 2.3, but suppose instead that the residues of the two strings being pulled over each are one apart, giving us extra terms. In particular, these terms all begin with crossings whose leftmost string has residue equal to l, which will often be used to show that terms are zero when performing calculations within Specht modules.

Lemma 2.7. *Suppose we have the crossings* $\left(\Psi_{x+2}{\uparrow}^{x+g+1}\right){\downarrow}_{x+1} \cdot \psi_{x+g+1}$ *with residues* l*,* m*, and* $r_1, \ldots, r_g$ *as shown in Figure 2.10. Also suppose that there are* $z_1 < z_2 < \ldots < z_k$ *with each* $z_j \in \{1, \ldots, g\}$ *such that* $l = r_{z_j}$ *for every* $j \in \{1, \ldots, k\}$ *and* $l \neq r_i$ *for* $i \notin \{z_1, z_2, \ldots, z_k\}$*. These residues are shown in Figure 2.10. Now suppose that either*

(i) $l \leftarrow m$*, or*

(ii) $l \rightarrow m$*.*

Then

$$\begin{aligned}&\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1}\cdot\psi_{x+g+1}\\&=\psi_{x+1}\cdot\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1}\pm\sum_{j=1}^{k}\Psi_{x+z_j+2}\!\uparrow^{x+g+1}\Psi_{x+2}\!\uparrow^{x+g}\Psi_{x+1}\!\uparrow^{x+z_j-1}\end{aligned}$$

where the $\pm$ is a plus in case (i) and a minus in case (ii).

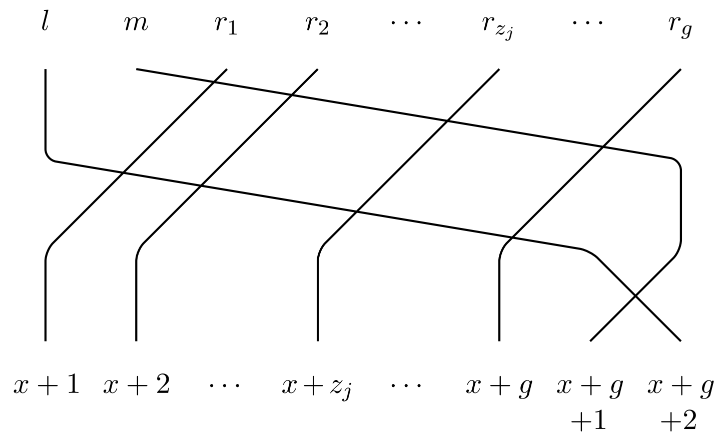


Figure 2.10: Crossings at the start of Lemma 2.7.

Proof. We have

$$\begin{aligned}&\left(\Psi_{x+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+1}\cdot\psi_{x+g+1}\\&\quad=\left(\Psi_{x+2}\!\uparrow^{x+z_k+1}\right)\!\downarrow_{x+1}\cdot\left(\Psi_{x+z_k+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+z_k+1}\cdot\psi_{x+g+1}\end{aligned}$$

which, by Lemma 2.3

$$\begin{aligned}&=\left(\Psi_{x+2}\!\uparrow^{x+z_k+1}\right)\!\downarrow_{x+1}\cdot\psi_{x+z_k+1}\cdot\left(\Psi_{x+z_k+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+z_k+1}\\&=\left(\Psi_{x+2}\!\uparrow^{x+z_k}\right)\!\downarrow_{x+1}\cdot\psi_{x+z_k+1}\psi_{x+z_k}\psi_{x+z_k+1}\cdot\left(\Psi_{x+z_k+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+z_k+1},\end{aligned}$$

then using the braid relation (1.11)

$$\begin{aligned}&=\left(\Psi_{x+2}\!\uparrow^{x+z_k}\right)\!\downarrow_{x+1}\cdot\psi_{x+z_k}\cdot\left(\Psi_{x+z_k+1}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+z_k}\\&\quad\pm\left(\Psi_{x+2}\!\uparrow^{x+z_k}\right)\!\downarrow_{x+1}\cdot\left(\Psi_{x+z_k+2}\!\uparrow^{x+g+1}\right)\!\downarrow_{x+z_k+1},\end{aligned}$$

where the $\pm$ is a plus in case (i) and a minus in case (ii). In the former term of the sum we repeat what we have just done using $(\Psi_{x+2}\uparrow^{x+z_k})\downarrow_{x+1} \cdot \psi_{x+z_k}$. That is, we rewrite $(\Psi_{x+2}\uparrow^{x+z_k})\downarrow_{x+1}$ as a product of two multiplicands to take the z_{k-1} into account, and use the braid relation to obtain two terms.

Repeating this k times in total we have

$$\begin{aligned}
&\left(\Psi_{x+2}\uparrow^{x+g+1}\right)\downarrow_{x+1} \cdot \psi_{x+g+1} \\
&\quad = \left(\Psi_{x+2}\uparrow^{x+z_1}\right)\downarrow_{x+1} \cdot \psi_{x+z_1} \cdot \left(\Psi_{x+z_1+1}\uparrow^{x+g+1}\right)\downarrow_{x+z_1} \\
&\qquad \pm \sum_{j=1}^{k} \left(\Psi_{x+2}\uparrow^{x+z_j}\right)\downarrow_{x+1} \cdot \left(\Psi_{x+z_j+2}\uparrow^{x+g+1}\right)\downarrow_{x+z_j+1} .
\end{aligned}$$

In the first term we can apply Lemma 2.3 to $(\Psi_{x+2}\uparrow^{x+z_1})\downarrow_{x+1} \cdot \psi_{x+z_1}$ to obtain $\psi_{x+1} \cdot (\Psi_{x+2}\uparrow^{x+z_1})\downarrow_{x+1}$ and in the terms of the sum we can slightly rearrange the entries so that all together we have

$$\begin{aligned}
&\left(\Psi_{x+2}\uparrow^{x+g+1}\right)\downarrow_{x+1} \cdot \psi_{x+g+1} \\
&= \psi_{x+1} \cdot \left(\Psi_{x+2}\uparrow^{x+g+1}\right)\downarrow_{x+1} \pm \sum_{j=1}^{k} \Psi_{x+z_j+2}\uparrow^{x+g+1} \; \Psi_{x+2}\uparrow^{x+g} \; \Psi_{x+1}\uparrow^{x+z_j-1}
\end{aligned}$$

as required. $\square$

The next lemma allows us to swiftly deal with multiple consecutively occurring cases of relation (1.10), as long as the relevant residues are sufficiently far apart.

Lemma 2.8. *Suppose we have the crossings*

$$\left(\Psi_{x+f}\uparrow^{x+f+h+g-1}\right)\downarrow_{x+1} \cdot \left(\Psi_{x+h+g}\uparrow^{x+f+h+g+k-1}\right)\downarrow_{x+h+1} \tag{2.4}$$

with residues $l_1, \dots, l_f, p_1, \dots, p_h, r_1, \dots, r_g, q_1, \dots, q_k$ as shown in Figure 2.11. Suppose that $l_1, l_2, \dots, l_f \not- r_1, r_2, \dots, r_g$. Then (2.4) is equal to

$$\left(\Psi_{x+f}\uparrow^{x+f+h-1}\right)\downarrow_{x+1} \cdot \left(\Psi_{x+f+h+g}\uparrow^{x+f+h+g+k-1}\right)\downarrow_{x+f+h+1} .$$

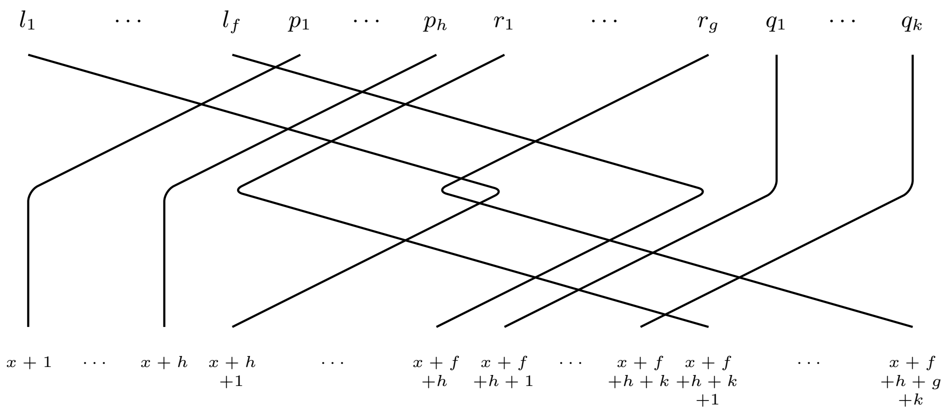


Figure 2.11: Crossings at the start of Lemma 2.8.

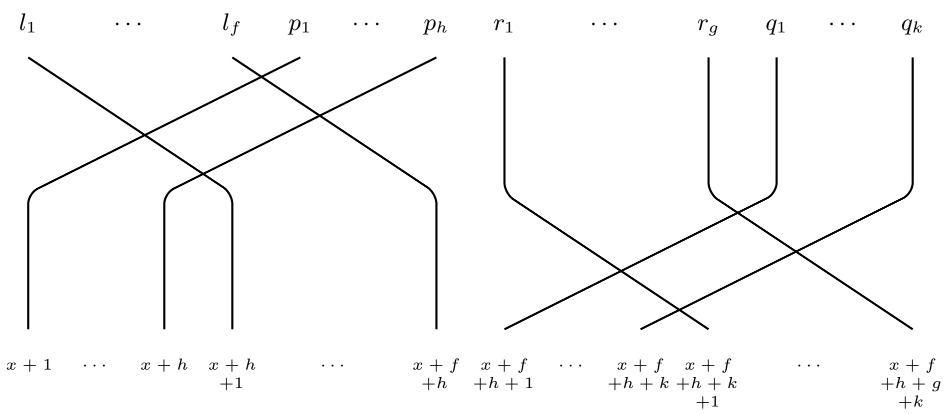


Figure 2.12: Crossings at the end of Lemma 2.8.

Proof. We have

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+h+g-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+h+g}\uparrow^{x+f+h+g+k-1}\right)\downarrow_{x+h+1}\\
&=\left(\Psi_{x+f}\uparrow^{x+f+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+h}\uparrow^{x+f+h+g-1}\right)\downarrow_{x+h+1}\\
&\qquad\cdot\left(\Psi_{x+h+g}\uparrow^{x+f+h+g-1}\right)\downarrow_{x+h+1}\cdot\left(\Psi_{x+f+h+g}\uparrow^{x+f+g+h+k-1}\right)\downarrow_{x+h+f-1}\\
&=\left(\Psi_{x+f}\uparrow^{x+f+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+h}\uparrow^{x+f+h+g-1}\right)\downarrow_{x+h+1}\\
&\qquad\cdot\left(\Psi^{x+h+g}\downarrow_{x+h+1}\right)\uparrow^{x+f+h+g-1}\cdot\left(\Psi_{x+f+h+g}\uparrow^{x+f+g+h+k-1}\right)\downarrow_{x+h+f-1}\,.
\end{aligned}$$

The second and third multiplicands here are equal to

$$\left(\Psi_{x+f+h}\uparrow^{x+f+h+g-1}\Psi_{x+f+h-1}\uparrow^{x+f+h+g-2}\cdots\Psi_{x+h+1}\uparrow^{x+h+g}\right)$$
$$\cdot\left(\Psi^{x+h+g}\downarrow_{x+h+1}\Psi^{x+h+g+1}\downarrow_{x+h+2}\cdots\Psi^{x+f+h+g-1}\downarrow_{x+f+h}\right)$$

and then since $l_1, l_2, \ldots, l_f \not- r_1, r_2, \ldots, r_g$ these brackets cancel each other out by forming squares using the relation (1.10). So we have that (2.4) is equal to

$$\left(\Psi_{x+f}\uparrow^{x+f+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+h+g}\uparrow^{x+f+g+h+k-1}\right)\downarrow_{x+h+f-1}.$$

□

The next lemma acts as an extension of Lemma 2.7, and as such exhibits when a fairly simple product may be equal to a large sum of terms. However, it is important to make note of the leading term in most of the summands, as usually it will be enough to then concern ourselves only with how this leading term interacts with the crossings above it in a diagram. Along with one 'regular' looking term, there will be f summands whose initial term is a crossing whose leftmost residue is l_i for $i \in \{1, \ldots, f\}$. In addition, there will be another sum of terms whose leading terms are crossings of various multiplicities, whose leftmost residues are always such that they are equal to an l_i.

Note that in the proof of the following lemma there is a $(*)$ in the margin that can be ignored for now and will be of use in a later chapter.

Lemma 2.9. *Suppose we have the crossings*

$$\Psi_{x+f+1}\uparrow^{x+2f+g}\cdot\left(\Psi_{x+f}\uparrow^{x+2f+g}\right)\downarrow_{x+1} \tag{2.5}$$

with residues $l_1, \ldots, l_f$, m and $r_1, \ldots r_{f+g}$ as shown in Figure 2.13. Suppose that the l_i are all distinct from each other and m, that $l_i = r_{g+i}$ for every $i \in \{1, \ldots, f\}$ and that

$$l_1 \leftarrow l_2 \leftarrow \cdots \leftarrow l_f \leftarrow m \rightarrow r_{f+g} \rightarrow r_{f+g-1} \rightarrow \cdots \rightarrow r_{g+1}.$$

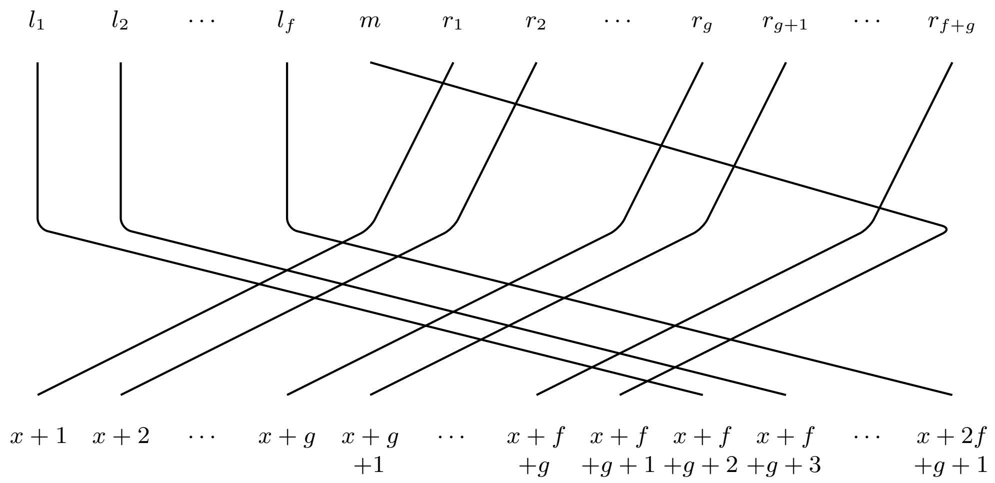


Figure 2.13: Crossings at the start of Lemma 2.9.

Then (2.5) is equal to

$$
\begin{aligned}
&\sum_{s=0}^{f-1} \psi_{x+f-s} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\cdot \left(\Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-1}\right)\downarrow_{x+f-s} \cdot \psi_{x+2f+g-2s} \\
&\quad \cdot \psi_{x+2f+g-2s-1} \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \\
&\qquad \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1} \\
&+ \sum_{s=0}^{f-1} \sum_{j=1}^{k_s} \Big[\psi_{x+f+z_{s_j}+1} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\quad \cdot \Psi_{x+f+z_{s_j}-s+2}\uparrow^{x+2f+g-2s-1} \cdot \Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-2} \\
&\quad \cdot \Psi_{x+f-s}\uparrow^{x+f+z_{s_j}-s-2} \cdot \psi_{x+2f+g-2s}\psi_{x+2f+g-2s-1} \\
&\quad \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1}\Big] \\
&+ \left(\Psi_{x+f+1}\uparrow^{x+f+g}\right)\downarrow_{x+1}
\end{aligned}
$$

for some constants $k_0, k_1, \ldots, k_{f-1} \geq 0$.

Proof. We will prove that for $0 \leq K \leq f$ we have that (2.5) is equal to

$$
\begin{aligned}
&\sum_{s=0}^{K-1} \psi_{x+f-s} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\cdot \left(\Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-1}\right)\downarrow_{x+f-s} \cdot \psi_{x+2f+g-2s} \\
&\quad \cdot \psi_{x+2f+g-2s-1} \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \\
&\qquad \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1} \\
&+\sum_{s=0}^{K-1}\sum_{j=1}^{k_s} \Big[\psi_{x+f+z_{s_j}+1} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\quad \cdot \Psi_{x+f+z_{s_j}-s+2}\uparrow^{x+2f+g-2s-1} \cdot \Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-2} \\
&\quad \cdot \Psi_{x+f-s}\uparrow^{x+f+z_{s_j}-s-2} \cdot \psi_{x+2f+g-2s}\psi_{x+2f+g-2s-1} \\
&\quad \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1}\Big] \\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-K}\right)\downarrow_{x+f+1-K} \cdot \left(\Psi_{x+f-K}\uparrow^{x+2f+g-K}\right)\downarrow_{x+1}
\end{aligned}
$$

for some constants $k_0, k_1, \ldots, k_{K-1} \geq 0$.We can see that if $K = 0$ we recover (2.5). So now by induction, suppose $\gamma \in \{0, 1, \ldots, f-1\}$ and that (2.5) is equal to

$$
\begin{aligned}
&\sum_{s=0}^{\gamma-1} \psi_{x+f-s} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\cdot \left(\Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-1}\right)\downarrow_{x+f-s} \cdot \psi_{x+2f+g-2s} \\
&\quad \cdot \psi_{x+2f+g-2s-1} \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \\
&\qquad \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1} \\
&+\sum_{s=0}^{\gamma-1}\sum_{j=1}^{k_s} \Big[\psi_{x+f+z_{s_j}+1} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\quad \cdot \Psi_{x+f+z_{s_j}-s+2}\uparrow^{x+2f+g-2s-1} \cdot \Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-2} \\
&\quad \cdot \Psi_{x+f-s}\uparrow^{x+f+z_{s_j}-s-2} \cdot \psi_{x+2f+g-2s}\psi_{x+2f+g-2s-1} \\
&\quad \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1}\Big] \\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+1-\gamma} \cdot \left(\Psi_{x+f-\gamma}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+1}\,. \qquad (2.6)
\end{aligned}
$$

We show some of the latter term of (2.6) in Figure 2.14. Rewriting the latter term

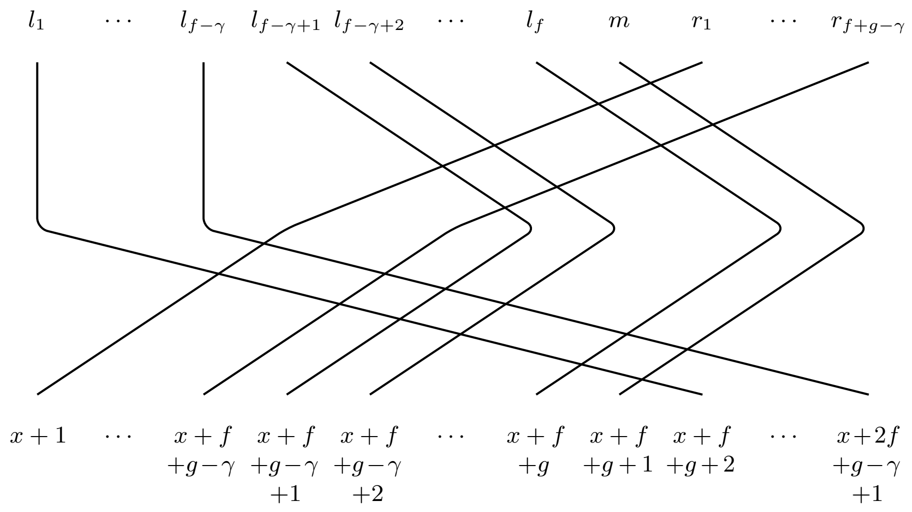


Figure 2.14: Crossings from the latter term of (2.6).

of (2.6) we have that it is equal to

$$\begin{aligned}
&\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\\
&\quad\cdot\psi_{x+2f+g-2\gamma}\psi_{x+2f+g-2\gamma-1}\psi_{x+2f+g-2\gamma}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\\
&\qquad\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1}\\
=&\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\\
&\quad\cdot\psi_{x+2f+g-2\gamma-1}\psi_{x+2f+g-2\gamma}\psi_{x+2f+g-2\gamma-1}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\\
&\qquad\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1}\\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\\
&\quad\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1},
\end{aligned}\tag{2.7}$$

by applying the braid relation (1.11) to $\psi_{x+2f+g-2\gamma}\psi_{x+2f+g-2\gamma-1}\psi_{x+2f+g-2\gamma}$ (since $l_{f-\gamma}\leftarrow l_{f-\gamma+1}\rightarrow r_{f+g-\gamma}$).

The latter term of (2.7) is equal to

$$\begin{aligned}
&\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\\
&\quad\cdot\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1},
\end{aligned}$$

so we can apply Lemma 2.8 to

$$\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}$$

since $l_{f-\gamma+2},\ldots,l_f \not\leftarrow r_{f+g-\gamma}$ (take $\bar{x}=x+f-\gamma+1, \bar{f}=\gamma, \bar{h}=f+g-\gamma-1, \bar{g}=1, \bar{k}=0$, where $\bar{x},\bar{f},\bar{g},\bar{h},\bar{k}$ are the x,f,g,h and k in the hypotheses of Lemma 2.8). So this term is equal to

$$\left(\Psi_{x+f+1}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+f+1-(\gamma+1)}\cdot\left(\Psi_{x+f-(\gamma+1)}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+1}.$$

Now also apply Lemma 2.7 to $\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\cdot\psi_{x+2f+g-2\gamma-1}$ in the former term of (2.7) since $l_{f-\gamma}\leftarrow l_{f-\gamma+1}$ giving

$$\begin{aligned}
&\psi_{x+f-\gamma}\cdot\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma}\\
&\qquad\cdot\psi_{x+2f+g-2\gamma}\cdot\psi_{x+2f+g-2\gamma-1}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\\
&\qquad\qquad\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1}\\
&+\sum_{j=1}^{k_\gamma}\Big[\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\psi_{x+f+z_{\gamma_j}+1-\gamma}\\
&\qquad\cdot\Psi_{x+f+z_{\gamma_j}-\gamma+2}\uparrow^{x+2f+g-2\gamma-1}\cdot\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-2}\\
&\qquad\cdot\Psi_{x+f-\gamma}\uparrow^{x+f+z_{\gamma_j}-\gamma-2}\cdot\psi_{x+2f+g-2\gamma}\\
&\qquad\cdot\psi_{x+2f+g-2\gamma-1}\cdot\Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma}\cdot\left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1}\Big]\\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+f+1-(\gamma+1)}\cdot\left(\Psi_{x+f-(\gamma+1)}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+1}
\end{aligned}$$

for some constant $k_\gamma\geq 0$, and we have $r_{z_{\gamma_j}}=l_{f-\gamma}$ for each $j\in\{1,\ldots,k_\gamma\}$.

In the terms arising in the sum from 1 to k_γ, apply Corollary 2.6 to $\left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\psi_{x+f+z_{\gamma_j}+1-\gamma}$ since $l_{f-\gamma+2},\ldots,l_f\not\leftarrow r_{z_\gamma}$. Thus (∗)

the former term of (2.7) is equal to

$$\begin{aligned}
&\psi_{x+f-\gamma} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma} \cdot \left(\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-1}\right)\downarrow_{x+f-\gamma} \\
&\qquad \cdot\ \psi_{x+2f+g-2\gamma} \cdot \psi_{x+2f+g-2\gamma-1} \cdot \Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma} \\
&\qquad\qquad \cdot \left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1} \\
&+\sum_{j=1}^{k_\gamma}\Big[\psi_{x+f+z_{\gamma_j}+1} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma} \\
&\quad \cdot\Psi_{x+f+z_{\gamma_j}-\gamma+2}\uparrow^{x+2f+g-2\gamma-1} \cdot\Psi_{x+f+1-\gamma}\uparrow^{x+2f+g-2\gamma-2} \cdot\Psi_{x+f-\gamma}\uparrow^{x+f+z_{\gamma_j}-\gamma-2} \\
&\qquad \cdot\psi_{x+2f+g-2\gamma} \cdot \psi_{x+2f+g-2\gamma-1} \cdot \Psi_{x+2f+g-2\gamma+1}\uparrow^{x+2f+g-\gamma} \\
&\qquad\qquad \cdot \left(\Psi_{x+f-\gamma-1}\uparrow^{x+2f+g-\gamma-1}\right)\downarrow_{x+1}\Big] \\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+f+1-(\gamma+1)} \cdot \left(\Psi_{x+f-(\gamma+1)}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+1}\ .
\end{aligned}$$

So using this we have that (2.6) is equal to

$$\begin{aligned}
&\sum_{s=0}^{\gamma}\psi_{x+f-s} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\cdot \left(\Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-1}\right)\downarrow_{x+f-s} \cdot \psi_{x+2f+g-2s} \\
&\quad \cdot \psi_{x+2f+g-2s-1} \cdot \Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \\
&\qquad \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1} \\
&+\sum_{s=0}^{\gamma}\sum_{j=1}^{k_s}\Big[\psi_{x+f+z_{s_j}+1} \cdot \left(\Psi_{x+f+1}\uparrow^{x+2f+g-s}\right)\downarrow_{x+f+2-s} \\
&\quad \cdot\Psi_{x+f+z_{s_j}-s+2}\uparrow^{x+2f+g-2s-1} \cdot\Psi_{x+f+1-s}\uparrow^{x+2f+g-2s-2} \\
&\quad \cdot\Psi_{x+f-s}\uparrow^{x+f+z_{s_j}-s-2} \cdot \psi_{x+2f+g-2s}\psi_{x+2f+g-2s-1} \\
&\quad \cdot\Psi_{x+2f+g-2s+1}\uparrow^{x+2f+g-s} \cdot \left(\Psi_{x+f-s-1}\uparrow^{x+2f+g-s-1}\right)\downarrow_{x+1}\Big] \\
&+\left(\Psi_{x+f+1}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+f+1-(\gamma+1)} \cdot \left(\Psi_{x+f-(\gamma+1)}\uparrow^{x+2f+g-(\gamma+1)}\right)\downarrow_{x+1}
\end{aligned}$$

and we have shown the inductive step. Taking $K = f$ we obtain the desired result.

□

With the next lemma, we consider attempting to pull one string over some amount of other strings, where at each step we immediately obtain an extra term due to relation (1.11). The result is a sum of f terms each beginning with a crossing whose leftmost residue is $l_i = r_i$ for $i \in \{1, \ldots, f\}$, plus one term beginning with a crossing whose leftmost residue is m.

Lemma 2.10. *Suppose that we have the crossings*

$$\left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+1} \cdot \Psi_{x+1}\uparrow^{x+g} \tag{2.8}$$

with residues $l_1, l_2, \ldots, l_f$, m, $r_1, r_2, \ldots, r_f, r_{f+1}, \ldots, r_g$ *as shown in Figure 2.15. Suppose that the* l_i *are pairwise distinct and not equal to* m*, that* $l_i = r_i$ *for every* $i \in \{1, \ldots, f\}$*, and that*

$$l_f \to l_{f-1} \to \cdots \to l_1 \to m \leftarrow r_1 \leftarrow r_2 \leftarrow \cdots \leftarrow r_f.$$

Then (2.8) is equal to

$$\begin{aligned}&\sum_{i=1}^{f}\Big[\psi_{x+f+i} \cdot \left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+i} \\ &\qquad \cdot \Psi_{x+2i}\uparrow^{x+g+i-1} \cdot \left(\Psi_{x+2i-2}\uparrow^{x+g+i-2}\right)\downarrow_{x+i}\Big] \\ &+ \left(\Psi_{x+2f+1}\uparrow^{x+f+g}\right)\downarrow_{x+f+1}.\end{aligned}$$

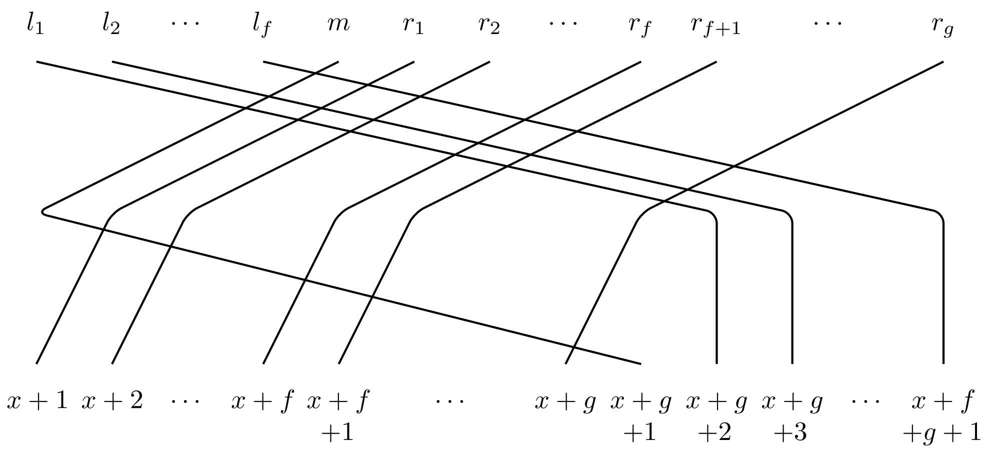


Figure 2.15: Crossings at the start of Lemma 2.10.

Proof. We will prove that for $0 \leq k \leq f$ we have that (2.8) is equal to

$$\sum_{i=1}^{k} \Big[\psi_{x+f+i} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+i} \cdot \Psi_{x+2i}{\uparrow}^{x+g+i-1} \cdot \left(\Psi_{x+2i-2}{\uparrow}^{x+g+i-2}\right){\downarrow}_{x+i}\Big] + \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+k+1} \cdot \left(\Psi_{x+2k+1}{\uparrow}^{x+g+k}\right){\downarrow}_{x+k+1} .$$

We can see that if $k = 0$ we recover (2.8). So now by induction, suppose $\gamma \in \{0, 1, \ldots, f-1\}$ and that (2.8) is equal to

$$\sum_{i=1}^{\gamma} \Big[\psi_{x+f+i} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+i} \cdot \Psi_{x+2i}{\uparrow}^{x+g+i-1} \cdot \left(\Psi_{x+2i-2}{\uparrow}^{x+g+i-2}\right){\downarrow}_{x+i}\Big] \tag{2.9}$$

$$+ \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+\gamma+1} \cdot \left(\Psi_{x+2\gamma+1}{\uparrow}^{x+g+\gamma}\right){\downarrow}_{x+\gamma+1} . \tag{2.10}$$

We will consider what happens to the latter term, i.e. (2.10), which is shown here in Figure 2.16. Rewriting the terms, we have that (2.10) is equal to

$$\left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+\gamma+2} \cdot \Psi_{x+\gamma+1}{\uparrow}^{x+2\gamma} \cdot \psi_{x+2\gamma+1}\psi_{x+2\gamma+2}\psi_{x+2\gamma+1} \cdot \Psi_{x+2\gamma+3}{\uparrow}^{x+g+\gamma+1} \cdot \Psi_{x+2\gamma+2}{\uparrow}^{x+g+\gamma} \cdot \left(\Psi_{x+2\gamma}{\uparrow}^{x+g+\gamma-1}\right){\downarrow}_{x+\gamma+1}$$

and then applying the braid relation (1.11) since $l_{\gamma+1} \to r_\gamma \leftarrow r_{\gamma+1}$,

$$= \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+\gamma+2} \cdot \Psi_{x+\gamma+1}{\uparrow}^{x+2\gamma} \cdot \psi_{x+2\gamma+2}\psi_{x+2\gamma+1}\psi_{x+2\gamma+2} \cdot \Psi_{x+2\gamma+3}{\uparrow}^{x+g+\gamma+1} \cdot \Psi_{x+2\gamma+2}{\uparrow}^{x+g+\gamma} \cdot \left(\Psi_{x+2\gamma}{\uparrow}^{x+g+\gamma-1}\right){\downarrow}_{x+\gamma+1} \tag{2.11}$$

$$+ \left(\Psi_{x+f}{\uparrow}^{x+f+g}\right){\downarrow}_{x+\gamma+2} \cdot \Psi_{x+\gamma+1}{\uparrow}^{x+2\gamma} \cdot \Psi_{x+2\gamma+3}{\uparrow}^{x+g+\gamma+1} \cdot \Psi_{x+2\gamma+2}{\uparrow}^{x+g+\gamma} \cdot \left(\Psi_{x+2\gamma}{\uparrow}^{x+g+\gamma-1}\right){\downarrow}_{x+\gamma+1} . \tag{2.12}$$

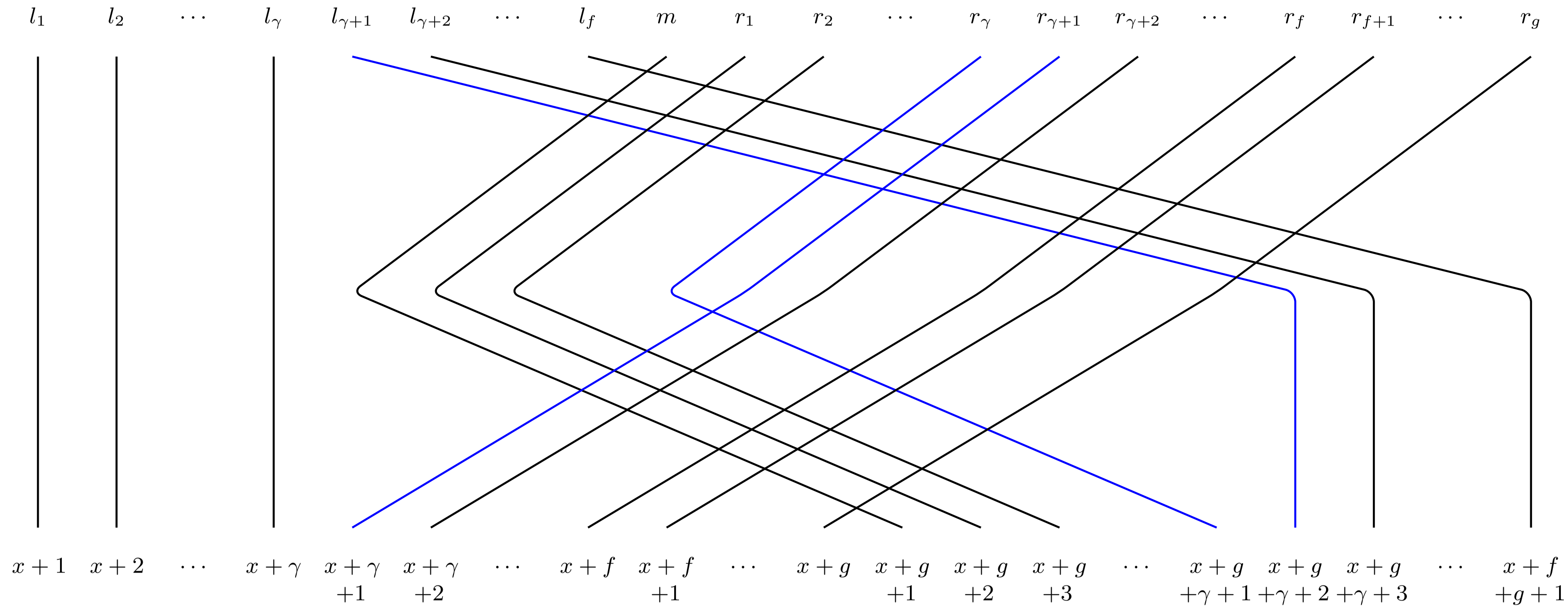


Figure 2.16: Crossings of the term (2.10) during the induction stage of Lemma 2.10. The strings that we will apply the braid relation (1.11) to are coloured blue.

Commuting $\psi_{x+2\gamma+2}$ with multiple crossings we have that (2.11) is equal to

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+\gamma+2}\cdot\psi_{x+2\gamma+2}\cdot\ \Psi_{x+\gamma+1}\uparrow^{x+g+\gamma+1}\\
&\qquad\cdot\Psi_{x+2\gamma+2}\uparrow^{x+g+\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}\\
=&\left(\Psi_{x+f}\uparrow^{x+f+\gamma-1}\right)\downarrow_{x+\gamma+2}\cdot\left(\Psi_{x+f+\gamma}\uparrow^{x+f+\gamma+1}\right)\downarrow_{x+2\gamma+2}\cdot\psi_{x+2\gamma+2}\\
&\qquad\cdot\left(\Psi_{x+f+\gamma+2}\uparrow^{x+f+g}\right)\downarrow_{x+2\gamma+4}\cdot\Psi_{x+\gamma+1}\uparrow^{x+g+\gamma+1}\\
&\qquad\qquad\cdot\Psi_{x+2\gamma+2}\uparrow^{x+g+\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}
\end{aligned}$$

and then applying Lemma 2.4 to $\left(\Psi_{x+f+\gamma}\uparrow^{x+f+\gamma+1}\right)\downarrow_{x+2\gamma+2}\cdot\psi_{x+2\gamma+2}$,

$$\begin{aligned}
=&\left(\Psi_{x+f}\uparrow^{x+f+\gamma-1}\right)\downarrow_{x+\gamma+2}\cdot\psi_{x+f+\gamma+1}\cdot\left(\Psi_{x+f+\gamma}\uparrow^{x+f+\gamma+1}\right)\downarrow_{x+2\gamma+2}\\
&\qquad\cdot\left(\Psi_{x+f+\gamma+2}\uparrow^{x+f+g}\right)\downarrow_{x+2\gamma+4}\cdot\Psi_{x+\gamma+1}\uparrow^{x+g+\gamma+1}\\
&\qquad\qquad\cdot\Psi_{x+2\gamma+2}\uparrow^{x+g+\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}\\
=&\ \psi_{x+f+\gamma+1}\cdot\left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+\gamma+1}\\
&\cdot\Psi_{x+2\gamma+2}\uparrow^{x+g+\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}\ .
\end{aligned}\tag{2.13}$$

Now consider (2.12), as shown in Figure 2.17. This is equal to

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+\gamma+2}\cdot\Psi_{x+2\gamma+3}\uparrow^{x+g+\gamma+1}\cdot\Psi_{x+2\gamma+2}\uparrow^{x+g+\gamma}\\
&\qquad\cdot\ \Psi_{x+\gamma+1}\uparrow^{x+2\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}
\end{aligned}$$

and we can apply Lemma 2.8 to $\Psi_{x+\gamma+1}\uparrow^{x+2\gamma}\cdot\left(\Psi_{x+2\gamma}\uparrow^{x+g+\gamma-1}\right)\downarrow_{x+\gamma+1}$. Then (2.12) is equal to

$$\left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+\gamma+2}\cdot\left(\Psi_{x+2\gamma+3}\uparrow^{x+g+\gamma+1}\right)\downarrow_{x+\gamma+2}\ .\tag{2.14}$$

So putting what we have done together, we have that (2.10) is the sum of

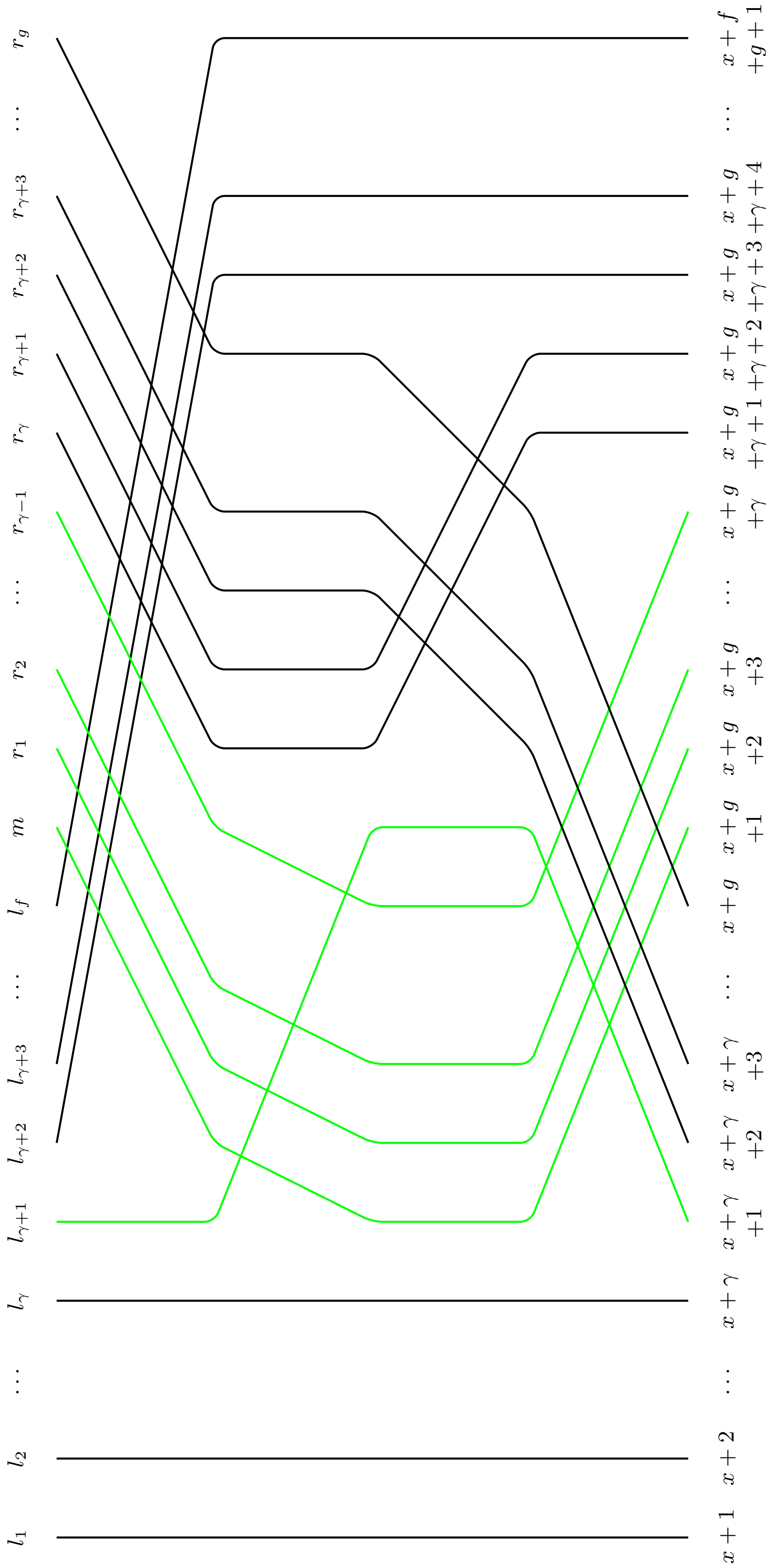


Figure 2.17: Crossings of the term (2.12). The strings to which we apply Lemma 2.8 are coloured green.

(2.13) and (2.14), i.e. (2.10) is equal to

$$\begin{aligned}
&\psi_{x+f+(\gamma+1)} \cdot \left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+(\gamma+1)} \\
&\quad \cdot \Psi_{x+2(\gamma+1)}\uparrow^{x+g+(\gamma+1)-1} \cdot \left(\Psi_{x+2(\gamma+1)-2}\uparrow^{x+g+(\gamma+1)-2}\right)\downarrow_{x+(\gamma+1)} \\
&+ \left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+(\gamma+1)+1} \cdot \left(\Psi_{x+2(\gamma+1)+1}\uparrow^{x+g+(\gamma+1)}\right)\downarrow_{x+(\gamma+1)+1}.
\end{aligned}$$

Finally, we can combine the latter term here with (2.9), so that (2.8) is equal to:

$$\begin{aligned}
&\sum_{i=1}^{\gamma+1}\Big[\psi_{x+f+i} \cdot \left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+i} \\
&\quad \cdot \Psi_{x+2i}\uparrow^{x+g+i-1} \cdot \left(\Psi_{x+2i-2}\uparrow^{x+g+i-2}\right)\downarrow_{x+i}\Big] \\
&+ \left(\Psi_{x+f}\uparrow^{x+f+g}\right)\downarrow_{x+(\gamma+1)+1} \cdot \left(\Psi_{x+2(\gamma+1)+1}\uparrow^{x+g+(\gamma+1)}\right)\downarrow_{x+(\gamma+1)+1},
\end{aligned}$$

proving the inductive step. Thus taking $k = f$, we obtain the desired result.

□

The next lemma speeds up the process of using relation (1.8) when we are required to move a dot through multiple crossings, whose residues may mean we encounter multiple additional terms. We obtain one term where the dot is pulled past all the crossings, along with a sum of terms which can be rearranged in order to have their leading term be a crossing whose leftmost residue is equal to l.

Lemma 2.11. *Suppose we have $\Psi_{x+1}\uparrow^{x+g} \cdot y_{x+g+1}$ with residues l and $r_1, \ldots, r_g$ as shown in Figure 2.18. Also suppose that there are $z_1 < z_2 < \ldots < z_k$ with*

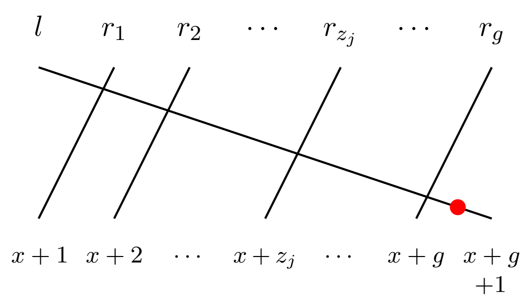


Figure 2.18: Crossings and dot at the start of Lemma 2.11.

each $z_j \in \{1, \ldots, g\}$ such that $l = r_{z_j}$ for every $j \in \{1, \ldots, k\}$, and $l \neq r_i$ for

$i \notin \{z_1, z_2, \ldots, z_k\}$. *Then*

$$\Psi_{x+1}{\uparrow}^{x+g} \cdot y_{x+g+1} = y_{x+1} \cdot \Psi_{x+1}{\uparrow}^{x+g} + \sum_{j=1}^{k} \Psi_{x+1}{\uparrow}^{x+z_j-1} \cdot \Psi_{x+z_j+1}{\uparrow}^{x+g} .$$

Proof. We have

$$\begin{aligned}
\Psi_{x+1}{\uparrow}^{x+g} \cdot y_{x+g+1} &= \Psi_{x+1}{\uparrow}^{x+z_k-1} \cdot \psi_{x+z_k} y_{x+z_k+1} \cdot \Psi_{x+z_k+1}{\uparrow}^{x+g} \\
&= \Psi_{x+1}{\uparrow}^{x+z_k-1} \cdot y_{x+z_k} \cdot \Psi_{x+z_k}{\uparrow}^{x+g} \\
&\qquad + \Psi_{x+1}{\uparrow}^{x+z_k-1} \cdot \Psi_{x+z_k+1}{\uparrow}^{x+g}
\end{aligned}$$

by relation (1.8). Repeating this k times in total we have

$$\begin{aligned}
&= \Psi_{x+1}{\uparrow}^{x+z_1-1} \cdot y_{x+z_1} \cdot \Psi_{x+z_1}{\uparrow}^{x+g} \\
&\qquad + \sum_{j=1}^{k} \Psi_{x+1}{\uparrow}^{x+z_j-1} \cdot \Psi_{x+z_j+1}{\uparrow}^{x+g} \\
&= y_{x+1} \cdot \Psi_{x+1}{\uparrow}^{x+g} + \sum_{j=1}^{k} \Psi_{x+1}{\uparrow}^{x+z_j-1} \cdot \Psi_{x+z_j+1}{\uparrow}^{x+g} .
\end{aligned}$$

□

The next lemma combines the pulling over of strings from Lemma 2.5 with the cancelling of square terms as in Lemma 2.8. We assume that the relevant residues are far enough apart so that we can pull multiple strings over each other before the ensuing squares disappear.

Lemma 2.12. *Suppose we have the crossings*

$$\left(\Psi_{x+f}{\uparrow}^{x+f+g+k+h-1}\right){\downarrow}_{x+1} \cdot \left(\Psi_{x+k+h}{\uparrow}^{x+f+g+k+h+t-1}\right){\downarrow}_{x+k+1} \tag{2.15}$$

with residues $l_1, \ldots, l_f$, $p_1, \ldots, p_k$, $m_1, \ldots, m_h$, $r_1, \ldots, r_g$, $q_1, \ldots, q_t$ *as shown in Figure 2.19. Suppose that* $l_1, \ldots, l_f \not\to m_1, \ldots, m_h$. *Then* (2.15) *is equal to*

$$\left(\Psi_{x+f+k+h}{\uparrow}^{x+f+g+k+h+t-1}\right){\downarrow}_{x+f+k+1} \cdot \left(\Psi_{x+f}{\uparrow}^{x+f+g+k-1}\right){\downarrow}_{x+1} .$$

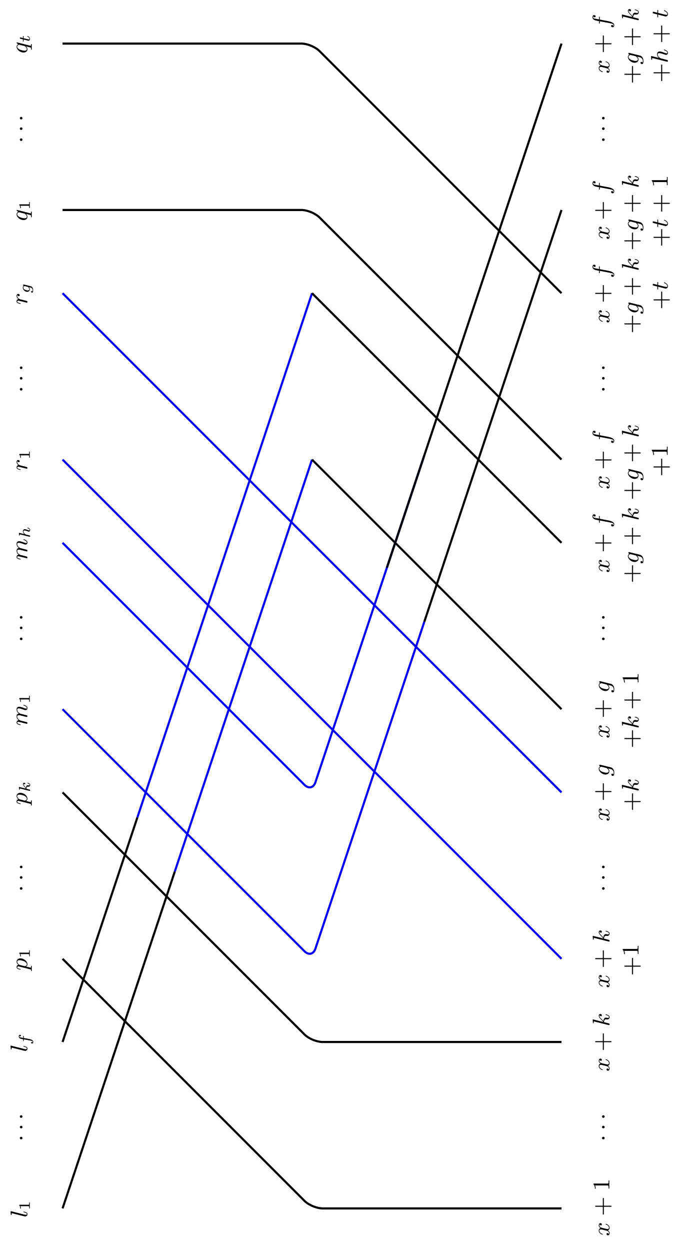


Figure 2.19: Crossings at the start of Lemma 2.12. The crossings to which we apply Lemma 2.5 are coloured blue.

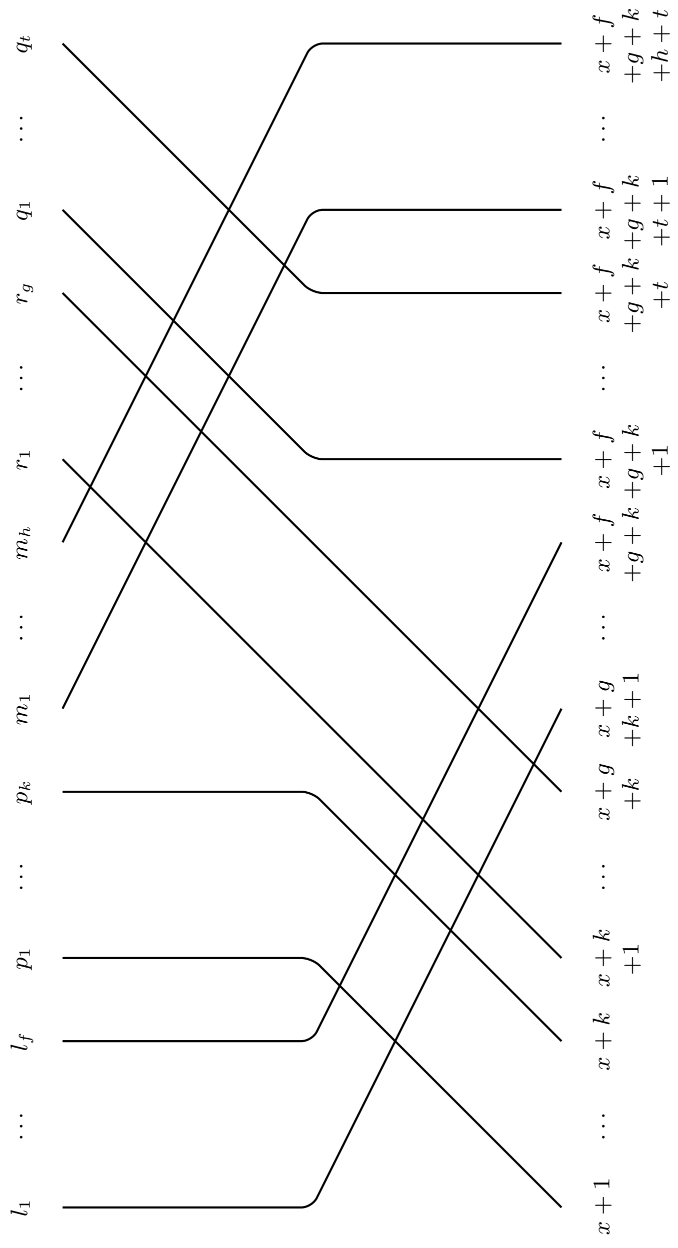


Figure 2.20: Crossings at the end of Lemma 2.12.

Proof. We have

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+g+k+h-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+k+1}\\
&=\left(\Psi_{x+f}\uparrow^{x+f+k-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+k}\uparrow^{x+f+g+k+h-1}\right)\downarrow_{x+k+1}\\
&\qquad\cdot\left(\Psi_{x+k+h}\uparrow^{x+g+k+h-1}\right)\downarrow_{x+k+1}\\
&\qquad\qquad\cdot\left(\Psi_{x+g+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+g+k+1}\,.
\end{aligned}$$

Apply Lemma 2.5 to

$$\left(\Psi_{x+f+k}\uparrow^{x+f+g+k+h-1}\right)\downarrow_{x+k+1}\cdot\left(\Psi_{x+k+h}\uparrow^{x+g+k+h-1}\right)\downarrow_{x+k+1}$$

since $l_1,\ldots,l_f \not\to m_1,\ldots,m_h$, so then (2.15) is equal to

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+k-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+k+h}\uparrow^{x+g+k+h-1}\right)\downarrow_{x+f+k+1}\\
&\qquad\cdot\left(\Psi_{x+f+k}\uparrow^{x+f+g+k+h-1}\right)\downarrow_{x+k+1}\\
&\qquad\qquad\cdot\left(\Psi_{x+g+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+g+k+1}
\end{aligned}\tag{2.16}$$

which is shown in Figure 2.21. Now we can apply Lemma 2.8 to

$$\left(\Psi_{x+f+k}\uparrow^{x+f+g+k+h-1}\right)\downarrow_{x+k+1}\cdot\left(\Psi_{x+g+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+g+k-1}$$

since $l_1,\ldots,l_f \not\to m_1,\ldots,m_h$. Thus (2.15) is equal to

$$\begin{aligned}
&\left(\Psi_{x+f}\uparrow^{x+f+k-1}\right)\downarrow_{x+1}\cdot\left(\Psi_{x+f+k+h}\uparrow^{x+g+k+h-1}\right)\downarrow_{x+f+k+1}\\
&\qquad\cdot\left(\Psi_{x+f+k}\uparrow^{x+f+g+k-1}\right)\downarrow_{x+k+1}\\
&\qquad\cdot\left(\Psi_{x+f+g+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+f+g+k+1}\\
&=\left(\Psi_{x+f+k+h}\uparrow^{x+f+g+k+h+t-1}\right)\downarrow_{x+f+k+1}\cdot\left(\Psi_{x+f}\uparrow^{x+f+g+k-1}\right)\downarrow_{x+1}\,.
\end{aligned}$$

□

The final two lemmas extend a couple of the earlier results to the case where there are multiple sets of strings to be negotiated, that is, we are able to apply

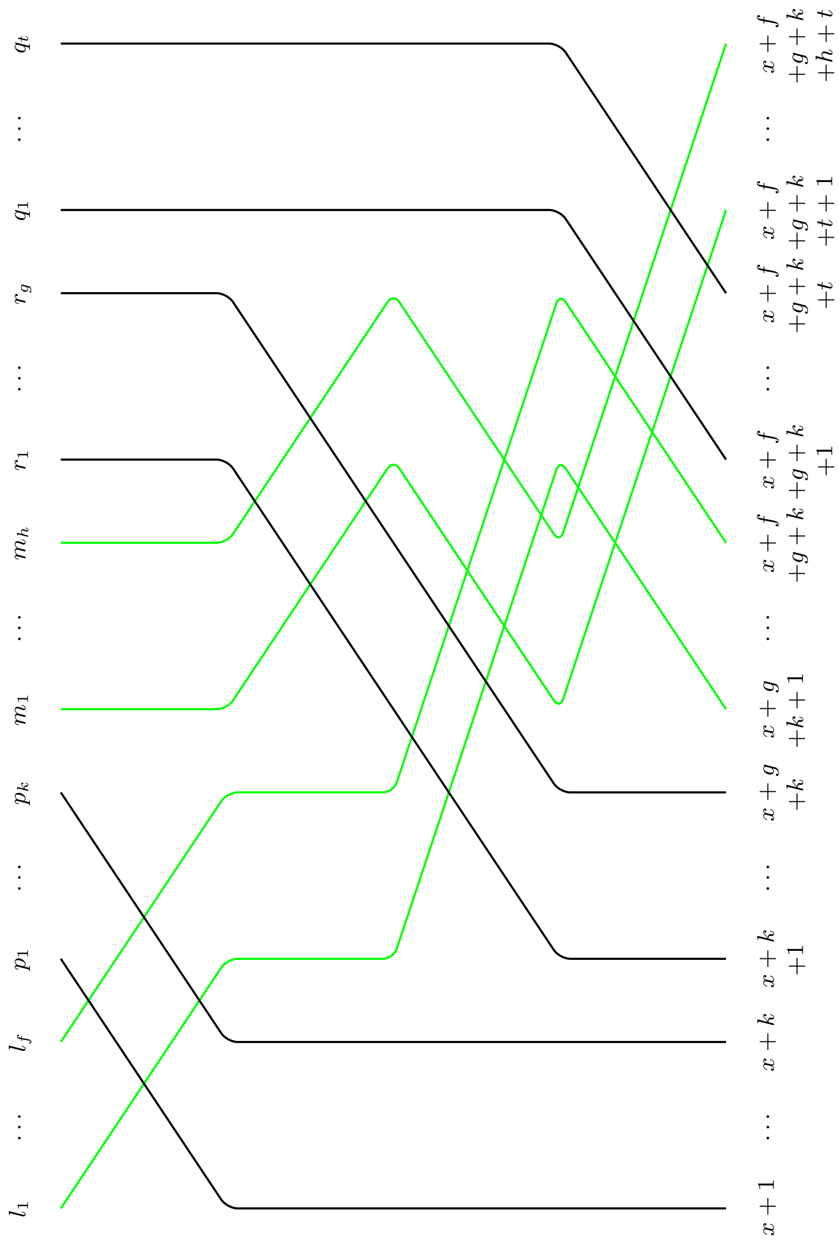


Figure 2.21: Crossings in (2.16). The strings to which we apply Lemma 2.8 are coloured green.

one of the earlier lemmas and then immediately find ourselves in another situation where we can apply the lemma again to a new set of crossings. To ease notation, we will write $\sum_a^b$ for $\sum_{i=1}^a b_i$. We wish here in particular to draw the reader's attention to the diagrams that accompany each lemma, as with this amount of crossings the written terms appear large and complex but the diagrammatic depictions of the crossings are comparatively straightforward.

The first of these lemmas extends Lemma 2.12. We encounter the setup of this lemma a times, each time having some residues $l_1^i, \dots, l_{f_i}^i$ which are sufficiently far apart from $m_1, \dots, m_h$.

Lemma 2.13. *Suppose we have the crossings*

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+h+\sum_{a-1}^g-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{a-2}^k+1}\\
&\qquad\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+\sum_1^g-1}\right)\downarrow_{x+1}\\
&\qquad\cdot\left(\Psi_{x+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_a^k+1}
\end{aligned}\tag{2.17}$$

with residues $l_1^i, \dots, l_{f_i}^i$*,* $p_1^i, \dots, p_{k_i}^i$*,* $m_1, \dots, m_h$*,* $r_1^i, \dots, r_{g_i}^i$*,* $q_1, \dots, q_t$ *for* $i \in \{1, \dots, a\}$ *as shown in Figure 2.22. Suppose that* $l_1^i, \dots, l_{f_i}^i \not\to m_1, \dots, m_h$ *for* $i \in \{1, \dots, a\}$*. Then* (2.17) *is equal to:*

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_a^f+\sum_a^k+1}\\
&\cdot\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+\sum_a^g-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+\sum_{a-1}^g-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{a-2}^k+1}\\
&\qquad\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+\sum_1^g-1}\right)\downarrow_{x+1}.
\end{aligned}$$

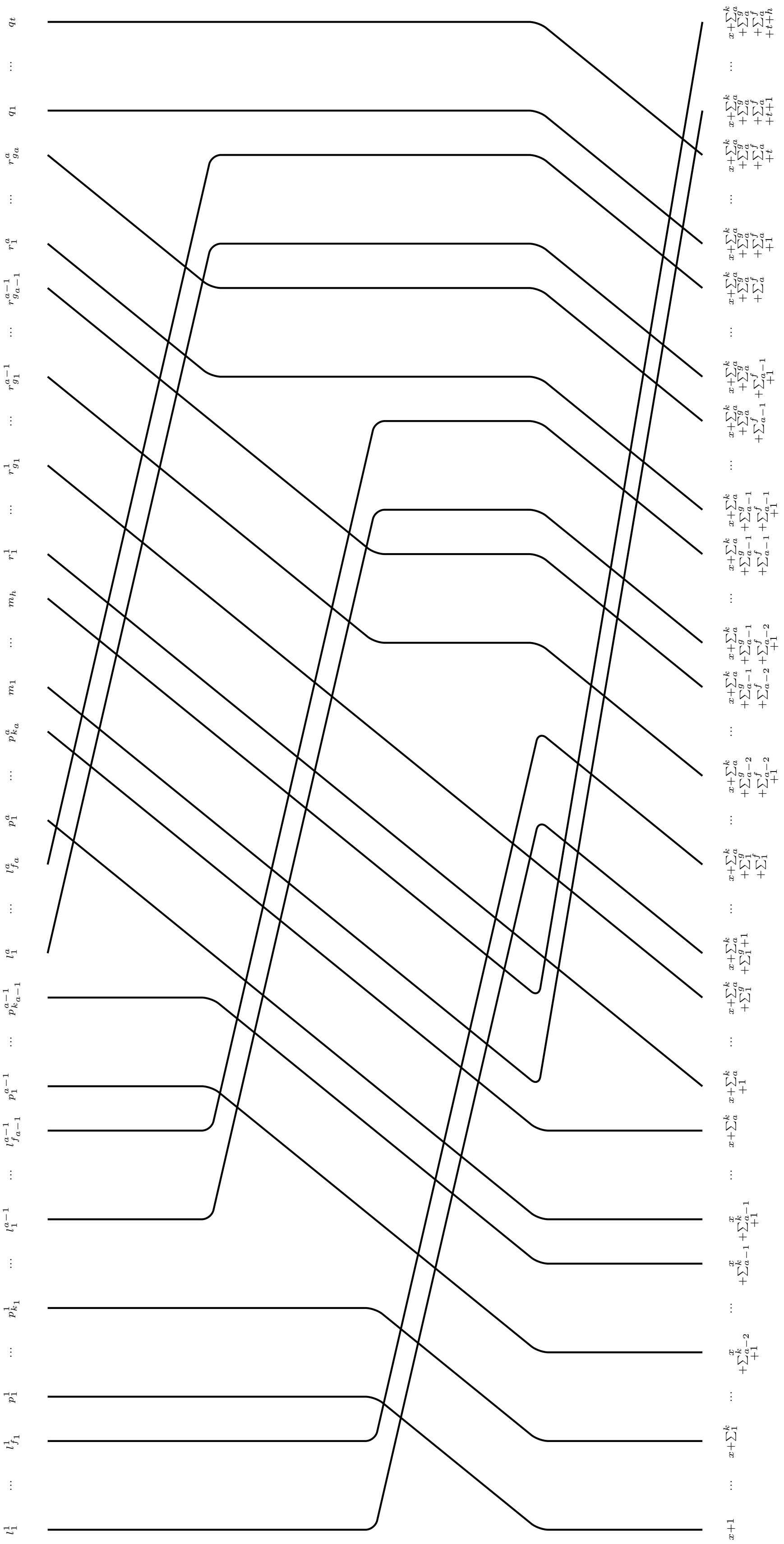


Figure 2.22: Crossings at the start of Lemma 2.13.

Proof. Suppose for $\gamma \in \{0, \dots, a-1\}$ we have

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1} \\
&\cdots\cdot\left(\Psi_{x+\sum_{\gamma+2}^f+\sum_{\gamma+1}^k}\uparrow^{x+\sum_{\gamma+2}^f+\sum_a^k+h+\sum_{\gamma+2}^g-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_{\gamma+1}^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_\gamma^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+\sum_{\gamma+1}^g-1}\right)\downarrow_{x+\sum_\gamma^f+\sum_\gamma^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_\gamma^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_\gamma^f+\sum_a^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_\gamma^f+\sum_{\gamma-1}^k}\uparrow^{x+\sum_\gamma^f+\sum_a^k+\sum_\gamma^g-1}\right)\downarrow_{x+\sum_{\gamma-1}^f+\sum_{\gamma-1}^k+1} \\
&\qquad\qquad\cdots\cdot\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+\sum_1^g-1}\right)\downarrow_{x+1}\,.
\end{aligned} \tag{2.18}$$

Apply Lemma 2.12 to

$$\begin{aligned}
&\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_\gamma^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+\sum_{\gamma+1}^g-1}\right)\downarrow_{x+\sum_\gamma^f+\sum_\gamma^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_\gamma^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_\gamma^f+\sum_a^k+1}
\end{aligned}$$

(take $\bar{x} = x + \sum_\gamma^f + \sum_\gamma^k$, $\bar{f} = f_{\gamma+1}$, $\bar{k} = \sum_a^k - \sum_\gamma^k$, $\bar{h} = h$, $\bar{g} = \sum_{\gamma+1}^g$, $\bar{t} = \sum_a^g - \sum_{\gamma+1}^g + \sum_a^f - \sum_{\gamma+1}^f + t$, where $\bar{x}, \bar{f}, \bar{g}, \bar{h}, \bar{k}, \bar{t}$ are the x, f, g, h, k and t in the hypotheses of Lemma 2.12). Then we can rewrite (2.18) as

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1} \\
&\cdots\cdot\left(\Psi_{x+\sum_{\gamma+2}^f+\sum_{\gamma+1}^k}\uparrow^{x+\sum_{\gamma+2}^f+\sum_a^k+h+\sum_{\gamma+2}^g-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_{\gamma+1}^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_a^k+1} \\
&\qquad\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_\gamma^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+\sum_{\gamma+1}^g-1}\right)\downarrow_{x+\sum_\gamma^f+\sum_\gamma^k+1} \\
&\qquad\cdot\left(\Psi_{x+\sum_\gamma^f+\sum_{\gamma-1}^k}\uparrow^{x+\sum_\gamma^f+\sum_a^k+\sum_\gamma^g-1}\right)\downarrow_{x+\sum_{\gamma-1}^f+\sum_{\gamma-1}^k+1} \\
&\qquad\qquad\cdots\cdot\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+\sum_1^g-1}\right)\downarrow_{x+1},
\end{aligned}$$

i.e. we can replace γ with $\gamma+1$. If $\gamma = 0$ we have that (2.18) is equal to (2.17), so then if $\gamma = a-1$, performing the above shows that (2.17) is equal to

$$
\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+\sum_a^g+t-1}\right)\downarrow_{x+\sum_a^f+\sum_a^k+1}\\
&\quad\cdot\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+\sum_a^g-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+\sum_{a-1}^g-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{\alpha-2}^k+1}\\
&\qquad\qquad\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+\sum_1^g-1}\right)\downarrow_{x+1}.
\end{aligned}
$$

□

The second such lemma extends Lemma 2.5. Again, we encounter the setup of this lemma a times, each time having some residues $l_1^i, \ldots, l_{f_i}^i$ which are sufficiently far apart from $m_1, \ldots, m_h$.

Lemma 2.14. *Suppose we have the crossings*

$$
\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+g+\sum_a^t-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+h+g+\sum_{a-1}^t-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{a-2}^k+1}\cdot\\
&\qquad\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+g+\sum_1^t-1}\right)\downarrow_{x+1}\\
&\qquad\qquad\cdot\left(\Psi_{x+\sum_a^k+h}\uparrow^{x+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_a^k+1}
\end{aligned}
\tag{2.19}
$$

with residues $l_1^i, \ldots, l_{f_i}^i$, $p_1^i, \ldots, p_{k_i}^i$, $m_1, \ldots, m_h$, $r_1, \ldots, r_g$, $q_1^i, \ldots, q_{t_i}^i$ *for* $i \in \{1, \ldots, a\}$ *as shown in Figure 2.23. Suppose that* $l_1^i, \ldots, l_{f_i}^i \not\leftrightarrow m_1, \ldots, m_h$ *for* $i \in \{1, \ldots, a\}$*. Then* (2.19) *is equal to*

$$
\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_a^f+\sum_a^k+1}\\
&\quad\cdot\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+g+\sum_a^t-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+h+g+\sum_{a-1}^t-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{a-2}^k+1}\cdot\\
&\qquad\qquad\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+g+\sum_1^t-1}\right)\downarrow_{x+1}.
\end{aligned}
$$

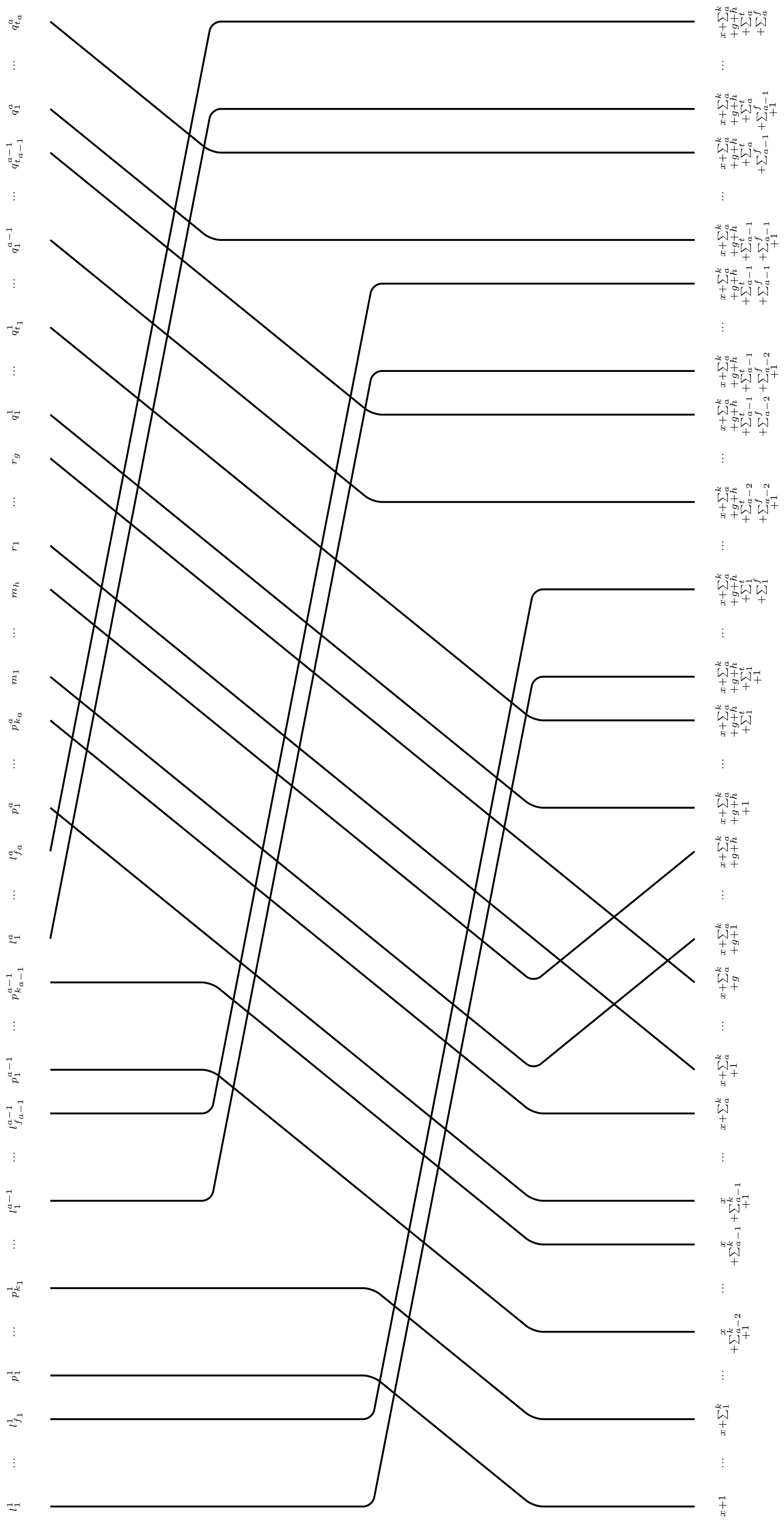


Figure 2.23: Crossings at the start of Lemma 2.14

Proof. Suppose for $\gamma \in \{0, \ldots, a-1\}$ we have

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+g+\sum_a^t-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\cdot\\
&\cdots\left(\Psi_{x+\sum_{\gamma+2}^f+\sum_{\gamma+1}^k}\uparrow^{x+\sum_{\gamma+2}^f+\sum_a^k+h+g+\sum_{\gamma+2}^t-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_{\gamma+1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_{\gamma}^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+g+\sum_{\gamma+1}^t-1}\right)\downarrow_{x+\sum_{\gamma}^f+\sum_{\gamma}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma}^f+\sum_a^k+h}\uparrow^{x+\sum_{\gamma}^f+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_{\gamma}^f+\sum_a^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma}^f+\sum_{\gamma-1}^k}\uparrow^{x+\sum_{\gamma}^f+\sum_a^k+h+g+\sum_{\gamma}^t-1}\right)\downarrow_{x+\sum_{\gamma-1}^f+\sum_{\gamma-1}^k+1}\cdot\\
&\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+g+\sum_1^t-1}\right)\downarrow_{x+1}.
\end{aligned}\tag{2.20}$$

Apply Corollary 2.6 to

$$\begin{aligned}
&\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_{\gamma}^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+g+\sum_{\gamma+1}^t-1}\right)\downarrow_{x+\sum_{\gamma}^f+\sum_{\gamma}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma}^f+\sum_a^k+h}\uparrow^{x+\sum_{\gamma}^f+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_{\gamma}^f+\sum_a^k+1}
\end{aligned}$$

(take $\bar{x} = x + \sum_\gamma^f + \sum_\gamma^k, \bar{f} = f_{\gamma+1}, \bar{k} = \sum_a^k - \sum_\gamma^k, \bar{h} = h, \bar{g} = g, \bar{t} = \sum_{\gamma+1}^t$, where $\bar{x}, \bar{f}, \bar{g}, \bar{h}, \bar{k}, \bar{t}$ are the x, f, g, h, k and t in the the hypotheses of Corollary 2.6). Then we can rewrite (2.20) as

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+g+\sum_a^t-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1}\cdot\\
&\cdots\left(\Psi_{x+\sum_{\gamma+2}^f+\sum_{\gamma+1}^k}\uparrow^{x+\sum_{\gamma+2}^f+\sum_a^k+h+g+\sum_{\gamma+2}^t-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_{\gamma+1}^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_a^k+h}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_{\gamma+1}^f+\sum_a^k+1}\\
&\cdot\left(\Psi_{x+\sum_{\gamma+1}^f+\sum_{\gamma}^k}\uparrow^{x+\sum_{\gamma+1}^f+\sum_a^k+h+g+\sum_{\gamma+1}^t-1}\right)\downarrow_{x+\sum_{\gamma}^f+\sum_{\gamma}^k+1}\cdot\\
&\cdots\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+g+\sum_1^t-1}\right)\downarrow_{x+1},
\end{aligned}$$

i.e. we can replace γ with $\gamma+1$. If $\gamma = 0$ we have that (2.20) is equal to (2.19), so then if $\gamma = a-1$, performing the above shows that (2.19) is equal to

$$\begin{aligned}
&\left(\Psi_{x+\sum_a^f+\sum_a^k+h}\uparrow^{x+\sum_a^f+\sum_a^k+h+g-1}\right)\downarrow_{x+\sum_a^f+\sum_a^k+1} \\
&\quad\cdot\left(\Psi_{x+\sum_a^f+\sum_{a-1}^k}\uparrow^{x+\sum_a^f+\sum_a^k+h+g+\sum_a^t-1}\right)\downarrow_{x+\sum_{a-1}^f+\sum_{a-1}^k+1} \\
&\cdot\left(\Psi_{x+\sum_{a-1}^f+\sum_{a-2}^k}\uparrow^{x+\sum_{a-1}^f+\sum_a^k+h+g+\sum_{a-1}^t-1}\right)\downarrow_{x+\sum_{a-2}^f+\sum_{a-2}^k+1}\cdot \\
&\qquad\qquad\cdots\cdot\left(\Psi_{x+\sum_1^f}\uparrow^{x+\sum_1^f+\sum_a^k+h+g+\sum_1^t-1}\right)\downarrow_{x+1}.
\end{aligned}$$

□

Chapter 3

Constructing homomorphisms

Now that we are armed with numerous lemmas from the previous chapter, we aim to utilise them. This chapter is devoted to proving the existence of explicit homomorphisms between certain Specht modules of KLR algebras. The 'certain' pairs of Specht modules that we are interested in will be indexed by multipartitions which differ by the moving of nodes. Results concerning decomposition numbers related to such pairs emerge from the results of Kleschev regarding partitions that differ by one node [Kle97]; in particular these have been generalised to the graded case of the Iwahori-Hecke algebra in papers by Chuang, Miyachi, Tan [CMT08] and Tan and Teo [TT13], and further to the case of diagrammatic Cherednik algebras by Bowman and Speyer [BS18]. Homomorphisms between Specht modules of KLR algebras have been studied by Lyle and Mathas in [LM14], where they define the notion of a *Carter-Payne pair*. In Sections 3.1 and 3.2 the homomorphisms we detail will arise directly between Specht modules indexed by Carter-Payne pairs, so their existence is already known, but due to our approach we will additionally explicitly describe where the generator of the domain Specht module is mapped to. In Section 3.3, we will build on the methods used in the previous sections in order to prove the existence of homomorphisms between Specht modules that are not indexed by Carter-Payne pairs, and explicitly describe the mapping also. Note that at no point will we make any assumptions about the characteristic of the base field $\mathbb{F}$.

3.1 One-node homomorphisms

To begin with, we will consider two bipartitions, λ and μ, where μ is formed from λ by moving a single node from the second component to the first. In order for our result to hold, we require that e is large enough so that within a given component of λ or μ, the nodes of constant residue will all appear along the same diagonal of nodes (the kth *diagonal* of component m being all nodes (r, c, m) such that $r - c = k$). In general, if this occurs within some partition ν, we shall say that ν has the *diagonal residue condition.* This will be satisfied if $e > h_{11}^{\nu}$, where h_{11}^{ν} is the hook length of the top left node in $[\nu]$, i.e. h_{11}^{ν} equals the sum of ν_1 and the number of rows of ν, minus one. One reason for requiring this condition is to control the Garnir relations: if e is large, we have to take into account e-bricks and we can obtain rather messy Garnir relations with numerous summands, causing our calculations to quickly get out of control. We will say that an l-multipartition has the diagonal residue condition if all l of its components do.

The diagonal residue condition also allows us to ensure that our homomorphisms keep their 'form'. In the one-node case, this means that the generator of S^{λ} is sent to just a single basis element indexed by a standard μ-tableau, and this tableau is that obtained by simply moving the "one-node" in $\mathbf{t}^{\lambda}$ to its position in $[\mu]$, keeping its value intact.

For the proof of the result, we aim to show that whenever some element of $\mathscr{H}_n^{\Lambda_\kappa}$ annihilates the generator of S^{λ}, it also annihilates its image under the proposed map. In particular, we must take the generating relations of S^{λ} and check that they still apply when we replace $v^{\mathbf{t}^{\lambda}}$ with its image. The y_r relations are fairly straightforward, since the generating relations for S^{λ} have a natural counterpart in S^{μ}, whilst the ψ_r relations are similar also, except for that which relates to the position where the "one-node" was removed. In the case of the Garnir relations, we have to check a few cases, since many natural counterparts for these relations are affected by the removed node.

When performing calculations within a Specht module S^{λ}, certain relations require an idempotent $e(\mathbf{i})$ (in particular, (1.8)–(1.11)), however we will almost always drop the idempotent from the algebraic expression of the terms, as it will only really serve to make things appear more convoluted when written down. For example, we will write $v^{\mathbf{t}^{\lambda}}\psi_r$ instead of $v^{\mathbf{t}^{\lambda}}e(\mathbf{i}^{\lambda})\psi_r = v^{\mathbf{t}^{\lambda}}\psi_r e(s_r\mathbf{i}^{\lambda})$ (using (1.4)). Nevertheless, it is important not to forget that it will influence calculations. With braid diagrams in mind, we will always be able to easily observe the relevant residues and this should hopefully alleviate any potential mystery.

Note that the proof of the following proposition has notes in the margin of the form (A•). These can be ignored for now and will become relevant when considering the proof of Corollary 3.3.

Proposition 3.1. *Let λ and μ be 2-multipartitions of n. Suppose*

$$e \geq \max\{h_{11}^{\lambda^{(1)}}+1, h_{11}^{\lambda^{(2)}}+1, h_{11}^{\mu^{(1)}}+1, h_{11}^{\mu^{(2)}}+1\}$$

and that $[\mu]$ is formed from $[\lambda]$ by moving one node from the second component to the first. Let $\mathfrak{s}$ be the μ-tableau defined by

$$\mathfrak{s}[i] = \begin{cases} \mathfrak{t}^\lambda[i], & \text{if } i \in [\lambda], \\ \mathfrak{t}^\lambda[j], & \text{if } i \notin [\lambda], \end{cases}$$

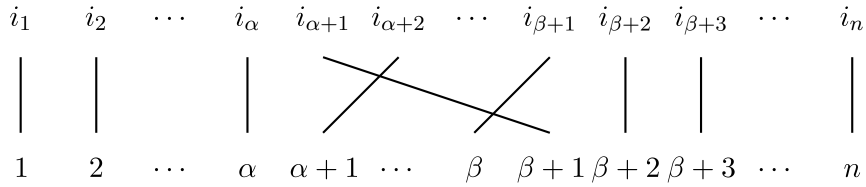
where j is the single node in $[\lambda] \setminus ([\lambda] \cap [\mu])$. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

Proof. First, observe that in both $[\lambda]$ and $[\mu]$ we have the diagonal residue condition. Let $\beta+1$ be the entry of the node j in $\mathfrak{t}^\lambda$. To obtain $[\mu]$ from $[\lambda]$, j is moved from $[\lambda^{(2)}]$ to $[\lambda^{(1)}]$. Let $\alpha+1$ be the entry in $\mathfrak{s}$ of the added node. Then $\psi^{\mathfrak{s}} = \Psi_{\alpha+1}\!\uparrow^{\beta}$ (note this may be equal to 1 if $\alpha+1 = \beta+1$). The following diagrams help to illustrate these definitions:

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & \cdots\cdots \\ \hline \multicolumn{3}{|c|}{\cdots\cdots\cdots} \\ \hline \cdots\cdots & \alpha & \\ \hline \alpha+1 & \cdots & \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|} \hline \multicolumn{3}{|c|}{\cdots\cdots\cdots} \\ \hline \cdots\cdots & \beta & \boxed{\beta+1} \\ \hline \beta+2 & \cdots & \\ \hline \cdots & n & \\ \hline \end{array} \right)$$

$$\mathfrak{s} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & \cdots\cdots \\ \hline \multicolumn{3}{|c|}{\cdots\cdots\cdots} \\ \hline \cdots\cdots & \alpha & \boxed{\beta+1} \\ \hline \alpha+1 & \cdots & \\ \hline \end{array} \;,\; \begin{array}{|c|c|} \hline \multicolumn{2}{|c|}{\cdots\cdots\cdots} \\ \hline \cdots\cdots & \beta \\ \hline \beta+2 & \cdots \\ \hline \cdots & n \\ \hline \end{array} \right)$$

It is clear from the definition of $\mathfrak{s}$ that the residue sequence of $\mathfrak{s}$ is the same as that of $\mathfrak{t}^\lambda$, so we must check that $\varphi(v^{\mathfrak{t}^\lambda})a = 0$ whenever $v^{\mathfrak{t}^\lambda}a = 0$ for $a \in \mathscr{H}_n^{\Lambda_\kappa}$. In particular, we must check that the generating relations of S^λ hold on the image of $v^{\mathfrak{t}^\lambda}$. If $(i_1, i_2, \ldots, i_n)$ is the residue sequence of $\mathfrak{t}^\mu$ then $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$ is represented by:



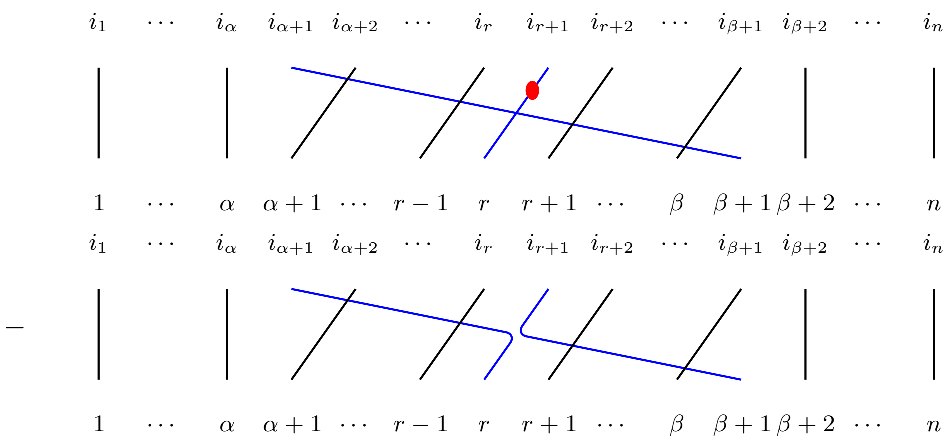
Checking y_r relations

We must check that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}y_r = 0$ for $r \in \{1, 2, \ldots, n\}$. Suppose first that

$$r \in \{1, 2, \ldots, \alpha\} \cup \{\beta+2, \beta+3, \ldots, n\},$$

then y_r commutes with $\psi^{\mathfrak{s}}$ hence $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}y_r = v^{\mathfrak{t}^\mu}y_r\psi^{\mathfrak{s}} = 0$ as y_r kills $v^{\mathfrak{t}^\mu}$ by the definition of S^μ.

Now suppose $r \in \{\alpha+1, \alpha+2, \ldots, \beta\}$. Then the dot corresponding to y_r has to pass through one crossing. Either $i_{\alpha+1} \neq i_{r+1}$ and the dot passes through (in particular this happens when $r = \beta$), giving $v^{\mathfrak{t}^\mu}y_{r+1}\psi^{\mathfrak{s}} = 0$, or instead we have $i_{\alpha+1} = i_{r+1}$ and then using the braid relation (1.9) we obtain the following sum of diagrams:



Clearly the first summand is zero since the dot has reached the top of the diagram. In the second summand, we have a ψ_{r+1} crossing at the top. Since we are assuming $i_{\alpha+1} = i_{r+1}$, then using the diagonal residue condition we must have that the node containing $r+1$ lies in the second component of $v^{\mathfrak{t}^\mu}$, above and to the left of where the removed node would have been. Due to this, we must have that ψ_{r+1} (A1) is a row relation for S^μ, hence this summand is zero also.

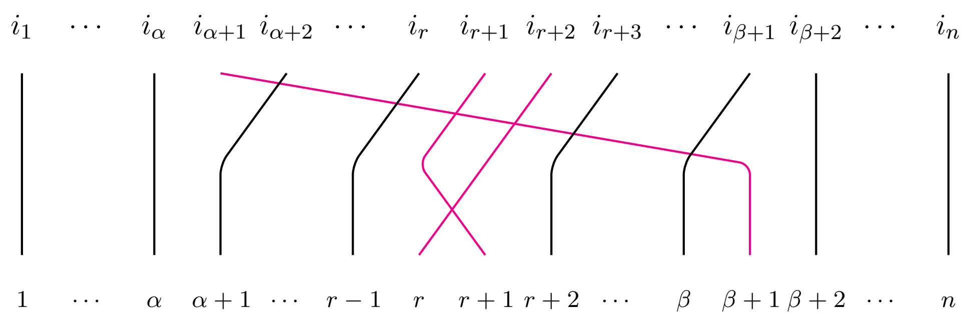
Finally, suppose $r = \beta + 1$. Then the dot corresponding to y_r will have to contend with multiple crossings, but by the previous case if any of these crossings 'split' using relation (1.9), then the resulting 'split' term is zero. Hence eventually we are only left with a term where the dot is at the top of the diagram, and hence this term is zero also.

So we have $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}y_r = 0$ for every $r \in \{1, 2, \ldots, n\}$.

Checking ψ_r relations

We must check that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r = 0$ for $r \in \{1, 2, \ldots, n-1\}$ where ψ_r is a row relation for S^λ. Suppose first that $r \in \{1, 2, \ldots, \alpha-1\} \cup \{\beta+2, \beta+3, \ldots, n-1\}$ and that ψ_r is a row relation for S^λ. Then ψ_r commutes with $\psi^{\mathfrak{s}}$, hence $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r = v^{\mathfrak{t}^\mu}\psi_r\psi^{\mathfrak{s}} = 0$ as ψ_r will also be a row relation for S^μ.

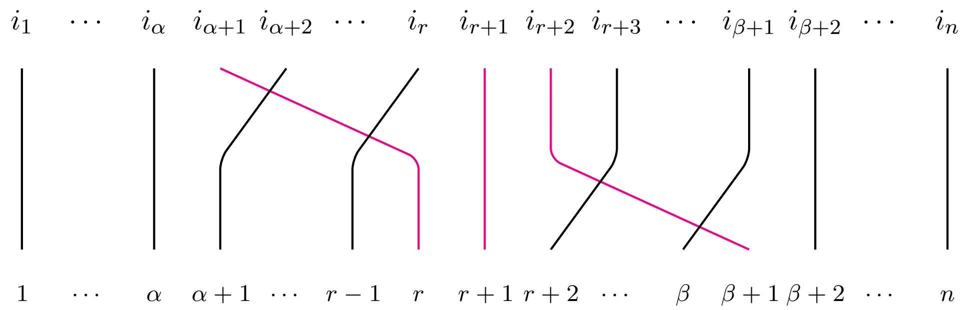
Note that if $r = \alpha$ or $r = \beta + 1$ then ψ_r will not be a row relation for S^λ since both the node containing α and the node containing $\beta + 1$ are at the ends of rows in $\mathfrak{t}^\lambda$. So now suppose $r \in \{\alpha + 1, \alpha + 2, \ldots, \beta - 1\}$ and that ψ_r is a row relation for S^λ. Then ψ_{r+1} will be a row relation for S^μ. For $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r$ we have the diagram:



As is clear in the diagram, we have within this the crossings $\psi_r\psi_{r+1}\psi_r$. Using the braid relation (1.11), regardless of the residues we will always obtain a term where we replace $\psi_r\psi_{r+1}\psi_r$ with $\psi_{r+1}\psi_r\psi_{r+1}$. This term will now be $v^{\mathfrak{t}^\mu}\psi_{r+1}\psi^{\mathfrak{s}} = 0$ since ψ_{r+1} is a row relation for S^μ.

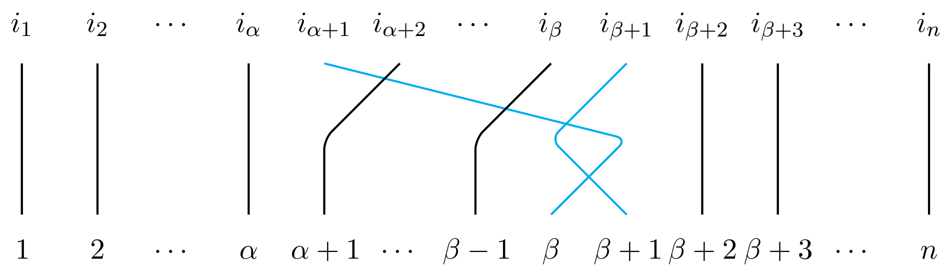
If the residues are such that $i_{\alpha+1} = i_{r+2} = i_{r+1} \pm 1$ then the braid relation

(1.11) will also give us the term $v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}\!\uparrow^{r-1}\,\Psi_{r+2}\!\uparrow^{\beta}$. In this case, we cannot have $r = \beta - 1$ since $i_{\alpha+1} \neq i_{\beta+1}$ as the nodes containing β and $\beta+1$ were not in the same diagonal in λ. So then we can rewrite this term as $v^{\mathfrak{t}^\mu}\Psi_{r+2}\!\uparrow^{\beta}\,\Psi_{\alpha+1}\!\uparrow^{r-1}$ and we have ψ_{r+2} at the top of the diagram:



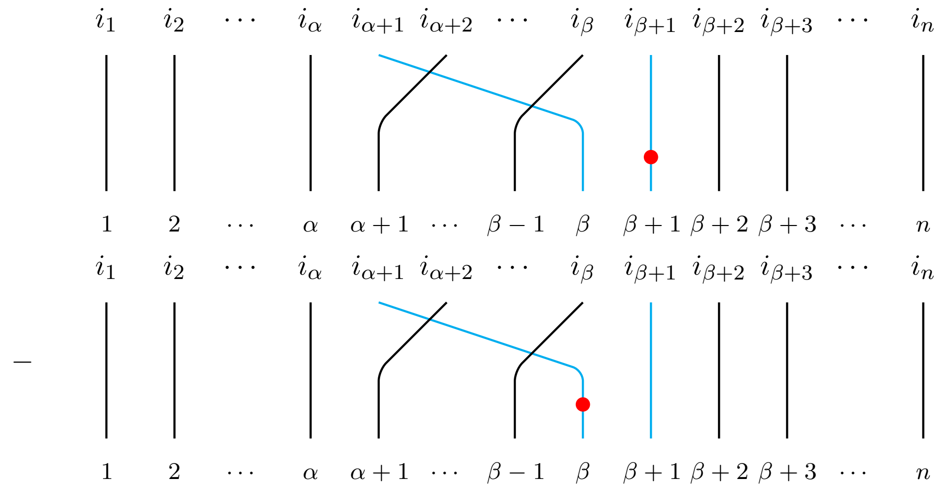
Since we are assuming $i_{\alpha+1} = i_{r+2}$, by the diagonal residue condition we must have that the node containing $r+2$ lies in the second component of $v^{\mathfrak{t}^\mu}$, above and to the left of where the removed node would have been. Hence ψ_{r+2} is a row (A2) relation for S^μ and so $v^{\mathfrak{t}^\mu}\Psi_{r+2}\!\uparrow^{\beta}\,\Psi_{\alpha+1}\!\uparrow^{r-1}= 0$. Thus all together we have that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r = 0$ for $r \in \{\alpha+1, \alpha+2, \ldots, \beta-1\}$.

So now we only need to consider when $r = \beta$ and ψ_β is a row relation for S^λ. In this case we have that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r = v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}\!\uparrow^{\beta}\,\psi^2_{\beta+1}$, which as a diagram is represented as:



Using the square relation (1.10) we can rewrite ψ^2_β. Since we are assuming ψ_β is a row relation for S^λ, the node containing β in $\mathfrak{t}^\lambda$ was adjacent and to the left of the node containing $\beta+1$ in $\mathfrak{t}^\lambda$, hence $i_{\alpha+1} = i_{\beta+1}+1$. So we replace ψ^2_β with

$(y_{\beta+1} - y_\beta)$, giving us the sum of diagrams:



The first summand is clearly zero since the dot is at the top of the diagram. The second summand will be zero following the same reasoning as to why $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}y_{\beta+1}$ is equal to zero.

So all together we have shown that for any row relation ψ_r in S^λ we have that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r = 0$ as we wanted.

Checking the Garnir relations

In $\mathfrak{t}^\lambda$ a Garnir belt will look like the following for some integers $r \geq 0$ and $s, t \geq 1$:

$$\begin{array}{cccccc} & & & \boxed{r+1} & \boxed{\cdots\cdots} & \boxed{r+s} \\ \boxed{r+s+1} & \boxed{\cdots\cdots} & \boxed{r+t} & & & \end{array}$$

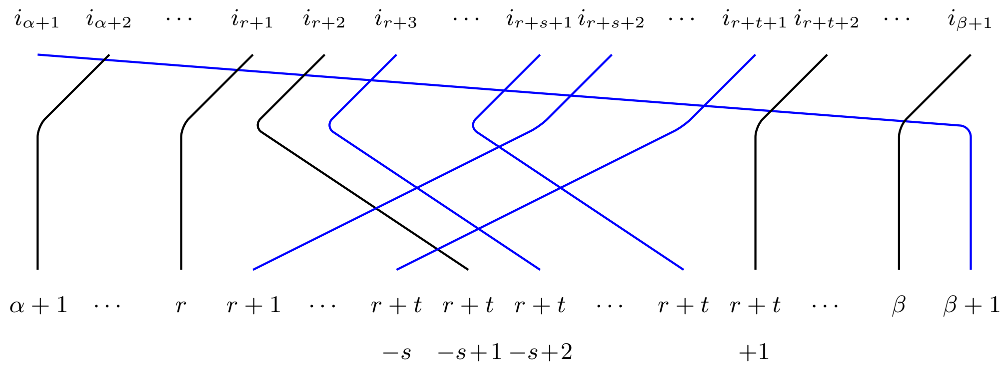
Note that due to the diagonal residue condition, $t \leq e-1$, so the corresponding Garnir relation for S^λ is $g = \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1}$. We split the checking of Garnir relations into cases based on the location of the Garnir node with respect to the moved node.

Case I: $r+t < \alpha+1$ or $r+1 > \beta+1$

In this case, the Garnir belt also exists in $[\mu]$ with the exact same entries, so the Garnir relation is also a generating relation for S^μ. Since in this case the relation also commutes with $\psi^{\mathfrak{s}}$, we have $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g = v^{\mathfrak{t}^\mu}g\psi^{\mathfrak{s}} = 0$.

Case II: $r+1>\alpha$ and $r+t<\beta+1$

In this case the Garnir belt also exists in $[\mu]$, however its entries are all raised by 1, giving the Garnir relation $h = \left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+2}$ for S^μ. Forgetting the extraneous strings, $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g$ as a diagram looks like:



By rewriting terms we have that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g$ is equal to

$$v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r+1}\cdot\Psi_{r+2}{\uparrow}^{r+t}\cdot\left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2}\cdot\Psi_{r+1}{\uparrow}^{r+t-s}\cdot\Psi_{r+t+1}{\uparrow}^{\beta}$$

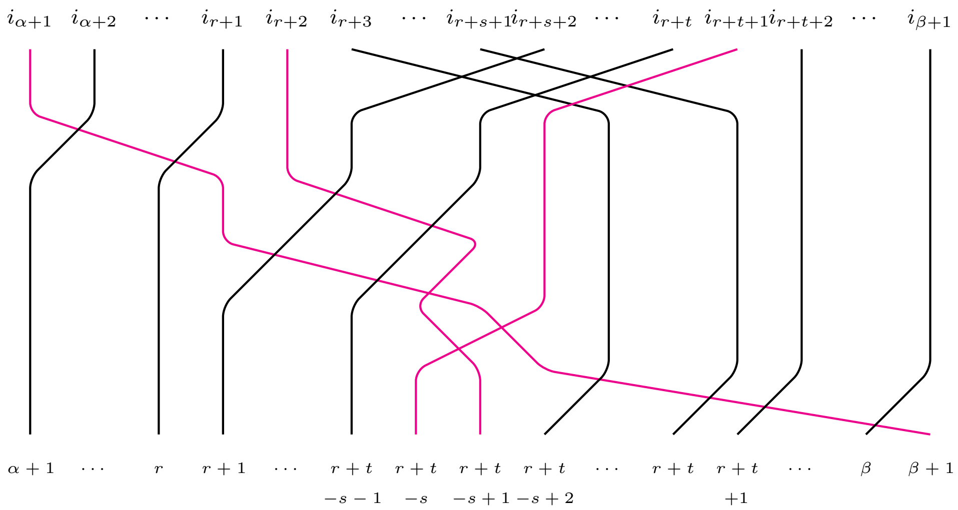
so that we can apply Lemma 2.5 to $\Psi_{r+2}{\uparrow}^{r+t}\cdot\left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2}$, which we can do since $i_{r+3}, i_{r+4}, \dots, i_{r+s+1} \not\leftarrow i_{r+s+2}, i_{r+s+3}, \dots, i_{r+t+1}$. The strings to which we apply Lemma 2.5 to are coloured blue in the figure above. This gives

$$\begin{aligned}
&v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r+1}\cdot\left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+3}\cdot\Psi_{r+2}{\uparrow}^{r+t}\cdot\Psi_{r+1}{\uparrow}^{r+t-s}\cdot\Psi_{r+t+1}{\uparrow}^{\beta}\\
&\quad= v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r}\cdot\left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+3}\cdot\Psi_{r+1}{\uparrow}^{r+t-s+1}\cdot\Psi_{r+1}{\uparrow}^{r+t-s}\cdot\Psi_{r+t-s+2}{\uparrow}^{\beta}\\
&\quad= v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+3}\cdot\Psi_{\alpha+1}{\uparrow}^{r}\cdot\Psi_{r+1}{\uparrow}^{r+t-s}\cdot\Psi_{r+1}{\uparrow}^{r+t-s-1}\\
&\qquad\qquad\cdot\Psi^{r+t-s+1}{\downarrow}_{r+t-s}\cdot\Psi_{r+t-s+2}{\uparrow}^{\beta}
\end{aligned}$$

and here we can apply Lemma 2.3 to $\Psi_{r+1}{\uparrow}^{r+t-s}\cdot\Psi_{r+1}{\uparrow}^{r+t-s-1}$ since we have $i_{r+2}\not\leftarrow i_{r+s+2}, i_{r+s+3}, \cdots, i_{r+t}$, so that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g$ is equal to

$$\begin{aligned}
v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+3}\cdot\Psi_{\alpha+1}{\uparrow}^{r}\cdot\Psi_{r+2}{\uparrow}^{r+t-s}\cdot\Psi_{r+1}{\uparrow}^{r+t-s}&\\
\cdot\Psi^{r+t-s+1}{\downarrow}_{r+t-s}\cdot\Psi_{r+t-s+2}{\uparrow}^{\beta},&
\end{aligned}$$

and now our diagram looks like:



Now within this we have $\psi_{r+t-s}\psi_{r+t-s+1}\psi_{r+t-s}$ with associated left residue $i_{\alpha+1}$, right residue i_{r+t+1} and center residue i_{r+2}. Regardless of what these residues are, we will always obtain the term where we pull the i_{r+2}-string over the $\psi_{r+t-s+1}$ crossings, which will be equal to

$$v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\uparrow^{r+t}\right)\downarrow_{r+2}\Psi_{\alpha+1}\uparrow^{\beta}= v^{\mathfrak{t}^\mu} h\psi^{\mathfrak{s}} = 0.$$

If $i_{\alpha+1} = i_{r+t+1}$, then as $i_{r+t+1} = i_{r+2} - 1$ using the braid relation (1.11) we obtain an extra term equal to

$$\begin{aligned} & v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\uparrow^{r+t}\right)\downarrow_{r+3}\cdot\Psi_{\alpha+1}\uparrow^{r}\cdot\left(\Psi_{r+2}\uparrow^{r+t-s}\right)\downarrow_{r+1}\cdot\Psi_{r+t-s+2}\uparrow^{\beta} \\ &= v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\uparrow^{r+t}\right)\downarrow_{r+3}\cdot\Psi_{r+t-s+2}\uparrow^{\beta}\cdot\Psi_{\alpha+1}\uparrow^{r}\cdot\left(\Psi_{r+2}\uparrow^{r+t-s}\right)\downarrow_{r+1} \end{aligned}$$

and now since $i_{r+3},\cdots,i_{r+s+1} \not\to i_{r+t+1}$ we can apply Lemma 2.8 to $\left(\Psi_{r+s+1}\uparrow^{r+t}\right)\downarrow_{r+3}\cdot\Psi_{r+t-s+2}\uparrow^{\beta}$ (take $x = r+2, f = s-1, h = t-s-1, g = 1, k = \beta - r - t$).

This leaves us with the extra term being equal to

$$v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\uparrow^{r+t-1}\right)\downarrow_{r+3}\cdot\Psi_{r+t+1}\uparrow^{\beta}\cdot\Psi_{\alpha+1}\uparrow^{r}\cdot\left(\Psi_{r+2}\uparrow^{r+t-s}\right)\downarrow_{r+1}$$

so finally we commute the $\Psi_{r+t+1}\uparrow^{\beta}$ to the left of the term, i.e. we have a ψ_{r+t+1}

crossing at the top of the corresponding diagram. Since we assumed $i_{\alpha+1} = i_{r+t+1}$, (A3) using the diagonal residue condition we conclude that ψ_{r+t+1} must be a row relation, hence this extra term is zero. Thus all together we have shown that $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g = 0$ in case II.

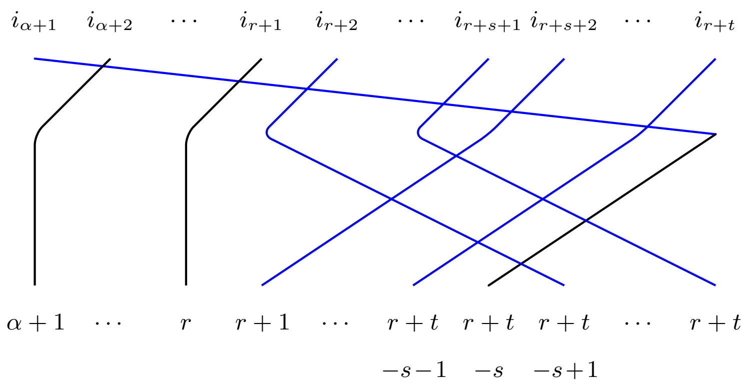
Case III: $r + t = \beta + 1$

In this case, the Garnir belt must lie in the second component of $[\lambda]$. This implies that $r + 1 > \alpha$. Unlike in the previous cases, this Garnir belt does not entirely exist in $[\mu]$; the removed node is the node beneath the Garnir node.

We can write $\psi^{\mathfrak{s}} = \Psi_{\alpha+1}{\uparrow}^{r+t-1}$ and so

$$\begin{aligned} v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g &= v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+1}{\uparrow}^{r+t-1} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \\ &= v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+1}{\uparrow}^{r} \cdot \Psi_{r+1}{\uparrow}^{r+t-1} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-2}\right){\downarrow}_{r+1} \cdot \Psi^{r+t-1}{\downarrow}_{r+t-s} \end{aligned}$$

which, forgetting extraneous strings, looks like:



Note that only the crossings where both strings are blue in the above diagram is where we shall apply Lemma 2.5. Apply Lemma 2.5 to

$$\Psi_{r+1}{\uparrow}^{r+t-1} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-2}\right){\downarrow}_{r+1}$$

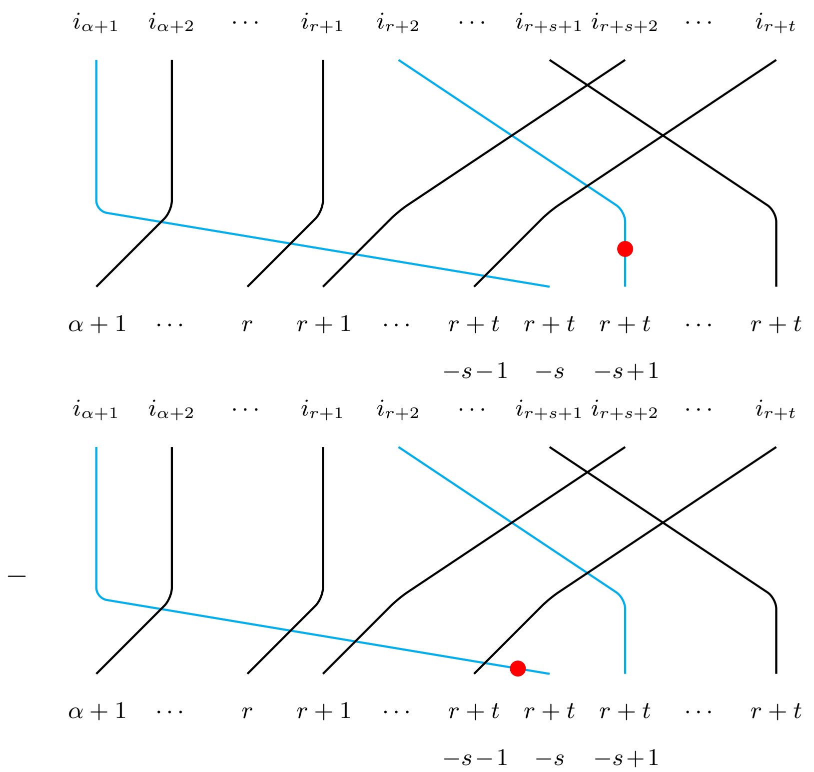
since $i_{\alpha+1} \neq i_{r+s+2}, \ldots, i_{r+t}$, giving

$$\begin{aligned} v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}} g &= v^{\mathfrak{t}^{\mu}} \Psi_{\alpha+1}{\uparrow}^{r} \cdot \left(\Psi_{r+s+1}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{r+1}{\uparrow}^{r+t-1} \cdot \Psi^{r+t-1}{\downarrow}_{r+t-s} \\ &= v^{\mathfrak{t}^{\mu}} \Psi_{\alpha+1}{\uparrow}^{r} \cdot \left(\Psi_{r+s+1}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{r+1}{\uparrow}^{r+t-2} \cdot \Psi^{r+t-2}{\downarrow}_{r+t-s} \\ &\vdots \\ &= v^{\mathfrak{t}^{\mu}} \Psi_{\alpha+1}{\uparrow}^{r} \cdot \left(\Psi_{r+s+1}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{r+1}{\uparrow}^{r+t-s+1} \cdot \Psi^{r+t-s+1}{\downarrow}_{r+t-s} \end{aligned}$$

by repeated use of relation (1.10) (since $i_{\alpha+1} \neq i_{r+s+1}, \ldots, i_{r+3}$)

$$= v^{\mathfrak{t}^{\mu}} \left(\Psi_{r+s+1}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{\alpha+1}{\uparrow}^{r+t-s-1} \cdot (y_{r+t-s+1} - y_{r+t-s})$$

since $i_{\alpha+1} = i_{r+2} - 1$. As a sum of diagrams (not including extraneous strings) this is:



In the former term, since $i_{r+2} \neq i_{r+t}, \ldots, i_{r+s+2}$, the dot moves straight to the top of the diagram hence this term is zero. In the latter term, we start by observing that we can move the dot some of the way towards the top since $i_{\alpha+1} \neq i_{r+t}, \ldots, i_{r+s+2}$.

Next we must compare $i_{\alpha+1}$ and i_{r+1}. If $t-s=1$ then there can be no nodes of residue $i_{\alpha+1}$ whose entry is $\alpha+2$ or greater in either component by the diagonal residue property. Thus $i_{\alpha+1} \neq i_{\alpha+2}, \ldots, i_{r+1}$ so the dot moves all the way to the (A4) top and the term is zero. However, if $t-s>1$, then $i_{\alpha+1}=i_{r+1}$ and in $[\mu]$ we will have the Garnir belt:

$$\begin{array}{|c|c|c|c|c|}
\cline{3-5}
\multicolumn{2}{c|}{} & r+1 & \cdots\cdots & r+s+1 \\
\hline
r+s+2 & \cdots\cdots & r+t & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}$$

giving the Garnir relation $h = \left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+1}$. In this instance we will have that the term is equal to

$$\begin{aligned}
&-v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+2} \cdot \Psi_{\alpha+1}\!\uparrow^{r-1} \cdot \psi_r y_{r+1} \cdot \Psi_{r+1}\!\uparrow^{r+t-s-1} \\
&= -v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+2} \cdot \Psi_{\alpha+1}\!\uparrow^{r-1} y_r \cdot \Psi_r\!\uparrow^{r+t-s-1} \qquad (3.1)\\
&\qquad - v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+2} \cdot \Psi_{\alpha+1}\!\uparrow^{r-1} \cdot \Psi_{r+1}\!\uparrow^{r+t-s-1} \qquad (3.2)
\end{aligned}$$

by applying relation (1.8). In (3.1), apply Lemma 2.11 to $\Psi_{\alpha+1}\!\uparrow^{r-1} y_r$. Then there is some $k \geq 0$ so that this term is equal to

$$\begin{aligned}
&- v^{\mathfrak{t}^\mu} y_{\alpha+1}\left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+2} \cdot \Psi_{\alpha+1}\!\uparrow^{r+t-s-1} \\
&+ \sum_{j=1}^{k} v^{\mathfrak{t}^\mu} \Psi_{\alpha+z_j+1}\!\uparrow^{r-1} \cdot \left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+2} \cdot \Psi_{\alpha+1}\!\uparrow^{\alpha+z_j-1} \cdot \Psi_r\!\uparrow^{r+t-s-1}
\end{aligned}$$

where $i_{\alpha+1} = i_{\alpha+z_j+1}$ for all $j \in \{1, \ldots, k\}$. Since $i_{\alpha+1} = i_{\alpha+z_j+1}$, all of the (A5) $\psi_{\alpha+z_j+1}$ crossings will be row relations by the diagonal residue condition. Thus (3.1) is equal to zero.

By rearranging, (3.2) is equal to

$$-v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}\!\uparrow^{r+t-1}\right)\!\downarrow_{r+1} \cdot \Psi_{\alpha+1}\!\uparrow^{r-1} = -v^{\mathfrak{t}^\mu} h \Psi_{\alpha+1}\!\uparrow^{r-1} = 0.$$

So all together we have $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g = 0$ in case III.

Case IV: $r+s=\beta+1$

In this case, the Garnir belt must lie in the second component of $[\lambda]$. This implies that $r+1 > \alpha$. The entire Garnir belt does not exist in $[\mu]$, since the removed node lies within the belt. In $[\mu]$ we have the Garnir belt

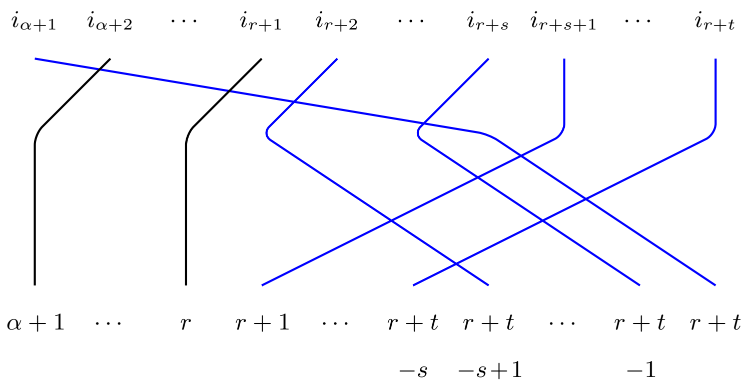
$$\begin{array}{|c|c|c|c|c|}
\cline{3-5}
\multicolumn{2}{c|}{} & r+2 & \cdots\cdots & r+s \\
\hline
r+s+1 & \cdots\cdots & r+t & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}$$

giving the Garnir relation $h = \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2}$.

We can write $\psi^{\mathfrak{s}} = \Psi_{\alpha+1}{\uparrow}^{r+s-1}$ and so

$$\begin{aligned}
v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g &= v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r+s-1} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \\
&= v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r} \cdot \Psi_{r+1}{\uparrow}^{r+t-1} \cdot \left(\Psi_{r+s-1}{\uparrow}^{r+t-2}\right){\downarrow}_{r+1}
\end{aligned}$$

which, when forgetting irrelevant strings, looks like:



Now since $i_{\alpha+1} \neq i_{r+s+1}, i_{r+s+2}\ldots, i_{r+t}$ we can apply Lemma 2.5 to

$$\Psi_{r+1}{\uparrow}^{r+t-1} \cdot \left(\Psi_{r+s-1}{\uparrow}^{r+t-2}\right){\downarrow}_{r+1}$$

giving us

$$\begin{aligned}
&v^{\mathfrak{t}^\mu}\Psi_{\alpha+1}{\uparrow}^{r} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{r+1}{\uparrow}^{r+t-1} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+2} \cdot \Psi_{\alpha+1}{\uparrow}^{r+t-1} \\
&= v^{\mathfrak{t}^\mu} h\, \Psi_{\alpha+1}{\uparrow}^{r+t-1} \\
&= 0,
\end{aligned}$$

so indeed $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g = 0$.

Case V: $r + s = \alpha$

In this case, the Garnir belt lies in the first component of $[\lambda]$. The removed node gets added onto this belt, meaning in $[\mu]$ we have the Garnir belt

		$r+1$	$\cdots\cdots$	$r+s$	$r+s+1$
$r+s+2$	$\cdots\cdots$	$r+t+1$			

giving the Garnir relation $h = \left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+1}$.

We can write $\psi^{\mathfrak{s}} = \Psi_{r+s+1}{\uparrow}^{\beta}$ and so

$$\begin{aligned}
v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g &= v^{\mathfrak{t}^\mu}\Psi_{r+s+1}{\uparrow}^{\beta} \cdot \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{r+s+1}{\uparrow}^{r+t}\right){\downarrow}_{r+1} \cdot \Psi_{r+t+1}{\uparrow}^{\beta} \\
&= v^{\mathfrak{t}^\mu} h \Psi_{r+t+1}{\uparrow}^{\beta} \\
&= 0.
\end{aligned}$$

Conclusion

Having checked that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} a = 0$ for every $a \in \mathscr{H}_n^{\Lambda_\kappa}$ we have indeed shown that there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

$\square$

We now extend Proposition 3.1 to work with multipartitions with more than two components. We now suppose that the moved node moves from one component, say $\lambda^{(q)}$, to an earlier one, say $\lambda^{(p)}$, potentially with other components either side and in the middle of these. As long as there are no removable nodes of the same residue as the moved node in the components labelled with $p+1, \ldots, q-1$, we can form a homomorphism in practically the same way as Proposition 3.1. The only real difference is that we have to make a few adjustments to the proof of the proposition in order to account for the extra components in between $\lambda^{(p)}$ and $\lambda^{(q)}$.

In addition to this, we show how the degree of such a homomorphism can be calculated in a combinatorial manner based upon counting the number of addable nodes of the same residue as the moved node amongst those components labelled by $p+1, \ldots, q-1$. In order to help state the corollary and its ensuing results clearly, we make the following definition for a pair of multipartitions λ and μ, where μ is formed from λ in the fashion described above.

Definition 3.2. Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n, where $[\mu]$ is formed from $[\lambda]$ by moving one node of residue ι from the qth component to the pth, for some p and q such that $p < q$. In addition suppose that

$$e \geq \max_{p \leq c \leq q} \{h_{11}^{\lambda^{(c)}} + 1, h_{11}^{\mu^{(c)}} + 1\}.$$

Suppose that amongst the components $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$, there are exactly $k \geq 0$ components containing addable ι-nodes. If $k > 0$, then we also require that e is large enough so that the diagonal residue condition holds when the ι-node is added to these components. Suppose that each component $\lambda^{(c')}$ contains no removable ι-nodes. Then we say that $(\lambda, \mu)_\iota$ is a *one node pair (of degree $k+1$)*.

Corollary 3.3. *Suppose that $(\lambda, \mu)_\iota$ is a one node pair of degree $k+1$. Let $\mathfrak{s}$ be the μ-tableau defined by*

$$\mathfrak{s}[i] = \begin{cases} \mathfrak{t}^\lambda[i], & \text{if } i \in [\lambda], \\ \mathfrak{t}^\lambda[j], & \text{if } i \notin [\lambda], \end{cases}$$

where j is the single node in $[\lambda] \setminus ([\lambda] \cap [\mu])$. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu} \psi^{\mathfrak{s}}$. This homomorphism has degree $k+1$ and can be written as a composition of $k+1$ homomorphisms, all of degree one.

Proof. Define α and β similarly to the proof of Proposition 3.1, so that the node to be moved contains $\beta+1$ in $\mathfrak{t}^\lambda$ and in $\mathfrak{t}^\mu$ the added node contains $\alpha+1$. Then $\psi^{\mathfrak{s}} = \Psi_{\alpha+1}\uparrow^\beta$. We need to check that the generating relations of S^λ hold on $\varphi\left(v^{\mathfrak{t}^\lambda}\right)$.

For each type of relation, our definitions of α and β allow us to follow the same methods as in Proposition 3.1, only now accounting for the additional nodes in between the first and last components of $[\mu]$ as well as those outside of these components. In checking each of the relations, apply the same reasoning as in Proposition 3.1, however there are a few changes to be made at the places annotated by numbers in the margins.

(A1) In the second summand, we have $\Psi_{r+1}\uparrow^{\beta}$ at the top of the diagram, and if ψ_{r+1} is not a row relation, by the diagonal residue condition we have that the node containing $r+1$ in $\mathfrak{t}^{\mu}$ must be a Garnir node, so the corresponding Garnir relation will be at the top of the diagram.

(A2) We have $\Psi_{r+2}\uparrow^{\beta}$ at the top of the diagram, and then by the diagonal residue condition, if ψ_{r+2} is not a row relation then the node containing $r+2$ in $\mathfrak{t}^{\mu}$ must be a Garnir node, so the corresponding Garnir relation will be at the top of the diagram.

(A3) By the diagonal residue condition, if ψ_{r+t+1} is not a row relation, the node containing $r+t+1$ in $\mathfrak{t}^{\mu}$ will be a Garnir node and we have the corresponding Garnir relation at the top of the diagram.

(A4) We may have some z such that $i_{\alpha+1} = i_z$ in which case by Lemma 2.11 we will obtain $\Psi_z\uparrow^{r}$ at the top of the diagram. Using the diagonal residue condition, either ψ_z is a row relation and we are done, or the node containing z in $\mathfrak{t}^{\mu}$ is a Garnir node and its corresponding Garnir relation will be at the top of the diagram.

(A5) By the diagonal residue condition, if $\psi_{\alpha+z_j+1}$ is not a row relation in this case, the node containing $\alpha+z_j+1$ in $\mathfrak{t}^{\mu}$ will be a Garnir node and we have the corresponding Garnir relation at the top of the diagram.

To describe the degree of φ, first suppose that $k=0$. By Proposition 1.34 we have that $\deg\left(v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\right) = \deg(\mathfrak{s})$. We wish to compute $\deg\left(v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\right) - \deg\left(v^{\mathfrak{t}^{\lambda}}\right) = \deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^{\lambda}\right)$. Using the recursive definition of the degree, the nodes containing $n, n-1, \dots, \beta+2$ in both tableaux contribute the same value to the respective degrees. Hence

$$\deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^{\lambda}\right) = \deg\left(\mathfrak{s}_{<\beta+2}\right) - \deg\left(\mathfrak{t}^{\lambda}_{<\beta+2}\right).$$

Let A be the node $\mathfrak{s}_{<\beta+2}^{-1}(\beta+1)$ and B be the node $\left(\mathfrak{t}_{<\beta+2}^{\lambda}\right)^{-1}(\beta+1)$. Then the number of addable ι-nodes below A is one more than that below B, since in $\mathfrak{s}_{<\beta+2}$ we also count the position where the node was removed from in the qth component. Thus

$$\begin{aligned}\deg\left(\mathfrak{s}_{<\beta+2}\right)-\deg\left(\mathfrak{t}_{<\beta+2}^{\lambda}\right)&=1+\deg\left(\mathfrak{s}_{<\beta+1}\right)-\deg\left(\mathfrak{t}_{<\beta+1}^{\lambda}\right)\\&=1,\end{aligned}$$

since $\mathfrak{s}_{<\beta+1}$ is identical to $\mathfrak{t}_{<\beta+1}^{\lambda}$. So the degree of a homomorphism when $k=0$ is 1.

Next, suppose $k>0$ and we will use induction. Let $\tilde{c}\in\{p+1,p+2,\ldots,q-1\}$ be maximal so that $\lambda^{(\tilde{c})}$ has an addable ι-node. Suppose that if we add the node to the diagram $\left[\lambda^{(\tilde{c})}\right]$ we obtain the diagram $\left[\nu^{(\tilde{c})}\right]$ and consider the multipartition

$$\nu:=\left(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(\tilde{c}-1)},\nu^{(\tilde{c})},\mu^{(\tilde{c}+1)},\mu^{(\tilde{c}+2)},\ldots,\mu^{(l)}\right).$$

Then by induction we have that there is a homomorphism $\varphi_1:S^{\lambda}\to S^{\nu}$ given by $v^{\mathfrak{t}^{\lambda}}\mapsto v^{\mathfrak{t}^{\nu}}\Psi_{\gamma+1}{\uparrow}^{\beta}$, where $\gamma+1$ is the value of the added node in $\mathfrak{t}^{\nu}$. By our choice of $\tilde{c}$, this homomorphism must have degree one. Similarly, we also obtain a homomorphism $\varphi_2:S^{\nu}\to S^{\mu}$ given by $v^{\mathfrak{t}^{\nu}}\mapsto v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+1}{\uparrow}^{\gamma}$. By induction, φ_2 has degree k and can be written as a composition of k degree one homomorphisms.

Composing, we see that $\varphi_2\circ\varphi_1:S^{\lambda}\to S^{\mu}$ is given by $v^{\mathfrak{t}^{\lambda}}\mapsto v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+1}{\uparrow}^{\beta}$ thus $\varphi=\varphi_2\circ\varphi_1$. Hence φ has degree $k+1$, and can be written as a composition of $k+1$ degree one homomorphisms as we wanted.

□

We have shown that a homomorphism exists when we move one node to form $[\mu]$ from $[\lambda]$, but what if we move two or more different unadjacent nodes? If we assume that the nodes are of residues at least one apart, and that we are able to form homomorphisms by moving the nodes independently of each other as in Corollary 3.3, then the homomorphisms obtained by moving the nodes one by one, in any order, always compose to give the same overall homomorphism from S^{λ} to

S^μ. We detail how this works for moving two different unadjacent nodes in the following Corollary. Note that results concerning decomposition numbers where multiple nodes are moved of 'unadjacent residues' have been given by Chuang and Tan [CT16] and Bowman and Speyer [BS18].

Corollary 3.4. *Let $l \geq 2$ and suppose that λ, ν_1, ν_2 and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving one node of residue ι_1 and one other node - not the same as or adjacent to the first - of residue ι_2. Suppose that $[\nu_1]$ is formed from $[\lambda]$ by moving just the ι_1-node and that $[\nu_2]$ is formed from $[\lambda]$ by moving just the ι_2-node, and that $\iota_1 \neq \iota_2$ and $\iota_1 \not\leftrightarrow \iota_2$. Suppose that $(\lambda, \nu_1)_{\iota_1}$, $(\lambda, \nu_2)_{\iota_2}$, $(\nu_1, \mu)_{\iota_2}$ and $(\nu_2, \mu)_{\iota_1}$ are all one node pairs. Then there are non-zero homomorphisms*

$$\varphi_{\lambda\nu_1} : S^\lambda \to S^{\nu_1}, \quad \varphi_{\nu_1\mu} : S^{\nu_1} \to S^\mu,$$
$$\varphi_{\lambda\nu_2} : S^\lambda \to S^{\nu_2}, \quad \varphi_{\nu_2\mu} : S^{\nu_2} \to S^\mu,$$

and we have that $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1} = \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2} \neq 0$.

In addition, if $(\lambda, \nu_1)_{\iota_1}$ and $(\nu_1, \mu)_{\iota_2}$ have degrees $k_1 + 1$ and $k_2 + 1$, we have that the degree of $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$ is $k_1 + k_2 + 2$.

Proof. Since $(\lambda, \nu_1)_{\iota_1}$, $(\lambda, \nu_2)_{\iota_2}$, $(\nu_1, \mu)_{\iota_2}$ and $(\nu_2, \mu)_{\iota_1}$ are all one node pairs, by Corollary 3.3 we have that there are non-zero homomorphisms

$$\varphi_{\lambda\nu_1} : S^\lambda \to S^{\nu_1}, \quad \varphi_{\nu_1\mu} : S^{\nu_1} \to S^\mu,$$
$$\varphi_{\lambda\nu_2} : S^\lambda \to S^{\nu_2}, \quad \varphi_{\nu_2\mu} : S^{\nu_2} \to S^\mu.$$

Write $\varphi_{\lambda\nu_j}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^{\nu_j}} \Psi_{\alpha_j+1}{\uparrow}^{\beta_j}$ for some $\alpha_j \in \{0, \ldots, n-1\}$ and $\beta_j \in \{1, \ldots, n-1\}$ with $\alpha_j \leq \beta_j$, for $j \in \{1, 2\}$. Without loss of generality, assume that $\beta_1 < \beta_2$. If $\beta_1 < \alpha_2$ then

$$\begin{aligned}
\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+1}{\uparrow}^{\beta_2}\ \Psi_{\alpha_1+1}{\uparrow}^{\beta_1} \\
&= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}{\uparrow}^{\beta_1}\ \Psi_{\alpha_2+1}{\uparrow}^{\beta_2} \\
&= \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda})
\end{aligned}$$

and we are done. Hence, assume that $\beta_1 \geq \alpha_2$. Then we have multiple cases.

Case I: The ι_1-node is moved to a position above the ι_2-node in $[\mu]$

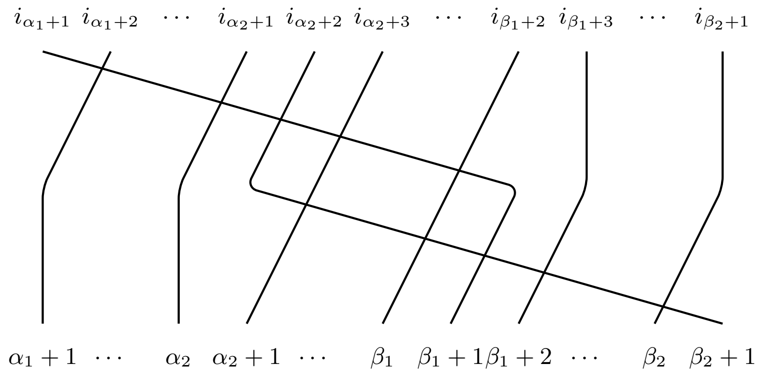
In this case, we have that

$$\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+2}\!\uparrow^{\beta_2}\ \Psi_{\alpha_1+1}\!\uparrow^{\beta_1}.$$

Now

$$\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}\!\uparrow^{\beta_1+1}\ \Psi_{\alpha_2+1}\!\uparrow^{\beta_2}$$

which as a diagram is:



Write

$$\begin{aligned}
\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}\!\uparrow^{\beta_1+1}\ \Psi_{\alpha_2+1}\!\uparrow^{\beta_2} \\
&= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}\!\uparrow^{\alpha_2+1} \cdot \left(\Psi_{\alpha_2+2}\!\uparrow^{\beta_1+1}\right)\!\downarrow_{\alpha_2+1} \cdot \psi_{\beta_1+1} \cdot \Psi_{\beta_1+2}\!\uparrow^{\beta_2} \\
&= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}\!\uparrow^{\alpha_2+1} \cdot \psi_{\alpha_2+1} \cdot \left(\Psi_{\alpha_2+2}\!\uparrow^{\beta_1+1}\right)\!\downarrow_{\alpha_2+1} \cdot \Psi_{\beta_1+2}\!\uparrow^{\beta_2}
\end{aligned}$$

by Lemma 2.3 since $i_{\alpha_1+1} \not\leftarrow i_{\alpha_2+2}$,

$$= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+1}\!\uparrow^{\alpha_2} \cdot \Psi_{\alpha_2+2}\!\uparrow^{\beta_1+1} \cdot \Psi_{\alpha_2+1}\!\uparrow^{\beta_1} \cdot \Psi_{\beta_1+2}\!\uparrow^{\beta_2}$$

since $\psi^2_{\alpha_2+1} = 0$ as $i_{\alpha_1+1} \not\to i_{\alpha_2+2}$,

$$\begin{aligned} &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+2}{\uparrow}^{\beta_2} \cdot \Psi_{\alpha_1+1}{\uparrow}^{\beta_1} \\ &= \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}). \end{aligned}$$

Thus $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = \varphi_{\nu_2\mu}\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda})$.

Case II: The ι_1-node is moved to a position below the ι_2-node in $[\mu]$

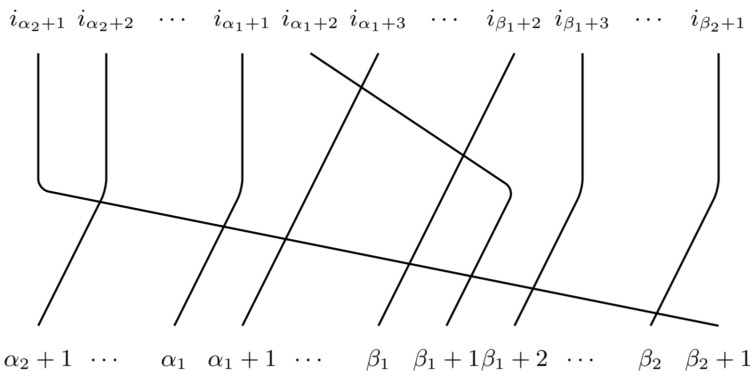
In this case, we have that

$$\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+1}{\uparrow}^{\beta_2}\ \Psi_{\alpha_1+1}{\uparrow}^{\beta_1}\ .$$

Now

$$\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+2}{\uparrow}^{\beta_1+1}\ \Psi_{\alpha_2+1}{\uparrow}^{\beta_2}$$

which as a diagram is:



Write

$$\begin{aligned} \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_1+2}{\uparrow}^{\beta_1+1}\ \Psi_{\alpha_2+1}{\uparrow}^{\beta_2} \\ &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+1}{\uparrow}^{\alpha_1} \cdot \left(\Psi_{\alpha_1+2}{\uparrow}^{\beta_1+1}\right){\downarrow}_{\alpha_1+1} \cdot \psi_{\beta_1+1} \cdot \Psi_{\beta_1+2}{\uparrow}^{\beta_2} \\ &= v^{\mathfrak{t}^\mu} \Psi_{\alpha_2+1}{\uparrow}^{\alpha_1} \cdot \psi_{\alpha_1+1} \cdot \left(\Psi_{\alpha_1+2}{\uparrow}^{\beta_1+1}\right){\downarrow}_{\alpha_1+1} \cdot \Psi_{\beta_1+2}{\uparrow}^{\beta_2} \end{aligned}$$

by Lemma 2.3 since $i_{\alpha_2+1} \not\rightarrow i_{\alpha_1+2}$,

$$
\begin{aligned}
&= v^{\mathrm{t}^{\mu}} \Psi_{\alpha_2+1}{\uparrow}^{\beta_2}\ \Psi_{\alpha_1+1}{\uparrow}^{\beta_1} \\
&= \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathrm{t}^{\lambda}}).
\end{aligned}
$$

Thus $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathrm{t}^{\lambda}}) = \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathrm{t}^{\lambda}})$.

Since the degree of $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$ is equal to the sum of degrees, the degree is $(k_1+1)+(k_2+1) = k_1+k_2+2$.

Note that in both cases, $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathrm{t}^{\lambda}})$ is given by some product of ψ_i corresponding to a reduced expression which is not zero thus $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathrm{t}^{\lambda}}) = \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathrm{t}^{\lambda}}) \neq 0$.

□

Now that we know that the homomorphisms from moving any two different unadjacent nodes in any order can be composed in either order to yield the same result, we now show that this means that for any number of moved nodes we obtain a similar result. Given any sequence of homomorphisms related to the gradual moving of individual nodes, we can swap the order of consecutive pairs of homomorphisms repeatedly using Corollary 3.4 until we obtain any other ordering.

Corollary 3.5. *Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving m distinct nodes $x_1, \ldots, x_m$, whose residues $\iota_1, \iota_2, \ldots, \iota_m$ are such that $\iota_i \neq \iota_j$ nor $\iota_i \not\rightarrow \iota_j$ for all $i \neq j$ with $1 \leq i, j \leq m$.*

Suppose that for each $X \subseteq \{1, \ldots, m\}$ we have an l-multipartition of n, ν_X, such that $[\nu_{\{i_1,\ldots,i_t\}}]$ is formed from $[\lambda]$ by moving just the nodes $x_{i_1}, \ldots, x_{i_t}$. In particular $\nu_\varnothing = \lambda$ and $\nu_{\{1,\ldots,m\}} = \mu$. Suppose that whenever $B \setminus A = \{r\}$, we have that $(\nu_A, \nu_B)_{\iota_r}$ is a one node pair, whose corresponding homomorphism is $\varphi_{\nu_A\nu_B}$.

Then there is a non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ and given any sequence of sets $\varnothing = X_0 \subsetneq X_1 \subsetneq X_2, \subsetneq \cdots \subsetneq X_m = \{1, \ldots, m\}$ we have that

$$\varphi = \varphi_{\nu_{X_{m-1}}\nu_{X_m}} \circ \varphi_{\nu_{X_{m-2}}\nu_{X_{m-1}}} \circ \cdots \circ \varphi_{\nu_{X_0}\nu_{X_1}}.$$

Proof. Without loss of generality suppose that node x_a is above node x_b for every $a < b$. Let $Y_j := \{1, 2, \ldots, j\}$ for $j \in \{0, \ldots, m\}$. Then $\varnothing = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m = \{1, 2, \ldots, m\}$. By assumption we have l-multipartitions of n, ν_{Y_j}, and non-zero homomorphisms $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}$ for each $j \in \{0, \ldots, m-1\}$. We may write $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}(v^{\mathfrak{t}^{\nu_{Y_j}}}) = v^{\mathfrak{t}^{\nu_{Y_{j+1}}}}\Psi_{\alpha_{j+1}+\zeta_{j+1}}\uparrow^{\beta_{j+1}}$ for some α_{j+1} and β_{j+1} related to the positions of the moved nodes, and ζ_{j+1} based on whether moved nodes are added above or below other moved nodes. Then

$$\varphi_{\nu_{Y_{m-1}}\nu_{Y_m}} \circ \cdots \circ \varphi_{\nu_{Y_0}\nu_{Y_1}}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu}\Psi_{\alpha_m+\zeta_m}\uparrow^{\beta_m} \cdots \Psi_{\alpha_1+\zeta_1}\uparrow^{\beta_1}. \tag{3.3}$$

Since $\beta_1 < \beta_2 < \cdots < \beta_m$ we must have that in the braid diagram for the above, no strings will cross twice and so by Proposition 1.18 the above will correspond to a reduced expression and so this composition of homomorphisms is not zero.

Now let $\varnothing = X_0 \subsetneq X_1 \subsetneq X_2, \subsetneq \cdots \subsetneq X_m = \{1, \ldots, m\}$ be a sequence of sets. Any such sequence of sets corresponds to a permutation $\sigma \in \mathfrak{S}_m$, given by $X_j \setminus X_{j-1} = \{\sigma(j)\}$, and given any permutation in $\mathfrak{S}_m$ we can define such a sequence of sets in the same fashion. For $j \in \{1, \ldots, m-1\}$, given $s_j = (j, j+1) \in \mathfrak{S}_m$ if we replace X_j with $\tilde{X}_j := X_{j-1} \cup (X_{j+1} \setminus X_j)$ then we have $\varnothing = X_0 \subsetneq X_1 \subsetneq X_2, \subsetneq \cdots \subsetneq X_{j-1} \subsetneq \tilde{X}_j \subsetneq X_{j+1} \cdots \subsetneq X_m = \{1, \ldots, m\}$ and this sequence corresponds to the permutation $s_j\sigma \in \mathfrak{S}_m$.

From our original sequence of sets corresponding to $\sigma \in \mathfrak{S}_m$ we have the homomorphism

$$\varphi = \varphi_{\nu_{X_{m-1}}\nu_{X_m}} \circ \cdots \circ \varphi_{\nu_{X_{j+1}}\nu_{X_{j+2}}} \circ \varphi_{\nu_{X_j}\nu_{X_{j+1}}} \circ \varphi_{\nu_{X_{j-1}}\nu_{X_j}} \circ \varphi_{\nu_{X_{j-2}}\nu_{X_{j-1}}} \circ \cdots \circ \varphi_{\nu_{X_0}\nu_{X_1}}.$$

By Corollary 3.4 we have that $\varphi_{\nu_{X_j}\nu_{X_{j+1}}} \circ \varphi_{\nu_{X_{j-1}}\nu_{X_j}} = \varphi_{\nu_{\tilde{X}_j}\nu_{X_{j+1}}} \circ \varphi_{\nu_{X_{j-1}}\nu_{\tilde{X}_j}}$ thus φ is equal to:

$$\varphi_{\nu_{X_{m-1}}\nu_{X_m}} \circ \cdots \varphi_{\nu_{X_{j+1}}\nu_{X_{j+2}}} \circ \varphi_{\nu_{\tilde{X}_j}\nu_{X_{j+1}}} \circ \varphi_{\nu_{X_{j-1}}\nu_{\tilde{X}_j}} \circ \varphi_{\nu_{X_{j-2}}\nu_{X_{j-1}}} \circ \cdots \circ \varphi_{\nu_{X_0}\nu_{X_1}},$$

that is, the homomorphism stemming from the sequence of sets corresponding to $s_j\sigma$. Since $\mathfrak{S}_m$ is generated by transpositions, given any sequence of sets we can show that the corresponding composition of homomorphisms can be permuted into any other. In particular any such homomorphism is equal to that given by (3.3), so is not zero.

□

3.2 One-row homomorphisms

Using Proposition 3.1 and its corollaries we are able to show that various homomorphisms between Specht modules exist related to moving individual nodes in a multipartition. The next step is to ask, whether we can obtain similar homomorphisms when moving a shape that is more than just a single node. We first consider moving a single row of nodes. In this case, homomorphisms will again arise between a pair Specht modules corresponding to a Carter-Payne pair of bipartitions, however we are able to explicitly describe the image of the generator of the domain Specht module by such a homomorphism.

The proof of Proposition 3.1 relies on directly checking the different generating relations for the Specht module S^λ, with the Garnir relation in Case III of the proof being the most convoluted to check, due to the node beneath the Garnir node being removed to form $[\mu]$. Since we are now removing a row of nodes, this sort of problem is going to happen more than once, and since we are looking to eventually find homomorphisms by moving more than just a single row of nodes we wish to avoid having to check a large variety of different relations in various ways. To this end, we use an inductive approach.

Consider bipartitions λ and μ, with $[\mu]$ formed from $[\lambda]$ by moving a row of at least 2 nodes from the second component to the first. Form $[\tilde{\lambda}]$ from $[\lambda]$ by removing the rightmost node in the row of nodes and moving it to a new third component. Then we can move what was left of the row in $[\tilde{\lambda}]$ to form $[\tilde{\lambda}_1]$, which is almost identical to $[\mu]$ except for the single node we placed in the third component. Then consider $[\tilde{\mu}]$ formed by moving the node from the third component to the end of the moved row; this will look almost identical to $[\mu]$ except for the empty extra component.

By induction on the number of nodes in the moved row, we obtain homomorphisms from $S^{\tilde{\lambda}}$ to $S^{\tilde{\lambda}_1}$ and we have a one-node homomorphism from $S^{\tilde{\lambda}_1}$ to $S^{\tilde{\mu}}$. These compose to give us a non-zero homomorphism from $S^{\tilde{\lambda}}$ to $S^{\tilde{\mu}}$ and we are able to deduce that generating relations from $S^{\tilde{\lambda}}$ must kill the image of $v^{\mathfrak{t}^{\tilde{\lambda}}}$ in $S^{\tilde{\mu}}$. Since this homomorphism is incredibly similar to that which we wish to prove exists from S^λ to S^μ, we can use the fact that nearly all the generating relations for S^λ have a direct counterpart in a generating relation for $S^{\tilde{\lambda}}$ to conclude that most of the generating relations for S^λ will immediately kill the image of $v^{\mathfrak{t}^\lambda}$ in S^μ. With this, we remove the need to directly check a large amount of relations, and can instead just mainly focus on the few generating relations for S^λ which do not have a counterpart for $S^{\tilde{\lambda}}$.

Now we shall state our result involving the moving of rows for bipartitions.

Proposition 3.6. *Let λ and μ be 2-multipartitions of n. Suppose*

$$e \geq \max\{h_{11}^{\lambda^{(1)}}+1, h_{11}^{\lambda^{(2)}}+1, h_{11}^{\mu^{(1)}}+1, h_{11}^{\mu^{(2)}}+1\}$$

and that $[\mu]$ is formed from $[\lambda]$ by moving a row containing a nodes from the second component to the first. Let $\mathfrak{s}$ be the μ-tableau defined by considering $\mathfrak{t}^\lambda$ and moving the row of a nodes from the second component to the first, keeping their values intact. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

We shall first discuss our strategy for the proof of Proposition 3.6 and expand on the inductive approach described in the introduction of this section. We can immediately observe that if $a = 1$ then by Proposition 3.1 we obtain the desired result. So we shall be able to suppose that $a \geq 2$. Also, as in Proposition 3.1 we have the diagonal residue condition. Let the row of a nodes that are removed from $\left[\lambda^{(2)}\right]$ be

$$\begin{array}{|c|c|c|c|}\hline \beta+1 & \beta+2 & \cdots\cdots & \beta+a \\ \hline\end{array}$$

and let $\alpha+1, \alpha+2, \ldots, \alpha+a$ be the entries of the added nodes in $\mathfrak{t}^\mu$. Then $\psi^{\mathfrak{s}} = \left(\Psi_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1}$ (note this may be equal to 1 if $\alpha+1 = \beta+1$, i.e. there are no nodes beneath the node containing $\alpha+a$ in $\left[\mu^{(1)}\right]$ and no nodes at all in $\left[\mu^{(2)}\right]$). The following diagrams help to illustrate these definitions.

$$\mathfrak{t}^\lambda = \left(\begin{array}{l} \begin{array}{|c|c|l|}\hline 1 & 2 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hline\end{array} \\ \begin{array}{|l|c|}\hline \cdots\cdots\cdots\cdots & \alpha \\ \hline\end{array} \\ \begin{array}{|c|l|}\hline \alpha+1 & \cdots\cdots \\ \hline\end{array} \\ \begin{array}{|l|}\hline \cdots\cdots\cdots \\ \hline\end{array} \end{array} \;\middle|\; \begin{array}{l} \begin{array}{|l|}\hline \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hline\end{array} \\ \begin{array}{|l|c|c|c|c|}\hline \cdots\cdots\cdots\cdots\cdots\cdots & \beta & \beta+1 & \cdots\cdots & \beta+a \\ \hline\end{array} \\ \begin{array}{|c|l|}\hline \beta+a+1 & \cdots\cdots\cdots \\ \hline\end{array} \\ \begin{array}{|l|c|}\hline \cdots\cdots\cdots\cdots\cdots & n \\ \hline\end{array} \end{array}\right)$$

$$\mathfrak{s} = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & & \\ \hline \cdots & \alpha & \beta+1 & \cdots & \beta+a \\ \hline \alpha+1 & \cdots & & & \\ \hline \cdots & & & & \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|}\hline \cdots & \\ \hline \cdots & \beta \\ \hline \beta+a+1 & \cdots \\ \hline \cdots & n \\ \hline \end{array} \right)$$

We now wish to work with some 3-multipartitions of n. For this, we need to define a new KLR algebra $\mathscr{H}_n^{\Lambda_{\tilde{\kappa}}}$ using quantum characteristic $\tilde{e} := e$ and multicharge $\tilde{\kappa} := (\kappa_1, \kappa_2, \mathrm{res}_\lambda(\beta+a))$. We write $\tilde{e}(\mathbf{i})$, $\tilde{y}_i$ and $\tilde{\psi}_i$ for the generators of this algebra. We also have the notation $\tilde{\Psi}_\bullet{\uparrow}^\bullet$, $\tilde{\Psi}^\bullet{\downarrow}_\bullet$, $\left(\tilde{\Psi}_\bullet{\uparrow}^\bullet\right){\downarrow}_\bullet$ and $\left(\tilde{\Psi}^\bullet{\downarrow}_\bullet\right){\uparrow}^\bullet$ formed by taking the corresponding relations in $\mathscr{H}_n^{\Lambda_\kappa}$ and replacing any $\psi_\bullet$ with $\tilde{\psi}_\bullet$. Note that the only place $\mathscr{H}_n^{\Lambda_\kappa}$ and $\mathscr{H}_n^{\Lambda_{\tilde{\kappa}}}$ will differ in their definition is in the relations of the form (1.12).

Recall that given a partition ν, let $\nu_{\hat{k}} := (\nu_1, \nu_2, \ldots, \nu_{k-1}, \nu_k - 1, \nu_{k+1}, \nu_{k+2}, \ldots)$, i.e. $[\nu_{\hat{k}}]$ is $[\nu]$ with the rightmost node on the kth row removed. So now suppose that the row of $[\lambda]$ to which the nodes will be added is the k_1th row and the row from which the nodes will be removed is the k_2th row. Then consider a 3-multipartition of n, $\tilde{\lambda}$, defined as

$$\tilde{\lambda} := \left(\lambda^{(1)}, \lambda^{(2)}_{\hat{k}_2}, (1)\right),$$

i.e. so that

$$\mathfrak{t}^{\tilde{\lambda}} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & \cdots \\ \hline \cdots & \alpha & \\ \hline \alpha+1 & \cdots & \\ \hline \cdots & & \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|}\hline \cdots & & & & \\ \hline \cdots & \beta & \beta+1 & \cdots & \beta+a-1 \\ \hline \beta+a & \cdots & & & \\ \hline \cdots & n-1 & & & \\ \hline \end{array} \;\middle|\; \begin{array}{|c|}\hline n \\ \hline \end{array} \right).$$

Also define $\tilde{\nu}$, a 3-multipartition of n, by

$$\tilde{\nu} := \left(\mu^{(1)}_{\hat{k}_1}, \mu^{(2)}, (1)\right)$$

with a $\tilde{\nu}$-tableau $\mathfrak{s}_1$ defined as

$$\mathfrak{s}_1 = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & & \\ \hline \cdots & \alpha & \beta+1 & \cdots & \beta+a-1 \\ \hline \alpha+1 & \cdots & & & \\ \hline \cdots & & & & \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|}\hline \cdots & \\ \hline \cdots & \beta \\ \hline \beta+a & \cdots \\ \hline \cdots & n-1 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|}\hline n \\ \hline \end{array} \right)$$

so that $\tilde{\psi}^{\mathfrak{s}_1} = \left(\tilde{\Psi}_{\alpha+a-1}{\uparrow}^{\beta+a-2}\right){\downarrow}_{\alpha+1}$.

Consider $\tilde{\lambda}^* = (\tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)})$ along with $\tilde{\nu}^* = (\tilde{\nu}^{(1)}, \tilde{\nu}^{(2)})$ and let $\mathfrak{s}_1^*$ be the $\tilde{\nu}^*$-tableau defined as the first two components of $\mathfrak{s}_1$. Then by induction on the number of nodes in the moved row, there exists a homomorphism $\varphi_1^* : S^{\tilde{\lambda}^*} \to S^{\tilde{\nu}^*}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}^*}} \mapsto v^{\mathfrak{t}^{\tilde{\nu}^*}} \psi^{\mathfrak{s}_1^*}$, and no generating relation for $S^{\tilde{\lambda}}$ kills $v^{\mathfrak{t}^{\tilde{\nu}^*}} \psi^{\mathfrak{s}_1^*}$ via a relation of the form (1.12). The base case for this is given by Proposition 3.1. Due to the definition of $\mathfrak{s}_1^*$, every generating relation for $S^{\tilde{\lambda}}$ except for y_n will correspond directly to a generating relation for $S^{\tilde{\lambda}^*}$ and, due to the existence of φ_1^* and the fact that a relation of the form (1.12) is never used, must kill $v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}$. In addition, y_n will kill $v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}$ since it will commute with $\tilde{\psi}^{\mathfrak{s}_1}$. In this way, we can see that there exists a homomorphism $\varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\nu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}$.

Now consider the 3-multipartition $\tilde{\mu} := (\mu^{(1)}, \mu^{(2)}, \varnothing)$ and the $\tilde{\mu}$-tableau $\mathfrak{s}_2$ defined by

$$\mathfrak{s}_2 = \left(\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \multicolumn{4}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ \hline \multicolumn{2}{|c|}{\cdots\cdots} & \alpha & \alpha+1 & \cdots\cdots & \alpha+a-1 \;\; \boxed{n} \\ \hline \alpha+a & \cdots\cdots & \multicolumn{4}{c}{} \\ \cline{1-2} \multicolumn{2}{|c|}{\cdots\cdots} & \multicolumn{4}{c}{} \\ \cline{1-2} \end{array} \;\middle|\; \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ \hline \cdots\cdots & \beta+a-1 \\ \hline \beta+a & \cdots\cdots \\ \hline \cdots\cdots & n-1 \\ \hline \end{array} \;\middle|\; \varnothing \right)$$

with $\tilde{\psi}^{\mathfrak{s}_2} = \tilde{\Psi}_{\alpha+a}\!\uparrow^{n-1}$. Using Corollary 3.3 we know that there is a homomorphism $\varphi_2 : S^{\tilde{\nu}} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\nu}}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}_2}$.

We have that $\tilde{\psi}^{\mathfrak{s}} = \left(\tilde{\Psi}_{\alpha+a}\!\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+1}$. Composing φ_2 with φ_1 we have a homomorphism $\tilde{\varphi} := \varphi_2 \circ \varphi_1 \colon S^{\tilde{\lambda}} \to S^{\tilde{\mu}}$ given by

$$\begin{aligned}
\tilde{\varphi}\left(v^{\mathfrak{t}^{\tilde{\lambda}}}\right) &= \varphi_2\left(\varphi_1\left(v^{\mathfrak{t}^{\tilde{\lambda}}}\right)\right) \\
&= \varphi_2\left(v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}\right) \\
&= \varphi_2\left(v^{\mathfrak{t}^{\tilde{\nu}}}\right) \tilde{\psi}^{\mathfrak{s}_1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}_2} \tilde{\psi}^{\mathfrak{s}_1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\Psi}_{\alpha+a}\!\uparrow^{n-1} \cdot \left(\tilde{\Psi}_{\alpha+a-1}\!\uparrow^{\beta+a-2}\right)\downarrow_{\alpha+1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \left(\tilde{\Psi}_{\alpha+a}\!\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+1} \cdot \tilde{\Psi}_{\beta+a}\!\uparrow^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}\!\uparrow^{n-1} .
\end{aligned}$$

Since the residue sequence of $\mathfrak{s}$ is the same as that of $\mathfrak{t}^{\lambda}$, in order to show that the homomorphism $\varphi : S^{\lambda} \to S^{\mu}$ exists we must show that $\varphi(v^{\mathfrak{t}^{\lambda}})a = 0$ whenever

$v^{\mathfrak{t}^\lambda} a = 0$ for $a \in \mathscr{H}_n^{\Lambda_\kappa}$. In particular, we must check that the generating relations of S^λ hold on the image of $v^{\mathfrak{t}^\lambda}$. We are required to check that:

(i) $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} y_r = 0$ for all $r \in \{1, \dots, n\}$.

(ii) $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} \psi_r = 0$ for all $r \in \{1, \dots, n-1\}$ such that r and $r+1$ are in the same row of $\mathfrak{t}^\lambda$.

(iii) $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g_\lambda^A = 0$ for every Garnir node A of $[\lambda]$.

Now as we know that $\tilde{\varphi}$ exists, we have the following facts:

(i*) $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\Psi_{\beta+a}\!\uparrow^{n-1} \tilde{y}_r = 0$ for all $r \in \{1, \dots, n\}$.

(ii*) $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\Psi_{\beta+a}\!\uparrow^{n-1} \tilde{\psi}_r = 0$ for all $r \in \{1, \dots, n-1\}$ such that r and $r+1$ are in the same row of $\mathfrak{t}^{\tilde{\lambda}}$.

(iii*) $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\Psi_{\beta+a}\!\uparrow^{n-1} g_{\tilde{\lambda}}^A$ for every Garnir node A of $\left[\tilde{\lambda}\right]$.

Since $\tilde{\mu}$ is identical to μ except for the empty third component, we will have that the generating relations for S^μ and $S^{\tilde{\mu}}$ are identical (up to swapping $e(\mathbf{i})$ for $\tilde{e}(\mathbf{i})$, y_i for $\tilde{y}_i$ and ψ_i for $\tilde{\psi}_i$). Note also that since the multicharge $\tilde{\kappa}$ is just κ with an additional component, any time that a relation of the form (1.12) is used to kill a term within $\mathscr{H}_n^{\Lambda_{\tilde{\kappa}}}$, we can use a corresponding relation of this form in the same way within $\mathscr{H}_n^{\Lambda_\kappa}$ since $(\Lambda_\kappa, \alpha_{i_1}) \le (\Lambda_{\tilde{\kappa}}, \alpha_{i_1})$. So since this is the only place that $\mathscr{H}_n^{\Lambda_\kappa}$ and $\mathscr{H}_n^{\Lambda_{\tilde{\kappa}}}$ differ, and the fact that as the diagrams for $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$ and $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}$ are also identical, any relation which kills $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}$ will also kill $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$. Hence our strategy is to show that using the relations (i*), (ii*) and (iii*), we can deduce many of the relations in (i), (ii) and (iii), leaving just a few extra cases to consider.

Note that the following proof has notes in the margin of the form (B•). These can be ignored for now and will become relevant when considering the proof of Corollary 3.9.

Proof of Proposition 3.6. As remarked above, we may suppose that $a \ge 2$. Let $(i_1, i_2, \dots, i_n)$ be the residue sequence of $\mathfrak{t}^{\tilde{\mu}}$ (which is identical to that of $\mathfrak{t}^\mu$), then

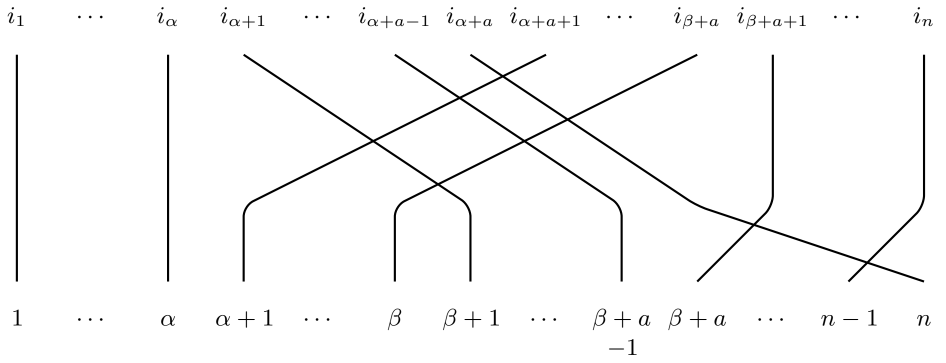


Figure 3.1: Braid diagram of $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}$.

$v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}$ is shown diagrammatically in Figure 3.1. We shall check that the relations in (i), (ii) and (iii) hold in separate sections below.

3.2.1 Relations in (i).

(B1)

If we are able to show that $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{y}_r = 0$ for every $r \in \{1, \dots, n\}$, then we will have that $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}y_r = 0$ for every such r. Take $r \in \{1, \dots, \beta+a-1\}$. Then

$$\begin{aligned} 0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}\,\tilde{y}_r \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{y}_r\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{y}_r, \end{aligned}$$

with the last equality following since this is equal to zero and we can apply $\tilde{\Psi}^{n-1}{\downarrow}_{\beta+a}$ to both sides and then use Lemma 2.8 since $i_{\alpha+a} \not\to i_{\beta+a+1}, \dots, i_n$.

Now suppose that $r \in \{\beta+a, \dots, n-1\}$. Then $i_{\alpha+a} \not\to i_{r+1}$ hence we have

$$\begin{aligned} 0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}\,\tilde{y}_r \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{y}_{r+1}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{y}_{r+1}. \end{aligned}$$

Finally suppose that $r = n$. Then as $i_{\alpha+a} \not\to i_{\beta+a+1}, \dots, i_n$ we have

$$
\begin{aligned}
0 &= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \, \tilde{y}_n \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{y}_{\beta+a} \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{y}_{\beta+a}.
\end{aligned}
$$

Putting all of these together, we have that $v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{y}_r = 0$ for every $r \in \{1, \dots, n\}$ as we wanted.

3.2.2 Relations in (ii).

We wish to show that $v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\psi}_r = 0$ for when ψ_r is a row relation for S^{λ} so that (B1)
then we have $v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}} \psi_r = 0$. All but one row relation for $S^{\tilde{\lambda}}$ will correspond to a row relation for S^{λ}, and we can use this to easily check a large amount of the relations in (ii).

Suppose that ψ_r is a row relation for S^{λ} with $r \in \{1, \dots, \beta + a - 2\}$. Then $\tilde{\psi}_r$ is also a row relation for S^{λ}, and will commute with $\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}$, hence

$$
\begin{aligned}
0 &= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \, \tilde{\psi}_r \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\psi}_r \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}
\end{aligned}
$$

and now as this is zero, multiply by $\tilde{\Psi}^{n-1}{\downarrow}_{\beta+a}$ on both sides:

$$
= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\psi}_r.
$$

Now suppose that $r \in \{\beta + a + 1, \dots, n-1\}$ and ψ_r is a row relation for S^{λ}. Then there is a corresponding row relation for $S^{\tilde{\lambda}}$, namely $\tilde{\psi}_{r-1}$. We know that we have

$$
\begin{aligned}
0 &= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \, \tilde{\psi}_{r-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}{\uparrow}^{r-2} \cdot \tilde{\psi}_{r-1} \tilde{\psi}_r \tilde{\psi}_{r-1} \cdot \tilde{\Psi}_{r+1}{\uparrow}^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\Psi}_{\beta+a}{\uparrow}^{r-2} \cdot \tilde{\psi}_r \tilde{\psi}_{r-1} \tilde{\psi}_r \cdot \tilde{\Psi}_{r+1}{\uparrow}^{n-1},
\end{aligned}
$$

as $i_{\alpha+a} \not\to i_{r+1}$. Then rearranging and applying $\tilde{\Psi}^{n-1}\downarrow_{\beta+a}$ to both sides

$$= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\psi}_r.$$

So far we have shown that $v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \tilde{\psi}_r = 0$ for $r \in \{1, \ldots, \beta + a - 2\} \cup \{\beta + a + 1, \ldots, n - 1\}$. All that is left to check is when $r = \beta + a - 1$. However, in this instance, there is not a corresponding row relation in $S^{\tilde{\lambda}}$, so we check that $v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}} \psi_{\beta+a-1}$ is equal to zero directly. We have

$$\begin{aligned} v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}} \psi_{\beta+a-1} &= v^{\mathfrak{t}^{\mu}} \left(\Psi_{\alpha+a}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+1} \cdot \psi_{\beta+a-1} \\ &= v^{\mathfrak{t}^{\mu}} \left(\Psi_{\alpha+a}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+a-1} \cdot \psi_{\beta+a-1} \cdot \left(\Psi_{\alpha+a-2}\uparrow^{\beta+a-3}\right)\downarrow_{\alpha+1}, \end{aligned}$$

and then apply Lemma 2.7 to $\left(\Psi_{\alpha+a}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+a-1} \cdot \psi_{\beta+a-1}$

$$\begin{aligned} = \quad & v^{\mathfrak{t}^{\mu}} \psi_{\alpha+a-1} \cdot \left(\Psi_{\alpha+a}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+a-1} \cdot \left(\Psi_{\alpha+a-2}\uparrow^{\beta+a-3}\right)\downarrow_{\alpha+1} \\ & + v^{\mathfrak{t}^{\mu}} \sum_{i=1}^{k} \Psi_{\alpha+a+z_j}\uparrow^{\beta+a-1} \cdot \Psi_{\alpha+a}\uparrow^{\beta+a-2} \cdot \Psi_{\alpha+a-1}\uparrow^{\alpha+a+z_j-3} \\ & \qquad \cdot \left(\Psi_{\alpha+a-2}\uparrow^{\beta+a-3}\right)\downarrow_{\alpha+1} \end{aligned}$$

for some $k \geq 0$, with z_j's arising from residues $i_{\alpha+a+z_j}$ which are equal to $i_{\alpha+a-1}$. In the former term, we have a $\psi_{\alpha+a-1}$ crossing at the top, and this is a row relation for S^{μ} hence this term is zero. In the latter term, every summand begins with a $\psi_{\alpha+a+z_j}$ crossing , which due to the diagonal residue condition must also (B2)
be a row relation for S^{μ}. Thus we have $v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}} \psi_{\beta+a-1} = 0$ as we wanted.

3.2.3 Relations in (iii).

Given a multipartition ν with the diagonal residue condition, Garnir relations in ν arise from Garnir belts of the form:

$$\begin{array}{cccccc} & & & \boxed{r+1} & \boxed{\cdots\cdots} & \boxed{r+s} \\ \boxed{r+s+1} & \boxed{\cdots\cdots} & \boxed{r+t} & & & \end{array}$$

The associated Garnir relation is then $\left(\Psi_{r+s}\uparrow^{r+t-1}\right)\downarrow_{r+1}$. If A is the node in ν containing $r+1$ then we write g_ν^A or $g_\nu\left(\boxed{r+1}\right)$ for the above Garnir relation. We also write $g_{\tilde{\nu}}^A = g_{\tilde{\nu}}\left(\boxed{r+1}\right) = \left(\tilde{\Psi}_{r+s}\uparrow^{r+t-1}\right)\downarrow_{r+1}$.

Let $\tilde{r}$ be the entry of the node directly above that containing $\beta+a-1$ in $\mathfrak{t}^\lambda$. The proof splits into cases depending on the location of a Garnir relation with respect to $\tilde{r}$. These cases are:

- $r \in \{0, 1, \dots, \tilde{r}-1\}$
- $r \in \{\tilde{r}+1, \dots, \beta+a-1\}$
- $r \in \{\beta+a, \beta+a+1, \dots, n-2\}$
- $r = \tilde{r}$

$\mathbf{r} \in \{\mathbf{0}, \mathbf{1}, \dots, \tilde{\mathbf{r}}-\mathbf{1}\}$

(B3)

Suppose $r \in \{0, 1, \dots, \tilde{r}-1\}$. Then $g_{\tilde{\lambda}}\left(\boxed{r+1}\right)$ is a Garnir relation of S^μ, and $g_{\tilde{\lambda}}\left(\boxed{r+1}\right)$ commutes with $\Psi_{\beta+a}\uparrow^{n-1}$, hence

$$\begin{aligned}
0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}\uparrow^{n-1} g_{\tilde{\lambda}}\left(\boxed{r+1}\right) \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} g_{\tilde{\lambda}}\left(\boxed{r+1}\right)\tilde{\Psi}_{\beta+a}\uparrow^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} g_{\tilde{\lambda}}\left(\boxed{r+1}\right)\tilde{\Psi}_{\beta+a}\uparrow^{n-1}\tilde{\Psi}^{n-1}\downarrow_{\beta+a} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} g_{\tilde{\lambda}}\left(\boxed{r+1}\right),
\end{aligned}$$

and then this implies that $0 = v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g_\lambda\left(\boxed{r+1}\right)$.

$\mathbf{r} \in \{\tilde{\mathbf{r}}+\mathbf{1}, \dots, \boldsymbol{\beta}+\mathbf{a}-\mathbf{1}\}$

(B3)

Next, suppose that $r \in \{\tilde{r}+1, \dots, \beta+a-1\}$. Since the nodes containing $\beta+1, \dots, \beta+a$ in $\mathfrak{t}^\lambda$ are removable, we may assume $r+1 < \beta+1$. Also, the nodes that are in the same row as $\tilde{r}+1$ but with a higher entry have no nodes beneath them. So overall, we need only consider such r where the node containing $r+1$ is on the same row as the removed nodes and $r+1 < \beta+1$. Let us write

$r+1 = \beta - j$ for some $j \geq 0$. Then the Garnir belt is

$$\begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & \beta - j & \cdots\cdots & \beta + a \\ \hline \beta+a+1 & \cdots\cdots & \beta + k & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array}$$

for some k. The Garnir relation is $g_\lambda\left(\boxed{r+1}\right) = \left(\Psi_{\beta+a}{\uparrow}^{\beta+k-1}\right){\downarrow}_{\beta-j}$.

In $\mathfrak{t}^{\tilde{\lambda}}$ we have the similar Garnir belt

$$\begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & \beta - j & \cdots\cdots & \beta + a - 1 \\ \hline \beta+a & \cdots\cdots & \beta + k - 1 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array}$$

giving the Garnir relation $\left(\tilde{\Psi}_{\beta+a-1}{\uparrow}^{\beta+k-2}\right){\downarrow}_{\beta-j}$ for $S^{\tilde{\lambda}}$. So we have

$$\begin{aligned} 0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \cdot \left(\tilde{\Psi}_{\beta+a-1}{\uparrow}^{\beta+k-2}\right){\downarrow}_{\beta-j} \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\Psi_{\beta+a}{\uparrow}^{\beta+k-1} \cdot \left(\tilde{\Psi}_{\beta+a-1}{\uparrow}^{\beta+k-2}\right){\downarrow}_{\beta-j} \cdot \tilde{\Psi}_{\beta+k}{\uparrow}^{n-1} \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\left(\tilde{\Psi}_{\beta+a}{\uparrow}^{\beta+k-1}\right){\downarrow}_{\beta-j} \cdot \tilde{\Psi}_{\beta+k}{\uparrow}^{n-1} \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\left(\tilde{\Psi}_{\beta+a}{\uparrow}^{\beta+k-1}\right){\downarrow}_{\beta-j}, \end{aligned}$$

by applying Lemma 2.8, implying that $0 = v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\left(\Psi_{\beta+a}{\uparrow}^{\beta+k-1}\right){\downarrow}_{\beta-j}$, i.e. $0 = v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{r+1}\right)$.

$\mathbf{r} \in \{\boldsymbol{\beta}+\mathbf{a}, \boldsymbol{\beta}+\mathbf{a}+\mathbf{1}, \ldots, \mathbf{n}-\mathbf{2}\}$ (B3)

Now suppose that $r \in \{\beta+a, \beta+a+1, \ldots, n-2\}$. Then if

$$g_\lambda\left(\boxed{r+1}\right) = \left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1},$$

we have that $g_{\tilde{\lambda}}\left(\boxed{r+1}\right) = \left(\tilde{\Psi}_{r+s-1}{\uparrow}^{r+t-2}\right){\downarrow}_r$. Figure 3.2 exhibits the important parts of the diagram for $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}\, g_{\tilde{\lambda}}\left(\boxed{r+1}\right)$. We have

$$\begin{aligned} 0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}\, g_{\tilde{\lambda}}\left(\boxed{r+1}\right) \\ &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{r-1} \cdot \tilde{\Psi}_r{\uparrow}^{r+t-1} \cdot \left(\tilde{\Psi}_{r+s-1}{\uparrow}^{r+t-2}\right){\downarrow}_r \cdot \tilde{\Psi}_{r+t}{\uparrow}^{n-1} \end{aligned}$$

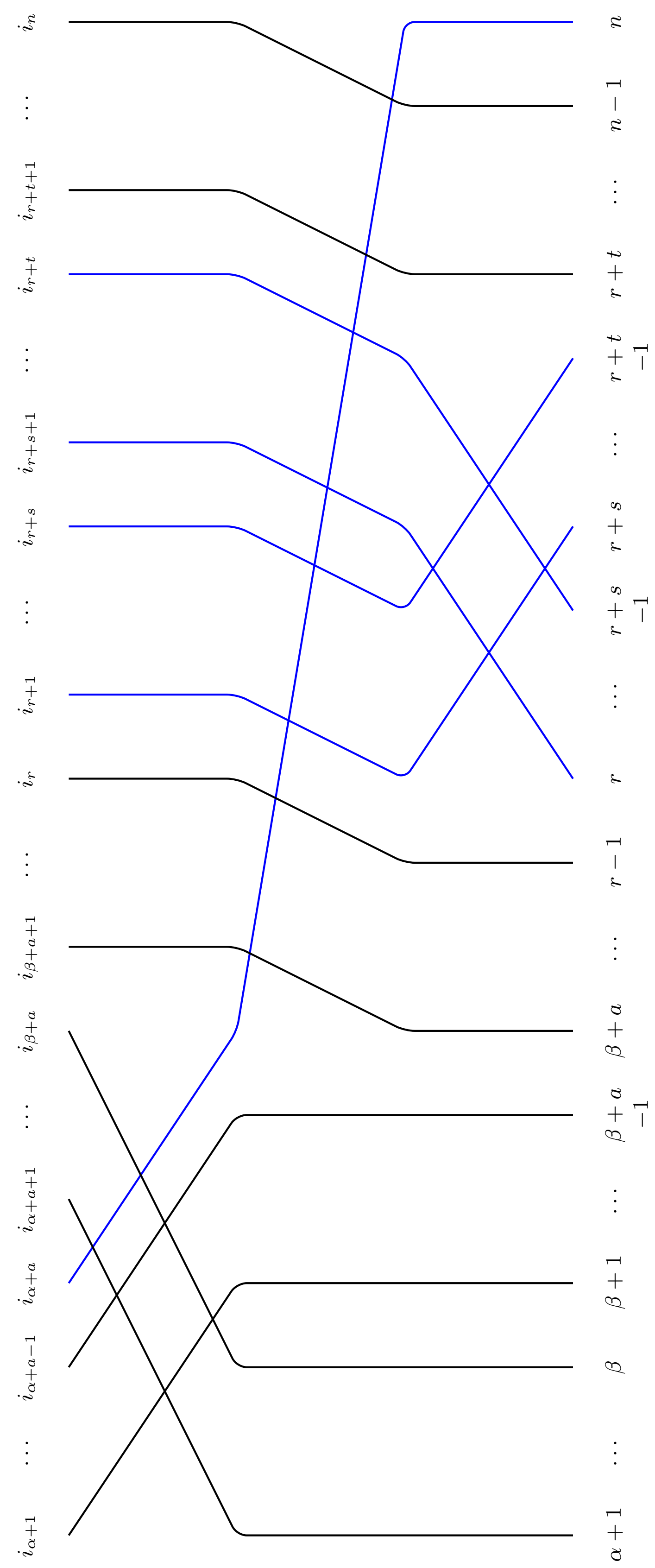


Figure 3.2: Part of the braid diagram for $v^{\mathbf{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathbf{s}}\tilde{\Psi}_{\beta+a}\!\uparrow^{n-1} g_{\bar{\lambda}}\left(\boxed{r+1}\right)$ when $r \in \{\beta+a, \dots, n-2\}$. The strings coloured blue are those to which Lemma 2.5 is applied.

and so we can apply Lemma 2.5 to $\tilde{\Psi}_r{\uparrow}^{r+t-1} \cdot \left(\tilde{\Psi}_{r+s-1}{\uparrow}^{r+t-2}\right){\downarrow}_r$ since $i_{\alpha+a} \neq i_{r+s+1}, i_{r+s+2}, \ldots, i_{r+t}$. This gives

$$\begin{aligned}
0 &= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{r-1} \cdot \left(\tilde{\Psi}_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \cdot \tilde{\Psi}_r{\uparrow}^{r+t-1} \cdot \tilde{\Psi}_{r+t}{\uparrow}^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} \left(\tilde{\Psi}_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \cdot \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} \left(\tilde{\Psi}_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1} \cdot \tilde{\Psi}_{\beta+a}{\uparrow}^{n-1}\ \tilde{\Psi}^{n-1}{\downarrow}_{\beta+a} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}} \left(\tilde{\Psi}_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1}.
\end{aligned}$$

This implies that $0 = v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\left(\Psi_{r+s}{\uparrow}^{r+t-1}\right){\downarrow}_{r+1}$, i.e. $0 = v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{r+1}\right)$ as we require.

$\mathbf{r = \tilde{r}}$

So now we are left to check the Garnir relation when $r = \tilde{r}$. In this case, there is not a corresponding Garnir relation in $S^{\tilde{\lambda}}$, so we check that $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$ is equal to zero directly. In $\mathfrak{t}^{\lambda}$ the Garnir belt is

$$\begin{array}{|c|c|c|c|c|}
\cline{3-5}
\multicolumn{2}{c|}{} & \tilde{r}+1 & \cdots\cdots & \tilde{r}+s \\
\hline
\tilde{r}+s+1 & \cdots\cdots & \beta+a & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}$$

giving the Garnir relation $g_\lambda\left(\boxed{\tilde{r}+1}\right) = \left(\Psi_{\tilde{r}+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1}$. The nodes which remain from the Garnir belt in $\mathfrak{t}^{\mu}$ are:

$$\begin{array}{|c|c|c|c|c|c|c|}
\cline{5-7}
\multicolumn{4}{c|}{} & \tilde{r}+a+1 & \cdots\cdots & \tilde{r}+a+s \\
\cline{1-3} \cline{5-7}
\tilde{r}+a+s+1 & \cdots\cdots & \beta+a & \multicolumn{4}{c}{} \\
\cline{1-3}
\end{array}$$

The important parts of the braid diagram of $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$ are shown in Figure 3.3.

We have

$$\begin{aligned}
v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right) &= v^{\mathfrak{t}^{\mu}}\left(\Psi_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1} \cdot \left(\Psi_{\tilde{r}+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1} \\
&= v^{\mathfrak{t}^{\mu}}\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}+a-1}\right){\downarrow}_{\alpha+1} \cdot \left(\Psi_{\tilde{r}+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1} \\
&\qquad \cdot \left(\Psi_{\tilde{r}+s}{\uparrow}^{\beta-1}\right){\downarrow}_{\tilde{r}+1} \cdot \left(\Psi_{\beta}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\beta-s+1},
\end{aligned}$$

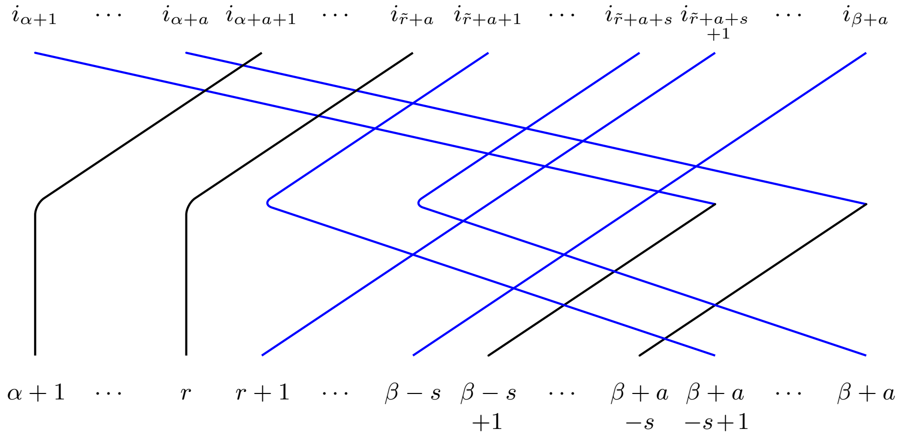


Figure 3.3: Part of the braid diagram for $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$. The blue parts of the strings show where Lemma 2.5 is applied.

and we can apply Lemma 2.5 to

$$\left(\Psi_{\tilde{r}+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1}\cdot\left(\Psi_{\tilde{r}+s}{\uparrow}^{\beta-1}\right){\downarrow}_{\tilde{r}+1}$$

since $i_{\tilde{r}+a+1},\dots,i_{\tilde{r}+a+s} \not\to i_{\tilde{r}+a+s+1},\dots,i_{\beta+a}$. This gives

$$\begin{aligned}
v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right) &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}+a-1}\right){\downarrow}_{\alpha+1}\cdot\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a+1}\\
&\qquad\cdot\left(\Psi_{\tilde{r}+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1}\cdot\left(\Psi_{\beta}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\beta-s+1}\\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a+1}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1}\\
&\qquad\cdot\left(\Psi_{\beta}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\beta-s+2}\cdot\Psi_{\beta-s+1}{\uparrow}^{\beta+a-s}
\end{aligned}\tag{3.4}$$

which is shown as a diagram in Figure 3.4. Since

$$i_{\alpha+1},\dots,i_{\alpha+a}\not\to i_{\tilde{r}+a+2},\dots,i_{\tilde{r}+a+s}$$

apply Lemma 2.8 to $\left(\Psi_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1}\cdot\left(\Psi_{\beta}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\beta-s+2}$ (take $x=\alpha$,

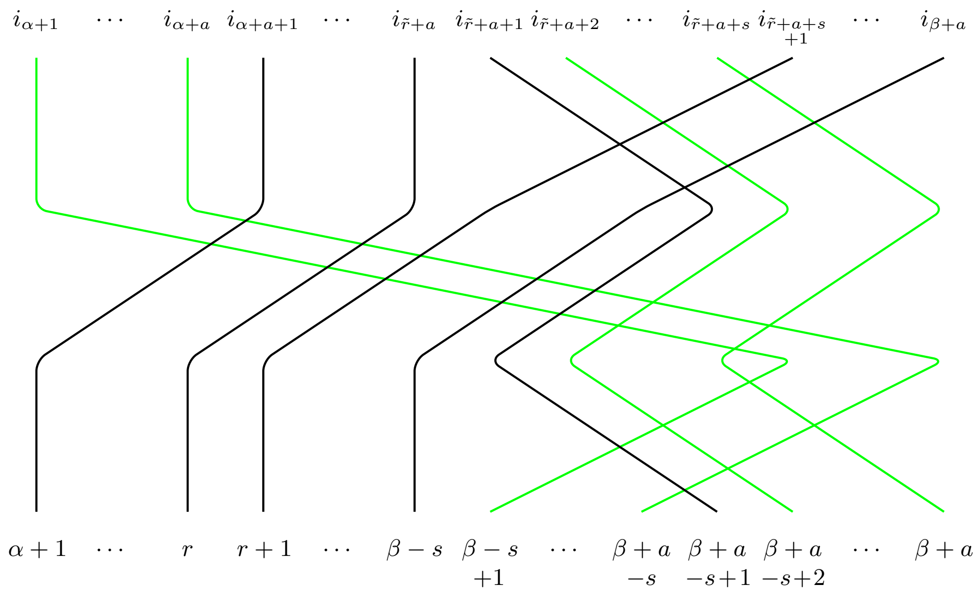


Figure 3.4: Part of the braid diagram for (3.4). The strings to which we apply Lemma 2.8 are coloured green.

$f = a, h = \beta + a - s + 1, g = s - 1, k = 0$), giving

$$\begin{aligned}
v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}} g_\lambda\left(\boxed{\tilde{r}+1}\right) &= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a+1}\cdot\left(\Psi_{\alpha+a}\uparrow^{\beta+a-s}\right)\downarrow_{\alpha+1} \\
&\qquad\cdot\Psi_{\beta-s+1}\uparrow^{\beta+a-s} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a+1}\cdot\left(\Psi_{\alpha+a}\uparrow^{\beta+a-s-1}\right)\downarrow_{\alpha+1} \\
&\qquad\cdot\Psi^{\beta+a-s}\downarrow_{\beta-s+1}\cdot\Psi_{\beta-s+1}\uparrow^{\beta+a-s} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a-1}\cdot\left(\Psi_{\alpha+a}\uparrow^{\beta+a-s-1}\right)\downarrow_{\alpha+1} \\
&\qquad\cdot(y_{\beta+a-s} - y_{\beta+a-s+1})
\end{aligned}$$

using relation (1.10) since $i_{\alpha+1}, \ldots, i_{\alpha+a-1} \not\rightarrow i_{\tilde{r}+a+1}$ but $i_{\alpha+a} + 1 = i_{\tilde{r}+a+1}$.

Now in terms of diagrams we have two summands, each with a dot within them to deal with. For that with the dot corresponding to $y_{\beta+a-s+1}$, we simply move the dot up through the crossings since $i_{\tilde{r}+a+1} \neq i_{\beta+a}, i_{\beta+a-1}, \ldots, i_{\tilde{r}+a+s+1}$, so with the dot now at the top we have that this summand is zero. For the other summand, we can move the dot up through some of the crossings since $i_{\alpha+a} \neq i_{\beta+a}, i_{\beta+a-1}, \ldots, i_{\tilde{r}+a+s+1}$. However, for the next crossing we have $i_{\alpha+a} = i_{\tilde{r}+a}$

and so using relation (1.8) we have that $v^{\mathsf{t}^\mu}\psi^{\mathfrak{s}} g_\lambda\left(\boxed{\tilde r + 1}\right)$ is equal to

$$v^{\mathsf{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a+1}\cdot\Psi_{\alpha+a}{\uparrow}^{\tilde r+a-2}\,y_{\tilde r+a-1}\cdot\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-1}\cdot\left(\Psi_{\alpha+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\alpha+1} \tag{3.5}$$

$$+\,v^{\mathsf{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a+1}\cdot\Psi_{\alpha+a}{\uparrow}^{\tilde r+a-2}\cdot\Psi_{\tilde r+a}{\uparrow}^{\beta+a-s-1}\cdot\left(\Psi_{\alpha+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\alpha+1}\,. \tag{3.6}$$

In (3.5), apply Lemma 2.11 to $\Psi_{\alpha+a}{\uparrow}^{\tilde r+a-2}\,y_{\tilde r+a-1}$. Then there is some $k\geq 0$ so that this term is equal to

$$\begin{aligned}&v^{\mathsf{t}^\mu}y_{\alpha+a}\cdot\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a+1}\cdot\Psi_{\alpha+a}{\uparrow}^{\tilde r+a-2}\cdot\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-1}\\&\qquad\cdot\left(\Psi_{\alpha+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\alpha+1}\\&+\sum_{j=1}^{k}v^{\mathsf{t}^\mu}\Psi_{\alpha+a+z_j}{\uparrow}^{\tilde r+a-2}\cdot\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a+1}\cdot\Psi_{\alpha+a}{\uparrow}^{\alpha+a+z_j-2}\\&\qquad\cdot\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-1}\cdot\left(\Psi_{\alpha+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\alpha+1}\end{aligned}$$

where $i_{\alpha+a}=i_{\alpha+a+z_j}$ for all $j\in\{1,\dots,k\}$. Thus as all of the $\psi_{\alpha+a+z_j}$ are row (B4) relations by the diagonal residue condition, (3.5) is equal to zero.

Now (3.6) (which is shown in Figure 3.5) can be rewritten as

$$\begin{aligned}&v^{\mathsf{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+1}\\&\cdot\Psi_{\tilde r+1}{\uparrow}^{\tilde r+a-2}\cdot\left(\Psi_{\tilde r}{\uparrow}^{\tilde r+a-2}\right){\downarrow}_{\tilde r-a+3}\cdot\Psi_{\tilde r-a+2}{\uparrow}^{\tilde r}\\&\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1}\,.\end{aligned}$$

We can then apply Lemma 2.9 to $\Psi_{\tilde r+1}{\uparrow}^{\tilde r+a-2}\cdot\left(\Psi_{\tilde r}{\uparrow}^{\tilde r+a-2}\right){\downarrow}_{\tilde r-a+3}$ since $i_{\alpha+j}=i_{\tilde r+j}$ for $j\in\{2,3,\dots,a-1\}$ (take $x=r-a+2, f=a-2, g=0$). So (3.6) is equal to

$$\begin{aligned}&\sum_{j=1}^{a-1}v^{\mathsf{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+1}\\&\cdot\left(\Psi_{\tilde r-j+1}{\uparrow}^{\tilde r+a-j-1}\right){\downarrow}_{\tilde r-a+3}\cdot\left(\Psi_{\tilde r-a+2+j}{\uparrow}^{\tilde r}\right){\downarrow}_{\tilde r-a+3}\cdot\Psi_{\tilde r-a+2}{\uparrow}^{\tilde r}\\&\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1}\,.\end{aligned}\tag{3.7}$$

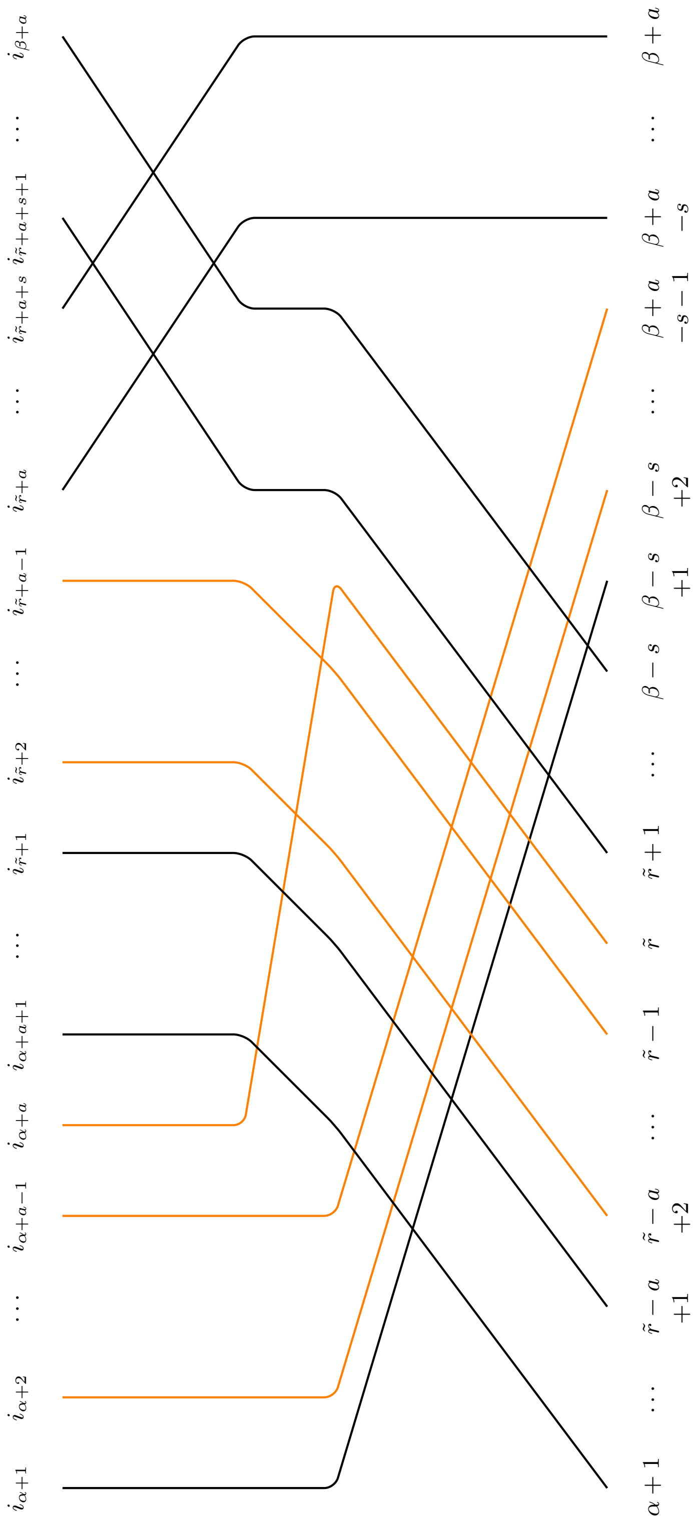


Figure 3.5: Part of the braid diagram for (3.6). The strings to which we apply Lemma 2.9 are shown in orange.

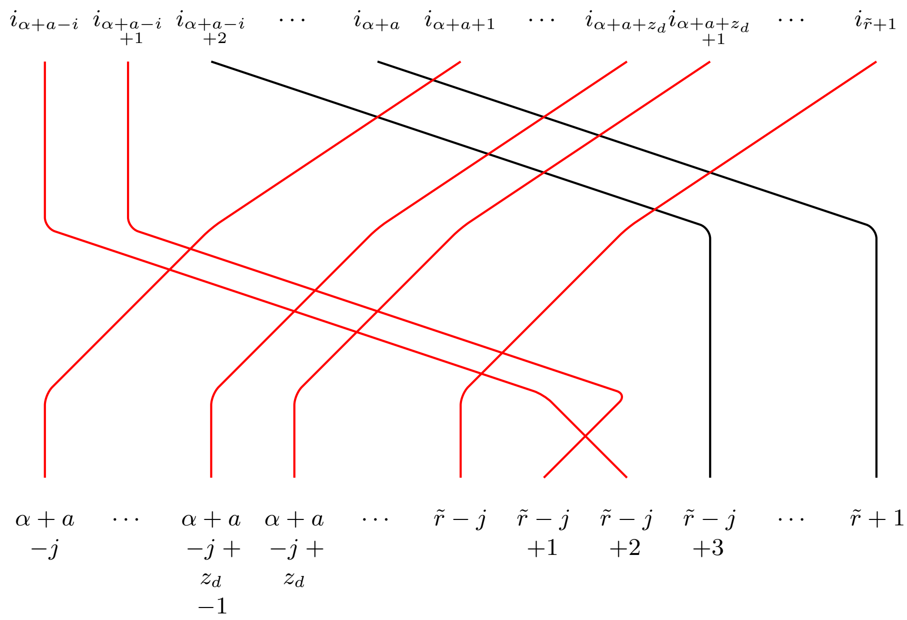


Figure 3.6: Part of the braid diagram of (3.8).

Given a summand of (3.7) for when $j \in \{1, \dots, a-2\}$, we have that this is:

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\!\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}\!\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+1}\cdot\psi_{\tilde{r}-j+1}\\
&\quad\cdot\left(\Psi_{\tilde{r}-j+2}\!\uparrow^{\tilde{r}+a-j-1}\right)\downarrow_{\tilde{r}-a+3}\cdot\left(\Psi_{\tilde{r}-a+2+j}\!\uparrow^{\tilde{r}}\right)\downarrow_{\tilde{r}-a+3}\cdot\Psi_{\tilde{r}-a+2}\!\uparrow^{\tilde{r}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\Psi_{\tilde{r}+a-1}\!\uparrow^{\beta+a-s-2}\right)\downarrow_{\tilde{r}+1}\\
&\quad= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\!\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}\!\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2}\\
&\qquad\cdot\left(\Psi_{\alpha+a-j+1}\!\uparrow^{\tilde{r}-j+1}\right)\downarrow_{\alpha+a-j}\cdot\psi_{\tilde{r}-j+1}\cdot\left(\Psi_{\alpha+a-j-1}\!\uparrow^{\tilde{r}-j-1}\right)\downarrow_{\alpha+1}\\
&\qquad\cdot\left(\Psi_{\tilde{r}-j+2}\!\uparrow^{\tilde{r}+a-j-1}\right)\downarrow_{\tilde{r}-a+3}\cdot\left(\Psi_{\tilde{r}-a+2+j}\!\uparrow^{\tilde{r}}\right)\downarrow_{\tilde{r}-a+3}\cdot\Psi_{\tilde{r}-a+2}\!\uparrow^{\tilde{r}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\Psi_{\tilde{r}+a-1}\!\uparrow^{\beta+a-s-2}\right)\downarrow_{\tilde{r}+1}
\end{aligned}
$$

We exhibit

$$
\left(\Psi_{\alpha+a}\!\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2}\cdot\left(\Psi_{\alpha+a-j+1}\!\uparrow^{\tilde{r}-j+1}\right)\downarrow_{\alpha+a-j}\cdot\psi_{\tilde{r}-j+1} \tag{3.8}
$$

in Figure 3.6. This allows us to apply Lemma 2.7 to

$$
\left(\Psi_{\alpha+a-j+1}\!\uparrow^{\tilde{r}-j+1}\right)\downarrow_{\alpha+a-j}\cdot\psi_{\tilde{r}-j+1}
$$

since $i_{\alpha+a-j} \leftarrow i_{a+a-j+1}$ (take $x = \alpha + a - j - 1, g = \tilde{r} - \alpha - a + 1$). Thus such a summand of (3.7) is equal to

$$\begin{aligned}
&v^{\mathrm{t}^\mu}\psi_{\alpha+a-j} \cdot \left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a} \cdot \left(\Psi_{\alpha+a}\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2} \\
&\quad \cdot \left(\Psi_{\alpha+a-j+1}\uparrow^{\tilde{r}-j+1}\right)\downarrow_{\alpha+a-j} \cdot \left(\Psi_{\alpha+a-j-1}\uparrow^{\tilde{r}-j-1}\right)\downarrow_{\alpha+1} \\
&\quad\quad \cdot \left(\Psi_{\tilde{r}-j+2}\uparrow^{\tilde{r}+a-j-1}\right)\downarrow_{\tilde{r}-a+3} \cdot \left(\Psi_{\tilde{r}-a+2+j}\uparrow^{\tilde{r}}\right)\downarrow_{\tilde{r}-a+3} \\
&\quad\quad\quad\quad \cdot \Psi_{\tilde{r}-a+2}\uparrow^{\tilde{r}} \cdot \left(\Psi_{\tilde{r}+a-1}\uparrow^{\beta+a-s-2}\right)\downarrow_{\tilde{r}+1} \cdot \\
&+ \sum_{d=1}^{k'} v^{\mathrm{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a} \cdot \left(\Psi_{\alpha+a}\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2} \cdot \Psi_{\alpha+a-j+z_d+1}\uparrow^{\tilde{r}-j+1} \\
&\quad\quad \Psi_{\alpha+a-j+1}\uparrow^{\tilde{r}-j} \cdot \Psi_{\alpha+a-j}\uparrow^{\alpha+a-j+z_d-2} \cdot \left(\Psi_{\alpha+a-j-1}\uparrow^{\tilde{r}-j-1}\right)\downarrow_{\alpha+1} \\
&\quad\quad \cdot \left(\Psi_{\tilde{r}-j+2}\uparrow^{\tilde{r}+a-j-1}\right)\downarrow_{\tilde{r}-a+3} \cdot \left(\Psi_{\tilde{r}-a+2+j}\uparrow^{\tilde{r}}\right)\downarrow_{\tilde{r}-a+3} \cdot \Psi_{\tilde{r}-a+2}\uparrow^{\tilde{r}} \\
&\quad\quad\quad\quad \cdot \left(\Psi_{\tilde{r}+a-1}\uparrow^{\beta+a-s-2}\right)\downarrow_{\tilde{r}+1}
\end{aligned} \tag{3.9}$$

for some $k' \geq 0$, where the z_d are such that $\psi_{\alpha+a-j} = \psi_{\alpha+a+z_d}$. The former term is rearranged to have leading term $\psi_{\alpha+a-j}$, thus is zero as this is a row relation. We have that (3.9) is equal to

$$\begin{aligned}
&\sum_{d=1}^{k'} v^{\mathrm{t}^\mu}\left(\Psi_{\tilde{r}+a+s}\uparrow^{\beta+a-1}\right)\downarrow_{\tilde{r}+a} \cdot \left(\Psi_{\alpha+a}\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2} \cdot \psi_{\alpha+a-j+z_d+1} \\
&\quad\quad \Psi_{\alpha+a-j+z_d+2}\uparrow^{\tilde{r}-j+1} \cdot \Psi_{\alpha+a-j+1}\uparrow^{\tilde{r}-j} \cdot \Psi_{\alpha+a-j}\uparrow^{\alpha+a-j+z_d-2} \\
&\cdot \left(\Psi_{\alpha+a-j-1}\uparrow^{\tilde{r}-j-1}\right)\downarrow_{\alpha+1} \cdot \left(\Psi_{\tilde{r}-j+2}\uparrow^{\tilde{r}+a-j-1}\right)\downarrow_{\tilde{r}-a+3} \cdot \left(\Psi_{\tilde{r}-a+2+j}\uparrow^{\tilde{r}}\right)\downarrow_{\tilde{r}-a+3} \\
&\quad\quad\quad\quad \cdot \Psi_{\tilde{r}-a+2}\uparrow^{\tilde{r}} \cdot \left(\Psi_{\tilde{r}+a-1}\uparrow^{\beta+a-s-2}\right)\downarrow_{\tilde{r}+1}
\end{aligned}$$

and then since $i_{\alpha+a+z_d} = i_{\alpha+a-j} \implies i_{\alpha+a+z_d} \not\leftarrow i_{\alpha+a-j+2}, \ldots, i_{\alpha+a}$ we can apply Corollary 2.6 to $\left(\Psi_{\alpha+a}\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2} \cdot \psi_{\alpha+a-j+z_d+1}$ (take $x = \alpha + a - j + 1, f = j - 1, k = z_d - 1, h = 1, g = 1, t = \tilde{r} - \alpha - a - z_d$). Then every term of the sum has leading term $\psi_{\alpha+a+z_d}$ which is a row relation by the diagonal residue (B5) condition, hence (3.9) is equal to zero.

Now we can go back to (3.7). We are left with considering the summand when

$j = a-1$, which is equal to

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+1}\cdot\Psi_{\tilde{r}-a+2}{\uparrow}^{\tilde{r}}\\
&\qquad\qquad\qquad\qquad\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1} \qquad (3.10)\\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+2}\right){\downarrow}_{\alpha+1}\cdot\psi_{\tilde{r}-a+2}\\
&\qquad\qquad\qquad\qquad\cdot\Psi_{\tilde{r}-a+3}{\uparrow}^{\tilde{r}}\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1}. \qquad (3.11)
\end{aligned}
$$

This is shown in Figure 3.7. Now we have two cases, depending on whether $i_{\alpha+1}$ (B6) is equal to $i_{\tilde{r}+1}$ or not. If they are not equal then we must have also that no residue $i_{\tilde{r}+1}, i_{\tilde{r}}, \ldots, i_{\alpha+a+1}$ will equal $i_{\alpha+1}$. So (3.11) is equal to

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\psi_{\alpha+1}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+2}\right){\downarrow}_{\alpha+1}\\
&\qquad\qquad\qquad\qquad\cdot\Psi_{\tilde{r}-a+3}{\uparrow}^{\tilde{r}}\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1}
\end{aligned}
$$

by applying Lemma 2.3 to $\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+2}\right){\downarrow}_{\alpha+1}\cdot\psi_{\tilde{r}-a+2}$ (take $x = \alpha$, $g = r-\alpha-a+1$). Then as $\psi_{\alpha+1}$ is a row relation we have that (3.10) is equal to zero as we wanted.

The second case is when $i_{\alpha+1} = i_{\tilde{r}+1}$. Then since $i_{\alpha+1} = i_{\tilde{r}+1} = i_{\alpha+2}-1$ we have that (3.11) is equal to:

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+1}\right){\downarrow}_{\alpha+1}\\
&\qquad\cdot\psi_{\tilde{r}-a+2}\psi_{\tilde{r}-a+1}\psi_{\tilde{r}-a+2}\cdot\Psi_{\tilde{r}-a+3}{\uparrow}^{\tilde{r}}\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1}\\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+1}\right){\downarrow}_{\alpha+1}\\
&\qquad\cdot\psi_{\tilde{r}-a+1}\psi_{\tilde{r}-a+2}\psi_{\tilde{r}-a+1}\cdot\Psi_{\tilde{r}-a+3}{\uparrow}^{\tilde{r}}\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1} \qquad (3.12)\\
&\quad+v^{\mathfrak{t}^\mu}\left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde{r}}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+1}\right){\downarrow}_{\alpha+1}\\
&\qquad\qquad\qquad\cdot\Psi_{\tilde{r}-a+3}{\uparrow}^{\tilde{r}}\cdot\left(\Psi_{\tilde{r}+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde{r}+1} \qquad (3.13)
\end{aligned}
$$

First consider (3.12); we follow the same method as we did for a summand of (3.7). Apply Lemma 2.7 to $\left(\Psi_{\alpha+2}{\uparrow}^{\tilde{r}-a+1}\right){\downarrow}_{\alpha+1}\cdot\psi_{\tilde{r}-a+1}$ (take $x = \alpha, g = \tilde{r}-\alpha-a$) so

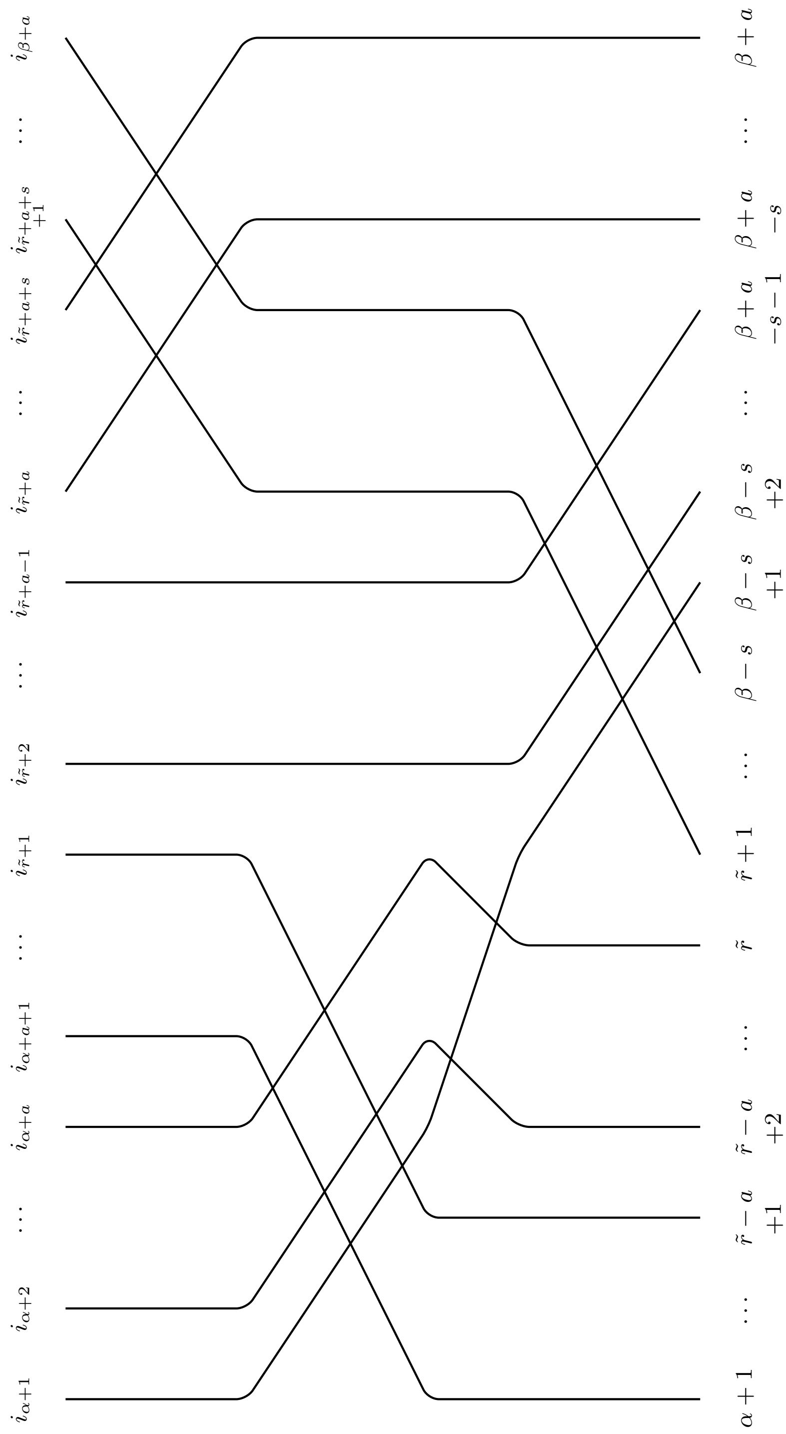


Figure 3.7: Part of the braid diagram for (3.10).

that (3.12) is equal to

$$\begin{aligned}
&v^{\mathtt{t}^\mu}\psi_{\alpha+1}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde r-a+1}\right){\downarrow}_{\alpha+1}\\
&\qquad\cdot\psi_{\tilde r-a+2}\cdot\psi_{\tilde r-a+1}\cdot\Psi_{\tilde r-a+3}{\uparrow}^{\tilde r}\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1}\\
&+\sum_{j=1}^{k''}v^{\mathtt{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+3}\cdot\Psi_{\alpha+z_j+2}{\uparrow}^{\tilde r-a+1}\\
&\qquad\cdot\Psi_{\alpha+2}{\uparrow}^{\tilde r-a}\cdot\Psi_{\alpha+1}{\uparrow}^{\tilde r-a}\cdot\psi_{\tilde r-a+2}\cdot\psi_{\tilde r-a+1}\\
&\qquad\qquad\cdot\Psi_{\tilde r-a+3}{\uparrow}^{\tilde r}\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1}
\end{aligned}$$

for some $k''\geq 0$, and the z_j are such that $i_{\alpha+1}=i_{\alpha+a+z_j}$. The former term is zero since $\psi_{\alpha+1}$ is a row relation. For latter terms in the sum, write such a term as

$$\begin{aligned}
&v^{\mathtt{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+3}\cdot\psi_{\alpha+z_j+2}\cdot\Psi_{\alpha+z_j+3}{\uparrow}^{\tilde r-a+1}\\
&\qquad\cdot\Psi_{\alpha+2}{\uparrow}^{\tilde r-a}\cdot\Psi_{\alpha+1}{\uparrow}^{\tilde r-a}\cdot\psi_{\tilde r-a+2}\cdot\psi_{\tilde r-a+1}\\
&\qquad\qquad\cdot\Psi_{\tilde r-a+3}{\uparrow}^{\tilde r}\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1},
\end{aligned}$$

and since $i_{\alpha+3},\dots,i_{\alpha+a}\not\to i_{\alpha+a+z_j}$, apply Corollary 2.6 to

$$\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+3}\cdot\psi_{\alpha+z_j+2}$$

(take $x=\alpha+2, f=a-2, k=z_j-1, h=1, g=1, t=\tilde r-\alpha-a-z_j$). Then such a term has $\psi_{\alpha+a+z_j}$ as its leading term and thus is zero, as this is a row relation (B7) by the diagonal residue condition.

Now we are left with (3.13), which is shown in Figure 3.8. Since $i_{\alpha+3},\dots,i_{\alpha+a}\not\to i_{\tilde r+1}$ we can apply Lemma 2.8 to $\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r}\right){\downarrow}_{\alpha+3}\cdot\Psi_{\tilde r-a+3}{\uparrow}^{\tilde r}$ (take $x=\alpha+a, f=a-2, h=\tilde r-\alpha-a, g=1$). Then (3.13) is equal to

$$\begin{aligned}
&v^{\mathtt{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+a}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r-1}\right){\downarrow}_{\alpha+3}\cdot\left(\Psi_{\alpha+2}{\uparrow}^{\tilde r-a+1}\right){\downarrow}_{\alpha+1}\\
&\qquad\cdot\left(\Psi_{\tilde r+a-1}{\uparrow}^{\beta+a-s-2}\right){\downarrow}_{\tilde r+1}\\
&=v^{\mathtt{t}^\mu}\left(\Psi_{\tilde r+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde r+1}\cdot\left(\Psi_{\alpha+a}{\uparrow}^{\tilde r-1}\right){\downarrow}_{\alpha+1}\,.
\end{aligned}\tag{3.14}$$

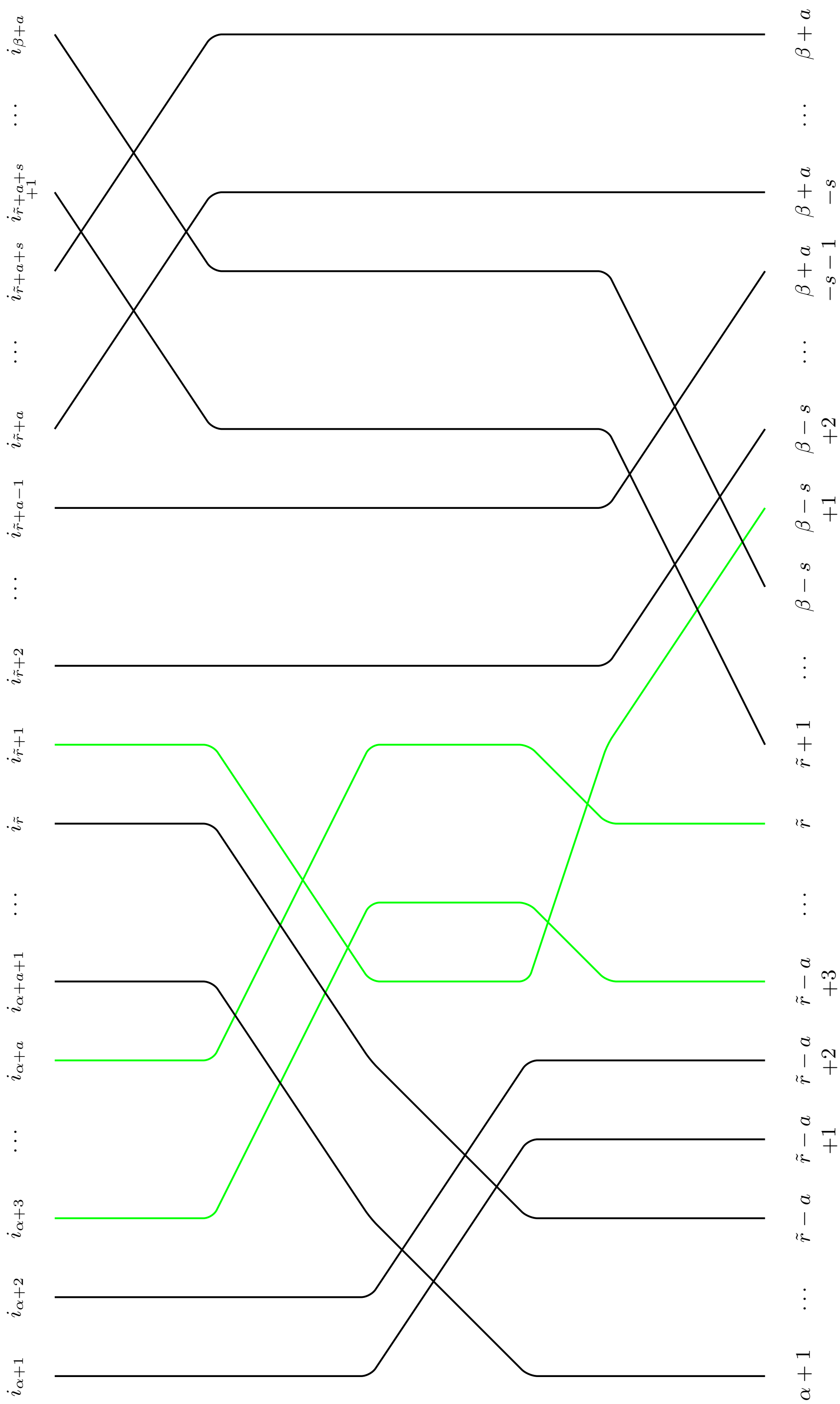


Figure 3.8: Braid diagram for (3.10). The strings which are involved in the cancelling using Lemma 2.8 are coloured green.

As we assumed that $i_{\alpha+1} = i_{\tilde{r}+1}$, we have that the following Garnir belt is in $[\mu]$:

$$\begin{array}{cc|c|c|c|c|c|}
\cline{3-7}
 & & \tilde{r}+1 & \cdots\cdots & \tilde{r}+a+1 & \cdots & \tilde{r}+a+s \\
\hline
\multicolumn{1}{|c|}{\tilde{r}+a+s+1} & \cdots & \beta+a & & & & \\
\cline{1-3}
\end{array}$$

This gives the Garnir relation $g_\mu\left(\boxed{\tilde{r}+1}\right) = \left(\Psi_{\tilde{r}+a+s}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\tilde{r}+1}$, and so (3.14) must be equal to zero. With this, we have finally shown that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right) = 0$ as we wanted.

Conclusion.

Having checked all of the relations in (i), (ii) and (iii) in the previous sections, we are done and so there indeed is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

□

3.2.4 Extending the result

We can extend Proposition 3.6 to multipartitions with greater than two components in a similar way to how we extended Proposition 3.1. This time, we define a one row pair and then exhibit the relevant changes to the proof of Proposition 3.6, whilst also describing the degree of such a homomorphism.

Definition 3.7. Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n, where $[\mu]$ is formed from $[\lambda]$ by moving a row of $a \geq 2$ nodes from the qth component to the pth, for some p and q such that $p < q$. Suppose that the residue of the leftmost node in the moved row is ι. In addition suppose that

$$e \geq \max_{p \leq c \leq q}\{h_{11}^{\lambda^{(c)}} + 1, h_{11}^{\mu^{(c)}} + 1\}.$$

Amongst the components $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$, suppose that there are exactly $k \geq 0$ such components to which a row of a nodes whose leftmost residue is ι can be added. If $k > 0$, then we also require that e is large enough so that the diagonal residue condition holds when the row is added to these k components. Suppose that amongst the components $\lambda^{(c')}$, there are no removable nodes of residues $\iota, \iota+1, \ldots, \iota+a-1$ and there are m_j addable nodes of residue $\iota+j$ for $j \in \{0, 1, \ldots, a-1\}$. Let $m = \sum_{j=0}^{a-1} m_j$. Then we say that $(\lambda, \mu)_\iota^k$ is a *one row pair (of degree $m+1$).*

Remark 3.8. Since we have the diagonal residue condition, if λ belongs to a one row pair, then in a component $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$, we can either have some individual addable nodes of the residues in the row *or* we can add only the entire row itself and not some other individual nodes of residues within the row also.

Corollary 3.9. *Suppose that $(\lambda, \mu)_\iota^k$ is a one row pair of degree $m+1$. Let $\mathfrak{s}$ be the μ-tableau defined by considering $\mathfrak{t}^\lambda$ and moving the row of a nodes from the qth component to the pth, keeping their values intact. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$. This homomorphism has degree $m+1$ and can be written as a composition of $k+1$ homomorphisms.*

Proof. If (λ, μ) is a one node pair, then we can simply use Corollary 3.3. Note that in this case, m will be equal to k and thus the degree matches that of Corollary 3.3. So instead we shall suppose that the shape moved is definitely a row of at least two nodes.

We shall begin by assuming that $k = 0$. Define α, β and a similarly as in the proof of Proposition 3.6, so that the nodes to be moved contain $\beta+1, \ldots, \beta+a$ in $\mathfrak{t}^\lambda$ from left to right, whilst in $\mathfrak{t}^\mu$ the added nodes contain $\alpha+1, \ldots, \alpha+a$ from left to right. Then $\psi^{\mathfrak{s}} = \left(\Psi_{\alpha+a}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+1}$. We need to check that the generating relations of S^λ hold on $\varphi\left(v^{\mathfrak{t}^\lambda}\right)$.

Similarly to Proposition 3.6, define a new KLR Algebra $\mathscr{H}_n^{\Lambda_{\tilde\kappa}}$ using quantum characteristic $\tilde{e} := e$ and multicharge

$$\tilde{\kappa} := (\kappa_1, \kappa_2, \ldots, \kappa_q, \mathrm{res}_\lambda(\beta+a), \kappa_{q+1}, \kappa_{q+2}, \ldots, \kappa_l),$$

and define $l+1$-multipartitions:

$$\begin{aligned}
\tilde{\lambda} &:= \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q-1)}, \lambda^{(q)}_{\hat{k}_2}, (1), \lambda^{(q+1)}, \lambda^{(q+2)}, \ldots, \lambda^{(l)}\right) \\
\tilde{\lambda}_1 &:= \Big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(p-1)}, \mu^{(p)}_{\hat{k}_1}, \mu^{(p+1)}, \mu^{(p+2)}, \ldots, \mu^{(q)}, (1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mu^{(q+1)}, \mu^{(q+2)}, \ldots, \mu^{(l)}\Big) \\
\tilde{\mu} &:= \left(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(q)}, \varnothing, \mu^{(q+1)}, \mu^{(q+2)}, \ldots, \mu^{(l)}\right)
\end{aligned}$$

We define a $\tilde{\lambda}_1$-tableau $\mathfrak{s}_1$ by $\tilde{\psi}^{\mathfrak{s}_1} = \left(\tilde{\Psi}_{\alpha+a-1}{\uparrow}^{\beta+a-2}\right){\downarrow}_{\alpha+1}$ and also a $\tilde{\mu}$-tableau $\mathfrak{s}_2$ by $\tilde{\psi}^{\mathfrak{s}_2} = \tilde{\Psi}_{\alpha+a}{\uparrow}^{Q-1}$, where $Q = \sum_{i=1}^{q}\left|\lambda^{(i)}\right|$. Then by induction on the number of nodes moved we have a homomorphism $\varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\lambda}_1}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\lambda}_1}}\tilde{\psi}^{\mathfrak{s}_1}$, and another $\varphi_2 : S^{\tilde{\lambda}_1} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}_1}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}_2}$. Defining $\tilde{\psi}^{\mathfrak{s}} := \left(\tilde{\Psi}_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1}$, the composition of φ_2 with φ_1 gives us a homomorphism $\tilde{\varphi} := \varphi_2 \circ \varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+a}{\uparrow}^{Q-1}$. From this we obtain relations (i*), (ii*), (iii*), just as in Proposition 3.6, and we can use these to check the relations (i), (ii) and (iii). Since $\tilde{\mu}$ is identical to μ except for the empty third component, the generating relations for S^{μ} and $S^{\tilde{\mu}}$ are identical up to changing the notation of the generators, and the diagrams for $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}$ and $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}$ are also identical, so any relation killing $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}$ will also kill $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}$.

For each type of relation, the above setup allows us to follow the same methods as in Proposition 3.6, only now accounting for the additional nodes in between the first and last components of $[\mu]$ as well as those outside of these components. In checking each of the relations, apply the same reasoning as in Proposition 3.6, however there are a few changes to be made at the places annotated by the following labels in the margins:

(B1) Replace n with Q throughout and note that $i_{\alpha+a} \not\to i_{\beta+a+1}, \ldots, i_Q$. For $r \in \{Q+1, \ldots, n-1\}$, follow the same reasoning as for $r \in \{1, \ldots, \beta+a-1\}$.

(B2) If $\psi_{\alpha+a+z_j}$ is not a row relation then by the diagonal residue condition the node containing $\alpha+a+z_j$ must be a Garnir node, so the corresponding Garnir relation will be at the top of the diagram for those terms in the sum.

(B3) Replace n with Q throughout and note that $i_{\alpha+a} \not\rightarrow i_{\beta+a+1}, \dots, i_Q$. For $r \in \{Q, \dots, n-2\}$, follow the same reasoning as for $r \in \{0, 1, \dots, \tilde{r}-1\}$.

(B4) If $\psi_{\alpha+a+z_j}$ is not a row relation then by the diagonal residue condition the node containing $\alpha + a + z_j$ must be a Garnir node, so the corresponding Garnir relation will be at the top of the diagram within $\Psi_{\alpha+a+z+j}\!\uparrow^{\tilde{r}+a-1}$.

(B5) If $\psi_{\alpha+a+z_d}$ is not a row relation then by the diagonal residue condition the node containing $\alpha+a+z_d$ in $\mathfrak{t}^\mu$ must be a Garnir node. Apply Corollary 2.6 to $\left(\Psi_{\alpha+a}\!\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+a-j+2} \cdot \Psi_{\alpha+a-j+z_d+1}\!\uparrow^{\tilde{r}-j+1}$ instead, then we have $\Psi_{\alpha+a+z_d}\!\uparrow^{\tilde{r}}$ at the top of the diagram, i.e. we have the Garnir relation for the node containing $\alpha + a + z_d$ in $\mathfrak{t}^\mu$ at the top of the diagram.

(B6) Even if $i_{\alpha+1} \neq i_{\tilde{r}+1}$ we may now have some $i_{\alpha+a+z}$ that is equal to $i_{\alpha+1}$ for $z \in \{1, \dots, \tilde{r}-\alpha-a\}$. In this case, follow the same method as for summands of (3.7), applying Lemma 2.7 and using (B5) to obtain a row or Garnir relation at the top of the diagram.

(B7) Treat this similarly to (B5). If $\psi_{\alpha+a+z_j}$ is not a row relation then by the diagonal residue condition the node containing $\alpha + a + z_j$ in $\mathfrak{t}^\mu$ must be a Garnir node. Apply Corollary 2.6 to $\left(\Psi_{\alpha+a}\!\uparrow^{\tilde{r}}\right)\downarrow_{\alpha+3} \cdot \Psi_{\alpha+z_j+2}\!\uparrow^{\tilde{r}-a+1}$ instead, then we have $\Psi_{\alpha+a+z_j}\!\uparrow^{\tilde{r}-1}$ at the top of the diagram, i.e. we have the Garnir relation for the node containing $\alpha + a + z_j$ in $\mathfrak{t}^\mu$ at the top of the diagram.

Now suppose that $k \geq 0$, then we wish to show that we can rewrite φ as a composition of $k+1$ homomorphisms. When $k = 0$ this is trivially true, so suppose that $k > 0$. Let $\tilde{c} \in \{p+1, p+2, \dots, q-1\}$ be maximal so that a row of a nodes whose leftmost residue is ι can be added to $\lambda^{(\tilde{c})}$. Suppose that if we add the row of a nodes to $\left[\lambda^{(\tilde{c})}\right]$ we obtain the diagram $\left[\nu^{(\tilde{c})}\right]$ and consider the multipartition

$$\nu := \left(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\tilde{c}-1)}, \nu^{(\tilde{c})}, \mu^{(\tilde{c}+1)}, \mu^{(\tilde{c}+2)}, \dots, \mu^{(l)}\right).$$

Then, noting Remark 3.8, by induction we have that there is a homomorphism $\varphi_1 : S^\lambda \to S^\nu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\nu} \left(\Psi_{\gamma+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\gamma+1}$, where $\gamma+1, \gamma+2, \ldots, \gamma+a$ are the values of the added nodes in $\mathfrak{t}^\nu$. Similarly, we also obtain a homomorphism $\varphi_2 : S^\nu \to S^\mu$ given by $v^{\mathfrak{t}^\nu} \mapsto v^{\mathfrak{t}^\mu} \left(\Psi_{\alpha+a}{\uparrow}^{\gamma+a-1}\right){\downarrow}_{\alpha+1}$. By induction, φ_2 can be written as a composition of k homomorphisms.

Composing, $\varphi_2 \circ \varphi_1 : S^\lambda \to S^\mu$ is given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu} \left(\Psi_{\alpha+a}{\uparrow}^{\beta+a-1}\right){\downarrow}_{\alpha+1}$ thus $\varphi = \varphi_2 \circ \varphi_1$. Hence φ can be written as a composition of $k+1$ homomorphisms as we wanted.

Finally, we shall describe the degree of φ. By Proposition 1.34 we have that $\deg\left(v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\right) = \deg(\mathfrak{s})$. We wish to compute $\deg\left(v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\right) - \deg\left(v^{\mathfrak{t}^\lambda}\right) = \deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^\lambda\right)$. Using the recursive definition of the degree, the nodes containing $n, n-1, \ldots, \beta+a+1$ in both tableaux contribute the same value to the respective degrees. Hence

$$\deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^\lambda\right) = \deg\left(\mathfrak{s}_{<\beta+a+1}\right) - \deg\left(\mathfrak{t}^\lambda_{<\beta+a+1}\right).$$

Let A_j be the node $\mathfrak{s}^{-1}_{<\beta+j+2}(\beta+j+1)$ and B_j be the node $\left(\mathfrak{t}^\lambda_{<\beta+j+2}\right)^{-1}(\beta+j+1)$. Then for $j \in \{1, 2, \ldots, a-1\}$ we have that the number of addable $\iota+j$-nodes below A_j is equal to that below B_j plus m_j more. The number of removable $\iota+j$-nodes below A_j is equal to that below B_j, since there are no removable $\iota+j-1$-nodes in the components indexed by $p+1, p+2, \ldots, q-1$. Hence we have

$$\deg\left(\mathfrak{s}_{<\beta+a+1}\right) - \deg\left(\mathfrak{t}^\lambda_{<\beta+a+1}\right) = \sum_{j=1}^{a-1} m_j + \deg\left(\mathfrak{s}_{<\beta+2}\right) - \deg\left(\mathfrak{t}^\lambda_{<\beta+2}\right).$$

Now the number of addable ι-nodes below A_0 is m_0+1 greater than below B_0, since in $\mathfrak{s}_{<\beta+2}$ we count the m_0 addable ι-nodes within the components labelled with $p+1, p+2, \ldots, q-1$ along with the position where the node was removed from in the qth component. The number of removable ι-nodes below A_0 is the same as that below B_0. Thus

$$\begin{aligned}\deg\left(\mathfrak{s}_{<\beta+2}\right)-\deg\left(\mathfrak{t}^{\lambda}_{<\beta+2}\right)&=m_0+1+\sum_{j=1}^{a-1}m_j+\deg\left(\mathfrak{s}_{<\beta+1}\right)-\deg\left(\mathfrak{t}^{\lambda}_{<\beta+1}\right)\\&=m+1,\end{aligned}$$

since $\mathfrak{s}_{<\beta+1}$ is identical to $\mathfrak{t}^{\lambda}_{<\beta+1}$. Thus the degree of φ is $m+1$.

□

As with the one-node homomorphisms, we can now consider what happens when we move two or more different rows of nodes to form $[\mu]$ from $[\lambda]$. We can naturally extend the hypotheses of Corollary 3.4 to consider rows instead of nodes, and with this we obtain a similar corollary.

Corollary 3.10. *Let $l \geq 2$ and suppose that λ, ν_1, ν_2 and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving one row of a_1 nodes whose leftmost residue is ι_1 and one other row - not the same as or adjacent in any way to the first - whose leftmost residue is ι_2. Suppose $[\nu_1]$ is formed from $[\lambda]$ by moving just the row of a_1 nodes, whilst $[\nu_2]$ is formed from $[\lambda]$ by moving just the row of a_2 nodes. Suppose that $\iota_1+j \not\!\!- \iota_2+k$ and $\iota_1+j \neq \iota_2+k$ for all $j \in \{0,1,\ldots,a_1-1\}$ and $k \in \{0,1,\ldots,a_2-1\}$. Suppose that $(\lambda,\nu_1)_{\iota_1}$, $(\lambda,\nu_2)_{\iota_2}$, $(\nu_1,\mu)_{\iota_2}$ and $(\nu_2,\mu)_{\iota_1}$ are all one row pairs. Then there are non-zero homomorphisms*

$$\begin{aligned}\varphi_{\lambda\nu_1}&:S^{\lambda}\to S^{\nu_1}, &\varphi_{\nu_1\mu}&:S^{\nu_1}\to S^{\mu},\\ \varphi_{\lambda\nu_2}&:S^{\lambda}\to S^{\nu_2}, &\varphi_{\nu_2\mu}&:S^{\nu_2}\to S^{\mu},\end{aligned}$$

and we have that $\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}=\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}\neq 0$.

In addition, if $(\lambda,\nu_1)_{\iota_1}$ and $(\nu_1,\mu)_{\iota_2}$ have degrees $m+1$ and $m'+1$, we have that the degree of $\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}$ is $m+m'+2$.

Proof. Since $(\lambda,\nu_1)_{\iota_1}$, $(\lambda,\nu_2)_{\iota_2}$, $(\nu_1,\mu)_{\iota_2}$ and $(\nu_2,\mu)_{\iota_1}$ are all one row pairs, by Corollary 3.9 we have that there are non-zero homomorphisms

$$\begin{aligned}\varphi_{\lambda\nu_1}&:S^{\lambda}\to S^{\nu_1}, &\varphi_{\nu_1\mu}&:S^{\nu_1}\to S^{\mu},\\ \varphi_{\lambda\nu_2}&:S^{\lambda}\to S^{\nu_2}, &\varphi_{\nu_2\mu}&:S^{\nu_2}\to S^{\mu}.\end{aligned}$$

Write $\varphi_{\lambda\nu_j}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^{\nu_j}}\left(\Psi_{\alpha_j+a_j}\uparrow^{\beta_j+a_j-1}\right)\downarrow_{\alpha_j+1}$ for some $\alpha_j \in \{0,\dots,n-1\}$ and $\beta_j \in \{0,\dots,n-1\}$ with $\alpha_j \leq \beta_j$, for $j \in \{1,2\}$. Without loss of generality, assume that $\beta_1 < \beta_2$. If $\beta_1 < \alpha_2$ then it must be the case that $\beta_1 + a_1 < \alpha_2 + 1$ hence

$$\begin{aligned}
\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1}\cdot\left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1}\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1} \\
&= \varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda})
\end{aligned}$$

and we are done. Hence, assume that $\beta_1 \geq \alpha_2$. Then we have multiple cases.

Case I: The row of a_1 nodes is moved to a position above the row of a_2 nodes in $[\mu]$

In this case, we have that

$$\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_2+a_1+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+a_1+1}\cdot\left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1}$$

whilst

$$\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_1+1}\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1}, \tag{3.15}$$

which as a diagram is shown in Figure 3.9.

Write

$$\begin{aligned}
\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_1+1}\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1} \\
&= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_1+a_1}\uparrow^{\alpha_2+a_1+a_2-1}\right)\downarrow_{\alpha_1+1} \\
&\quad\cdot\left(\Psi_{\alpha_2+a_1+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+a_2+1} \\
&\quad\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+1} \\
&\quad\cdot\left(\Psi_{\beta_1+a_1+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\beta_1+a_1+1}\cdot
\end{aligned}$$

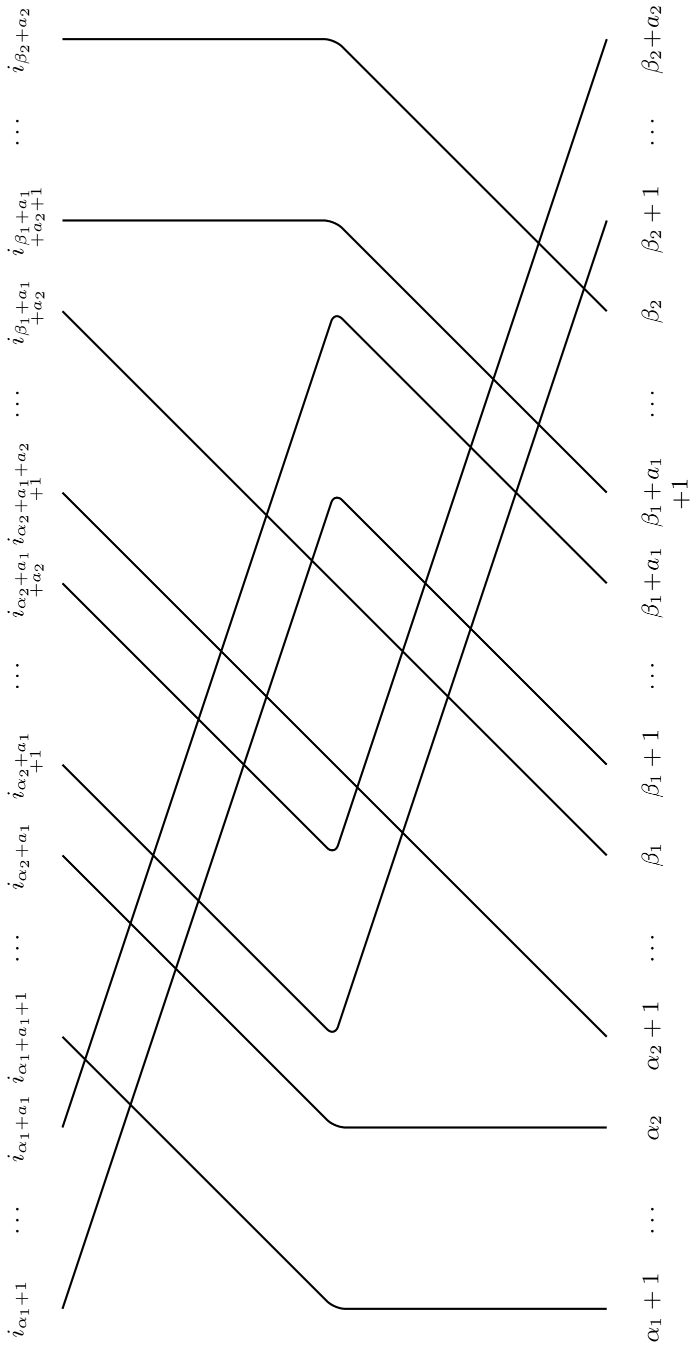


Figure 3.9: Part of the braid diagram for (3.15).

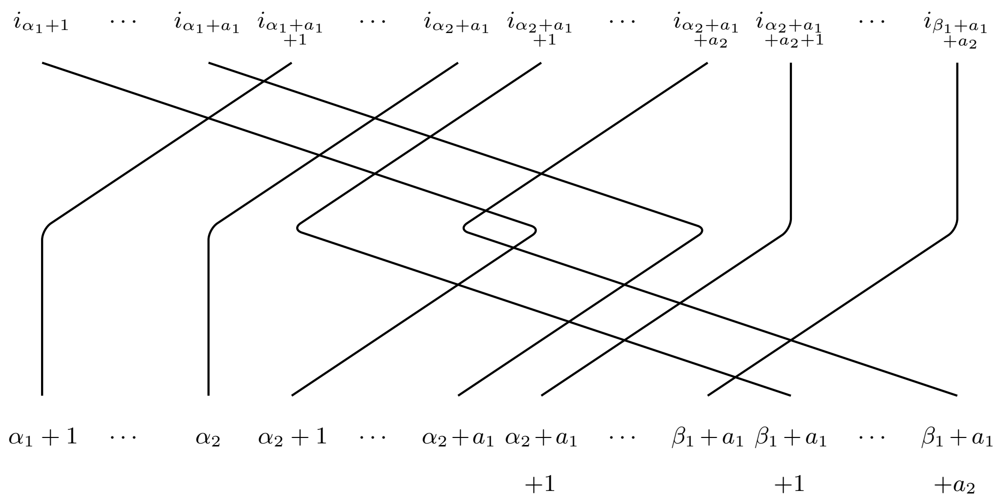
Now apply Lemma 2.5 to

$$\left(\Psi_{\alpha_2+a_1+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_1+a_2+1}\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+1}$$

since $i_{\alpha_1+j} \not\!\!- i_{\alpha_2+a_2+1},\ldots,i_{\alpha_2+a_1+a_2}$ for $j \in \{1,\ldots,a_1\}$. So then we have

$$\begin{aligned}\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu}&\left(\Psi_{\alpha_1+a_1}\uparrow^{\alpha_2+a_1+a_2-1}\right)\downarrow_{\alpha_1+1}\\ &\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+1}\\ &\cdot\left(\Psi_{\alpha_2+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_2+1}\\ &\cdot\left(\Psi_{\beta_1+a_1+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\beta_1+a_1+1}.\end{aligned}$$

We show $v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_1+1}\uparrow^{\alpha_2+a_1+a_2-1}\right)\downarrow_{\alpha_1+1}\cdot\left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+1}$ in the following diagram:



Now apply Lemma 2.8 to these crossings since $i_{\alpha_1+j} \not\!\!- i_{\alpha_2+a_2+1},\ldots,i_{\alpha_2+a_1+a_2}$

for $j \in \{1, \ldots, a_1\}$. So then we have

$$\begin{aligned}\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha_1+a_1}\uparrow^{\alpha_2+a_1-1}\right)\downarrow_{\alpha_1+1} \\ &\cdot \left(\Psi_{\alpha_2+a_1+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_2+a_1+1} \\ &\cdot \left(\Psi_{\alpha_2+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_2+1} \\ &\cdot \left(\Psi_{\beta_1+a_1+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\beta_1+a_1+1} \\ &= \left(\Psi_{\alpha_2+a_1+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+a_1+1} \cdot \left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1} \\ &= \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}).\end{aligned}$$

Thus $\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2} = \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$.

Case II: The row of a_1 nodes is moved to a position below the row of a_2 nodes in $[\mu]$

In this case, we have that

$$\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1} \cdot \left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1}.$$

Now

$$\begin{aligned}\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha_1+a_1+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_1+a_2+1} \\ &\cdot \left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1}\end{aligned} \tag{3.16}$$

which as a diagram is shown in Figure 3.10.

Now apply Corollary 2.6 to

$$\left(\Psi_{\alpha_1+a_1+a_2}\uparrow^{\beta_1+a_1+a_2-1}\right)\downarrow_{\alpha_1+a_2+1} \cdot \left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1}$$

since $i_{\alpha_2+j} \not\leftarrow i_{\alpha_1+a_2+1}, \ldots, i_{\alpha_1+a_1+a_2}$ for $j \in \{1, \ldots, a_2\}$, so that we have

$$\begin{aligned}\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) &= v^{\mathfrak{t}^\mu} \left(\Psi_{\alpha_2+a_2}\uparrow^{\beta_2+a_2-1}\right)\downarrow_{\alpha_2+1} \cdot \left(\Psi_{\alpha_1+a_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+1} \\ &= \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}).\end{aligned}$$

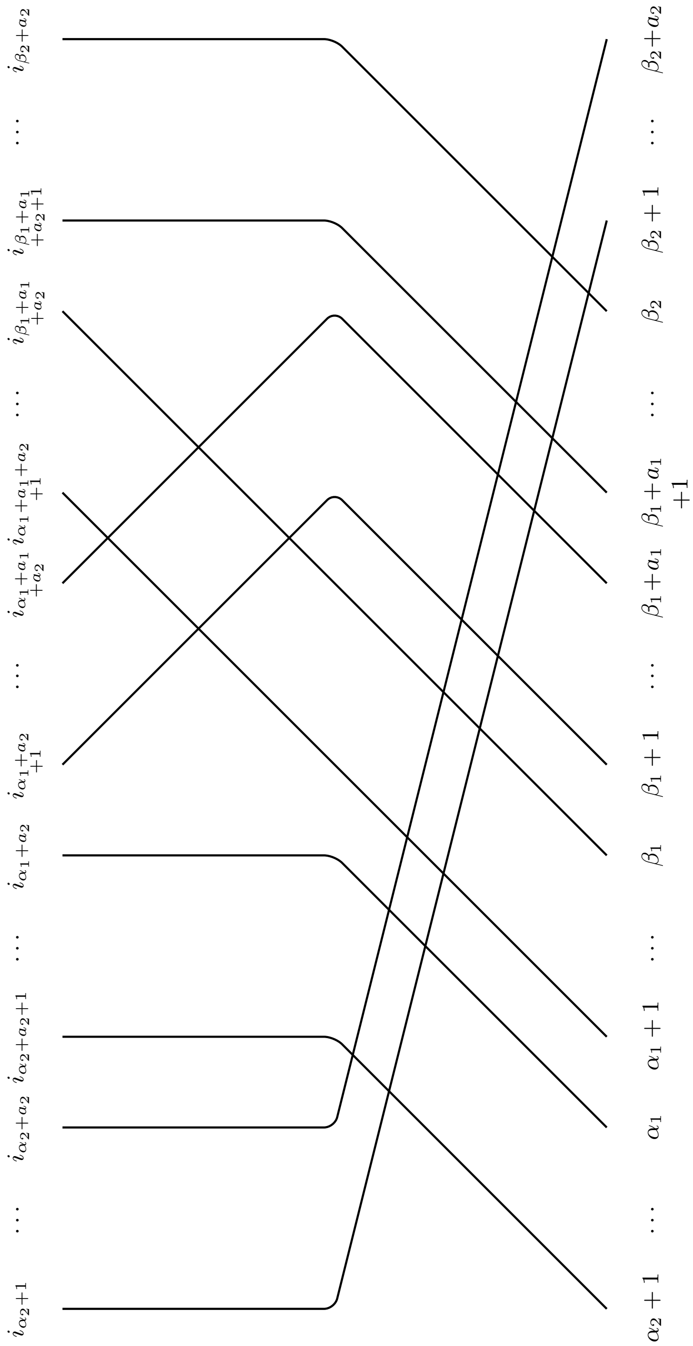


Figure 3.10: Part of the braid diagram for (3.16).

Thus $\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2} = \varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$.

Since the degree of $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$ is equal to the sum of the degrees, the degree is $(m+1) + (m'+1) = m + m' + 2$.

Note that in both cases, $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda})$ is given by some product of ψ_i corresponding to a reduced expression (using Proposition 1.18 as no strings cross twice) which is not zero, thus $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1} = \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2} \neq 0$.

□

We also have an analogue to Corollary 3.5, that is, that if $[\mu]$ is formed from $[\lambda]$ by moving multiple rows of nodes whose residues are sufficiently spread apart, then we can move the rows in any order to get various homomorphisms which always compose to give the same overall homomorphism.

Corollary 3.11. *Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving m distinct rows of nodes $R_1, \ldots, R_m$, whose leftmost residues are $\iota_1, \iota_2, \ldots, \iota_m$ and whose residues amongst the rows are such that none are equal or adjacent between any two given rows.*

Suppose that for each $X \subseteq \{1, \ldots, m\}$ we have an l-multipartition of n, ν_X, such that $[\nu_{\{i_1,\ldots,i_t\}}]$ is formed from $[\lambda]$ by moving just the rows $R_{i_1}, \ldots, R_{i_t}$. In particular $\nu_\varnothing = \lambda$ and $\nu_{\{1,\ldots,m\}} = \mu$. Suppose that whenever $B \setminus A = \{r\}$, we have that $(\nu_A, \nu_B)_{\iota_r}$ is a one row pair, whose corresponding homomorphism is $\varphi_{\nu_A\nu_B}$.

Then there is a non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ and given any sequence of sets $\varnothing = X_0 \subsetneq X_1 \subsetneq X_2, \subsetneq \cdots \subsetneq X_m = \{1, \ldots, m\}$ we have that

$$\varphi = \varphi_{\nu_{X_{m-1}}\nu_{X_m}} \circ \varphi_{\nu_{X_{m-2}}\nu_{X_{m-1}}} \circ \cdots \circ \varphi_{\nu_{X_0}\nu_{X_1}}.$$

Proof. Without loss of generality suppose that the row R_a is above R_b whenever $a < b$. Let $Y_j \coloneqq \{1, 2, \ldots, j\}$ for $j \in \{0, \ldots, m\}$. Then

$$\varnothing = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m = \{1, 2, \ldots, m\}.$$

By assumption we have l-multipartitions of n, ν_{Y_j}, and non-zero homomorphisms $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}$ for each $j \in \{0, \ldots, m-1\}$. Suppose that row R_t has length a_t. We may write $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}(v^{\mathfrak{t}^{\nu_{Y_j}}}) = v^{\mathfrak{t}^{\nu_{Y_{j+1}}}} \left(\Psi_{\alpha_{j+1}+a_{j+1}+\zeta_{j+1}}\uparrow^{\beta_{j+1}+a_{j+1}-1}\right)\downarrow_{\alpha_{j+1}+\zeta_{j+1}+1}$ for some α_{j+1} and β_{j+1} related to the positions of the moved nodes, and ζ_{j+1} based on whether moved nodes are added above or below other moved nodes. Then

$$\varphi_{\nu_{Y_{m-1}}\nu_{Y_m}} \circ \cdots \circ \varphi_{\nu_{Y_0}\nu_{Y_1}}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^\mu} \left(\Psi_{\alpha_m+a_m+\zeta_m}\uparrow^{\beta_m+a_m-1}\right)\downarrow_{\alpha_m+\zeta_m+1} \cdots \cdot \left(\Psi_{\alpha_1+a_1+\zeta_1}\uparrow^{\beta_1+a_1-1}\right)\downarrow_{\alpha_1+\zeta_1+1} .$$

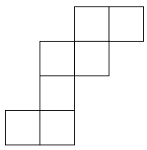
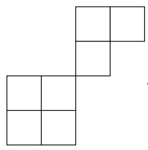
Since $\beta_{j+1}+1 > \beta_j + a_j$ for $j \in \{1,\ldots,m-1\}$ we must have in the braid diagram for the above, no strings will cross twice and so by Proposition 1.18 the above will correspond to a reduced expression, and the associated tableau will be standard, so this composition of homomorphisms is not zero.

The rest of the proof is the same as that for Corollary 3.5, replacing the use of Corollary 3.4 with Corollary 3.10. □

3.3 Skew homomorphisms

Now that we have shown the existence of homomorphisms between Specht modules arising from moving rows of nodes in a multipartition, our final step is to take this yet further and consider moving some arbitrary connected shape of nodes. To be precise, we say that a diagram is *connected* if any two nodes in it are connected by a path going through edges which connect two nodes in the diagram. Given multipartitions ν and ρ, if the diagram $[\nu]$ contains the diagram $[\rho]$, then the *skew diagram* $[\nu \setminus \rho]$ is the set-theoretic difference of $[\nu]$ and $[\rho]$. We define a *skew shape* to be a connected skew diagram of the form $[\nu \setminus \rho]$.

Example 3.12. Given $\nu = (4, 3, 2^2)$, $\rho_1 = (2, 1^2)$ and $\rho_2 = (2^2)$, we see that $[\nu \setminus \rho_1]$ is a skew shape whilst $[\nu \setminus \rho_2]$ is not.

$\nu \setminus \rho_1 =$  whilst $\nu \setminus \rho_2 =$  .

$\diamondsuit$

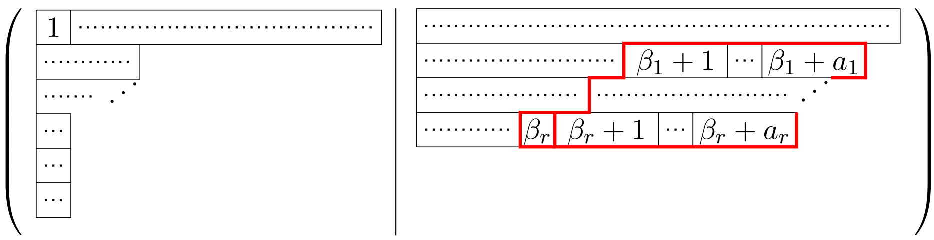
We want to consider forming $[\mu]$ from $[\lambda]$ by moving a skew shape from one component to another. In order to prove an explicit homomorphism $\varphi : S^\lambda \to S^\mu$ exists, we are able to take a similar approach to Proposition 3.6: removing the 'last' node of the skew shape and using induction to obtain different homomorphisms and relations, which we can use to help check that many of the generating relations for S^λ hold on the image of $v^{\mathfrak{t}^\lambda}$.

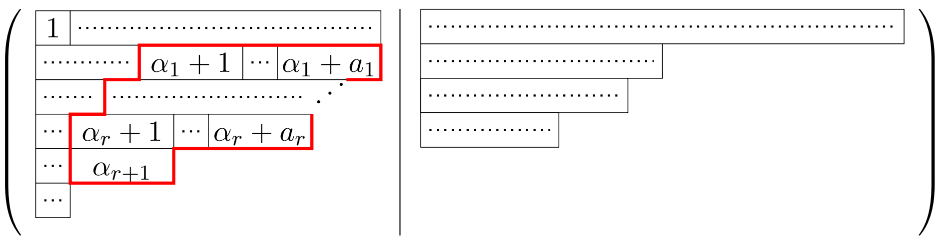
The following lemma will at first seem rather technical and out of place. It is concerned with showing that in a specific setup related to moving a skew shape to form μ from λ, there will be no standard μ-tableaux with the same residue sequence as that of $\mathfrak{t}^\lambda$. This will turn out to be useful during the proof of Theorem 3.14 when showing that a certain term is zero. The use of this lemma is out of place with the rest of our working, but will remove the need to do yet more braid diagram combinatorics. In particular it will ensure that we will not need to deal with crossings within the product R that we describe in the proof of

Theorem 3.14.

Lemma 3.13. *Let λ be an l-multipartition of n that contains a skew shape of r rows in the lth component. Label the rows in the skew shape from top to bottom by 1 to r and let the nodes in the jth row of the skew shape in $\mathfrak{t}^\lambda$ contain the entries $\beta_j + 1, \ldots, \beta_j + a_j$. Suppose that the node containing β_r is directly to the left of that containing $\beta_r + 1$. Suppose that we can form an l-multipartition μ of n by considering λ and moving the skew shape - keeping its shape intact - to the first component, as well as the node containing β_r in $\mathfrak{t}^\lambda$ to the position directly below that where the node containing $\beta_r + 1$ in $\mathfrak{t}^\lambda$ is moved. Suppose that both λ and μ satisfy the diagonal residue condition and that amongst the components $\lambda^{(2)}, \ldots, \lambda^{(l-1)}$ there are no removable nodes of residues of any of the residues in the skew shapes. Then there are no standard μ-tableaux whose residue resequence is the same as that of $\mathfrak{t}^\lambda$.*

Proof. Let the nodes in the jth row of the skew shape in $\mathfrak{t}^\mu$ contain the values $\alpha_j + 1, \ldots, \alpha_j + a_j$ from left to right. Let the entry of the node beneath the skew shape be $\alpha_r + 1$. Then in the bipartition case, we have the following tableaux (for $l \geq 2$ there are simply some other components between these):

$\mathfrak{t}^\lambda =$ 

$\mathfrak{t}^\mu =$ 

For $j \in \{1, \ldots, r\}$ let m_j be the number of nodes in the jth row of the skew shape which have no node directly above them in the skew shape. Now we shall try to create a standard μ-tableau, $\mathfrak{s}$, with the same residue sequence as $\mathfrak{t}^\lambda$. Write

$\mathbf{i}^\lambda = (i_1, \ldots, i_n)$, and $\mathbf{i}^\mu = (j_1, \ldots, j_n)$. Given $s \in \{1, \ldots, n\}$, let $N[s]$ denote the node in $[\mu]$ that contains s in $\mathfrak{t}^\mu$. Firstly, consider which node β_r will fill. For $\mathfrak{s}$ to be standard we must have that the node containing β_r has no node directly beneath it, since there are no nodes in $\mathfrak{t}^\lambda$ whose entries are greater than β_r but of residue $i_{\beta_r} - 1$. If such a node exists within the components labelled $2, \ldots l-1$, then there must be at least $a_r + 1$ nodes to the right of it in that row, otherwise we have a removable node whose residue exists in the skew shape, but then these nodes cannot all be filled whilst keeping $\mathfrak{s}$ standard. Thus the only node satisfying this description is $N[\alpha_{r+1}]$.

Let $k \in \{0, 1, \ldots, r-2\}$. Suppose in addition to filling in a node with β_r we have filled in nodes with the entries $\beta_c + 1, \ldots, \beta_c + a_c$ for $c \in \{r, r-1, \ldots, r-k+1\}$. It will be clear where these entries go after reading the below, but in effect if an entry was in the node in position (r, c, l) in $[\lambda]$ then it will be put in the node in position $(r-1, c-1, l)$ if this node exists, otherwise it will be put in the first component in the node in the skew shape where it originated from. Then consider $\beta_{r-k} + 1$. The only nodes of residue $i_{\beta_{r-k}+1}$ in $\mu^{(1)}$ have nodes directly beneath them, which then would not be able to be filled in with any value greater than $\beta_{r-k} + 1$. Amongst those components labelled $2, \ldots, l-1$, we may have a node of this residue with no node directly beneath it, but then there must be at least a_{r-k} nodes to the right of it in that row; otherwise we have a removable node whose residue exists in the skew shape, but these cannot all be filled whilst keeping $\mathfrak{s}$ standard. So then in the last component there is then only one suitable node, namely $N[\beta_{r-(k+1)} - m_{r-k}]$. Then the nodes $N[\beta_{r-(k+1)} - m_{r-k} + 1], \ldots, N[\beta_{r-(k+1)}]$ must be filled with the values $\beta_{r-k} + 2, \ldots, \beta_{r-k} + m_{r-k} + 1$ respectively. Note that these nodes lie directly above and one to the left of where the nodes with values $\beta_{r-k} + 1, \ldots, \beta_{r-k} + m_{r-k} + 1$ were in $\mathfrak{t}^\lambda$. Now if $a_{r-k} > m_{r-k} + 1$, then we must fill in some nodes with $\beta_{r-k} + m_{r-k} + 2, \ldots, \beta_{r-k} + a_{r-k}$. Now the only nodes suitable to be filled with $\beta_{r-k} + m_{r-k} + 2$ lie in $\mu^{(1)}$, since one such node in the last component is that above the node we just filled with $\beta_{r-k} + m_{r-k} + 1$, and the rest all have an empty node directly beneath them, or are in the middle component and must have at least $a_{r-k} + m_{r-k} - 1$ nodes to the right of them in that row in order to prevent the existence of a removable node whose residue exists in the skew shape, and then these nodes cannot all be filled whilst keeping $\mathfrak{s}$ standard. There is then only one suitable node in the first component which does not have an empty node directly beneath it, $N[\alpha_{r-k} + m_{r-k} + 2]$, so fill this with $\beta_{r-k}+m_{r-k}+2$. Then we must fill in the nodes $N[\alpha_{r-k}+m_{r-k}+3], \dots, N[\alpha_r+a_r]$ with $\beta_{r-k} + m_{r-k} + 3, \dots \beta_{r-k} + a_{r-k}$ respectively. Note that these nodes are just the 'moved versions' of the nodes that contained these values in $\mathfrak{t}^\lambda$, i.e. they are the nodes in the skew shape added to the first component that would have had these values in $\mathfrak{t}^\lambda$.

Now we have filled in nodes with β_r and entries $\beta_c + 1, \dots, \beta_c + a_c$ for $c \in \{r, r-1, \dots, 2\}$. To begin with, we can follow the same idea as above. Consider $\beta_1 + 1$. As before the only nodes of residue i_{β_1+1} in the first component have nodes directly beneath them, and any in the components labelled with $2, \dots, l-1$ must have at least a_1 nodes to the right of them in that row otherwise we have a removable node whose residue exists in the skew shape, and then these nodes cannot all be filled in whilst keeping $\mathfrak{s}$ standard. So the only suitable nodes will be in the last component. Suppose that the row containing $\beta_1 + 1$ in $\mathfrak{t}^\lambda$ is the top row of λ. Then there will be no suitable nodes in the last component either, and we can conclude that there is no standard μ-tableau of the same residue sequence as $\mathfrak{t}^\lambda$. So instead suppose that the row containing $\beta_1 + 1$ in $\mathfrak{t}^\lambda$ is not the top row of λ. Let the residue of the node in $[\mu]$ lying directly above that which contained $\beta_1 + a_1$ in $\mathfrak{t}^\lambda$ be j_{β_0}. Then there is now a suitable node in the last component, i.e. $N[\beta_0 - a_1]$. Then the nodes $N[\beta_0 - a_1 + 1], \dots, N[\beta_0 - 1]$ must be filled with the values $\beta_1 + 2, \dots, \beta_1 + a_1$ respectively. But now there is no value with which the node $N[\beta_0]$ can take, since there is no value x such that $x > \beta_1 + a_1$ and $i_x = i_{\beta_1+a_1} + 1$. Thus we can conclude that there is no standard μ-tableau of the same residue sequence as $\mathfrak{t}^\lambda$ in this case either.

□

Now we can state the main theorem of this chapter.

Theorem 3.14. *Let λ and μ be 2-multipartitions of n. Suppose*

$$e \geq \max\{h_{11}^{\lambda^{(1)}} + 1, h_{11}^{\lambda^{(2)}} + 1, h_{11}^{\mu^{(1)}} + 1, h_{11}^{\mu^{(2)}} + 1\}$$

and that $[\mu]$ is formed from $[\lambda]$ by moving a skew shape from the second component to the first, without changing their shape. Let $\mathfrak{s}$ be the μ-tableau defined by considering $\mathfrak{t}^\lambda$ and moving the skew shape from the second component to the first, keeping their tableau values intact. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

We shall discuss the strategy for the proof of Theorem 3.14 in much the same way as for Proposition 3.6. Firstly, note that if the skew shape moved is just a row of nodes then by Proposition 3.6 we have the desired result. So we may assume that the skew shape has nodes in at least two rows of $\left[\lambda^{(2)}\right]$.

As in Proposition 3.1 we have the diagonal residue condition. Consider the bottom two rows of the skew shape in $\left[\lambda^{(2)}\right]$; suppose that there are $a \geq 1$ nodes in the bottom such row, with $q \geq 0$ nodes to the left of this which are not removed. Suppose that in the higher of the two rows, there are $a_2 \geq 1$ nodes, $b \geq 0$ of which are removable. Let the entry of the node on the end of this row be β. Then the two rows look like so:

$$\begin{array}{cc|c|c|c|c|c|} \cline{3-8} & & \beta - a_2 + 1 & \cdots\cdots & \beta - b & \beta - b + 1 & \cdots\cdots & \beta \\ \hline \multicolumn{1}{|c|}{\beta + q + 1} & \multicolumn{3}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots} & \beta + q + a \\ \cline{1-5} \end{array} \tag{3.17}$$

Now consider adding the skew shape to $\left[\mu^{(1)}\right]$; suppose that in $\mathfrak{t}^\mu$ there are $p \geq 0$ nodes to the left of where the bottom such row is added, and that the entry of the node on the end of the row above this is α. The following diagram shows the two rows of $\mathfrak{t}^\mu$ to which the bottom two rows of the skew shape have been added, with the nodes that have been added highlighted.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \multicolumn{3}{|c|}{\cdots\cdots\cdots\cdots\cdots} & \alpha - a_2 & \alpha - a_2 + 1 & \cdots & \alpha - b & \alpha - b + 1 & \cdots & \alpha \\ \hline \alpha + 1 & \cdots & \alpha + p & \alpha + p + 1 & \multicolumn{2}{c|}{\cdots\cdots\cdots\cdots} & \alpha + p + a \\ \cline{1-7} \end{array}$$

Then $\psi^{\mathfrak{s}} = \left(\Psi_{\alpha+p+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+1} \cdot \left(\Psi_{\alpha}\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R$, where R is a product of crossings coming from the rows higher than the bottom two in the skew shape. Note that some or both of the brackets in the product may be zero depending on the values of α, β, p and q.

As in Proposition 3.6, we now wish to work with some 3-multipartitions of n. So we define a new KLR algebra $\mathscr{H}_n^{\Lambda_{\tilde{\kappa}}}$ just as before.

Suppose that the row of $[\lambda]$ which the bottom row of the skew shape extends by being added to it is the k_1th row and the row which this bottom row shortens by being removed from it is the k_2th row. Then consider a 3-multipartition of n, λ, defined as

$$\tilde{\lambda} := \left(\lambda^{(1)}, \lambda^{(2)}_{\hat{k}_2}, (1)\right),$$

i.e. so that $\mathfrak{t}^{\tilde{\lambda}}$ is formed from $\mathfrak{t}^{\lambda}$ by removing the node containing $\beta + q + a$, subtracting one from the entry of all the nodes containing $\beta + q + a + 1, \beta + q + a + 2, \ldots, n$, and placing one node in the third component which will have label n.

Also define $\tilde{\nu}$, a 3-multipartition of n, by

$$\tilde{\nu} := \left(\mu^{(1)}_{\hat{k}_1}, \mu^{(2)}, (1)\right),$$

with a $\tilde{\nu}$-tableau $\mathfrak{s}_1$ defined by considering $\mathfrak{s}$ and removing the node containing $\beta + q + a$, subtracting one from the entry of all the nodes containing $\beta + q + a + 1$, $\beta + q + a + 2, \ldots, n$, and placing one node in the third component which will have label n. Then $\tilde{\psi}^{\mathfrak{s}_1} = \left(\tilde{\Psi}_{\alpha+p+a-1}\uparrow^{\beta+q+a-2}\right)\downarrow_{\alpha+p+1} \cdot \left(\tilde{\Psi}_{\alpha}\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot \tilde{R}$, where $\tilde{R}$ is just R with every ψ replaced by a $\tilde{\psi}$. Using induction on the number of nodes moved in a similar way to the strategy for Proposition 3.6, we can assume that there exists a non-zero homomorphism $\varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\nu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}$, and no generating relation for $S^{\tilde{\lambda}}$ kills $v^{\mathfrak{t}^{\tilde{\nu}}} \tilde{\psi}^{\mathfrak{s}_1}$ via a relation of the form (1.12). The base case for this is given by Proposition 3.1.

Now consider the 3-multipartition $\tilde{\mu} := \left(\mu^{(1)}, \mu^{(2)}, \varnothing\right)$ and the $\tilde{\mu}$-tableau $\mathfrak{s}_2$ defined by considering $\mathfrak{t}^{\tilde{\mu}}$ and changing the entry of the node containing $\alpha + p + a$ to n whilst subtracting one from the entry of every node containing $\alpha + p + a + 1$ or greater. We have $\tilde{\psi}^{\mathfrak{s}_2} = \tilde{\Psi}_{\alpha+p+a}\uparrow^{n-1}$. Using Corollary 3.3 we know that there is a non-zero homomorphism $\varphi_2 : S^{\tilde{\nu}} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\nu}}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}_2}$.

Let $\tilde{\psi}^{\mathfrak{s}} = \left(\tilde{\Psi}_{\alpha+p+a}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+1} \cdot \left(\tilde{\Psi}_{\alpha}\!\uparrow^{\beta-1}\right)\!\downarrow_{\alpha-a_2+1} \cdot \tilde{R}$. Composing φ_2 with φ_1 we have a homomorphism $\tilde{\varphi} := \varphi_2 \circ \varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\mu}}$ given by

$$\begin{aligned}
\tilde{\varphi}\left(v^{\mathfrak{t}^{\tilde{\lambda}}}\right) &= \varphi_2\left(\varphi_1\left(v^{\mathfrak{t}^{\tilde{\lambda}}}\right)\right) \\
&= \varphi_2\left(v^{\tilde{\nu}} \tilde{\psi}^{\mathfrak{s}_1}\right) \\
&= \varphi_2\left(v^{\tilde{\nu}}\right) \tilde{\psi}^{\mathfrak{s}_1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}_2} \tilde{\psi}^{\mathfrak{s}_1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\Psi}_{\alpha+p+a}\!\uparrow^{n-1} \cdot \left(\tilde{\Psi}_{\alpha+p+a-1}\!\uparrow^{\beta+q+a-2}\right)\!\downarrow_{\alpha+p+1} \\
&\qquad\qquad\qquad \cdot \left(\tilde{\Psi}_{\alpha}\!\uparrow^{\beta-1}\right)\!\downarrow_{\alpha-a_2+1} \cdot \tilde{R} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \left(\tilde{\Psi}_{\alpha+p+a}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+1} \cdot \left(\tilde{\Psi}_{\alpha}\!\uparrow^{\beta-1}\right)\!\downarrow_{\alpha-a_2+1} \cdot \tilde{R} \cdot \tilde{\Psi}_{\beta+q+a}\!\uparrow^{n-1} \\
&= v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}} \cdot \tilde{\Psi}_{\beta+q+a}\!\uparrow^{n-1}.
\end{aligned}$$

Just as in the strategy for Proposition 3.6, the residue sequence of $\mathfrak{s}$ is identical to that of $\mathfrak{t}^{\lambda}$ and so to prove the existence of $\varphi : S^{\lambda} \to S^{\mu}$ we must show that $\varphi\left(v^{\mathfrak{t}^{\lambda}}\right) a = 0$ whenever $v^{\mathfrak{t}^{\lambda}} a = 0$ for $a \in \mathscr{H}_n^{\Lambda_\kappa}$. In particular, we must check that the generating relations of S^{λ} hold on the image of $v^{\mathfrak{t}^{\lambda}}$, and so we are required to check (i), (ii) and (iii) just as in Proposition 3.6. Since $\tilde{\varphi}$ exists we also have identical-looking facts (i*), (ii*) and (iii*), just with $\beta + a$ replaced with with $\beta + q + a$.

For the same reasoning as in Proposition 3.6, any relation which kills $v^{\mathfrak{t}^{\tilde{\mu}}} \tilde{\psi}^{\mathfrak{s}}$ will also kill $v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}}$. Thus our strategy will once again be to use the relations (i*), (ii*) and (iii*) in order to deduce many of the relations given by (i), (ii) and (iii), leaving a few additional cases.

Note that the following proof has notes in the margin of the form (C•). These can be ignored for now and will become relevant when considering the proof of Corollary 3.18.

Proof. As we have remarked above, we may suppose that the skew shape moved has nodes in at least two rows of $\left[\lambda^{(2)}\right]$. Let $(i_1, i_2, \ldots, i_n)$ be the residue sequence of $\mathfrak{t}^{\tilde{\mu}}$ (which is identical to that of $\mathfrak{t}^{\mu}$), then

$$v^{\mathfrak{t}^{\tilde{\mu}}}\left(\tilde{\Psi}_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+1}\cdot\left(\tilde{\Psi}_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot\tilde{\Psi}_{\beta+q+a}{\uparrow}^{n-1}$$

is shown diagrammatically in Figure 3.11.

3.3.1 Relations in (i).

(C1)

Every relation here is checked identically to that in Proposition 3.1, only we replace α with $\alpha+p$ and β with $\beta+q$.

3.3.2 Relations in (ii).

(C1)

We can show that $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\psi}_r=0$ for $r\in\{1,\dots,\beta+q+a-2\}\cup\{\beta+q+a+1,\dots,n-1\}$ in an identical matter to Proposition 3.1, only we replace α with $\alpha+p$ and β with $\beta+q$. All that is left to check is when $r=\beta+q+a-1$. However, in this instance there is not a corresponding row relation in $S^{\tilde{\lambda}}$, so we check that $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\psi_{\beta+q+a-1}$ is equal to zero directly.

First, suppose that $a=1$. If $q=0$, then ψ_β is not a row relation so there is nothing to check. So we must suppose that $q>0$ and we have $r=\beta+q$. Then we have

$$\begin{aligned}
v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\psi_{\beta+q}&=v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+1}{\uparrow}^{\beta+q}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot R\cdot\psi_{\beta+q}\\
&=v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+1}{\uparrow}^{\beta+q}\cdot\psi_{\beta+q}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot R\\
&=v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+1}{\uparrow}^{\beta+q-1}\cdot(y_{\beta+q+1}-y_{\beta+q})\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot R
\end{aligned}$$

by relation (1.10),

$$=-v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+1}{\uparrow}^{\beta+q-1}\cdot y_{\beta+q}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot R$$

since $v^{\mathfrak{t}^{\mu}}y_{\beta+q+1}$ equals zero,

$$=-v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+1}{\uparrow}^{\beta}\,y_{\beta+1}\cdot\Psi_{\beta+1}{\uparrow}^{\beta+q-1}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1}\cdot R$$

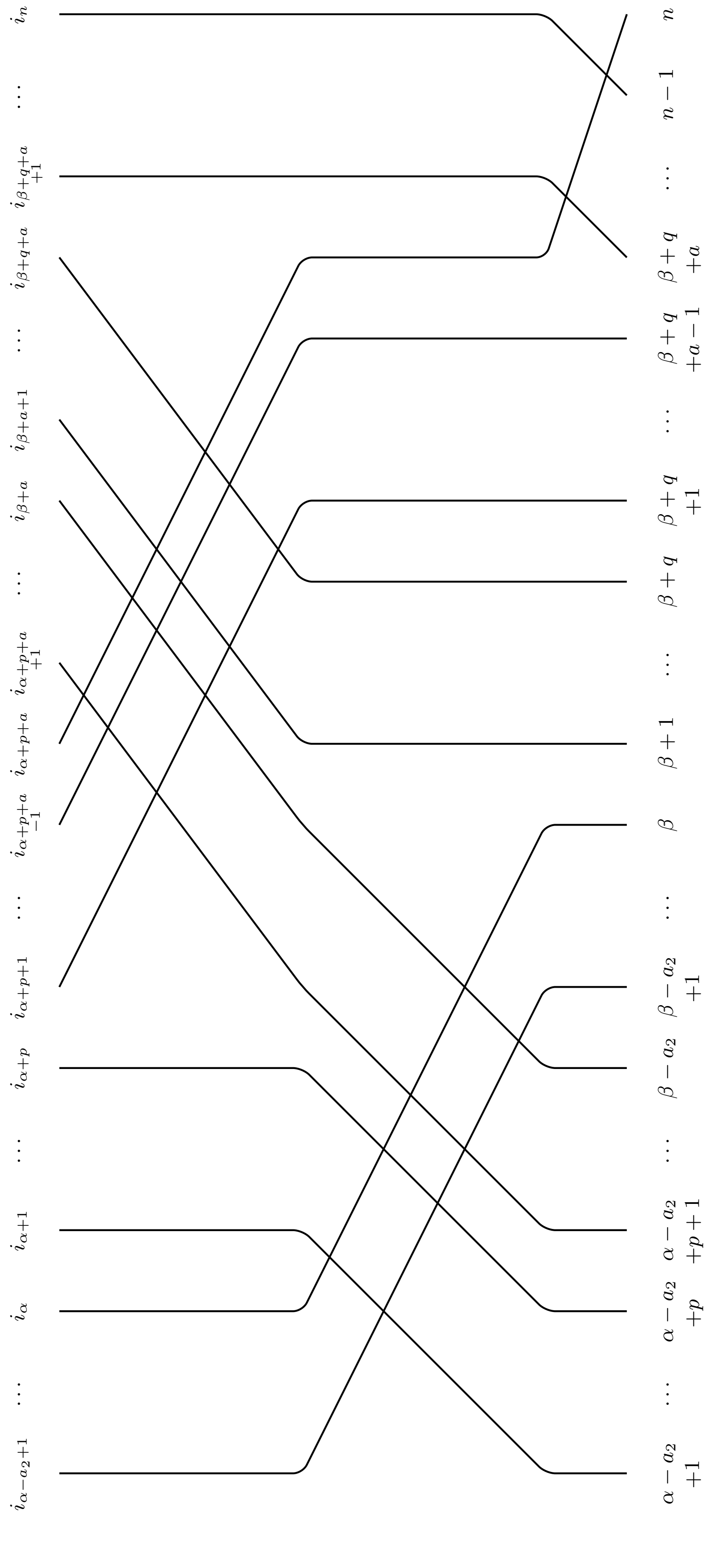


Figure 3.11: Part of the braid diagram for $v^{\mathbf{t}^{\tilde{\mu}}}\left(\tilde{\Psi}_{\alpha+p+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+1}\cdot\left(\tilde{\Psi}_{\alpha}\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1}\cdot\tilde{\Psi}_{\beta+q+a}\uparrow^{n-1}$.

since $i_{\alpha+p+1} \not\leftarrow i_{\beta+q}, i_{\beta+q-1}, \ldots, i_{\beta+2}$. Now $i_{\alpha+p+1} = i_{\beta+1}$, so using relation (1.8) we replace $\psi_\beta y_{\beta+1}$ with $y_\beta \psi_\beta + 1$. So we have

$$v^{\mathfrak{t}^\mu} \psi^{\mathfrak{s}} \psi_{\beta+q} = -v^{\mathfrak{t}^\mu} \Psi_{\alpha+p+1}\uparrow^{\beta-1} y_\beta \cdot \Psi_\beta \uparrow^{\beta+q-1} \cdot \left(\Psi_\alpha \uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R \tag{3.18}$$

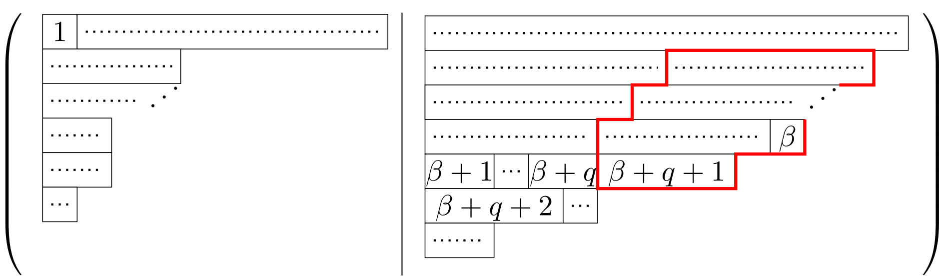
$$- v^{\mathfrak{t}^\mu} \Psi_{\alpha+p+1}\uparrow^{\beta-1} \cdot \Psi_{\beta+1} \uparrow^{\beta+q-1} \cdot \left(\Psi_\alpha \uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R. \tag{3.19}$$

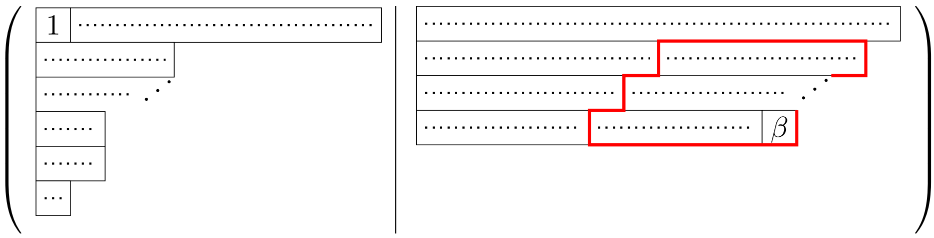
In (3.18), apply Lemma 2.11 to $\Psi_{\alpha+p+1}\uparrow^{\beta-1} y_\beta$. Then (3.18) is equal to

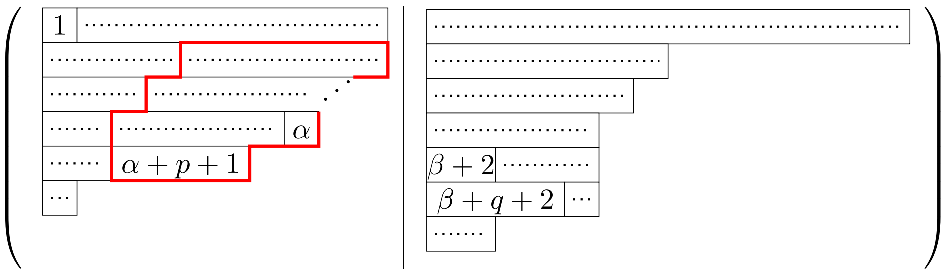
$$- v^{\mathfrak{t}^\mu} y_{\alpha+p+1} \Psi_{\alpha+p+1}\uparrow^{\beta-1} \cdot \Psi_\beta \uparrow^{\beta+q-1} \cdot \left(\Psi_\alpha \uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R$$
$$- \sum_{j=1}^{k} v^{\mathfrak{t}^\mu} \Psi_{\alpha+p+1}\uparrow^{\alpha+p+z_j-2} \cdot \Psi_{\alpha+p+z_j} \uparrow^{\beta-1} \cdot \Psi_\beta \uparrow^{\beta+q-1} \cdot \left(\Psi_\alpha \uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R$$

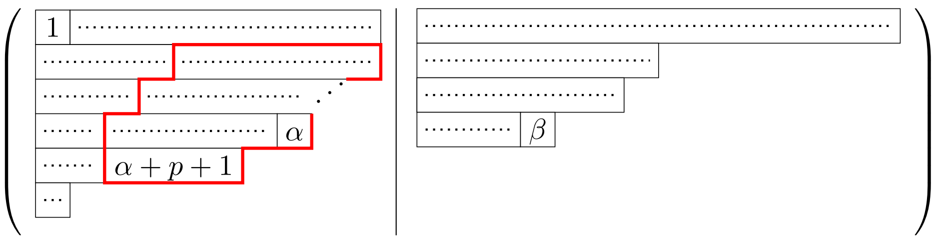
for some $k \geq 0$ and where the z_j are such that $i_{\alpha+p+1} = i_{\alpha+p+z_j}$. Then all of this is zero, since $\psi_{\alpha+p+z_j}$ is a row relation by the diagonal residue condition. (C2)

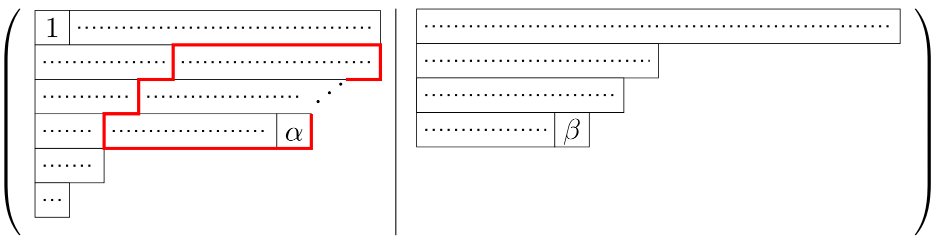
We show that (3.19) is zero in a different way. Consider the multipartitions (C3) $\bar{\lambda}$ and $\bar{\mu}$ defined by considering λ and μ respectively, and removing the nodes containing $\beta+1, \beta+2, \ldots, n$. Now define $\bar{\nu}$ by considering $\bar{\lambda}$ and moving the remaining rows of the skew shape from the second component to the first, as they were moved from λ to form μ. Let $\mathfrak{t}_1$ be the ν-tableau defined by considering $\mathfrak{t}^{\bar{\lambda}}$ and moving the skew shape from the second component to the first, keeping their tableau values intact. Let $\mathfrak{t}_2$ be the μ-tableau defined by considering $\mathfrak{t}^{\bar{\nu}}$ and moving the node containing β from the second component to the first (to the only possible position based on its residue), keeping its value intact. The following pictures help exhibit some of these tableaux.

$\mathfrak{t}^\lambda =$ 

$\mathfrak{t}^{\bar{\lambda}} =$ 

$\mathfrak{t}^{\mu} =$ 

$\mathfrak{t}^{\bar{\mu}} =$ 

$\mathfrak{t}^{\bar{\nu}} =$ 

Now by induction on the number of nodes moved, we know that there is a non-zero homomorphism $\varphi_1 : S^{\bar{\lambda}} \to S^{\bar{\nu}}$ given by $v^{\mathfrak{t}^{\bar{\lambda}}} \mapsto v^{\mathfrak{t}^{\bar{\lambda}}} \left(\Psi_\alpha \uparrow^{\beta-1}\right) \downarrow_{\alpha-a_2+1} \cdot R$. We also have that there is a non-zero homomorphism $\varphi_2 : S^{\bar{\nu}} \to S^{\bar{\mu}}$ given by $v^{\mathfrak{t}^{\bar{\nu}}} \mapsto v^{\mathfrak{t}^{\bar{\mu}}} \Psi_{\alpha+p+1} \uparrow^{\beta-1}$. So composing, we know that there is a homomorphism $\varphi_2 \circ \varphi_1 : S^{\bar{\lambda}} \to S^{\bar{\mu}}$ given by $v^{\mathfrak{t}^{\bar{\lambda}}} \mapsto v^{\mathfrak{t}^{\bar{\mu}}} \Psi_{\alpha+p+1} \uparrow^{\beta-1} \cdot \left(\Psi_\alpha \uparrow^{\beta-1}\right) \downarrow_{\alpha-a_2+1} \cdot R$. However, we may apply Lemma 3.13 to $\bar{\lambda}$ and $\bar{\mu}$ and thus there cannot be a standard $\bar{\mu}$-tableaux of the same residue sequence as $\mathfrak{t}^{\bar{\lambda}}$, meaning that the homomorphism $\varphi_2 \circ \varphi_1$ is zero, hence $v^{\mathfrak{t}^{\bar{\mu}}} \Psi_{\alpha+p+1}{\uparrow}^{\beta-1} \cdot \left(\Psi_\alpha{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1} \cdot R$ is zero. Note that any generating relation for $S^{\bar{\mu}}$ corresponds to a generating relation for S^μ. Thus this means that $v^{\mathfrak{t}^\mu} \Psi_{\alpha+p+1}{\uparrow}^{\beta-1} \cdot \left(\Psi_\alpha{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1} \cdot R$ is zero, i.e. that since $\Psi_{\beta+1}{\uparrow}^{\beta+q-1}$ commutes with R, that (3.19) is zero.

Now suppose that $a \geq 2$. We have

$$\begin{aligned} v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_{\beta+q+a-1} &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+1}\left(\Psi_\alpha{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1} \\ &\qquad \cdot R \cdot \psi_{\beta+q+a-1} \\ &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a-1} \cdot \psi_{\beta+q+a-1} \\ &\qquad \cdot \left(\Psi_{\alpha+p+a-2}{\uparrow}^{\beta+q+a-3}\right){\downarrow}_{\alpha+p+1} \cdot R_2. \end{aligned}$$

where $R_2 = \left(\Psi_\alpha{\uparrow}^{\beta-1}\right){\downarrow}_{\alpha-a_2+1} \cdot R$. Since $i_{\alpha+p+a-1} \leftarrow i_{\alpha+p+a}$, we can apply Lemma 2.7 to $\left(\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a-1} \cdot \psi_{\beta+q+a-1}$ (take $x = \alpha+p+a-2, g = \beta+q-\alpha-p$). Then

$$\begin{aligned} v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_{\beta+q+a-1} &= \sum_{j=1}^{k} \Psi_{\alpha+p+a+z_j}{\uparrow}^{\beta+q+a-1}\,\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-2} \\ &\qquad \Psi_{\alpha+p+a-1}{\uparrow}^{\alpha+p+a+z_j-3} \cdot \left(\Psi_{\alpha+p+a-2}{\uparrow}^{\beta+q+a-3}\right){\downarrow}_{\alpha+p+1} \cdot R_2 \end{aligned}$$

for some $k \geq 0$ and $z_1 < z_2 < \cdots < z_k$ such that $i_{\alpha+p+a+z_j} = i_{\alpha+p+a-1}$. By the diagonal residue property, $\psi_{\alpha+p+a+z_j}$ will certainly be a row relation for $j \in \{1, \ldots, k-1\}$. Thus

$$\begin{aligned} v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_{\beta+q+a-1} &= \Psi_{\alpha+p+a+z_k}{\uparrow}^{\beta+q+a-1}\,\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-2} \\ &\qquad \Psi_{\alpha+p+a-1}{\uparrow}^{\alpha+p+a+z_k-3} \cdot \left(\Psi_{\alpha+p+a-2}{\uparrow}^{\beta+q+a-3}\right){\downarrow}_{\alpha+p+1} \cdot R_2. \end{aligned} \tag{3.20}$$

If $\psi_{\alpha+p+a+z_k}$ is a row relation, we are done. However this need not be the case, i.e. if the node to the right of the node containing $\alpha+p+a+z_k$ was removed as part of the skew shape. A diagram of part of (3.20) is shown in Figure 3.12.

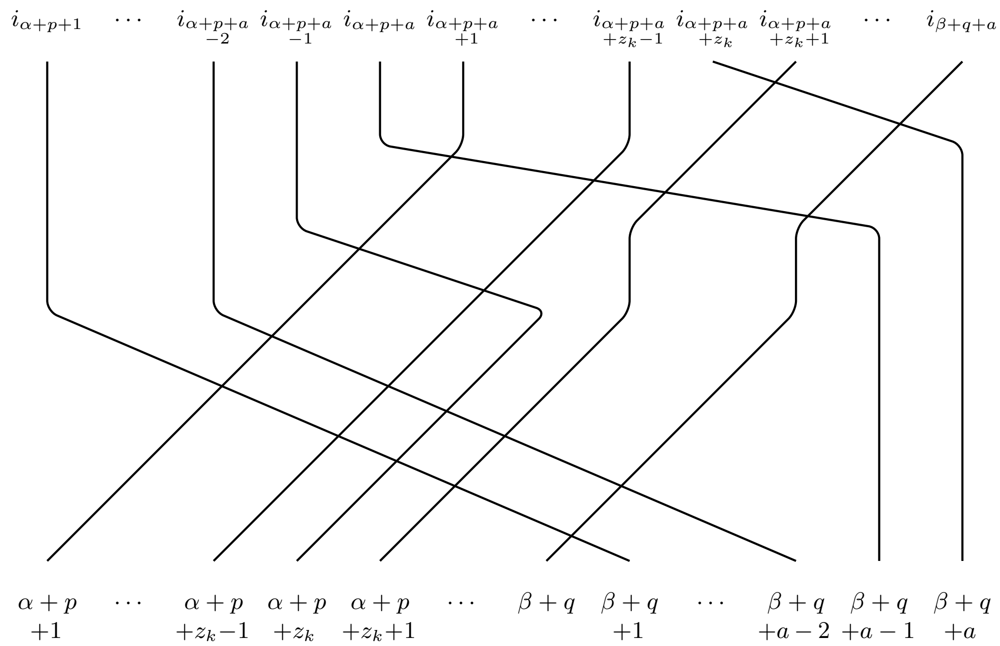


Figure 3.12: Part of the braid diagram for (3.20).

Now, if possible, take $d \in \{0, 1, \dots, a-3\}$ maximal so that

$$i_{\alpha+p+a-2-d} \not\leftarrow i_{\alpha+p+a-1-d} \leftarrow i_{\alpha+p+a-d} \leftarrow \cdots \leftarrow i_{\alpha+p+a-1} \rightarrow i_{\alpha+p+a+z_k-1} \rightarrow \\ \rightarrow i_{\alpha+p+a+z_k-2} \rightarrow \cdots \rightarrow i_{\alpha+p+a+z_k-d}.$$

We can interpret this as meaning that the node containing $\alpha+p+a+z_k-d$ in $[\mu]$ has no node to the left of it, and has the same residue as the node containing $\alpha+p+a-1-d$, so we know which diagonal it lies within. If we cannot take such a d, take $d = a-2$. Now rewrite what we have as

$$\begin{aligned} &v^{t^{\mu}}\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-2} \cdot \left(\Psi_{\alpha+p+a-1}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+1} \\ &\cdot \Psi_{\alpha+p+a+z_k-d-2}\uparrow^{\alpha+p+a+z_k-3} \cdot \left(\Psi_{\alpha+p+a+z_k-d-3}\uparrow^{\alpha+p+a+z_k-3}\right)\downarrow_{\alpha+p+a+z_k-2d-2} \\ &\qquad \cdot \left(\Psi_{\alpha+p+a+z_k-2d-3}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+z_k-d} \\ &\qquad\qquad \cdot \left(\Psi_{\alpha+p+a+z_k}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+z_k+1} \cdot R_2, \end{aligned} \tag{3.21}$$

some of which is shown in this form in Figure 3.13. Then we can apply Lemma 2.9

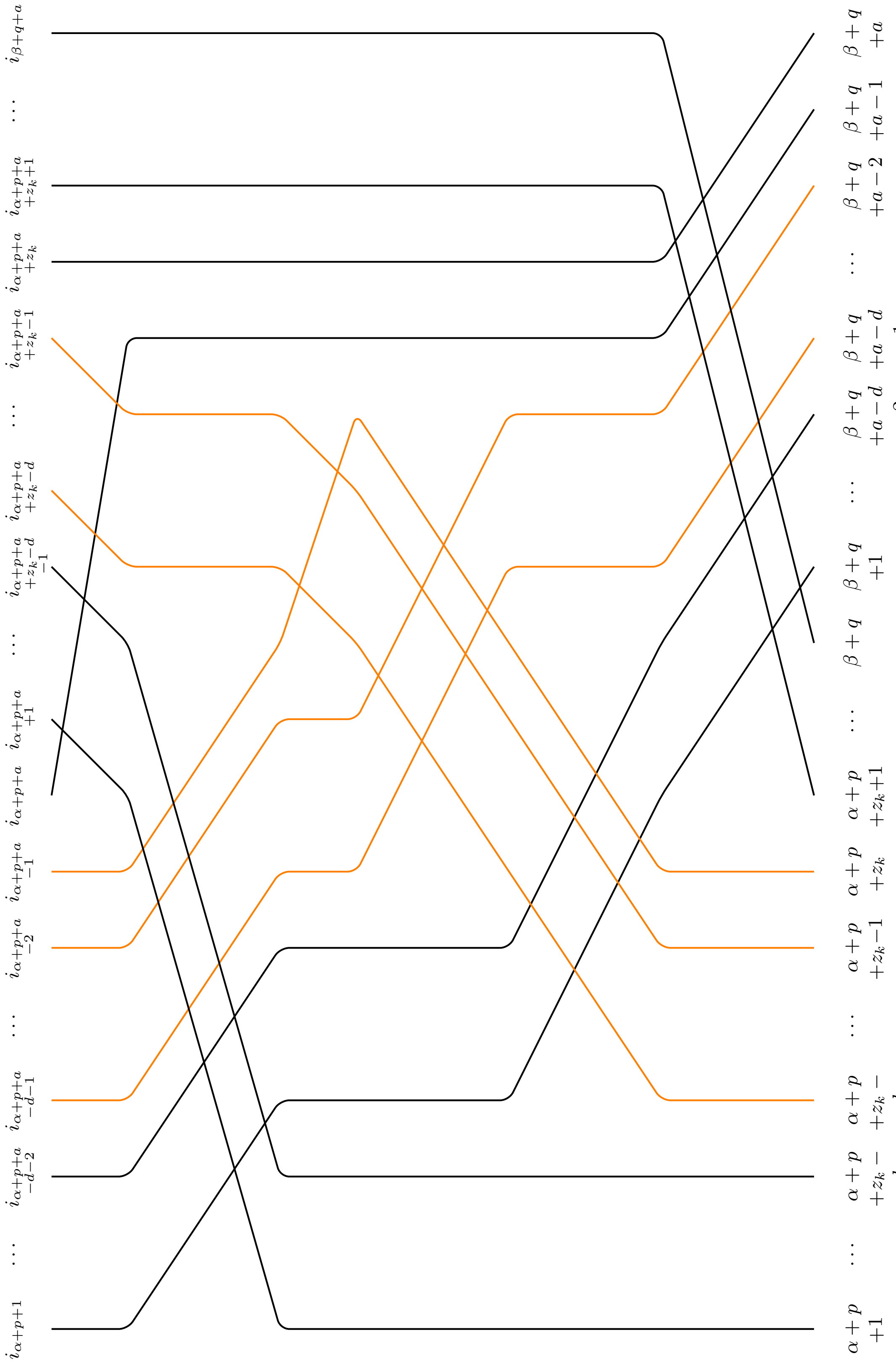


Figure 3.13: Part of the braid diagram for (3.21) excluding R_2. The strings to which Lemma 2.9 is applied are coloured orange.

to $\Psi_{\alpha+p+a+z_k-d-2}\uparrow^{\alpha+p+a+z_k-3} \cdot (\Psi_{\alpha+p+a+z_k-d-3}\uparrow^{\alpha+p+a+z_k-3})\downarrow_{\alpha+p+a+z_k-2d-2}$ (take $x = \alpha+p+a+z_k-2d-3, f = d, g = 0$), to replace it with

$$\begin{aligned}\sum_{m=1}^{d+1} &(\Psi_{\alpha+p+a+z_k-d-2-m}\uparrow^{\alpha+p+a+z_k-2-m})\downarrow_{\alpha+p+a+z_k-2d-2} \\ &\cdot \left(\Psi_{\alpha+p+a+z_k-2d-3+m}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+a+z_k-2d-2} \cdot\end{aligned} \tag{3.22}$$

The terms where $m \in \{1, \ldots, d\}$ are of the form:

$$\begin{aligned} &v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2}\right)\downarrow_{\alpha+p+1} \cdot \psi_{\alpha+p+a+z_k-d-2-m} \cdot R_3 \\ &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2}\right)\downarrow_{\alpha+p+a-m+1} \\ &\quad \cdot \left(\Psi_{\alpha+p+a-m}\uparrow^{\alpha+p+a+z_k-d-m-2}\right)\downarrow_{\alpha+p+a-m-1} \cdot \psi_{\alpha+p+a+z_k-d-m-2} \\ &\qquad \cdot \left(\Psi_{\alpha+p+a-m-2}\uparrow^{\alpha+p+a+z_k-d-m-4}\right)\downarrow_{\alpha+p+1} \cdot R_3 \end{aligned} \tag{3.23}$$

where R_3 consists of later terms which are no longer needed. Some of (3.23) this is shown in Figure 3.14. Now since $i_{\alpha+p+a-m-1} \leftarrow i_{\alpha+p+a-m}$ apply Lemma 2.7 to $(\Psi_{\alpha+p+a-m}\uparrow^{\alpha+p+a+z_k-d-m-2})\downarrow_{\alpha+p+a-m-1} \cdot \psi_{\alpha+p+a+z_k-d-m-2}$ (take $x = \alpha+p+a-m-2, g = z_k-d-1$) so that (3.23) is equal to:

$$\begin{aligned} &v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2}\right)\downarrow_{\alpha+p+a-m+1} \\ &\quad \cdot \psi_{\alpha+p+a-m-1} \cdot \left(\Psi_{\alpha+p+a-m}\uparrow^{\alpha+p+a+z_k-d-m-2}\right)\downarrow_{\alpha+p+a-m-1} \\ &\qquad \cdot \left(\Psi_{\alpha+p+a-m-2}\uparrow^{\alpha+p+a+z_k-d-m-4}\right)\downarrow_{\alpha+p+1} \cdot R_3 \\ &+ \sum_{j'=1}^{k'} v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2}\right)\downarrow_{\alpha+p+a-m+1} \cdot \psi_{\alpha+p+a+z'_{j'}-m} \cdot R_4 \end{aligned}$$

for some $k' \geq 0$ and $z'_{j'}$ such that $i_{\alpha+p+a+z'_{j'}} = i_{\alpha+p+a-m-1}$, and where R_4 consists of later terms which are no longer needed. The former term will be zero since $\psi_{\alpha+p+a-m-1}$ is a row relation. For a term in the sum, apply Corollary 2.6 to $(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2})\downarrow_{\alpha+p+a-m+1} \cdot \psi_{\alpha+p+a+z'_{j'}-m}$ since $i_{\alpha+p+a+z'_{j'}+1} \neq i_{\alpha+p+a-m+1}, i_{\alpha+p+a-m+2}, \ldots, i_{\alpha+p+a}$ (take $x = \alpha+p+a-m, f = m, k = z'_{j'} -$

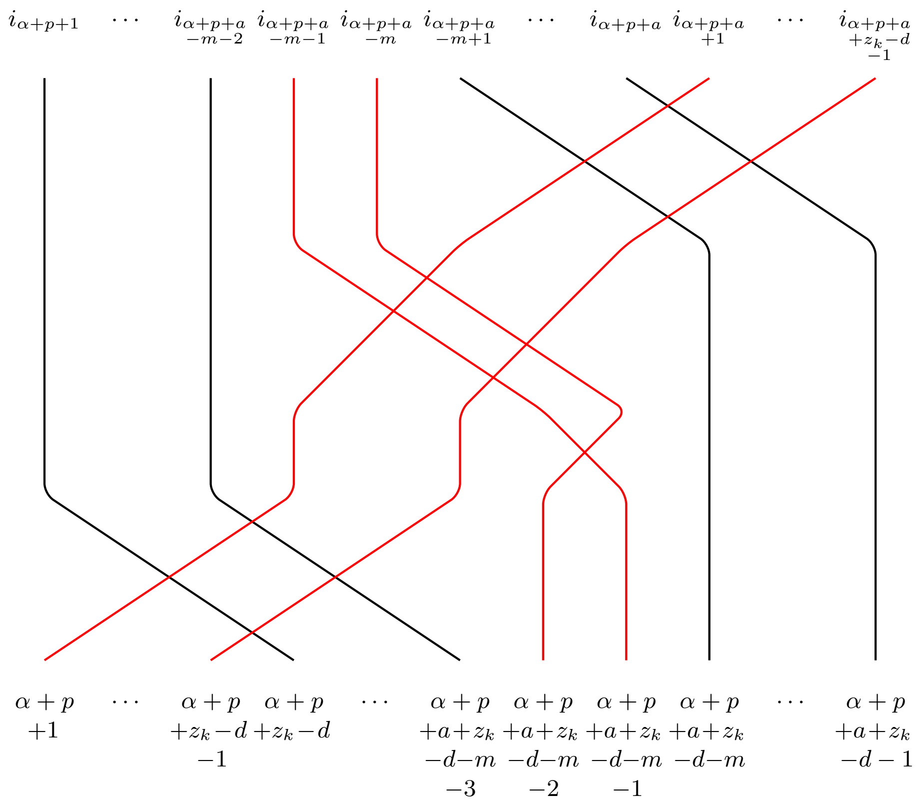


Figure 3.14: Part of the braid diagram for (3.23) excluding R_3. The strings to which Lemma 2.7 is applied are coloured red

$1, h = 1, g = 1, t = z_k - d - 2 - z'_{j'}$). Then (3.23) is equal to:

$$\sum_{j'=1}^{k'} v^{\mathfrak{t}^{\mu}} \psi_{\alpha+p+a+z'_{j'}} \cdot \left(\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-d-2}\right)\downarrow_{\alpha+p+a-m+1}$$
$$= 0$$

since $\psi_{\alpha+p+a+z'_{j'}}$ is a row relation by the diagonal residue condition. We are definitely able to always write the term like this and apply Corollary 2.6 since $i_{\alpha+p+a+z_k-d-1}$ is never equal to $i_{\alpha+p+a-m-1}$ for $m \in \{1, \dots, d\}$.

So now we are just left with the term arising from when $m = d + 1$ in the sum in (3.22), i.e. (3.21) is equal to

$$
\begin{aligned}
&v^{\mathsf{t}^\mu}\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-2}\cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+1}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a+z_k-2d-3}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+z_k-d}\\
&\qquad\qquad\cdot\left(\Psi_{\alpha+p+a+z_k}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+z_k+1}\cdot R_2.\\
&=v^{\mathsf{t}^\mu}\Psi_{\alpha+p+a}\uparrow^{\alpha+p+a+z_k-2}\cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+a-d}\\
&\qquad\cdot\Psi_{\alpha+p+a-d-1}\uparrow^{\alpha+p+a+z_k-2d-3}\cdot\left(\Psi_{\alpha+p+a-d-2}\uparrow^{\alpha+p+a+z_k-d-3}\right)\downarrow_{\alpha+p+1}\\
&\qquad\qquad\cdot\left(\Psi_{\alpha+p+a+z_k}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+z_k+1}\cdot R_2.
\end{aligned}
\tag{3.24}
$$

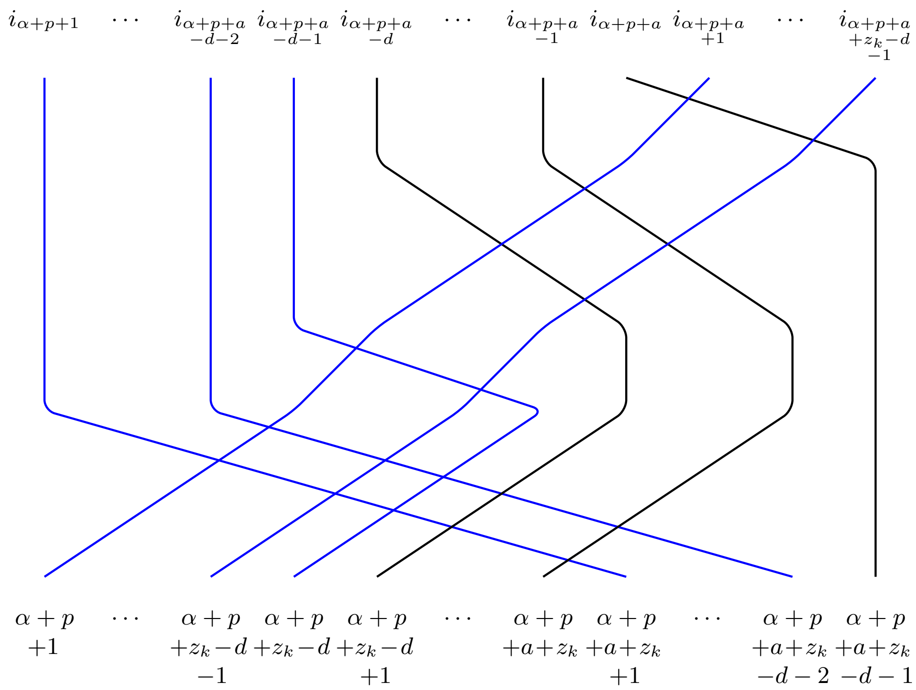


Figure 3.15: Part of the braid diagram for (3.24) excluding $\Psi_{\alpha+p+a+z_k-d-1}\uparrow^{\alpha+p+a+z_k-2}\cdot\left(\Psi_{\alpha+p+a+z_k}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+z_k+1}\cdot R_2$. The strings to which we apply Corollary 2.6 are coloured blue.

First assume that $d\neq a-2$, then we know that

$$i_{\alpha+p+a-d-2}\neq i_{\alpha+p+a+1},i_{\alpha+p+a+2},\ldots,i_{\alpha+p+a+z_k-d-1}$$

hence apply Corollary 2.6 to

$$\Psi_{\alpha+p+a-d-1}{\uparrow}^{\alpha+p+a+z_k-2d-3} \cdot \left(\Psi_{\alpha+p+a-d-2}{\uparrow}^{\alpha+p+a+z_k-d-3}\right){\downarrow}_{\alpha+p+1}$$

(take $x = \alpha + p, f = a - d - 2, k = 0, h = 1, g = z_k - d - 1, t = d$) and then we have a $\psi_{\alpha+p+a-d-2}$ crossing which kills v^{t^μ} as this is a row relation.

So now assume instead that $d = a - 2$. Then (3.21) is equal to:

$$\begin{aligned} v^{\mathrm{t}^\mu}\Psi_{\alpha+p+a}{\uparrow}^{\alpha+p+a+z_k-2} \cdot \left(\Psi_{\alpha+p+a-1}{\uparrow}^{\alpha+p+z_k-1}\right){\downarrow}_{\alpha+p+1}& \\ \cdot \left(\Psi_{\alpha+p+a+z_k}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+z_k+1} \cdot R_2&. \end{aligned} \tag{3.25}$$

If the node containing $\alpha+p+a+z_k$ in $[\mu]$ is a Garnir node, then we are done since the Garnir relation for this node will be contained within the fourth multiplicand.

So suppose the node containing $\alpha + p + a + z_k$ in $[\mu]$ is not a Garnir node, and let $\delta \in \{0, 1, \ldots, a-2\}$ be as small as possible so that the node containing $\alpha+p+a+z_k-\delta$ is a Garnir node in $[\mu]$, whilst the node containing $\alpha+p+a+z_k-\delta+1$ is not. Such a node is guaranteed to exist by the fact that $d = a - 2$. Note that we now must have $a \geq 3$ in order to be in this situation. Rewrite (3.25) as

$$\begin{aligned} v^{\mathrm{t}^\mu}\Psi_{\alpha+p+a+z_k}{\uparrow}^{\beta+q+a-1} \cdot \left(\Psi_{\alpha+p+a}{\uparrow}^{\alpha+p+z_k}\right){\downarrow}_{\alpha+p+1} \cdot \Psi_{\alpha+p+z_k+1}{\uparrow}^{\alpha+p+a+z_k-\delta-2}& \\ \cdot \Psi_{\alpha+p+a+z_k-\delta-1}{\uparrow}^{\beta+q+a-2} \cdot \left(\Psi_{\alpha+p+a+z_k-2}{\uparrow}^{\beta+q+a-3}\right){\downarrow}_{\alpha+p+z_k-\delta-1}& \\ \cdot \left(\Psi_{\alpha+p+a+z_k-\delta-2}{\uparrow}^{\beta+q+a-\delta-3}\right){\downarrow}_{\alpha+p+z_k+1} \cdot R_2&. \end{aligned} \tag{3.26}$$

Some of this is shown in Figure 3.16.

Now, since $i_{\alpha+p+a} \not\to i_{\alpha+p+a+z_k-1}, i_{\alpha+p+a+z_k-2}, \ldots, i_{\alpha+p+a+z_k-\delta}$ apply Lemma 2.5 to $\Psi_{\alpha+p+a+z_k-\delta-1}{\uparrow}^{\beta+q+a-2} \cdot \left(\Psi_{\alpha+p+a+z_k-2}{\uparrow}^{\beta+q+a-3}\right){\downarrow}_{\alpha+p+z_k-\delta-1}$ (take $x = \alpha + p + a + z_k - \delta - 2, f = 1, g = \beta + q - \alpha - p - z_k, h = \delta$). So (3.25)

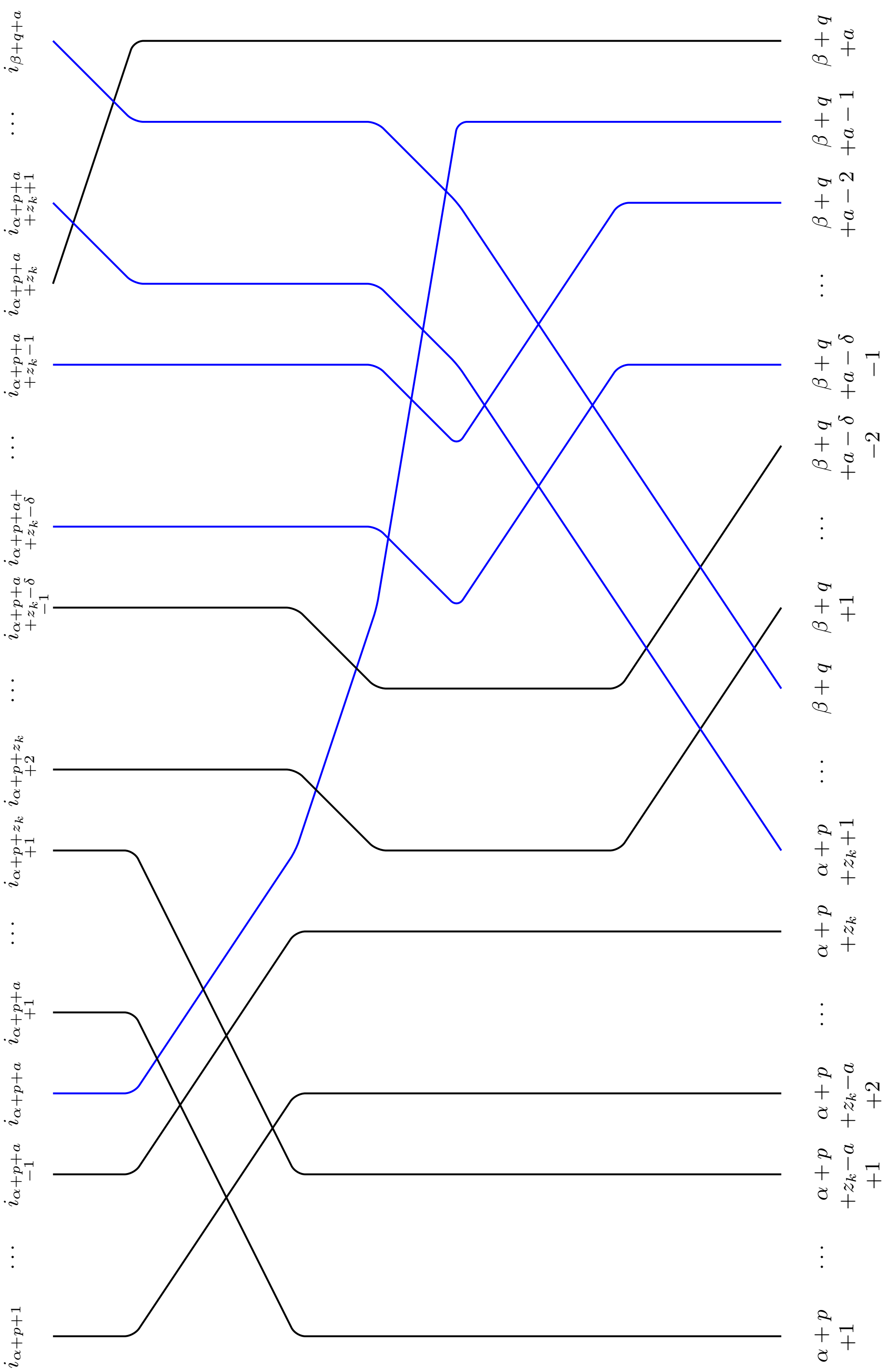


Figure 3.16: Part of the braid diagram for 3.26. The strings to which Lemma 2.5 is applied are coloured blue.

is equal to:

$$
\begin{aligned}
&v^{\mathfrak{t}^{\mu}}\Psi_{\alpha+p+a+z_k}{\uparrow}^{\beta+q+a-1}\cdot\left(\Psi_{\alpha+p+a}{\uparrow}^{\alpha+p+z_k}\right){\downarrow}_{\alpha+p+1}\cdot\Psi_{\alpha+p+z_k+1}{\uparrow}^{\alpha+p+a+z_k-\delta-2}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a+z_k-1}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p+a+z_k-\delta}\cdot\Psi_{\alpha+p+a+z_k-\delta-1}{\uparrow}^{\beta+q+a-2}\\
&\qquad\qquad\cdot\left(\Psi_{\alpha+p+a+z_k-\delta-2}{\uparrow}^{\beta+q-\delta-3}\right){\downarrow}_{\alpha+p+z_k+1}\cdot R_2.
\end{aligned}
\tag{3.27}
$$

Now with a bit of rearranging, we can see that we have $\left(\Psi_{\alpha+p+a+z_k}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a+z_k-\delta}$ at the top of the diagram, thus the Garnir relation corresponding to the node containing $\alpha+p+a+z_k-\delta$ will be at the top of the diagram, making the whole term zero.

So we are done for this section, having shown $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}\psi_{\beta+q+a-1}=0$ in all cases.

3.3.3 Relations in (iii).

We use the same notation as at the beginning of this section in Proposition 3.6. The proof splits into the same cases depending on the location of a Garnir relation with respect to $\tilde{r}$ just as before. For the cases:

- $r\in\{0,1,\dots,\tilde{r}-1\}$
- $r\in\{\tilde{r}+1,\dots,\beta+q+a-1\}$
- $r\in\{\beta+q+a,\beta+q+a+1,\dots,n-2\}$

we follow the same method as in Proposition 3.6, replacing α with $\alpha+p$ and β with $\beta+q$. Then we are left with only one case left to check.

$\mathbf{r=\tilde{r}}$

We must check the Garnir relation when $r=\tilde{r}$. In this case, there is not a corresponding Garnir relation in $S^{\tilde{\lambda}}$, so we check that $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}g_{\lambda}\left(\boxed{\tilde{r}+1}\right)$ is equal to zero directly. Using our notation, we can write $\tilde{r}+1=\beta-b$ for some $b\geq 0$. Then in $\mathfrak{t}^{\lambda}$ the Garnir belt is

$$
\begin{array}{ccccc}
 & & \beta-b & \cdots\cdots & \beta\\
\beta+1 & \cdots\cdots & \beta+q+a & &
\end{array}
$$

giving the Garnir relation $g_\lambda\left(\boxed{\tilde{r}+1}\right) = (\Psi_\beta{\uparrow}^{\beta+q+a-1}){\downarrow}_{\beta-b}$. In Figure 3.17 we display the important parts of the braid diagram of $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$.

We have that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$ is equal to

$$\begin{aligned}
&v^{\mathfrak{t}^\mu}\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\cdot\left(\Psi_\alpha{\uparrow}^{\alpha+p-1}\right){\downarrow}_{\alpha-b}\\
&\cdot\left(\Psi_{\alpha+p+a-1}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p+1}\cdot\left(\Psi_{\alpha+p}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p-b}\\
&\quad\cdot\Psi^{\beta+q+a-1}{\downarrow}_{\beta+q+a-b-1}\cdot\left(\Psi_{\alpha-b-1}{\uparrow}^{\beta-b-2}\right){\downarrow}_{\alpha-a_2+1}\cdot R,
\end{aligned}\tag{3.28}$$

and then we can apply Lemma 2.5 to

$$\left(\Psi_{\alpha+p+a-1}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p+1}\cdot\left(\Psi_{\alpha+p}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p-b}$$

(take $x = \alpha+p-b-1, f = b+1, g = \beta+q-\alpha-p, h = a-1$) since $i_{\alpha+p+1},\dots,i_{\alpha+p+a-1} \not\multimap i_{\alpha-b},\dots,i_\alpha$. Figure 3.18 helps demonstrate this.

Thus (3.28) is equal to

$$\begin{aligned}
&v^{\mathfrak{t}^\mu}\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\cdot\left(\Psi_\alpha{\uparrow}^{\alpha+p-1}\right){\downarrow}_{\alpha-b}\\
&\cdot\left(\Psi_{\alpha+p}{\uparrow}^{\beta+q+a-2}\right){\downarrow}_{\alpha+p-b}\cdot\left(\Psi_{\alpha+p+a-b-2}{\uparrow}^{\beta+q+a-b-3}\right){\downarrow}_{\alpha+p-b}\\
&\quad\cdot\Psi^{\beta+q+a-1}{\downarrow}_{\beta+q+a-b-1}\cdot\left(\Psi_{\alpha-b-1}{\uparrow}^{\beta-b-2}\right){\downarrow}_{\alpha-a_2+1}\cdot R\\
&= v^{\mathfrak{t}^\mu}\left(\Psi_\alpha{\uparrow}^{\alpha+p+a-2}\right){\downarrow}_{\alpha-b}\cdot\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\\
&\quad\cdot\left(\Psi_{\alpha+p+a-1}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a-b}\cdot\Psi_{\alpha+p+a-b-1}{\uparrow}^{\beta+q+a-b-1}\\
&\quad\cdot\left(\Psi_{\alpha+p+a-b-2}{\uparrow}^{\beta+q+a-b-3}\right){\downarrow}_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}{\uparrow}^{\beta-b-2}\right){\downarrow}_{\alpha-a_2+1}\cdot R,
\end{aligned}\tag{3.29}$$

some of which is shown in Figure 3.19. Since $i_{\alpha+p+a} \not\multimap i_{\alpha-b+1},\dots,i_\alpha$, we can apply Lemma 2.4 to $\Psi_{\alpha+p+a}{\uparrow}^{\beta+q+a-1}\cdot\left(\Psi_{\alpha+p+a-1}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a-b}$ (take $x = \alpha+p+a-b-1, f = b, g = \beta+q-\alpha-p$). Then (3.29) is equal to

$$\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_\alpha{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha-b+1}\cdot\Psi_{\alpha-b}{\uparrow}^{\alpha+p+a-b-2}\\
&\quad\cdot\left(\Psi_{\alpha+p+a-b}{\uparrow}^{\beta+q+a-b-1}\right){\downarrow}_{\alpha+p+a-b-1}\cdot\psi_{\beta+q+a-b-1}\\
&\cdot\left(\Psi_{\alpha+p+a-b-2}{\uparrow}^{\beta+q+a-b-3}\right){\downarrow}_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}{\uparrow}^{\beta-b-2}\right){\downarrow}_{\alpha-a_2+1}\cdot R
\end{aligned}\tag{3.30}$$

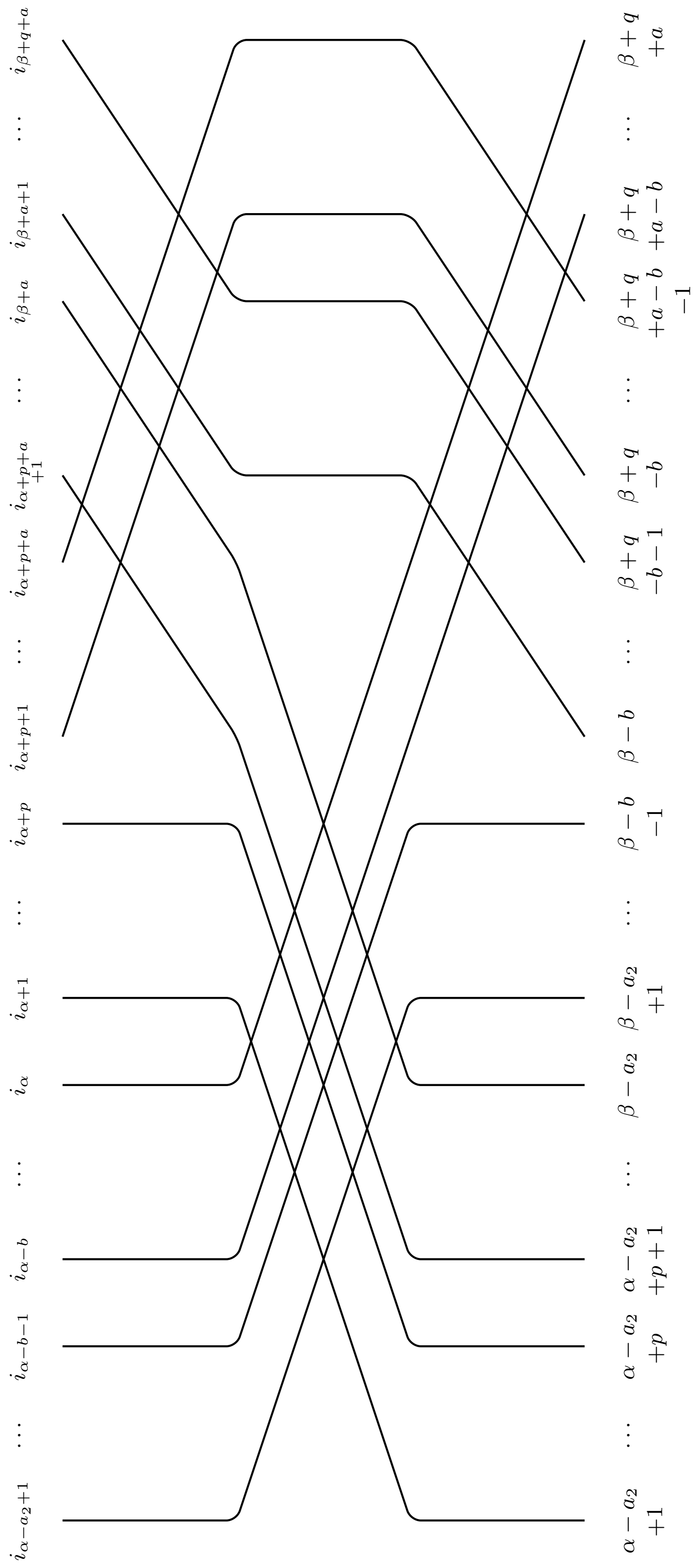


Figure 3.17: Part of the braid diagram for $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$ excluding R.

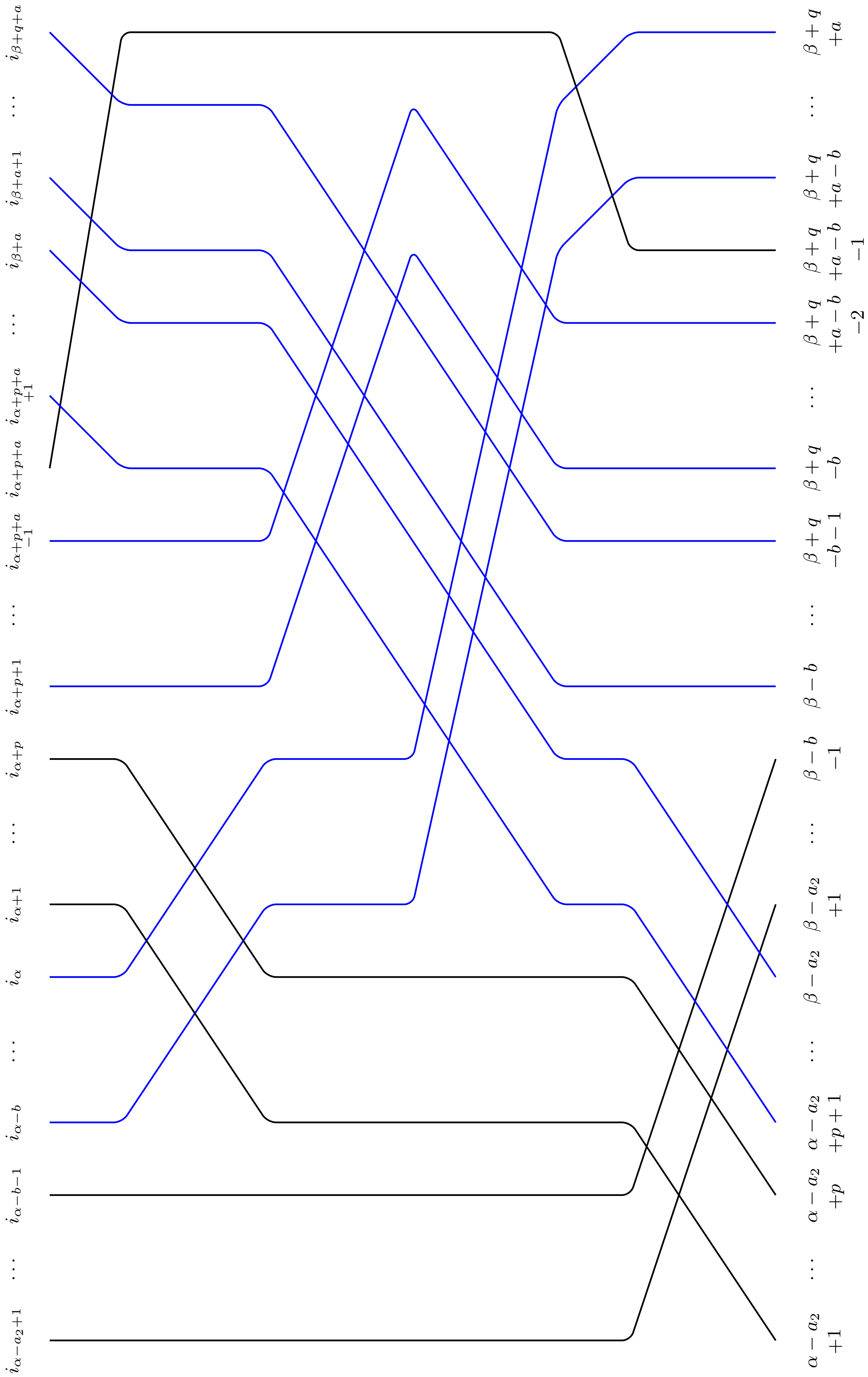


Figure 3.18: Part of the braid diagram for (3.28) excluding R. The strings to which we apply Lemma 2.5 are coloured blue.

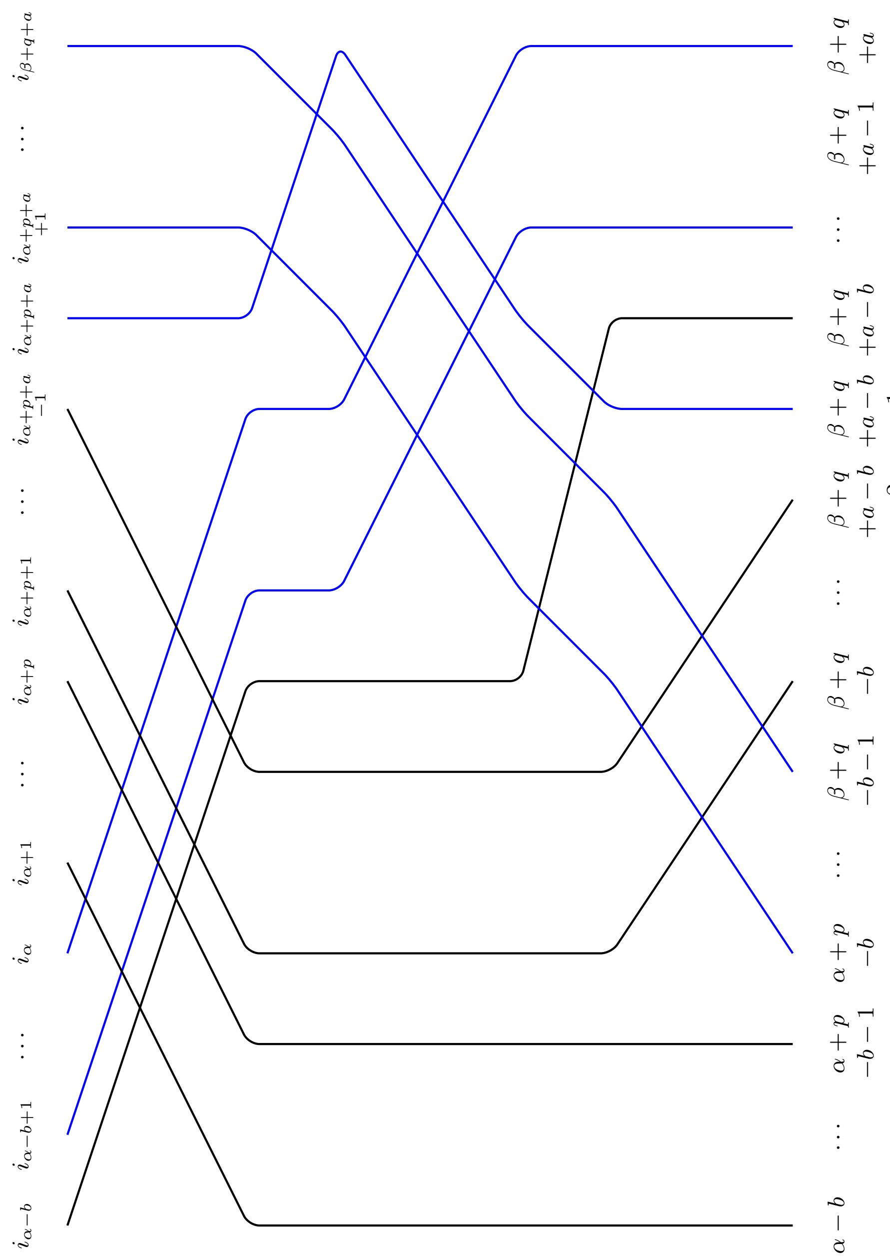


Figure 3.19: Part of the braid diagram for (3.29) excluding $\left(\Psi_{\alpha+p-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R$. The strings to which we apply Lemma 2.4 are coloured blue.

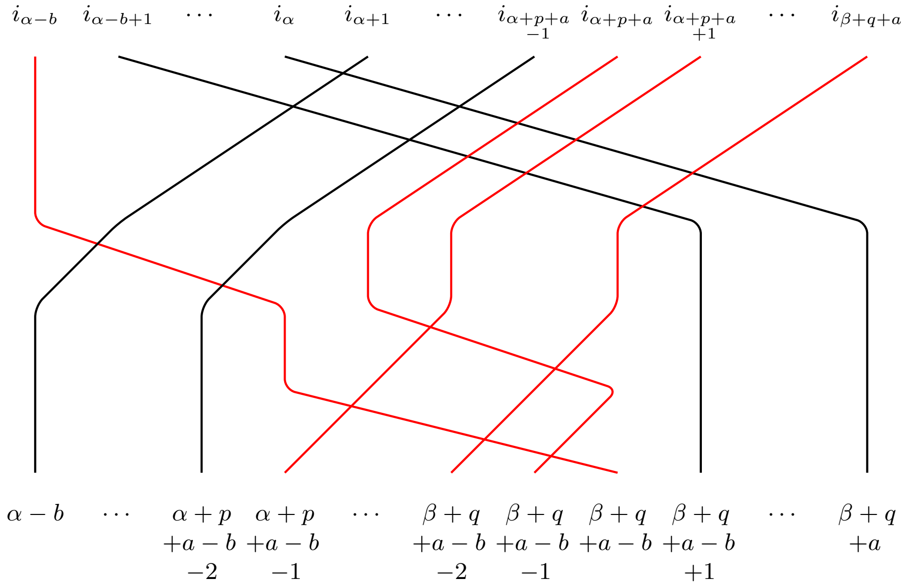


Figure 3.20: Part of the braid diagram for (3.30) excluding the last three multiplicands. The strings to which we apply Lemma 2.7 are coloured red.

some of which is shown in Figure 3.20.

Since $i_{\alpha-b} = i_{\alpha+p+a} + 1$, we can apply Lemma 2.7 to

$$\left(\Psi_{\alpha+p+a-b}\uparrow^{\beta+q+a-b-1}\right)\downarrow_{\alpha+p+a-b-1} \cdot \psi_{\beta+q+a-b-1}$$

(take $x = \alpha+p+a-b-2, g = \beta+q-\alpha-p$). Then (3.30) is equal to

$$v^{\mathfrak{t}^\mu}\left(\Psi_\alpha\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b} \cdot \Psi_{\alpha+p+a-b-1}\uparrow^{\beta+q+a-b-2}$$
$$\cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b} \cdot \left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1} \cdot R \tag{3.31}$$

$$+\sum_{j=1}^{k} v^{\mathfrak{t}^\mu}\left(\Psi_\alpha\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b} \cdot \left(\Psi_{\alpha+p+a-1}\uparrow^{\alpha+p+a+z_j-2}\right)\downarrow_{\alpha+p+a-b}$$
$$\cdot\left(\Psi_{\alpha+p+a+z_j-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z_j-b} \cdot \Psi_{\alpha+p+a+z_j-b}\uparrow^{\beta+q+a-b-1} \tag{3.32}$$
$$\cdot\Psi_{\alpha+p+a-b}\uparrow^{\beta+q+a-b-2} \cdot \Psi_{\alpha+p+a-b-1}\uparrow^{\alpha+p+a+z_j-b-3}$$
$$\cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b} \cdot \left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1} \cdot R$$

for some $k \geq 0$, with z_j's arising from residues $i_{\alpha+p+a+z_j}$ which are equal to $i_{\alpha-b}$. When considering an arbitrary term in (3.32) we shall just write z for z_j. Part of such a term in the sum of (3.32) is shown in Figure 3.21. Note that

$$\left(\Psi_\alpha\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b} = \left(\Psi_\alpha\uparrow^{\alpha+p+a-1}\right)\downarrow_{\alpha-b}\cdot\left(\Psi_{\alpha+p+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b}$$
$$= g_\mu\left(\boxed{\ \alpha-b\ }\right)\cdot\left(\Psi_{\alpha+p+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b}$$

so that (3.31) is equal to zero.

Take the greatest $\delta \in \{0,1,\dots,b\}$ such that $\psi_{\alpha+p+a+z+m}$ is a row relation for S^μ for each $m \in \{0,1,\dots,\delta-1\}$. If $\delta = 0$, then note that most of the following does not apply and we can move straight to considering (3.33) (as it is equal to a given term of (3.32)). Figure 3.22 helps to illustrate the residues of the nodes related to these row relations.

Consider the fourth and fifth multiplicand in (3.32), and rewrite these as

$$\begin{aligned}&\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z-b}\cdot\Psi_{\alpha+p+a+z-b}\uparrow^{\beta+q+a-b-1}\\&=\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z-b+\delta}\\&\quad\cdot\left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b+\delta-1}\right)\downarrow_{\alpha+p+a+z-b}\cdot\Psi_{\alpha+p+a+z-b}\uparrow^{\beta+q+a-b-1}.\end{aligned}$$

In Figure 3.23 we show the relevant part of (3.32) with the corresponding residues.

Now since

$$i_{\alpha-b+\delta}\to i_{\alpha-b+\delta-1}\to\cdots\to i_{\alpha-b+1}\to i_{\alpha+p+a+z}\leftarrow i_{\alpha+p+a+z+1}\leftarrow$$
$$\cdots\leftarrow i_{\alpha+p+a+z+\delta},$$

we can apply Lemma 2.10 to

$$\left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b+\delta-1}\right)\downarrow_{\alpha+p+a+z-b}\cdot\Psi_{\alpha+p+a+z-b}\uparrow^{\beta+q+a-b-1}$$

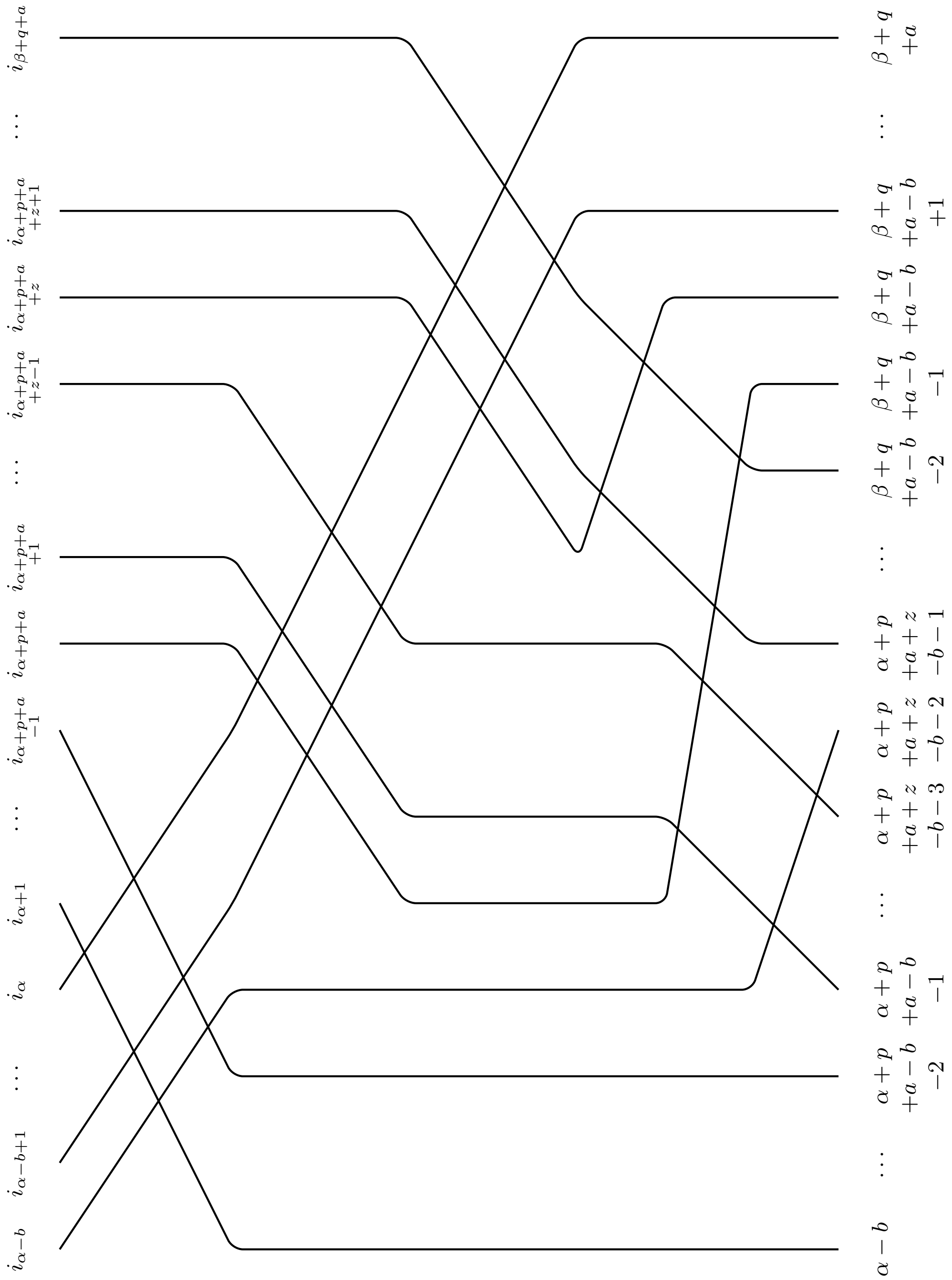


Figure 3.21: Part of the braid diagram for (3.32) excluding $\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b} \cdot \left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1} \cdot R$.

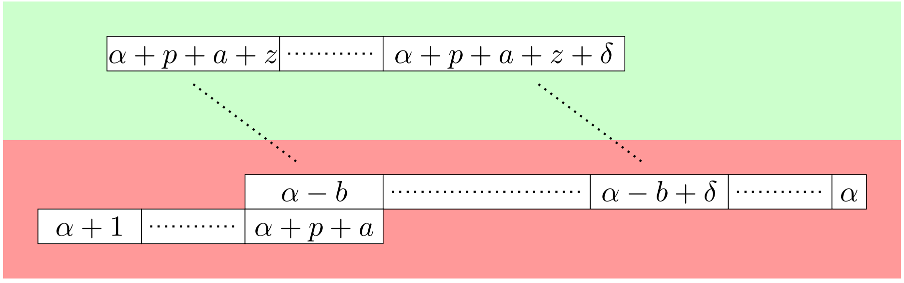


Figure 3.22: Diagram to show equality of residues between the different components of μ. The top half shows nodes in the second component of $\mathfrak{t}^\mu$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue.

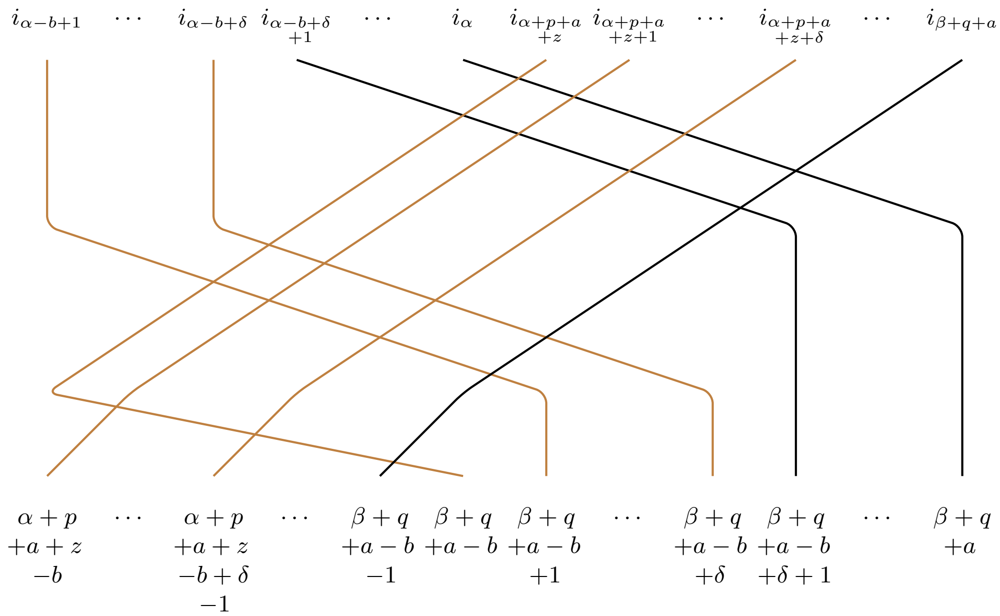


Figure 3.23: Braid diagram of the crossings $\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z-b+\delta} \cdot \left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b+\delta-1}\right)\downarrow_{\alpha+p+a+z-b} \cdot \Psi_{\alpha+p+a+z-b}\uparrow^{\beta+q+a-b-1}$ with the associated residues from (3.32). The strings to which we apply Lemma 2.10 are coloured brown.

(take $x = \alpha + p + a + z - b - 1, f = \delta, g = \beta + q - \alpha - p - z$), replacing it with

$$\begin{aligned}\sum_{j'=1}^{\delta} \Big[&\psi_{\alpha+p+a+z-b+\delta-1+j'} \\ &\cdot \left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b+\delta-1}\right)\downarrow_{\alpha+p+a+z-b-1+j'} \\ &\cdot \Psi_{\alpha+p+a+z-b-1+2j'}\uparrow^{\beta+q+a-b-2+j'} \\ &\cdot \left(\Psi_{\alpha+p+a+z-b-3+2j'}\uparrow^{\beta+q+a-b-3+j'}\right)\downarrow_{\alpha+p+a+z-b-1+j'}\Big] \\ + &\left(\Psi_{\alpha+p+a+z-b+2\delta}\uparrow^{\beta+q+a-b+\delta-1}\right)\downarrow_{\alpha+p+a+z-b+\delta}.\end{aligned}$$

So consider a term of (3.32), then this will consist of terms corresponding to the above sum for $j' \in \{1, 2, \ldots, \delta\}$ and these will each be equal to

$$v^{\mathrm{t}^{\mu}} \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot \psi_{\alpha+p+a+z-b+\delta-1+j'} \cdot R''$$

where R'' consists of later terms which are no longer needed in calculations. Since $i_{\alpha-b+\delta+1}, i_{\alpha-b+\delta+2}, \ldots, i_{\alpha} \neq i_{\alpha+p+a+z+j'}$ we can apply Corollary 2.6 to the above (take $x = \alpha-b+\delta, f = b-\delta, k = p+a+z+\delta-1, h = 1, g = 1, t = \beta+q-\alpha-p-z-\delta$), giving

$$v^{\mathrm{t}^{\mu}} \psi_{\alpha+p+a+z-1+j'} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot R''$$

which will be zero.

So now we need only look at the term which corresponds to when $j' = \delta + 1$. If $\psi_{\alpha+p+a+z+\delta}$ is a row relation then we can follow the same method as for the terms when $j' \in \{1, 2, \ldots, \delta\}$ and annihilate $v^{\mathrm{t}^{\mu}}$ with a $\psi_{\alpha+p+a+z+\delta}$ crossing. So suppose instead that $\psi_{\alpha+p+a+z+\delta}$ is not a row relation. Then overall we have that

(3.32) is equal to

$$\begin{aligned} v^{\mathfrak{t}^\mu} & \left(\Psi_\alpha\!\uparrow^{\alpha+p+a-2}\right)\!\downarrow_{\alpha-b} \cdot \left(\Psi_{\alpha+p+a-1}\!\uparrow^{\alpha+p+a+z-2}\right)\!\downarrow_{\alpha+p+a-b} \\ & \cdot \left(\Psi_{\alpha+p+a+z-1}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+a+z-b+\delta} \\ & \cdot \left(\Psi_{\alpha+p+a+z-b+2\delta}\!\uparrow^{\beta+q+a-b+\delta-1}\right)\!\downarrow_{\alpha+p+a+z-b+\delta} \cdot \Psi_{\alpha+p+a-b}\!\uparrow^{\beta+q+a-b-2} \\ & \cdot \Psi_{\alpha+p+a-b-1}\!\uparrow^{\alpha+p+a+z-b-3} \\ & \cdot \left(\Psi_{\alpha+p+a-b-2}\!\uparrow^{\beta+q+a-b-3}\right)\!\downarrow_{\alpha+p-b} \cdot \left(\Psi_{\alpha-b-1}\!\uparrow^{\beta-b-2}\right)\!\downarrow_{\alpha-a_2+1} \cdot R, \end{aligned} \tag{3.33}$$

some of which is shown in Figure 3.24. Since none of $i_{\alpha-b+\delta+1}, i_{\alpha-b+\delta+2}, \ldots, i_\alpha$ are equal to any of $i_{\alpha+p+a+z+\delta+1}, i_{\alpha+p+a+z+\delta+2}, \ldots, i_{\beta+q+a}$ we can apply Lemma 2.5 to the fourth and fifth multiplicands (take $x = \alpha+p+a+z-b+\delta-1, f = b-\delta$, $g = \beta+q-\alpha-p-z-\delta, h = \delta+1$). This gives us $\left(\Psi_{\alpha+p+a+z+\delta}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+a+z}$ at the top of the diagram. If the node containing $\alpha+p+a+z$ in $\mathfrak{t}^\mu$ is a Garnir node, then we have the corresponding Garnir relation at the top of the diagram giving us zero. So instead, assume it is not a Garnir node. (C4)

So now (3.33) is equal to

$$\begin{aligned} v^{\mathfrak{t}^\mu} & \left(\Psi_{\alpha+p+a+z+\delta}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+a+z} \cdot \left(\Psi_\alpha\!\uparrow^{\alpha+p+a-2}\right)\!\downarrow_{\alpha-b} \\ & \cdot \left(\Psi_{\alpha+p+a-1}\!\uparrow^{\beta+q+a-1}\right)\!\downarrow_{\alpha+p+a-b+\delta} \\ & \cdot \left(\Psi_{\alpha+p+a-b+\delta-1}\!\uparrow^{\alpha+p+a+z-b+\delta-2}\right)\!\downarrow_{\alpha+p+a-b} \cdot \Psi_{\alpha+p+a-b}\!\uparrow^{\alpha+p+a+z-b-2} \\ & \cdot \Psi_{\alpha+p+a+z-b-1}\!\uparrow^{\beta+q+a-b-2} \cdot \Psi_{\alpha+p+a-b-1}\!\uparrow^{\alpha+p+a+z-b-3} \\ & \cdot \left(\Psi_{\alpha+p+a-b-2}\!\uparrow^{\beta+q+a-b-3}\right)\!\downarrow_{\alpha+p-b} \cdot \left(\Psi_{\alpha-b-1}\!\uparrow^{\beta-b-2}\right)\!\downarrow_{\alpha-a_2+1} \cdot R, \end{aligned} \tag{3.34}$$

some of which is shown in Figure 3.25. Since

$$i_{\alpha-b+1}, i_{\alpha-b+2}, \ldots, i_{\alpha-b+\delta} \nrightarrow i_{\alpha+p+a}$$

we can apply Lemma 2.5 to the fourth and fifth multiplicands (take $x = \alpha+p+a-b-1, f = \delta, g = z-1, h = 1$), replacing them with

$$\Psi_{\alpha+p+a-b+\delta}\!\uparrow^{\alpha+p+a+z-b+\delta-2} \cdot \left(\Psi_{\alpha+p+a-b+\delta-1}\!\uparrow^{\alpha+p+a+z-b+\delta-2}\right)\!\downarrow_{\alpha+p+a-b} \cdot$$

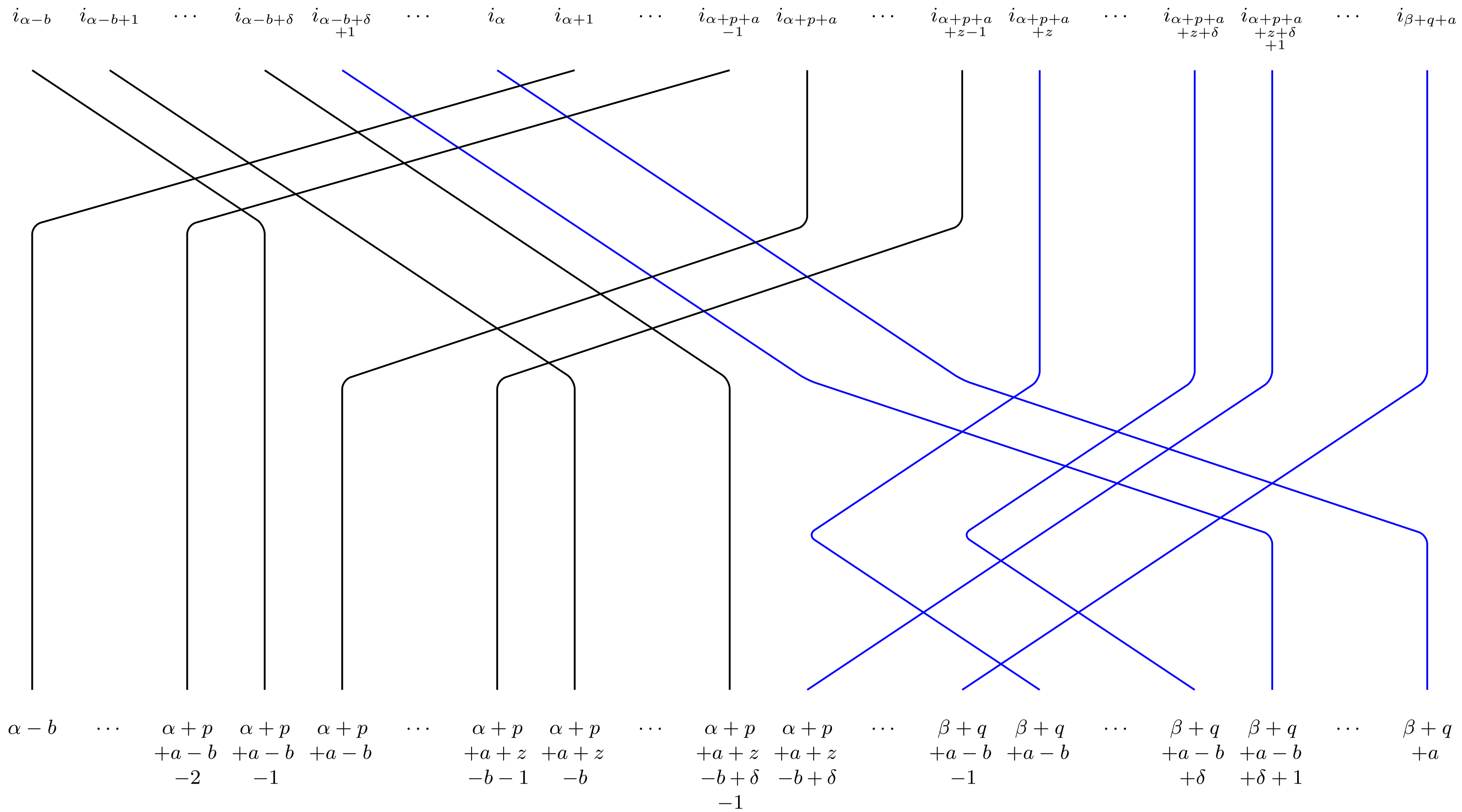


Figure 3.24: Part of the braid diagram for the first 5 multiplicands of (3.33). The strings to which we apply Lemma 2.5 are coloured blue.

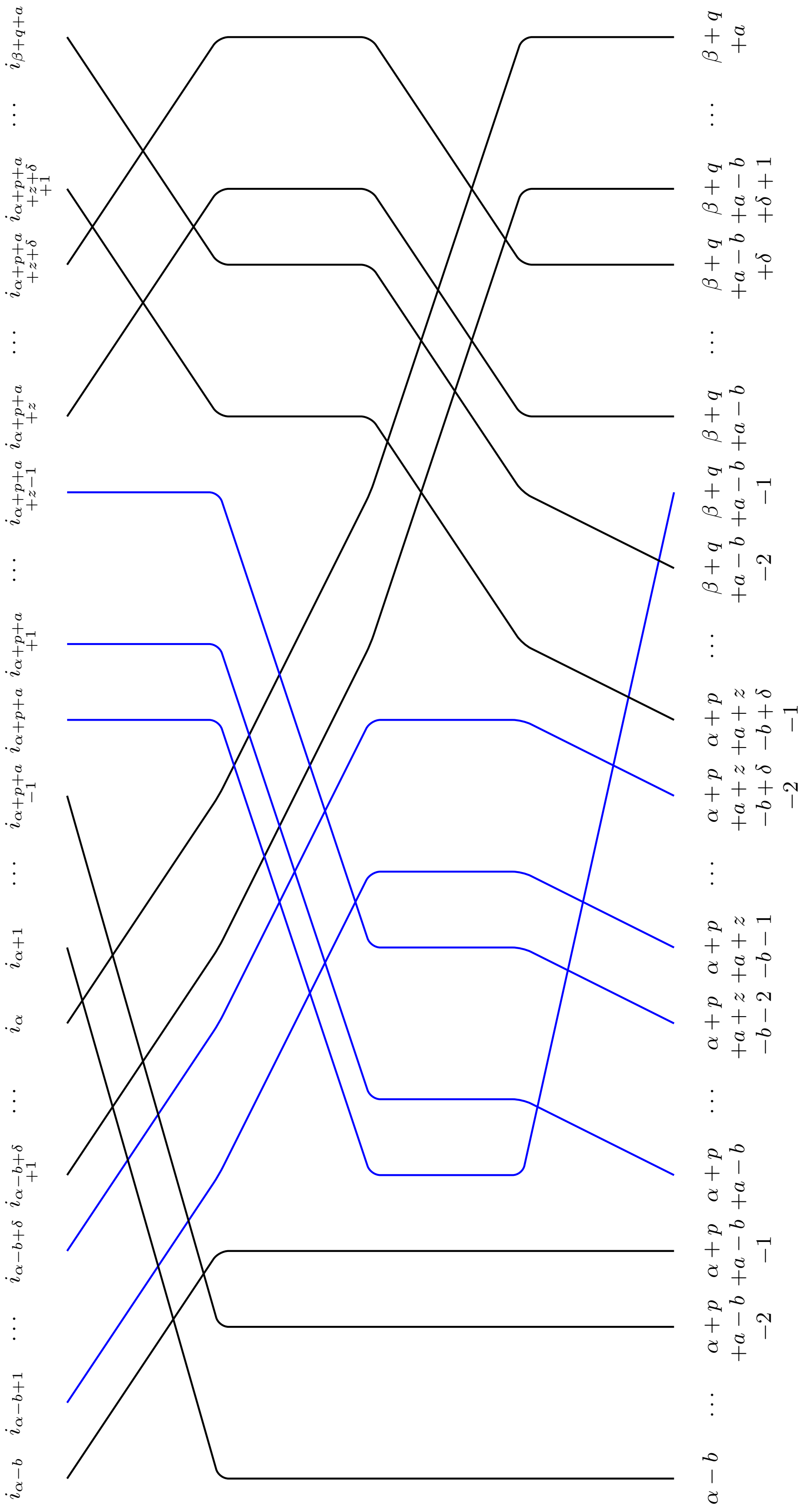


Figure 3.25: Part of the braid diagram for (3.34). The strings to which we apply Lemma 2.5 are coloured blue.

Now rewrite

$$\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\alpha+p+a+z-b+\delta-2}\right)\downarrow_{\alpha+p+a-b}\cdot\Psi_{\alpha+p+a+z-b-1}\uparrow^{\beta+q+a-b-2}$$

as

$$\begin{aligned}\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\alpha+p+a+z-b+\delta-3}\right)\downarrow_{\alpha+p+a-b}\cdot\Psi^{\alpha+p+a+z-b+\delta-2}\downarrow_{\alpha+p+a+z-b-1}\\ \cdot\Psi_{\alpha+p+a+z-b-1}\uparrow^{\alpha+p+a+z-b+\delta-2}\cdot\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b-2}\end{aligned}$$

and then since we still have $i_{\alpha-b+1}, i_{\alpha-b+2}, \ldots, i_{\alpha-b+\delta} \not\to i_{\alpha+p+a}$, apply Lemma 2.8 to $\Psi^{\alpha+p+a+z-b+\delta-2}\downarrow_{\alpha+p+a+z-b-1}\cdot\Psi_{\alpha+p+a+z-b-1}\uparrow^{\alpha+p+a+z-b+\delta-2}$ (take $x=\alpha+p+a+z-b-2, f=\delta, g=1, k=0$). We can see how this is applied in Figure 3.26. So now (3.34) is equal to

$$\begin{aligned}v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_\alpha\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b}\\ \cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b+\delta}\\ \cdot\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+p+a+z-b+\delta-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\alpha+p+a+z-b+\delta-3}\right)\downarrow_{\alpha+p+a-b}\\ \cdot\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+q+a-b-2}\cdot\Psi_{\alpha+p+a-b-1}\uparrow^{\alpha+p+a+z-b-3}\\ \cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R,\end{aligned}\tag{3.35}$$

some of which is shown in Figure 3.27.

Rearranging, (3.35) is equal to

$$\begin{aligned}v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_\alpha\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b}\\ \cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b+\delta}\\ \cdot\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\alpha+p+a+z-b+\delta-3}\right)\downarrow_{\alpha+p+a-b-1}\\ \cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R,\end{aligned}\tag{3.36}$$

and then since $i_{\alpha-b+j}\not\to i_{\alpha+p+1}, i_{\alpha+p+2}, \ldots, i_{\alpha+p+a-1}$ for $j\in\{0,1,\ldots,\delta\}$, we

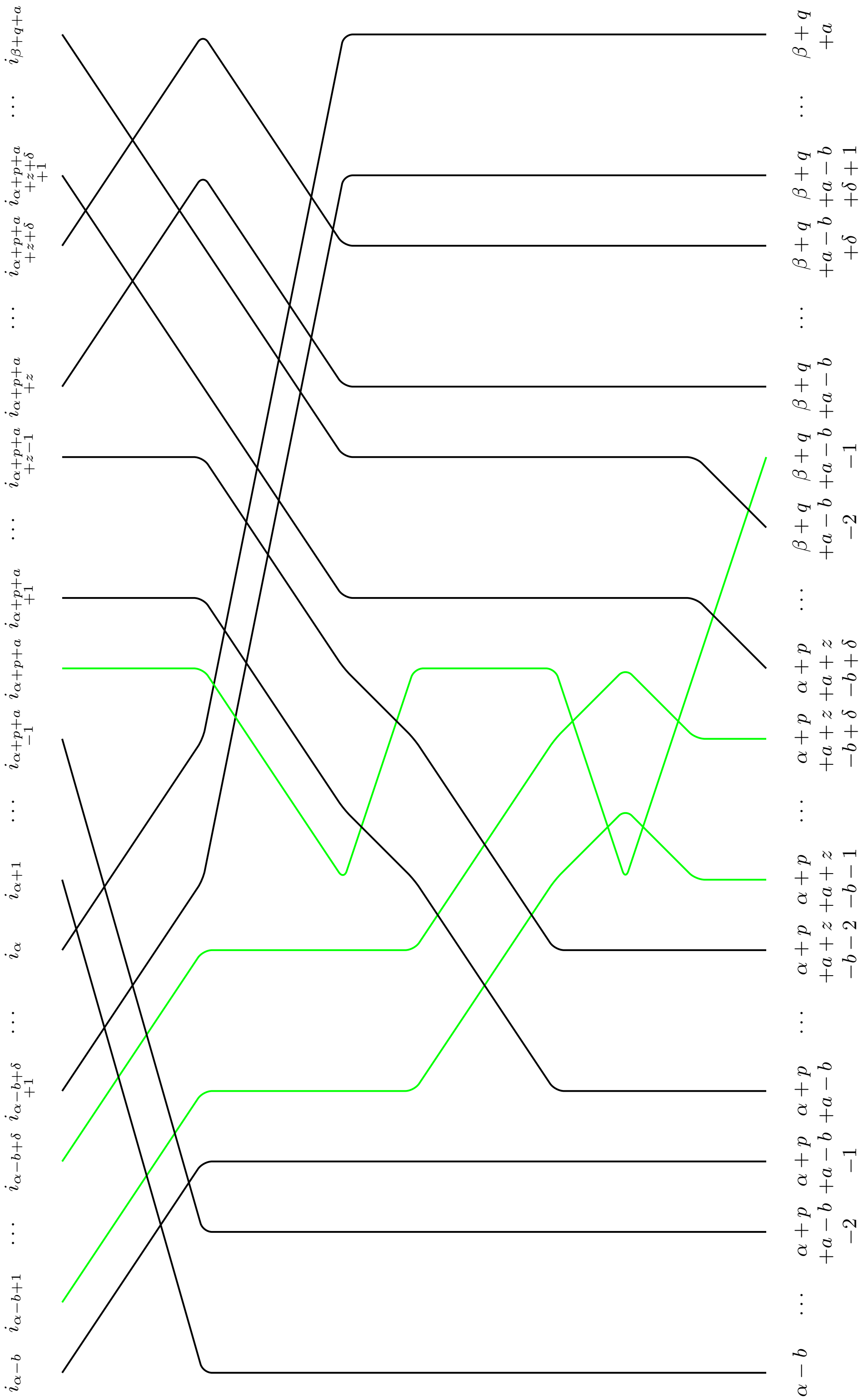


Figure 3.26: Part of the braid diagram for (3.34) after applying Lemma 2.5. The strings to which we apply Lemma 2.8 are coloured green.

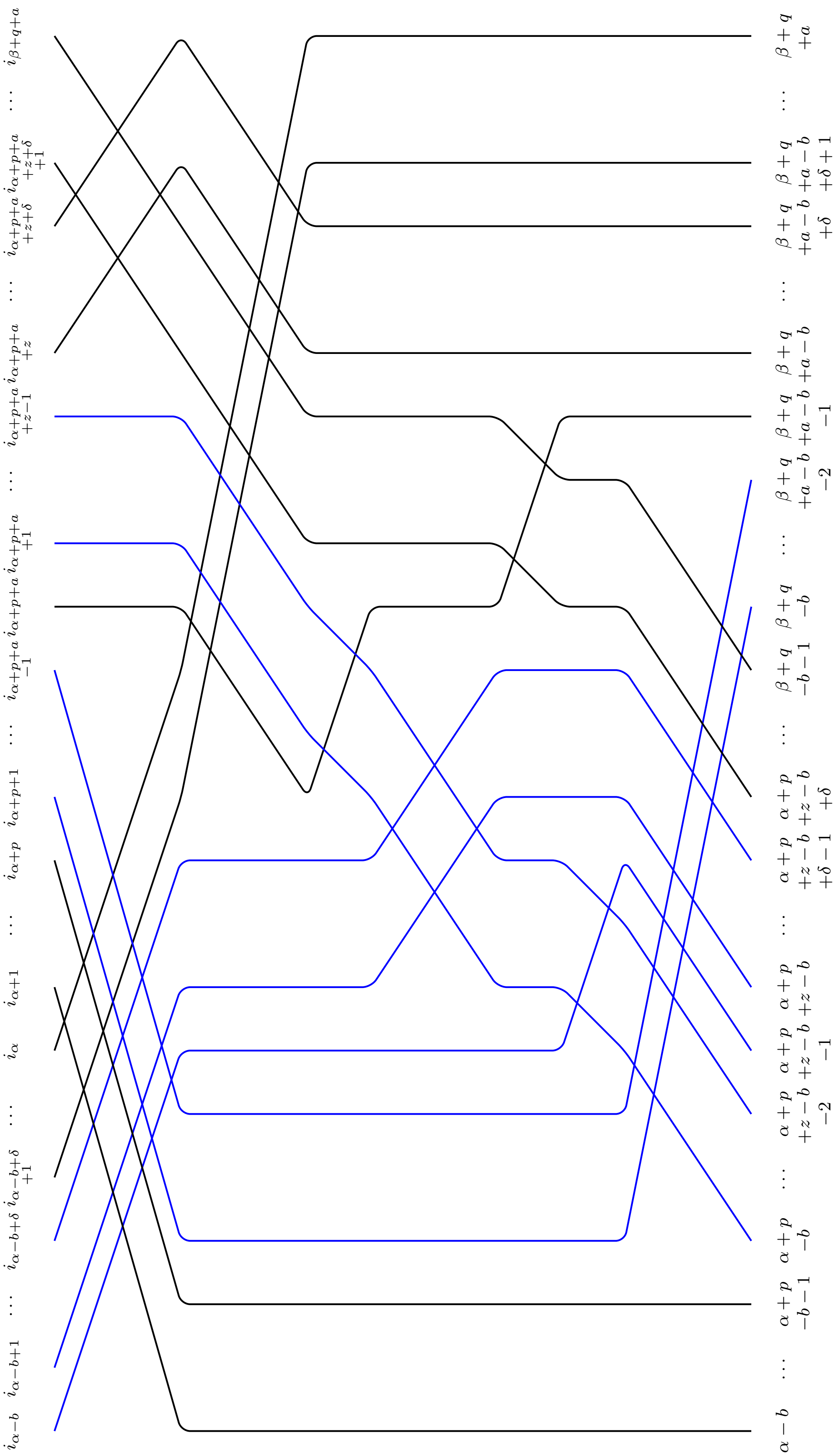


Figure 3.27: Part of the braid diagram for (3.35). The strings to which we apply Corollary 2.6 are coloured blue.

can apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\alpha+p+a+z-b+\delta-3}\right)\downarrow_{\alpha+p+a-b-1} \\ \cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}$$

(take $x=\alpha+p-b-1, f=a-1, k=0, h=\delta+1, g=z-1, t=\beta+q-\alpha-p-z-\delta$), so that (3.36) is equal to

$$\begin{aligned}
&v^{t^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_\alpha\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b+\delta}\\
&\quad\cdot\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}\\
&\quad\cdot\left(\Psi_{\alpha+p-b+\delta}\uparrow^{\alpha+p+z-b+\delta-2}\right)\downarrow_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R\\
&=v^{t^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_\alpha\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b+\delta+1}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b+\delta}\cdot\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\\
&\qquad\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+a-b+\delta-2}\right)\downarrow_{\alpha-b}\cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}\\
&\qquad\cdot\left(\Psi_{\alpha+p-b+\delta}\uparrow^{\alpha+p+z-b+\delta-2}\right)\downarrow_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}\tag{3.37}$$

Since $i_{\alpha-b+j}\not\to i_{\alpha+p+1}, i_{\alpha+p+2},\dots,i_{\alpha+p+a-1}$ for $j\in\{0,1,\dots,\delta\}$, apply Lemma 2.8 to $\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+a-b+\delta-2}\right)\downarrow_{\alpha-b}\cdot\left(\Psi_{\alpha+p+a-b-2}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b}$ (take $x=\alpha-b-1, f=\delta+1, h=p, g=a-1, k=\beta+q-\alpha-p-\delta-1$). So

then (3.37) is equal to

$$
\begin{aligned}
&v^{\mathrm{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\alpha+p+a-2}\right)\downarrow_{\alpha-b+\delta+1}\\
&\quad\cdot\left(\Psi_{\alpha+p+a-1}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a-b+\delta}\cdot\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p-b+\delta-1}\right)\downarrow_{\alpha-b}\cdot\left(\Psi_{\alpha+p+a-b+\delta-1}\uparrow^{\beta+q+a-b-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\quad\cdot\left(\Psi_{\alpha+p-b+\delta}\uparrow^{\alpha+p+z-b+\delta-2}\right)\downarrow_{\alpha+p-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R\\
&=v^{\mathrm{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\qquad\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-2}\right)\downarrow_{\alpha-b}\cdot\left(\Psi_{\alpha-b-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
\tag{3.38}
$$

Some of this is shown in Figure 3.28.

Suppose $m\in\{1,\dots,a_2-b-1\}$ and consider

$$
\begin{aligned}
&v^{\mathrm{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\qquad\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\qquad\cdot\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-m+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m+1}\cdot\left(\Psi_{\alpha-b-m}\uparrow^{\beta-b-1-m}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
\tag{3.39}
$$

Note that if $m=1$ we recover (3.38). Consider the bottom line of (3.39). Rewrite this as

$$
\begin{aligned}
&\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m+2}\cdot\left(\Psi_{\alpha-b-m+1}\uparrow^{\alpha+p+z-b-2m}\right)\downarrow_{\alpha-b-m}\\
&\qquad\cdot\psi_{\alpha+p+z-b-2m}\cdot\Psi_{\alpha+p+z-b-2m+1}\uparrow^{\beta-b-1-m}\\
&\qquad\qquad\cdot\left(\Psi_{\alpha-b-m-1}\uparrow^{\beta-b-2-m}\right)\downarrow_{\alpha-a_2+1}\cdot R
\end{aligned}
$$

and then apply Lemma 2.7 to

$$\left(\Psi_{\alpha-b-m+1}\uparrow^{\alpha+p+z-b-2m}\right)\downarrow_{\alpha-b-m}\cdot\psi_{\alpha+p+z-b-2m}$$

since $i_{\alpha-b-m}\leftarrow i_{\alpha-b-m+1}$ (take $x=\alpha-b-m$, $g=p+z-m$). Then (3.39) is

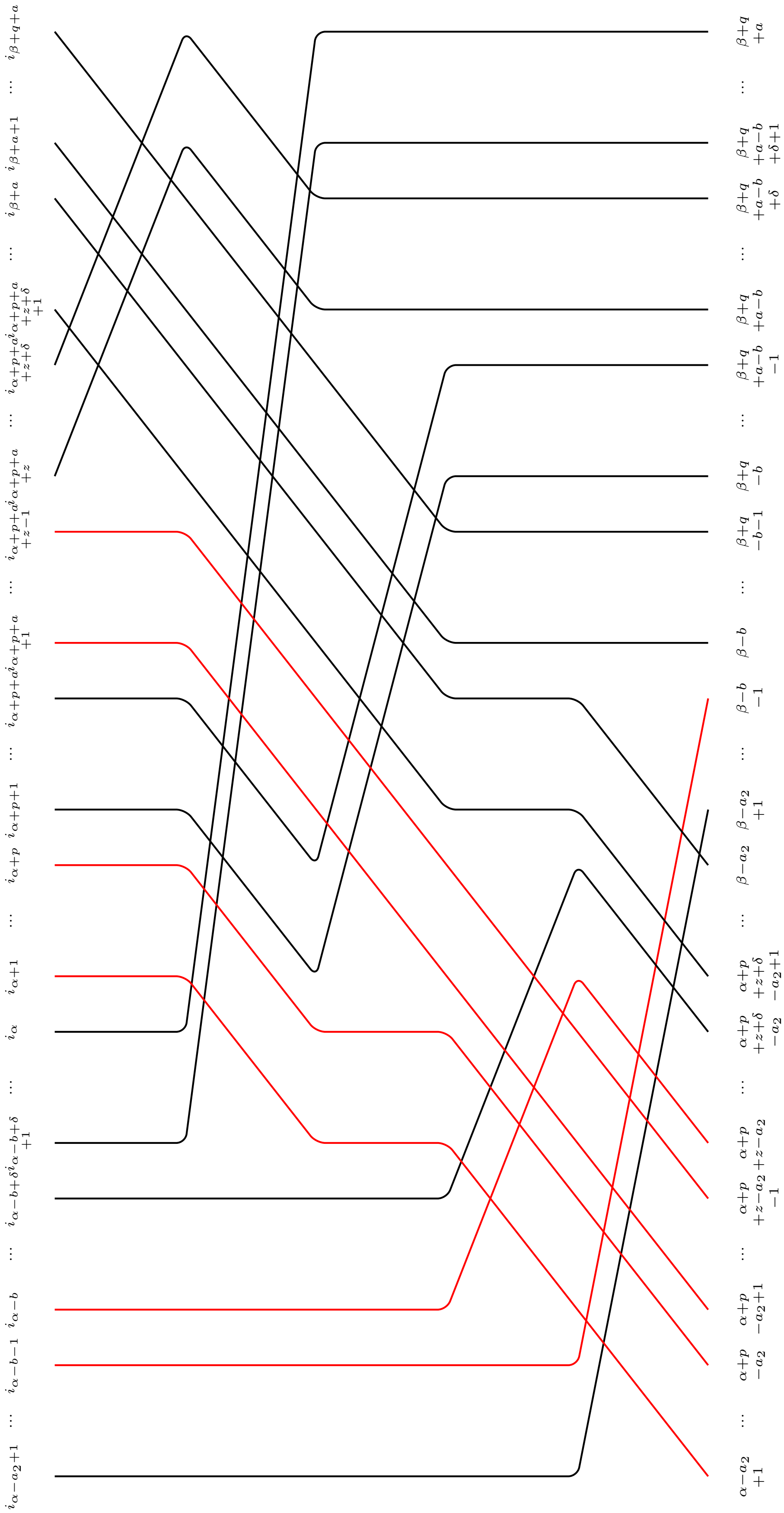


Figure 3.28: Part of the braid diagram for (3.38) excluding R. The strings to which we apply Lemma 2.7 are coloured red.

equal to

$$
\begin{aligned}
v^{\mathsf{t}^\mu}\psi_{\alpha-b-m}\cdot\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-m+1}\\
\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m}\\
\cdot\Psi_{\alpha+p+z-b-2m+1}\uparrow^{\beta-b-1-m}\cdot\left(\Psi_{\alpha-b-m-1}\uparrow^{\beta-b-2-m}\right)\downarrow_{\alpha-a_2+1}\cdot R\\
+\sum_{t=1}^{k'}v^{\mathsf{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-m+1}\\
\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m+2}\cdot\Psi_{\alpha-b-m+\zeta_t+1}\uparrow^{\alpha+p+z-b-2m}\\
\cdot\Psi_{\alpha-b-m+1}\uparrow^{\alpha+p+z-b-2m-1}\cdot\Psi_{\alpha-b-m}\uparrow^{\alpha-b-m+\zeta_t-2}\\
\cdot\Psi_{\alpha+p+z-b-2m+1}\uparrow^{\beta-b-1-m}\cdot\left(\Psi_{\alpha-b-m-1}\uparrow^{\beta-b-2-m}\right)\downarrow_{\alpha-a_2+1}\cdot R
\end{aligned}
\tag{3.40}
$$

for some $k'\geq 0$ and $\zeta_t\in\{p+1,\dots,p+z-m\}$ such that $i_{\alpha+\zeta_t+a}$ equals $i_{\alpha-b-m}$. The first term will be zero since $\psi_{\alpha-b-m}$ is a row relation. Given a term in (3.40), some of which is shown in Figure 3.29, if $\zeta_t\neq p+z-m$ then since $i_{\alpha-b-m+2},i_{\alpha-b-m+3},\dots,i_{\alpha-b+\delta}\not\rightarrow i_{\alpha+\zeta_t+a}$, apply Corollary 2.6 to $\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m+2}\cdot\Psi_{\alpha-b-m+\zeta_t+1}\uparrow^{\alpha+p+z-b-2m}$. In Figure 3.30 we illustrate the relationships between the residues of the nodes in question here.

So now such a term in (3.40) is equal to

$$
\begin{aligned}
v^{\mathsf{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
\cdot\Psi_{\alpha-b+\delta+\zeta_t}\uparrow^{\alpha+p+z-b+\delta-m-1}\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-m-1}\right)\downarrow_{\alpha-b-m+2}\cdot R_{\zeta_t}
\end{aligned}
$$

where R_{ζ_t} consists of terms that we no longer need. Now the nodes containing (C5) $\alpha+\zeta_t+a,\dots,\alpha+\zeta_t+a+m-1$ all belong to the same row, to the left of the node

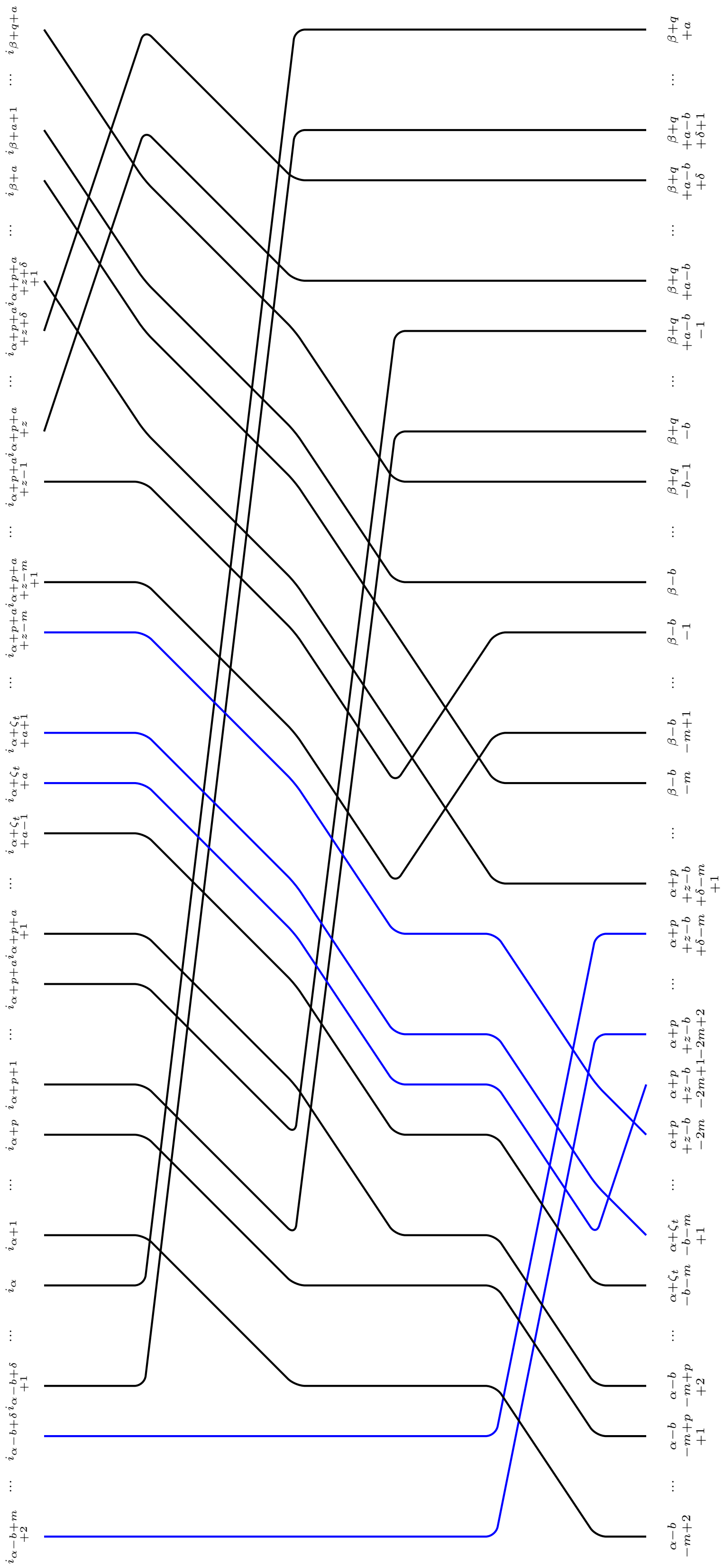


Figure 3.29: Part of the braid diagram for (3.40). The strings to which we apply Corollary 2.6 are coloured blue.

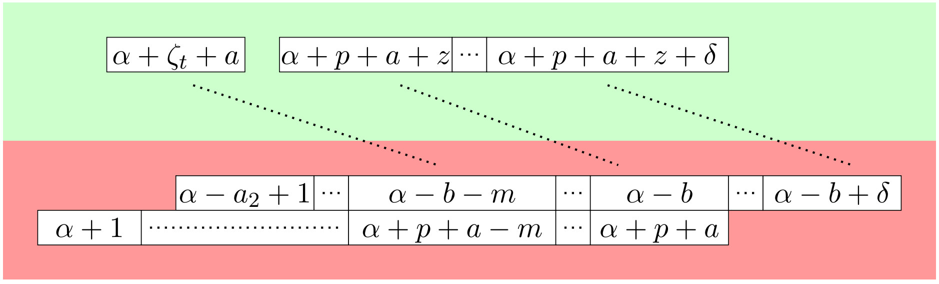


Figure 3.30: Diagram to show equality of residues between the different components of μ with the introduction of $\alpha+\zeta_t+a$. The top half shows nodes in the second component of $\mathfrak{t}^\mu$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue.

containing $\alpha+p+a+z$, so $\psi_{\alpha+\zeta_t+a}$ will be a row relation by the diagonal residue condition. If $m=1$ then $i_{\alpha+\zeta_t+a}=i_{\alpha+p+a}$ so $i_{\alpha+\zeta_t+a+1}\neq i_{\alpha+p+1},\dots,i_{\alpha+p+a}$, and $i_{\alpha+\zeta_t+a}\not\to i_\alpha,\dots i_{\alpha-b+\delta+1}$ so we can apply Corollary 2.6 to pull the $\psi_{\alpha-b+\delta+\zeta_t}$ crossing to the top, obtaining $\psi_{\alpha+\zeta+a}$ at the top of the diagram meaning our term will be zero. Now suppose $m>1$, then since

$$i_{\alpha+p+1},\dots,i_{\alpha+p+a-m+1}\neq i_{\alpha+\zeta_t+a+1},\dots,i_{\alpha+\zeta_t+a+m-1},$$

rewrite $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\Psi_{\alpha-b+\delta+\zeta_t}\uparrow^{\alpha+p+z-b+\delta-m-1}$ as

$$\begin{aligned}&\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2}\\ &\cdot\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+\zeta_t-b+\delta+m-2}\\ &\qquad\cdot\Psi_{\alpha+\zeta_t-b+\delta+m-1}\uparrow^{\alpha+p+z-b+\delta-m-1}\end{aligned}\tag{3.41}$$

and apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\Psi_{\alpha-b+\delta+\zeta_t}\uparrow^{\alpha+\zeta_t-b+\delta+m-2}$$

giving

$$\begin{aligned}&\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-1}\\ &\quad\cdot\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\end{aligned}$$

Then (3.41) is equal to

$$\begin{aligned}&\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2}\\ &\cdot\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+m-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1}\\ &\qquad\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-1}\\ &\cdot\left(\Psi_{\alpha+\zeta_t+a-b+\delta+m-1}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta+1}\\ &\cdot\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1}\\ &\qquad\cdot\Psi_{\alpha+\zeta_t-b+\delta+m-1}\uparrow^{\alpha+p+z-b+\delta-m-1}.\end{aligned}$$

In Figure 3.31 we show some of the crossings at this stage. Now apply Lemma 2.10 to

$$\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+m-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1}\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-1}$$

since $i_{\alpha+p+a}\to\cdots\to i_{\alpha+p+a-m+2}\to i_{\alpha+\zeta_t+a}\leftarrow i_{\alpha+\zeta_t+a+1}\leftarrow\cdots\leftarrow i_{\alpha+\zeta_t+a+m-1}$ (take $x=\alpha+\zeta_t+a-b+\delta-m$, $f=m-1$, $g=m-1$).

Thus replace

$$\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+m-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1}\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-1}$$

with a sum of terms, which each begin with the crossing $\psi_{\alpha+\zeta_t+a-b+\delta-1+j}$ for $j\in\{1,\dots,m-1\}$, along with one other term where the crossings in question disappear. In the former case, these crossings $\psi_{\alpha+\zeta_t+a-b+\delta-1+j}$ commute with $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2}$, and since

$$i_{\alpha-b+\delta+1},\dots,i_\alpha\not- i_{\alpha+\zeta_t+a}$$

we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\psi_{\alpha+\zeta_t+a-b+\delta-1+j}$$

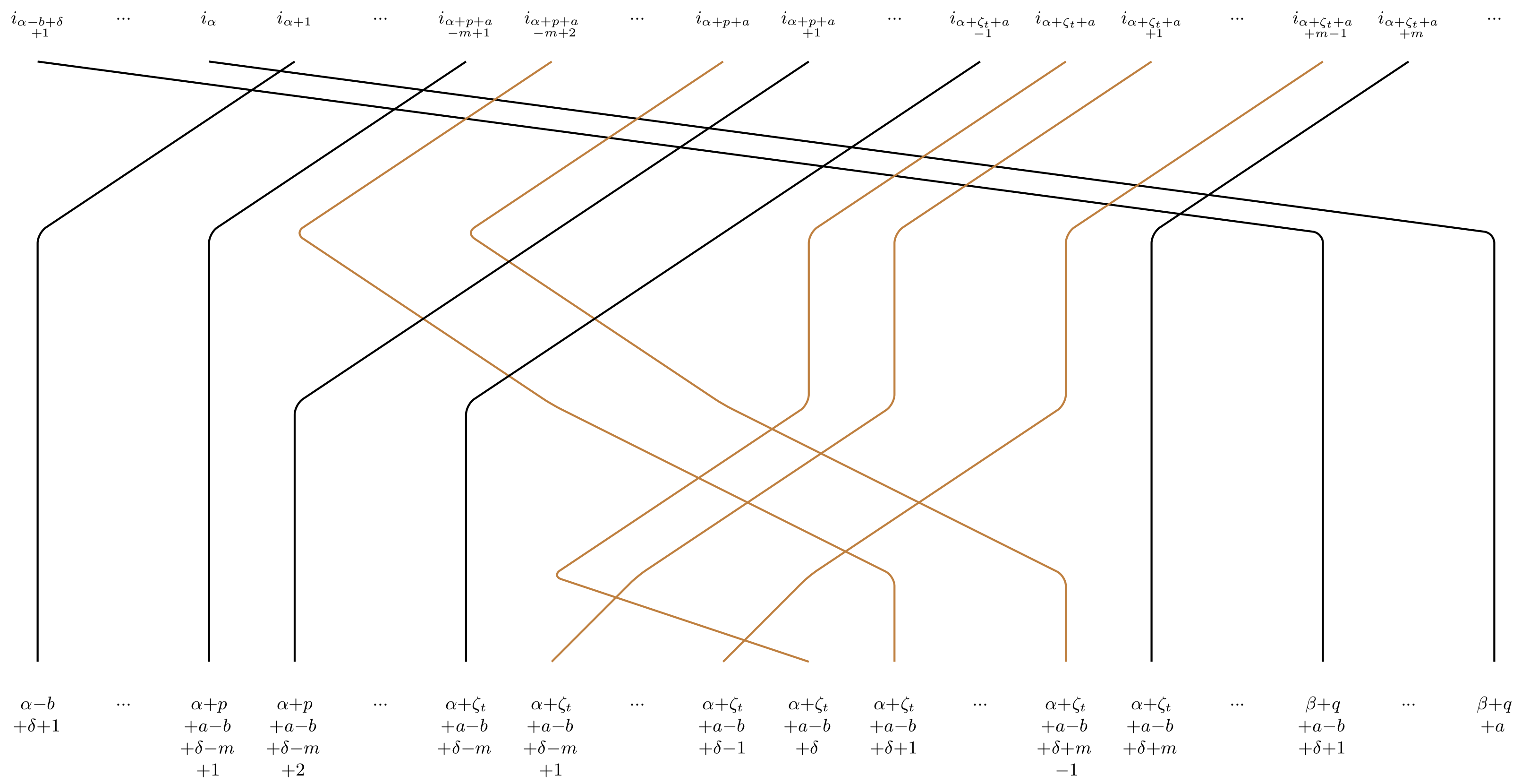


Figure 3.31: Part of the braid diagram obtained after applying Corollary 2.6 in (3.41). When $m > 1$, the strings to which we apply Corollary 2.10 are coloured brown.

to obtain a sum of terms which have the crossings $\psi_{\alpha+\zeta_t+a-1+j}$ at the top for $j \in \{1,\ldots,m\}$. In the latter case, we do the same but with the crossing $\psi_{\alpha+\zeta_t+a-b+\delta+m-1}$ coming from $\left(\Psi_{\alpha+\zeta_t+a-b+\delta+m-1}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta+1}$. In either case, the crossings we obtain at the top of the diagram are all row relations, since they will all occur to the left of the node containing $\alpha+p+a+z$ in $\mathfrak{t}^\mu$, and so all the corresponding terms are zero.

So instead suppose that we have a term in (3.40) where $\zeta_t = p+z-m$. Then (3.39) is equal to

$$
\begin{aligned}
v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
\cdot\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-m+1}\\
\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-m-1}\right)\downarrow_{\alpha-b-m+2}\cdot\Psi_{\alpha-b-m+1}\uparrow^{\alpha+p+z-b-2m-1}\\
\cdot\Psi_{\alpha-b-m}\uparrow^{\alpha+p+z-b-2m-2}\cdot\Psi_{\alpha+p+z-b-2m+1}\uparrow^{\beta-b-1-m}\\
\cdot\left(\Psi_{\alpha-b-m-1}\uparrow^{\beta-b-2-m}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
$$

Rearrange terms and apply Lemma 2.8 to

$$
\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-1-m}\right)\downarrow_{\alpha-b-m+2}\cdot\Psi_{\alpha+p+z-b-2m+1}\uparrow^{\beta-b-1-m}
$$

(take $x=\alpha-b-m+1$, $f=\delta+m-1$, $h=p+z-m-1$, $g=1$, $k=\beta-\alpha-p-z-\delta$)

since $i_{\alpha-b-m+2}, \ldots, i_{\alpha-b+\delta} \neq i_{\alpha+p+a+z-m}$, so that we have

$$\begin{aligned}
v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-m+1} \\
&\cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-m-2}\right)\downarrow_{\alpha-b-m+2} \cdot \Psi_{\alpha+p+z-b+\delta-m}\uparrow^{\beta-b-1-m} \\
&\cdot \Psi_{\alpha-b-m+1}\uparrow^{\alpha+p+z-b-2m-1} \cdot \Psi_{\alpha-b-m}\uparrow^{\alpha+p+z-b-2m-2} \\
&\cdot \left(\Psi_{\alpha-b-m-1}\uparrow^{\beta-b-2-m}\right)\downarrow_{\alpha-a_2+1} \cdot R \\
= v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-(m+1)+1} \\
&\cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z-b+\delta-(m+1)-1}\right)\downarrow_{\alpha-b-(m+1)+1} \\
&\cdot \left(\Psi_{\alpha-b-(m+1)}\uparrow^{\beta-b-1-(m+1)}\right)\downarrow_{\alpha-a_2+1} \cdot R.
\end{aligned}$$

So we are able to repeat the above process multiple times for successive values of m, assuming we have relevant ζ_t (otherwise we obtain zero and are done). This leaves us with

$$\begin{aligned}
v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\
&\cdot \left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\
&\cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R.
\end{aligned} \tag{3.42}$$

If we did have relevant ζ_t, then we have that

$$i_{\alpha+p+a+z+b-a_2+1} \leftarrow i_{\alpha+p+a+z+b-a_2+2} \leftarrow \cdots \leftarrow i_{\alpha+p+a+z-1}$$

and that the nodes corresponding to these residues all belong to the same row of $\mathfrak{t}^\mu$. Using the diagonal residue condition, we know that $i_{\alpha+p+a+z+b-a_2+1} = i_{\alpha-a_2+1}$ implies that the node containing $\alpha+p+a+z+b-a_2+1$ in $\mathfrak{t}^\mu$ is a Garnir node for $[\mu]$. So there will be some $\gamma \in \{1, \dots, a_2 - b - 1\}$ such that the node containing $\alpha + p + a + z - \gamma$ in $[\mu]$ is a Garnir node whilst the node directly to the right of it is not. We show some of the braid diagram for (3.42) in Figure 3.32.

Write $(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2})\downarrow_{\alpha+p+z+\delta-a_2+1}$ as

$$\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-\gamma+1}$$
$$\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-1}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}$$

and apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}$$
$$\cdot\left(\Psi_{\alpha+p+z-b+\delta-1}\uparrow^{\beta-b-2}\right)\downarrow_{\alpha+p+z-b+\delta-\gamma+1}$$

(take $x = \alpha + p - b + \delta$, $f = a$, $k = z - \gamma$, $h = \gamma - 1$, $g = \beta - \alpha - p - z - \delta$, $t = q$) since $i_{\alpha+p+a+z-j} \not\to i_{\alpha+p+a+z+\delta+1}, i_{\alpha+p+a+z+\delta+2}, \dots, i_{\beta+a}$ for $j \in \{1, \dots, \gamma - 1\}$. This gives us all together:

$$v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}$$
$$\cdot\left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+a-b-2}\right)\downarrow_{\alpha+p+a+z-b+\delta-\gamma+1}$$
$$\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}$$
$$\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-1}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}$$
$$\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$$

Now apply Corollary 2.6 to

$$\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a+z-b+\delta-1}\uparrow^{\beta+a-b-2}\right)\downarrow_{\alpha+p+a+z-b+\delta-\gamma+1}$$

(take $x = \alpha - b + \delta$, $f = b + \delta$, $k = p + a + z - \gamma$, $h = \gamma - 1$, $g = \beta - \alpha - p - z - \delta$, $t = q + \delta + 1$) since $i_{\alpha+p+a+z-j} \not\to i_{\alpha+p+a+z+\delta+1}, i_{\alpha+p+a+z+\delta+2}, \dots, i_{\beta+a}$ for

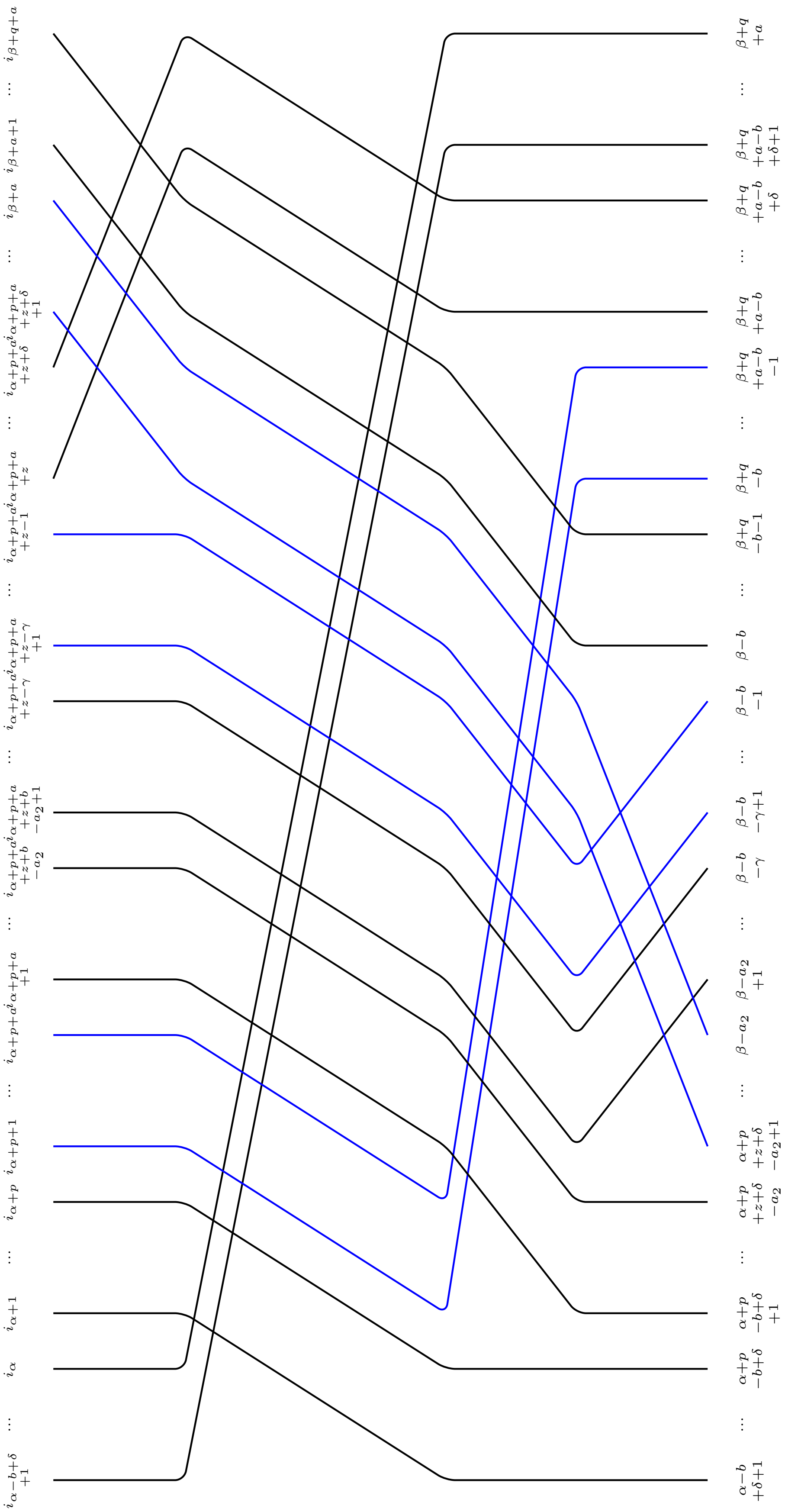


Figure 3.32: Part of the braid diagram for (3.42) excluding $\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$. The strings to which we apply Corollary 2.6 are coloured blue.

$j \in \{1, \ldots, \gamma - 1\}$. Then we have

$$\begin{aligned}v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\ &\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\ &\qquad\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\ &\qquad\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-1}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\ &\qquad\qquad\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.\end{aligned} \tag{3.43}$$

Some of this is shown in Figure 3.33.

Let the value of the node underneath the one containing $\alpha + p + a + z - \gamma$ be η. Figure 3.34 helps to illustrate the positions of certain nodes including the one containing η. We have that $i_{\alpha+p+a+z-\gamma} = i_{\alpha-b-\gamma}$ and $i_\eta = i_{\alpha+p+a-\gamma}$. Write $\left(\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-1}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}$ as

$$\begin{aligned}&\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\eta-b-\gamma-a-2}\cdot\Psi_{\eta-b-\gamma-a-1}\uparrow^{\beta-b-\gamma-1} \\ &\qquad\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\end{aligned}$$

and then as we have that $i_{\alpha+p+a+z-\gamma} \not\leftarrow i_{\alpha+p+a+z+\delta+1}, \ldots, i_{\eta-1}$, we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\Psi_{\alpha+p+z-b+\delta-\gamma}\uparrow^{\eta-b-\gamma-a-2}$$

(take $x = \alpha + p - b + \delta$, $f = a$, $k = z - \gamma - 1$, $h = 1$, $g = \eta - \alpha - p - a - z - \delta - 1$, $t = \beta + a - \eta + \gamma + q$). This gives us

$$\begin{aligned}v^{\mathfrak{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\ &\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\ &\cdot\Psi_{\alpha+p+a+z-b+\delta-\gamma}\uparrow^{\eta-b-\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\ &\cdot\Psi_{\eta-b-\gamma-a-1}\uparrow^{\beta-b-\gamma-1}\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\ &\qquad\qquad\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.\end{aligned} \tag{3.44}$$

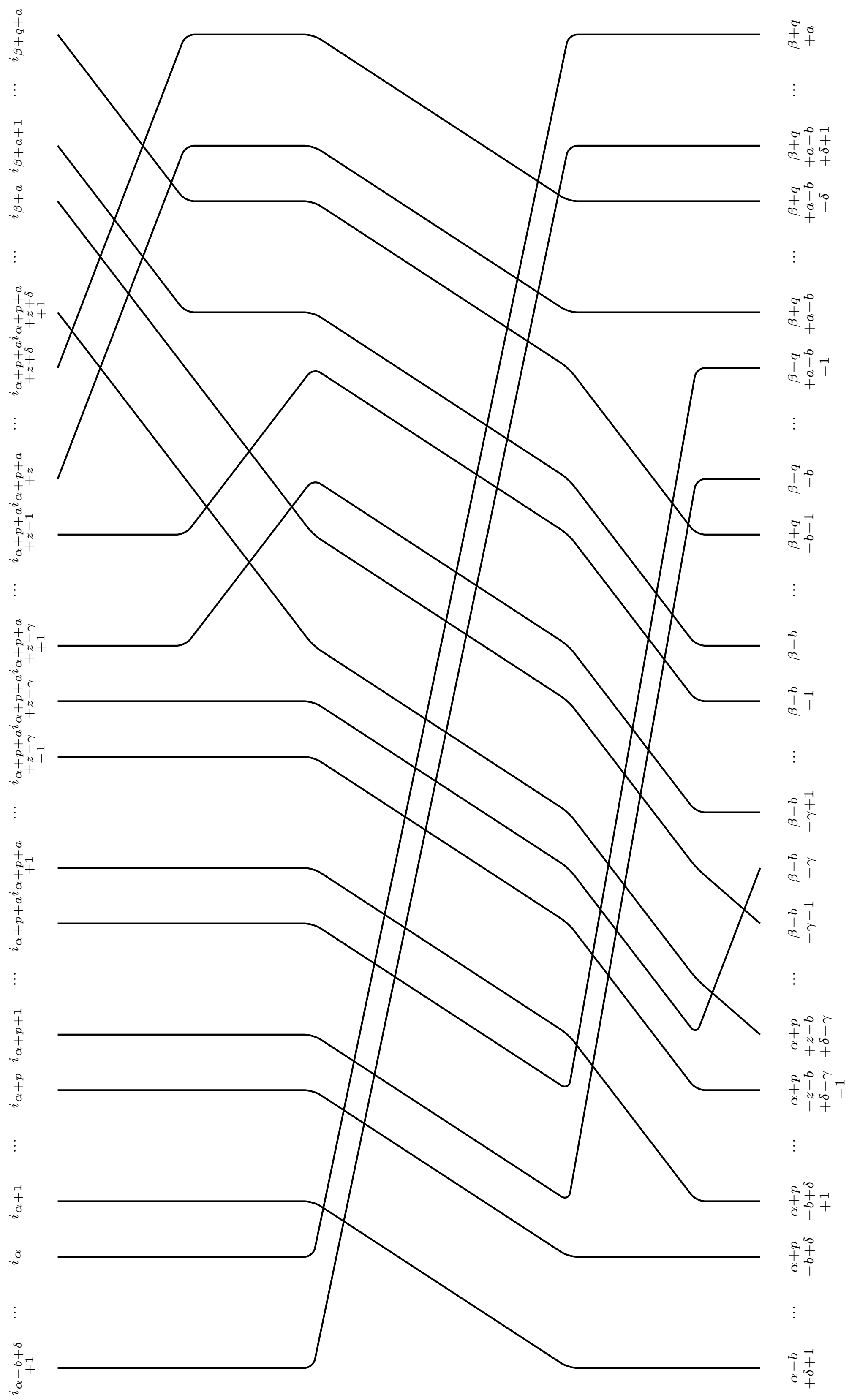


Figure 3.33: Part of the braid diagram for (3.43) excluding $\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$.

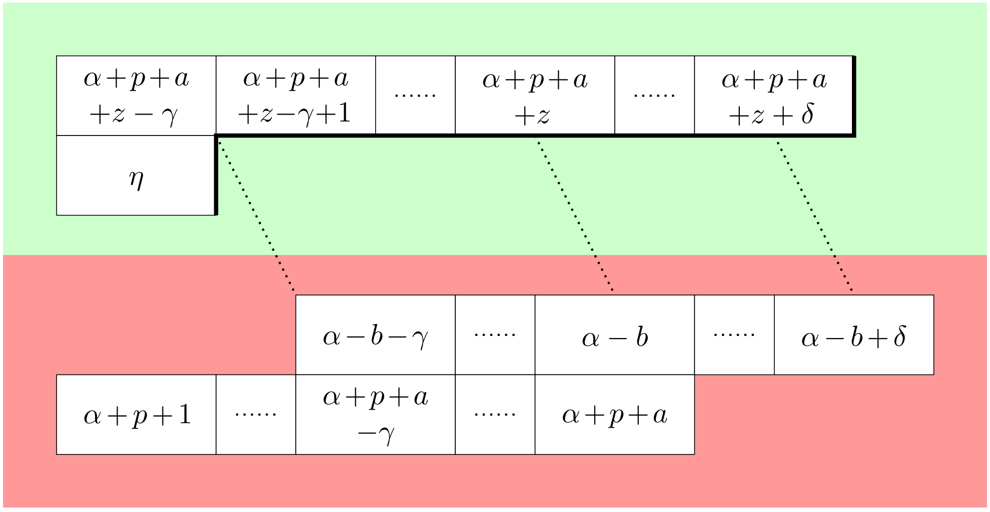


Figure 3.34: Diagram to show equality of residues between the different components of μ with the introduction of η. The top half shows nodes in the second component of $\mathfrak{t}^\mu$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue. The bold line along the top nodes illustrates the border of the component.

Now write $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1}$ as

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma}$$
$$\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\alpha+p-b+\delta+1}$$

and then since $i_{\alpha+p+a+z-\gamma} \not\to i_{\alpha+p+1}, \dots, i_{\alpha+p+a-\gamma-1}$ we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\alpha+p-b+\delta+1}\cdot\Psi_{\eta-b-\gamma-a-1}\uparrow^{\beta-b-\gamma-1}$$

(take $x=\alpha+p-b+\delta$, $f=a-\gamma-1$, $k=\eta-\alpha-p-a-\gamma-\delta-2$, $h=1$, $g=\beta+a-\eta+1$, $t=\gamma+q-1$). The use of this corollary is demonstrated in

Figure 3.35. So then we have

$$
\begin{aligned}
v^{t^{\mu}} & \left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\
& \cdot \left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\
& \cdot \Psi_{\alpha+p+a+z-b+\delta-\gamma}\uparrow^{\eta-b-\gamma-2} \cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma} \\
& \cdot \Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-2} \cdot \left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\alpha+p-b+\delta+1} \\
& \cdot \left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\
& \cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R.
\end{aligned} \tag{3.45}
$$

Writing $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma} \cdot \Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-2}$ as

$$
\begin{aligned}
& \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\eta-b-\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma} \cdot \left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\eta-b-2\gamma-2} \\
& \qquad \cdot \Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-2} \\
= & \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\eta-b-\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma} \\
& \cdot \left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\eta-b-2\gamma-1} \cdot \psi_{\eta-b-2\gamma-2}\psi_{\eta-b-2\gamma-1}\psi_{\eta-b-2\gamma-2} \\
& \cdot \Psi_{\eta-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2} \cdot \Psi_{\eta-b-2\gamma-1}\uparrow^{\beta+a-b-2\gamma-2}
\end{aligned}
$$

we can then use the braid relation (1.11) on $\psi_{\eta-b-2\gamma-2}\psi_{\eta-b-2\gamma-1}\psi_{\eta-b-2\gamma-2}$ since $i_{\alpha+p+a-\gamma} \leftarrow i_{\alpha+p+a+z-\gamma} \rightarrow i_{\eta}$.

We now obtain a sum of two terms, one where we replace the crossings $\psi_{\eta-b-2\gamma-2}\psi_{\eta-b-2\gamma-1}\psi_{\eta-b-2\gamma-2}$ with $\psi_{\eta-b-2\gamma-1}\psi_{\eta-b-2\gamma-2}\psi_{\eta-b-2\gamma-1}$ and one where these crossings disappear. We will deal with each term separately. Consider the former case, then instead of just replacing these crossings we could apply Corollary 2.6 to

$$
\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma} \cdot \Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-2}
$$

in (3.45) since $i_{\alpha+p+a+z-\gamma} \not\rightarrow i_{\eta+1},\dots,i_{\beta+a}$ and ignoring the fact that $i_{\alpha+p+a+z-\gamma} \rightarrow i_{\eta}$ (take $x = \alpha+p+a-b+\delta-\gamma-1$, $f = \gamma+1$, $k = \eta-\alpha-p-a-\gamma-\delta-2$, $h = 1$, $g = \beta+a-\eta+1$, $t = \gamma+q-1$),

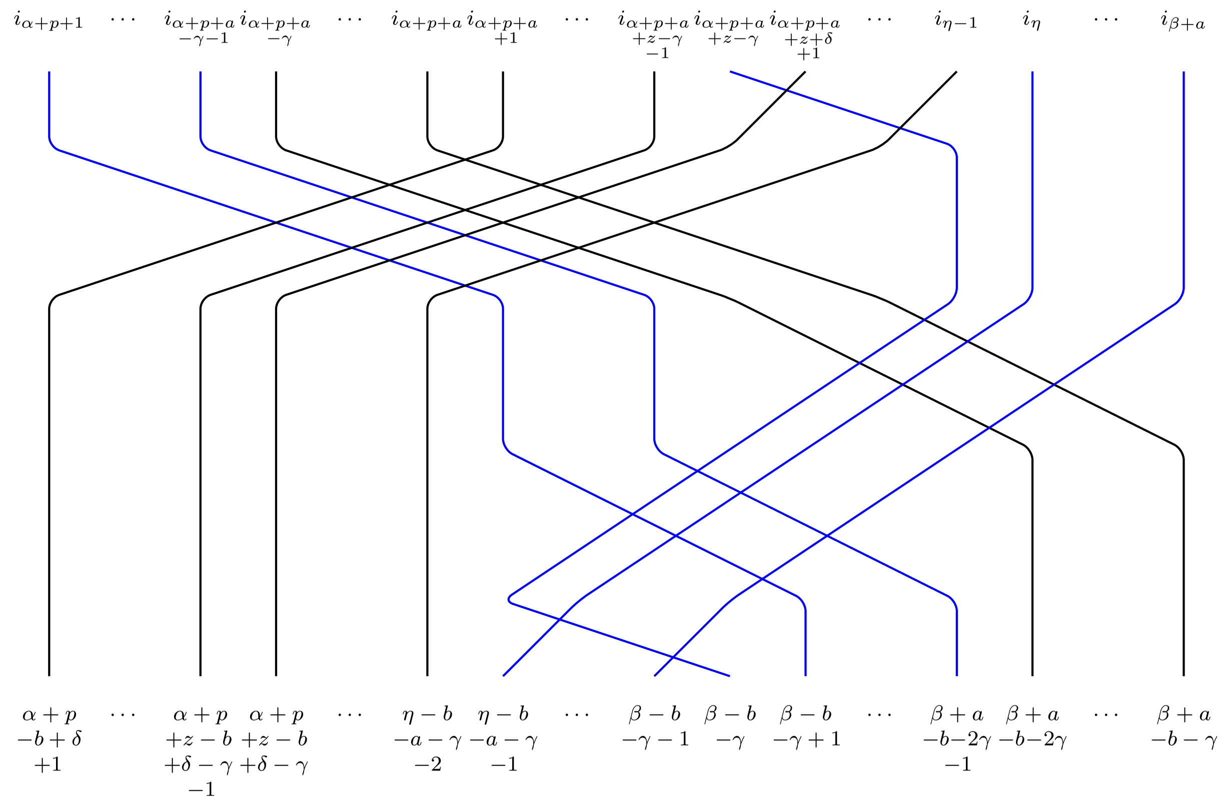


Figure 3.35: Braid diagram showing some crossings in (3.44) which are relevant for the application of Corollary 2.6. The strings to which we apply the corollary are coloured blue.

giving us

$$v^{\mathrm{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot\Psi_{\alpha+p+a+z-b+\delta-\gamma}\uparrow^{\beta+a-b-\gamma-1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R. \tag{3.46}$$

Some of this is shown in Figure 3.36.

Now apply Corollary 2.6 to

$$\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\Psi_{\alpha+p+a+z-b+\delta-\gamma}\uparrow^{\beta+a-b-\gamma-1}$$

since $i_{\alpha+p+a+z-\gamma} \nrightarrow i_{\alpha-b+\delta+1},\ldots,i_\alpha$ (take $x=\alpha-b+\delta$, $f=b-\delta$, $k=p+a+z-\gamma-1$, $h=1$, $g=\beta-\alpha-p-z-\delta$, $t=\gamma+q+\delta$). Then we have

$$v^{\mathrm{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma} \cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R. \tag{3.47}$$

Note that this means that in terms of the diagram we have the crossings $\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+a-1}\right)\downarrow_{\alpha+p+a+z-\gamma}$ at the top, and this will certainly contain the Garnir relation corresponding to the Garnir node containing $\alpha+p+a+z-\gamma$ in $[\mu]$. So (3.47) will be zero.

So now consider the other term arising from the application of braid relation

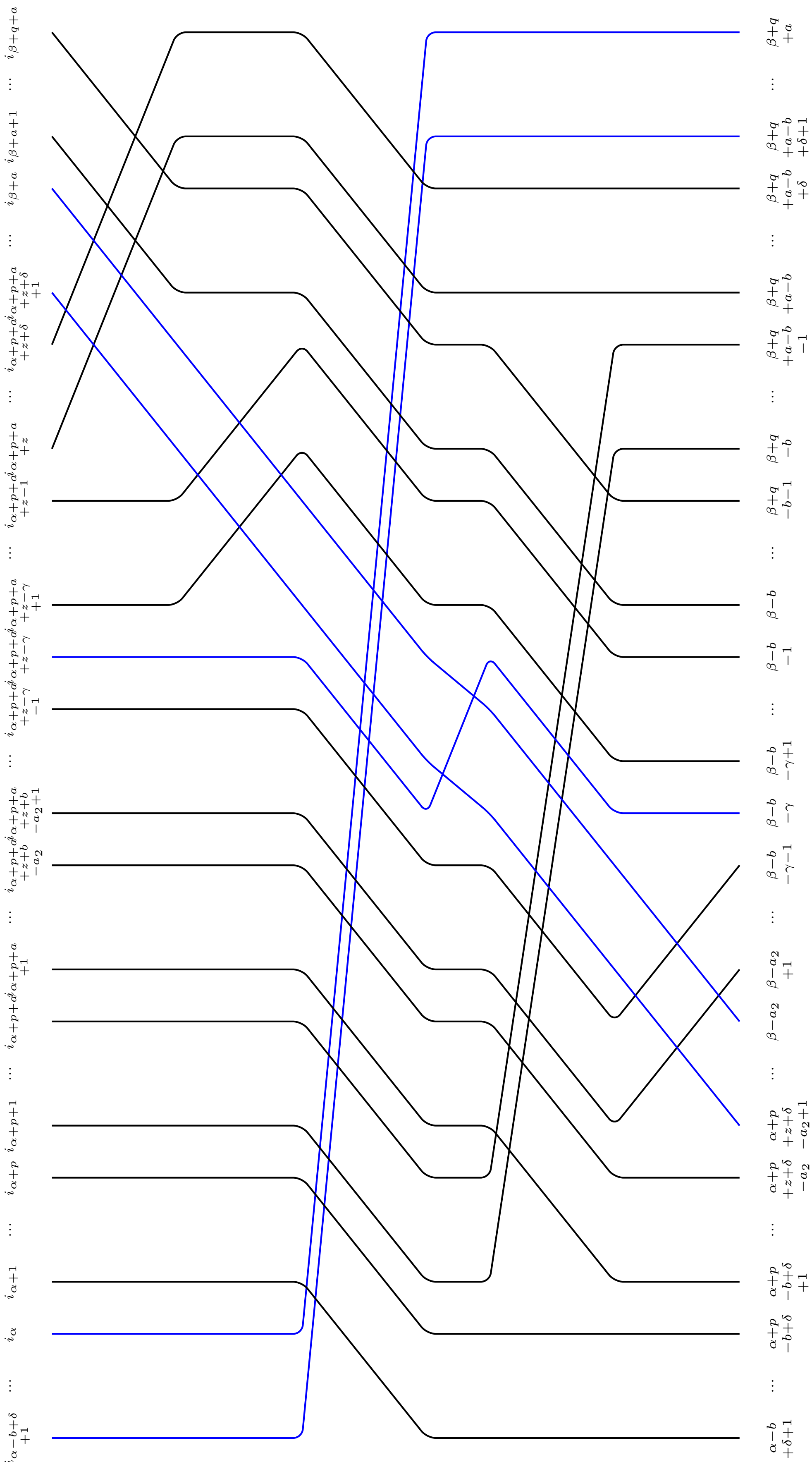


Figure 3.36: Part of the braid diagram for (3.46) excluding $\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$. The strings to which we apply Corollary 2.6 are coloured blue.

(1.11) to (3.45). This is

$$
\begin{aligned}
&v^{\mathfrak{t}^{\mu}}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\quad\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\quad\cdot\Psi_{\alpha+p+a+z-b+\delta-\gamma}\uparrow^{\eta-b-\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\eta-b-\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma}\\
&\qquad\cdot\left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\eta-b-2\gamma-1}\cdot\Psi_{\eta-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\\
&\cdot\Psi_{\eta-b-2\gamma-1}\uparrow^{\beta+a-b-2\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\qquad\qquad\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\qquad\qquad\qquad\qquad\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
\tag{3.48}
$$

We show most of the braid diagram for (3.48) in Figure 3.37. Write $\Psi_{\eta-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}$ as $\Psi_{\eta-b-2\gamma}\uparrow^{\beta+a-b-2\gamma-1}\cdot\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}$ and then since $i_{\alpha+p+a-\gamma+j}\neq i_{\eta+1},\dots,i_{\beta+a}$ for $j\in\{1,\dots,\gamma\}$ apply Corollary 2.6 to

$$\left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\eta-b-2\gamma-1}\cdot\Psi_{\eta-b-2\gamma}\uparrow^{\beta+a-b-2\gamma-1}$$

(take $x=\eta-b-2\gamma-2$, $f=\gamma$, $k=1$, $h=1$, $g=\beta+a-\eta$, $t=\gamma+q-1$). Then since $i_{\alpha-b+\delta+j}\neq i_{\eta+1},\dots,i_{\beta+a}$ for $j\in\{1,\dots,b-\delta\}$ apply Corollary 2.6 to

$$\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\Psi_{\eta-b-\gamma}\uparrow^{\beta+a-b-\gamma-1},$$

then as $i_{\alpha+p+a+z-\gamma+1},\dots,i_{\alpha+p+a+z-1}\not\leftarrow i_{\eta}$ apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\eta-\delta-\gamma}\uparrow^{\beta+a-\delta-\gamma-1}$$

and then finally as $i_{\alpha+p+a+z},\dots,i_{\alpha+p+a+z+\delta}\not\leftarrow i_{\eta}$ apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\Psi_{\eta-\delta-1}\uparrow^{\beta+a-\delta-2}.$$

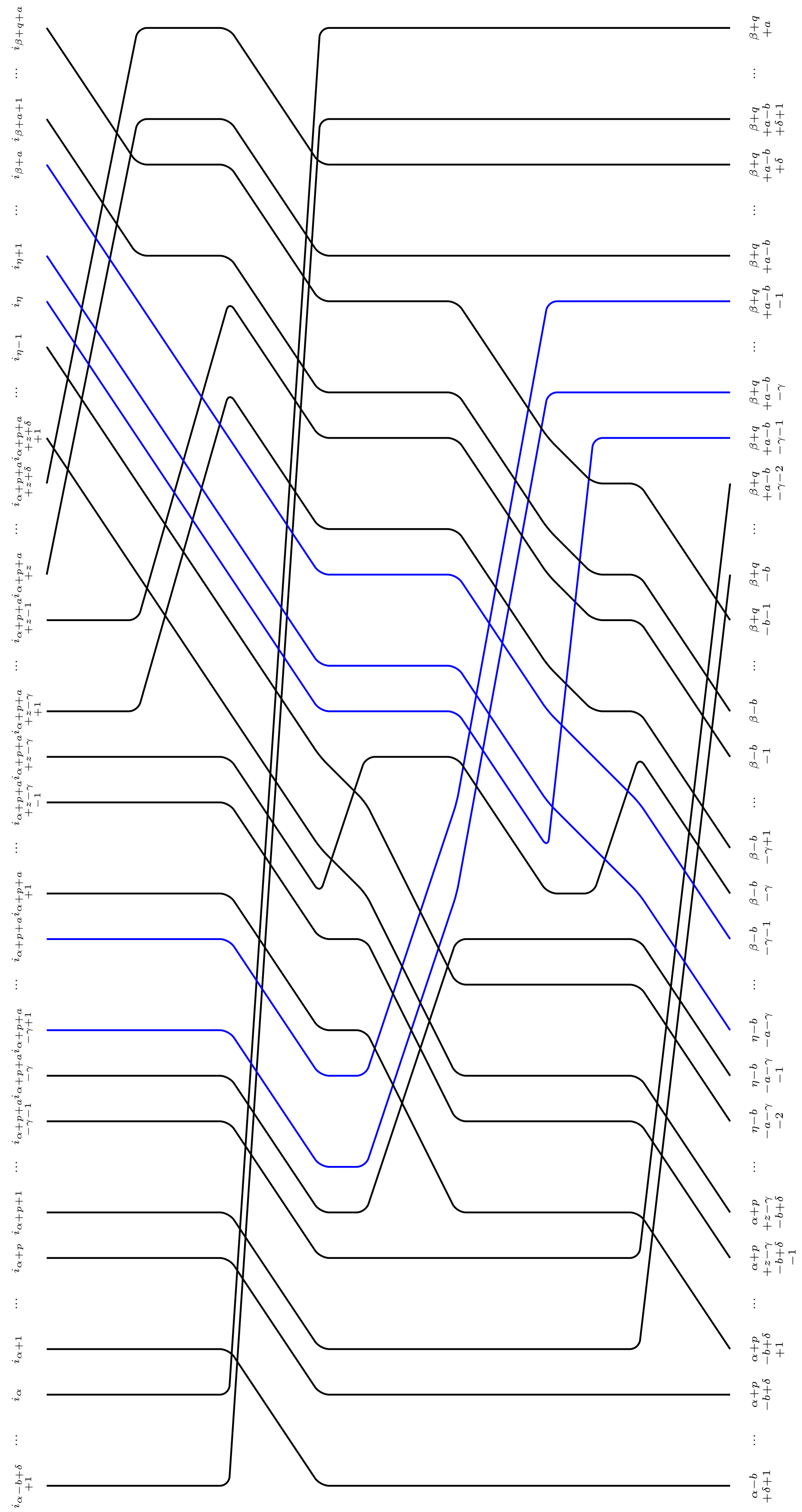


Figure 3.37: Part of the braid diagram for (3.48) excluding $\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$. The strings to which we apply Corollary 2.6 are coloured blue.

Then we have

$$
\begin{aligned}
&v^{\mathfrak{t}^{\mu}}\Psi_{\eta}{\uparrow}^{\beta+a-1}\cdot\left(\Psi_{\alpha+p+a+z+\delta}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}{\uparrow}^{\beta+a-\delta-2}\right){\downarrow}_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha-b+\delta+1}\\
&\cdot\ \Psi_{\alpha+p+a+z-b+\delta-\gamma}{\uparrow}^{\eta-b-\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta}{\uparrow}^{\eta-b-\gamma-3}\right){\downarrow}_{\alpha+p+a-b+\delta-\gamma}\\
&\cdot\left(\Psi_{\eta-b-\gamma-2}{\uparrow}^{\beta+q+a-b-2}\right){\downarrow}_{\eta-b-2\gamma-1}\cdot\Psi_{\beta+a-b-2\gamma}{\uparrow}^{\beta+q+a-b-\gamma-2}\\
&\cdot\Psi_{\eta-b-2\gamma-1}{\uparrow}^{\beta+a-b-2\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}{\uparrow}^{\beta+q+a-b-\gamma-3}\right){\downarrow}_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}{\uparrow}^{\beta-b-\gamma-2}\right){\downarrow}_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}{\uparrow}^{\alpha+p+z+\delta-1-a_2}\right){\downarrow}_{\alpha-a_2+1}\cdot R.
\end{aligned}
$$

Now apply Corollary 2.6 to

$$
\left(\Psi_{\eta-b-\gamma-2}{\uparrow}^{\beta+q+a-b-2}\right){\downarrow}_{\eta-b-2\gamma-1}\cdot\Psi_{\eta-b-2\gamma-1}{\uparrow}^{\beta+a-b-2\gamma-2}
$$

since $i_{\alpha+p+a+z-\gamma} \not\rightarrow i_{\eta+1},\dots,i_{\beta+a}$ (take $x=\eta-b-2\gamma-2$, $f=\gamma$, $k=0$, $h=1$, $g=\beta+a-\eta$, $t=\gamma+q$), giving us

$$
\begin{aligned}
&v^{\mathfrak{t}^{\mu}}\Psi_{\eta}{\uparrow}^{\beta+a-1}\cdot\left(\Psi_{\alpha+p+a+z+\delta}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}{\uparrow}^{\beta+a-\delta-2}\right){\downarrow}_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\alpha}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha-b+\delta+1}\\
&\cdot\ \Psi_{\alpha+p+a+z-b+\delta-\gamma}{\uparrow}^{\beta+a-b-\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta}{\uparrow}^{\eta-b-\gamma-3}\right){\downarrow}_{\alpha+p+a-b+\delta-\gamma}\\
&\cdot\left(\Psi_{\eta-b-\gamma-2}{\uparrow}^{\beta+q+a-b-2}\right){\downarrow}_{\eta-b-2\gamma-1}\cdot\Psi_{\beta+a-b-2\gamma}{\uparrow}^{\beta+q+a-b-\gamma-2}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}{\uparrow}^{\beta+q+a-b-\gamma-3}\right){\downarrow}_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}{\uparrow}^{\beta-b-\gamma-2}\right){\downarrow}_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}{\uparrow}^{\alpha+p+z+\delta-1-a_2}\right){\downarrow}_{\alpha-a_2+1}\cdot R.
\end{aligned}
$$

Then we can apply Corollary 2.6 to

$$
\left(\Psi_{\alpha}{\uparrow}^{\beta+q+a-1}\right){\downarrow}_{\alpha-b+\delta+1}\cdot\Psi_{\alpha+p+a+z-b+\delta-\gamma}{\uparrow}^{\beta+a-b-\gamma-2}
$$

since $i_{\alpha+p+a+z-\gamma} \not\rightarrow i_{\alpha+p+a+z+\delta+1},\dots,i_{\eta-1},i_{\eta+1},\dots,i_{\beta+a}$ (take $x=\alpha-b+\delta$,

$f = b-\delta$, $k = p+a+z-\gamma-1$, $h = 1$, $g = \beta-\alpha-p-z-\delta-1$, $t = \gamma+q+\delta+1$). Then we have

$$
\begin{aligned}
&v^{\mathrm{t}^\mu}\Psi_\eta\uparrow^{\beta+a-1}\cdot\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\\
&\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\eta-b-\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma}\\
&\cdot\left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\eta-b-2\gamma-1}\cdot\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
\tag{3.49}
$$

Some of the above is shown in Figure 3.38.

Take $d \in \{0, 1, \ldots, a-\gamma-2\}$ to be maximal such that the node containing $\eta-d$ in $[\mu]$ is in the same row as the node containing η, whilst the node containing $\eta-d-1$ is not. If such a d does not exist, let $d = a-\gamma-1$. Then $i_{\eta-d} = i_{\alpha+p+a-\gamma-d}$. We illustrate some relevant residues in Figure 3.39. Write (3.49) as

$$
\begin{aligned}
&v^{\mathrm{t}^\mu}\Psi_\eta\uparrow^{\beta+a-1}\cdot\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\\
&\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\\
&\cdot\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\cdot\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-3}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\eta-b-2\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma-d}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\eta-b-a-\gamma}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R
\end{aligned}
\tag{3.50}
$$

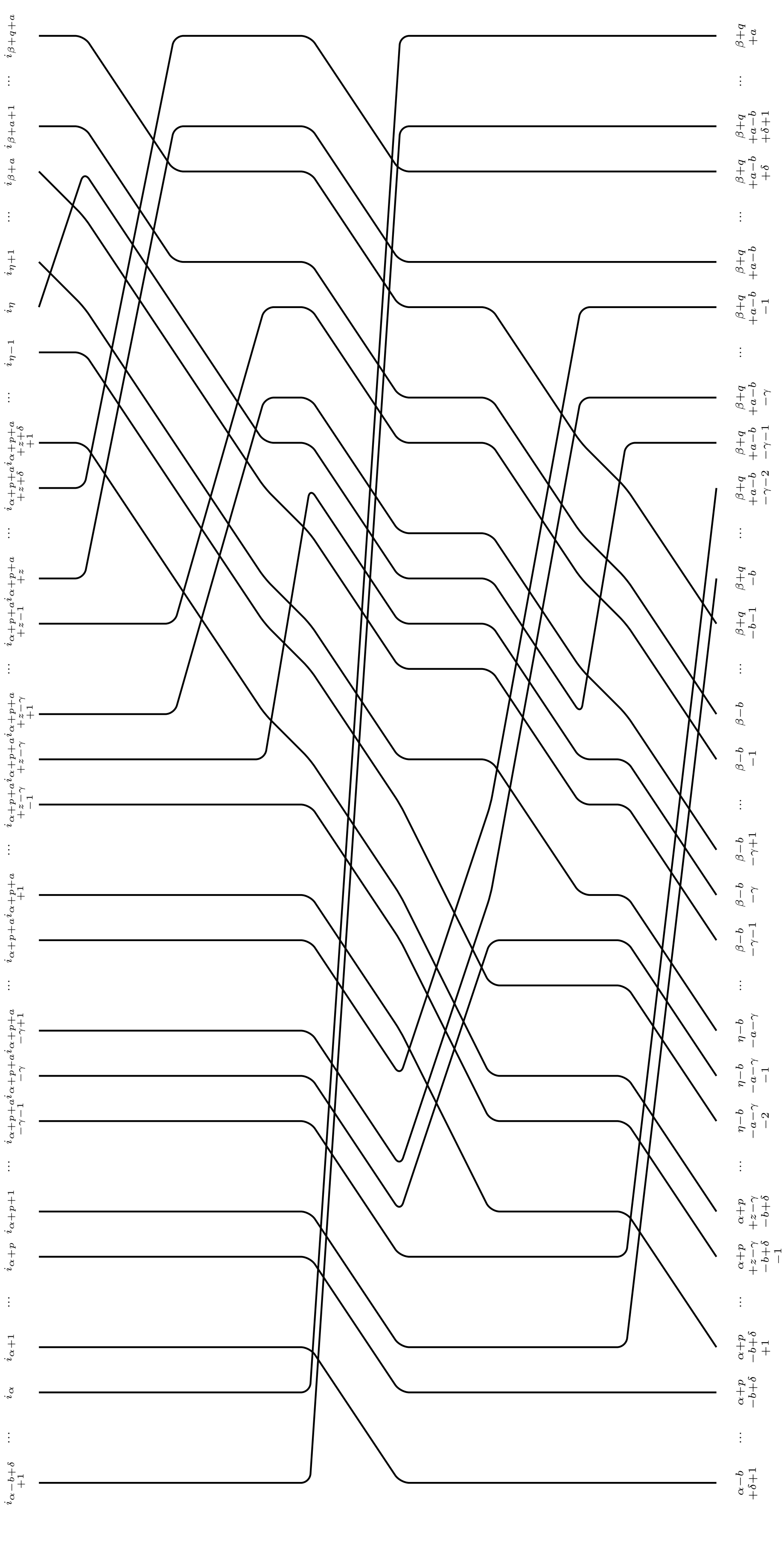


Figure 3.38: Part of the braid diagram for (3.49) excluding $\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R$.

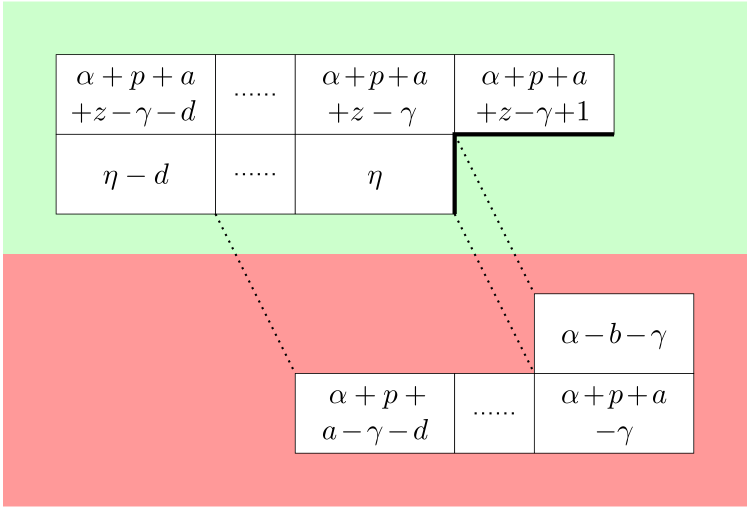


Figure 3.39: Diagram to show equality of residues between the different components of μ with the introduction of d. The top half shows nodes in the second component of $\mathfrak{t}^\mu$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue. The short bold line along the top nodes illustrates the border of the component.

so that we can apply Lemma 2.9 to

$$\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-3}\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-1}\uparrow^{\eta-b-2\gamma-3}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma-d} \tag{3.51}$$

since

$$\begin{aligned} &i_{\alpha+p+a-\gamma-d}\leftarrow i_{\alpha+p+a-\gamma-d+1}\leftarrow\cdots\\ &\quad\cdots\leftarrow i_{\alpha+p+a-\gamma-1}\leftarrow i_{\alpha+p+a-\gamma}\rightarrow i_{\eta-1}\rightarrow i_{\eta-2}\rightarrow\cdots\rightarrow i_{\eta-d}\end{aligned}$$

(take $x=\alpha+p+a-b+\delta-d-\gamma-1, f=d, g=\eta-\alpha-p-a-\delta-\gamma-d-2$). This is shown in Figure 3.40. So we must replace (3.51) with a large sum of terms, whose summands belong to three different types.

The first type are terms which begin with a $\psi_{\alpha+p+a-b+\delta-\gamma-1-s}$ crossing for $s\in\{0,\dots,d-1\}$. Then we can apply Corollary 2.6 to

$$\left(\Psi_\alpha\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\psi_{\alpha+p+a-b+\delta-\gamma-1-s}$$

since $i_{\alpha-b+\delta+1},\dots,i_\alpha\not\rightarrow i_{\alpha+p+j}$ for $j\in\{1,\dots,a-\gamma-1\}$ (take $x=\alpha-b+\delta$,

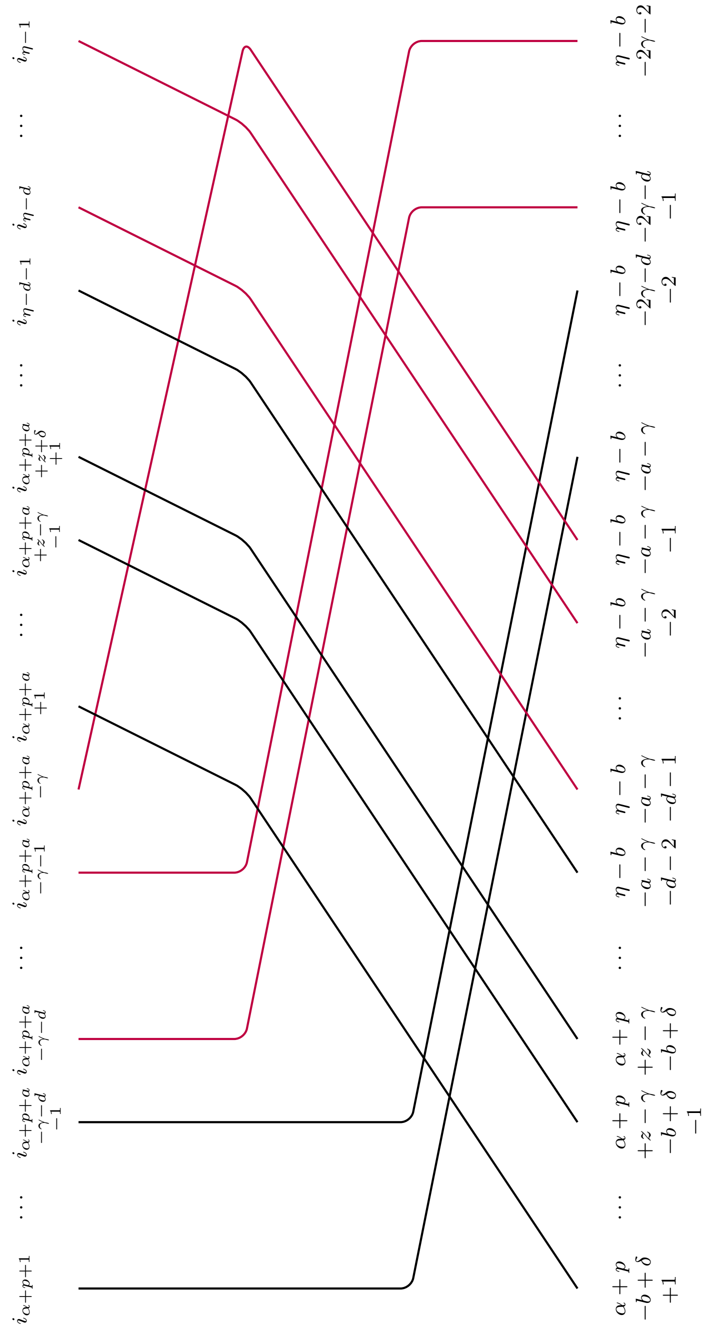


Figure 3.40: Braid diagram of the crossings from (3.51) along with $\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p-b+\delta+1}$ with the associated residues from (3.49). The strings to which we apply Lemma 2.9 are coloured purple.

$f = b - \delta$, $k = p + a - \gamma - 2 - s$, $h = 1$, $g = 1$, $t = \beta + q - \alpha - p + \gamma + s - 1$), giving us $\psi_{\alpha+p+a-\gamma-1-s}$ at the top of the diagram, which is a row relation and so terms of this type will be zero.

The second type are terms which begin with $\psi_{\alpha+p+a-b+\delta-\gamma+Z}$ where (C6) Z belongs to $\{1, \ldots, z - \gamma - 1\}$ and $i_{\alpha+p+a+Z} = i_{\alpha+p+a-\gamma-j}$ for some $j \in \{1, \ldots, d\}$ (note that $i_{\alpha+p+a+z+\delta+1}, \ldots, i_{\eta-d-1} \neq i_{\alpha+p+a-\gamma-j}$ for such j). Since $i_{\alpha+p+a-\gamma+Z} \not\to i_{\alpha+p+a-\gamma+1}, \ldots, i_{\alpha+p+a}$, we may apply Corollary 2.6 to $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \cdot\psi_{\alpha+p+a-b+\delta-\gamma+Z}$ and then since $i_{\alpha+p+a-\gamma+Z} \not\to i_{\alpha-b+\delta+1}, \ldots, i_{\alpha}$ we may apply Corollary 2.6 to $\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot\psi_{\alpha+p+a-b+\delta+Z}$ giving us $\psi_{\alpha+p+a+Z}$ at the top of the diagram, which is a row relation by the diagonal residue condition and so terms of this type will be zero.

So then all that is left is to consider the term where we have replaced (3.51) with

$$\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p+a-b+\delta-d-\gamma}\cdot$$

Overall we have that (3.50) is equal to

$$\begin{aligned}
v^{\mathfrak{t}^{\mu}}\Psi_{\eta}\uparrow^{\beta+a-1}\cdot&\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\\
&\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\\
&\cdot\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p+a-b+\delta-d-\gamma}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\eta-b-a-\gamma}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned} \tag{3.52}$$

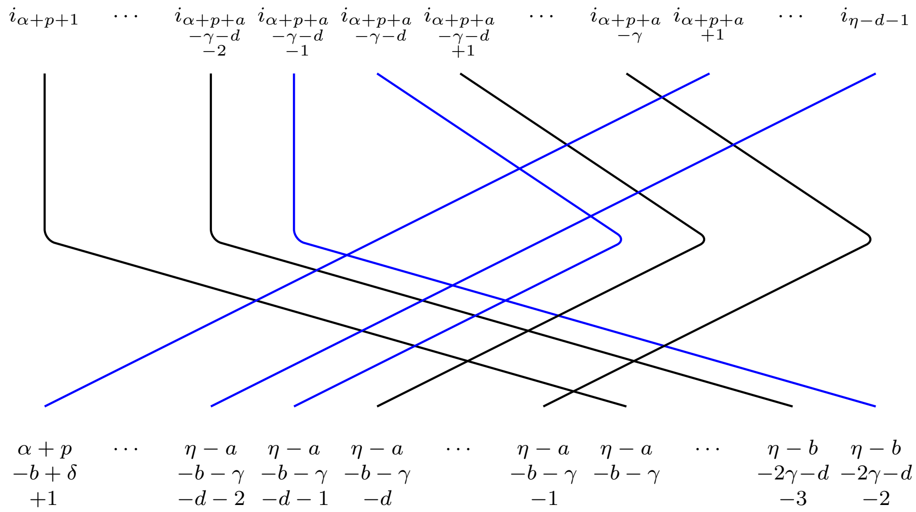


Figure 3.41: Braid diagram of the crossings in (3.53) with the associated residues from (3.52). The strings to which we apply Corollary 2.6 are coloured blue.

We show

$$\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p+a-b+\delta-d-\gamma} \cdot \left(\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p-b+\delta+1} \tag{3.53}$$

in Figure 3.41.

If $d \in \{0,1,\ldots,a-\gamma-2\}$ then $\eta-d = \alpha+p+a+z+\delta+1$ and

$$i_{\alpha+p+a-d-\gamma-1} \not\to i_{\alpha+p+a+1},\ldots i_{\alpha+p+a+z-\gamma-1}, i_{\alpha+p+a+z+\delta+1},\ldots,i_{\eta-d-1}$$

using the diagonal residue condition. Then we can apply Corollary 2.6 to (C7)

$$\Psi_{\alpha+p+a-b+\delta-d-\gamma}\uparrow^{\eta-b-2\gamma-2d-3}\cdot\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}$$

(take $x=\alpha+p+a-b+\delta-\gamma-d-2$, $f=1$, $k=0$, $h=1$, $g=\eta-\gamma-\alpha-p-a-d-\delta-2$, $t=d$). Then we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha}\uparrow^{\alpha-b+\delta+1}\right)\downarrow_{\beta+q+a-1}\cdot\psi_{\alpha+p+a-b+\delta-\gamma-d-1}$$

since $i_{\alpha+p+a-d-\gamma-1} \not\to i_{\alpha-b+\delta+1},\ldots,i_{\alpha}$ giving us $\psi_{\alpha+p+a-\gamma-d-1}$ at the top of the diagram, which is a row relation and so we have zero.

So instead suppose that we had to take $d = a - \gamma - 1$. We show a diagram for some of (3.52) when $d = a - \gamma - 1$ in Figure 3.42. Figure 3.43 illustrates the residues of some relevant nodes. If we suppose that $\eta < \beta + a$, then write $\left(\Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\eta-b-a-\gamma}$ as

$$\left(\Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-3}\right)\downarrow_{\eta-b-a-\gamma} \cdot \left(\Psi_{\beta+a-b-2\gamma-2}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\beta-b-\gamma} .$$

Note that

$$\begin{aligned} i_{\eta-a+\gamma+j} \not\sim i_{\alpha+p+a-\gamma+1}, \dots, i_{\alpha+p+a}, i_{\alpha-b+\delta+1}, \dots \\ \dots, i_{\alpha}, i_{\alpha+p+a+z-\gamma}, \dots, i_{\alpha+p+a+z+\delta} \end{aligned}$$

for $j \in \{1, \dots, a - \gamma - 1\}$. So we can apply Corollary 2.6 to

$$\begin{aligned} \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\ \cdot \left(\Psi_{\eta-b-2\gamma-2}\uparrow^{\beta+a-b-2\gamma-3}\right)\downarrow_{\eta-b-a-\gamma}, \end{aligned}$$

and then we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot \left(\Psi_{\eta-b-\gamma-2}\uparrow^{\beta+a-b-\gamma-3}\right)\downarrow_{\eta-b-a} .$$

Next, we can apply Corollary 2.6 to

$$\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2} \cdot \left(\Psi_{\eta-\delta-\gamma-2}\uparrow^{\beta+a-\delta-\gamma-3}\right)\downarrow_{\eta-a-\delta},$$

then to

$$\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1} \cdot \left(\Psi_{\eta-\delta-\gamma-1}\uparrow^{\beta+a-\delta-\gamma-2}\right)\downarrow_{\eta-a-\delta+1},$$

then to

$$\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \cdot \left(\Psi_{\eta-\delta-2}\uparrow^{\beta+a-\delta-3}\right)\downarrow_{\eta-a-\delta+\gamma},$$

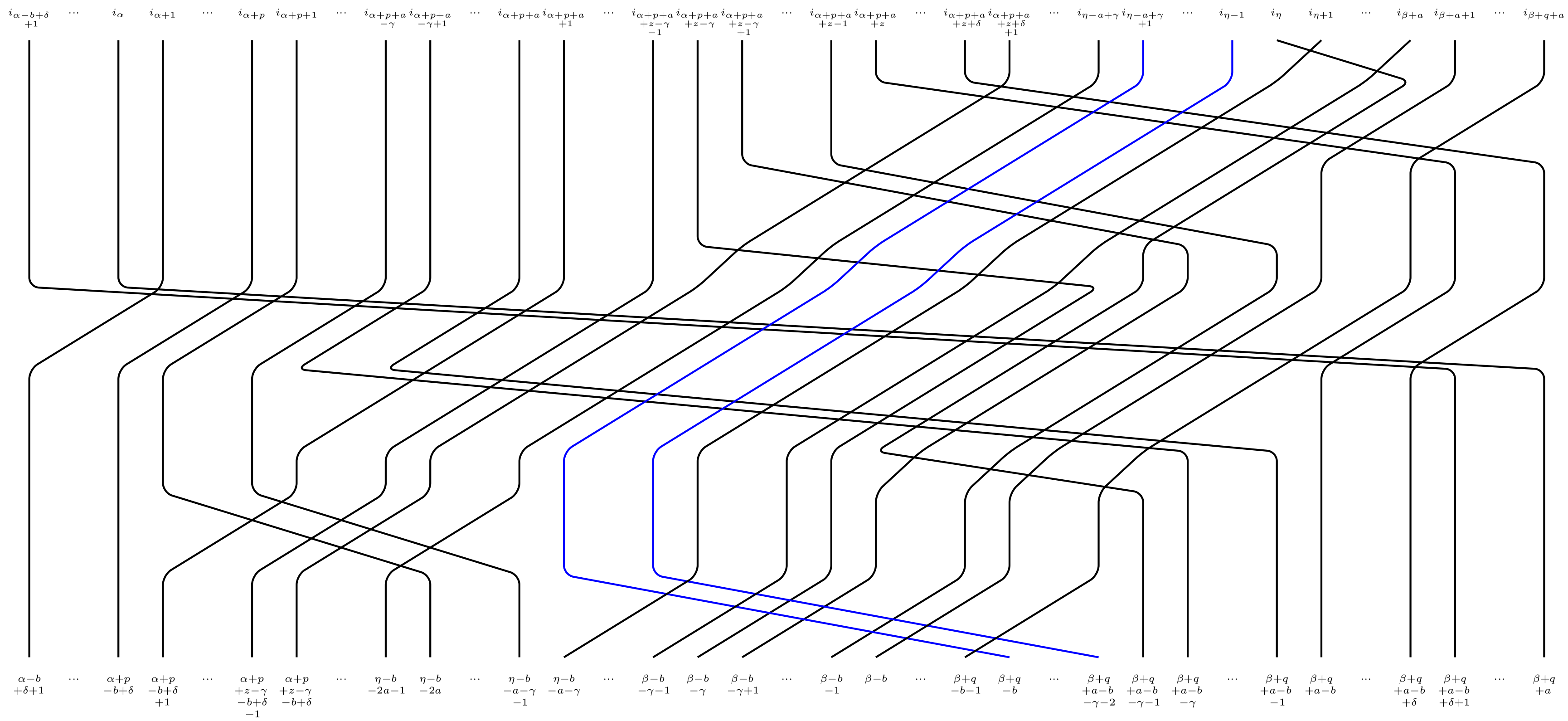


Figure 3.42: Part of the braid diagram for (3.52) when $d = a - \gamma - 1$, excluding $\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R$. The strings coloured blue are those which are pulled over multiple crossings using Corollary 2.6 to help obtain $\left(\Psi_{\eta}\uparrow^{\beta+a-1}\right)\downarrow_{\eta-a+\gamma+1}$.

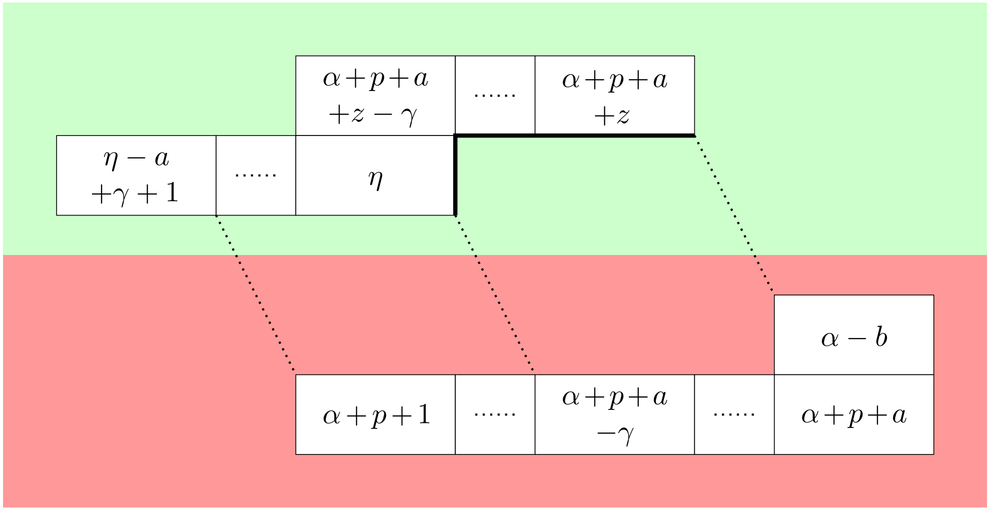


Figure 3.43: Diagram to show equality of residues between the different components of μ when $d = a - \gamma - 1$. The top half shows nodes in the second component of $\mathfrak{t}^{\mu}$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue. The bold line along the top nodes illustrates the border of the component.

and then this gives us

$$\Psi_{\eta}\!\uparrow^{\beta+a-1} \cdot \left(\Psi_{\eta-1}\!\uparrow^{\beta+a-2}\right)\downarrow_{\eta-a+\gamma+1} = \left(\Psi_{\eta}\!\uparrow^{\beta+a-1}\right)\downarrow_{\eta-a+\gamma+1}$$

at the top of the diagram. The node containing $\eta - a + \gamma + 1$ in $\mathfrak{t}^{\mu}$ must be a Garnir node since it has the same residue as $i_{\alpha+p+a-\gamma-d} = i_{\alpha+p+1}$, so we have the Garnir relation corresponding to this node at the top of the diagram and thus we have zero.

Finally we can suppose then that $\eta = \beta + a$. Figure 3.44 helps to illustrate some of the nodes and their associated residues in this case. Rewriting (3.52) in

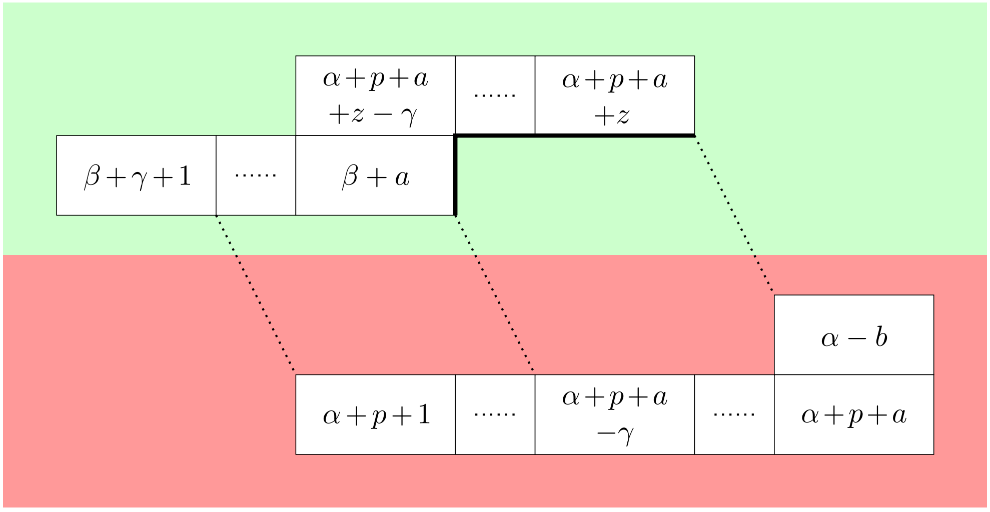


Figure 3.44: Diagram to show equality of residues between the different components of μ when $\eta = \beta + a$. The top half shows nodes in the second component of $\mathfrak{t}^{\mu}$ whilst the bottom shows nodes in the first, with the dotted lines connecting nodes of equal residue. The bold line along the top nodes illustrates the border of the component.

this case gives us the following:

$$\begin{aligned}
v^{\mathfrak{t}^{\mu}} & \left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\
& \cdot \left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1} \\
& \cdot \Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2} \cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\
& \cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\
& \cdot \Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2} \cdot \left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\
& \cdot \left(\Psi_{\beta+a-b-2\gamma-2}\uparrow^{\beta+q+a-b-\gamma-3}\right)\downarrow_{\beta-b-\gamma} \\
& \cdot \left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\
& \cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R
\end{aligned} \tag{3.54}$$

This is shown in Figure 3.45. By rearranging we have that (3.54) is equal to

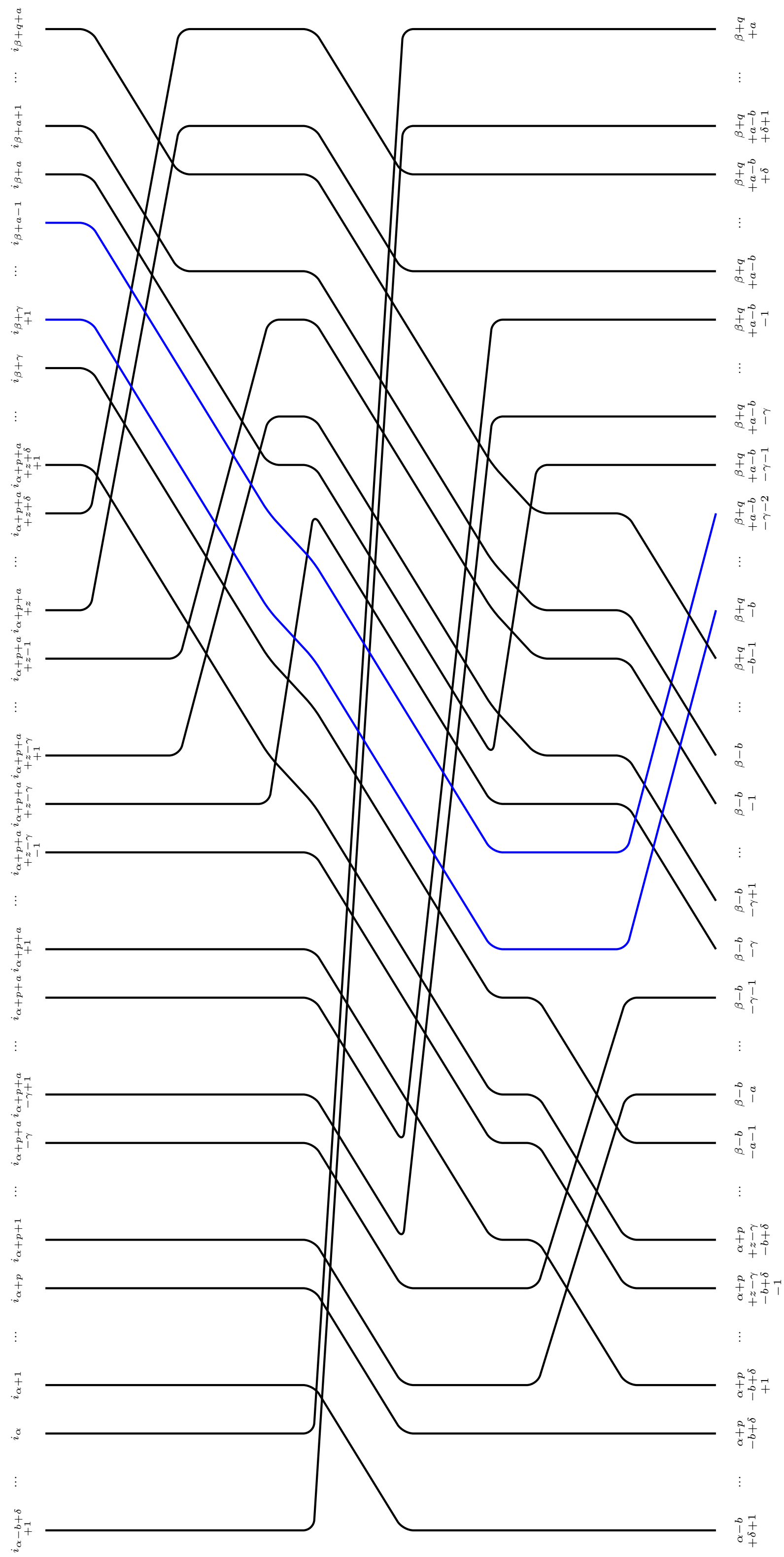


Figure 3.45: Part of the braid diagram for (3.54), excluding $(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2})\downarrow_{\alpha+p+z+\delta-a_2+1}\cdot(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2})\downarrow_{\alpha-a_2+1}\cdot R$. The strings coloured blue are those which we pull over multiple crossings using Corollary 2.6.

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\\
&\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\\
&\cdot\Psi^{\beta+a-b-2\gamma-2}\downarrow_{\beta-b-\gamma}\cdot\left(\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\right)\downarrow_{\beta-b-\gamma+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R
\end{aligned}
$$

and now we can apply Corollary 2.6 to

$$
\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\cdot\Psi^{\beta+a-b-2\gamma-2}\downarrow_{\beta-b-\gamma}
$$

(take $x=\alpha+p+a-b+\delta-\gamma$, $f=\gamma$, $k=\beta-\alpha-p-a-\delta-1$, $h=a-\gamma-1$, $g=1$, $t=\gamma+q$) since $i_{\alpha+p+a-\gamma+j}\not\to i_{\beta+\gamma+1},\dots,i_{\beta+a-1}$ for $j\in\{1,\dots,\gamma\}$. Then as $i_{\alpha-b+\delta+j}\not\to i_{\beta+\gamma+1},\dots,i_{\beta+a-1}$ for $j\in\{1,\dots,b-\delta\}$ we can also apply Corollary 2.6 to $\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\Psi^{\beta+a-b-\gamma-2}\downarrow_{\beta-b}$ and so (3.54) is equal to

$$
\begin{aligned}
&v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\\
&\cdot\Psi^{\beta+a-\delta-\gamma-2}\downarrow_{\beta-\delta}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\\
&\cdot\left(\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\right)\downarrow_{\beta-b-\gamma+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
$$

Now apply Lemma 2.8 to $\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta+a-\delta-\gamma-2}\cdot\Psi^{\beta+a-\delta-\gamma-2}\downarrow_{\beta-\delta}$ (take $x=\alpha+p+a+z-\gamma-1$, $f=1$, $h=\beta+\gamma-\alpha-p-a-z-\delta$, $g=a-\gamma-1$,

$k = 0$) since $i_{\alpha+p+a+z-\gamma} \not\to i_{\beta+\gamma+1}, \ldots, i_{\beta+a-1}$. This leaves us with

$$\begin{aligned} v^{\mathrm{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\ &\cdot \left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1} \cdot \Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta-\delta-1} \\ &\cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \\ &\cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\ &\cdot \left(\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\right)\downarrow_{\beta-b-\gamma+1} \\ &\cdot \left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\ &\cdot \left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\ &\cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R, \end{aligned} \tag{3.55}$$

most of which is shown in Figure 3.46. Since $i_{\alpha+p+a+z-\gamma+j} \not\to i_{\beta+\gamma+1}, \ldots, i_{\beta+a}$ for $j \in \{1, \ldots, \gamma-1\}$ we can apply Corollary 2.6 to

$$\begin{aligned} &\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\ &\quad\cdot \left(\Psi_{\beta+a-b-2\gamma}\uparrow^{\beta+q+a-b-\gamma-2}\right)\downarrow_{\beta-b-\gamma+1}, \end{aligned}$$

and then again to $\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot \left(\Psi_{\beta+a-b-\gamma}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\beta-b+1}$ since $i_{\alpha+p+a+z-\gamma+j} \not\to i_{\beta+\gamma+1}, \ldots, i_{\beta+a}$ for $j \in \{1, \ldots, \gamma-1\}$, so that (3.55) equals

$$\begin{aligned} v^{\mathrm{t}^\mu} &\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\ &\cdot \left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1} \cdot \left(\Psi_{\beta+a-\delta-\gamma}\uparrow^{\beta+q+a-\delta-2}\right)\downarrow_{\beta-\delta+1} \\ &\cdot \Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta-\delta-1} \\ &\cdot \left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\ &\cdot \left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\ &\cdot \left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\ &\cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R. \end{aligned} \tag{3.56}$$

Now since $i_{\alpha+p+a+z-\gamma+j} \not\to i_{\beta-\gamma+1}, \ldots, i_{\beta+a}$ for $j \in \{1, \ldots, \gamma-1\}$ we can apply

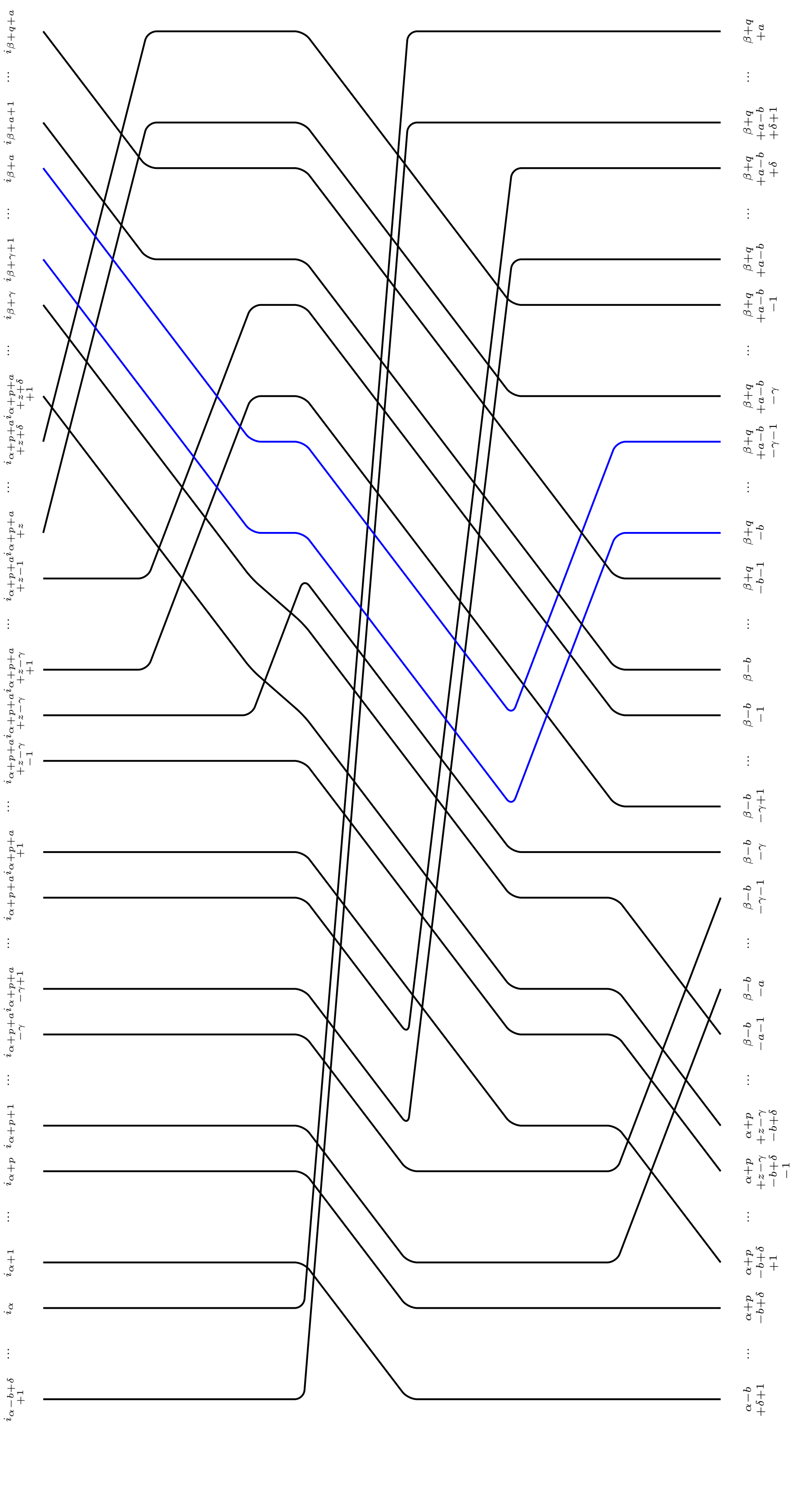


Figure 3.46: Part of the braid diagram for (3.55), excluding $\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \cdot \left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1} \cdot R$. The strings coloured blue are those which we pull over multiple crossings using Corollary 2.6.

Lemma 2.8 to

$$\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+a-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\beta+a-\delta-\gamma}\uparrow^{\beta+q+a-\delta-2}\right)\downarrow_{\beta-\delta+1}$$

(take $x=\alpha+p+a+z-\gamma$, $f=\gamma-1$, $h=\beta+\gamma-\alpha-p-a-z-\delta$, $g=a-\gamma$, $k=q$) so that (3.56) is equal to

$$\begin{aligned}
v^{\mathfrak{t}^{\mu}}&\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\\
&\cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+\gamma-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\left(\Psi_{\beta+a-\delta-1}\uparrow^{\beta+q+a-\delta-2}\right)\downarrow_{\beta+\gamma-\delta}\\
&\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta-\delta-1}\cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1}\\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1}\\
&\cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1}\\
&\cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}\tag{3.57}$$

We show the crossings corresponding to the first few multiplicands of (3.57) in Figure 3.47. Since $i_{\alpha+p+a+z+j}\not\to i_{\beta+\gamma+1},\dots,i_{\beta+a}$ for $j\in\{0,1,\dots,\delta\}$ we can apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z}\cdot\left(\Psi_{\beta+a-\delta-1}\uparrow^{\beta+q+a-\delta-2}\right)\downarrow_{\beta+\gamma-\delta}$$

(take $x=\alpha+p+a+z-1$, $f=\delta+1$, $k=\beta+\gamma-\alpha-p-a-z-\delta$, $h=a-\gamma$,

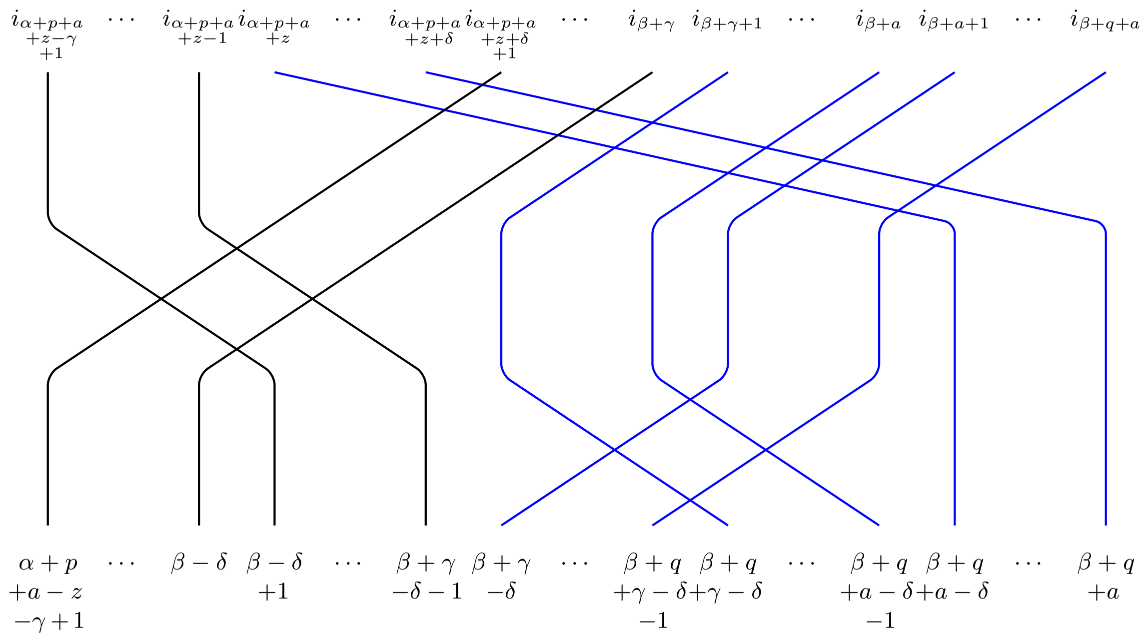


Figure 3.47: Braid diagram for the second, third and fourth multiplicands of (3.57) using the reisudes as they would be in (3.57). The strings to which we apply Lemma 2.4 are coloured blue.

$g = q$, $t = 0$). Then we have that (3.57) is equal to

$$
\begin{aligned}
v^{\mathfrak{t}^\mu} & \left(\Psi_{\beta+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\beta+\gamma+1}\cdot\left(\Psi_{\alpha+p+a+z+\delta}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+a+z} \\
& \cdot\left(\Psi_{\alpha+p+a+z-1}\uparrow^{\beta+\gamma-\delta-2}\right)\downarrow_{\alpha+p+a+z-\gamma+1}\cdot\Psi_{\alpha+p+a+z-\gamma}\uparrow^{\beta-\delta-1} \\
& \cdot\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \\
& \cdot\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p-b+\delta+1} \\
& \cdot\left(\Psi_{\alpha+p+z-b+\delta-\gamma-1}\uparrow^{\beta-b-\gamma-2}\right)\downarrow_{\alpha+p+z+\delta-a_2+1} \\
& \cdot\left(\Psi_{\alpha-b+\delta}\uparrow^{\alpha+p+z+\delta-1-a_2}\right)\downarrow_{\alpha-a_2+1}\cdot R.
\end{aligned}
$$

Since the node containing $\beta+\gamma+1$ in $\mathfrak{t}^\mu$ will be a Garnir node (otherwise we could not have taken $d = a-\gamma-1$), we have a Garnir relation at the top of our diagram, giving us zero. With this we have finally shown that $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}g_\lambda\left(\boxed{\tilde{r}+1}\right)$ is zero and we are done checking relations in (iii).

Conclusion

Having checked all of the relations in (i), (ii) and (iii), we are done and so there is indeed a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}$.

□

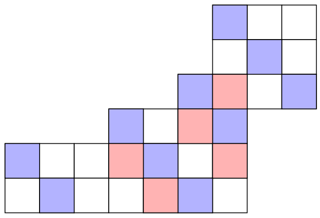
3.3.4 Extending the result

In order to describe the degree of a homomorphism arising from moving a skew shape, we will need to use the following definitions given two partitions λ and μ and a skew shape of the form $[\lambda \setminus \mu]$.

- Given a node $(x, y) \in [\lambda\setminus\mu]$ such that $(x-1, y), (x-1, y-1), (x, y-1) \notin [\lambda\setminus\mu]$, call the nodes $(x+j, y+j) \in [\lambda \setminus \mu]$ for $j \geq 0$ a *positive diagonal.*
- Given a node $(x, y) \in [\lambda \setminus \mu]$ such that $(x-1, y), (x, y-1) \in [\lambda \setminus \mu]$ whilst $(x-1, y-1) \notin [\lambda \setminus \mu]$, call the nodes $(x+j, y+j) \in [\lambda \setminus \mu]$ such that $j \geq 0$ a *negative diagonal.*

This definition extends naturally to components of multipartitions. Let a_+ be the number of nodes in positive diagonals, and a_- be the number of nodes in negative diagonals. Then the *base degree* of $\lambda \setminus \mu$ is defined as $a_+ - a_-$.

Example 3.15. For the given skew shape below, the positive diagonals are shown in blue whilst the negative diagonals are shown in red.



The base degree is $10 - 5 = 5$.

$\Diamond$

Definition 3.16. Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n, where $[\mu]$ is formed from $[\lambda]$ by moving a skew shape of base degree b from the qth component to the pth, for some p and q such that $p < q$. In addition suppose that

$$e \geq \max_{p \leq c \leq q} \{h_{11}^{\lambda^{(c)}} + 1, h_{11}^{\mu^{(c)}} + 1\}.$$

Suppose that amongst each component $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$, there are exactly $k \geq 0$ such components to which the same skew shape of the same residues can be added. If $k > 0$, then we also require that e is large enough so that the diagonal residue condition holds when the skew shape is added to these k components. Suppose that amongst the components $\lambda^{(c')}$ that are not one of these k components, there are no removable nodes of any of the residues in the skew shape, and that there are m_ι addable nodes of residue ι, with a_ι instances of the residue ι within the skew shape. Then we say that $(\lambda, \mu)^k$ is a *skew pair, of degree* $(k+1)b + \sum_\iota a_\iota m_\iota$, where the sum runs over all residues ι in the skew shape.

Remark 3.17. Since we have the diagonal residue condition, if λ is one half of a skew pair, then in a component $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$, either we can either have some individual addable nodes of the residues in the skew shape *or* we can add only the entire skew shape itself and not some other individual nodes of residues within the skew shape.

We may sometimes refer to components $\lambda^{(c')}$ with $c' \in \{p+1, p+2, \ldots, q-1\}$ as the *middle components*.

Corollary 3.18. *Suppose that $(\lambda, \mu)^k$ is a skew pair of degree $(k+1)b + \sum_\iota a_\iota m_\iota$. Let $\mathfrak{s}$ be the μ-tableau defined by considering $\mathfrak{t}^\lambda$ and moving the skew shape from the qth component to the pth, keeping their values intact. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu} \psi^{\mathfrak{s}}$. This homomorphism has degree $(k+1)b + \sum_\iota a_\iota m_\iota$ and can be written as a composition of $k+1$ homomorphisms.*

Proof. If (λ, μ) is a row pair, then we can simply use Corollary 3.9. Note that in this case $b = 1$, $a_\iota \leq 1$ for every residue ι and that m_ι is the number of addable nodes of residue ι across *all* components $\lambda^{(p+1)}, \ldots, \lambda^{(q-1)}$ for every ι except that which is the leftmost residue in the row, in which case $k + m_\iota$ counts this value instead. Then the degree $(k+1)b + \sum_\iota a_\iota m_\iota$ matches that of Corollary 3.9. So instead we shall suppose that the shape moved is definitely a skew shape of at least two rows worth of nodes.

We shall begin by assuming that $k = 0$. Define α, β, q, a, a_2 and b similarly to Theorem 3.14, so that the bottom two rows of the skew shape to be moved are as in (3.17). Then $\psi^{\mathfrak{s}} = \left(\Psi_{\alpha+p+a}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+1} \cdot \left(\Psi_\alpha\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot R$ where R is a product of crossings coming from the rows higher than the bottom two in the skew shape. We need to check that the generating relations of S^λ hold on $\varphi\left(v^{\mathfrak{t}^\lambda}\right)$.

Similarly to Theorem 3.14, define a new KLR algebra $\mathscr{H}_n^{\Lambda_\kappa}$ using quantum characteristic $\tilde{e} := e$ and multicharge

$$\tilde{\kappa} := (\kappa_1, \kappa_2, \ldots, \kappa_q, \mathrm{res}_\lambda(\beta + q + a), \kappa_{q+1}, \kappa_{q+2}, \ldots, \kappa_l)$$

and define $l+1$-multipartitions:

$$\begin{aligned}
\tilde{\lambda} &:= \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q-1)}, \lambda^{(q)}_{\hat{k}_2}, (1), \lambda^{(q+1)}, \lambda^{(q+2)}, \ldots, \lambda^{(l)}\right) \\
\tilde{\lambda}_1 &:= \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(p-1)}, \mu^{(p)}_{\hat{k}_1}, \mu^{(p+1)}, \mu^{(p+2)}, \ldots, \mu^{(q)}, (1),\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\mu^{(q+1)}, \mu^{(q+2)}, \ldots, \mu^{(l)}\right) \\
\tilde{\mu} &:= \left(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(q)}, \varnothing, \mu^{(q+1)}, \mu^{(q+2)}, \ldots, \mu^{(l)}\right)
\end{aligned}$$

We define a $\tilde{\lambda}_1$-tableau $\mathfrak{s}_1$ by

$$\tilde{\psi}^{\mathfrak{s}_1} = \left(\Psi_{\alpha+p+a-1}\!\uparrow^{\beta+q+a-2}\right)\downarrow_{\alpha+p+1} \left(\Psi_{\alpha}\!\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot\tilde{R},$$

where $\tilde{R}$ is just R but every ψ is replaced by $\tilde{\psi}$, and also a $\tilde{\mu}$-tableau $\mathfrak{s}_2$ by $\tilde{\psi}^{\mathfrak{s}_2} = \Psi_{\alpha+p+a}\!\uparrow^{Q-1}$ where $Q = \sum_{i=1}^{q} \left|\lambda^{(i)}\right|$. Then by induction we have a homomorphism $\varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\lambda}_1}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\lambda}_1}}\tilde{\psi}^{\mathfrak{s}_1}$, and another $\varphi_2 : S^{\tilde{\lambda}_1} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}_1}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}_2}$. Defining

$$\tilde{\psi}^{\mathfrak{s}} := \left(\tilde{\Psi}_{\alpha+p+a}\!\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha+p+1} \cdot \left(\tilde{\Psi}_{\alpha}\!\uparrow^{\beta-1}\right)\downarrow_{\alpha-a_2+1} \cdot\tilde{R},$$

the composition of φ_2 with φ_1 gives us a homomorphism $\tilde{\varphi} := \varphi_2 \circ \varphi_1 : S^{\tilde{\lambda}} \to S^{\tilde{\mu}}$ given by $v^{\mathfrak{t}^{\tilde{\lambda}}} \mapsto v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}\tilde{\Psi}_{\beta+q+a}\!\uparrow^{Q-1}$. This gives us relations (i*), (ii*), (iii*), just as in Theorem 3.14, and we can use these to check the relations (i), (ii) and (iii), since the diagrams for $v^{\mathfrak{t}^{\mu}}\psi^{\mathfrak{s}}$ and $v^{\mathfrak{t}^{\tilde{\mu}}}\tilde{\psi}^{\mathfrak{s}}$ are identical.

For each type of relation, the above setup allows us to follow the same methods as in Theorem 3.14, only now accounting for the additional nodes in between the first and last components of $[\mu]$ as well as those outside of these components. In checking each of the relations, we apply the same reasoning as in Theorem 3.14; however, there are a few changes to be made at the places annotated by the following labels in the margins:

(C1) Replace n with Q throughout and note that $i_{\alpha+p+a} \not\rightarrow i_{\beta+q+a+1}, \ldots, i_Q$. For $r \in \{Q+1, \ldots, n-1\}$, we may follow the same reasoning as for $r \in \{1, \ldots, \beta+q+a-1\}$.

(C2) If $\psi_{\alpha+p+z_j}$ is not a row relation then by the diagonal residue condition the node containing $\alpha+p+z_j$ in $\mathfrak{t}^\mu$ must be a Garnir node, so the corresponding Garnir relation will be at the top of the diagram for those terms in the sum.

(C3) We can define $\bar{\lambda}, \bar{\mu}$ in the same way, only in addition dropping the components labelled from 1 to $p-1$. Note that we should really relabel the tableaux entries here by a shift, but this would only serve to make things more confusing. We can then define $\bar{\nu}$ and the tableaux $\mathfrak{t}_1$ and $\mathfrak{t}_2$ and we will still get homomorphisms φ_1 and φ_2 in the same way.

(C4) Note that if $\psi_{\alpha+p+a+z+\delta}$ is not a row relation then the node containing $\alpha+p+a+z$ in $\mathfrak{t}^\mu$ will always be a Garnir node if it occurs in the middle components, otherwise there will be a removable node of a residue which occurs in the skew shape, which we have assumed do not exist. Hence from now on we can assume the node containing $\alpha+p+a+z$ cannot lie in a middle component.

(C5) The node containing $\alpha+\zeta_t+a$ may belong to a middle component. Take $D \in \{0,1,\dots,m-1\}$ maximal so that $\psi_{\alpha+\zeta_t+a+j}$ is a row relation for $j \in \{0,1,\dots,D-1\}$. If $D = m-1$ and $\psi_{\alpha+\zeta_t+a+m-1}$ is a row relation then we can follow the reasoning just as before (note in particular this will happen if the node containing $\alpha+\zeta_t+a$ is on the bottom row of a middle component). So suppose otherwise. If $D=0$ then let $\alpha+\zeta_t+a+X$ be the value of the node beneath that containing $\alpha+\zeta_t+a$ and write $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot \Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+p+z-b+\delta-m-1}$ as

$$\begin{aligned}&\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-m}\\ &\qquad\cdot\left(\Psi_{\alpha+p+a-b+\delta-m-1}\uparrow^{\beta+q+a-b-m-3}\right)\downarrow_{\alpha+p-b+\delta+1}\\ &\quad\cdot\Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+\zeta_t-b+\delta+X-1}\cdot\Psi_{\alpha+\zeta_t-b+\delta+X}\uparrow^{\alpha+p+z-b+\delta-m-1}\end{aligned}$$

and since $i_{\alpha+\zeta_t+a} \nrightarrow i_{\alpha+p+1},\dots,i_{\alpha+p+a-m-1}$ apply Corollary 2.6 to the second and third multiplicand in the above. Then we need only consider

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-m} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta-m-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+X-m-2}.$$

Now, if we ignore the fact that $i_{\alpha+\zeta_t+a+X} = i_{\alpha+p+a-m}$ we could apply Corollary 2.6 here since $i_{\alpha+\zeta_t+a} \not\to i_{\alpha+\zeta_t+a+1}, \dots, i_{\alpha+\zeta_t+a+X-1}$. Then we could apply Corollary 2.6 again since $i_{\alpha+\zeta_t+a} \not\to i_{\alpha-b+\delta+1}, \dots, i_\alpha$, giving $\Psi_{\alpha+\zeta_t+a}\uparrow^{\alpha+\zeta_t+a+X-1}$ at the top of the diagram, which is the Garnir relation for the node containing $\alpha+\zeta_t+a$ so this is would be zero. However, we have that $i_{\alpha+\zeta_t+a+X} = i_{\alpha+p+a-m}$, so in fact we have to also take this into consideration. We instead use Corollary 2.6 to pull over all of the crossings except the last. Then we have

$$\Psi_{\alpha+\zeta_t+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta+X-2} \\ \cdot \left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta+X-1}\right)\downarrow_{\alpha+p+a-b+\delta-m} \\ \cdot \psi_{\alpha+\zeta_t+a-b+\delta+X-m-2} \\ \cdot \left(\Psi_{\alpha+\zeta_t+a-b+\delta+X}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta+X-m}.$$

Since $i_{\alpha+p+a-m} \leftarrow i_{\alpha+\zeta_t+a} \to i_{\alpha+\zeta_t+a+X}$ we apply the braid relation to $\psi_{\alpha+\zeta_t+a-b+\delta+X-m-2}\psi_{\alpha+\zeta_t+a-b+\delta+X-m-1}\psi_{\alpha+\zeta_t+a-b+\delta+X-m-2}$. If we pull the crossing over we actually just apply the reasoning above where we applied Corollary 2.6 to the whole thing. If not, then consider the crossings

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+1} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta+X-m}\uparrow^{\alpha+\zeta_t+a-b+\delta+X+X_2-m-1} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta+X+X_2-m}\uparrow^{\beta+q+a-b-m-2},$$

where $\alpha+\zeta_t+a+X+X_2$ is the value of the node beneath that containing $\alpha+\zeta_t+a+X$. Apply Corollary 2.6 to the first two multiplicands in question, since

$$i_{\alpha+p+a-m+1}, \ldots, i_{\alpha+p+a} \not\to i_{\alpha+\zeta_t+a+X+1}, \ldots, i_{\alpha+\zeta_t+a+X+X_2}.$$

Then pull the resulting crossings $\Psi_{\alpha+\zeta_t+a-b+\delta+X} \uparrow^{\alpha+\zeta_t+a-b+\delta+X+X_2-1}$ over the next set of strings using Corollary 2.6 since $i_{\alpha+\zeta_t+a+X} \not\to i_{\alpha-b+\delta+1}, \ldots, i_\alpha$. Thus we have the Garnir relation for the node containing $\alpha+\zeta_t+a+X$ at the top of the diagram so this is zero.

Now suppose $D > 0$, then since

$$i_{\alpha+p+1} \ldots i_{\alpha+p+a-m+1} \neq i_{\alpha+\zeta_t+a+1} \ldots i_{\alpha+\zeta_t+a+D},$$

rewrite $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot \Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+p+z-b+\delta-m-1}$ as

$$\begin{aligned}
&\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2} \\
&\qquad \cdot \left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1} \\
&\quad \cdot \Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+\zeta_t-b+\delta+D-1} \cdot \Psi_{\alpha+\zeta_t-b+\delta+D}\uparrow^{\alpha+p+z-b+\delta-m-1}
\end{aligned} \tag{3.58}$$

and apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot \Psi_{\alpha+\zeta_t-b+\delta}\uparrow^{\alpha+\zeta_t-b+\delta+D-1}$$

giving

$$\begin{aligned}
&\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+D} \\
&\quad \cdot \left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1} \cdot
\end{aligned}$$

Then (3.58) is equal to

$$
\begin{aligned}
&\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2} \\
&\cdot\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+D-1}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1} \\
&\qquad\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+D} \\
&\cdot\left(\Psi_{\alpha+\zeta_t+a-b+\delta+D}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+2} \\
&\cdot\left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1} \\
&\qquad\cdot\Psi_{\alpha+\zeta_t-b+\delta+D}\uparrow^{\alpha+p+z-b+\delta-m-1}.
\end{aligned}
\tag{3.59}
$$

In Figure 3.48 we show some of the crossings at this stage. Now write the second and third multiplicands of the above as

$$
\begin{aligned}
&\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+D-1}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+1} \\
&\cdot\left(\Psi_{\alpha+\zeta_t+a-b+\delta-m+D}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+2D}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1} \\
&\qquad\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+D}
\end{aligned}
$$

and then as

$$
\begin{aligned}
&i_{\alpha+p+a-m+D+1}\to\cdots\to i_{\alpha+p+a-m+2}\to i_{\alpha+\zeta_t+a}\leftarrow i_{\alpha+\zeta_t+a+1}\leftarrow\cdots \\
&\qquad\cdots\leftarrow i_{\alpha+\zeta_t+a+D}
\end{aligned}
$$

apply Lemma 2.10 to

$$
\begin{aligned}
&\left(\Psi_{\alpha+\zeta_t+a-b+\delta-m+D}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+2D}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+1} \\
&\qquad\cdot\Psi_{\alpha+\zeta_t+a-b+\delta-m+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+D}
\end{aligned}
$$

(take $x=\alpha+\zeta_t+a-b+\delta-m$, $f=D$, $g=D$) and replace this with a sum of terms, each which begin with $\psi_{\alpha+\zeta_t+a-b+\delta-m+D+j}$ for $j\in\{1,\dots,D\}$, along with another term where these crossings all disappear. In the former

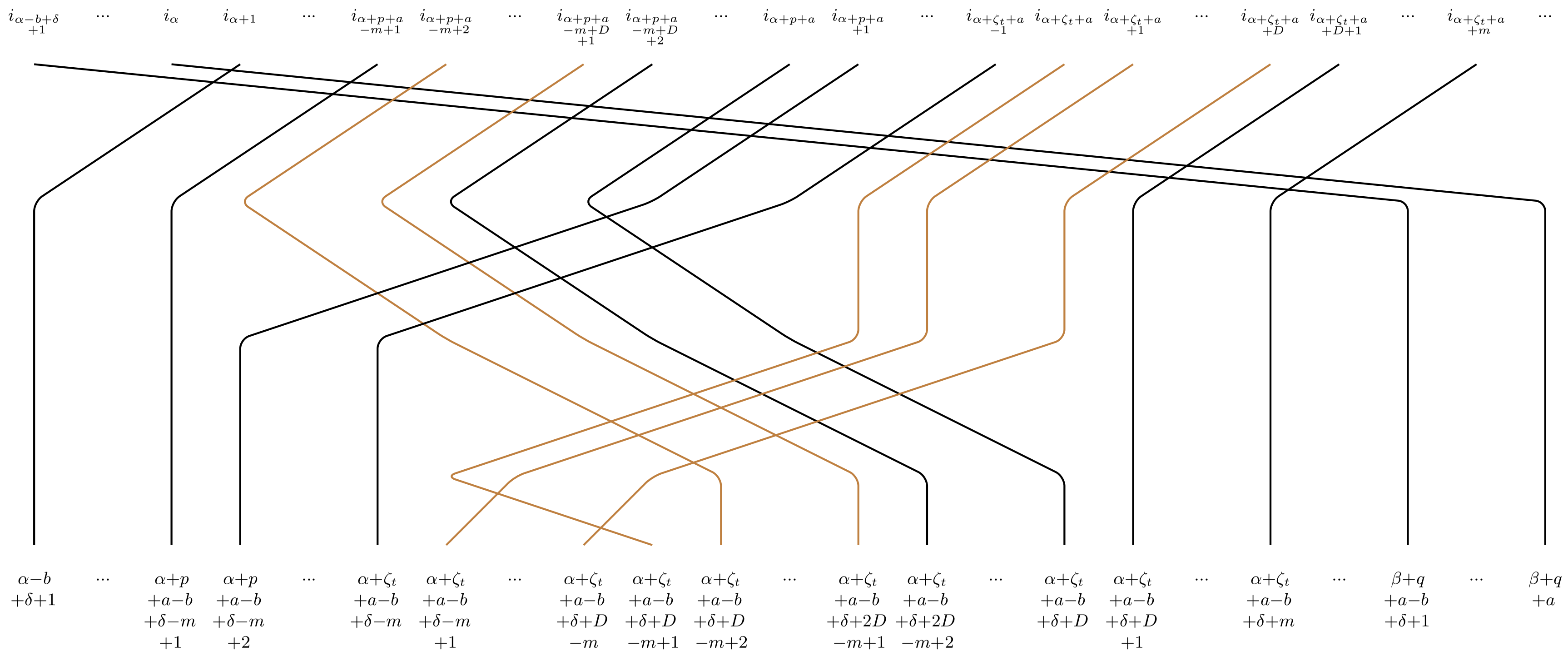


Figure 3.48: Part of the braid diagram obtained after applying Corollary 2.6 in (3.58). The strings to which we will apply Lemma 2.10 are coloured brown.

cases, as $i_{\alpha+\zeta_t+a+j-1} \not\to i_{\alpha+p+a-m+D+2}, \ldots, i_{\alpha+p+a}$, apply Corollary 2.6 to

$$\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+D-1}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+1} \\ \cdot \psi_{\alpha+\zeta_t+a-b+\delta-m+D+j},$$

then as $i_{\alpha+\zeta_t+a_j-1} \not\to i_{\alpha-b+\delta+1}, \ldots, i_\alpha$ we can apply Corollary 2.6 to $\left(\Psi_\alpha\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot \psi_{\alpha+\zeta_t+a-b+\delta+j-1}$. Then we have a $\psi_{\alpha+\zeta_t+a+j-1}$ crossing at the top of the diagram. This will be a row relation by assumption. In the latter case, where the crossings disappear, we have that (3.59) is equal to

$$\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\alpha+\zeta_t+a-b+\delta-2}\right)\downarrow_{\alpha+p+a-b+\delta-m+2} \\ \cdot \left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\alpha+\zeta_t+a-b+\delta+D-1}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+1} \\ \cdot \left(\Psi_{\alpha+\zeta_t+a-b+\delta+D}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+2} \\ \cdot \left(\Psi_{\alpha+p+a-b+\delta-m+1}\uparrow^{\beta+q+a-b-m-1}\right)\downarrow_{\alpha+p-b+\delta+1} \\ \cdot \Psi_{\alpha+\zeta_t-b+\delta+D}\uparrow^{\alpha+p+z-b+\delta-m-1}.$$

Figure 3.49 shows some of the term we are dealing with now. Since we are assuming that $\alpha+\zeta_t+a+D$ is not a row relation then write $g_\mu\left(\boxed{\alpha+\zeta_t+a+D}\right) = \Psi_{\alpha+\zeta_t+a+D}\uparrow^{\alpha+\zeta_t+a+D+X}$ for some $X \geq 0$. Rewrite the second and third multiplicands here in the form

$$\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+1} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta-m+2D+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+2D+X+1} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta-m+2D+X+2}\uparrow^{\beta+q+a-b-m+D-1} \\ \cdot \left(\Psi_{\alpha+\zeta_t+a-b+\delta-m+2D}\uparrow^{\beta+q+a-b-m+D-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+2}$$

and now apply Corollary 2.6 to

$$\left(\Psi_{\alpha+\zeta_t+a-b+\delta-1}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+\zeta_t+a-b+\delta-m+D+1} \\ \cdot \Psi_{\alpha+\zeta_t+a-b+\delta-m+2D+1}\uparrow^{\alpha+\zeta_t+a-b+\delta-m+2D+X+1}$$

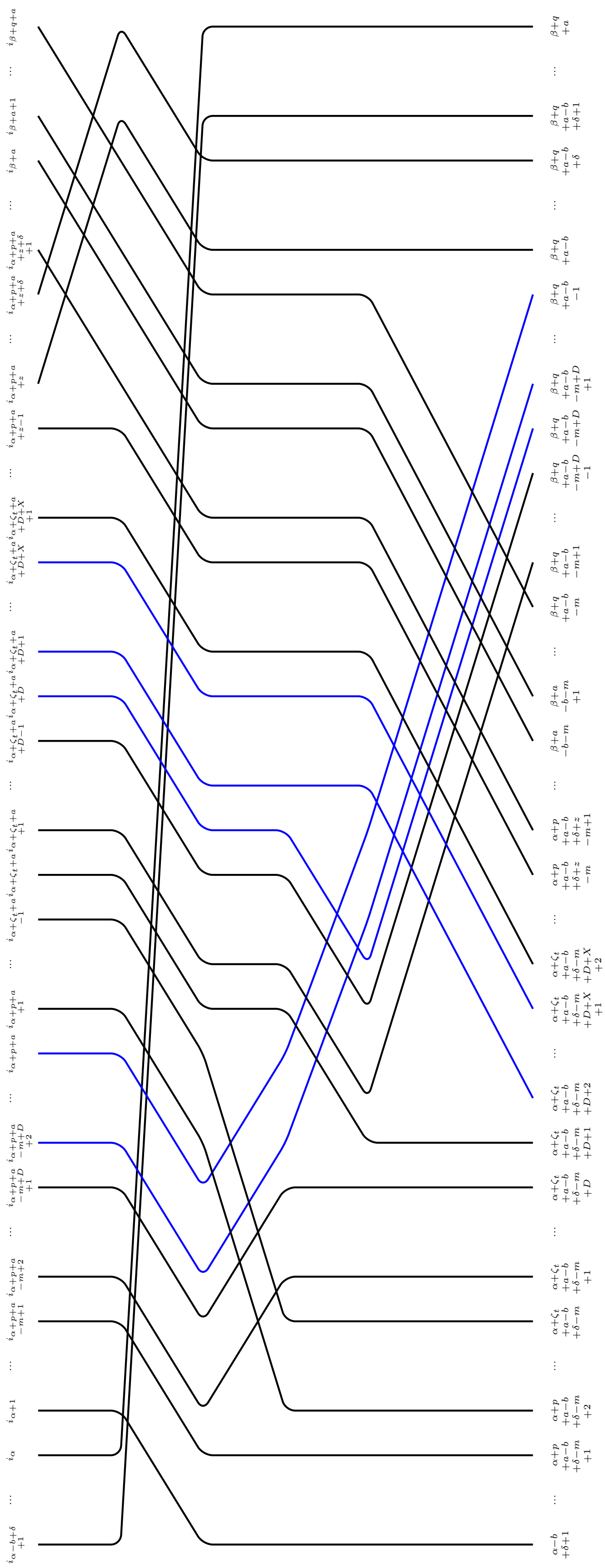


Figure 3.49: Part of the braid diagram for (3.59) when the crossings disappear. The strings coloured blue are those to which we apply Corollary 2.6.

since

$$i_{\alpha+p+a-m+D+2},\ldots,i_{\alpha+p+a}\neq i_{\alpha+\zeta_t+a+D+1},\ldots,i_{\alpha+\zeta_t+a+D+X},$$

and then apply Corollary 2.6 again to

$$\left(\Psi_\alpha\!\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1}\cdot\Psi_{\alpha+\zeta_t+a-b+\delta+D}\!\uparrow^{\alpha+\zeta_t+a-b+\delta+D+X}.$$

This gives us $g_\mu\left(\boxed{\alpha+\zeta_t+a+D}\right)$ at the top of the diagram, and thus we will obtain zero.

(C6) We may also have terms which begin with $\psi_{Z-b+\delta-\gamma}$ where Z is the entry of a node in a middle component of μ such that $i_Z=i_{\alpha+p+a-\gamma-j}$ for some $j\in\{1,\ldots,d\}$. In this case either ψ_Z is a row relation, and we can pull $\psi_{Z-b+\delta-\gamma}$ to the top for the same reasons as we could $\psi_{\alpha+p+a-b+\delta-\gamma+Z}$, or the node containing Z is a Garnir node (otherwise it is a removable node of a forbidden residue) and so in the application of Lemma 2.9, note that we could have instead applied Corollary 2.6 at $(*)$ to

$$\left(\Psi_{x+f+1}\!\uparrow^{x+2f+g-\gamma}\right)\downarrow_{x+f+2-\gamma}\cdot\Psi_{x+f+z_{\gamma_j}+1-\gamma}\!\uparrow^{x+2f+g-2\gamma-1},$$

which would then instead correspond to a term that begins with the crossings $\Psi_{Z-b+\delta-\gamma}\!\uparrow^{\eta-b-2\gamma-3}$. Then apply Corollary 2.6 in the same way as above to pull $\Psi_{Z-b+\delta-\gamma}\!\uparrow^{\eta-b-2\gamma-3}$ to the top, giving $\Psi_Z\!\uparrow^{\eta-\delta-\gamma-3}$ at the top of the diagram which will contain the Garnir relation corresponding to the node containing Z (in this case $d\geq 1$ thus the node containing $\eta-\delta-\gamma-2$ is either that containing $\alpha+p+a+z-\gamma$ or to the right of it).

(C7) We may have $Z\in\{1,\ldots,z-\gamma-1\}$ such that the node containing $\alpha+p+a+Z$ lies in a middle component and $i_{\alpha+p+a+Z}=i_{\alpha+p+a-\gamma-d-1}$. In this case, write $\Psi_{\alpha+p+a-b+\delta-d-\gamma}\uparrow^{\eta-b-2\gamma-2d-3} \cdot\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-d-3}$ as

$$\Psi_{\alpha+p+a-b+\delta-d-\gamma}\uparrow^{\eta-b-2\gamma-2d-3} \cdot\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-2d-3} \\ \cdot\Psi_{\eta-b-2\gamma-2d-2}\uparrow^{\eta-b-2\gamma-d-3}$$

and apply Lemma 2.7 to

$$\Psi_{\alpha+p+a-b+\delta-d-\gamma}\uparrow^{\eta-b-2\gamma-2d-3} \cdot\Psi_{\alpha+p+a-b+\delta-\gamma-d-1}\uparrow^{\eta-b-2\gamma-2d-3}$$

(take x and g as in the application of Corollary 2.6). We deal with the term beginning with $\psi_{\alpha+p+a-b+\delta-\gamma-d-1}$ in the same way as in the original proof. In addition there will be terms beginning with $\Psi_{\alpha+p+a-b+\delta-d-\gamma+Z}\uparrow^{\eta-b-2\gamma-2d-3}$. Then apply Corollary 2.6 to

$$\left(\Psi_{\alpha+p+a-b+\delta-\gamma}\uparrow^{\eta-b-2\gamma-d-3}\right)\downarrow_{\alpha+p+a-b+\delta-d-\gamma} \\ \cdot\Psi_{\alpha+p+a-b+\delta-d-\gamma+Z}\uparrow^{\eta-b-2\gamma-2d-3}$$

since $i_{\alpha+p+a+Z} \not- i_{\alpha+p+a-\gamma-d+1}, \dots, i_{\alpha+p+a-\gamma}$, then apply Corollary 2.6 to $\left(\Psi_{\alpha+p+a-b+\delta}\uparrow^{\beta+q+a-b-2}\right)\downarrow_{\alpha+p+a-b+\delta-\gamma+1} \cdot\Psi_{\alpha+p+a-b+\delta-\gamma+Z}\uparrow^{\eta-b-2\gamma-d-3}$ as $i_{\alpha+p+a+Z} \not- i_{\alpha+p+a-\gamma+1}, \dots, i_{\alpha+p+a}$, and then apply Corollary 2.6 to $\left(\Psi_{\alpha}\uparrow^{\beta+q+a-1}\right)\downarrow_{\alpha-b+\delta+1} \cdot\Psi_{\alpha+p+a-b+\delta+Z}\uparrow^{\eta-b-\gamma-d-3}$. But now using the fact that $\eta-d=\alpha+p+a+z+\delta+1$ we will have $\Psi_{\alpha+p+a+Z}\uparrow^{\alpha+p+a+z-\gamma-2}$ at the top of the diagram, and either $\psi_{\alpha+p+a+Z}$ will be a row relation and this is zero, or the node containing $\alpha+p+a+Z$ will be a Garnir node (otherwise it is a removable node of a forbidden residue) this will contain the corresponding Garnir relation $g_\mu\left(\boxed{\alpha+p+a+Z}\right)$.

Now suppose that $k \geq 0$, then we wish to show that we can rewrite φ as a composition of $k+1$ homomorphisms. When $k=0$ this is trivially true, so suppose that $k>0$. Let $\tilde{c} \in \{p+1, p+2, \dots, q-1\}$ be maximal so that the skew shape (with residues intact) can be added to $\lambda^{(\tilde{c})}$. Suppose that if we add the skew shape to $\left[\lambda^{(\tilde{c})}\right]$ we obtain the diagram $\left[\nu^{(\tilde{c})}\right]$ and consider the multipartition

$$\nu := \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\tilde{c}-1)}, \nu^{\tilde{c}}, \mu^{(\tilde{c}+1)}, \mu^{(\tilde{c}+2)}, \ldots, \mu^{(l)}\right).$$

Let $\mathfrak{u}$ be the ν-tableau defined by considering $\mathfrak{t}^{\lambda}$ and moving the skew shape from the qth component to the $\tilde{c}$th, keeping its tableau values intact. Then, noting Remark 3.17, by induction we have that there is a homomorphism $\varphi_1 : S^{\lambda} \to S^{\nu}$ given by $v^{\mathfrak{t}^{\lambda}} \mapsto v^{\mathfrak{t}^{\nu}} \psi^{\mathfrak{u}}$. Similarly, we also obtain a homomorphism $\varphi_2 : S^{\nu} \to S^{\mu}$ given by $v^{\mathfrak{t}^{\nu}} \mapsto v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{v}}$ where $\mathfrak{v}$ is the μ-tableau defined by considering $\mathfrak{t}^{\nu}$ and moving the skew shape from the $\tilde{c}$th component to the pth.

Note that $d(\mathfrak{v})$ maps the entries of the skew shape in $\mathfrak{t}^{\mu}$ to the values of the corresponding entries as they were in $\mathfrak{t}^{\nu}$, whilst $d(\mathfrak{u})$ maps the entries of the skew shape in $\mathfrak{t}^{\nu}$ to the values of the corresponding entries as they were in $\mathfrak{t}^{\lambda}$. Thus performing $d(\mathfrak{u})$ followed by $d(\mathfrak{v})$ will map the entries of the skew shape in $\mathfrak{t}^{\mu}$ to the values of the corresponding entries as they were in $\mathfrak{t}^{\lambda}$, and we have that $d(\mathfrak{v}) \cdot d(\mathfrak{u}) = d(\mathfrak{s})$ and this is a reduced expression, thus $\psi^{\mathfrak{v}} \cdot \psi^{\mathfrak{u}} = \psi^{\mathfrak{s}}$. With this in mind, $\varphi_2 \circ \varphi_1 : S^{\lambda} \to S^{\mu}$ is given by $v^{\mathfrak{t}^{\lambda}} \mapsto v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{v}} \cdot \psi^{\mathfrak{u}} = v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}}$ thus $\varphi = \varphi_2 \circ \varphi_1$. Hence φ can be written as a composition of $k+1$ homomorphisms as we wanted.

Finally, we shall describe the degree of φ. By Proposition 1.34 we have that $\deg\left(v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}}\right) = \deg(\mathfrak{s})$. We wish to compute $\deg\left(v^{\mathfrak{t}^{\mu}} \psi^{\mathfrak{s}}\right) - \deg\left(v^{\mathfrak{t}^{\lambda}}\right) = \deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^{\lambda}\right)$. Using the recursive definition of the degree, the nodes containing $n, n-1, \ldots, \beta+q+a+1$ in both tableaux contribute the same value to the respective degrees. Hence

$$\deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^{\lambda}\right) = \deg(\mathfrak{s}_{<\beta+q+a+1}) - \deg\left(\mathfrak{t}^{\lambda}_{<\beta+q+a+1}\right).$$

Now, when calculating the change in degree due to those nodes within the skew shape, most nodes will simply be of such a residue ι that there are m_ι addable nodes below them and no removable nodes below them amongst the components indexed by $p+1, p+2, \ldots, q$. However, if such a node belongs to a positive or negative diagonal, there will still be m_ι addable and no removable nodes below it amongst the components indexed by $p+1, p+2, \ldots, q-1$. In addition, in any of the k components to which the skew shape of the same residues can be added along with the component indexed by q, if the node in question lies in a positive diagonal then there will in addition be an addable node of residue ι, whilst conversely if the node belongs to a negative diagonal then there will in addition be an removable node of residue ι.

Thus we find that as we count over the nodes in the skew shape, the degree is obtained by summing $(k+1)b$ with $\sum_\iota a_\iota m_\iota$. The first summand arising due to the additional addable or removable nodes corresponding to those in the positive or negative diagonals, and the second arising simply from the miscellaneous addable nodes amongst the components indexed by $p+1, p+2, \ldots, q-1$. Thus

$$\deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^\lambda\right) = (k+1)b + \sum_\iota a_\iota m_\iota + \deg(\mathfrak{s}_{<x+1}) - \deg\left(\mathfrak{t}^\lambda_{<x+1}\right)$$

where $x+1$ is the least value present in the skew shape within the tableaux $\mathfrak{s}$ and $\mathfrak{t}^\lambda$. Then since $\mathfrak{s}_{<x+1}$ and $\mathfrak{t}^\lambda_{<x+1}$ are identical, we have that

$$\deg(\mathfrak{s}) - \deg\left(\mathfrak{t}^\lambda\right) = (k+1)b + \sum_\iota a_\iota m_\iota,$$

i.e. the degree of φ is $(k+1)b + \sum_\iota a_\iota m_\iota$.

□

As before, we are now in the position to consider what happens when we move two or more different skew shapes to form $[\mu]$ from $[\lambda]$. We extend the hypothesis of Corollary 3.10 to consider skew shapes instead of rows, and with this we obtain another similar corollary.

Corollary 3.19. *Let $l \geq 2$ and suppose that λ, ν_1, ν_2 and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving two separate skew shapes. Suppose $[\nu_1]$ is formed from $[\lambda]$ by moving just one of the skew shapes, whilst $[\nu_2]$ is formed from $[\lambda]$ by moving just the other skew shape. Suppose that given one of the skew shapes, the residues contained within it are not equal nor adjacent to any of those contained within the other skew shape. Suppose that (λ, ν_1), (λ, ν_2),*

(ν_1,μ), (ν_2,μ) are all skew pairs. Then there are non-zero homomorphisms

$$\varphi_{\lambda\nu_1} : S^\lambda \to S^{\nu_1}, \quad \varphi_{\nu_1\mu} : S^{\nu_1} \to S^\mu,$$
$$\varphi_{\lambda\nu_2} : S^\lambda \to S^{\nu_2}, \quad \varphi_{\nu_2\mu} : S^{\nu_2} \to S^\mu,$$

and we have that $\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1} = \varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2} \neq 0$.

In addition, if (λ,ν_1) and (ν_1,μ) have degrees d_1 and d_2 respectively, we have that the degree of $\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}$ is d_1+d_2.

Proof. Since (λ,ν_1), (λ,ν_2), (ν_1,μ), (ν_2,μ) are all skew pairs, by Corollary 3.18 we have that there are non-zero homomorphisms

$$\varphi_{\lambda\nu_1} : S^\lambda \to S^{\nu_1}, \quad \varphi_{\nu_1\mu} : S^{\nu_1} \to S^\mu,$$
$$\varphi_{\lambda\nu_2} : S^\lambda \to S^{\nu_2}, \quad \varphi_{\nu_2\mu} : S^{\nu_2} \to S^\mu.$$

We shall label the tableau entries in the skew shapes being moved differently to the labelling used in Theorem 3.14. Suppose the first skew shape being moved has k rows, with r_j nodes in each row (for $j\in\{1,\dots,k\}$). We also label the first node in row j of the shape as β_j+1. Within $\mathfrak{t}^\lambda$, this skew shape will look as follows:

$$\begin{array}{cccccc}
 & & & \boxed{\beta_1+1} & \cdots\cdots & \boxed{\beta_1+r_1} \\
 & & \boxed{\beta_2+1} & \cdots\cdots & \boxed{\beta_2+r_2} & \\
 & \iddots & & \iddots & & \\
\boxed{\beta_k+1} & \cdots\cdots & \boxed{\beta_k+r_k} & & &
\end{array}$$

Label nodes in the component to which the skew shape will be added to so that if $\mathfrak{u}$ is the tableau formed by moving the skew shape, we have

$$\psi^{\mathfrak{u}} = \left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdot\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1} \tag{3.60}$$

for some $a_1,\dots,a_k$, where $\sum_j := \sum_{i=1}^j r_j$ to ease notation.

Similarly we suppose that the second skew shape being moved has k' rows, with r'_j nodes in each row (for $j\in\{1,\dots,k'\}$). We label the first node in row j as β'_j+1. If $\mathfrak{v}$ is the tableau formed by moving this skew shape, we have

$$\psi^{\mathfrak{v}} = \left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdot\cdots\cdot\left(\Psi_{\alpha'_1+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1},$$

where $\sum_j' \coloneqq \sum_{i=1}^{j} r_j'$.

With this in mind, we can write

$$\varphi_{\lambda\nu_1}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^{\nu_1}} \left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}$$

and

$$\begin{aligned}\varphi_{\lambda\nu_2}(v^{\mathfrak{t}^\lambda}) = v^{\mathfrak{t}^{\nu_2}} \left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdots\cdot \\ \cdot\left(\Psi_{\alpha'_1+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}.\end{aligned}$$

Without loss of generality, assume that $\beta_1 < \beta_1'$. If $\beta_1 < \alpha_1'$ then we will in fact have that $\beta_k + r_k < \alpha_1' + 1$ and so

$$\begin{aligned}\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1} &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdots\cdot \\ &\quad\cdot\left(\Psi_{\alpha'_1+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\cdot\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1} \\ &\quad\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1} \\ &= v^{\mathfrak{t}^\mu}\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\cdot \\ &\quad\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}\cdot\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1} \\ &\quad\cdots\cdot\left(\Psi_{\alpha'_1+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1} \\ &= \varphi_{\lambda\nu_1}\circ\varphi_{\nu_1\mu}\end{aligned}$$

and we are done. Hence, assume that $\beta_1 \geq \alpha_1'$. Then we have multiple cases.

Case I: The skew shape containing β_1 is moved to a position above the other skew shape in $[\mu]$

In this case, we have that

$$\begin{aligned}\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1} = &\left(\Psi_{\alpha'_{k'}+\sum_k+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum_k+\sum'_{k'-1}+1}\cdots\cdot \\ &\cdot\left(\Psi_{\alpha'_1+\sum_k+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+\sum_k+1} \\ &\cdot\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}\end{aligned}$$

whilst

$$\begin{aligned}\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2} = \left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+\sum'_{k'}+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\\ \cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+\sum'_{k'}+r_1-1}\right)\downarrow_{\alpha_1+1}\\ \cdot\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdots\\ \cdot\left(\Psi_{\alpha'_1+\sum'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}.\end{aligned}$$

Now observe that for $\gamma\in\{0,1,\dots,k'-1\}$, if we have

$$\begin{aligned}&\left(\Psi_{\alpha'_{k'}+\sum'_{k'}+\sum_k}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum_k+\sum'_{k'-1}+1}\cdots\\ &\cdot\left(\Psi_{\alpha'_{k'-(\gamma-1)}+\sum'_{k'-(\gamma-1)}+\sum_k}\uparrow^{\beta'_{k'-(\gamma-1)}+r'_{k'-(\gamma-1)}-1}\right)\downarrow_{\alpha'_{k'-(\gamma-1)}+\sum_k+\sum'_{k'-\gamma}+1}\\ &\cdot\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+\sum'_{k'-\gamma}+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\\ &\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+\sum'_{k'-\gamma}+r_1-1}\right)\downarrow_{\alpha_1+1}\\ &\cdot\left(\Psi_{\alpha'_{k'-\gamma}+\sum'_{k'-\gamma}}\uparrow^{\beta'_{k'-\gamma}+r'_{k'-\gamma}-1}\right)\downarrow_{\alpha'_{k'-\gamma}+\sum'_{k'-(\gamma+1)}+1}\cdots\\ &\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\end{aligned}$$

then we can apply Lemma 2.13 to

$$\begin{aligned}\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+\sum'_{k'-\gamma}+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+\sum'_{k'-\gamma}+r_1-1}\right)\downarrow_{\alpha_1+1}\\ \cdot\left(\Psi_{\alpha'_{k'-\gamma}+\sum'_{k'-\gamma}}\uparrow^{\beta'_{k'-\gamma}+r'_{k'-\gamma}-1}\right)\downarrow_{\alpha'_{k'-\gamma}+\sum'_{k'-(\gamma+1)}+1}\end{aligned}$$

(take: $x=\alpha_1,\ f_i=r_i,\ k_i=\alpha_{i+1}-\alpha_i,(i\in\{1,\dots,k-1\}),\ k_k=\alpha'_{k'-\gamma}-\alpha_k+\sum_{k'-(\gamma+1)},\ h=r'_{k'-\gamma},\ g_1=\beta_1-\alpha'_{k'-\gamma},\ g_i=\beta_i-\beta_{i-1}-r_{i-1},(i\in\{2,\dots,k\}),\ t=\beta'_{k'-\gamma}-\beta_k-\sum'_{k'-(\gamma+1)}-r_k)$. Then $l^j_c=i_{\alpha_j+\sum_{j-1}+c}$ for $j\in\{1,\dots,k\}$ and $c\in\{1,\dots,f_j\}$, whilst $m_b=i_{\alpha'_{k'-\gamma}+\sum_k+\sum'_{k'-(\gamma+1)}+b}$ for $b\in\{1,\dots,r'_{k'-\gamma}\}$ and we have that $l^j_c\not\leftrightarrow m_b$ for admissible j,c and b since the nodes of residues l^j_c belong to one of the moved sets of nodes whilst the nodes of residues m_b belong to the other.

After applying the lemma we have

$$\begin{aligned}&\left(\Psi_{\alpha'_{k'}+\sum'_{k'}+\sum_k}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum_k+\sum'_{k'-1}+1}\cdot\\ &\quad\cdots\left(\Psi_{\alpha'_{k'-\gamma}+\sum'_{k'-\gamma}+\sum_k}\uparrow^{\beta'_{k'-\gamma}+r'_{k'-\gamma}-1}\right)\downarrow_{\alpha'_{k'-\gamma}+\sum_k+\sum'_{k'-(\gamma+1)}+1}\\ &\qquad\cdot\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+\sum'_{k'-(\gamma+1)}+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\\ &\qquad\quad\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+\sum'_{k'-(\gamma+1)}+r_1-1}\right)\downarrow_{\alpha_1+1}\\ &\cdot\left(\Psi_{\alpha'_{k'-(\gamma+1)}+\sum'_{k'-(\gamma+1)}}\uparrow^{\beta'_{k'-(\gamma+1)}+r'_{k'-(\gamma+1)}-1}\right)\downarrow_{\alpha'_{k'-(\gamma+1)}+\sum'_{k'-(\gamma+2)}+1}\cdot\\ &\qquad\qquad\cdots\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\cdot\end{aligned}$$

All we have in effect done is 'replaced' γ with $\gamma+1$.

So apply the lemma repeatedly to the terms corresponding to $\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}$. Eventually we obtain the expression for $\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}$, thus $\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}=\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}$.

Case II: The skew shape containing β_1 is moved to a position below the other skew shape in $[\mu]$

In this case we have that

$$\begin{aligned}\varphi_{\nu_1\mu}\circ\varphi_{\lambda\nu_1}=&\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdots\\ &\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\cdot\left(\Psi_{\alpha_k+\sum_k}\uparrow^{\beta_k+r_k-1}\right)\downarrow_{\alpha_k+\sum_{k-1}+1}\cdots\\ &\qquad\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}\end{aligned}$$

whilst

$$\begin{aligned}\varphi_{\nu_2\mu}\circ\varphi_{\lambda\nu_2}=&\left(\Psi_{\alpha_k+\sum'_{k'}+\sum_k}\uparrow^{\beta_k+\sum'_{k'}+r_k-1}\right)\downarrow_{\alpha_k+\sum'_{k'}+\sum_{k-1}+1}\cdot\\ &\cdots\left(\Psi_{\alpha_1+\sum'_{k'}+r_1}\uparrow^{\beta_1+\sum'_{k'}+r_1-1}\right)\downarrow_{\alpha_1+\sum'_{k'}+1}\\ &\cdot\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdots\\ &\qquad\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\cdot\end{aligned}$$

Now observe that for $\gamma \in \{0, 1, \ldots, k-1\}$, if we have

$$\begin{aligned}&\left(\Psi_{\alpha_k+\sum'_{k'}+\sum_k}\uparrow^{\beta_k+\sum'_{k'}+r_k-1}\right)\downarrow_{\alpha_k+\sum'_{k'}+\sum_{k-1}+1}\cdot\\ &\cdots\cdot\left(\Psi_{\alpha_{k-\gamma+1}+\sum'_{k'}+\sum_{k-\gamma+1}}\uparrow^{\beta_{k-\gamma+1}+\sum'_{k'}+r_{k-\gamma+1}-1}\right)\downarrow_{\alpha_{k-\gamma+1}+\sum'_{k'}+\sum_{k-\gamma}+1}\\ &\cdot\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdot\cdots\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\\ &\cdot\left(\Psi_{\alpha_{k-\gamma}+\sum_{k-\gamma}}\uparrow^{\beta_{k-\gamma}+r_{k-\gamma}-1}\right)\downarrow_{\alpha_{k-\gamma}+\sum_{k-(\gamma+1)}+1}\cdots\\ &\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}\end{aligned}$$

then we can apply Lemma 2.14 to

$$\begin{aligned}&\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdot\cdots\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\\ &\cdot\left(\Psi_{\alpha_{k-\gamma}+\sum_{k-\gamma}}\uparrow^{\beta_{k-\gamma}+r_{k-\gamma}-1}\right)\downarrow_{\alpha_{k-\gamma}+\sum_{k-\gamma-1}+1}\end{aligned}$$

(take: $x = \alpha'_1$, $f_i = r'_i$, $k_i = \alpha'_{i+1} - \alpha'_i, (i \in \{1, \ldots, k-1\})$, $k_k = \alpha_{k-\gamma} - \alpha'_{k'} + \sum_{k-(\gamma+1)}$, $h = r_{k-\gamma}$, $g = \beta_{k-\gamma} - \alpha_{k-\gamma} - \sum_{k-(\gamma+1)}$, $t_1 = \beta'_1 - \beta_{k-\gamma} - r_{k-\gamma}$, $t_i = \beta'_i - \beta'_{i-1} - r'_{i-1}, (i \in \{2, \ldots, k\})$). Then $l^j_c = i_{\alpha'_j+\sum'_{j-1}+c}$ for $j \in \{1, \ldots, k'\}$ and $c \in \{1, \ldots, f_j\}$, whilst $m_b = i_{\alpha_{k-\gamma}+\sum'_{k'}+\sum_{k-(\gamma+1)}+b}$ for $b \in \{1, \ldots, r_{k-\gamma}\}$ and we have that $l^j_c \not\!\!- m_b$ for admissible j, c, and b since the nodes of residues l^j_c belong to one of the moved sets of nodes whilst the nodes of residues m_b belong to the other. After applying the lemma we have:

$$\begin{aligned}&\left(\Psi_{\alpha_k+\sum'_{k'}+\sum_k}\uparrow^{\beta_k+\sum'_{k'}+r_k-1}\right)\downarrow_{\alpha_k+\sum'_{k'}+\sum_{k-1}+1}\cdot\\ &\cdots\cdot\left(\Psi_{\alpha_{k-\gamma}+\sum'_{k'}+\sum_{k-\gamma}}\uparrow^{\beta_{k-\gamma}+\sum'_{k'}+r_{k-\gamma}-1}\right)\downarrow_{\alpha_{k-\gamma}+\sum'_{k'}+\sum_{k-\gamma-1}+1}\\ &\cdot\left(\Psi_{\alpha'_{k'}+\sum'_{k'}}\uparrow^{\beta'_{k'}+r'_{k'}-1}\right)\downarrow_{\alpha'_{k'}+\sum'_{k'-1}+1}\cdot\cdots\cdot\left(\Psi_{\alpha'_1+r'_1}\uparrow^{\beta'_1+r'_1-1}\right)\downarrow_{\alpha'_1+1}\\ &\cdot\left(\Psi_{\alpha_{k-(\gamma+1)}+\sum_{k-(\gamma+1)}}\uparrow^{\beta_{k-(\gamma+1)}+r_{k-(\gamma+1)}-1}\right)\downarrow_{\alpha_{k-(\gamma+1)}+\sum_{k-(\gamma+2)}+1}\cdot\\ &\cdots\cdot\left(\Psi_{\alpha_1+r_1}\uparrow^{\beta_1+r_1-1}\right)\downarrow_{\alpha_1+1}\end{aligned}$$

What we have in effect done is 'replaced' γ with $\gamma + 1$.

So apply the lemma repeatedly to the terms corresponding to $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1}$. Eventually we obtain the expression for $\varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}$, thus $\varphi_{\nu_1\mu} \circ \varphi_{\lambda\nu_1} = \varphi_{\nu_2\mu} \circ \varphi_{\lambda\nu_2}$.

$\square$

We can again make an analogue to Corollaries 3.5 and 3.11, that is that if $[\mu]$ is formed from $[\lambda]$ by moving multiple skew shapes of nodes whose residues are sufficiently spread apart, then we can move the rows in any order to get various homomorphisms which always compose to give the same overall homomorphism.

Corollary 3.20. *Let $l \geq 2$ and suppose that λ and μ are l-multipartitions of n. Suppose that $[\mu]$ is formed from $[\lambda]$ by moving m distinct skew shapes of nodes $S_1, \dots, S_m$, whose residues amongst the skew shapes are such that none are equal or adjacent between any two given skew shapes.*

Suppose that for each $X \subseteq \{1, \dots, m\}$ we have an l-multipartition of n, ν_X, such that $[\nu_{\{i_1,\dots,i_t\}}]$ is formed from $[\lambda]$ by moving just the skew shapes $S_{i_1}, \dots, S_{i_t}$. In particular $\nu_\varnothing = \lambda$ and $\nu_{\{1,\dots,m\}} = \mu$. Suppose that whenever $|B \setminus A| = 1$, we have that (ν_A, ν_B) is a skew pair, whose corresponding homomorphism is $\varphi_{\nu_A\nu_B}$.

Then there is a non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ and given any sequence of sets $\varnothing = X_0 \subsetneq X_1 \subsetneq X_2, \subsetneq \cdots \subsetneq X_m = \{1, \dots, m\}$ we have that

$$\varphi = \varphi_{\nu_{X_{m-1}}\nu_{X_m}} \circ \varphi_{\nu_{X_{m-2}}\nu_{X_{m-1}}} \circ \cdots \circ \varphi_{\nu_{X_0}\nu_{X_1}}.$$

Proof. Without loss of generality suppose that the shape S_a is above S_b whenever $a < b$. Let $Y_j \coloneqq \{1, 2, \dots, j\}$ for $j \in \{0, \dots, m\}$. Then $\varnothing = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m = \{1, 2, \dots, m\}$. By assumption we have l-multipartitions of n, ν_{Y_j}, and non-zero homomorphisms $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}$ for each $j \in \{0, \dots, m-1\}$. As in Corollary 3.19, we may write $\varphi_{\nu_{Y_j}\nu_{Y_{j+1}}}(v^{\mathfrak{t}^{\nu_{Y_j}}})$ as in (3.60) and then similarly to Corollary 3.11 the product $\varphi_{\nu_{Y_{m-1}}\nu_{Y_m}} \circ \cdots \circ \varphi_{\nu_{Y_0}\nu_{Y_1}}(v^{\mathfrak{t}^\lambda})$ will correspond to a reduced expression, since in $\mathfrak{t}^\lambda$, the smallest entry within S_j is strictly greater than the largest entry within S_{j-1} for every $j \in \{2, \dots, m\}$. Hence no strings will cross twice, and thus as the associated tableau will be standard, the composition of homomorphisms is not zero.

The rest of the proof is the same as that for Corollary 3.5, replacing the use of Corollary 3.4 with Corollary 3.19. $\square$

To conclude this section, we give a conjecture that is concerned with relaxing the conditions that define a skew pair. Given a partition λ, write $\hat{\lambda} \to \lambda$ if $\hat{\lambda}$ is a partition formed form λ by removing a node. Write $\hat{\lambda} \xrightarrow{m} \lambda$ if there is a sequence $\hat{\lambda} = \lambda_0 \to \lambda_1 \to \cdots \to \lambda_m = \lambda$ for some $m \geq 0$. Now given a skew shape S of the form $[\lambda \setminus \mu]$ whose nodes have associated residues, define the set X_S as follows:

$$X_S \coloneqq \Big\{X \mid X \text{ is a non-empty connected skew shape with associated residues from } S \text{ of the form } [\hat{\lambda} \setminus \mu] \text{ where } \hat{\lambda} \xrightarrow{m} \lambda \text{ for some } m \geq 0\Big\}$$

Now for l-multipartitions λ and μ, define $(\lambda, \mu)^*$ to be a *skew* pair* if the requirements of Definition 3.16 are satisfied, except that amongst each component $\lambda^{(c')}$ that are not of the k components to which a skew shape of the same residues can be added, there are no removable shapes X such that X belongs to X_S, where S is the skew shape to be moved to form $[\mu]$ from $[\lambda]$. Then we have the following conjecture, which is an adaptation of Corollary 3.18.

Conjecture 3.21. *Suppose that $(\lambda, \mu)^*$ is a skew* pair. Let $\mathfrak{s}$ be the μ-tableau defined by considering $\mathfrak{t}^\lambda$ and moving the skew shape from the qth component to the pth, keeping their values intact. Then there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{t}^\mu} \psi^{\mathfrak{s}}$.*

The difficulty in proving this conjecture relates to the fact that in proving Theorem 3.14 we only ever rely on using terms arising from the bottom two rows of the skew shape (i.e. we never utilise the term R in $\psi^{\mathfrak{s}}$). When dealing with skew* pairs, we find that it is necessary to deal with terms arising from other rows in addition to these, and given the already unwieldy nature of our combinatorics, this would appear to be a step too far.

We also conjecture that there are similar adaptations of Corollaries 3.19 and 3.20. We give some examples below of homomorphisms which exist, but cannot be proved to exist using Corollary 3.18 and instead fall under those covered by the conjecture.

Example 3.22. Let $e = 4$, $\kappa = (0, 1, 0)$, $\lambda = (\varnothing, (3), (4))$ and $\mu = ((4), (3), \varnothing)$. Then $X_S \coloneqq \left\{ \boxed{0}\,, \boxed{0}\boxed{1}\,, \boxed{0}\boxed{1}\boxed{2}\,, \boxed{0}\boxed{1}\boxed{2}\boxed{3} \right\}$ and there is a homomorphism

$\varphi : S^\lambda \to S^\mu$ given by

$$\mathfrak{t}^\lambda = \left(\varnothing \;\middle|\; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 4&5&6&7\\\hline\end{array} \right) \mapsto \left(\begin{array}{|c|c|c|c|}\hline 4&5&6&7\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \;\middle|\; \varnothing \right).$$

Note that $\lambda^{(2)}$ contains a removable 3 node, but $\boxed{3} \notin X_S$. ◊

Example 3.23. Let $e = 4$, $\kappa = (0, \kappa_2, 0)$, $\lambda = (\varnothing, (1), (2, 2))$, and $\mu = ((2, 2), (1), \varnothing)$. Then $X_S := \left\{ \begin{array}{|c|}\hline 0\\\hline\end{array}, \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}, \begin{array}{|c|}\hline 0\\\hline 3\\\hline\end{array}, \begin{array}{|c|c|}\hline 0&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 0&1\\\hline 3&0\\\hline\end{array} \right\}$ and then for any $\kappa_2 \neq 0$ there is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$\mathfrak{t}^\lambda = \left(\varnothing \;\middle|\; \begin{array}{|c|}\hline 1\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array} \right) \mapsto \left(\begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array} \;\middle|\; \begin{array}{|c|}\hline 1\\\hline\end{array} \;\middle|\; \varnothing \right).$$

◊

3.4 Relaxing the diagonal residue condition

We now exhibit some examples which demonstrate some possible effects when relaxing the diagonal residue condition, that is, working with small e.

Example 3.24. Let $e = 3$, $\lambda = ((1), (6, 5))$, $\mu = ((4, 3), (3, 2))$, $\kappa = (0, 1)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|}\hline 1\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|c|}\hline 2&3&4&5&6&7\\\hline 8&9&10&11&12\\\cline{1-5}\end{array} \right).$$

Then there is exactly one non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}}$$

where

$$\mathfrak{s} = \left(\begin{array}{|c|c|c|c|}\hline 1&5&6&7\\\hline 10&11&12\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 8&9\\\cline{1-2}\end{array} \right)$$

and

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 10&11&12\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 5&6&7\\\hline 8&9\\\cline{1-2}\end{array} \right).$$

◊

Note that the tableau $\mathfrak{s}$ arises in the same way as we expect from Theorem 3.14, however we also have a term indexed by the tableau $\mathfrak{t}$, and naively speaking this is formed by acting on $\mathfrak{s}$ by the permutation $(2,5)(3,6)(4,7)$.

Example 3.25. Let $e = 3$, $\lambda = ((1),(7,6))$, $\mu = ((4,3),(4,3))$, $\kappa = (0,0)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|}\hline 1\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|c|c|}\hline 2&3&4&5&6&7&8\\\hline 9&10&11&12&13&14\\\cline{1-6}\end{array} \right).$$

Then there is exactly one non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}} + 2v^{\mathfrak{u}} + 4v^{\mathfrak{v}}$$

where

$$\mathfrak{s} = \left(\begin{array}{|c|c|c|c|}\hline 1&6&7&8\\\hline 12&13&14\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 9&10&11\\\cline{1-3}\end{array} \right),$$

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|c|}\hline 1&6&7&8\\\hline 9&10&11\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 12&13&14\\\cline{1-3}\end{array} \right),$$

$$\mathfrak{u} = \left(\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 12&13&14\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 2&6&7&8\\\hline 9&10&11\\\cline{1-3}\end{array} \right),$$

and

$$\mathfrak{v} = \left(\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 9&10&11\\\cline{1-3}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 2&6&7&8\\\hline 12&13&14\\\cline{1-3}\end{array} \right).$$

$\Diamond$

Similarly to the previous example, the tableau $\mathfrak{s}$ arises as we expect, and we have

$$\begin{aligned}
\mathfrak{t} &= \mathfrak{s}(9,12)(10,13)(11,14)\\
\mathfrak{u} &= \mathfrak{s}(3,6)(4,7)(6,8), \text{ and}\\
\mathfrak{v} &= \mathfrak{s}(3,6)(4,7)(6,8)(9,12)(10,13)(11,14).
\end{aligned}$$

Example 3.26. Let $e = 3$, $\lambda = ((1),(9,8))$, $\mu = ((4,3),(6,5))$, $\kappa = (0,1)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|}\hline 1\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 2&3&4&5&6&7&8&9&10\\\hline 11&12&13&14&15&16&17&18\\\cline{1-8}\end{array} \right).$$

Then there is exactly one non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}} + 2v^{\mathfrak{u}} + 4v^{\mathfrak{v}} + 3v^{\mathfrak{w}} + 6v^{\mathfrak{x}}$$

where

$$\mathfrak{s} = \left(\begin{array}{cccc} 1 & 8 & 9 & 10 \\ 16 & 17 & 18 & \end{array} \;\middle|\; \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \\ 11 & 12 & 13 & 14 & 15 & \end{array} \right),$$

$$\mathfrak{t} = \left(\begin{array}{cccc} 1 & 8 & 9 & 10 \\ 13 & 14 & 15 & \end{array} \;\middle|\; \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \\ 11 & 12 & 16 & 17 & 18 & \end{array} \right),$$

$$\mathfrak{u} = \left(\begin{array}{cccc} 1 & 5 & 6 & 7 \\ 16 & 17 & 18 & \end{array} \;\middle|\; \begin{array}{cccccc} 2 & 3 & 4 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & \end{array} \right),$$

$$\mathfrak{v} = \left(\begin{array}{cccc} 1 & 5 & 6 & 7 \\ 13 & 14 & 15 & \end{array} \;\middle|\; \begin{array}{cccccc} 2 & 3 & 4 & 8 & 9 & 10 \\ 11 & 12 & 16 & 17 & 18 & \end{array} \right),$$

$$\mathfrak{w} = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 16 & 17 & 18 & \end{array} \;\middle|\; \begin{array}{cccccc} 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & \end{array} \right),$$

and

$$\mathfrak{x} = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & \end{array} \;\middle|\; \begin{array}{cccccc} 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 16 & 17 & 18 & \end{array} \right).$$

$\Diamond$

Again, the tableau $\mathfrak{s}$ arises as we expect, and we have

$$\mathfrak{t} = \mathfrak{s}(13,16)(14,17)(15,18),$$

$$\mathfrak{u} = \mathfrak{s}(5,8)(6,9)(7,10),$$

$$\mathfrak{v} = \mathfrak{s}(5,8)(6,9)(7,10)(13,16)(14,17)(15,18),$$

$$\mathfrak{w} = (5,8)(6,9)(7,10)(2,5)(3,6)(4,7) = \mathfrak{s}(2,5,8)(3,6,9)(4,7,10), \text{ and}$$

$$\mathfrak{x} = \mathfrak{s}(2,5,8)(3,6,9)(4,7,10)(13,16)(14,17)(15,18).$$

These examples appear to exhibit a pattern in the images of the homomorphisms. However, if we work with larger components we see that this does not work quite as nicely as in the last examples.

Example 3.27. Let $e = 3$, $\lambda = ((3,2),(7,6))$, $\mu = ((6,5),(4,3))$, $\kappa = (0,2)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \middle| \begin{array}{ccccccc} 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 & \end{array} \right).$$

Then there is exactly one non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}} + 2v^{\mathfrak{u}} + 4v^{\mathfrak{v}}$$

where

$$\mathfrak{s} = \left(\begin{array}{cccccc} 1 & 2 & 3 & 5 & 11 & 12 \\ 4 & 10 & 16 & 17 & 18 & \end{array} \middle| \begin{array}{cccc} 6 & 7 & 8 & 9 \\ 13 & 14 & 15 & \end{array} \right),$$

$$\mathfrak{t} = \left(\begin{array}{cccccc} 1 & 2 & 3 & 5 & 11 & 12 \\ 4 & 10 & 13 & 14 & 15 & \end{array} \middle| \begin{array}{cccc} 6 & 7 & 8 & 9 \\ 16 & 17 & 18 & \end{array} \right),$$

$$\mathfrak{u} = \left(\begin{array}{cccccc} 1 & 2 & 3 & 5 & 8 & 9 \\ 4 & 7 & 16 & 17 & 18 & \end{array} \middle| \begin{array}{cccc} 6 & 10 & 11 & 12 \\ 13 & 14 & 15 & \end{array} \right),$$

and

$$\mathfrak{v} = \left(\begin{array}{cccccc} 1 & 2 & 3 & 5 & 8 & 9 \\ 4 & 7 & 13 & 14 & 15 & \end{array} \middle| \begin{array}{cccc} 6 & 10 & 11 & 12 \\ 16 & 17 & 18 & \end{array} \right).$$

$\diamondsuit$

This time, none of the tableaux are of the ‘expected form’. However, note that

$$\begin{aligned} \mathfrak{t} &= \mathfrak{s}(13,16)(14,17)(15,18), \\ \mathfrak{u} &= \mathfrak{s}(7,10)(8,11)(9,12), \text{ and} \\ \mathfrak{v} &= \mathfrak{s}(7,10)(8,11)(9,12)(13,16)(14,17)(15,18), \end{aligned}$$

displaying a similar relationship between the tableaux as before.

From a naive point of view, the tableau we expect (as in Theorem 3.14) is prevented from appearing because there are now other nodes of the residues that we are moving, that lie between the position where we remove these nodes in $\lambda^{(2)}$ and where we add them in $\mu^{(1)}$, and that in some cases we can move these to the positions in $\mu^{(1)}$ before moving the nodes in the skew shape there.

The patten becomes obfuscated when we begin to deal with multipartitions containing a component consisting of at least e rows, as in the following example:

Example 3.28. Let $e = 3$, $\lambda = ((3,2,1),(7,6,5))$, $\mu = ((6,5,4),(4,3,2))$, $\kappa = (0,2)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array} \;\middle|\; \begin{array}{ccccccc} 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 14 & 15 & 16 & 17 & 18 & 19 & \\ 20 & 21 & 22 & 23 & 24 & & \end{array} \right).$$

Then there is exactly one homomorphism $\varphi : S^\lambda \to S^\mu$, whose image consists of 16 terms and whose coefficients belong to the set $\{-4,-2,1,2,4,\}$. $\lozenge$

A similar, but more explicit example is as follows.

Example 3.29. Let $e = 3$, $\lambda = ((1),(2^2,1))$, $\mu = ((2^3,1),(2^4,1))$, $\kappa = (0,1)$. The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{array}{c} 1 \end{array} \;\middle|\; \begin{array}{cc} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 & 9 \\ 10 & 11 \\ 12 & 13 \\ 14 & 15 \\ 16 & \end{array} \right).$$

There is exactly one non-zero homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda}$ maps to:

$$\left(\begin{array}{cc} 1 & 11 \\ 9 & 13 \\ 14 & 15 \\ 16 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 & 12 \\ 10 & \end{array} \right) + \left(\begin{array}{cc} 1 & 8 \\ 3 & 10 \\ 5 & 12 \\ 7 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 6 \\ 4 & 11 \\ 9 & 13 \\ 14 & 15 \\ 16 & \end{array} \right) - \left(\begin{array}{cc} 1 & 5 \\ 6 & 7 \\ 8 & 12 \\ 10 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 3 \\ 4 & 11 \\ 9 & 13 \\ 14 & 15 \\ 16 & \end{array} \right)$$
$$+ \left(\begin{array}{cc} 1 & 5 \\ 3 & 7 \\ 14 & 15 \\ 16 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 6 \\ 4 & 8 \\ 9 & 10 \\ 11 & 12 \\ 13 & \end{array} \right) + 3 \left(\begin{array}{cc} 1 & 5 \\ 3 & 7 \\ 11 & 12 \\ 13 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 6 \\ 4 & 8 \\ 9 & 10 \\ 14 & 15 \\ 16 & \end{array} \right) - 2 \left(\begin{array}{cc} 1 & 5 \\ 3 & 7 \\ 8 & 12 \\ 10 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 9 \\ 4 & 11 \\ 6 & 13 \\ 14 & 15 \\ 16 & \end{array} \right)$$
$$+3 \left(\begin{array}{cc} 1 & 5 \\ 3 & 7 \\ 8 & 9 \\ 10 & \end{array} \;\middle|\; \begin{array}{cc} 2 & 6 \\ 4 & 11 \\ 12 & 13 \\ 14 & 15 \\ 16 & \end{array} \right) - 1 \left(\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 8 & 12 \\ 10 & \end{array} \;\middle|\; \begin{array}{cc} 5 & 6 \\ 7 & 11 \\ 9 & 13 \\ 14 & 15 \\ 16 & \end{array} \right)$$

$\lozenge$

In the following Chapter, we will reinvestigate the pattern observed here in the context of multipartitions within core blocks in level 2, and we will claim that the preventions which restrict the pattern from occurring will not arise.

Chapter 4

Core blocks

Within the previous chapter we have demonstrated the existence of certain homomorphisms between Specht modules. In this chapter, we will use these homomorphisms to prove Theorem 4.26, which, provided e is large enough, allows us to describe the entire set of *dominated* homomorphism spaces between Specht modules that lie in *core blocks* when $l = 2$. Once again, all results will be independent of the characteristic of the base field $\mathbb{F}$. We shall first need to state a few relevant definitions so that we can understand core blocks precisely.

Let A be a finite-dimensional algebra over $\mathbb{F}$, and suppose $A = B_1 \oplus \cdots \oplus B_c$ is a decomposition of A into a direct sum of indecomposable two-sided ideals. Then we call the B_i the *blocks* of A. As remarked following Definition 1.14, the blocks of $\mathscr{H}_n^{\Lambda_\kappa}$ are given by the algebras $\mathscr{H}_\alpha^{\Lambda_\kappa}$, that is, two Specht modules S^λ and S^μ belong to the same block of $\mathscr{H}_n^{\Lambda_\kappa}$ if and only if the multipartitions λ and μ have the same content. We will say that a multipartition λ lies in a block B of $\mathscr{H}_n^{\Lambda_\kappa}$ if S^λ lies in B.

Let λ be a partition of n. The *rim* of $[\lambda]$ is defined to be the set of nodes

$$\{(i,j) \in [\lambda] \mid (i+1, j+1) \notin [\lambda]\}.$$

For $e \in \{2,3,4,\dots\}$ define an *e-rim hook* to be a connected subset R of the rim containing exactly e nodes such that $[\lambda] \setminus R$ is the diagram of a partition. If λ has no e-rim hooks, or if $e = \infty$, then we say that λ is an *e-core*. If we can remove w e-rim hooks from $[\lambda]$ to produce an e-core, then we say that λ has *e-weight* w. In particular, an e-core has weight 0.

Note that given the abacus configuration for λ, removing an e-rim hook from $[\lambda]$ corresponds to sliding a bead up one row on the abacus. So an abacus configuration for an e-core has all the beads pushed up the runners to their highest possible positions. Using this we obtain the next result which demonstrates that the definition of e-weight is well-defined.

Lemma 4.1. *[Jam78b] [Mat99, Lemma 5.35] Let λ be a partition. Then the e-core and e-weight of λ depend only on λ and e.*

Now we can state when two Specht modules lie in the same block of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$.

Theorem 4.2 (The Nakayama conjecture)**.** *[DJ87, Corollary 4.4] [JM97, Theorem 4.29] Suppose that λ and μ are partitions of n. Then the Specht modues S^λ and S^μ belong to the same block of $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ if and only if λ and μ have the same e-core.*

Of course, we wish to study core blocks for Ariki-Koike algebras as well as KLR algebras. Due to Theorem 1.16, and since the theory we detail is the same in both settings, we simultaneously develop both cases. We use the following definition to extend the notion of an e-core to these algebras.

Definition 4.3. An l-multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ is an *e-multicore* if $\lambda^{(i)}$ is an e-core for each $i \in \{1, \ldots, l\}$.

Note that when $e = \infty$, every multipartition is an e-core. For $l = 1$, an e-multicore is an e-core. There is also an analogous definition for the weight of a multipartition, however as we will deal only with *core blocks* we shall not be required to state the long setup. The relevant details are given in [Fay06] and [Fay07].

As we have seen, the weight and core of a partition λ play an important role in determining the block that S^λ belongs to and its properties within $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$. In particular, Theorem 4.2 states that two Specht modules S^λ and S^μ belong to the same block if and only if λ and μ have the same core. However, for $l > 1$ the natural generalisation of this is not necessarily true; S^λ and S^μ may belong to the same block yet λ may be a multicore whilst μ is not.

Example 4.4. Let $e = 4$, $\kappa = (3,1)$ and consider $\lambda = ((3,1^2),(3))$ and $\mu = ((1^3),(5))$. Then λ and μ have the same content and thus belong to the same block, but λ is a multicore whilst the second component of μ has a removable 4-rim hook. $\Diamond$

In this chapter we study certain core blocks when $l = 2$. Following the work of Fayers in [Fay06] and [Fay07], we shall define core blocks for arbitrary l and use the results of the previous chapters in order to study homomorphisms within these blocks.

Suppose $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ is a multicore and $\mathfrak{a} = (a_1, \dots, a_l) \in \mathbb{Z}^l$. If $e < \infty$ then we define $\mathfrak{b}^{\mathfrak{a}}_{ij}(\lambda)$ to be the position of the lowest bead on runner i of the abacus display for $\lambda^{(j)}$ with respect to $\mathfrak{a}$. In other words, $\mathfrak{b}^{\mathfrak{a}}_{ij}(\lambda)$ is the largest element of $\beta_{a_j}(\lambda)$ that is congruent to i modulo e.

Now we can state the definition of a core block for Ariki-Koike algebras (and hence KLR algebras) using the following theorem.

Theorem 4.5. *[Fay07, Theorem 3.1] Suppose $e \in \{2,3,4,\dots\}$ and that λ is a multipartition with S^λ lying in a block B of $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. Let κ be the multicharge associated to this algebra. Then the following are equivalent.*

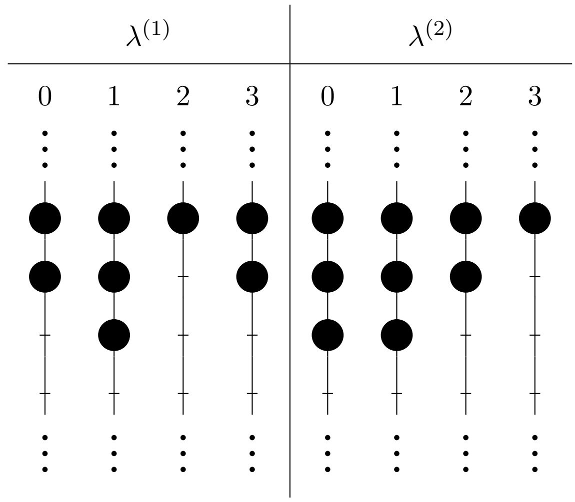
1. *λ is a multicore, and there exists $\mathfrak{a} = (a_1, \dots, a_l) \in \mathbb{Z}^l$ such that $a_i \equiv \kappa_i \mod e$ and integers $\alpha_0, \dots, \alpha_{e-1}$ such that for each $i \in I$ and $j \in \{1, \dots, l\}$, $\mathfrak{b}^{\kappa}_{ij}(\lambda)$ equals either α_i or $\alpha_i + e$.*

2. *Every multipartition in B is a multicore.*

Definition 4.6. Suppose B is a block of $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. Then we say B is a *core block* if

- $e \in \{2,3,4,\dots\}$ and the conditions of Theorem 4.5 are satisfied for any λ in B, or
- $e = \infty$.

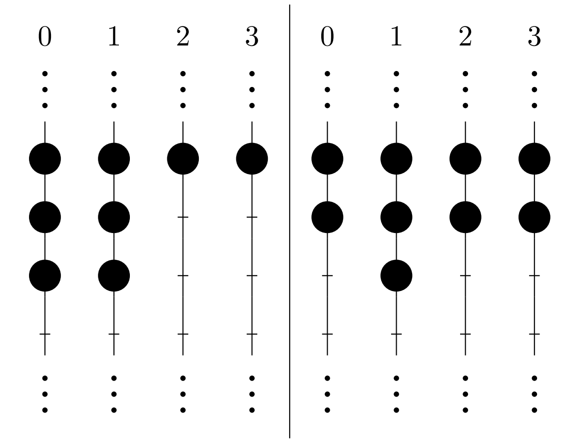
Example 4.7. Let $e = 4$, $\kappa = (0,1)$, and let $\lambda = ((2,1),(1^2))$. The abacus configuration for λ is:



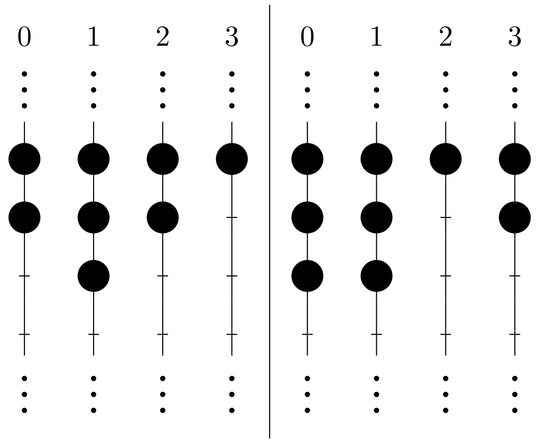
Let B be the block containing S^λ. Then since

$$\begin{aligned} \mathfrak{b}^{\kappa}_{01} &= -4, & \mathfrak{b}^{\kappa}_{02} &= 0, \\ \mathfrak{b}^{\kappa}_{11} &= 1, & \mathfrak{b}^{\kappa}_{12} &= 1, \\ \mathfrak{b}^{\kappa}_{21} &= -6, & \mathfrak{b}^{\kappa}_{22} &= -2, \\ \mathfrak{b}^{\kappa}_{31} &= -1, & \mathfrak{b}^{\kappa}_{32} &= -5, \end{aligned}$$

we may take $\alpha_0 = -4$, $\alpha_1 = 1$, $\alpha_2 = -6$ and $\alpha_3 = -5$ to see that B is a core block. The other multipartitions in the block are $((2,2),(1))$ and $((2),(1^3))$ with respective abacus configurations



and



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◊

4.1 Core blocks in level 2 and plus minus sequences

Now we will consider core blocks for Ariki-Koike algebras in level 2, i.e. when $l = 2$. In this section we shall use Theorems 4.14 and 4.15 that are from personal communication with Lyle. They supplement results of Brundan and Stroppel [BS11] and Hu and Mathas [HM10] which state that when $e = \infty$ or $e > n$, the decomposition numbers are independent of the characteristic of the base field, no matter the weight of the block.

To begin, let $e < \infty$ and consider the abacus configuration for a partition λ. The set $\beta_a(\lambda)$ used to define the abacus is an infinite set, and as such we have an infinite amount of beads in the abacus configuration, in particular there is a point where every row to the north of this point is completely full of beads. Instead we can consider a *truncated* abacus configuration which has only finitely many beads on each runner, which we associate to a multipartition by filling in all the rows north of the highest beads with beads in every position.

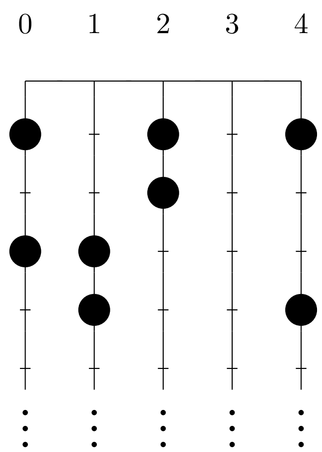
Conversely, if we are given a partition λ we can fix a truncated abacus configuration associated to it. Let N be maximal so that $x \in \beta_a(\lambda)$ whenever $x < Ne$. Then we define the truncated abacus configuration for λ to be the one corresponding to the set $\beta_a(\lambda) \cap \{Ne, Ne + 1, \ldots\}$. In the same way we can associate an l-tuple of truncated abacus configurations to an l-multipartition.

Example 4.8. Let $e = 5$, $a = 1$ and $\lambda = (12, 10, 6^2, 4, 2, 1)$, as in Example 1.25.

Then we take $N = -1$, so that

$$\beta_a(\lambda) \cap \{-5, -4, -3, \ldots\} = \{14, 11, 6, 5, 2, -1, -3, -5\}$$

and the truncated abacus configuration is:



$\Diamond$

Now let λ be a multipartition, $e \in \{2, 3, 4, \ldots\}$, κ an e-multicharge, $\mathfrak{a} = (a_1, \ldots, a_l) \in \mathbb{Z}^l$ such that $a_i \equiv \kappa_i \mod e$ and define $\mathsf{b}^{\mathfrak{a}}_{ij}(\lambda)$ to be the number of beads on runner i of the truncated abacus display for $\lambda^{(j)}$ with respect to $\mathfrak{a}$. Using Theorem 4.5, we see that for λ corresponding to S^λ in a core block, we have that there are integers $b_0, b_1, \ldots, b_{e-1}$ such that for each $i \in I$ and $j \in \{1, \ldots, l\}$, $\mathsf{b}^{\mathfrak{a}}_{ij}(\lambda)$ equals either b_i or $b_i + 1$. We call such an e-tuple $(b_0, b_1, \ldots, b_{e-1})$ a *base tuple* for λ. Adapting Theorem 4.5 we have the following result.

Proposition 4.9. *Suppose $l = 2$, $e \in \{2, 3, 4, \ldots\}$, λ is a multicore and $\kappa = (\kappa_1, \kappa_2)$ is a 2-multicharge for $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$. Then S^λ lies in a core block of $\mathcal{H}_{\mathbb{F},q,Q}(\mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n)$ if and only if there is $\mathfrak{a} = (a_1, a_2) \in \mathbb{Z}^2$ such that $a_i \equiv \kappa_i \mod e$ and an abacus configuration for λ such that $|\mathsf{b}^{\mathfrak{a}}_{i2}(\lambda) - \mathsf{b}^{\mathfrak{a}}_{i1}(\lambda)| \leq 1$ for each $i \in I$.*

Suppose $l = 2$ and we have a base tuple $B = (b_0, b_1, \ldots, b_{e-1})$ for λ lying in a core block along with κ and $\mathfrak{a}$ as above. Then we define a total order $\prec$ on

$\{0, 1, \ldots, e-1\}$ by

$$i \prec j \iff b_i < b_j \text{ or } b_i = b_j \text{ and } i < j$$

and we let π be the permutation of $\{0, 1, \ldots, e-1\}$ such that

$$\pi(0) \prec \pi(1) \prec \cdots \prec \pi(e-1).$$

Then define

$$d_i := \begin{cases} + & \text{if } \mathsf{b}^{\mathfrak{a}}_{\pi(i)2}(\lambda) - \mathsf{b}^{\mathfrak{a}}_{\pi(i)1}(\lambda) = 1, \\ 0 & \text{if } \mathsf{b}^{\mathfrak{a}}_{\pi(i)2}(\lambda) - \mathsf{b}^{\mathfrak{a}}_{\pi(i)1}(\lambda) = 0, \\ - & \text{if } \mathsf{b}^{\mathfrak{a}}_{\pi(i)2}(\lambda) - \mathsf{b}^{\mathfrak{a}}_{\pi(i)1}(\lambda) = -1, \end{cases}$$

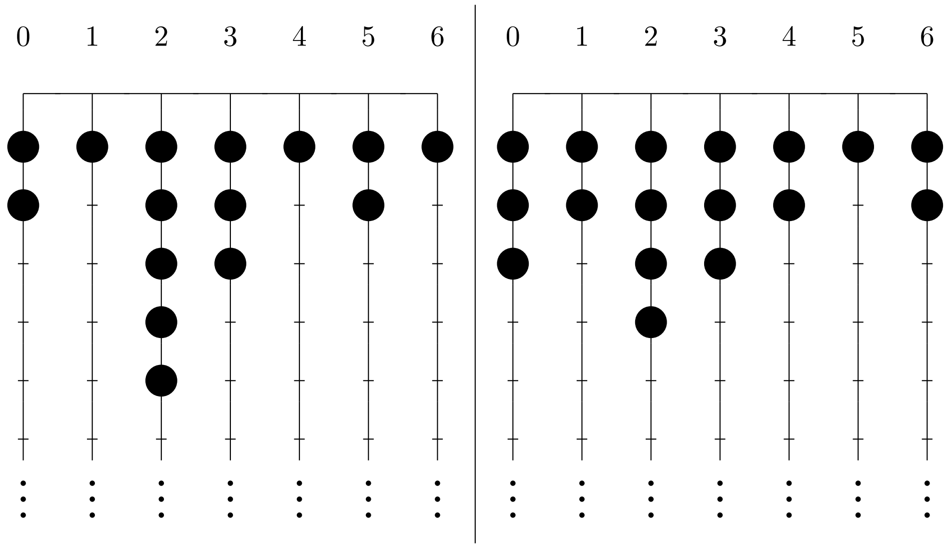
and so for a multipartition λ, and $e \in \{2, 3, 4, \ldots\}$ we get the *plus minus sequence*

$$d(\lambda) = (d_0, d_1, \ldots, d_{e-1}).$$

Naively, we obtain $d(\lambda)$ by ordering the runners so that a runner with a lower base tuple entry precedes one with a greater entry, and if these are the same then they are ordered from left to right. Then d_i equals the symbol $+$ if the ith runner in this order contains one more bead in the second component than the first, it equals the symbol $-$ if this ith runner contains one less bead in the second component than the first, and it equals 0 if this ith runner contains the same number of beads in both components.

The *reduced plus minus sequence* $\hat{d}(\lambda)$ is obtained by removing all 0s and recursively cancelling adjacent pairs $-, +$ within the sequence $d(\lambda)$. If such a pair $-, +$ can be cancelled in this way we say they are *linked by an arc* or call them a *linked pair*, and draw an arc between them. Notation-wise, to distinguish a reduced sequence $\hat{d}(\lambda)$ from a sequence $d(\lambda)$ we also remove the commas.

Example 4.10. Consider the truncated abacus configuration for a multipartition λ given below



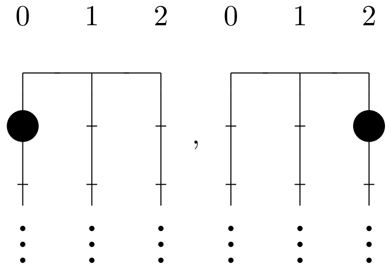
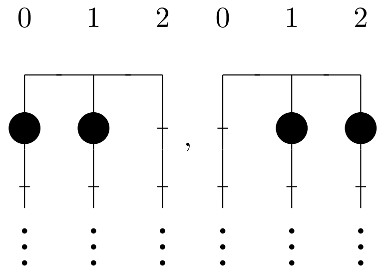
with base tuple $(2, 1, 4, 3, 1, 1, 1)$. Then $\pi = (0, 1, 4)(2, 5, 3, 6)$ and

$$d(\lambda) = (+, +, \overset{\frown}{-\ ,\ +}\ , +, 0, -).$$

The reduced sequence is $\hat{d}(\lambda) = (+ + + -)$. ◊

Note that when an entry of $d(\lambda)$ is zero, it is not clear as to whether the number of beads on the corresponding runner is equal to the corresponding entry of the base tuple, or if it is one more than it. Thus we cannot necessarily uniquely construct a multipartition given just a plus minus sequence and a base tuple. However, we can write 0_N to signify that we intend the corresponding runner to have 'no' extra bead, i.e. the number of beads on the runner is equal to the corresponding entry of the base tuple, and 0_B when the number of beads is one more than the corresponding entry of the base tuple.

Example 4.11. Let $e = 3$ and $B = (0, 0, 0)$. Then the following table illustrates an example of the difference between 0_N and 0_B.

$d(\lambda)$	Pair of abacuses	Multipartition
$(-, 0_N, +)$		$(\varnothing, (2))$
$(-, 0_B, +)$		$(\varnothing, (1^2))$

$\Diamond$

With this notation, since we can now determine the multipartition along with its residues from the abacus, if we are given a plus minus sequence and a base tuple we can calculate the corresponding multicharge κ.

Proposition 4.12. *Suppose we have a plus minus sequence and a base tuple* $(b_0, b_1, \ldots, b_{e-1})$ *giving a truncated abacus configuration and a* 2*-multipartition* $\lambda = \left(\lambda^{(1)}, \lambda^{(2)}\right)$*. Let* p *be the number of plusses in the sequence,* m *be the number of minuses,* z_N *be the number of zeroes corresponding to no bead added and* z_B *be the number of zeroes corresponding to a bead added. Then the* e*-multicharge* $\kappa = (\kappa_1, \kappa_2)$ *associated to* λ *is given by*

$$\kappa_1 \equiv \sum_{k=0}^{e-1} b_k + m + z_B \mod e,$$
$$\kappa_2 \equiv \sum_{k=0}^{e-1} b_k + p + z_B \mod e.$$

Proof. Suppose the truncated abacus configuration we obtain for $\lambda^{(c)}$ has beads at positions $\hat{\beta}^c_j$ for $j \geq 1$, where position 0 is the position at the top left on runner 0 and the $\hat{\beta}^c_j$'s are ordered so that $\hat{\beta}^c_j > \hat{\beta}^c_{j+1}$ for every $j \geq 1$. If the truncated abacus configuration has no beads then we let $\hat{\beta}_1^c = -1$. These positions satisfy

$$\hat{\beta}_j^c \equiv \beta_j^c \mod e.$$

Since by definition, $\beta_1^c = \lambda_1^{(c)} - 1 + \kappa_c$, we have

$$\kappa_c \equiv \hat{\beta}_1^c - \lambda_1^{(c)} + 1 \mod e, \text{ for } c \in \{1, 2\}.$$

Now let runner r_c be the runner with position $\hat{\beta}_1^c$ on it. If such a runner does not exist then we have $\hat{\beta}_1^c = -1$ and $\lambda_1^{(c)} = 0$ hence $\kappa_c \equiv 0 \mod e$. Otherwise, $\hat{\beta}_1^c = r_c + (x-1)e$ where x is the number of beads on runner r_c. To find $\lambda_1^{(c)}$, we need to count all the empty spaces preceding the bead at position $\hat{\beta}_1^c$.

Define δ as follows:

$$\delta = \begin{cases} 1, & \text{if the entry of the } -+ \text{ sequence corresponding to runner } r_c \text{ is a } +, \\ 0, & \text{if the entry of the } -+ \text{ sequence corresponding to runner } r_c \text{ is a } -. \end{cases}$$

First, consider when $c = 1$. Then $\hat{\beta}_1^1 = r_1 + (b_{r_1} - \delta)e$. We can write $\lambda_1^{(1)}$ as

$$\lambda_1^{(1)} = \sum_i (b_{r_1} - b_i + 1 - \delta) + \sum_j (b_{r_1} - b_j - \delta) - (e - 1 - r_1),$$

where the first sum is over all i such that b_i corresponds to a $+$ or a 0_N runner, the second is over all j such that b_j corresponds to a $-$ or a 0_B runner. The third summand accounts for those empty spaces in the $e - 1 - r_1$ positions greater than $\hat{\beta}_1^1$ that have been overcounted. Grouping terms we have

$$\lambda_1^{(1)} = b_{r_1} e - \sum_{k=0}^{e-1} b_k + p + z_N - \delta e - e + 1 + r_1$$

and so

$$\begin{aligned}\kappa_1 &\equiv \hat{\beta}_1^1 - \lambda_1^{(1)} + 1\\ &\equiv r_1 + b_{r_1}e - \delta e - b_{r_1}e + \sum_{k=0}^{e-1} b_k - p - z_N + \delta e + e - 1 - r_1 + 1\\ &\equiv \sum_{k=0}^{e-1} b_k + e - p - z_N\\ &\equiv \sum_{k=0}^{e-1} b_k + m + z_B\end{aligned}$$

modulo e.

Now consider when $c = 2$. Then $\hat{\beta}_1^2 = r_2 + (b_{r_2} + \delta - 1)e$. We can write $\lambda_1^{(2)}$ as

$$\lambda_1^{(2)} = \sum_i (b_{r_2} - b_i - 1 + \delta) + \sum_j (b_{r_2} - b_j + \delta) - (e - 1 - r_2),$$

where the first sum is over all i such that b_i corresponds to a $+$ or a 0_B runner, the second is over all j such that b_j corresponds to a $-$ or a 0_B runner. Grouping terms we have

$$b_{r_2}e - \sum_{k=0}^{e-1} b_k - p - z_B + \delta e - e + 1 + r_2$$

and so

$$\begin{aligned}\kappa_2 &\equiv \hat{\beta}_1^2 - \lambda_1^{(2)} + 1\\ &\equiv r_2 + b_{r_2}e + \delta e - e - b_{r_2}e + \sum_{k=0}^{e-1} b_k + p + z_B - \delta e + e - 1 - r_2 + 1\\ &\equiv \sum_{k=0}^{e-1} b_k + p + z_B\end{aligned}$$

modulo e.

□

With this, we can note the following.

Remark 4.13. Given a plus minus sequence $d(\lambda)$ and a base tuple B, we can extract a value for n based on the associated λ, a value for e from the numbers of runners on the abacuses, and a value for κ by Proposition 4.12. Thus given a base field $\mathbb{F}$ we can uniquely define an algebra $\mathscr{H}_n^{\Lambda_\kappa}$ based solely on $d(\lambda)$ and B.

It turns out that the sequence $d(\lambda)$ is incredibly useful, and holds a lot of information about the core block that S^λ lies in.

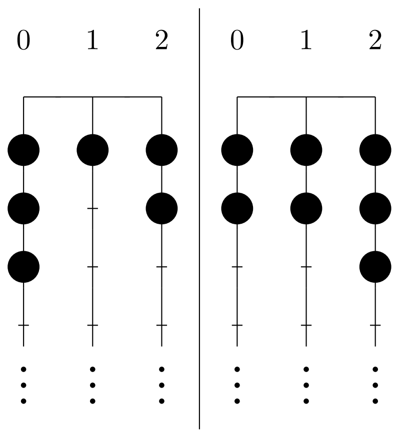
Theorem 4.14. *[Lyle] Let λ be a multipartition arising from a plus minus sequence. Then λ is Kleshchev if and only if $\hat{d}(\lambda)$ consists only of plusses, or only of minuses, or is empty.*

Given a multicore λ lying in a core block, we obtain all the other multipartitions in the block by permuting the plusses and minuses in the sequence $d(\lambda)$ to obtain other multicores with respect to the same base tuple.

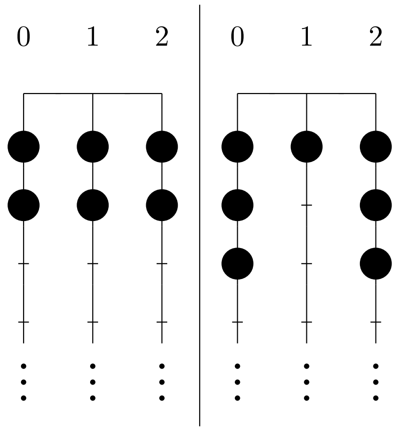
Suppose λ is a multicore, Kleshchev, and lying in a core block with plus minus sequence $d(\lambda)$, and that μ is a multicore also lying in the same core block where $d(\mu)$ is obtained from $d(\lambda)$ by swapping each pair in some subset of pairs $-,+$ that are each linked by an arc to obtain $+,-$. Then we say that μ is formed from λ by a *process of arcs*, and write $\lambda \frown \mu$. Any Kleshchev multicore λ is formed from itself by a trivial process of arcs so $\lambda \frown \lambda$. This notion allows us to state the graded decomposition number for the relevant Specht and irreducible modules.

Theorem 4.15. *[Lyle] Let λ and μ be multicores lying in the same block with λ Kleshchev. Then $[S^\mu : D^\lambda] \neq 0 \iff \lambda \frown \mu$. Moreover, if $\lambda \frown \mu$, then $[S^\mu : D^\lambda]_v = v^i$ where i is the number of $-,+$ pairs that have been swapped to obtain $d(\mu)$ from $d(\lambda)$.*

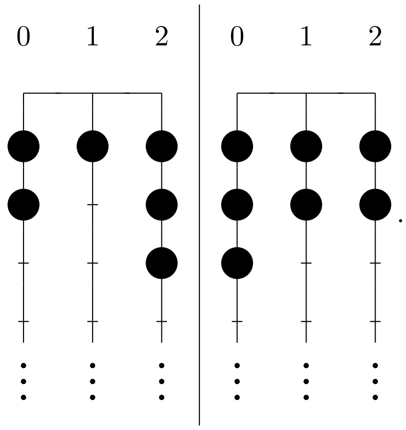
Example 4.16. Let $e = 3$, and consider the abacus configuration



which corresponds to the multicore $\lambda = ((1^2), (2))$ with base tuple $(2, 1, 2)$ and multicharge $\kappa = (0, 1)$. We have $d(\lambda) = (+, -, +)$ and so $\hat{d}(\lambda) = (+)$ hence λ is Kleshchev. To get the other multicores in the block we permute the plusses and minuses in $d(\lambda)$. So we will have μ with $d(\mu) = (-, +, +)$, $\hat{d}(\mu) = (+)$ and ν with $d(\nu) = (+, +, -)$, $\hat{d}(\nu) = (+ + -)$. More precisely, μ is the Kleshchev multipartition $(\varnothing, (2, 1^2))$ with abacus configuration



and ν is the (not Kleshchev) multipartition $((3, 1), \varnothing)$ with abacus configuration



We see that $\mu \frown \lambda$ by swapping one $-, +$ pair:

$$d(\mu) = (- , + , +) \text{ whilst } (+, -, +) = d(\lambda),$$

and $\lambda \frown \nu$ by swapping one $-, +$ pair:

$$d(\lambda) = (+, - , +) \text{ whilst } (+, +, -) = d(\nu).$$

Hence using Theorem 4.15, we obtain the decomposition matrix for the multicores in the block:

	D^μ	D^λ
S^μ	1	0
S^λ	v	1
S^ν	0	v

$\lozenge$

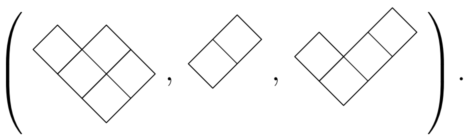
4.2 Homomorphisms within core blocks

Let $e \in \{2, 3, 4, \ldots\}$ and suppose from now on that we have the base tuple $B = (0, 0, \ldots, 0)$. Then if we have a bipartition λ given by a plus minus sequence $d(\lambda)$, we must have that the components of λ (and all other multipartitions in the corresponding block) obey the diagonal residue condition. Thus we can use Theorem 3.14 to find homomorphisms between Specht modules in the block containing λ. We first exhibit some combinatorics related to our plus minus sequence using the Russian convention for drawing partitions, and then exhibit a result of Hu and Mathas [HM10] in this setting.

Given the diagram of an l-multicomposition λ:

$$[\lambda] = \{(r, c, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\} \mid c \leq \lambda_r^{(m)}\}$$

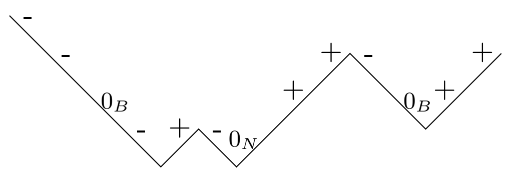
we can draw its diagram in the *Russian convention* by drawing each node as a box, with the r coordinate increasing from south-east to north-west and the c coordinate increasing from south-west to north-east. For example, the Russian convention diagram of $((2, 2, 1), (2), (3, 1))$ is drawn as



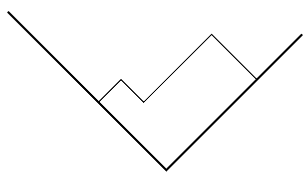
Now given our bipartition λ and its plus minus sequence $d(\lambda)$, we can construct paths corresponding to the two components of λ by reading along the plus minus sequence. To draw the path for $\lambda^{(1)}$, whenever we encounter a $-$ we draw a line $\diagdown$ whilst for $+$ we draw $\diagup$. To draw $\lambda^{(2)}$ we do the opposite: for $-$ draw $\diagup$ and for $+$ draw $\diagdown$. For either component, if we encounter a zero we draw $\diagdown$ if this corresponds to a bead (i.e. it is a 0_B), or draw $\diagup$ if it does not (i.e. it is a 0_N). We can place the path for $\lambda^{(1)}$ in a 'trough' of e lines long, consisting of at first #(minuses) + #(bead zeroes) lines of the form $\diagdown$ followed by #(plusses) + #(no bead zeroes) lines of the form $\diagup$. We place the path for $\lambda^{(2)}$ in a 'trough' of e lines long, consisting of #(plusses) + #(bead zeroes) lines of the form $\diagdown$ followed by #(minuses) + #(no bead zeroes) lines of the form $\diagup$.

This constructs the diagram for λ since if we observe the abacus corresponding to the first component plus minus sequence, every time we encounter a minus (or a bead zero) this corresponds to a bead which corresponds to the end of a row (which may be empty), whilst every time we encounter a plus (or a no bead zero) this corresponds to an empty runner on the abacus which corresponds to a column. Similarly we obtain the diagram of the second component since the roles of the plus and minus swap with respect to where we place beads on the abacus, whilst the roles of the zeroes stay fixed.

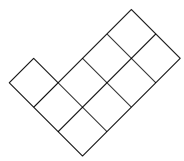
Example 4.17. Let $e = 13$, and $d(\lambda) = (-, -, 0_B, -, +, -, 0_N, +, +, -, 0_B, +, +)$. The path for $\lambda^{(1)}$ is:



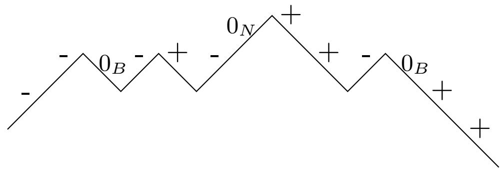
The path for $\lambda^{(1)}$ sits in a trough whose left hand side is 7 lines long and whose right hand side is 6 lines long, which is:



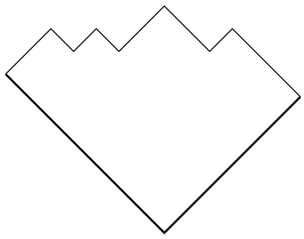
giving:



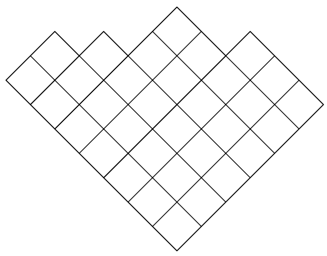
Meanwhile, the path for $\lambda^{(2)}$ is:



and this sits in a trough whose left hand side is 7 lines long and whose right hand side is 6 lines long, which is:



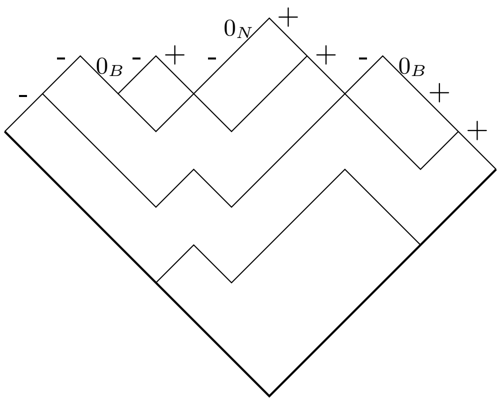
giving:



$\lozenge$

We can pair up edges of the path in an analogous way to how we pair entries of the plus minus sequence. If a pair $-, +$ is linked by an arc in the plus minus sequence, we link the corresponding edges of the second component of the tableau by a tile. To be precise, starting at the node adjacent to the edge labelled by the $-$, if this is also adjacent to the edge labelled by the $+$ we are done and our tile consists of just the one node. Otherwise, we also incorporate the node north-east of this, unless this is already within another tile, in which case we incorporate the node south-east of this into the tile. We then repeat this until we reach the node adjacent to the edge labelled by the $+$. Note that in practice, we must pair up the edges by starting with those $-, +$ pairs that are contained within other pairs and then work outwards.

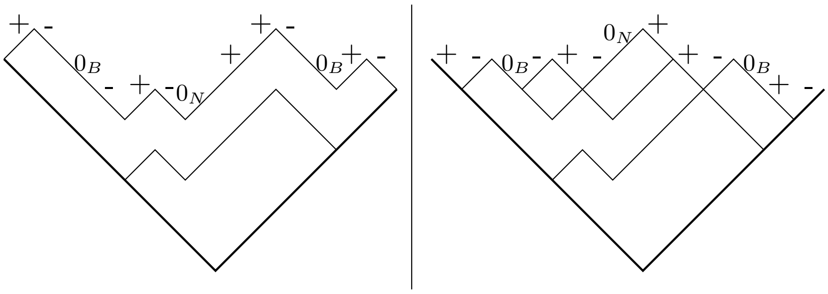
Example 4.18. We exhibit the tiling for $\lambda^{(2)}$ as in the previous example.



◊

Now, if we swap a linked $-,+$ pair in $d(\lambda)$, we obtain a new plus minus sequence which will correspond to a bipartition μ. Using the path construction of the Russian convention diagram, we see that to obtain $\mu^{(1)}$ we add the tile corresponding to the $-,+$ pair in $\lambda^{(2)}$ to $\lambda^{(1)}$, whilst to obtain $\mu^{(2)}$ we remove this tile from $\lambda^{(2)}$.

Example 4.19. If $d(\mu) = (+,-,0_B,-,+,-,0_N,+,+,-,0_B,+,-)$ (i.e. we have swapped the outer $-,+$ pair in the previous $d(\lambda)$), then the shape of μ is as follows:



◊

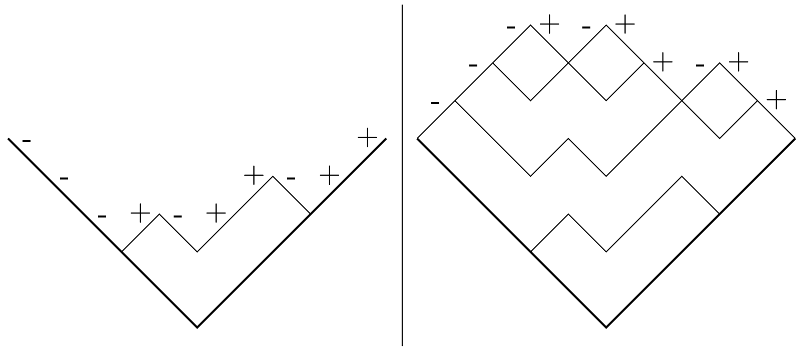
For any such μ obtained by swapping some linked $-,+$ pairs, we can construct a μ-tableau $\mathfrak{s}$ from $\mathfrak{t}^\lambda$ by considering values of the corresponding tiles in $\mathfrak{t}^\lambda$ and simply filling in the tiles of μ with the same values as they had in $\mathfrak{t}^\lambda$. In this way, we construct a standard μ-tableau whose residue sequence is the same as that of $\mathfrak{t}^\lambda$. In fact, due to Hu and Mathas, the following result tells us that this is the only such tableau with $\mathfrak{s} \trianglerighteq \mathfrak{t}^\lambda$. Let e be large enough so that λ and μ obey the diagonal residue condition, and define $\mathrm{Std}^\lambda(\mu) := \{\mathfrak{s} \in \mathrm{Std}(\mu) \mid \mathfrak{s} \trianglerighteq \mathfrak{t}^\lambda \text{ and } \mathrm{res}(\mathfrak{s}) = \mathrm{res}(\mathfrak{t}^\lambda)\}$. We will say that a tableau $\mathfrak{t}$ is *regular* if its entries increase along the diagonals in each component (recall that the kth diagonal of component m consists of the nodes (r, c, m) such that $r - c = k$).

Proposition 4.20. *[HM10, Lemma B1 & Corollary B2] Suppose that λ and μ are 2-multipartitions of n and let e be large enough so that λ and μ both obey the diagonal residue condition. Then $\#\mathrm{Std}^\lambda(\mu) \leq 1$.*

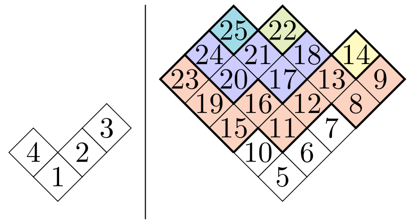
Proof. Suppose that $\mathfrak{t} \in \mathrm{Std}^\lambda(\mu)$ such that $\mathfrak{t} \triangleright \mathfrak{t}^\lambda$ (otherwise $\mathfrak{t} = \mathfrak{t}^\lambda$ and we are done). Note that the tableau $\mathfrak{t}$ is uniquely determined by its residue sequence and the sets $\mathfrak{t}^{(1)}$ and $\mathfrak{t}^{(2)}$. Let X be the set of nodes in $\mu^{(1)} \setminus \lambda^{(1)}$ that are either (horizontally, vertically, or diagonally) adjacent to a node in $\lambda^{(1)}$ or are in the first row or the first column of $\mu^{(1)}$. Let $A := \{\mathfrak{t}(x) \mid x \in X\}$. Define $\mathfrak{t}_A$ to be the unique regular tableau with $\mathrm{res}(\mathfrak{t}) = \mathrm{res}(\mathfrak{t}^\lambda)$ such that $\mathfrak{t}_A^{(1)} = \mathfrak{t}^{(1)} \setminus A$ and $\mathfrak{t}_A^{(2)} = \mathfrak{t}^{(2)} \cup A$. In other words, we form $\mathfrak{t}_A$ from $\mathfrak{t}$ by moving the numbers in A from the first component of $\mathfrak{t}$ to the second without changing their 'shape', whilst 'sliding' numbers along the diagonals in order to fill in the gaps from where A was in the first component and create gaps for A in the second component. As $\mathfrak{t}$ and $\mathfrak{t}^\lambda$ are both standard we must have that $\mathfrak{t}_A$ is standard.

Now we show how we can uniquely form $\mathfrak{t}$ given only λ and μ. We have that λ and μ uniquely determine the set X, and so they also uniquely determine $\mathrm{Shape}(\mathfrak{t}_A)$. Note that $\mathrm{Shape}(\mathfrak{t}_A) \triangleleft \mu$ and so by induction $\#\mathrm{Std}^\lambda(\mathrm{Shape}(\mathfrak{t}_A)) \leq 1$. The basis case of the induction is that $\#\mathrm{Std}^\lambda(\lambda) = 1$, and any tableau in $\mathrm{Std}^\lambda(\nu)$ for some ν must have been formed using the above construction in reverse, hence there is only at most one candidate for such a standard tableau. In particular, given the tableau $\mathfrak{t}_A$, we can recover $\mathfrak{t}$. Thus $\#\mathrm{Std}^\lambda(\mu) \leq 1$. □

Example 4.21. Consider the plus minus sequence $(-,-,-,+,-,+,+,-,+,+)$ associated to the multipartition $\lambda = ((3,1),(5^2,4^2,3))$. We exhibit λ below with the tiling made clear in $\lambda^{(2)}$.

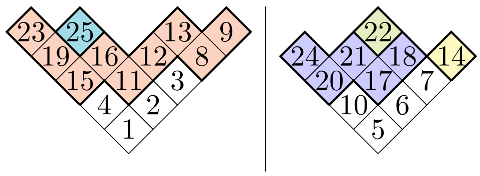


The tableau $\mathfrak{t}^{\lambda}$ is as follows:

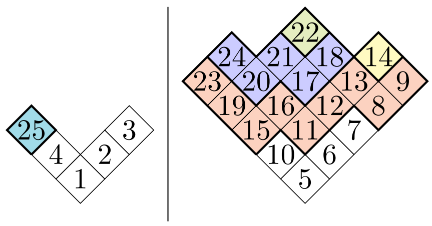


Let $\mu = ((5,4,2^2,1),(4,3^2,1))$ be the multipartition corresponding to the plus minus sequence $d(\mu) = (+,-,+,-,-,+,+,-,+,-)$, obtained from that for λ by swapping some $-,+$ pairs linked by arcs.

The μ-tableau $\mathfrak{t}$ is as follows:



The set X consists of the red nodes, and $A = \{8,9,11,12,13,15,16,19,23\}$. In this case, $\mathfrak{t}_A$ is the tableau:



$\Diamond$

Now we wish to determine the homomorphisms between Specht modules indexed by bipartitions arising from plus minus sequences. We will need the following definition.

Definition 4.22. Suppose λ and μ are l-multipartitions of n. If $\varphi \in \mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu)$, we say that φ is *dominated* if $\varphi(v^{\mathfrak{t}^\lambda}) \in \langle v^{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}^\lambda(\mu)\rangle_{\mathbb{F}}$. We write $\mathrm{DHom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu)$ for the space of dominated homomorphisms from S^λ to S^μ.

Due to the first part of the following theorem (which we state for arbitrary l) we have that when $e \neq 2$ and $\kappa_1 \neq \kappa_2$ it will be enough to concern ourselves with studying dominated homomorphisms.

Theorem 4.23. *[FS16, Theorem 3.13] Suppose $e \neq 2$ and that $\kappa_1, \ldots, \kappa_l$ are distinct, and λ and μ are l-multipartitions of n. Then the set $\mathrm{DHom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu)$ is equal to $\mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu)$. Hence $\mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu) \neq \{0\}$ only if $\lambda \trianglelefteq \mu$, $\mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\lambda)$ is one-dimensional and S^λ is indecomposable.*

Example 4.24. The following two examples demonstrate what happens upon relaxing the hypotheses of Theorem 4.23.

1. Let $e = 2$, $l = 1$, $\lambda = (2)$ and $\mu = (1,1)$. Then there is a non-zero homomorphism $\varphi : S^\mu \to S^\lambda$, $v^{\mathfrak{t}^\mu} \mapsto v^{\mathfrak{t}^\lambda}$, but this is not a dominated homomorphism.

2. Let $e = 3$, take $\kappa = (1,1)$, $\lambda = (\varnothing, (2))$, $\mu = ((2), \varnothing)$. Then there is a non-zero homomorphism $\varphi : S^\mu \to S^\lambda$, $v^{\mathfrak{t}^\mu} \mapsto v^{\mathfrak{t}^\lambda}$, but this is not a dominated homomorphism. Note that λ corresponds to $d(\lambda) = (-, 0_N, +)$ whilst $d(\mu) = (+, 0_N, -)$ for the base tuple $B = (0,0,0)$.

Note that both of these homomorphisms are in fact isomorphisms. $\diamondsuit$

Let us again suppose that we have the base tuple $B = (0, 0, \ldots, 0)$ and a bipartition λ given by a plus minus sequence $d(\lambda)$. We wish to find bipartitions μ so that $\mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu) \neq \{0\}$. Of course, we will only need to consider bipartitions μ that belong to the same block as λ. So in order to find the bipartitions λ and μ such that $\mathrm{DHom}_{\mathscr{H}_n^{\Lambda_\kappa}}(S^\lambda, S^\mu) \neq \{0\}$ we will first need to find the sets $\mathrm{Std}^\lambda(\mu)$ that are non-empty. Define a *tile* of nodes to be a finite set of nodes that can be ordered as $\{N_1, N_2, \dots, N_k\}$ such that given $N_i = (r, c, m)$, we have $N_{i+1} \in \{(r+1, c, m), (r, c+1, m)\}$. The following proposition allows us to disregard many different bipartitions μ for a given λ. Note that here we are in effect utilising our combinatorial setting in order to adapt [HM10, Theorem B3] to our needs.

Proposition 4.25. *Let $e \in \{2, 3, 4, \dots\}$ and suppose we have a base tuple $B = (0, 0, \dots, 0)$. Suppose also that λ and μ are obtained from plus minus sequences $d(\lambda)$ and $d(\mu)$. Then if $\mathrm{Std}^\lambda(\mu) \neq \varnothing$ then $d(\mu)$ is obtained from $d(\lambda)$ by swapping some $-, +$ pairs that are linked by arcs in $d(\lambda)$.*

Proof. Let $\mathfrak{t} \in \mathrm{Std}^\lambda(\mu)$. Then following the construction in Proposition 4.20, we can form $\mathfrak{t}$ from a tableau $\mathfrak{t}_A$ by sliding some nodes from the second component to the first. By induction, assume that $d(\mathrm{Shape}(\mathfrak{t}_A))$ is obtained from $d(\lambda)$ by swapping some $-, +$ pairs that are linked by arcs in $d(\lambda)$ (the base case being when $\mathrm{Shape}(\mathfrak{t}_A)$ is just λ). So we wish to show that $d(\mu)$ is formed from $d(\mathrm{Shape}(\mathfrak{t}_A))$ by swapping some $-, +$ pairs that are linked by arcs in $d(\lambda)$.

Note that since $\mathrm{Std}^\lambda(\mu) \neq \varnothing$, λ and μ must lie in the same block. $d(\lambda)$ and $d(\mu)$ must contain the same number of plusses and minuses. Consider $d(\mu)$. In order to recover $\mathrm{Shape}(\mathfrak{t}_A)$ we need to remove some tiles of nodes, that are adjacent to $\lambda^{(1)}$. Removing a tile from $[\mu]$ will correspond to swapping the positions of a $+$ and a $-$ in $d(\mu)$, where $+$ occurs before the $-$. We cannot swap any $+$ and $-$ corresponding to a tile which also occur in the exact same positions as in $d(\lambda)$, since no such tile can have its rightmost (in the Russian convention) residue being equal to that of a residue at the end of a row of $\lambda^{(1)}$, as we would then be removing nodes from $\lambda^{(1)}$ which is not allowed as we must have that $\lambda^{(1)}$ is contained within $\mu^{(1)}$.

From now on, we refer to the $+$ and $-$ to be swapped as a *backwards pair*. Consider the entries of $d(\mu)$ that fall between the backwards pair. We must have that amongst these entries, there are the same number of $-$'s and $+$'s, since otherwise the tile removed from the first component of μ will not be the same as that added to the second component to form Shape($\mathfrak{t}_A$). Not only this, but if these entries correspond to any nodes, these nodes will sit 'above' (in the Russian convention) the tile corresponding to the backwards pair, and so these entries will pair up as either a pair $+,-$ that was a linked $-,+$ pair in $d(\lambda)$, or as a pair $-,+$ that corresponds to a linked $-,+$ pair in $d(\lambda)$. But so this means that the backwards pair $+,-$ must be the result of swapping a pair $-,+$ that is linked by an arc in $d(\lambda)$.

Thus $d(\mu)$ arises from $d(\mathrm{Shape}(\mathfrak{t}_A))$ by swapping some pairs $-,+$ that were linked by an arc in $d(\lambda)$, hence by induction, the whole of $d(\mu)$ arises by swapping some such pairs $-,+$ in $d(\lambda)$. $\square$

So we now need only consider those μ whose plus minus sequence $d(\mu)$ arises from that of λ by swapping $-,+$, pairs that are linked by arcs. Suppose that we obtain $d(\mu)$ by swapping linked pairs $-,+$ along with all linked pairs $-,+$ that are contained within these pairs when reading the plus minus sequence. We shall denote this by writing $\lambda \stackrel{\frown}{\frown} \mu$. Trivially, we have $\lambda \stackrel{\frown}{\frown} \lambda$. If $\mu \neq \lambda$, then $\lambda \stackrel{\frown}{\frown} \mu$ will correspond to removing skew shapes from the right hand component and adding them to the left hand component. Now we can state the main theorem of this chapter.

Theorem 4.26. *Let $e \in \{2,3,4,\ldots\}$ and suppose we have the base tuple $B = (0,0,\ldots,0)$. Suppose that λ and μ are obtained from plus minus sequences $d(\lambda)$ and $d(\mu)$ respectively, and that $\mathscr{H}_n^{\Lambda_\kappa}$ is the uniquely determined algebra as in Remark 4.13. Then*

$$\mathrm{Dim}_{\mathbb{F}}\left(\mathrm{DHom}_{\mathscr{H}_n^{\Lambda_\kappa}}\left(\mathrm{S}^\lambda, \mathrm{S}^\mu\right)\right) = \begin{cases} 1 & \text{if } \lambda \stackrel{\frown}{\frown} \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, when this dimension equals 1, we can explicitly describe the homomorphism in $\mathrm{DHom}_{\mathscr{H}_n^{\Lambda_\kappa}}\left(S^\lambda, S^\mu\right)$ and its degree is equal to the number of $-,+$ pairs swapped to obtain μ from λ.

Proof. If $\lambda \stackrel{\frown}{\frown} \mu$, then by Proposition 4.20 we can construct a unique standard tableau so that using Corollary 3.18 we have that there is exactly one non-zero dominated homomorphism from S^λ to S^μ. When $\mu = \lambda$, the only such homomorphism from S^λ to itself is the trivial homomorphism.

Now suppose that we do not have $\lambda \frown \mu$, and so in order to obtain $d(\mu)$ we have to swap a linked pair $-, +$ but do not swap some linked $-, +$ pair that is contained within this pair. Then in view of the tile construction from above, we will have that there is a unique μ-tableau $\mathfrak{s} \in \mathrm{Std}^\lambda(\mu)$, that is constructed by removing some tiles from the second component of $\mathfrak{t}^\lambda$ and adding them to the first component, without also removing every tile that sits above them (in the Russian convention). But then there will be some value r in the moved tiles such that $r + 1$ belongs to an unmoved tile, and that ψ_r is a row relation for S^λ. However, $v^{\mathfrak{t}^\mu}\psi^{\mathfrak{s}}\psi_r \neq 0$ since swapping r and $r + 1$ in $\mathfrak{s}$ still gives us a standard μ-tableau. Thus there is no non-zero dominated homomorphism from S^λ to S^μ in this case.

Now we are left with proving the statement about calculating the degree. So first suppose that $\lambda \frown \mu$ and that we swap just one linked $-, +$ pair to obtain $d(\mu)$, then we have moved one tile from the second component of λ to the first. Note that in the language associated to Corollary 3.18, any tile contains one more positive diagonal than negative diagonals hence the base degree associated to a tile will be 1.

If we instead are required to swap a linked $-, +$ pair along with any completely contained linked $-, +$ pairs then note that each completely contained pair will simply add 1 to the associated base degree, as the corresponding tile will have one more positive diagonal than negative diagonals and these will line up directly with positive and negative diagonals associated with the outer $-, +$ pair.

Thus the degree of a homomorphism $S^\lambda \to S^\mu$ will be equal to the number of linked $-, +$ pairs that are swapped. □

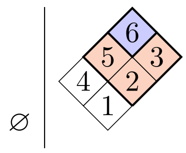
We are now able to put everything we have done together in order to compute every homomorphism space between the Specht modules lying in a core block of a level 2 KLR algebra when $e \in \{3, 4, 5, \ldots\}$ and $\kappa_1 \neq \kappa_2$. Adapting Theorem 4.26 in light of Theorem 4.23 we have the following.

Theorem 4.27. *Suppose the assumptions of Theorem 4.26 hold, and further suppose that $e \neq 2$ and that the number of plusses in $d(\lambda)$ or $d(\mu)$ is not equal to the number of minuses. Then $\kappa_1 \neq \kappa_2$ and so we have that*

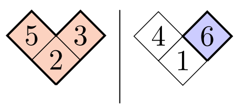
$$\mathrm{Dim}_{\mathbb{F}}\left(\mathrm{Hom}_{\mathscr{H}_n^{\Lambda_\kappa}}\left(\mathrm{S}^{\lambda}, \mathrm{S}^{\mu}\right)\right) = \begin{cases} 1 & \text{if } \lambda \frown \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Given a plus minus sequence $d(\lambda)$ corresponding to some multipartition λ, we shall write $S^{d(\lambda)}$ to mean S^{λ}. For the following example we shall exhibit how when $e \in \{3, 4, 5, \ldots\}$ and $\kappa_1 \neq \kappa_2$ we can compute the entire set of homomorphism spaces between Specht modules in a core block for which $B = (0, 0, \ldots, 0)$.

Example 4.28. Let $d(\lambda) = (-, -, -, +, +)$, $e = 5$, $B = (0, 0, 0, 0, 0)$. Then the decomposition matrix for the corresponding block is shown in Table 4.1. Using the facts we have outlined above, we can complete Table 4.2. We can fill in most of the homomorphism table purely on the basis that homomorphisms only arise from swapping linked $-, +$ pairs. The only two entries that we have to worry about are those that related are related to homomorphisms $S^{(-,-,-,+,+)} \to S^{(-,+,-,+,-)}$ and $S^{(-,-,+,+,-)} \to S^{(+,-,+,-,-)}$. Note that in the former case, when $d(\mu) = (-, +, -, +, -)$, we have that $\mathfrak{t}^{\lambda}$ is:



whilst the unique tableau $\mathfrak{t}$ in $\mathrm{Std}^{\lambda}(\mu)$ is:



and a row relation for S^{λ} that does not annihilate $v^{\mathfrak{t}^{\mu}} v^{\mathfrak{t}}$ is ψ_5. We can follow a similar argument for the latter case, and thus the two entries of the homomorphism table in question are both zero. $\diamondsuit$

In the case that the entries of the multicharge are not all distinct, we cannot

	$(-,-,-,+,+)$	$(-,-,+,-,+)$	$(-,-,+,+,-)$	$(-,+,-,-,+)$	$(-,+,-,+,-)$	$(-,+,+,-,-)$	$(+,-,-,-,+)$	$(+,-,-,+,-)$	$(+,-,+,-,-)$	$(+,+,-,-,-)$
$(-,-,-,+,+)$	1	v	0	0	v	v^2	0	0	0	0
$(-,-,+,-,+)$	0	1	v	v	v^2	0	0	0	0	0
$(-,-,+,+,-)$	0	0	1	0	v	0	0	0	v	v^2
$(-,+,-,-,+)$	0	0	0	1	v	0	v	v^2	0	0
$(-,+,-,+,-)$	0	0	0	0	1	v	0	v	v^2	0

Table 4.1: Decomposition matrix for the block corresponding to $d(\lambda)$. The columns represent Specht modules associated to those $-,+$ sequences whilst the rows correspond to the irreducible modules associated to those $-,+$ sequences. The entry whose column heading is y and row heading is x is thus the graded decomposition number $[S^y : D^x]_v$.

	$(-,-,-,+,+)$	$(-,-,+,-,+)$	$(-,-,+,+,-)$	$(-,+,-,-,+)$	$(-,+,-,+,-)$	$(-,+,+,-,-)$	$(+,-,-,-,+)$	$(+,-,-,+,-)$	$(+,-,+,-,-)$	$(+,+,-,-,-)$
$(-,-,-,+,+)$	1	v	0	0	0	v^2	0	0	0	0
$(-,-,+,-,+)$	0	1	v	v	v^2	0	0	0	0	0
$(-,-,+,+,-)$	0	0	1	0	v	0	0	0	0	v^2
$(-,+,-,-,+)$	0	0	0	1	v	0	v	v^2	0	0
$(-,+,-,+,-)$	0	0	0	0	1	v	0	v	v^2	0
$(-,+,+,-,-)$	0	0	0	0	0	1	0	0	v	0
$(+,-,-,-,+)$	0	0	0	0	0	0	1	v	0	0
$(+,-,-,+,-)$	0	0	0	0	0	0	0	1	v	0
$(+,-,+,-,-)$	0	0	0	0	0	0	0	0	1	v
$(+,+,-,-,-)$	0	0	0	0	0	0	0	0	0	1

Table 4.2: Table to show the graded dimension of homomorphism spaces for the block corresponding to $d((-,-,-,+,+))$. The entry whose row heading is x and column heading is y is the graded dimension of the space of homomorphisms from S^x to S^y.

	$(-,-,+,+)$	$(-,+,-,+)$	$(-,+,+,-)$	$(+,-,-,+)$	$(+,-,+,-)$	$(+,+,-,-)$
$(-,-,+,+)$	1	v	0	0	0	v^2
$(-,+,-,+)$	0	1	v	v	v^2	0
$(-,+,+,-)$	0	0	1	1	v	0
$(+,-,-,+)$	0	0	1	1	v	0
$(+,-,+,-)$	v^{-1}	1	0	0	1	v
$(+,+,-,-)$	v^{-2}	v^{-1}	0	0	0	1

Table 4.3: Table of graded dimensions of homomorphism spaces for the block corresponding to $d((-,-,+,+))$. The entry whose row heading is x and column heading is y is the graded dimension of the space of homomorphisms from S^x to S^y.

claim that every homomorphism is a dominated homomorphism, and so in turn we cannot determine the possible standard tableaux in the image of a homomorphism as we could when considering 'sliding tiles'. In general, we are thus unable to determine the entire set of homomorphism spaces precisely without checking each tableaux individually.

Example 4.29. Let $d(\lambda) = (-,-,+,+)$, $e = 4$, $B = (0,0,0,0,0)$. Then an example of a homomorphism which is not a dominated homomorphism is $\varphi : S^{(+,-,-,+)} \to S^{(-,+,+,-)}$ given by

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \right) \mapsto \left(\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right)$$

The homomorphism table for the associated block is shown in Table 4.3. ♢

4.3 Different base tuples

We now detail some examples of homomorphisms between Specht modules that arise from plus minus sequences for base tuples other than just $(0,0,\ldots,0)$, and discuss a potential pattern seen in the images of these homomorphisms. It will be useful to refer back to Chapter 3, Section 3.4.

Firstly, we note simply that if we have a base tuple other than $(0,0,\ldots,0)$ then our homomorphisms may not be indexed by a single tableau.

Example 4.30. Let $e = 3$, $d(\lambda) = (-,-,+)$, $d(\mu) = (-,+,-)$ and $B = (0,0,2)$.

The initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{ytableau} 1 & 2 \end{ytableau} \;\middle|\; \begin{ytableau} 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 \end{ytableau} \right).$$

There is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by $v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}}$ where

$$\mathfrak{s} = \left(\begin{ytableau} 1 & 2 & 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 \end{ytableau} \;\middle|\; \begin{ytableau} 3 & 4 & 5 \\ 9 \\ 13 \end{ytableau} \right)$$

and

$$\mathfrak{t} = \left(\begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 10 & 11 & 12 \\ 14 \end{ytableau} \;\middle|\; \begin{ytableau} 6 & 7 & 8 \\ 9 \\ 13 \end{ytableau} \right).$$

$\lozenge$

The tableau $\mathfrak{s}$ arises in the same way as we expect, but we also have a term indexed by the tableau $\mathfrak{t}$, and this is formed by acting on $\mathfrak{s}$ by the permutation $(3,6)(4,7)(5,8)$.

Next, note that if we make the differences between the base tuple entries bigger, we obtain even more terms in the image of a homomorphism.

Example 4.31. Consider the setup of the previous example but now suppose that $B = (0,0,3)$. Then the initial tableau $\mathfrak{t}^\lambda$ is

$$\mathfrak{t}^\lambda = \left(\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 \end{ytableau} \;\middle|\; \begin{ytableau} 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \\ 25 & 26 \end{ytableau} \right).$$

There is a homomorphism $\varphi : S^\lambda \to S^\mu$ given by

$$v^{\mathfrak{t}^\lambda} \mapsto v^{\mathfrak{s}} + 2v^{\mathfrak{t}} + 2v^{\mathfrak{u}} + 4v^{\mathfrak{v}}$$

where

$$\mathfrak{s} = \left(\begin{ytableau} 1 & 2 & 3 & 4 & 12 & 13 & 14 \\ 5 & 6 & 18 & 19 & 20 \\ 22 & 23 & 24 \\ 26 \end{ytableau} \;\middle|\; \begin{ytableau} 7 & 8 & 9 & 10 & 11 \\ 15 & 16 & 17 \\ 21 \\ 25 \end{ytableau} \right),$$

$$\mathfrak{t} = \left(\begin{ytableau} 1 & 2 & 3 & 4 & 9 & 10 & 11 \\ 5 & 6 & 18 & 19 & 20 \\ 22 & 23 & 24 \\ 26 \end{ytableau} \;\middle|\; \begin{ytableau} 7 & 8 & 12 & 13 & 14 \\ 15 & 16 & 17 \\ 21 \\ 25 \end{ytableau} \right),$$

$$\mathfrak{u} = \left(\begin{ytableau} 1 & 2 & 3 & 4 & 12 & 13 & 14 \\ 5 & 6 & 15 & 16 & 17 \\ 22 & 23 & 24 \\ 26 \end{ytableau} \;\middle|\; \begin{ytableau} 7 & 8 & 9 & 10 & 11 \\ 18 & 19 & 20 \\ 21 \\ 25 \end{ytableau} \right),$$

and

$$\mathfrak{v} = \left(\begin{ytableau} 1 & 2 & 3 & 4 & 9 & 10 & 11 \\ 5 & 6 & 15 & 16 & 17 \\ 22 & 23 & 24 \\ 26 \end{ytableau} \;\middle|\; \begin{ytableau} 7 & 8 & 12 & 13 & 14 \\ 18 & 19 & 20 \\ 21 \\ 25 \end{ytableau} \right).$$

$\Diamond$

Similarly, to before, we obtain the tableau $\mathfrak{s}$ as expected and then

$$\mathfrak{t} = \mathfrak{s}(9,12)(10,13)(11,14),$$

$$\mathfrak{u} = \mathfrak{s}(15,18)(16,19)(17,20), \text{ and}$$

$$\mathfrak{v} = \mathfrak{s}(9,12)(10,13)(11,14)(15,18)(16,19)(17,20).$$

Note that $v^{\mathfrak{t}}$ and $v^{\mathfrak{u}}$ both have coefficient 2, whilst $v^{\mathfrak{v}}$ has a coefficient of 4.

In the previous two examples, we may observe the same pattern in the coefficients that was also exhibited in Examples 3.24 – 3.26. Whereas some of the other examples in Section 3.4 did not follow this pattern, we conjecture that in the current setting this pattern will always appear. In an attempt to motivate this, suppose that λ and μ are multipartitions arising from plus minus sequences with some arbitrary base tuple B, and that $\lambda \frown \mu$, where μ is formed just by moving a single i-node x in λ (for some residue i). It will be useful to consider the abacus configurations of λ and μ here. First, consider $\lambda^{(2)}$, then there are no removable i-nodes above x in this component, since otherwise removing x will not leave an e-core. Now consider $\lambda^{(1)}$; there can be no removable i-nodes below where we shall add x to form $\mu^{(1)}$, since otherwise we cannot add x in the first place. Thus in terms of the naive point of view discussed at the very end of Section 3.4, there are no removable i-nodes lying between the position from which node x is removed and added, and so we expect that the 'expected tableau' will appear in the image.

If $\lambda \frown \mu$, where μ is formed from λ by moving a skew shape S, then we claim that the restrictions of the abacus afforded by working with such multipartitions ensure that the shapes in X_S cannot be removed from anywhere higher in the second component, or from anywhere lower in the first component. Thus we conjecture that we will always obtain homomorphisms which arise in the same way as the examples above, following a pattern based around permuting sets of e entries, and moreover that these homomorphisms are the only ones that arise, so that we have the following:

Conjecture 4.32. *Theorem 4.26 holds for any arbitrary base tuple.*

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