Cuspidal Representations of Dyadic Classical Groups

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Abstract

Let G be a Symplectic group or a Split Special Orthogonal group defined over a dyadic field. We begin by classifying the reductive quotients of most maximal parahoric subgroups of G so that we can explicitly describe its irreducible cuspidal depth-zero representations in terms of their local data. By a result of Blondel we compute the reducibility points of a parabolically induced representation from a cuspidal representation of a maximal Levi subgroup. These reducibility points are described by certain parameters of a spherical Hecke algebra occuring in the construction of a Bushnell–Kutzko cover. Using classical Deligne–Lusztig theory for finite reductive groups, we verify an equality due to Mœglin which (conjecturally) allows one to identify the Langlands parameter associated to an irreducible cuspidal depth-zero representation of G through the local Langlands correspondence.

We then begin an exhaustive investigation into positive-depth cuspidal representations of $\mathrm{Sp}_4(F)$ over a dyadic field. By using both the languages of Bushnell–Kutzko and Moy–Prasad we show that any irreducible representation of $\mathrm{Sp}_4(F)$ contains a G-fundamental stratum. We then take the first steps towards the computation of intertwining of G-fundamental strata by explicitly describing the distinguished double-coset representatives of the maximal parahoric subgroups.

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Chapter 1

Introduction

1.1 Overview

Let F be a non-archimedean local field of residual characteristic p. Let $\mathbb{G}$ be a connected reductive algebraic group with $G = \mathbb{G}(F)$ the F-points of $\mathbb{G}$, which we call a *p-adic group*. The local Langlands correspondence (LLC), which is now known to hold in many cases, predicts a relationship between two different mathematical objects. Denote by $\mathcal{R}(G)$ the category of smooth complex representations of G, with $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible representations in $\mathcal{R}(G)$. On the p-adic side of the LLC we have $\mathrm{Irr}(G)$. On the other, we have certain analogues of Galois representations which we call *Langlands parameters* (these are certain homomorphisms from the Weil–Deligne group $\mathcal{W}'_F$ into the Langlands dual group ${}^LG$ of G). The LLC then says that there is a surjective map from $\mathrm{Irr}(G)$ to the set of Langlands parameters of G (which preserves certain arithmetical properties). The fibre of a given Langlands parameter is finite and is called an *L-packet*. The beauty of the LLC is that it allows one to transfer questions from one side to the other, where they may be easier to answer. There are certain cases where explicit constructions of $\mathrm{Irr}(G)$ is known. It is then hoped that knowing explicitly the LLC in these cases means that one may transfer across arithmetical information about p-adic groups to previously unknown information about the associated Galois representation.

When $G = \mathrm{GL}_n(F)$, the LLC was proved independently by Harris–Taylor [HT01] and Henniart [Hen00], in which they show that this map is a bijection (and so the L-packets are singletons). While they prove the existence of the LLC, they do not give an explicit description of the correspondence. Bushnell–Henniart, in a series of papers [BH05a, BH05b, BH10, BH14] prove many results which works towards making this description explicit using the construction of $\mathrm{Irr}(G)$ due to Bushnell–Kutzko [BK93a]. The LLC is also proven to exist in other cases: for $\mathrm{SL}_n(F)$ [GK82, HS12], quasi-split Orthogonal and Symplectic groups [Art13], quasi-split Unitary groups [Mok15] and both $\mathrm{GSp}_4(F)$ and $\mathrm{Sp}_4(F)$ [GT11, GT10]. In these cases the LLC is proven to not be a bijection.

The representation theory of p-adic groups relies on understanding $\mathrm{Irr}(G)$. In particular, one would like to know precisely how one can obtain all irreducible representations in $\mathrm{Irr}(G)$. For G connected reductive there is a general procedure to do this. Take $\mathcal{P}$ a parabolic subgroup of G with Levi factor $\mathcal{M}$. Since $\mathcal{M}$ is of smaller semisimple rank compared to G, its representation theory is moderately simpler. One takes an irreducible representation of $\mathcal{M}$, and through a process called *parabolic induction* obtains a finite length representation of G, which one can decompose into irreducibles. This does not capture all irreducible representations of G; the irreducibles which do not appear as subquotients of parabolically induced representations are called *supercuspidal* representations. One obtains all irreducible representations of G in the following way. First one takes an irreducible cuspidal representation of a Levi subgroup (including G itself), and then decompose the parabolically induced representation into irreducibles. Therefore the problem of understanding $\mathrm{Irr}(G)$ begins with understanding the construction of supercuspidal representations of a Levi subgroup $\mathcal{M}$.

We can interpret this in the LLC as follows. For $\mathrm{GL}_n(F)$, we have that irreducible cuspidal representations of $\mathrm{GL}_n(F)$ are in bijection with *irreducible* n-dimensional representations of the Weil group $\mathcal{W}_F$. This simple description becomes more complicated for classical groups, by which we mean Symplectic, Special Orthogonal or Unitary groups. Here L-packets are no longer singletons, and they can contain both cuspidal and non-cuspidal representations. However, in [Mg14], Mœglin gives a description of those Langlands parameters whose packets contain cuspidal representations, including the expected number in the packet. Let $\mathrm{Cusp}(G)$ denote the set of equivalence classes of irreducible cuspidal representations of G. Given $\sigma \in \mathrm{Cusp}(G)$ and $\pi \in \mathrm{Cusp}(\mathrm{GL}_n(F))$, we can view $\mathcal{M} \simeq \mathrm{GL}_n(F) \times G$ as a maximal Levi subgroup of a classical group G' of the same type as G. Mœglin's work, which uses the language of *Jordan sets*, then gives a description of the Langlands parameter associated to σ through the LLC in terms of reducibility points of the parabolically induced representation

$$\mathrm{Ind}_{\mathcal{M},\mathcal{P}}^{G'} \pi|\det|^r \otimes \sigma, \qquad r \in \mathbb{R}$$

for $|\cdot|$ the normalized absolute value on F and $\mathcal{P}$ any parabolic subgroup containing $\mathcal{M}$. One looks at the self-dual π which gives reducibility at some $r > 1/2$, as these are precisely the ones which contribute to the Jordan set/Langlands parameter. In order to compute these points of reducibility we need to understand the construction of σ.

Originating with the work of Howe, the structure of an irreducible cuspidal representation of G is long conjectured to be of the following form. Given $\sigma \in \mathrm{Cusp}(G)$, there should exist an open compact-modulo-centre subgroup $\bar{J}$ of G and an irreducible representation Λ of $\bar{J}$ such that

$$\sigma \simeq \mathrm{ind}_{\bar{J}}^{G} \Lambda$$

where ind denotes the functor of *compact induction*. While this problem remains open for arbitrary connected reductive algebraic groups G, it is known to be true in many cases:

- $G = \mathrm{GL}_n(F), \mathrm{SL}_n(F)$ due to Bushnell–Kutzko [BK93a, BK93b, BK94];
- G arbitrary, but σ of “depth-zero”, due to Moy–Prasad and Morris [MP94, MP96, Mor99];

– G arbitrary, but σ "tamely ramified", due to Yu and Kim [Yu01, Kim07];

– G an inner form of GL due to Sécherre and Stevens [Séc05, SS08] ;

– G a classical group (i.e. Symplectic, Special Orthogonal or Unitary) provided $p \neq 2$, due to Stevens [Ste08];

– G a connected reductive algebraic group which splits over a tamely ramified extension of F and p does not divide the order of the Weyl group of G, due to Fintzen [Fin19].

Here we see the first stratification of cuspidal representations, that is the notion of depth. A representation $\sigma \in \mathrm{Irr}(G)$ is said to be of *depth-zero* if σ has fixed vectors under the pro-unipotent radical of a parahoric subgroup of G. The classification of depth-zero cuspidal representations of an arbitrary connected reductive algebraic group, as given by Moy–Prasad and Morris, is characteristic free. Using this concrete description of $\mathrm{Irr}(G)$, DeBacker and Reeder [DR09] constructed an explicit map from a large class of irreducible cuspidal depth-zero representations of G to a certain subset of Langlands parameters satisfying the conditions of the LLC. Namely, they considered *tame* regular discrete Langlands parameters. These are parameters which are trivial upon restriction to the wild inertia subgroup of $\mathcal{W}_F$.

Lust and Stevens [LS15] build upon this work by considering tame Langlands parameters and all irreducible cuspidal depth-zero representations of G, whilst imposing that G be a classical group defined over a non-archimedean local field of *odd* residual characteristic instead of an arbitrary connected reductive group. Their method involves computing the reducibility points of the parabolically induced representation via a result of Blondel by looking at the Hecke algebra of a cover (in the sense of Bushnell–Kutzko). This relies on knowing the local data which describes the representations π and σ. In this thesis, we do the same for *dyadic fields* (finite extensions of $\mathbb{Q}_2$) for the Symplectic group and most irreducible cuspidal depth-zero representations of a Split Special Orthogonal group. This amounts to showing that, if for all self-dual irreducible cuspidal depth-zero representations of $\mathrm{GL}_{m_\pi}(F)$ we write $r = r_\pi$ for the unique non-negative real number r such that the parabolically induced representation

$$\mathrm{Ind}_{\mathcal{M},\mathcal{P}}^{G'} \pi|\det|^r \otimes \sigma$$

is reducible, that the sum

$$\sum_{\pi \text{ self-dual cuspidal}} m_\pi \cdot \max\{2r_\pi - 1, 0\}$$

is equal to $N_{^L G}$, the dimension of the natural representation of the Langlands dual group $^L G$ of G. While this sum does not require that π is of depth-zero, we show that this equality holds for depth-zero representations π already, so that no other representations contribute to the sum.

For positive-depth cuspidal representations of a classical group G, we have seen that the construction of Stevens is exhaustive and complete in the sense that given $\sigma \in \mathrm{Cusp}(G)$, one can describe the local datum associated to σ. The only requirement is that the residual characteristic p is odd. Unlike the depth-zero case, trying to emulate these results here for dyadic fields is much more difficult because at almost every stage the construction due to Stevens fundamentally requires that $p \neq 2$. Here we restrict ourselves to the group $G = \mathrm{Sp}_4(F)$ and take the first steps towards an exhaustive construction of positive-depth cuspidal representations.

1.2 Summary of Chapters

In Chapter 2 we start by recalling the necessary material needed to define our classical groups G, by which we mean G is either a Symplectic group or a Split Special Orthogonal group. We then move on to prove new results about the reductive quotients of maximal parahoric subgroups of G. For the Symplectic group, we show that the description of the maximal parahorics and their reductive quotients is uniform for all p:

Proposition. *(2.9.2) Let K be a maximal parahoric subgroup of G stabilizing an almost self-dual lattice L with $\dim_{k_F}(L/L^{\#}) = 2m$. Then the reductive quotient K/K^1 is*

$$K/K^1 \simeq \mathrm{Sp}_{2m}(k_F) \times \mathrm{Sp}_{2(n-m)}(k_F).$$

For the Split Special Orthogonal group we restrict ourselves to dyadic fields. Even in this case, we are not able to consider all maximal parahoric subgroups, only those that arise from certain almost self-dual lattices L_m (Proposition 2.10.5 and Proposition 2.10.6). We then obtain the following description of their reductive quotients.

Corollary. *(2.10.7) Let G_i be a Split Special Orthogonal group with $i = \dim V_{\mathrm{an}}$. Let K_i denote the stabilizer of the lattice L_m define above and K_i° denote the maximal parahoric associated to K_i. Suppose $m \neq 1, 2, n-2, n-1$ for $i = 0$ and $m \neq n-2, n-1$ for $i = 1$. Then*

$$K_0^\circ/K_0^1 \simeq \mathrm{SO}^+_{2m}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F)$$

and

$$\begin{aligned} K_1^\circ/K_1^1 &\simeq \mathrm{SO}_{2m+1}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F) \\ &\simeq \mathrm{Sp}_{2m}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F). \end{aligned}$$

In addition we give a classification of the isometry classes of anisotropic quadratic forms over $\mathbb{Q}_2$. We do this because in order to try and give a full classification of the reductive quotients for an arbitrary Special Orthogonal group, we need to have a complete understanding of the Witt group of F. For $p \neq 2$, Morris uses the structure of the Witt group to classify the possible symmetric bilinear forms which arise [Mor91, Section 1.8], which in turn classifies the reductive quotients for the Special Orthogonal group. We note how the Witt group of F a dyadic fields depends on the degree of the field extension $F/\mathbb{Q}_2$, and so one would need to understand this fully to classify the reductive quotients in general.

In Chapter 3 we recall the representation theory of p-adic groups needed to state and prove our results. In Chapter 4 we consider G a Symplectic or Split Special Orthogonal group. For most irreducible cuspidal depth-zero representations σ of G we describe the Langlands parameter associated to σ through the local Langlands correspondence by appealing to work of Mœglin. We do this by proving the following Theorem.

Theorem. *(4.6.1) If G is a Symplectic group, let π be an arbitrary irreducible cuspidal depth-zero representation. If G is a Split Special Orthogonal group, let π be an irreducible cuspidal depth-zero representation arising from a maximal parahoric subgroup as considered in Corollary 2.10.7. Then*

$$\sum_{\substack{(\pi,n)\in\mathrm{Jord}(\sigma)\\ \pi\in\mathrm{Cusp}(F)\ \textit{of depth zero}}} \left\lfloor s_\sigma(\pi)^2 \right\rfloor m_\pi = N_{{}^LG}.$$

This requires us to prove a statement of Blondel (Proposition 4.4.1), which readily extends to dyadic fields, that allows us to interpret the reducibility points of a parabolically induced representation of a maximal Levi subgroup in terms of quadratic parameters arising in certain spherical Hecke algebras of a cover. We also need the relevant Deligne–Lusztig theory of (unipotent) cuspidal representations of finite classical (Symplectic, Special Orthogonal and Unitary) groups and general linear groups in characteristic 2 in order to calculate these quadratic parameters.

In Chapter 5 we begin an exhaustive investigation into the description of irreducible cuspidal representations of dyadic $G = \mathrm{Sp}_4(F)$. We note that Asmuth–Keys [AK91] also started this investigation for $\mathrm{GSp}_4(F)$ but they do not use the language of types, nor did they claim to construct all cuspidals. Our intentions were to give a construction of cuspidal representations of G in terms of the theory of types, as used by Bushnell–Kutzko and Stevens, but we do not get that far. We do manage to reprove a result of Moy–Prasad which says that any irreducible representation of G contains a G-fundamental stratum (Theorem 5.5.6). Note that the correct definition of G-fundamental requires the language of Moy–Prasad which uses filtrations on the dual of the Lie algebra $\mathfrak{g}$ of G.

We show that interpreting the definition in terms of the Moy–Prasad filtration is necessary by way of Example 5.4.5; this is because we obtain our characters ψ_β of our G-fundamental strata of $\mathrm{Sp}_4(F)$ by restriction of characters of strata on $\mathrm{GL}_4(F)$. We then move onto deriving a complete description of the distinguished (i.e. shortest) double-coset representatives for the three conjugacy classes of maximal parahoric subgroups of G (Theorem 5.6.3). This could be used in further work to compute the intertwining of the characters corresponding to G-fundamental strata (which are the building blocks for cuspidal representations).

Chapter 2

Classical Groups

2.1 Bilinear Forms

For a full treatise on bilinear forms and quadratic forms over arbitrary fields, we recommend [KL90] and [EKM08]. In particular, the book of Elman–Karpenko–Merkurjev adopts a characteristic free approach.

Let V be a finite-dimensional vector space over a field F of arbitrary characteristic. A *bilinear form* h is a map $h : V \times V \to F$ such that for all $u, v, w \in V$ and $\lambda \in F$,

$$\begin{aligned} h(u+v, w) &= h(u, w) + h(v, w); \\ h(u, v+w) &= h(u, v) + h(u, w); \\ h(\lambda u, v) &= h(u, \lambda v) = \lambda h(u, v). \end{aligned}$$

The bilinear form h is said to be *symmetric* if $h(u, v) = h(v, u)$ for all $u, v \in V$, *skew-symmetric* if $h(u, v) = -h(v, u)$ and *alternating* if $h(u, u) = 0$ for all $u \in V$. Alternating forms are skew-symmetric, since

$$0 = h(u+v, u+v)$$

$$= h(u,u) + h(u,v) + h(v,u) + h(v,v)$$
$$= h(u,v) + h(v,u).$$

If the characteristic of F is not 2 then the converse is also true: every skew-symmetric bilinear form is alternating since $h(u,u) = -h(u,u)$.

If the characteristic of F is 2, we need only consider symmetric and alternating bilinear forms since the notions of symmetric and skew-symmetric coincide. Moreover, by the calculation above, every alternating bilinear form is symmetric. However, the converse is not true: there exist symmetric bilinear forms which are not alternating.

Example 2.1.1. Let $V = \mathbb{F}_2^2$ with basis e_1, e_2. Let $h(e_1,e_1) = h(e_1,e_2) = h(e_2,e_1) = 1$ and $h(e_2,e_2) = 0$. Then h is a symmetric bilinear form which is not alternating.

The *Gram matrix* of h, with respect to the basis $\{e_i\}$ of V, is the matrix A_h whose ij^{th} entry is $h(e_i,e_j)$. The Gram matrix encodes all the properties of h which we wish to know. A form h is alternating if $(A_h)_{ii} = 0$ for all i. Similarly, a bilinear form h is symmetric if A_h is a symmetric matrix. In the example above, the Gram matrix of h is

$$A_h = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

If h, h' are bilinear forms on F-vector spaces V, V' respectively, an *isometry* is an invertible linear map $f : V \to V'$ which preserves the bilinear form i.e. $h(u,v) = h'(f(u), f(v))$ for all $u, v \in V'$. Equivalently, h and h' are isometric if there exist bases with respect to which their Gram matrices coincide. A vector v which satisfies $h(v,v) = 0$ is called *isotropic*. Note that for h an alternating form every vector is isotropic. Denote by $\dim h$ the *dimension* of h which is equal to $\dim V$.

Let $V^* = \mathrm{Hom}(V,F)$ denote the dual vector space of V. Consider the map $f_u : V \to V^*$ which sends v to $f_u(v) := h(u,v)$, for non-zero $u \in V$. If $u \mapsto f_u$ is an isomorphism between V and V^* then h is called *non-degenerate*, otherwise h is said to be *degenerate*. A *symplectic form* h is a non-degenerate alternating bilinear form. In practice, we will be able to test degeneracy of bilinear forms in the following way.

Two vectors u and v are *orthogonal* if $h(u, v) = 0$. For W a subspace of V, define the *orthogonal complement* $W^\perp$ of W by

$$W^\perp = \{v \in V \mid h(v, W) = 0\}.$$

For U, W subspaces of V, if $W \subseteq U^\perp$ then we say that W is *orthogonal* to U. The subspace $\operatorname{rad} h := V^\perp$ of V is called the *radical* of h. The bilinear form h is non-degenerate if and only if $\operatorname{rad} h = 0$.

Suppose $V = U \oplus W$ with $W \subseteq U^\perp$. We write $h = h\mid_U \perp h\mid_W$ and say h is the *orthogonal sum* of the forms $h\mid_U$ and $h\mid_W$. If $v = u + w, v' = u' + w'$, with $u, u' \in U$ and $w, w' \in W$, then

$$h(v, v') = h\mid_U (u, u') + h\mid_W (w, w').$$

Proposition 2.1.2. *Let h be a bilinear form on V. Let W be a subspace of V such that $V = \operatorname{rad} h \oplus W$. Then*

$$h = 0\mid_{\operatorname{rad} h} \perp h\mid_W$$

with $h\mid_W$ non-degenerate.

Proof. Note that we need only show that the restriction of h to W is non-degenerate. Suppose $w \in \operatorname{rad}(h\mid_W)$. Then $w \in W^\perp$; since $w \in W \subseteq (\operatorname{rad} h)^\perp$ also, we have $w \in (W + \operatorname{rad} h)^\perp = V^\perp$ so $w \in \operatorname{rad} h$. Therefore $w \in W \cap \operatorname{rad} h = \{0\}$. $\square$

2.2 Quadratic Forms

Let V be a finite-dimensional vector space over a field F of arbitrary characteristic. A *quadratic form* Q on V is a map $Q : V \to F$ satisfying:

1) $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V, \lambda \in F$;

2) $h : V \times V \to F$ given by $h(u,v) := Q(u+v) - Q(u) - Q(v)$ is a bilinear form.

The bilinear form h associated to any quadratic form is automatically symmetric since

$$h(u,v) = Q(u+v) - Q(u) - Q(v) = h(v,u).$$

Furthermore, it is alternating if $\operatorname{char} F = 2$ because

$$h(u,u) = Q(u+u) - Q(u) - Q(u) = 4Q(u) - 2Q(u) = 2Q(u) = 0.$$

Let A_h denote the Gram matrix of the bilinear form h associated to Q. The upper triangular matrix A_Q satisfying

$$A_Q + A_Q^T = A_h$$

is called the *Gram matrix of* Q.

An *isometry* between two quadratic forms Q and Q', defined over V and V' respectively, is an invertible linear map $f : V \to V'$ such that $Q(v) = Q'(f(v))$ for all $v \in V$. If there exists an isometry between Q and Q' then the two forms are *isometric*. Note that if f is an isometry for Q, then it is also an isometry for the corresponding form h, but the converse is false in general. If $V = V'$ then the two forms Q and Q' above are said to be *equivalent* if there exists an invertible matrix C such that $Q(v) = Q'(Cv)$ for all $v \in V$. We see from the definitions that the equivalence classes of quadratic forms correspond to the isometry classes of quadratic spaces.

Let Q be a quadratic form over V and $a \in F$. We say Q *represents* a if there exists some $v \in V$ such that $Q(v) = a$. We call $Q(v)$ the *norm* of v. If Q represents every $a \in F^\times$ then Q is said to be *universal*. We denote by $\operatorname{Im}(Q)$ the *image of* Q, which is the set of all possible norms of Q, i.e.

$$\operatorname{Im}(Q) = \{Q(v) : v \in V\}.$$

Proposition 2.2.1. *[dSP11, Partie 1.III] Suppose* $\operatorname{char} F \neq 2$. *Two quadratic forms are equivalent if and only if they have the same image.*

The *dimension* of Q, denoted $\dim Q$, is the dimension of V. A non-zero vector $v \in V$ is *singular* if v has norm 0, otherwise it is *anisotropic*. A subspace W of V is *anisotropic* if W contains no singular vectors. A quadratic form Q is *anisotropic* if V is anisotropic.

Remark 2.2.2. If a vector v is singular then it is isotropic for h. The converse is true when $\operatorname{char} F \neq 2$ since $h(v,v) = 2Q(v)$. When $\operatorname{char} F = 2$ the converse is false, see Example 2.2.4.

The *radical* of Q, denoted $\operatorname{rad} Q$, is the subset of vectors of $\operatorname{rad} h$ of norm 0 i.e.

$$\operatorname{rad} Q = \{v \in \operatorname{rad} h \mid Q(v) = 0\}.$$

Recall that h is non-degenerate if $\operatorname{rad} h = 0$. The quadratic form Q is *regular* if $\operatorname{rad} Q = 0$. We say Q is *non-degenerate* if Q is regular and $\dim \operatorname{rad} h \leq 1$. Thus we see that if h is non-degenerate then Q is non-degenerate, but the converse is not always true.

Remark 2.2.3. Some sources say that Q is non-degenerate if its associated bilinear form is non-degenerate. While this definition coincides with the definition above when $\operatorname{char} F \neq 2$ (as $\operatorname{rad}\ h = \operatorname{rad} Q$), it is too restrictive in our case in the sense that it will omit many quadratic forms from consideration. Consider the following example.

Example 2.2.4. Let Q be the 3-dimensional quadratic form defined by $Q(x,y,z) = x^2 + y^2 + yz + z^2$ over $V = \mathbb{F}_2^3$ with basis e_1, e_2, e_3. Q has associated bilinear form h with Gram matrix

$$A_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which is degenerate with radical $\langle (1,0,0) \rangle_{\mathbb{F}_2}$. The only singular vector $v \in \operatorname{rad} h$ is $v = 0$, and so Q is regular. Since its associated bilinear form h is degenerate with a one-dimensional radical, Q is in fact non-degenerate.

Lemma 2.2.5. *Let Q be a regular quadratic form over a finite field F of characteristic 2. Then Q is non-degenerate.*

Proof. Suppose $\operatorname{rad} h \neq \{0\}$ and let $v \in \operatorname{rad} h$ be non-zero. Since $x \mapsto x^2$ is an automorphism of F, we may scale v so that $Q(v) = 1$. For non-zero $u \in \operatorname{rad} h$, we have $Q(u) \neq 0$ by regularity of Q. For some $\delta \in F^\times$ we have $Q(u) = \delta^2 = Q(\delta v)$. By definition, $Q(u + \delta v) = Q(u) + h(u, \delta v) + Q(\delta v) = h(u, \delta v) = 0$, with the last equality holding since $u, v \in \operatorname{rad} h$. By regularity $Q(u + \delta v) = 0$ implies $u = \delta v$. Hence $\dim \operatorname{rad} h = 1$. $\square$

Suppose that $\operatorname{char} F \neq 2$, then there exists a basis $\{e_1, \ldots, e_n\}$ of V such that Q is diagonal, i.e. we write $Q = \langle \lambda_1, \ldots, \lambda_n \rangle$ for the form

$$Q\left(\sum_{i=1}^{n} a_i e_i\right) = \lambda_1 a_1^2 + \cdots \lambda_n a_n^2.$$

Unless otherwise stated, if the characteristic of F is not 2, then we assume that our quadratic form is diagonal.

The *hyperbolic form* $\mathbb{H}(V) = Q_{\mathbb{H}}$ on $V \oplus V^*$ is defined as

$$Q_{\mathbb{H}}(v, f) := f(v)$$

for all $v \in V$ and $f \in V^*$. If Q is a quadratic form isometric to $\mathbb{H}(V')$ for some vector space V', we say Q is a *hyperbolic form*. We call $\mathbb{H}(F)$ the *hyperbolic plane* and denote it by $\mathbb{H}$. If Q is isometric to $\mathbb{H}$, then two vectors u, v satisfying $Q(u) = Q(v) = 0$ and $h(u, v) = 1$ are called a *hyperbolic pair*.

We now turn to the question of classifying quadratic forms (up to isometry). In order to this, we make use of the following Theorem.

Theorem 2.2.6 (Witt's Decomposition Theorem)**.** *Let Q be a quadratic form on V. There exist subspaces V_1 and V_2 of V such that $Q = Q|_{\operatorname{rad} Q} \perp Q|_{V_1} \perp Q|_{V_2}$ with $Q|_{V_1}$ anisotropic and $Q|_{V_2}$ hyperbolic. Moreover, $Q|_{V_1}$ and $Q|_{V_2}$ are uniquely determined up to isometry by (V, Q).*

Remark 2.2.7. In Example 2.2.4 above, we have $V = \operatorname{rad} Q \oplus V_1$ where $\operatorname{rad} Q = \langle e_1 \rangle_{\mathbb{F}_2}$ and $V_1 = \langle e_2, e_3 \rangle_{\mathbb{F}_2}$ anisotropic.

In reality, the quadratic forms we consider will be non-degenerate. In particular, we will be interested in the group of isometries of such forms. If the dimension of the radical is zero, then Witt's Decomposition Theorem simplifies to the following.

Theorem 2.2.8. *Let V be a finite dimensional F-vector space. Then there exists an n such that*

$$V = V_{an} \oplus n\mathbb{H},$$

where $n\mathbb{H}$ denotes n copies of the hyperbolic plane and V_{an} denotes the anisotropic subspace uniquely determined by V (up to isometry).

We see that in order to understand the group of isometries of a quadratic form we now need to understand the isometry classes of anisotropic quadratic forms. The study of such spaces is dependent on the choice of underlying field; for our purposes we only consider finite fields of characteristic 2 and dyadic fields.

A *quadratic space* $X = (V_X, Q_X)$ is a vector space V_X endowed with a quadratic form Q_X. Let X be an anisotropic quadratic space. Denote by $[X]$ the class of quadratic spaces (V, Q_V) such that the anisotropic subspace V_{an} of V is isometric to V_X. We call $[X]$ the *Witt class of* X. The set of Witt classes has a natural group structure which is defined as follows.

The identity is the zero class $[0]$, which corresponds to the zero form $Q_0 = 0$ defined over the zero space $V_0 = \{0\}$. This is trivially anisotropic. Given two Witt classes $[X]$ and $[Y]$, their sum $[X + Y]$ is the class of quadratic forms which contains $V_X \perp V_Y$, equipped with the quadratic form $Q_X \perp Q_Y$; this is independent of the choice of representatives X, Y. Given $[X]$, its inverse $[-X]$ is the Witt class whose anisotropic subspace is isometric to V_X with quadratic form $-Q_X$.

2.3 Anisotropic Quadratic Forms over Dyadic Fields

Let F be a dyadic field. The following Theorem sheds light on a bound for the dimension of anisotropic forms.

Theorem 2.3.1. *[Lam05, Chapter 6] Any five-dimensional quadratic form Q over F is isotropic.*

Whilst the Theorem above tells us that any anisotropic space is at most 4-dimensional, it does not shed any light on any of their other properties. It is natural to ask how many isometry classes of anisotropic forms there are for a given field F. It turns out that the number of isometry classes is closely related to the degree of the field extension $F/\mathbb{Q}_2$.

Proposition 2.3.2. *[Lam05, Chapter 6] If F is a finite extension of $\mathbb{Q}_2$ of degree n, then F has 2^{n+4} anisotropic forms (up to isometry).*

We now have an explicit formula for the number of anisotropic forms, but in order to understand their nature, we must understand the Witt group $W(F)$ of F. While we will not be working explicitly with $W(F)$, we will need to make use of its structure, which is described in the following Theorem.

Theorem 2.3.3. *[Lam05, Chapter 6] Let F be a dyadic field of degree n over $\mathbb{Q}_2$.*

(i) If $-1 \in \left(F^\times\right)^2$ then $W(F) \simeq (\mathbb{Z}/2\mathbb{Z})^{n+4}$;

(ii) If $-1 \notin \left(F^\times\right)^2$, but -1 is the sum of two squares in F, then $W(F) \simeq (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^n$;

(iii) If -1 is not the sum of two squares in F, then $W(F) \simeq (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{n+1}$.

2.3.1 Classification of Anisotropic Forms over $\mathbb{Q}_2$

We explicitly study the case that $F = \mathbb{Q}_2$, so $n = [F : \mathbb{Q}_2] = 1$. By the Theorems above, we know that there are $2^{1+4} = 32$ anisotropic forms up to isometry, including the 0-form. Moreover, since -1 is not a sum of two squares in $\mathbb{Q}_2$, we know that Witt group of $\mathbb{Q}_2$ is of the form $W(\mathbb{Q}_2) \simeq (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. We now classify the anisotropic forms by their dimension, starting with the one-dimensional forms.

Suppose we have two one-dimensional quadratic forms Q_1 and Q_2 defined over $V_1 = V_2 = F$, spanned by some fixed vector v. They are isometric precisely when there exists an isometry $f : Q_1 \to Q_2$ such that $Q_1(v) = Q_2(f(v))$. Writing $f(v) = \lambda v$ for some $\lambda \in \mathbb{Q}_2^\times$ we have that Q_1 and Q_2 are isometric when $Q_1(v) = \lambda^2 Q_2(v)$ i.e. when Q_1 and Q_2 differ by a square in $\mathbb{Q}_2^\times$. This shows that the one-dimensional anisotropic forms are in bijection with $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$. Thus the inequivalent one-dimensional forms are

$$\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \langle 7\rangle, \langle 2\rangle, \langle 6\rangle, \langle 10\rangle, \langle 14\rangle.$$

From [Sch85, Chapter 5] in $\mathbb{Q}_2$ there is a unique four-dimensional anisotropic quadratic form, namely $Q = \langle 1, 1, 1, 1\rangle$. Moreover, this form is *universal*. Using this fact, alongside Theorem 2.3.1, we are able to find the three-dimensional forms immediately.

Since every five-dimensional form is isotropic, we can find a hyperbolic space, and, using Witt's Decomposition Theorem, we find a three-dimensional anisotropic subform. In this way, the anisotropic subform of the five-dimensional isotropic form $\langle 1, 1, 1, 1\rangle \perp \langle -1\rangle$ is $\langle 1, 1, 1\rangle$. Since $\langle 1, 1, 1, 1\rangle$ is universal it is isometric to $\langle \lambda, \lambda, \lambda, \lambda\rangle$ for any $\lambda \in \mathbb{Q}_2^\times$. By this procedure above, for all isometry classes of one-dimensional forms we get the following list of three-dimensional anisotropic forms:

$$\langle 1, 1, 1\rangle, \langle 3, 3, 3\rangle, \langle 5, 5, 5\rangle, \langle 7, 7, 7\rangle, \langle 2, 2, 2\rangle, \langle 6, 6, 6\rangle, \langle 10, 10, 10\rangle, \langle 14, 14, 14\rangle.$$

Since we only have the two-dimensional forms to find, we know that there are $32 - 1 - 1 - 8 - 8 = 14$ such forms.

For any $a, b \in \mathbb{Q}_2^\times$ when is the quadratic form $Q = \langle a, b\rangle$ anisotropic? Recall that Q is anisotropic if the only vector with norm 0 is the zero vector. Therefore Q is anisotropic if and only if the only solution to $ax^2 + by^2 = 0$ is $x = y = 0$. If (both) x, y are non-zero, we rearrange to get $-a/b = y^2/x^2$, so non-zero vectors can have zero norm if and only if $-a$ and b differ by a square in $\mathbb{Q}_2$. Using this criterion, we find that every anisotropic quadratic form is isometric to one of the following:

$$\begin{aligned}
&\langle 1,1\rangle, \langle 1,3\rangle, \langle 1,5\rangle, \langle 1,2\rangle, \langle 1,6\rangle, \langle 1,10\rangle, \langle 1,14\rangle,\\
&\langle 2,3\rangle, \langle 2,5\rangle, \langle 2,7\rangle, \langle 2,2\rangle, \langle 2,6\rangle, \langle 2,10\rangle,\\
&\langle 3,3\rangle, \langle 3,7\rangle, \langle 3,6\rangle, \langle 3,10\rangle, \langle 3,14\rangle,\\
&\langle 5,5\rangle, \langle 5,7\rangle, \langle 5,6\rangle, \langle 5,10\rangle, \langle 5,14\rangle,\\
&\langle 6,7\rangle, \langle 6,6\rangle, \langle 6,14\rangle,\\
&\langle 7,7\rangle, \langle 7,10\rangle, \langle 7,14\rangle,\\
&\langle 10,10\rangle, \langle 10,14\rangle,\\
&\langle 14,14\rangle.
\end{aligned}$$

We now need only classify the isometry classes of these forms. Two forms are isometric if and only if they represent the same numbers. One way to identify the isometry classes could be to calculate the square classes which each form represents, but this is a cumbersome method. Instead, we make use of work of [Sch85, Chapter 5]. Here the author gives a list of the square classes represented by quadratic forms of the form $\langle 1, a\rangle$, where $a \in \mathbb{Q}^\times/(\mathbb{Q}_2^\times)^2$. Using properties of the tensor product of quadratic forms, and the necessary condition that two forms are isometric if they have the same determinant, we are able to identify the square classes represented by all the two-dimension forms above immediately.

Example 2.3.4. We ask if the forms $\langle 5,6\rangle$ and $\langle 1,14\rangle$ are isometric. With multiplication defined over $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$, since the representatives for the square classes satisfy $5\cdot 6 = 14 = 1\cdot 14$, we have that the forms have the same determinant. We now check if they represent the same square classes in $\mathbb{Q}_2$, with the understanding that the multiplication above is of square classes in $\mathbb{Q}_2/(\mathbb{Q}_2^\times)^2$. We write $\langle 5,6\rangle$ as $\langle 5\rangle \otimes \langle 1,14\rangle$. Since $\langle 1,14\rangle$ represents $1, 2, 7, 14$, the form $\langle 5,6\rangle \simeq \langle 5\rangle \otimes \langle 1,14\rangle$ represents $5\cdot 1 = 5, 5\cdot 2 = 10, 5\cdot 7 = 3, 5\cdot 14 = 6$. Since $\{1, 2, 7, 14\} \neq \{3, 5, 6, 10\}$ we have that $\langle 5,6\rangle$ is not isometric to $\langle 1,14\rangle$.

A routine calculation in this fashion gives the following isometry classes of two-dimensional forms over $\mathbb{Q}_2$:

$$\langle 1,1\rangle, \langle 3,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 2,3\rangle, \langle 1,10\rangle,$$
$$\langle 1,14\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 2,10\rangle, \langle 3,10\rangle, \langle 5,10\rangle.$$

Table 2.1 summarises the results above to give a classification of all anisotropic quadratic forms over $\mathbb{Q}_2$.

0-dimensional	$\langle 0\rangle$
1-dimensional	$\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \langle 7\rangle,$ $\langle 2\rangle, \langle 6\rangle, \langle 10\rangle, \langle 14\rangle$
2-dimensional	$\langle 1,1\rangle, \langle 3,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle,$ $\langle 1,5\rangle, \langle 1,6\rangle, \langle 2,3\rangle, \langle 1,10\rangle,$ $\langle 1,14\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 2,10\rangle,$ $\langle 3,10\rangle, \langle 5,10\rangle$
3-dimensional	$\langle 1,1,1\rangle, \langle 3,3,3\rangle, \langle 5,5,5\rangle,$ $\langle 7,7,7\rangle, \langle 2,2,2\rangle, \langle 6,6,6\rangle,$ $\langle 10,10,10\rangle, \langle 14,14,14\rangle$
4-dimensional	$\langle 1,1,1,1\rangle$

Table 2.1: Isometry classes of anisotropic forms over $\mathbb{Q}_2$.

2.4 Symplectic Groups over Finite Fields of Characteristic 2

Let V be a finite-dimensional vector space over F a finite field of characteristic 2 and let h be a symplectic form on V. Recall that since $\operatorname{char} F = 2$ the form h is symmetric. The *Symplectic group* $\mathrm{Sp}(V)$ is the group of isometries of h, i.e.

$$\mathrm{Sp}(V) = \{g \in GL(V) \mid h(gu, gv) = h(u, v) \text{ for all } u, v \in V\ \}.$$

A *symplectic basis* of V is a basis $\{e_{-i}, e_j : 1 \leq i, j \leq n\}$ of V satisfying $h(e_{-i}, e_j) = \delta_{ij}$. Given any symplectic form h, we find a symplectic basis of V inductively as follows.

Pick two vectors u and v such that $h(u,v) = \lambda \neq 0$. Set $e_{-1} := u$, $e_1 := \lambda^{-1}v$ and $U = \langle e_{-1}, e_1 \rangle_F$. With respect to the basis $\{e_{-1}, e_1\}$ the symplectic form $h\ |_U$ satisfies $h(e_{-1}, e_{-1}) = h(e_1, e_1) = 0$ and $h(e_{-1}, e_1) = 1$. Since h is non-degenerate we have $V = U \perp U^\perp$ and $h\ |_{U^\perp}$ is non-degenerate. We restrict h to $U^\perp$ and repeat. In this way we obtain a symplectic basis $\{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ for V.

Given non-zero $v \in V$, the *symplectic transvection* associated to v is the linear map $t_v : V \to V$ given by $t_v(u) = u + h(u,v)v$ for all $u \in V$. Since we are in characteristic 2, the symplectic transvections are in fact involutions:

$$\begin{aligned} t_v(t_v(u)) &= t_v(u + h(u,v)v) \\ &= u + h(u,v)v + h(u + h(u,v)v, v)v \\ &= u + h(u,v)v + h(u,v)v + h(u,v)h(v,v)v \\ &= u. \end{aligned}$$

By [O'M78, Chapter 2], the Symplectic group is generated by symplectic transvections, i.e.

$$\mathrm{Sp}(V) = \langle t_v \mid v \in V \rangle.$$

2.5 Special Orthogonal Groups over Finite Fields of Characteristic 2

Let V be a finite-dimensional vector space over F a finite field of characteristic 2. Let Q be a quadratic form defined over V with associated bilinear form h. The *Orthogonal group* $\mathrm{O}(Q)$ is the group of isometries of Q i.e.

$$\mathrm{O}(Q) = \{g \in \mathrm{GL}(V) \mid Q(gv) = Q(v) \text{ for all } v \in V\}.$$

Let $v \in V$ be a non-singular vector. The *reflection in* v is the map $r_v : V \to V$ given by

$$u \mapsto u - \frac{h(u,v)}{Q(v)}v, \qquad \text{for } u \in V.$$

The map r_v truly is an involution:

$$\begin{aligned} r_v(r_v(u)) &= r_v\left(u - \frac{h(u,v)}{Q(v)}v\right) \\ &= u - \frac{h(u,v)}{Q(v)}v - \frac{h\left(u - \frac{h(u,v)}{Q(v)}v, v\right)}{Q(v)}v \\ &= u - \frac{h(u,v)}{Q(v)}v - \frac{h(u,v)}{Q(v)}v + \frac{h(u,v)}{Q(v)^2}h(v,v)v \\ &= u, \end{aligned}$$

as h is alternating and $\operatorname{char} F = 2$. Moreover, we have that $r_v(u) = r_{\lambda v}(u)$ for all $\lambda \in F^\times$ since

$$\begin{aligned} r_{\lambda v}(u) &= u - \frac{h(u,\lambda v)}{Q(\lambda v)}\lambda v \\ &= u - \frac{\lambda^2 h(u,v)}{\lambda^2 Q(v)}v \\ &= u - \frac{h(u,v)}{Q(v)}v \\ &= r_v(u). \end{aligned}$$

Remark 2.5.1. Note that the reflection defined above is not a Euclidean reflection $\sigma_v(u)$, which is of the form

$$\sigma_v(u) = u - 2\frac{h(u,v)}{Q(u)}v.$$

Proposition 2.5.2. *Suppose Q is a regular quadratic form on a vector space V of dimension $2n+1$. Then $\mathrm{O}(Q) \cong \mathrm{Sp}_{2n}(F)$.*

Proof. We know by Proposition 2.2.5 that the radical of h is at most one-dimensional. Since V is of odd dimension, the bilinear form associated to Q is alternating and degenerate which means that $V_0 = \operatorname{rad} h$ is precisely 1-dimensional. By scaling if necessary, we may assume that $v_0 \in V$ spans V_0 and has norm 1.

Let G be the group of isometries of Q, and $\bar{G}$ be the group of isometries of the form induced by h on V/V_0. This form is non-degenerate so V/V_0 is a symplectic space of dimension $2n$, and so $\bar{G} \simeq \mathrm{Sp}_{2n}(F)$. If $v \in V$ then the other elements of $v + V_0$ are of the form $v + \lambda v_0$, where $\lambda \in F$; then

$$\begin{aligned} Q(v + \lambda v_0) &= Q(v) + \lambda^2 Q(v_0) - h(v, \lambda v_0) \\ &= Q(v) + \lambda^2 \end{aligned}$$

since $v_0 \in V_0$ and v_0 has norm 1. As squaring is a bijection on F, every coset in V/V_0 contains a vector of every possible norm. Moreover, there is a unique vector of each norm, since if $Q(v + \lambda_1 v_0) = Q(v + \lambda_2 v_0)$ the calculation above shows that $\lambda_1 = \lambda_2$.

Let K denote the kernel of the homomorphism $G \to \bar{G}$ and let $k \in K$. Not only must k fix each coset of V/V_0, but it must map an element in the coset to another element of the same norm. Since there is precisely one vector of each norm in every coset, we must have that k is the identity. Thus $K = \{\mathrm{Id}\}$ and $G \hookrightarrow \bar{G}$.

Lastly, we must show that we can lift every isometry of $\bar{G}$ to an isometry of G. The Symplectic group is generated by symplectic transvections $t_{\bar{v}}$, so we need only prove that any $t_{\bar{v}}$ can be lifted. Since we have an element of every possible norm in each coset, we may choose a lift $v \in V$ of $\bar{v}$ with norm $Q(v) = 1$. Then the reflection $r_v \in G$ is a lift of $t_{\bar{v}}$, as required. □

Thus, if V is odd-dimensional we can view the orthogonal group of a regular quadratic form on V as a Symplectic group of smaller dimension. We therefore restrict ourselves to the case that V is even-dimensional and regular, so h is non-degenerate. We now find an orthogonal basis for V, which will depend on Q. We find a basis for V inductively in the same way as we do for the Symplectic group, except that we choose our basis vectors to be singular whenever possible.

If $\dim V > 2$, the vector space V is isotropic and so we can remove a hyperbolic space (by application of Witt's Decomposition Theorem), reducing ourselves to the case that $\dim V = 2$ [KL90, Lemma 2.5.2].

Up to equivalence, there are two quadratic forms of dimension 2. The first is of *plus type*, which means there exists a basis of V such that $Q(x,y) = xy$. In characteristic 2, this form is equivalent to a hyperbolic space. The second is of *minus type*, which means that there exists a basis of V such that $Q(x,y) = x^2 + xy + \lambda y^2$, where $\lambda \in F^\times$ is such that $X^2 + X + \lambda \in F[X]$ is irreducible.

For any vector space associated to Q, we can find a basis of the following kind:

1. $\mathcal{B}_+ = \{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ where $Q\mid_{\langle e_{-i}, e_i\rangle_F}$ is hyperbolic and the spaces $\langle e_{-i}, e_i\rangle_F$ are pairwise orthogonal;

2. $\mathcal{B}_- = \{e_{-(n-1)}, \ldots, e_{-1}, e_{-0}, e_{+0}, e_1, \ldots, e_{n-1}\}$ where $Q\mid_{\langle e_{-i}, e_i\rangle_F}$ is hyperbolic, $Q\mid_{\langle e_{-0}, e_{+0}\rangle_F}$ is of minus type and the spaces $\langle e_{-i}, e_i\rangle_F$ are pairwise orthogonal.

We define the *sign* of Q as

$$\operatorname{sgn} Q = \begin{cases} +, & \text{if } V \text{ has basis } \mathcal{B}_+, \\ -, & \text{if } V \text{ has basis } \mathcal{B}_-. \end{cases}$$

If $\operatorname{sgn} Q = +$ the orthogonal space (V,Q) is *hyperbolic* or of *Witt defect* 0. If $\operatorname{sgn} Q = -$, we refer to (V,Q) as *anisotropic* or of *Witt defect* 1. We therefore have two classes of Orthogonal groups, namely $\mathrm{O}^+_{2n}(F)$ if Q is hyperbolic and $\mathrm{O}^-_{2n}(F)$ if Q is anisotropic.

Proposition 2.5.3. *[KL90, Proposition 2.5.6] Let F be a finite field of characteristic 2. Provided $(n,F) \neq (2,\mathbb{F}_2)$, the Orthogonal group $\mathrm{O}^\pm_{2n}(F)$ is generated by the set of reflections $\{r_v : Q(v) \neq 0\}$.*

While we would like to define the Special Orthogonal group $\mathrm{SO}^\pm_{2n}(k)$ as the index two subgroup of $\mathrm{O}^\pm_{2n}(k)$, we see below that this is not always well-defined. Provided $(n,F) \neq (2,\mathbb{F}_2)$,

$$\mathrm{SO}^\pm_{2n}(F) = \left\{ g \in \mathrm{O}^\pm_{2n}(F) : \begin{array}{l} g \text{ can be written as a product of an} \\ \text{even number of reflections} \end{array} \right\}.$$

Suppose now we are in the case $(n, F) = (2, \mathbb{F}_2)$. Let Q be a 4-dimensional non-degenerate quadratic form which has no singular vectors, which has Gram matrix

$$A_Q = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 0 & & \\ 0 & & & \end{pmatrix}$$

with respect to the fixed basis $\mathcal{B}_+$ above. Let $\mathcal{U}$ denote the set of all maximal singular subspaces of V. Define an equivalence relation on $\mathcal{U}$ by saying that two subspaces $\mathcal{W}, \mathcal{W}'$ are related, written $\mathcal{W} \sim \mathcal{W}'$, if $\dim(\mathcal{W} \cap \mathcal{W}')$ is even. There are precisely two equivalence classes under this relation, which we denote $\mathcal{U}_i$ for $i = 1, 2$. The Orthogonal group $\mathrm{O}_4^+(\mathbb{F}_2)$ preserves this equivalence relation, which gives a homomorphism φ from the orthogonal group to the symmetric group on $\{\mathcal{U}_1, \mathcal{U}_2\}$. The Special Orthogonal group $\mathrm{SO}_4^+(\mathbb{F}_2)$ is then defined as the kernel of φ.

Proposition 2.5.4. *[KL90, Proposition 2.5.9] Let Q be the quadratic form defined above. There are three distinct subgroups of index 2 of $\mathrm{O}(Q) = \mathrm{O}_4^+(\mathbb{F}_2)$:*

(i) the subgroup of $\mathrm{O}(Q)$ generated by reflections;

(ii) the subgroup of $\mathrm{O}(Q)$ consisting of elements which induce an even permutation of $\mathcal{U}$;

(iii) the subgroup $\ker \varphi$.

Remark 2.5.5. If we are not in the case $(n, F) = (2, \mathbb{F}_2)$, then the three subgroups defined above coincide.

Remark 2.5.6. Similar to the symplectic case, it can be shown that for arbitrary fields F with $\operatorname{char} F \neq 2$ that the Orthogonal group of dimension n is generated by reflections [Gro02, Theorem 6.6]

2.6 Parabolic Subgroups

We refer the reader to Chapter 1 III - Paraboliques of [MgVW87] for more information on the following section. Let F be a field of characteristic different from 2 and V be an n-

dimensional F-vector space endowed with either a symplectic form h or a non-degenerate quadratic form Q. A *self-dual flag* in V is a flag of isotropic subspaces

$$\{0\} = V_r \subsetneq V_{r-1} \subsetneq \cdots \subsetneq V_1 \subsetneq V_0.$$

We then define

$$V_{-i} = \{v \in V : h(v, w) = 0 \text{ for all } w \in V_i\} = V_i^{\perp}.$$

The stabilizers of the self-dual flags are *parabolic* subgroups of G. Parabolic subgroups $\mathcal{P}$ admit a Levi decomposition $\mathcal{P} = \mathcal{M} \ltimes \mathcal{N}$, where $\mathcal{M} \simeq \mathcal{P}/\mathcal{N}$ is a Levi subgroup which is reductive and $\mathcal{N}$ its unipotent radical. While there is no canonical Levi subgroup $\mathcal{M}$, any two Levi subgroups of $\mathcal{P}$ are conjugate within $\mathcal{P}$. In order to explicitly describe $\mathcal{M}$, we first must choose a decomposition

$$V_0^{\perp} = W_0 \oplus W_1 \oplus \cdots \oplus W_r$$

such that $V_i = \bigoplus_{j>i} W_j$. The stabilizer of this decomposition is then a Levi subgroup and we get an isomorphism

$$\mathcal{M} \simeq G_0' \times \prod_{i=1}^{r} \mathrm{GL}(W_i),$$

where G_0' is the classical group of $(W_0, h\,|_{W_0})$ (resp. $(W_0, Q\,|_{W_0})$). The unipotent radical $\mathcal{N}$ is the set of elements of $\mathcal{P}$ which act trivially on all quotient spaces V_i/V_{i+1} for $-r < i \leq r$.

The parabolic subgroup $\mathcal{B}$ associated to a maximal self-dual flag is called a *Borel* subgroup. It has a Levi decomposition $\mathcal{B} = \mathcal{T} \ltimes \mathcal{N}_0$ where $\mathcal{T}$ is the centralizer of a maximal F-split torus. If we fix such a group $\mathcal{B}$, then we say that any parabolic subgroup $\mathcal{P}$ containing $\mathcal{B}$ is *standard*. Moreover, if we fix $\mathcal{T}$ then any Levi subgroup containing $\mathcal{T}$ is called standard.

2.7 Parahoric Subgroups

Let F be a dyadic field with $\mathfrak{o}_F$ its ring of integers of F and $\mathfrak{p}_F$ its unique maximal ideal so that the residue field $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ is finite of cardinality $q = p^r$ for some $r \in \mathbb{N}$. Fix

ϖ_F a uniformizer of F. When there is no ambiguity we will drop the subscript F from the notation above.

Let V be a finite-dimensional vector space defined over F. Let G be a classical group, by which we mean either V has Symplectic form h and $G = G_h = \mathrm{Sp}(V)$ is a Symplectic group or V has a non-degenerate quadratic form Q and $G = G_Q = \mathrm{SO}(V)$ is a Special Orthogonal group. In the latter case, we let h denote the associated bilinear form to Q:

$$h(u, v) = Q(u + v) - Q(u) - Q(v).$$

An $\mathfrak{o}_F$*-lattice* in V is a compact open $\mathfrak{o}_F$-submodule of V. Let $\mathcal{L}$ denote the set of lattices in V. For $L \in \mathcal{L}$, the lattice

$$L^{\#} = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$$

is called the *dual lattice* of L. The notion of dual lattice defined here is a duality, i.e. $\left(L^{\#}\right)^{\#} = L$ and $(L \cap M)^{\#} = L^{\#} + M^{\#}$ for all lattices $L, M \in \mathcal{L}$. A lattice L is said to be *almost self-dual* if

$$L \supseteq L^{\#} \supseteq \mathfrak{p}_F L.$$

An $\mathfrak{o}_F$*-lattice sequence* is a function $\Lambda : \mathbb{Z} \to \mathcal{L}$ satisfying:

(i) $\Lambda(n) \supseteq \Lambda(n+1)$ for all $n \in \mathbb{N}$;

(ii) there exists an $e(\Lambda) \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, we have $\varpi\Lambda(n) = \Lambda(n + e(\Lambda))$.

The integer $e = e(\Lambda)$ is called the $\mathfrak{o}_F$*-period* of Λ. An $\mathfrak{o}_F$*-lattice chain* is an injective lattice sequence. The notion of duality carries over to lattice sequences. The *dual lattice sequence* $\Lambda^{\#}$ of Λ is the lattice sequence $\Lambda^{\#}$ satisfying

$$\Lambda^{\#}(n) = (\Lambda(-n))^{\#}$$

for all $n \in \mathbb{N}$. We say that Λ is *self-dual* if there exists $k \in \mathbb{Z}$ such that $\Lambda(n) = \Lambda^{\#}(n + k)$ for all $n \in \mathbb{N}$.

For a self-dual lattice sequence Λ and $l \in \mathbb{Z}$, let Λ_l denote the lattice sequence which is a translate of Λ defined by $\Lambda_l(n) = \Lambda(n+l)$ for all $n \in \mathbb{Z}$. By considering an appropriate translate, we may assume that $k = 0$ or 1 for any self-dual lattice sequence.

For Λ a lattice sequence and $m \in \mathbb{Z}$ let

$$\mathfrak{A}_m = \mathfrak{A}_m(\Lambda) = \{x \in \mathrm{End}_F(V) : x\Lambda(n) \subseteq \Lambda(n+m) \text{ for all } n \in \mathbb{Z}\}.$$

The additive subgroup $\mathfrak{A} = \mathfrak{A}_0(\Lambda)$ is a *hereditary order.*

An $\mathfrak{o}_F$*-order* is a unital subring of $\mathrm{End}_F(V)$ which is itself an $\mathfrak{o}_F$-lattice. After fixing a suitable basis for V the hereditary order $\mathfrak{A}$ is identified as a block matrix which has entries on and above the diagonal in $\mathfrak{o}_F$, and matrices below the diagonal with entries in $\mathfrak{p}_F$. The *Jacobson radical* $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$ is the maximal two-sided invertible fractional ideal of $\mathfrak{A}$. It consists of block matrices which has entries in $\mathfrak{o}_F$ above the diagonal, and entries in $\mathfrak{p}_F$ along and below the diagonal. The Jacobson radical satisfies $\mathfrak{P}^{e(\Lambda)} = \varpi\mathfrak{A} = \mathfrak{A}_{e(\Lambda)}$.

We momentarily restrict ourselves to the case $G = \mathrm{GL}_N(F)$. The unit group

$$U(\Lambda) = U(\mathfrak{A}) = \mathfrak{A}^{\times}$$

is a *parahoric* subgroup of $\mathrm{GL}_N(F)$. If we let $\mathfrak{A}$ be a minimal hereditary order (i.e. $e = n$) then the unit group $\mathfrak{I} = U(\mathfrak{A})$ is called an *Iwahori* subgroup. For arbitrary Λ the unit group comes with a natural filtration by normal compact open subgroups

$$U^n(\Lambda) = 1 + \mathfrak{A}_n,$$

for $n \geq 1$. Since $\mathfrak{A}$ has blocks along the diagonal with entries in $\mathfrak{o}_F$ of size n_i such that $\sum_i^e n_i = N$, the quotient $U(\Lambda)/U^1(\Lambda)$ is isomorphic to the group $\prod_i^e \mathrm{GL}_{n_i}(k_F)$ defined over the residue field k_F.

The *normalizer* $\mathfrak{K}(\Lambda) = \{g \in \mathrm{GL}_N(F) : g\Lambda = \Lambda\}$ of Λ is an open, compact-mod-centre subgroup of $\mathrm{GL}_N(F)$. It normalizes $U(\Lambda)$ and contains $U(\Lambda)$ as its maximal compact subgroup. The normalizer of $U(\Lambda)$ is $EU(\Lambda)$ for some field extension E of F with $e(E/F) = e(\Lambda)$ where $e(E/F)$ is the ramification degree of E/F. Therefore, the normalizer modulo $U(\Lambda)$ is isomorphic to $\mathbb{Z}$, generated by a uniformizer ϖ_E of E.

We now return to the case of Λ being a self-dual lattice sequence. The subgroup

$$K(\Lambda) = U(\Lambda) \cap G$$

of G is compact open, with a filtration by normal subgroups

$$K^n(\Lambda) = U^n(\Lambda) \cap G$$

for $n \geq 1$. When the meaning is clear, we omit Λ from the notation and write $K = K(\Lambda)$ and $K^n = K^n(\Lambda)$. The *pro-p-radical* K^1 of K is the maximal normal pro-p subgroup. The *reductive quotient* $\mathcal{G} = K/K^1$ is a reductive group defined over k_F which need not be connected. We write $\mathcal{G}^\circ$ for the connected component of K/K^1 and denote by K° the inverse image of $\mathcal{G}^\circ$ in K. We call K° a *parahoric subgroup* of G.

In order to explicitly describe the parahoric subgroups of G we must return to the study of almost self-dual lattices. Let K be a compact subgroup of G. Since K is compact, it must stabilize some lattice: if we take a basis $\mathfrak{B}$ of V the $\mathfrak{o}_F$-linear span of the image of the action of K on $\mathfrak{B}$ defines such a lattice. We now work towards showing that every compact subgroup is the stabilizer of some almost self-dual lattice.

Proposition 2.7.1. *Let K be a compact subgroup of G and Σ be the set of all $\mathfrak{o}_F$-lattices stabilized by K.*

(1) If $L \in \Sigma$ then $L^\# \in \Sigma$;

(2) If $L_1, L_2 \in \Sigma$ then $L_1 \cap L_2 \in \Sigma$;

(3) If $L_1, L_2 \in \Sigma$ then $L_1 + L_2 \in \Sigma$.

Proof. (1) If $x \in L^\#$ then $h(x, L) \subseteq \mathfrak{p}_F$. For $k \in K$, $h(k\cdot x, L) = h(k\cdot x, k\cdot L) = h(x, L) \subseteq \mathfrak{p}_F$, with the last equality holding since $k \in G$ preserves h. Thus $k \cdot x \in L^\#$.

(2) Take $y \in L_1 \cap L_2$ and $k \in K$. Then $y \in L_1$ and $y \in L_2$. As each $L_i \in \Sigma$ we have $k \cdot y \in L_1$ and $k \cdot y \in L_2$ for all $k \in K$. Thus $k \cdot y \in L_1 \cap L_2$.

(3) By definition $L_1 + L_2 = \{v \in V : \exists a \in L_1, \exists b \in L_2 \text{ such that } v = a + b\}$. Take $z \in L_1 + L_2$, so $z = a + b$ for some $a \in L_1$ and $b \in L_2$. As each $L_1, L_2 \in \Sigma$ we have that $k \cdot a = a'$ and $k \cdot b = b'$ for some $a' \in L_1, b' \in L_2$ and $k \in K$. Then $k \cdot z = k \cdot (a + b) = k \cdot a + k \cdot b = a' + b'$. Thus $k \cdot z \in L_1 + L_2$.

□

Theorem 2.7.2. *Every compact subgroup K stabilizes an almost self-dual lattice.*

Proof. Since K is compact it must stabilize some lattice, which we denote by L. If $L \not\supseteq L^\#$ then we replace L by $M = L + L^\#$ which contains its dual $M^\#$ because $M^\# = (L + L^\#)^\# = L^\# \cap (L^\#)^\# = L^\# \cap L$. Since the set of lattices Σ stabilized by K is closed under taking duals, intersections and sums, K stabilizes M.

Thus K stabilizes some lattice L with the property $L \supseteq L^\#$ and, amongst all such, we choose L such that $\dim_{k_F}(L/L^\#)$ is minimal. Take $n \in \mathbb{N}$ minimal such that $L^\# \supseteq \varpi^n L$. We claim that $n = 1$. If not, form the lattice $M = L \cap \varpi^{1-n} L^\#$ which has dual $M^\# = L^\# + \varpi^{n-1} L$. We have $M \supseteq M^\#$ since $L \supset L^\#$ by assumption and $\varpi^{1-n} L^\# \supseteq \varpi^{n-1} L$ because, rearranging, this is equivalent to $L^\# \supseteq \varpi^{2(n-1)} L$ which is true as $n > 1$. This gives the following chain of inclusions

$$L \supsetneq M \supseteq M^\# \supsetneq L^\#,$$

which shows that $\dim_{k_F}(M/M^\#) < \dim_{k_F}(L/L^\#)$, contradicting our choice of lattice L. Therefore $L^\# \supseteq \varpi L$ and so we have found an almost self-dual lattice stabilized by K. □

2.8 Classification of Reductive Quotients

In the classification of depth-zero cuspidal representations of both $\mathrm{GL}_N(F)$ and classical groups G over non-archimedean local fields of odd residue characteristic, the starting point is to take a cuspidal representation of the reductive quotient of a maximal parahoric subgroup. For positive-depth representations, one similarly needs a cuspidal representation of the reductive quotient of a maximal parahoric subgroup of G. It is therefore important to know precisely what the reductive quotients are in these cases.

If $G = \mathrm{GL}_N(F)$ the reductive quotient of a maximal parahoric subgroup is $\mathrm{GL}_N(k_F)$, a finite reductive group defined over the residue field k_F. If G is a classical group defined over a p-adic field of odd residue characteristic, then even though the classification is more complicated, it is known and described in [LS15, Section 1].

2.9 Reductive Quotients of the Symplectic Group

In this subsection we let F be a dyadic field. Let h be a symplectic form defined on an F-vector space V of dimension $2n$. Let $G = \mathrm{Sp}(V)$ be the Symplectic group. We now describe the maximal parahoric subgroups of G and their reductive quotients.

Proposition 2.9.1. *Let L be an almost self-dual lattice. Then there exist a Witt basis $\{e_{-n}, \dots, e_{-1}, e_1, \dots, e_n\}$ and a non-negative integer m with $0 \leq m \leq n$ such that*

$$L = \mathfrak{o}_F e_{-n} \oplus \cdots \oplus \mathfrak{o}_F e_{-1} \oplus \mathfrak{o}_F e_1 \oplus \cdots \oplus \mathfrak{o}_F e_m \oplus \mathfrak{p}_F e_{m+1} \oplus \cdots \oplus \mathfrak{p}_F e_n,$$

and

$$L^{\#} = \mathfrak{o}_F e_{-n} \oplus \cdots \oplus \mathfrak{o}_F e_{-m-1} \oplus \mathfrak{p}_F e_{-m} \oplus \cdots \oplus \mathfrak{p}_F e_{-1} \oplus \mathfrak{p}_F e_1 \oplus \cdots \oplus \mathfrak{p}_F e_n.$$

Proof. We proceed by induction on the Witt index n. Suppose first $L \neq L^{\#}$ and take $e_1 \in L \setminus L^{\#}$. On $\overline{V}_1 := L/L^{\#}$ we have the induced form

$$\overline{h}_1(u + L^{\#}, v + L^{\#}) := h(u, v) + \mathfrak{p}_F.$$

The form $\overline{h}_1$ is non-degenerate, so there exists a $e_{-1} \in L \setminus L^{\#}$ such that $\overline{h}_1(e_{-1}, e_1) = 1$. Since $h(e_{-1}, e_1) \in \mathfrak{o}_F^{\times}$, we may replace e_{-1} with $h(e_{-1}, e_1)^{-1} e_{-1}$ and assume that

$h(e_{-1}, e_1) = 1$. Set $X = \langle e_{-1}, e_1 \rangle_F$ and $Y = X^\perp$.

For any $z \in L$, since $e_{-1}, e_1 \in L \subseteq \mathfrak{p}_F^{-1} L^\#$, we have both $h(e_{-1}, z), h(e_1, z) \in \mathfrak{o}_F$. We put $x = h(e_1, z)e_{-1} + h(e_{-1}, z)e_1 \in L \cap X$ and $y = z - x$. Then

$$\begin{aligned} h(y, e_1) = h(z - x, e_1) &= h(z, e_1) - h(x, e_1) \\ &= h(z, e_1) - h(e_{-1}, z)h(e_1, e_{-1}) - h(e_{-1}, z)h(e_{-1}, e_{-1}) \\ &= 0, \end{aligned}$$

since $h(e_1, e_{-1}) = -1$ and $h(u, u) = 0$. Similarly, we find that $h(y, e_{-1}) = 0$ and so $y \in L \cap Y$. Therefore we have

$$L = (L \cap X) \oplus (L \cap Y).$$

Similarly, if $z \in L^\#$ then we write $z = x + y$ with $x \in X$, $y \in Y$. For any $w \in L$ we write $w = x_w + y_w$ with $x_w \in L \cap X$ and $y_w \in L \cap Y$. This gives

$$h(x, w) = h(x, x_w) = h(z, x_w) \in \mathfrak{p}_F,$$

and so $x \in L^\# \cap X$. It follows that $y \in L^\# \cap Y$ and

$$L^\# = (L^\# \cap X) \oplus (L^\# \cap Y).$$

Applying the inductive hypothesis to $L \cap Y$ in Y, and adjoining the basis elements e_{-1} and e_1, we achieve a Witt basis as required.

Now suppose $L = L^\#$. We apply the same argument to $L^\# \setminus \mathfrak{p}_F L$. Take $e_2' \in L^\# \setminus \mathfrak{p}_F L$. On $\overline{V}_2 := L^\# / \mathfrak{p}_F L$ we have the induced form

$$\overline{h}_2(u + \mathfrak{p}_F L, v + \mathfrak{p}_F L) := \varpi^{-1} h(u, v) + \mathfrak{p}_F.$$

The form $\bar{h}_2$ is non-degenerate, so there exist $e_{-2}' \in L^\# \setminus \mathfrak{p}_F L$ such that $\overline{h}_2(e_{-2}', e_2') = 1$. Since $\varpi^{-1} h(e_{-2}', e_2') \in \mathfrak{o}_F^\times$, we may replace e_{-2}' by $\varpi h(e_{-2}', e_2')^{-1} e_{-2}'$ and assume $h(e_{-2}', e_2') = \varpi$. Put $e_2 = \varpi^{-1} e_2'$, $e_{-2} = e_{-2}'$, $X = \langle e_{-2}, e_2 \rangle_F = \langle e_{-2}', e_2' \rangle_F$ and $Y = X^\perp$.

For any $z \in L^{\#}$ we have both $h(e_{-2}, z), h(e_2, z) \in \mathfrak{p}_F$. Put $x = h(e_2, z)e_{-2} + h(e_{-2}, z)e_2 \in L^{\#} \cap X$ and $y = z - x$. Then the same calculation as above gives $h(y, e_{-2}) = h(y, e_2) = 0$ which shows $y \in L^{\#} \cap Y$. Therefore

$$L^{\#} = (L^{\#} \cap X) \oplus (L^{\#} \cap Y).$$

As in the first case we deduce that

$$L = (L \cap X) \oplus (L \cap Y).$$

Applying the inductive hypothesis to $L \cap Y$ in Y, and adjoining the basis elements e_{-2}, e_2, we achieve a Witt basis as required. $\square$

It follows directly from the Proposition above that for K a maximal parahoric subgroup of G with pro-unipotent radical K^1, the reductive quotient

$$K/K^1 \hookrightarrow \mathrm{Sp}_{2m}(k_F) \times \mathrm{Sp}_{2(n-m)}(k_F).$$

Here $2m = \dim_{k_F}(L/L^{\#})$, where L is the almost self-dual lattice stabilized by K.

Proposition 2.9.2. *Let K be a maximal parahoric subgroup of G stabilizing an almost self-dual lattice L with $\dim_{k_F}(L/L^{\#}) = 2m$. Then the reductive quotient K/K^1 is*

$$K/K^1 \simeq \mathrm{Sp}_{2m}(k_F) \times \mathrm{Sp}_{2(n-m)}(k_F).$$

Proof. We know that in arbitrary characteristic the Symplectic group is generated by symplectic transvections, which are maps of the form $t_u(v) = u + h(u, v)v$ for $u, v \in V$. Therefore it is enough to show that we can lift symplectic transvections through the quotient.

Using Proposition 2.9.1 we obtain a Witt basis for V so that L decomposes nicely with respect to this basis. We write $U = \mathrm{Span}\{e_i : L \cap Fe_i \neq L^{\#} \cap Fe_i\}$ and $W = \mathrm{Span}\{e_j : L^{\#} \cap Fe_j \neq \mathfrak{p}_F L \cap Fe_j\}$ so that $V = U \oplus W$.

Let $\bar{t}_{\bar{u}}$ be a transvection in $\mathrm{Sp}_{2m}(k_F)$, acting on $L/L^{\#}$. We lift $\bar{u} \in L/L^{\#}$ to an element $u \in L \cap U$ and denote by t_u the transvection associated to u defined on U. Therefore

t_u is a lift of $\bar{t}_{\bar{u}}$. Similarly, for $\bar{t}_{\bar{w}}$ a transvection in $\mathrm{Sp}_{2(n-m)}(k_F)$ acting on $L^\#/\mathfrak{p}_F L$, let $w \in L^\# \cap W$ denote a lift of $\bar{w}$ so that the transvection t_w defined on W is a lift of $\bar{t}_{\bar{w}}$.

Let $g = t_u + t_w$ be the automorphism of V defined by

$$g(u' + w') = t_u(u') + t_w(w')$$

for $u' \in U, w' \in W$. Then g is the required lift of the pair of transvections $(\bar{t}_{\bar{u}}, \bar{t}_{\bar{w}})$. $\square$

It remains to show that the stabilizers of the almost self-dual lattices above are maximal compact.

Proposition 2.9.3. *Let L be the standard almost-self dual lattice defined above with $K =$ $\mathrm{Stab}(L)$. Then K is maximal compact.*

Proof. Suppose $K \subsetneq K'$ is compact. Then K' stabilizes some almost self-dual lattice $L' \neq L$. Since $K \subset K'$, K also stabilizes L'.

We put

$$M = L \cap (L^\# + L') = L^\# + (L \cap L')$$

so

$$M^\# = (L \cap (L^\# + L'))^\# = L^\# + (L \cap L'^\#) \subseteq M.$$

Then we have the containments

$$L \supseteq M \supseteq M^\# \supseteq L^\#$$

so M is another almost self-dual lattice stabilized by K.

We put

$$N = L + (\mathfrak{p}_F^{-1} L^\# \cap L') = \mathfrak{p}_F^{-1} L^\# \cap (L + L')$$

so

$$N^{\#} = \mathfrak{p}_F L + (L^{\#} \cap L'^{\#}) = L^{\#} \cap (\mathfrak{p}_F L + L'^{\#}) \subseteq N.$$

Moreover, $\mathfrak{p}_F N = L^{\#} \cap (\mathfrak{p}_F L + \mathfrak{p}_F L') \subseteq N^{\#}$ and so we have

$$N \supseteq L \supseteq M \supseteq M^{\#} \supseteq L^{\#} \supseteq N^{\#} \supseteq \mathfrak{p}_F N.$$

Suppose $L = M = N$. Then $L = M = L^{\#} + (L \cap L')$ and $L^{\#} = N^{\#} = \mathfrak{p}_F L + (L^{\#} \cap L'^{\#}) \subseteq \mathfrak{p}_F L + (L \cap L')$ so $L = \mathfrak{p}_F L + (L \cap L')$. We deduce $L = L \cap L'$ and so $L \subseteq L'$. Since $L' \supseteq \mathfrak{p}_F^{-1} L'^{\#} \supseteq \mathfrak{p}_F^{-1} L^{\#}$ we get $N = L + (\mathfrak{p}_F^{-1} L^{\#} \cap L') = L + L'$, and since $L = N$, we see that $L = L'$ which is absurd.

Therefore at least one of M, N is not L and so we have found an almost self-dual lattice L'' stabilized by K such that either

$$L \supsetneq L'' \supset L''^{\#} \supsetneq L^{\#} \qquad \text{or} \qquad L'' \supsetneq L \supseteq L^{\#} \supsetneq L''^{\#}.$$

Then (the image of) K stabilizes the non-trivial subspaces

$$0 \neq L''^{\#}/L^{\#} \subsetneq L/L^{\#} \qquad \text{or} \qquad 0 \neq \mathfrak{p}_F L''/\mathfrak{p}_F L \subsetneq L^{\#}/\mathfrak{p}_F L.$$

But K surjects onto the connected component of the group of isometries of $L/L^{\#}$ and of $L^{\#}/\mathfrak{p}_F L$ and this group of isometries acts irreducibly, giving a contradiction. □

Remark 2.9.4. The classification of the reductive quotient for the Symplectic group as given in Proposition 2.9.2 coincides with the description when p is odd. Therefore, the description is uniform for *all* primes p.

2.10 Reductive Quotients of the Special Orthogonal Group

In [Mor91, 1.8] Morris gives a classification of all possible anisotropic symmetric bilinear forms h in odd residual characteristic. Moreover, in each case, he gives a description of the unique almost self-dual lattice which it stabilizes. This information is all that is needed to extrapolate the classification of the reductive quotients in the case $2 \in \mathfrak{o}_F^\times$. This is expected since the description of the Witt ring is uniform for all such fields (it depends on whether -1 is a square or not). However, we have seen that the Witt group for dyadic fields depends on the degree of the field extension (as well as whether -1 is a square or a sum of two squares), and we only know the full classification of the isometry classes of the anisotropic quadratic forms for the case $F = \mathbb{Q}_2$. It is for this reason that we restrict ourselves so that $F = \mathbb{Q}_2$, and even in this the simplest case, there are issues which arise.

Let Q be a non-degenerate quadratic form defined over F. Using Witt's Decomposition Theorem we can write

$$Q = Q|_{\operatorname{rad} Q} \perp Q|_{V_1} \perp Q|_{V_2}$$

with $Q|_{V_1}$ anisotropic and $Q|_{V_2}$ hyperbolic. Since Q is non-degenerate we have $\operatorname{rad} Q = \{0\}$. Therefore, in order to understand the possible reductive quotients for the maximal parahorics of G, we need to understand what reductive quotients arise for Q anisotropic and Q hyperbolic.

2.10.1 Anisotropic Orthogonal Groups

In this section we restrict ourselves so that $F = \mathbb{Q}_2$. Let Q be an anisotropic quadratic form over $\mathbb{Q}_2$ on a vector space V, let h be the associated bilinear form and denote by K the group of isometries of (V, Q). Suppose L is an almost self-dual lattice in V stabilized by K. On one hand, we have

$$\begin{aligned} Q(L) = \{Q(v) : v \in L\} &= \left\{\frac{1}{2}h(v,v) : v \in L\right\} \\ &\supseteq \left\{\frac{1}{2}h(v,v') : v \in V \text{ and } v' \in L^\#\right\} \qquad \text{since } L \supseteq L^\# \\ &= \frac{1}{2}\mathfrak{p}_F \\ &= \mathfrak{o}_F. \end{aligned}$$

On the other,

$$\begin{aligned} Q(L) = \{Q(v) : v \in L\} &= \left\{\frac{1}{2}h(v,v) : v \in L\right\} \\ &= \left\{\frac{1}{4}h(v,2v) : v \in L\right\} \\ &\subseteq \frac{1}{4}\mathfrak{p}_F \qquad \text{since } 2v \in \mathfrak{p}_F L \subseteq L^{\#} \\ &= \mathfrak{p}_F^{-1}. \end{aligned}$$

Thus any almost self-dual lattice L in V must satisfy

$$\mathfrak{o}_F \subseteq Q(L) \subseteq \mathfrak{p}_F^{-1}.$$

We write $N = \{v \in V : Q(v) \in \mathfrak{p}_F^{-1}\}$ and $M = \{v \in V : Q(v) \in \mathfrak{o}_F\}$. It follows from the definition that every $g \in K$ stabilizes both N and M:

$$\begin{aligned} gN &= \{gv : v \in V, Q(v) \in \mathfrak{p}_F^{-1}\} \\ &= \{gv : v \in V, Q(gv) \in \mathfrak{p}_F^{-1}\} \\ &= \{u : u \in V, Q(u) \in \mathfrak{p}_F^{-1}\} \\ &= N, \end{aligned}$$

and similarly $gM = M$. An analogous argument shows that g stabilizes both $N^{\#}$ and $M^{\#}$. Therefore any $g \in K$ must stabilize all of $N, M, N^{\#}$ and $M^{\#}$.

We now consider quadratic forms, characterized by their dimension, starting with the 1-dimensional anisotropic form $Q = \langle a \rangle$ for $a \in \mathbb{Q}_2^{\times}/(\mathbb{Q}_2^{\times})^2$. It is sufficient to consider the form $Q = \langle 1 \rangle$ since any other 1-dimensional quadratic form is just a scalar multiple of Q and so their groups of isometries coincide.

Case 1 : $Q = \langle 1 \rangle$

Write $V = \langle e_1 \rangle_F$ so that $Q(\lambda_1 e_1) = \lambda_1^2$. Since Q is 1-dimensional, we immediately see that $N = M = \mathfrak{o}_F e_1$ which is self-dual. Therefore, on $M/\mathfrak{p}_F N$ we get an induced form $\bar{Q}$

given by

$$\bar{Q}(v + \mathfrak{p}_F M) := Q(v) + \mathfrak{p}_F$$

which is non-degenerate anisotropic. Therefore, we have

$$K/K^1 \hookrightarrow \mathrm{O}_1(\mathbb{F}_2).$$

Since $\mathrm{O}_1(\mathbb{F}_2)$ is trivial, we see that K is itself a pro-2 group.

For the 2-dimensional forms, recall that we can write $Q = \langle \lambda a, \lambda b \rangle = \langle \lambda \rangle \otimes \langle a, b \rangle$ and so the group of isometries of Q is the same as the group of isometries of $Q' = \langle a, b \rangle$. Thus we need only consider forms $\langle 1, b \rangle$, for b in a set of representatives for $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$.

Suppose $Q = \langle 1, b \rangle$ with $\mathrm{val}_F(b) = 0$. Then for $v = (\lambda_1, \lambda_2) \in V$, since we wish to describe the lattices N and M, we are interested in the quantity $\mathrm{val}_F(\lambda_1^2 + b\lambda_2^2)$. By scaling if necessary, we may assume $\lambda_i \in \mathfrak{o}_F$, and so $\lambda_i^2 \equiv 0, 1 \bmod 4$ for $i = 1, 2$. Therefore $\lambda_1^2 + b\lambda_2^2 \equiv 0, 1, b, b+1 \bmod 4$. We deduce that the isometry groups of the forms $\langle 1, b \rangle$ and $\langle 1, b+4 \rangle$ stabilize the "same" lattices and will have the same reductive quotient. Similarly, if $\mathrm{val}_F(b) = 1$ then the lattices M and N are "independent" of the choice of b. Therefore, we need only consider the forms

$$Q \in \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle\}.$$

Case 2 : $Q = \langle 1, 1 \rangle$

We write $V = \langle e_1, e_2 \rangle_F$ so $Q(\lambda_1 e_1 + \lambda_2 e_2) = \lambda_1^2 + \lambda_2^2$. Recall $N = \{v = \lambda_1 e_1 + \lambda_2 e_2 \in V : Q(v) = \lambda_1^2 + \lambda_2^2 \in \mathfrak{p}_F^{-1}\}$. Take $v = \lambda_1 e_1 + \lambda_2 e_2 \in N$ arbitrary. If $\mathrm{val}_F(\lambda_1) = -n < 0$ then $\mathrm{val}_F(\lambda_1^2 + \lambda_2^2) \geq -1$ implies that $\mathrm{val}_F(\lambda_2) = -n < 0$. Writing $\mu_i := \varpi_F^n \lambda_i$ gives $\mu_i \in \mathfrak{o}_F^\times$ such that $\mu_1^2 + \mu_2^2 \in \mathfrak{p}_F^{2n-1}$.

Suppose $n > 1$. Reducing mod $\mathfrak{p}_F^2$ we get $\mu_1^2 = \mu_2^2 \equiv 0 \bmod 4$. However, squares in $\mathbb{Z}_2^\times$ are congruent to 1 mod 8, which in turn are congruent to 1 mod 4, and so $\mu_1^2 + \mu_2^2 \equiv 2 \bmod 4$

a contradiction. Therefore $n = 1$ and $\mu_i \in \mathfrak{o}_F^\times$ such that $\mu_1^2 + \mu_2^2 \in \mathfrak{p}_F$. This last condition is equivalent to $\mu_1 + \mu_2 \in \mathfrak{p}_F$ since the $\mu_1^2 + \mu_2^2 \equiv (\mu_1 + \mu_2)^2 \bmod 2$ and the valuation of a square is even. Scaling back gives

$$\begin{aligned} N &= \{\lambda_1 e_1 + \lambda_2 e_2 \in V : \lambda_1, \lambda_2 \in \mathfrak{p}_F^{-1} \text{ such that } \lambda_1 + \lambda_2 \in \mathfrak{o}_F\} \\ &= \left\langle \frac{1}{2}(e_1 + e_2), e_2 \right\rangle_{\mathfrak{o}_F}. \end{aligned}$$

Writing $v = \mu_1 e_1 + \mu_2 e_2 \in V$ and $u = \lambda_1(\frac{1}{2}(e_1 + e_2)) + \lambda_2 e_2 \in N^\#$ we have

$$\begin{aligned} v = \mu_1 e_1 + \mu_2 e_2 \in N^\# &\iff h(v, N) \subseteq \mathfrak{p}_F \\ &\iff h(v, u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff Q(v + u) - Q(v) - Q(u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_1\lambda_1 + 2\mu_2\lambda_2 + \mu_2\lambda_1 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \lambda_1(\mu_1 + \mu_2) + 2\lambda_2\mu_2 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_2 \in \mathfrak{o}_F \text{ and } \mu_1 + \mu_2 \in \mathfrak{p}_F \\ &\iff \mu_1, \mu_2 \in \mathfrak{o}_F \text{ such that } \mu_1 + \mu_2 \in \mathfrak{p}_F. \end{aligned}$$

Hence

$$N^\# = \{\mu_1 e_1 + \mu_2 e_2 \in V : \mu_1, \mu_2 \in \mathfrak{o}_F \text{ such that } \mu_1 + \mu_2 \in \mathfrak{p}_F\} = \mathfrak{p}_F N.$$

We now consider $M = \{v \in V : Q(v) \in \mathfrak{o}_F\}$. Note that if either $\lambda_1, \lambda_2 \in \mathfrak{p}_F^{-1} \setminus \mathfrak{o}_F$ then $Q(v) \in \mathfrak{p}_F^{-1} \not\subseteq \mathfrak{o}_F$ and so $\lambda_i \in \mathfrak{o}_F$ with no other restrictions. Thus $M = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2$ and a direct calculation shows that $M^\# = M$. Therefore K stabilizes $N \supset M = M^\# \supset N^\# = \mathfrak{p}_F N$, and both M, N are almost self-dual.

The group K/K^1 acts on the the 1-dimensional k_F space $\bar{V}_1 := N/M$, which is spanned by the image of $v_1 = \frac{1}{2}(e_1 + e_2)$ in the quotient N/M. We have the induced form

$$\bar{Q}_1(v + M) := \varpi Q(v) + \mathfrak{p}_F,$$

given by

$$\bar{Q}_1\left(\lambda_1\left(\frac{1}{2}(e_1 + e_2)\right)\right) = 2Q\left(\lambda_1\left(\frac{1}{2}(e_1 + e_2)\right)\right) + \mathfrak{p}_F$$

$$= 2\left(\frac{\lambda_1}{4} + \frac{\lambda_1}{4}\right) + \mathfrak{p}_F$$
$$= \lambda_1 + \mathfrak{p}_F,$$

which is non-degenerate anisotropic. Similarly we get a 1-dimensional non-degenerate anisotropic form on the quotient $\bar{V}_2 := M/\mathfrak{p}_F N$, which is spanned by the image of $v_2 = e_2$, given by

$$\bar{Q}_2(v + N^{\#}) := Q(v) + \mathfrak{p}_F.$$

Therefore

$$K/K^1 \hookrightarrow \mathrm{O}_1(\mathbb{F}_2) \times \mathrm{O}_1(\mathbb{F}_2).$$

As in Case 1 above, this is the trivial group so K is in fact a pro-2 group.

Case 3 : $Q = \langle 1, 2\rangle$

We write $V = \langle e_1, e_2\rangle_F$ so $Q(\lambda_1 e_1 + \lambda_2 e_2) = \lambda_1^2 + 2\lambda_2^2$. Writing $N = \{v \in V : Q(v) \in \mathfrak{p}_F^{-1}\}$ we have

$$\begin{aligned} v = \lambda_1 e_1 + \lambda_2 e_2 \in N \iff & \lambda_1^2 + 2\lambda_2^2 \in \mathfrak{p}_F^{-1} \\ \iff & \text{both } \lambda_1^2, 2\lambda_2^2 \in \mathfrak{p}_F^{-1} \text{ since } \mathrm{val}_F(\lambda_1^2) \text{ is even and} \\ & \mathrm{val}_F(2\lambda_2^2) \text{ is odd} \\ \iff & \lambda_1 \in \mathfrak{o}_F \text{ and } \lambda_2 \in \mathfrak{p}_F^{-1}. \end{aligned}$$

Thus

$$N = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F^{-1} e_2.$$

Similarly, we have $M = \{v \in V : Q(v) \in \mathfrak{o}_F\}$ which gives

$$\begin{aligned} v = \lambda_1 e_1 + \lambda_2 e_2 \in M \iff & \lambda_1^2 + 2\lambda_2^2 \in \mathfrak{o}_F \\ \iff & \lambda_1, \lambda_2 \in \mathfrak{o}_F, \end{aligned}$$

and so

$$M = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2.$$

A direct calculation shows that $M = N^{\#}$ and so N is the unique almost self-dual lattice stabilized by K with $N \supsetneq N^{\#} \supsetneq \mathfrak{p}_F N$. Writing $\bar{V}_1 = N/M$ and $\bar{V}_2 = M/\mathfrak{p}_F N$, spanned by the image of the vectors $v_1 = \frac{1}{2}e_2$ and $v_2 = e_1$ respectively, we have induced non-trivial non-degenerate k_F-quadratic forms

$$\bar{Q}_1(v + M) := 2Q(v) + \mathfrak{p}_F$$

for all $v \in N$, and

$$\bar{Q}_2(w + \mathfrak{p}_F N) := Q(w) + \mathfrak{p}_F$$

for all $w \in M$. Therefore

$$K/K^1 \hookrightarrow \mathrm{O}_1(\mathbb{F}_2) \times \mathrm{O}_1(\mathbb{F}_2)$$

is again trivial and so K is a pro-2 group.

Case 4 : $Q = \langle 1, 3 \rangle$

Write $V = \langle e_1, e_2 \rangle_F$ so that $Q(\lambda_1 e_1 + \lambda_2 e_2) = \lambda_1^2 + 3\lambda_2^2$. Suppose $\lambda_i \in \mathfrak{p}_F^{-n} \setminus \mathfrak{p}_F^{1-n}$. Then writing $\lambda_i = \mu_i \varpi^{-n}$ with $\mu_i \in \mathfrak{o}_F^\times$ we have

$$Q(\lambda_1 e_1 + \lambda_2 e_2) = \mu_1^2 \varpi^{-2n} + 3\mu_2^2 \varpi^{-2n} \in \mathfrak{p}_F^{-2n+2} \setminus \mathfrak{p}_F^{-2n+3}$$

since $\mu_i^2 \equiv 1 \bmod 8 \implies \mu_1^2 + 3\mu_2^2 \equiv 4 \bmod 4$. If both λ_i have the same valuation, then $Q(v)$ has an even valuation, which shows $N = M$. Carrying through the same analysis as in Case 2 gives

$$\begin{aligned} N = M &= \{\lambda_1 e_1 + \lambda_2 e_2 \in V : \lambda_i \in \mathfrak{p}_F^{-1} \text{ such that } \lambda_1 + \lambda_2 \in \mathfrak{o}_F\} \\ &= \left\langle \frac{1}{2}(e_1 + e_2), e_2 \right\rangle_{\mathfrak{o}_F}. \end{aligned}$$

Put $v = \mu_1 e_1 + \mu_2 e_2 \in V$ and $u = \lambda_1(\frac{1}{2}(e_1 + e_2)) + \lambda_2 e_2 \in M^{\#}$. Therefore

$$\begin{aligned} v = \mu_1 e_1 + \mu_2 e_2 \in M^{\#} &\iff h(v, M) \subseteq \mathfrak{p}_F \\ &\iff h(v, u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff Q(v+u) - Q(v) - Q(u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_1\lambda_1 + 6\mu_2\lambda_2 + 3\mu_2\lambda_1 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \lambda_1(\mu_1 + 3\mu_2) + 6\lambda_2\mu_2 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_2 \in \mathfrak{o}_F \text{ and } \mu_1 + \mu_2 \in \mathfrak{p}_F \\ &\iff \mu_1, \mu_2 \in \mathfrak{o}_F \text{ such that } \mu_1 + \mu_2 \in \mathfrak{p}_F. \end{aligned}$$

Therefore $M^{\#} = \mathfrak{p}_F M$ and so the space $M/\mathfrak{p}_F N$, spanned by the image of the vectors $v_1 = \frac{1}{2}(e_1 + e_2)$, $v_2 = e_2$ in the quotient, is 2-dimensional with induced form $\bar{Q}$ given by

$$\bar{Q}(v + \mathfrak{p}_F N) := Q(v) + \mathfrak{p}_F.$$

The space $M/\mathfrak{p}_F N$ contains the image of the vectors $v_1 = \frac{1}{2}(e_1 + e_2)$, $v_2 = e_2$ in the quotient. The form $\bar{Q}_2$ is

$$\begin{aligned} \bar{Q}_2(\lambda_1 v_1 + \lambda_2 v_2) &= Q(\lambda_1 v_1 + \lambda_2 v_2) + \mathfrak{p}_F \\ &= \frac{\lambda_1^2}{4} + 3\left(\frac{\lambda_1}{2} + \lambda_2\right)^2 + \mathfrak{p}_F \\ &= \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 + \mathfrak{p}_F, \end{aligned}$$

which is non-degenerate anisotropic. Therefore

$$K/K^1 \hookrightarrow \mathrm{O}_2^-(\mathbb{F}_2).$$

As abstract groups, we have $\mathrm{O}_2^-(\mathbb{F}_2) = \mathrm{GL}_2(\mathbb{F}_2) = \mathfrak{S}_3$.

For the 3-dimensional anisotropic forms, we have seen that every form is isometric to $\langle \lambda, \lambda, \lambda \rangle = \langle \lambda \rangle \otimes \langle 1, 1, 1 \rangle$ for $\lambda \in \mathbb{Q}_2/(\mathbb{Q}_2^\times)^2$. Therefore they all have the same group of isometries, so we need only analyze $Q = \langle 1, 1, 1 \rangle$.

Case 5 : $Q = \langle 1,1,1\rangle$

Write $V = \langle e_1, e_2, e_3\rangle_F$ so $Q(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Recall $N = \{v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in V : Q(v) \in \mathfrak{p}_F^{-1}\}$. Let $v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in N$ and suppose $-n = \mathrm{val}_F(\lambda_1) \leq \mathrm{val}_F(\lambda_2) \leq \mathrm{val}_F(\lambda_3)$. By the same analysis as in Case 3 we get $\mu_1, \mu_2 \in \mathfrak{o}_F^\times, \mu_3 \in \mathfrak{o}_F$ such that $\mu_1^2 + \mu_2^2 + \mu_3^2 \in \mathfrak{p}_F^{2n-1}$ when $-n < 0$. Assuming $n > 1$ we have $\mu_1^2 + \mu_2^2 + \mu_3^2 \in \mathfrak{p}_F^3$. However, $\mu_1^1 \equiv \mu_2^2 \equiv 1 \bmod 8$ and $\mu_3^2 \equiv 0,1,4 \bmod 8$, which implies $\mu_1^2 + \mu_2^2 + \mu_3^2 \not\equiv 0 \bmod 8$, contradicting $n > 1$. Therefore

$$\begin{aligned} N &= \{v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in V : Q(v) \in \mathfrak{p}_F^{-1}\} \\ &= \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in V : \lambda_1, \lambda_2, \lambda_3 \in \mathfrak{p}_F^{-1} \text{ such that } \lambda_1 + \lambda_2 + \lambda_3 \in \mathfrak{o}_F\} \\ &= \left\langle \frac{1}{2}(e_1 + e_2), \frac{1}{2}(e_1 + e_3), e_3 \right\rangle_{\mathfrak{o}_F}. \end{aligned}$$

Writing $v = \mu_1 e_1 + \mu_2 e_2 + \mu_4 e_3 \in V$ and $u = \lambda_1(\frac{1}{2}(e_1 + e_2)) + \lambda_2(\frac{1}{2}(e_1 + e_3)) + \lambda_3 e_3 \in N$ gives

$$\begin{aligned} v = \mu_1 e_1 + \mu_2 e_2 + \mu_4 e_3 \in N^\# &\iff h(v, N) \subseteq \mathfrak{p}_F \\ &\iff h(v, u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff Q(v+u) - Q(v) - Q(u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \lambda_1(\mu_1 + \mu_2) + \lambda_2(\mu_2 + \mu_3) + 2\lambda_3\mu_3 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_3 \in \mathfrak{o}_F, \mu_1 + \mu_2 \in \mathfrak{p}_F, \mu_2 + \mu_3 \in \mathfrak{p}_F \\ &\iff \mu_i \in \mathfrak{o}_F \text{ such that } \mu_1 + \mu_2 \equiv \mu_1 + \mu_3 \equiv \mu_2 + \mu_3 \equiv 0 \bmod \mathfrak{p}_F. \end{aligned}$$

Therefore

$$\begin{aligned} N^\# &= \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in V : \lambda_1, \lambda_2, \lambda_3 \in \mathfrak{o}_F \text{ such that } \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3 \in \mathfrak{p}_F\} \\ &= \langle 2e_1, 2e_2, e_1 + e_2 + e_3\rangle_{\mathfrak{o}_F} \neq \mathfrak{p}_F N. \end{aligned}$$

Similarly, we find that

$$M = \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in V : Q(v) \in \mathfrak{o}_F\} = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2 \oplus \mathfrak{o}_F e_3$$

and $M = M^\#$ by a direct calculation. Unlike the previous cases, we do not have a chain of inclusions. However, we do have the following diagram of containments:

$$\begin{tikzcd}
 & N \arrow[d, dash, "2"] & \\
 & M = M^{\#} \arrow[dl, dash, "2"'] \arrow[dr, dash, "1"] & \\
N^{\#} \arrow[dr, dash, "1"'] & & \mathfrak{p}_F N \arrow[dl, dash, "2"] \\
 & \mathfrak{p}_F M &
\end{tikzcd}.$$

First consider the 2-dimensional space $\bar{V}_1 = N/M$. It contains the image of the vectors $v_1 = \frac{1}{2}(e_1 + e_2)$ and $v_2 = \frac{1}{2}(e_1 + e_3)$, so the induced form $\bar{Q}_1$ is given by

$$\bar{Q}_1(v + M) := \varpi Q(v) + \mathfrak{p}_F,$$

where $v = \lambda_1 v_1 + \lambda_2 v_2$. Calculating the induced form gives

$$\begin{aligned}
\bar{Q}_1(\lambda_1 v_1 + \lambda_2 v_2) &= 2Q(\lambda_1 v_1 + \lambda_2 v_2) + \mathfrak{p}_F \\
&= 2Q\left(\frac{(\lambda_1 + \lambda_2)}{2} e_1 + \frac{\lambda_1}{2} e_2 + \frac{\lambda_2}{2} e_3\right) + \mathfrak{p}_F \\
&= \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 + \mathfrak{p}_F
\end{aligned}$$

which is a 2-dimensional non-degenerate anisotropic form.

Now consider the 1-dimensional space $\bar{V}_2 = M/\mathfrak{p}_F N$ spanned by the image of the vector $v_3 = e_1$ in the quotient. We have the induced form $\bar{Q}_2$ given by

$$\bar{Q}_2(v + \mathfrak{p}_F N) := Q(v) + \mathfrak{p}_F$$

is 1-dimensional non-degenerate anisotropic. Therefore

$$K/K^1 \hookrightarrow O_2^-(\mathbb{F}_2) \times O_1(\mathbb{F}_2).$$

Case 6 : $Q = \langle 1, 1, 1, 1 \rangle$

Write $V = \langle e_1, e_2, e_3, e_4 \rangle_F$ so $Q(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2$. By the same analysis as in previous cases, we find

$$N = \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \in V : \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathfrak{p}_F^{-1} \text{ such that } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \in \mathfrak{o}_F\}$$

$$= \left\langle \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \frac{1}{2}(e_1 + e_2), \frac{1}{2}(e_1 + e_3), e_4 \right\rangle_{\mathfrak{o}_F}.$$

One would expect to find that M is the obvious lattice $\mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2 \oplus \mathfrak{o}_F e_3 \oplus \mathfrak{o}_F e_4$. However, in this particular case, we find that M also contains vectors of the form $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4$ with $\lambda_i \in \mathfrak{p}_F^{-1} \setminus \mathfrak{o}_F$. Therefore

$$\begin{aligned} M &= \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \in V : \text{ either } \lambda_i \in \mathfrak{p}_F^{-1} \setminus \mathfrak{o}_F \text{ for all } i \text{ or } \lambda_i \in \mathfrak{o}_F\} \\ &= \left\langle \frac{1}{2}(e_1 + e_2 + e_3 + e_4), e_2, e_3, e_4 \right\rangle_{\mathfrak{o}_F}. \end{aligned}$$

Writing $u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \in N$ gives

$$\begin{aligned} v = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 \in N^{\#} &\iff h(v, N) \in \mathfrak{p}_F \\ &\iff h(v, u) \in \mathfrak{p}_F \quad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff Q(v + u) - Q(v) - Q(u) \subseteq \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_1\lambda_1 + \mu_1\lambda_2 + \mu_1\lambda_3 + \mu_2\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_1 + \\ &\qquad \mu_3\lambda_3 + \mu_4\lambda_1 + 2\mu_4\lambda_4 \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \lambda_1(\mu_1 + \mu_2 + \mu_3 + \mu_4) + \lambda_2(\mu_1 + \mu_2) + \lambda_3(\mu_1 + \mu_3) + \\ &\qquad 2\lambda_4(\mu_4) \in \mathfrak{p}_F \qquad \text{for all } \lambda_i \in \mathfrak{o}_F \\ &\iff \mu_1 + \mu_2 + \mu_3 + \mu_4 \in \mathfrak{p}_F, \mu_1 + \mu_2 \in \mathfrak{p}_F, \mu_1 + \mu_3 \in \mathfrak{p}_F, \\ &\qquad \mu_4 \in \mathfrak{o}_F \\ &\iff \mu_1, \mu_2, \mu_3, \mu_4 \in \mathfrak{o}_F \text{ such that } \mu_1 + \mu_2 + \mu_3 + \mu_4 \in \mathfrak{p}_F, \\ &\qquad \mu_1 + \mu_2 \in \mathfrak{p}_F, \mu_1 + \mu_3 \in \mathfrak{p}_F. \end{aligned}$$

By symmetry

$$\begin{aligned} N^{\#} &= \{\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 \in V : \mu_i \in \mathfrak{o}_F \text{ such that } \mu_i + \mu_j \in \mathfrak{p}_F \text{ for } i \neq j\} \\ &= \mathfrak{p}_F M. \end{aligned}$$

In the same way one could calculate $M^{\#}$ explicitly to show $M^{\#} = \mathfrak{p}_F N$, but one could also use properties of duality:

$$N = (N^{\#})^{\#} = (\mathfrak{p}_F M)^{\#} = \mathfrak{p}_F^{-1} M^{\#} \implies M^{\#} = \mathfrak{p}_F N.$$

First consider the 2-dimensional space $\bar{V}_1 = N/M$. It contains the image of the vectors $v_1 = \frac{1}{2}(e_1 + e_2), v_2 = \frac{1}{2}(e_1 + e_3)$ and so the reduced form $\bar{Q}_1$ is given by

$$\bar{Q}_1(v + M) := \varpi Q(v) + \mathfrak{p}_F.$$

By writing $v = \lambda_1 v_1 + \lambda_2 v_2$ we find that $\bar{Q}_1$ is

$$\begin{aligned}
\bar{Q}_1(\lambda_1 v_1 + \lambda_2 v_2) &= 2Q(\lambda_1 v_1 + \lambda_2 v_2) + \mathfrak{p}_F \\
&= 2\left(Q\left(\frac{\lambda_1}{2}(e_1 + e_2) + \frac{\lambda_2}{2}(e_1 + e_3)\right)\right) + \mathfrak{p}_F \\
&= 2\left(Q\left(\frac{(\lambda_1 + \lambda_2)}{2}e_1 + \frac{\lambda_1}{2}e_2 + \frac{\lambda_2}{2}e_3\right)\right) + \mathfrak{p}_F \\
&= \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 + \mathfrak{p}_F,
\end{aligned}$$

which is non-degenerate anisotropic. Similarly, the 2-dimensional space $\bar{V}_2 = M/\mathfrak{p}_F N$ has induced form $\bar{Q}_2$ given by

$$\bar{Q}_2(v + \mathfrak{p}_F N) := Q(v) + \mathfrak{p}_F.$$

By writing $v_3 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ and $v_4 = e_4$ we find that $\bar{Q}_2$ is

$$\begin{aligned}
\bar{Q}_2(v + \mathfrak{p}_F N) &= Q(v) + \mathfrak{p}_F \\
&= Q\left(\lambda_3 v_3 + \lambda_4 v_4\right) + \mathfrak{p}_F \\
&= Q\left(\frac{\lambda_3}{2}e_1 + \frac{\lambda_3}{2}e_2 + \frac{\lambda_3}{2}e_3 + \frac{(\lambda_3 + 2\lambda_4)}{2}e_4\right) + \mathfrak{p}_F \\
&= {\lambda_3}^2 + \lambda_3\lambda_4 + {\lambda_4}^2 + \mathfrak{p}_F,
\end{aligned}$$

which is non-degenerate anisotropic. Therefore

$$K/K^1 \hookrightarrow O_2^-(\mathbb{F}_2) \times O_2^-(\mathbb{F}_2).$$

In all cases above we have shown that given K the group of isometries of (V, Q) that K/K^1 injects into the groups $U(\bar{Q}_1) \times U(\bar{Q}_2)$. However, we do not know if we have an isomorphism. It turns out that in most cases this is true.

Proposition 2.10.1. *Let Q_{an} be an anisotropic non-degenerate quadratic over $\mathbb{Q}_2$ with K the stabilizer of . Then K/K^1 is of the following form:*

Case	Q_{an}	K/K^1
1	1-dimensional	trivial
2	$\langle 1,1\rangle, \langle 1,5\rangle, \langle 3,3\rangle, \langle 2,10\rangle$	trivial
3	$\langle 1,2\rangle, \langle 1,6\rangle, \langle 1,10\rangle, \langle 1,14\rangle,$ $\langle 2,3\rangle, \langle 2,5\rangle, \langle 3,10\rangle, \langle 5,10\rangle$	trivial
4	$\langle 1,3\rangle, \langle 2,6\rangle$	$\mathrm{O}_2^-(\mathbb{F}_2)$
5	3-dimensional	$\mathrm{O}_2^-(\mathbb{F}_2)$
6	4-dimensional	$\mathrm{O}_2^-(\mathbb{F}_2) \times \mathrm{O}_2^-(\mathbb{F}_2)$

.

Proof. Since there is no occurrence of $\mathrm{O}_4^+(\mathbb{F}_2)$ we know that the Orthogonal groups are generated by reflections [Gro02, Theorem 6.6] and [KL90, Proposition 2.5.6]. In the same way as Proposition 2.9.2, we show that we can lift reflections through the quotient.

Let $N, M, v_i, \bar{Q}_i$ be as in Cases $1-6$ above. Let $U = \mathrm{Span}\{v_i : N \cap Fv_i \neq M \cap Fv_i\}$ and $W = \mathrm{Span}\{v_j : M \cap Fv_j \neq \mathfrak{p}_F N \cap Fv_j\}$ so that $V = U \perp W$. With respect to these vectors we have that the lattices N and M decompose nicely i.e.

$$N = (N \cap U) \oplus (N \cap W),$$
$$M = (M \cap U) \oplus (M \cap W).$$

Let

$$r_{\bar{u}} = \bar{x} - \frac{h(\bar{x}, \bar{u})}{\bar{Q}_1(\bar{u})}\bar{u}$$

be a reflection in $\mathrm{O}(\bar{Q}_1)$ acting on N/M with $\bar{u}$ non-singular and $\bar{x} \in N/M$. We choose any lift $u \in N \cap U$ of $\bar{u}$ so that the reflection

$$r_u(x) = x - \frac{h(u,x)}{Q(u)}u, \qquad x \in N$$

is a lift of $r_{\bar{u}}$, which is possible by the non-degeneracy of $\bar{Q}_1$. Similarly, for $r_{\bar{w}}$ a reflection in $\mathrm{O}(\bar{Q}_2)$ acting on $M/\mathfrak{p}_F N$, we choose a lift $w \in M \cap W$ of $\bar{w}$ so that the reflection r_w is a lift of $r_{\bar{w}}$. Define $g = r_u + r_w$ by

$$g(u' + w') = r_u(u') + r_w(w')$$

for all $u' \in U, w' \in W$, which is an element of K. □

2.10.2 Split Special Orthogonal Groups

Let F be a dyadic field and G be a Split Special Orthogonal group, Then, given a non-degenerate quadratic form Q defined over V, we can write $Q = nQ_{\mathbb{H}} \perp Q_{\text{an}}$ where $nQ_{\mathbb{H}}$ is the orthogonal sum of n-copies of the hyperbolic form $Q_{\mathbb{H}}$ and Q_{an} is is either zero or 1-dimensional anisotropic.

Ideally one would like to be able to have an analogous version of Proposition 2.9.1 for the case of Orthogonal groups. In the proof of the Proposition, the key point is that given $v \in L \setminus L^{\#}$, we can find a $u \in L \setminus L^{\#}$ with $h(u, v) = 1$. However, in our case if we take $v \in L \setminus L^{\#}$ with $Q(v) = 0$, we are not guaranteed that we can find a $u \in L \setminus L^{\#}$ with $Q(u) = 0$ and $h(u, v) = 1$. Therefore, given a maximal compact subgroup K of G, even though it stabilizes some almost self-dual lattice L, we can not find a Witt basis with respect to which L nicely decomposes. This is already visible with the following example of $K = \mathrm{O}(Q_{\mathbb{H}})$.

Suppose $Q = Q_{\mathbb{H}}$ and e_1, e_2 is a Witt basis with respect to the symmetric bilinear form h associated to Q. A direct matrix calculation shows that

$$\mathrm{O}(Q) \simeq F^{\times} \rtimes C_2 \simeq \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix} : x \in F^{\times} \right\}.$$

The cyclic group C_2 is generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which acts on $F^{\times}$ by $x \mapsto x^{-1}$, and the Special Orthogonal group $\mathrm{SO}(Q)$ is just $F^{\times}$.

Consider the lattice $L = \mathfrak{o}_F(e_1 + e_2) + \mathfrak{o}_F e_2$ stabilized by K. One can check that it is self-dual, and so on $L/\mathfrak{p}_F L$ we have an induced form

$$\bar{Q}(v + \mathfrak{p}_F L) := Q(v) + \mathfrak{p}_F$$

which is degenerate with a 1-dimensional radical. The stabilizer of L also stabilizes the pre-image of $\mathrm{rad}(\bar{Q})$ in L, which is the dual of the almost self-dual lattice L' spanned by

e_1 and e_2. Therefore we chose the 'wrong' almost self-dual lattice: the compact subgroup which is the stabilizer of L stabilizes more that one almost self-dual lattice.

We return to the general setting. Put $V = V_{n\mathbb{H}} \perp V_{\mathrm{an}}$ with $V_{n\mathbb{H}}$ hyperbolic of dimension $2n$ and V_{an} anisotropic of dimension at most 1. Let h be the non-degenerate symmetric bilinear form associated to Q. We write $Q = nQ_{\mathbb{H}} \perp Q_{\mathrm{an}}$ and $h = nh_{\mathbb{H}} \perp h_{\mathrm{an}}$.

We choose a Witt basis for $V_{n\mathbb{H}}$ so that $V_{n\mathbb{H}} = \langle e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n \rangle_F$ with $h(e_{-i}, e_j) = \delta_{ij}$. We now consider two cases.

Firstly, suppose $V = V_{n\mathbb{H}}$ so $V_{\mathrm{an}} = \{0\}$ and $Q = Q_{n\mathbb{H}}$ is hyperbolic. Then G is an even Split Special Orthogonal group. For some $0 \leq m \leq n$ with $m \neq 1, 2, n-2, n-1$, let L_m be the almost self-dual lattice

$$L_m = \bigoplus_{i=-n}^{-1} \mathfrak{o}_F e_i \oplus \bigoplus_{j=1}^{m} \mathfrak{o}_F e_j \oplus \bigoplus_{k=m+1}^{n} \mathfrak{p}_F e_k$$

with dual

$$L_m^{\#} = \bigoplus_{i=-n}^{-(m+1)} \mathfrak{o}_F e_i \oplus \bigoplus_{j=-m}^{-1} \mathfrak{p}_F e_j \oplus \bigoplus_{k=1}^{n} \mathfrak{p}_F e_k.$$

Let $\bar{V}_1 := L_m/L_m^{\#}$ and $\bar{V}_2 := L_m^{\#}/\mathfrak{p}_F L_m$. The space $\bar{V}_1$ is $2m$-dimensional with induced induced form

$$\bar{Q}_1(v + L^{\#}) := Q(v) + \mathfrak{p}_F$$

which has basis the image of the vectors $e_{-m}, \ldots, e_{-1}, e_1, \ldots, e_m$ in the quotient. This form is non-degenerate and hyperbolic, so $\bar{Q}_1$ is the orthogonal sum of m-copies of the hyperbolic form $\bar{Q}_{\mathbb{H}}$ over the residue field k_F.

Similarly, the space $\bar{V}_2$ is $2(n-m)$-dimensional with induced form

$$\bar{Q}_2(v + \mathfrak{p}_F L_m) := \varpi^{-1} Q(v) + \mathfrak{p}_F$$

which has basis the image of the vectors $e_{-n}, \ldots, e_{-(m+1)}, \varpi e_{m+1}, \ldots, \varpi e_n$ in the quotient. This form is non-degenerate hyperbolic, so $\bar{Q}_2$ is the orthogonal sum of $(n-m)$ copies of the hyperbolic form $\bar{Q}_{\mathbb{H}}$ over the residue field k_F.

Let K be the stabilizer of L_m, which is compact open. Then K/K^1 is a subgroup of

$$\mathrm{O}(\bar{Q}_1) \times \mathrm{O}(\bar{Q}_2) \simeq \mathrm{O}^+_{2m}(k_F) \times \mathrm{O}^+_{2(n-m)}(k_F). \tag{$\dagger$}$$

Remark 2.10.2. The reason why we require $m \neq 2, n-2$ is that in this instance there would be a factor of $\mathrm{O}^+_4(k_F)$, which we have seen is not generated by reflections if $F = \mathbb{Q}_2$. This then begs the question as to what the reductive quotient would be in this case. Since we can lift reflections through the quotient, we expect that the reductive quotient would have as a factor the index 2 subgroup of $\mathrm{O}^+_4(\mathbb{F}_2)$ generated by an even number of reflections, which is not $\mathrm{SO}^+_4(\mathbb{F}_2)$.

Secondly, suppose $V_{\mathrm{an}} = \langle e_0 \rangle_F$ so G is an odd Split Special Orthogonal group. Then the form Q_{an} is isometric to $\langle \lambda \rangle$ for some $\lambda \in \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$. Moreover, by choosing instead our basis for $V_{n\mathbb{H}}$ so that $h\mid_{V_{n\mathbb{H}}}(e_{-i}, e_j) = \lambda\delta_{ij}$, and then rescaling our form h, we may assume that $\lambda = 1$ since forms which differ by an element of $F^\times$ have isomorphic isometry groups.

For some $0 \leq m \leq n$ with $m \neq 1$, let L_m be the almost self-dual lattice

$$L_m = \bigoplus_{i=-n}^{-1} \mathfrak{o}_F e_i \oplus L_{\mathrm{an}} \oplus \bigoplus_{j=1}^{m} \mathfrak{o}_F e_j \oplus \bigoplus_{k=m+1}^{n} \mathfrak{p}_F e_k$$

with dual

$$L_m^\# = \bigoplus_{i=-n}^{-(m+1)} \mathfrak{o}_F e_i \oplus \bigoplus_{j=-m}^{-1} \mathfrak{p}_F e_j \oplus L_{\mathrm{an}} \oplus \bigoplus_{k=1}^{n} \mathfrak{p}_F e_k,$$

where $L_{\mathrm{an}} = \mathfrak{o}_F e_0$ is the unique self-dual lattice stabilized by Q_{an}.

Put $N_m = \{v \in L_m^\# : Q(v) \in \mathfrak{p}_F\}$, which is a lattice stabilized by $K = \mathrm{Stab}(L_m)$: if $k \in K$ and $v \in N_m$ then $kv \in L_m^\#$ as $v \in L_m^\#$ and $Q(kv) = Q(v) \in \mathfrak{p}_F$ so $kv \in N_m$. Now compute

$Q(\sum_{i=1}^n \lambda_i v_i) = \sum_{i=1}^n \lambda_i \lambda_{-i} + \lambda_0^2$ so we get

$$N_m = = \bigoplus_{i=-n}^{-(m+1)} \mathfrak{o}_F e_i \oplus \bigoplus_{j=-m}^{-1} \mathfrak{p}_F e_j \oplus \mathfrak{p}_F L_{\text{an}} \oplus \bigoplus_{k=1}^{n} \mathfrak{p}_F e_k$$

Then K stabilizes

$$L_m \supseteq L_m^{\#} \supset N_m \supseteq \mathfrak{p}_F L_m.$$

Let $\bar{V}_1 := L_m/N_m$ and $\bar{V}_2 := N_m/\mathfrak{p}_F L_m$. The space $\bar{V}_1$ is $(2m+1)$-dimensional with induced form

$$\bar{Q}_1(v + N_m) := Q(v) + \mathfrak{p}_F$$

which has basis the image of the vectors $e_{-m}, \dots, e_{-1}, e_0, e_1, \dots, e_m$ in the quotient. This form is non-degenerate isotropic, so $\bar{Q}_1$ is the orthogonal sum of m copies of the hyperbolic form $\bar{Q}_{\mathbb{H}}$ and the one-dimensional anisotropic form $\bar{Q}_{\text{an}}$ over the residue field k_F.

The space $\bar{V}_2$ is $2(n-m)$-dimensional with induced form

$$\bar{Q}_2(v + \mathfrak{p}_F L_m) := \varpi^{-1} Q(v) + \mathfrak{p}_F$$

which has basis the image of the vectors $e_{-n}, \dots, e_{-(m+1)}, \varpi e_{m+1}, \dots, \varpi e_n$ in the quotient. This form is non-degenerate and hyperbolic, so $\bar{Q}_2$ is the orthogonal sum of $(n-m)$-copies of the hyperbolic form $\bar{Q}_{\mathbb{H}}$ over the residue field k_F.

Proposition 2.10.3. *Let G be a Split Special Orthogonal group and let L_m be the almost self-dual lattice defined above with stabilizer K with $m \neq 1, 2, n-2, n-1$. Suppose that we cannot lift the identity of the reductive quotient K/K^1 to an element of $O(Q)$ of determinant -1. Then*

$$K/K^1 \simeq \left\{ (g_1, g_2) \in \mathrm{O}(\bar{Q}_1) \times \mathrm{O}(\bar{Q}_2) : \begin{array}{c} \textit{either both} \;\; g_i \in \mathrm{SO}(\bar{Q}_i) \;\; \textit{or both} \\ g_i \in \mathrm{O}(\bar{Q}_i) \setminus \mathrm{SO}(\bar{Q}_i) \end{array} \right\} = H.$$

Proof. We have seen in Proposition 2.10.1 that we can lift orthogonal reflections over the residue field to a reflections over the p-adic field. Moreover, an orthogonal reflection over the p-adic field has determinant -1. Let $\bar{V}_i$ be the spaces above with induced forms $\bar{Q}_i$.

Suppose $\bar{g}_i \in \mathrm{O}(\bar{Q}_i)$ lifts to $g = g_1 + g_2 \in \mathrm{O}(Q)$ such that $g \mid_{\bar{V}_i} = \bar{g}_i$. Then $g \in \mathrm{SO}(Q)$ if and only if $\det(g) = \det(g_1)\det(g_2) = 1$ which immediately shows that $K/K^1 \supseteq H$. The assumption forces equality. □

Remark 2.10.4. The hypothesis that we cannot lift the identity to an element of determinant -1 is not always necessary. For unramified extensions F of $\mathbb{Q}_2$ the proof of this is as follows. Suppose g is such a lift with determinant -1. We can write $g = 1 + 2X$ with $\det(g) = -1$ and $X \in \mathrm{Mat}_{2n}(\mathfrak{o}_F)$. On one hand, we have $\det(g) = -1 \in 1 + 2\mathrm{tr}(X) + \mathfrak{p}_F^2$ which holds if and only if $\mathrm{tr}(X) \equiv 1 \bmod \mathfrak{p}_F$. On the other, if $g \in \mathrm{SO}(Q)$ then, by writing $A_h = \mathrm{antidiag}(1, \ldots, 1)$ for the Gram matrix of the bilinear form h associated to Q, we have $g^T A_h g = A_h$ implies $\mathrm{tr}(X) \equiv 0 \bmod \mathfrak{p}_F$ a contradiction.

It now remains to show that the stabilizers of the lattices L_m above are maximal compact. We first consider the even Split Special Orthogonal groups.

Proposition 2.10.5. *Let G be an Even Split Special Orthogonal group so Q is a non-degenerate hyperbolic quadratic form on V with associated bilinear form h. With respect to h, let $\{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ be a Witt basis for V. Write L_m for the almost self-dual lattice*

$$L_m = \bigoplus_{i=-n}^{-1} \mathfrak{o}_F e_i \oplus \bigoplus_{j=1}^{m} \mathfrak{o}_F e_j \oplus \bigoplus_{k=m+1}^{n} \mathfrak{p}_F e_k.$$

with stabilizer K. Suppose $m \neq 1, n-1$. Then K is maximal compact.

Proof. This is identical to the proof of Proposition 2.9.3 by taking $L = L_m$. □

Proposition 2.10.6. *Let G be an Odd Split Special Orthogonal group so $Q = Q_{V_{n\mathbb{H}}} \perp Q_{\mathrm{an}}$ is a non-degenerate anisotropic quadratic form on $V = V_{n\mathbb{H}} \perp V_{\mathrm{an}}$ with associated bilinear form $h = h \mid_{V_{n\mathbb{H}}} \perp h \mid_{V_{\mathrm{an}}}$. With respect to h, let $\{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ be a Witt basis for $V_{n\mathbb{H}}$ and e_0 be a basis for V_{an}. Write L_m for the almost self-dual lattice*

$$L_m = \bigoplus_{i=-n}^{-1} \mathfrak{o}_F e_i \oplus L_{\mathrm{an}} \oplus \bigoplus_{j=1}^{m} \mathfrak{o}_F e_j \oplus \bigoplus_{k=m+1}^{n} \mathfrak{p}_F e_k.$$

with stabilizer K, where $L_{\mathrm{an}} = \mathfrak{o}_F e_0$ is self-dual. Write N_m for lattice

$$N_m = \bigoplus_{i=-n}^{-1} \mathfrak{o}_F e_i \oplus \mathfrak{p}_F L_{\mathrm{an}} \oplus \bigoplus_{j=1}^{m} \mathfrak{o}_F e_j \oplus \bigoplus_{k=m+1}^{n} \mathfrak{p}_F e_k.$$

Suppose $m \neq n-1$. Then K is maximal compact.

Proof. Write $L = L_m$. The same argument as in Proposition 2.9.3 shows the existence of another self-dual lattice L'' stabilized by K. We write $N_m = L^{\#} \cap Q^{-1}(\mathfrak{p}_F)$ and $N''_m = L''^{\#} \cap Q^{-1}(\mathfrak{p}_F)$. We then have either

$$L \supsetneq L'' \supseteq N''_m \supsetneq N_m$$

or

$$L'' \supsetneq L \supsetneq N_m \supsetneq N''_m \supseteq \mathfrak{p}_F L'' \supsetneq \mathfrak{p}_F L.$$

In the former case the stabilizer of L''/N_m in L/N_m is a proper parabolic subgroup of G. Similarly, in the latter case the stabilizer of $\mathfrak{p}_F L''/\mathfrak{p}_F L$ in $N_m/\mathfrak{p}_F L$ is a proper parabolic subgroup of G. In either case, K surjects onto the connected component of L/N_m and $N_m/\mathfrak{p}_F L$, and this group of isometries acts irreducibly, giving the required contradiction. $\square$

Recall that a maximal parahoric subgroup of G is the inverse image in K/K^1 of the connected component of K/K^1 for K a maximal compact open subgroup of G. This immediately gives the following.

Corollary 2.10.7. *Let G_i be a Split Special Orthogonal group with $i = \dim V_{\mathrm{an}}$. Let K_i denote the stabilizer of the lattice L_m define above and K_i° denote the maximal parahoric associated to K_i. Suppose $m \neq 1, 2, n-2, n-1$ for $i = 0$ and $m \neq n-2, n-1$ for $i = 1$. Then*

$$K_0^{\circ}/K_0^1 \simeq \mathrm{SO}^+_{2m}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F)$$

and

$$\begin{aligned} K_1^{\circ}/K_1^1 &\simeq \mathrm{SO}_{2m+1}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F) \\ &\simeq \mathrm{Sp}_{2m}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F). \end{aligned}$$

Remark 2.10.8. In our situation, we have given a description of the reductive quotients for the maximal parahoric subgroups corresponding to the stabilizers of certain almost self-dual lattices. This coincides with the description of the reductive quotient for G a Split Special Orthogonal group when p is *odd* (albeit "swapped around" in the sense that the same groups appear in the direct product but in a reverse order as m ranges over its possible values). We recall this description below.

Suppose p is odd and ϖ_F is a fixed uniformizer of F. We define $G = \mathrm{SO}(h)$ as the group of isometries of a symmetric bilinear form h. Note that we need not consider quadratic forms since $h(u,v) = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right)$ in this case. Let L_m be an almost self-dual lattice for some m subject to $0 \leq m \leq n$. Write $L = L_m$ with dual $L^{\#} = L_m^{\#}$. On $L/L^{\#}$ we have the induced form

$$h(v + L^{\#}, w + L^{\#}) := h(v,w) + \mathfrak{p}_F, \qquad \text{for } v, w \in L.$$

Similarly, on $L^{\#}/\mathfrak{p}_F L$ the induced form is

$$h(v' + \mathfrak{p}_F L, w' + \mathfrak{p}_F L) := \varpi_F^{-1} h(v', w') + \mathfrak{p}_F, \qquad \text{for } v', w' \in L^{\#}.$$

Write K for the maximal parahoric corresponding to the lattice L, with pro-unipotent radical K^1:

- If $G = \mathrm{SO}_{2n}$ then
$$K/K^1 \simeq \mathrm{SO}^{+}_{2n}(k_F) \times \mathrm{SO}^{+}_{2(n-m)}(k_F)$$
for $0 \leq m \leq n$ with $m \neq 1, n-1$.

- If $G = \mathrm{SO}_{2n+1}$ then
$$K/K^1 \simeq \mathrm{SO}^{+}_{2m}(k_F) \times \mathrm{SO}_{2(n-m)+1}(k_F)$$
for $0 \leq m \leq n$ with $m \neq 1$.

The only difference here is that in odd characteristic we no longer have the isomorphism between odd-dimensional Special Orthogonal groups and Symplectic groups of codimension 1 over finite fields. Moreover, in our work we have the added caveat that we do not consider maximal parahorics which have a factor of $\mathrm{SO}_4^+(\mathbb{F}_2)$ appearing in their reductive quotient.

Chapter 3

Representation Theory

Let F be a non-archimedean local field and G be a locally profinite group, by which we mean G is a topological group in which every open neighbourhood of the identity contains a compact open subgroup. Our aim is to study the representation theory of G, in particular, we will be interested in *complex representations*.

A *representation* of G is a pair $(\pi, \mathcal{V})$ where $\pi : G \to \mathrm{GL}(\mathcal{V})$ is a homomorphism of groups and $\mathcal{V}$ is a $\mathbb{C}$-vector space. We omit the use of complex and simply talk of representations of G. A representation $(\pi, \mathcal{V})$ of G is *smooth* if, for every vector $v \in \mathcal{V}$, the stabilizer of v

$$\mathrm{stab}_G(v) = \{g \in G : \pi(g)v = v\}$$

is open. For $(\pi_1, \mathcal{V}_1), (\pi_2, \mathcal{V}_2)$ smooth representations of G, we write $\mathrm{Hom}_G(\pi_1, \pi_2)$ for the space of G-homomorphisms between $(\pi_1, \mathcal{V}_1)$ and $(\pi_2, \mathcal{V}_2)$. We denote by $\mathcal{R}(G)$ the category of smooth representations of G.

A representation $(\pi, \mathcal{V})$ is *irreducible* if there are no proper submodules of $\mathcal{V}$ which are stable under G. Let $\mathrm{Irr}(G)$ denote the set of equivalence classes of irreducible smooth representations of G. For ease of notation we often simply write π for the representation $(\pi, \mathcal{V})$.

Let $(\pi, \mathcal{V}) \in \mathcal{R}(G)$. We call $(\pi, \mathcal{V})$ *admissible* if the space $\mathcal{V}^H = \{v \in \mathcal{V} : \pi(h)v = v\}$ is finite-dimensional for all open subgroups H of G. Admissible representations admit nice properties which are useful in the study of the representation theory of G. One would hope that admissible representations encompass a large class of objects in $\mathcal{R}(G)$. This turns out to be the case.

Theorem 3.0.1. *[Jac75] Let $(\pi, \mathcal{V}) \in \mathrm{Irr}(G)$. Then $(\pi, \mathcal{V})$ is admissible.*

A classical result in representation theory which will be of use to us is Schur's Lemma.

Theorem 3.0.2. *[BH06, Chapter 1] Let $\pi_1, \pi_2 \in \mathrm{Irr}(G)$. Then $\mathrm{Hom}_G(\pi_1, \pi_2) \neq 0$ if and only if $\pi_1 \simeq \pi_2$. Moreover $\mathrm{End}_G(\pi_1) = \mathbb{C}$.*

Suppose $(\pi, \mathcal{V})$ is an irreducible smooth representation of G. Let $Z(G)$ denote the centre of G. It follows from Schur's Lemma that Z acts on $\mathcal{V}$ via a character $\omega_\pi : Z(G) \to \mathbb{C}^\times$. We call ω_π the *central character of* G.

3.1 Hecke Algebras

If G is a finite group, then the study of representations of G is equivalent to studying modules over the group algebra $\mathbb{C}G$. This is no longer true in our setting. Instead, we get an analogous result if we replace the group algebra $\mathbb{C}G$ with what is called the Hecke algebra of G.

Let $C_c^\infty(G)$ denote the space of functions $\phi : G \to \mathbb{C}$ which are locally constant and have compact support. The group G acts on $C_c^\infty(G)$ by left and right translation:

$$\begin{aligned}\pi_g\phi :&h \longmapsto \phi(g^{-1}h),\\ \pi'_g\phi :&h \longmapsto \phi(hg),\end{aligned}$$

for all $g, h \in G$ and all $\phi \in C_c^\infty(G)$. A function $\phi \in C_c^\infty(G)$ is said to be *positive* if $f(g) \geq 0$ for all $g \in G$, in which case we write $\phi \geq 0$. A *left Haar integral* I on G is a non-zero linear functional $I : C_c^\infty \to \mathbb{C}$ which is invariant under left translation and is positive on positive functions $\phi \in C_c^\infty(G)$; it is unique up to multiplication by a positive real scalar. One defines a *right Haar integral* in the same way. A group G is said to be *unimodular* if left Haar integrals and right Haar integrals coincide. Any reductive p-adic group is unimodular [Ren10, Proposition V.5.4].

Let X be a Hausdorff topological space. A *Borel set* S is a set which can be formed by countable unions, countable intersections or complementations of open subsets of X. A *σ-algebra* Σ on X is a subset of the power set $\mathcal{P}(X)$ such that

i) $X \in \Sigma$;

ii) Σ is closed under taking complements;

iii) Σ is closed under taking countable unions.

A *Radon measure* μ on G is a measure μ on the σ-algebra of Borel sets of G which is finite on compact sets, outer-regular on Borel sets i.e.

$$\mu(S) = \inf\{\mu(U) : S \subseteq U,\, U \text{ open}\},$$

and inner-regular on open sets i.e.

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}.$$

A *(left)-Haar measure* μ is a non-zero Radon measure which is invariant under left-translation i.e. $\mu(gS) = \mu(S)$ for all Borel sets $S \subseteq G$ and $g \in G$ [BH06, Chapter 1.3]. Moreover it is unique up to multiplication by positive scalars. An immediate consequence of this is that there exists a function $\delta : G \to (0, \infty)$ called the *modulus character of G* satisfying $\mu(Sx) = \delta(x)\mu(S)$ for all Borel sets S. This character is unique and has the property that G is unimodular if and only if $\delta(G) = 1$.

Haar measures enable us to define a convolution product on $C_c^\infty(G)$ as follows. Let μ be a left Haar measure on G. For any $f_1, f_2 \in C_c^\infty(G)$ we define the convolution product $f_1 * f_2$ as

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g) \mathrm{d}\mu(h).$$

With respect to the convolution product we can view $C_c^\infty(G)$ as an associative algebra. We call the algebra $\mathcal{H}(G) = (C_c^\infty(G), *)$ the *Hecke algebra of* G. While $\mathcal{H}(G)$ is non-unital, it contains many idempotents. For any compact open subgroup K of G the function

$$e_K(g) = \begin{cases} \frac{1}{\mu(K)} & \text{if } g \in K, \\ 0 & \text{otherwise,} \end{cases}$$

is an idempotent in $\mathcal{H}(G)$. Once can easily form a unital subalgebra of $\mathcal{H}(G)$, namely the algebra $e_K * \mathcal{H}(G) * e_K$ with unit e_K. This is the space of functions $\phi \in C_c^\infty(G)$ which satisfy $\phi(k_1 g k_2) = \phi(g)$ for all $k_1, k_2 \in K$. These subalgebras have the property that

$$\mathcal{H}(G) = \bigcup_K e_K * \mathcal{H}(G) * e_K,$$

where K runs over all compact open subgroups of G. An $\mathcal{H}(G)$-module M is *smooth* if for all $m \in M$, there exists a compact open subgroup K of G such that $e_K \cdot m = m$. Let M_1, M_2 be smooth $\mathcal{H}(G)$-modules. We write $\mathrm{Hom}_{\mathcal{H}(G)}(M_1, M_2)$ for the space of all $\mathcal{H}(G)$-homomorphisms from M_1 to M_2. If we take objects to be smooth $\mathcal{H}(G)$-modules and morphisms to be $\mathcal{H}(G)$-homomorphisms then we can construct the category $\mathcal{H}(G)$-Mod of smooth $\mathcal{H}(G)$-modules.

Theorem 3.1.1. *There is an equivalence of categories between* $\mathcal{R}(G)$*, the category of smooth representations of* G*, and* $\mathcal{H}(G)$-Mod*, the category of smooth* $\mathcal{H}(G)$*-modules.*

The action of $\varphi \in \mathcal{H}(G)$ on a representation $\mathcal{V} \in \mathcal{R}(G)$ is given by

$$\varphi \cdot v = \int_G \varphi(g) \pi(g) v \, dg,$$

which gives $\mathcal{V}$ the structure of a left $\mathcal{H}(G)$-module. This is a finite sum since both φ and v are smooth. On the other side, for a smooth $\mathcal{H}(G)$-module M, $g \in G$ acts in the following way:

$$\pi(g) \cdot m = \frac{1}{\mu(K)} \mathbb{1}_{gK} \cdot m$$

for $m \in M$ satisfying $e_K \cdot m = m$. Here $\mathbb{1}_{gK}$ denotes the characteristic function of gK.

3.2 Induction and Restriction

Let H be a closed subgroup of G and $\mathcal{R}(H)$ denote the category of smooth representations of H. Let $(\rho, \mathcal{W}) \in \mathcal{R}(H)$. Then one can construct from $(\rho, \mathcal{W})$ a representation $(\operatorname{Ind}_H^G \rho, \operatorname{Ind}_H^G \mathcal{W}) \in \mathcal{R}(G)$ as follows. Let $\operatorname{Ind}_H^G \mathcal{W}$ denote the vector space of functions $f : G \to W$ satisfying

i) $f(hg) = \rho(h)f(g)$ for all $h \in H, g \in G$;

ii) there exists a compact open subgroup K of G such that $f(gk) = f(g)$ for all $k \in K$.

The group G acts on $\operatorname{Ind}_H^G \mathcal{W}$ by right translation. The induction functor Ind_H^G is right adjoint to the restriction functor

$$\operatorname{Res}_H^G : \mathcal{R}(G) \longrightarrow \mathcal{R}(H)$$

which restricts representations and morphisms from G to H in the natural way. We can interpret this property in the following classical result.

Theorem 3.2.1 (Frobenius Reciprocity)**.** *Suppose H is a closed subgroup of G. Let $(\rho, \mathcal{W}) \in \mathcal{R}(H)$ and $(\pi, \mathcal{V}) \in \mathcal{R}(G)$. Then*

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \rho) \simeq \operatorname{Hom}_H(\operatorname{Res}_H^G \pi, \rho).$$

If H is also open, Res_H^G has a left adjoint which is compact induction, denoted ind_H^G. In terms of functions, $\operatorname{ind}_H^G W$ is the subspace of $\operatorname{Ind}_H^G W$ consisting of functions which are compactly supported modulo H. In this case, there is an analogous version of Frobenius Reciprocity.

Theorem 3.2.2 (Frobenius Reciprocity). *Suppose H is an open subgroup of G. Let $(\rho, \mathcal{W}) \in \mathcal{R}(H)$ and $(\pi, \mathcal{V}) \in \mathcal{R}(G)$. Then*

$$\mathrm{Hom}_G(\mathrm{ind}_H^G \pi, \rho) \simeq \mathrm{Hom}_H(\pi, \mathrm{Res}_H^G \, \rho).$$

This says that compact induction ind_H^G is left-adjoint to the restriction functor.

3.3 Parabolic Induction and Cuspidal Representations

Let G be a reductive p-adic group. We have seen above that we can obtain representations of G by the process of induction. When we start with a representation of a Levi subgroup and induce, this is known as *parabolic induction*. It is this method which we now describe.

Let $\mathcal{P}$ be a parabolic subgroup with Levi decomposition $\mathcal{P} = \mathcal{M} \ltimes \mathcal{N}$. Let $(\rho, \mathcal{W})$ be a representation of $\mathcal{M}$. Since $\mathcal{M} \simeq \mathcal{P}/\mathcal{N}$ we can *inflate* ρ to a representation of $\mathcal{P}$, which we denote $\mathrm{Infl}_{\mathcal{M}}^{\mathcal{P}} \rho$. We abuse notation and also refer to this inflated representation as ρ. This gives a functor $\mathrm{Infl}_{\mathcal{M}}^{\mathcal{P}} : \mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{P})$.

We can then induce from $\mathcal{P}$ to G to obtain a representation of G. The composition of these two functors gives a functor $\mathrm{Ind}_{\mathcal{M},\mathcal{P}}^G$ which we call *parabolic induction*. The space $\mathrm{Ind}_{\mathcal{M},\mathcal{P}}^G \, \rho$ consists of all locally constant functions $f : G \to \mathcal{W}$ such that

$$f(pg) = \rho(p) f(g)$$

for all $p \in \mathcal{P}, g \in G$. There is also a variant of parabolic induction called *normalized parabolic induction*. This is again a functor $\iota_{\mathcal{M},\mathcal{P}}^G : \mathcal{R}(\mathcal{M}) \to \mathcal{R}(G)$. However, $\iota_{\mathcal{M},\mathcal{P}}^G \mathcal{W}$ is now the space of locally constant functions $f : G \to \mathcal{W}$ such that

$$f(pg) = \delta_P^{\frac{1}{2}}(p) \rho(p) f(g)$$

for all $p \in \mathcal{P}$ and $g \in G$. Here δ_P is the *modulus character* of $\mathcal{P}$. By definition, we have

$$\iota_{\mathcal{M},\mathcal{P}}^G(\rho) = \mathrm{Ind}_{\mathcal{M},\mathcal{P}}^G(\delta_P^{\frac{1}{2}} \otimes \rho).$$

Whilst the definitions of the two functors are similar, normalized parabolic induction has the added benefit that it preserves unitary representations i.e. if ρ is a unitary representation then $\iota^G_{\mathcal{M},\mathcal{P}}(\rho)$ is a unitary representation.

One can ask if by ranging over all proper parabolic subgroups of G whether all irreducible representations of G appear as an irreducible subquotient of a (normalized) parabolically induced representation. This turns out to be false, and leads to the definition of a *cuspidal representation*.

Let π be an irreducible representation of G. We say that π is *cuspidal* if it is not a quotient of $\iota^G_{\mathcal{M},\mathcal{P}}\rho$, for any proper parabolic subgroup $\mathcal{P}$ of G with Levi factor $\mathcal{M}$ and σ an irreducible representation of $\mathcal{M}$. We call π *supercuspidal* if is not a subquotient of $\iota^G_{\mathcal{M},\mathcal{P}}\rho$, for any parabolic subgroup $\mathcal{P}$ of G with Levi factor $\mathcal{M}$ and σ an irreducible representation of $\mathcal{M}$.

In order to understand the representation theory of G, we therefore need to understand all cuspidal representations of G. This is a difficult problem which, although it has not yet been answered in full generality, has been resolved in many cases. The known results all suggest that the following long-standing conjecture is true, although not everyone in the mathematical community believes that this is the case.

Conjecture 3.3.1. *Let π be an irreducible cuspidal representation of G. There exist an open, compact mod-centre subgroup $\tilde{J}$ of G and an irreducible representation Λ of $\tilde{J}$ such that*

$$\pi \simeq \mathrm{ind}_{\tilde{J}}^{G}\Lambda.$$

3.4 Intertwining

We have seen above that in order to understand $\mathcal{R}(G)$ for a reductive p-adic group G, we need to understand the irreducible cuspidal representations of G. A powerful concept which is used when giving an explicit construction of such representations is that of *intertwining*.

Let J, J' be compact open subgroups of G, and let ρ, ρ' be representations of J, J' respectively. Let $g \in G$. We say that g *intertwines* λ with λ' if

$$\mathrm{Hom}_{J \cap {}^g J'}(\lambda, {}^g\lambda') \neq 0.$$

Here ${}^g J' = gJ'g^{-1}$ and ${}^g\lambda : j \mapsto \lambda(g^{-1}jg)$ for $j \in {}^g J'$. We denote the set of $g \in G$ which intertwine λ with λ' by $\mathcal{I}_G(\lambda, \lambda')$. Furthermore, if $\lambda = \lambda'$ we say that g *intertwines* λ and write $\mathcal{I}_G(\lambda)$ for the set of g which intertwine λ. While intertwining is both reflexive and symmetric, it falls short of being an equivalence relation because transitivity is not guaranteed.

The following Theorem due to Carayol lies at the heart of all proofs concerning classification Theorems of cuspidal representations. It highlights the importance of intertwining in these instances.

Theorem 3.4.1. *[Car84, Proposition 1.5] Let $\tilde{J}$ be an open, compact mod-centre subgroup of G. Let λ be an irreducible representation of $\tilde{J}$. If $\mathcal{I}_G(\lambda) = \tilde{J}$, then $\mathrm{ind}_{\tilde{J}}^G \lambda$ is irreducible and cuspidal.*

3.5 Bernstein Decomposition

We note that while we have an equivalence of categories in Theorem 3.1.1, the categories are too large to work with. One would hope that it is possible to decompose both categories into pieces and that there is an analogous result for each piece. This decomposition is known as the *Bernstein Decomposition*. For a more comprehensive treatise of the following material, see [BK98].

An *unramified character* of G is a character of the form $g \mapsto \mid \phi(g) \mid^s$ where ϕ is an F-rational character of G and $s \in \mathbb{C}$. Let $X_0(G)$ denote the group of unramified characters of G. Unramified characters have the property that they are trivial on every compact subgroup of G and are determined by their valuation on a uniformizer of F.

Let π be an irreducible representation of G. Let $\mathcal{M}$ be a Levi subgroup of a parabolic subgroup $\mathcal{P}$ of G and σ be an irreducible cuspidal representation of G. We call the pair $(\mathcal{M}, \sigma)$ a *cuspidal pair*. If the representation π is equivalent to a subquotient of the (normalized) parabolically induced representation $\iota_{\mathcal{M},\mathcal{P}}^{G}\sigma$, we refer to the cuspidal pair $(\mathcal{M}, \sigma)$ as the *cuspidal support* of π, which is unique up to conjugacy.

Two cuspidal pairs $(\mathcal{M}, \sigma)$ and $(\mathcal{M}', \sigma')$ are *inertially equivalent* if there exist a $g \in G$ and $\chi \in X_0(\mathcal{M}')$ satisfying $\mathcal{M}' = {}^g\mathcal{M}$ and $\sigma' = {}^g\sigma \otimes \chi$. The *inertial support* of π is the inertial equivalence class of its cuspidal support. We denote the *inertial equivalence class* of $(\mathcal{M}, \sigma)$ by $[\mathcal{M}, \sigma]_G$ and the set of inertial equivalence classes of G by $\mathbf{B}(G)$. We call $\mathbf{B}(G)$ the *Bernstein spectrum* of G.

We can now state the Bernstein Decomposition which describes a decomposition of the category $\mathcal{R}(G)$ of smooth representations of G.

Theorem 3.5.1 (Bernstein)**.** *[Ber84] For each $\mathfrak{s} \in \mathbf{B}(G)$, let $\mathcal{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathcal{R}(G)$ consisting of representations whose irreducible subquotients have inertial support contained in $\mathfrak{s} \in \mathbf{B}(G)$. Then*

$$\mathcal{R}(G) = \prod_{\mathfrak{s} \in \mathcal{B}(G)} \mathcal{R}^{\mathfrak{s}}(G).$$

That is, if $(\pi, \mathcal{V}), (\rho, \mathcal{W})$ are representations in $\mathcal{R}(G)$ then $\mathcal{V} = \bigoplus_{\mathfrak{s}} \mathcal{V}^{\mathfrak{s}}$ where $\mathcal{V}^{\mathfrak{s}}$ is the space associated to the block $\mathcal{R}^{\mathfrak{s}}(G)$ and $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) = \prod_{\mathfrak{s}} \mathrm{Hom}_G(\mathcal{V}^{\mathfrak{s}}, \mathcal{W}^{\mathfrak{s}})$.

Given such a decomposition, the task is now to find a nice description of each $\mathcal{R}^{\mathfrak{s}}(G)$, which we call a *block*.

Bernstein affords one way of splitting up $\mathcal{R}(G)$, but we shall now consider another which uses idempotents.

Let $(\rho, \mathcal{W})$ be a smooth irreducible representation of K, a compact open subgroup of G, and $(\pi, \mathcal{V})$ be a smooth representation of G. We write $\mathcal{V}^\rho$ for the sum of all irreducible K-subspaces of V which are equivalent to ρ and call it the *ρ-isotypic component* of $\mathcal{V}$. Since K is compact, the restriction of π to K is semisimple, so we can write

$$\mathcal{V} = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathcal{V}^\rho.$$

We say $\mathcal{V}$ *contains* ρ if $\mathcal{V}^\rho \neq 0$. Let $\mathcal{R}_\rho(G)$ denote the full subcategory of $\mathcal{R}(G)$ consisting of representations which are generated (as representations of G) by their ρ-isotypic vectors.

Now fix a Haar measure μ on G. Define $e_\rho \in \mathcal{H}(G)$ by

$$e_\rho = \begin{cases} \frac{\dim \rho}{\mu(K)} \mathrm{tr}_W(\rho(x^{-1})) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

This provides the projection of $\mathcal{V}$ onto each piece $\mathcal{V}^\rho$. Given two representations ρ and ρ' one then has $e_\rho * e_{\rho'} = e_\rho$ if and only if $\rho \simeq \rho'$, otherwise $e_\rho * e_{\rho'} = 0$. Moreover $e_\rho \cdot \mathcal{V} = \mathcal{V}^\rho$. This construction is important because we obtain the *scalar Hecke algebra* $\mathcal{H}_\rho(G) := e_\rho * \mathcal{H}(G) * e_\rho$ which is a subalgebra of $\mathcal{H}(G)$ with unit e_ρ. The ρ-isotypic component V^ρ is then a left $\mathcal{H}_\rho(G)-$module.

Whilst the scalar Hecke algebra gives a nice splitting of the category $\mathcal{H}(G)$, the subalgebra $\mathcal{H}_\rho(G)$ is still too large to work with. Instead, we turn to another type of Hecke algebra which is Morita equivalent to $\mathcal{H}_\rho(G)$, the *spherical Hecke algebra* $\mathcal{H}(G, \rho)$, defined as

$$\mathcal{H}(G, \rho) = \mathrm{End}_G(\mathrm{ind}_K^G \rho).$$

The following result of Bushnell–Kutzko gives a pair of functors which describe an equivalence of categories between $\mathcal{R}_\rho(G)$ and the categories of smooth modules over the two Hecke algebras described above, under certain conditions.

Theorem 3.5.2. *[BK98, Proposition 3.3, Theorem 4.3] Let $\mathcal{R}_\rho(G)$, $\mathcal{H}(G, \rho)$ and $\mathcal{H}_\rho(G)$ be as above. The following are equivalent:*

i) $\mathcal{R}_\rho(G)$ *is closed under subquotients;*

ii) *The functor* $\mathbf{M}^\rho : \mathcal{R}(G) \to \mathcal{H}_\rho(G)\text{-Mod}$ *which maps a smooth representation to its* ρ*-isotypic component induces an equivalence of categories* $\mathcal{R}_\rho(G) \simeq \mathcal{H}_\rho(G)\text{-Mod}$*;*

iii) *The functor* $\mathbf{m}_\rho : \mathcal{R}(G) \to \mathcal{H}(G,\rho)\text{-Mod}$ *which maps the representation* π *to* $\mathrm{Hom}_K(\rho,\pi)$ *induces an equivalence of categories* $\mathcal{R}_\rho(G) \simeq \mathcal{H}(G,\rho)\text{-Mod}$*;*

iv) *Every irreducible subquotient of* $\mathrm{ind}_K^G \rho$ *contains* ρ*;*

v) *There exists a finite subset* $\mathfrak{S} \subset \mathcal{B}(G)$ *such that*

$$\mathcal{R}_\rho(G) = \prod_{\mathfrak{s}\in\mathfrak{S}} \mathcal{R}^{\mathfrak{s}}(G).$$

We say that (K,ρ) is an $\mathfrak{S}$-*type* if satisfies the properties of Theorem 3.5.2. If $\mathfrak{S}$ is a singleton then we simple refer to (K,ρ) as a *type*. Types are important because if we have an explicit description of a type (K,ρ) and of its spherical Hecke algebra $\mathcal{H}(G,\rho)$ for each block $\mathcal{R}^{\mathfrak{s}}(G)$, then using the Bernstein Decomposition above we have an explicit description of the category $\mathcal{R}(G)$.

3.6 Representation Theory of $\mathrm{GL}_N(F)$

3.6.1 Notation

In this section we recall the classification of irreducible cuspidal representations as given by Bushnell–Kutzko in [BK93a]. The treatment given here is by no means complete, we shall only give the most basic details of the construction. We refer the reader to the original source above, or to the notes of Conley [Con09] which give a more comprehensive exposition.

Let F be a p-adic field, with no restriction on the residue characteristic. We write $\mathfrak{o}_F$ for its ring of integers, $\mathfrak{p}_F$ for the unique maximal ideal, and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ for the residue field which is finite of characteristic $q_F = p^r$ for some $r \in \mathbb{N}$. Fix ϖ_F a uniformizer of F. Let V be an N-dimensional F-vector space. Write $A = \mathrm{End}_F(V)$ and $G = \mathrm{Aut}_F(V)$ which, after fixing a basis for V, is isomorphic to $\mathrm{GL}_N(F)$.

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A with Jacobson radical $\mathfrak{P}$. The unit group

$$U(\mathfrak{A}) = \mathfrak{A}^\times$$

is a *parahoric* subgroup of $\mathrm{GL}_N(F)$, and every parahoric subgroup is the unit group of some hereditary order. The group $U(\mathfrak{A})$ comes with a natural filtration by normal compact open subgroups

$$U^n(\mathfrak{A}) = 1 + \mathfrak{P}^n,$$

for $n \geq 1$, where $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$. With respect to a suitable choice of basis $\mathfrak{A}$ consists of block matrices which are upper triangular mod $\mathfrak{p}$, with block sizes $N_1, \ldots, N_e$ satisfying $\sum_{i=1}^e N_i = N$. The quotient $U(\mathfrak{A})/U^1(\mathfrak{A})$ is isomorphic to the group $\prod_i^e \mathrm{GL}_{N_i}(k_F)$.

The *normalizer* $\mathfrak{K}(\mathfrak{A}) = \{g \in \mathrm{GL}_N(F) : g^{-1}\mathfrak{A}g = \mathfrak{A}\}$ of $\mathfrak{A}$ is an open, compact-mod-centre subgroup of $\mathrm{GL}_N(F)$. It normalizes $U^n(\mathfrak{A})$ for each n and contains $U(\mathfrak{A})$ as its maximal compact subgroup.

We fix ψ_F an additive character of F with conductor $\mathfrak{p}_F$ i.e. ψ_F is trivial on $\mathfrak{p}_F$ but nontrivial on $\mathfrak{o}_F$. We write tr for the trace map $\mathrm{tr} : A \to F$ so that $\psi_A = \psi_F \circ \mathrm{tr}$ is a character of A. For integers $1 \leq m \leq n \leq 2m$ we have the canonical isomorphism

$$\mathfrak{P}^m/\mathfrak{P}^n \simeq U^m(\mathfrak{A})/U^n(\mathfrak{A})$$

induced by $x \mapsto 1 + x$. For S a subset of A we write $S^* = \{a \in A : \psi_A(aS) = 1\}$. Then $(\mathfrak{P}^n)^* = \mathfrak{P}^{1-n}$ and we get an isomorphism

$$\mathfrak{P}^{-n}/\mathfrak{P}^{-m} \simeq \left(\mathfrak{P}^{m+1}/\mathfrak{P}^{n+1}\right)^\wedge$$

between cosets $\mathfrak{P}^{-n}/\mathfrak{P}^{-m}$ and characters of $\mathfrak{P}^{m+1}$ trivial on $\mathfrak{P}^{n+1}$. If we impose $0 \leq m \leq n \leq 2m + 1$ then we have an isomorphism

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\simeq \left(U^{m+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A})\right)^\wedge \\ \beta + \mathfrak{P}^{-m} &\mapsto \psi_\beta, \end{aligned}$$

where ψ_β is the character given by $\psi_\beta(1 + x) = \psi_A(\beta(x))$ for $1 + x \in U^{m+1}(\mathfrak{A})$, which is trivial on $U^{n+1}(\mathfrak{A})$. If we let $\nu_\mathfrak{A} : A \to \mathbb{Z}$ be the map $\nu_\mathfrak{A}(x) = \sup\{k \in \mathbb{Z} : x \in \mathfrak{P}^k\}$ then ψ_β is nontrivial on $U^n(\mathfrak{A})$ provided $\nu_\mathfrak{A}(\beta) = -n$.

We call the four-tuple $[\mathfrak{A}, n, m, \beta]$ a *stratum* if

1. $\mathfrak{A}$ is a hereditary order;

2. $m < n$ are nonnegative integers;

3. $\beta \in \mathfrak{P}^{-n}$.

We say that any two strata $[\mathfrak{A}_1, n_1, m_1, \beta_1]$ and $[\mathfrak{A}_2, n_2, m_2, \beta_2]$ are *equivalent* if

$$\beta_1 + \mathfrak{P}_1^{-m_1} = \beta_2 + \mathfrak{P}_2^{-m_2},$$

where $\mathfrak{P}_i$ is the Jacobson radical of $\mathfrak{A}_i$. We write $[\mathfrak{A}_1, n_1, m_1, \beta_1] \sim [\mathfrak{A}_2, n_2, m_2, \beta_2]$ to denote this equivalence. One can show that if the two strata above are equivalent, then

$\mathfrak{A}_1 = \mathfrak{A}_2$, $m_1 = m_2$ and $n_1 = n_2$.

We say that $g \in G$ *intertwines* the strata $[\mathfrak{A}_1, n_1, m_1, \beta_1]$ and $[\mathfrak{A}_2, n_2, m_2, \beta_2]$ if

$$g^{-1}(\beta_1 + \mathfrak{P}_1^{-m_1})g \cap (\beta_2 + \mathfrak{P}_2^{-m_2}) \neq \varnothing.$$

If $0 \leq m_i \leq n_i \leq 2m_i - 1$, for $i = 1, 2$, then this is equivalent to saying that the two strata intertwine if and only if, on the level of the characters ψ_{β_i},

$$^g\psi_{\beta_1} \mid_{^gU^{m_1}(\mathfrak{A}_1) \cap U^{m_2}(\mathfrak{A}_2)} = \psi_{\beta_2} \mid_{^gU^{m_1}(\mathfrak{A}_1) \cap U^{m_2}(\mathfrak{A}_2)} .$$

We denote the set of $g \in G$ which intertwine the two strata by

$$\mathcal{I}_G([\mathfrak{A}_1, n_1, m_1, \beta_1], [\mathfrak{A}_2, n_2, m_2, \beta_2]),$$

which we abbreviate to $\mathcal{I}_G([\mathfrak{A}, n, m, \beta])$ when both strata are equivalent to $[\mathfrak{A}, n, m, \beta]$.

3.6.2 Fundamental Strata

Before we move on and give the necessary definitions in order to review the construction of cuspidal representations of G, we first detour and look at a certain class of strata. Let $[\mathfrak{A}, n, n-1, \beta]$ be a stratum with no condition on n. We say that $[\mathfrak{A}, n, n-1, \beta]$ is *fundamental* if the coset $\beta + \mathfrak{P}^{1-n}$ does not contain a nilpotent element of A. A stratum which is not fundamental is called *non-fundamental*.

Remark 3.6.1. In practice, to identify if a stratum is fundamental or not, we use the following equivalent condition. Let $[\mathfrak{A}, n, n-1, \beta]$ be a stratum in A and write $y_\beta = \varpi^{n/g}\beta^{e/g} \in \mathfrak{A}$ with $e = e(\mathfrak{A})$ and $g = \gcd(n, e)$. Let $\Phi(X)$ be the characteristic polynomial of y_β and write $\varphi_\beta(X) \in k_F[X]$ for the reduction mod $\mathfrak{p}_F$ of $\Phi(X)$. We call $\varphi_\beta(X)$ the *characteristic polynomial of the stratum*. A stratum $[\mathfrak{A}, n, n-1, \beta]$ is then said to be fundamental if $\varphi_\beta(X) \neq X^n$.

The following Proposition gives one a way to identify non-fundamental strata.

Proposition 3.6.2. *Let $[\mathfrak{A}, n, n-1, \beta]$ be a stratum in A. The following are equivalent:*

(i) *the coset* $\beta + \mathfrak{P}^{1-n}$ *contains a nilpotent element;*

(ii) *there exists* $m \geq 1$ *such that* $\beta^m \in \mathfrak{P}^{1-mn}$.

We use the second condition to identify non-fundamental stratum, which says that the stratum is non-fundamental if β is nilpotent mod $\mathfrak{p}_F$. This is because $\beta \in \mathfrak{P}^{-n}$ and so while we expect $\beta^m \in \mathfrak{P}^{-mn}$, β^m actually lies one step further in the filtration. The reason why fundamental strata are important is seen in the following key result.

Theorem 3.6.3. *Let* $[\mathfrak{A}, n, n-1, \beta]$ *be a stratum in* A. *Write* $\mathfrak{P}$ *for the Jacobson radical of* $\mathfrak{A}$ *and* e *for the period of* $\mathfrak{A}$. *The following are equivalent:*

(i) *the stratum* $[\mathfrak{A}, n, n-1, \beta]$ *is non-fundamental;*

(ii) *there exists a stratum* $[\mathfrak{A}', n', n'-1, \beta']$ *such that*

$$\beta + \mathfrak{P}^{1-n} \subseteq \mathfrak{P}'^{-n'} \qquad \textit{and} \qquad \frac{n'}{e'} < \frac{n}{e},$$

where $\mathfrak{P}'$ *is the Jacobson radical of* $\mathfrak{A}'$, *which has period* e'.

The implications of this result for the representation theory of G are as follows.

Theorem 3.6.4. *Let* π *be an irreducible smooth representation of* G. *Then precisely one of the following occurs:*

(i) *there exists a hereditary order* $\mathfrak{A}$ *in* A *such that* π *contains the trivial character of* $U^1(\mathfrak{A})$*;*

(ii) *there exists a fundamental stratum* $[\mathfrak{A}, n, n-1, \beta]$ *with* $n \geq 1$ *such that* π *contains the character* ψ_β *of* $U^n(\mathfrak{A})$.

Moreover, if we are in the latter case, then for any other stratum $[\mathfrak{A}', n', n'-1, \beta']$ *with* $n' \geq 1$ *such that* π *contains the character* $\psi_{\beta'}$ *of* $U^{n'}(\mathfrak{A}')$, *we have*

$$\frac{n}{e} \leq \frac{n'}{e'},$$

where e *(resp.* e'*) is the period of* $\mathfrak{A}$ *(resp.* $\mathfrak{A}'$*).*

This Theorem implies that the fundamental strata in π can be categorised as the strata for which n/e is minimal amongst all strata in π. We call the invariant n/e the *depth* or *normalized level* of π.

We note that in the setting of Theorem 3.6.4, if we are in case (i) then we say π is of *depth-zero*; this means that π has fixed vectors under the pro-unipotent radical of the maximal parahoric subgroup $\mathrm{GL}_N(\mathfrak{o}_F)$ of G. If we are in the latter case then π is of *positive-depth*.

We note that the classification of depth-zero cuspidal representations of G is easier than that for positive-depth cuspidals, and so we split out attention into the two cases below.

3.6.3 Construction of Depth-Zero Cuspidal Representations

Here we give the outline of the construction of an irreducible cuspidal representation of G of depth-zero. Let $\mathfrak{A}$ be a principal hereditary order over A with period $e = e(\mathfrak{A})$. Set $N_e = N/e$. Then there exists a basis for V such that

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{o}_F \\ \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}$$

where each entry is a block of size $N_e \times N_e$. Moreover, the Jacobson radical $\mathfrak{P}$ has the form

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{o}_F \\ \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}.$$

The groups $U(\mathfrak{A})$ and $U^1(\mathfrak{A})$ have the property that

$$U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq \prod_{i=1}^{e} \mathrm{GL}_{N_e}(k_F).$$

A representation σ_0 of $\mathrm{GL}_{N_e}(k_F)$ is *cuspidal* if, for any proper parabolic subgroup $\mathcal{P}$ of $\mathrm{GL}_{N_e}(k_F)$ with unipotent radical $\mathcal{N}$, the restriction of σ_0 to $\mathcal{N}$ does not contain the trivial character of $\mathcal{N}$. We take σ_0 an irreducible cuspidal representation of $\mathrm{GL}_{N_e}(k_F)$ and form the tensor representation $\sigma = \sigma_0^{\otimes e}$, a representation of $U(\mathfrak{A})/U^1(\mathfrak{A})$. We inflate σ to a representation of $U(\mathfrak{A})$ which we also denote by σ. We then extend σ to a representation λ of the compact mod-center subgroup $J = \mathfrak{K}(\mathfrak{A})$ of G. The representation

$$\pi = \mathrm{ind}_J^G \lambda$$

is irreducible and cuspidal if and only if $e = 1$ i.e. if and only if $\mathfrak{A}$ is a maximal order and $\mathfrak{K}(\mathfrak{A}) = F^\times U(\mathfrak{A})$.

3.6.4 Construction of Positive-Depth Cuspidal Representations

Let $[\mathfrak{A}, n, m, \beta]$ be an arbitrary stratum in A. We say that $[\mathfrak{A}, n, m, \beta]$ is *pure* if

(i) the algebra $E = F[\beta]$ is a field;

(ii) $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$;

(iii) $\nu_{\mathfrak{A}}(\beta) = -n$.

If $[\mathfrak{A}, n, m, \beta]$ is pure, then we can consider V as an E-vector space. It is then natural to consider $B_\beta = \mathrm{End}_E(V)$ the centralizer of β in A. We write $\mathfrak{B}_\beta = \mathfrak{A} \cap B_\beta$ and $\mathfrak{Q}_\beta = \mathrm{rad}(\mathfrak{B}_\beta) = \mathfrak{P} \cap B_\beta$. Note that $\mathfrak{B}_\beta$ is a hereditary $\mathfrak{o}_E$-order with Jacobson radical $\mathfrak{Q}_\beta$. For fixed β, we define the map $a_\beta : A \to A$ by

$$a_\beta(x) = \beta x - x\beta, \text{ for } x \in A,$$

which is a (B_β, B_β)-bimodule homomorphism with kernel B_β. For $k \in \mathbb{Z}$, define

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} : a_\beta(x) \in \mathfrak{P}^k\}.$$

Then $\mathfrak{N}_k(\beta, \mathfrak{A})$ is a lattice in A since $\mathfrak{A} \supseteq \mathfrak{N}_k(\beta, \mathfrak{A}) \supseteq \mathfrak{P}^{k+n}$. Moreover, $\mathfrak{N}_k(\beta, \mathfrak{A}) \cap B_\beta = \mathfrak{B}_\beta$. For sufficiently large k we have $\mathfrak{N}_k(\beta, \mathfrak{A}) \subseteq \mathfrak{B}_\beta + \mathfrak{P}$. On the other hand, if k is sufficiently small, we have $\mathfrak{N}_k(\beta, \mathfrak{A}) = \mathfrak{A}$. This leads to the following important definition:

$$k_0(\beta, \mathfrak{A}) = \begin{cases} \max\{k \in \mathbb{Z} : \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subseteq \mathfrak{B}_\beta + \mathfrak{P}\} & \text{if } E \neq F, \\ -\infty & \text{if } E = F. \end{cases}$$

The reason why we set $k_0(\beta, \mathfrak{A}) = -\infty$ in the latter case is because $\mathfrak{A} = \mathfrak{B}_\beta + \mathfrak{P} = \mathfrak{N}_k(\beta, \mathfrak{A})$, for all $k \in \mathbb{Z}$. Suppose that the stratum $[\mathfrak{A}, n, m, \beta]$ is pure. If it satisfies $-m > k_0(\beta, \mathfrak{A})$ then we call $[\mathfrak{A}, n, m, \beta]$ *simple*. While we have a concrete definition for a simple stratum, calculating $k_0(\beta, \mathfrak{A})$ is difficult to do. Instead, we use the following alternate characterization of a simple stratum which does not rely on the value $k_0(\beta, \mathfrak{A})$ [BK93a, (2.4.1)(i)].

A pure stratum $[\mathfrak{A}, n, m, \beta]$ is called *simple* if, amongst all pure strata $[\mathfrak{A}, n, m, \beta']$ equivalent to $[\mathfrak{A}, n, m, \beta]$, the field extension $F[\beta]/F$ has minimal degree i.e. $[F[\beta] : F] \leq [F[\beta'] : F]$ for all equivalent pure strata $[\mathfrak{A}, n, m, \beta']$.

The first class of examples of simple strata is given by strata in which β is *minimal* over F. Let ν_E be the normalized additive valuation on $E = F[\beta]$ and write $\nu = \nu_E(\beta)$. Let $e(E \mid F)$ denote the ramification index of the field extension E/F. We say β is *minimal over* F if

(i) $\gcd(\nu, e(E \mid F)) = 1$;

(ii) the element $\varpi_F^{-\nu} \beta^{e(E|F)} + \mathfrak{p}_E \in k_E$ generates the residue class field extension k_E/k_F.

Moreover, the second condition is independent of the choice of uniformizer ϖ_F. If $E = F$, then β is always minimal over F. If β is minimal over F with $E = F[\beta]$ then it is possible to choose a hereditary order $\mathfrak{A}$ with the property $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$. One then simply sets $n = -\nu_{\mathfrak{A}}(\beta)$ to obtain a simple stratum $[\mathfrak{A}, n, m, \beta]$. We call this class of simple strata *minimal strata*. Note that the authors of [BK93a] call these strata *alfalfa*.

For brevity we now always consider β minimal over F (unless otherwise stated). This affords us the luxury of not having to define important (but superfluous to our needs) notions like defining sequences and the $\mathfrak{o}_F$-orders $\mathfrak{H}(\beta,\mathfrak{A}) \subseteq \mathfrak{J}(\beta,\mathfrak{A}) \subseteq \mathfrak{A}$. Let $[\mathfrak{A}, n, 0, \beta]$ be a minimal stratum with $\nu_{\mathfrak{A}}(\beta) = -n$. We define the groups

$$H = U(\mathfrak{B}_\beta)U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}),$$
$$J = U(\mathfrak{B}_\beta)U^{\lfloor \frac{n+1}{2} \rfloor}(\mathfrak{A}),$$

which have filtration subgroups

$$H^k = H \cap U^k(\mathfrak{A}),$$
$$J^k = J \cap U^k(\mathfrak{A}).$$

We have seen that for any stratum $[\mathfrak{A}, n, m, \beta]$ with $2m + 1 \geq n \geq m \geq 0$ we obtain a character ψ_β of $U^{m+1}(\mathfrak{A})$ trivial on $U^{n+1}(\mathfrak{A})$. We now wish to define characters θ of the group $H^{m+1}(\beta)$ which contain ψ_β.

Let $[\mathfrak{A}, n, 0, \beta]$ be a minimal stratum with $\nu_{\mathfrak{A}}(\beta) = -n$. Let $\det_{B_\beta} : B_\beta \to E^\times$ denote the determinant map. For $0 \leq m < n$, let $\mathcal{C}(\mathfrak{A}, m, \beta)$ denote the set of characters θ of $H^{m+1}(\beta)$ such that

(i) $\theta \mid_{H^{m+1}(\beta) \cap U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A})} = \psi_\beta$;

(ii) $\theta \mid_{H^{m+1}(\beta) \cap B_\beta^\times}$ factors through $\det_{B_\beta}$.

Note that the restriction of ψ_β to $U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$ and that the G-normalizer $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes ψ_β. We call such $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ *simple characters.*

Remark 3.6.5. In the case that β is not minimal over F, the authors of [BK93a] give an inductive definition of simple characters and an algorithm to compute them. This relies on the notion of defining sequence which we have not covered here. For our purposes we need only know that these characters exist and have the properties (i) and (ii) above.

We started with characters ψ_β of $U^{m+1}(\mathfrak{A})$ which we could then extend to simple characters $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ of the group $H^1(\beta)$. By definition we see that we have inclusions

$$H^1(\beta) \subseteq J^1(\beta) \subseteq J^0(\beta) = J(\beta).$$

Thus it follows that we wish to extend further our simple characters of $H^1(\beta)$ to the groups $J^1(\beta)$ and $J(\beta)$. The first of these steps is the simpler of the two.

Proposition 3.6.6. *[BK93a, (5.1.1)] Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A and $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. There exists a unique irreducible representation $\eta(\theta)$ of $J^1(\beta)$ with the property that $\eta(\theta) \mid_{H^1(\beta)} = \theta^{\bigoplus t}$ where $t = [J^1(\beta) : H^1(\beta)]$. Moreover, the G-intertwining of $\eta(\theta)$ is $J^1(\beta)B_\beta^\times J^1(\beta)$.*

We call $\eta(\theta)$ a *Heisenberg extension* of θ. Now given a representation $\eta(\theta)$ of $J^1(\beta)$, we no longer have such a nice choice of extension as we did in the previous step. There are many possible extensions of $\eta(\theta)$ to a representation of $J(\beta)$, not all having the desired properties. This leads us to the notion of a β-extension of $\eta(\theta)$.

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ a simple character and η the Heisenberg extension of θ. A *β-extension of* η is a representation κ of $J(\beta)$ satisfying

(i) $\kappa \mid_{J^1(\beta)} = \eta$;

(ii) $B_\beta^\times \subseteq \mathcal{I}_G(\kappa)$.

If we take χ any character of $\mathfrak{o}_E^\times$ trivial on $1 + \mathfrak{p}_E$ then $\chi \circ \det_{B_\beta}$ defines a character of the quotient $U(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$. Using the canonical isomorphism between $U(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$ and $J(\beta)/J^1(\beta)$, which follows from the definitions and theorems, we can view $\chi \circ \det_{B_\beta}$ as a character of $J(\beta)/J^1(\beta)$. This leads to the following Theorem.

Theorem 3.6.7. *[BK93a, (5.2.2)] Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ a simple character with Heisenberg extension η.*

(i) There exists a β-extension κ of η.

(ii) If κ is a β-extension of η, then all other β-extensions are of the form $\kappa \otimes (\chi \circ \det_{B_\beta})$ for some character $(\chi \circ \det_{B_\beta})$ of $U(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$.

(iii) Distinct characters χ give rise to non-isomorphic representations $\kappa \otimes (\chi \circ \det_{B_\beta})$ which do not intertwine.

We are now able to define simple types, which will result in us being able to state the main theorems of [BK93a], namely the classification of cuspidal representations of $\mathrm{GL}_N(F)$. Let J be a compact open subgroups of G and λ and irreducible representation of J. We call the pair (J, λ) a *simple type* if it is one of the following [BK93a, (5.5.10)]:

(1) $(J, \lambda) = (J^0(\beta), \kappa \otimes \sigma)$ where

  (i) the stratum $[\mathfrak{A}, n, 0, \beta]$ is simple with $\mathfrak{A}$ a principal hereditary $\mathfrak{o}_F$-order in A;

  (ii) for some simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, κ is a β-extension of the Heisenberg representation $\eta(\theta)$;

  (iii) let $E = F[\beta]$ and $e = e(\mathfrak{B}_\beta)$ so that σ is the inflation to $J^0(\beta)$ of $\sigma^{\otimes e}$ where σ_0 is an irreducible cuspidal representation of $\mathrm{GL}_{N/[E:F]}(k_E)$.

(2) $(J, \lambda) = (U(\mathfrak{A}), \sigma)$ where $e = e(\mathfrak{B}_\beta)$, $E = F$, $\mathfrak{A}$ is a principal hereditary $\mathfrak{o}_F$-order and σ is the inflation of $\sigma_0^{\otimes e}$ for σ_0 an irreducible cuspidal representation of $\mathrm{GL}_{N/e(\mathfrak{A})}(k_F)$.

In fact, the distinction here is not necessary. We can view case (2) as a special case of case (1) by setting $E = F$, $\mathfrak{B} = \mathfrak{A}$, $J^n(\beta, \mathfrak{A}) = U^n(\mathfrak{A})$ and taking θ, η, κ all trivial.

Furthermore, a *maximal simple type* is a simple type (J, λ) for which $e(E \mid F) = e(\mathfrak{A})$. We are now ready to state the follow Theorem which summarises the main results of [BK93a].

Theorem 3.6.8. *(1) Let π be an irreducible cuspidal representation of G. Then π contains some simple type (J, λ).*

(2) If π is an irreducible cuspidal representation of G then π contains a maximal simple type (J, λ) with multiplicity 1. Moreover, if π contains two maximal simple types (J_1, λ_1) and (J_2, λ_2), then (J_1, λ_1) and (J_2, λ_2) are conjugate in G.

(3) Let (J, λ) be a maximal simple type. If π is an irreducible representation of G which contains λ, then π is cuspidal. Moreover, if π' is another irreducible representation of G which contains λ, then $\pi' \simeq \pi \otimes (\chi \circ \det_G)$ for χ an unramified character of $F^\times$ and $\det_G : G \to F^\times$ the determinant map i.e. π and π' are inertially equivalent.

(4) Let (J, λ) be a maximal simple type and π an irreducible cuspidal representation of G containing λ. Then there exists a unique representation Λ of $E^\times J$ extending λ such that

$$\pi \simeq \mathrm{ind}_{E^\times J}^{G} \Lambda.$$

Chapter 4

Depth-Zero L-parameters of Classical Groups

4.1 Depth Zero Cuspidal Representations

In this section we concern ourselves with recalling the classification of depth zero irreducible cuspidal representations of $\mathrm{GL}_N(F)$ and a classical group G, by which we mean a Symplectic group or Split Special Orthogonal group.

We write $\mathrm{Cusp}_N(F)$ for the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_N(F)$. We set $\mathrm{Cusp}(F) = \bigcup_{N\geq 1} \mathrm{Cusp}_N(F)$ with the understanding that $\pi \in \mathrm{Cusp}(F)$ is an irreducible cuspidal representation of some $\mathrm{GL}_N(F)$.

A representation π is said to be *self-dual* if π is isomorphic to its dual representation; we write $\mathrm{Cusp}^*_N(F)$ for the set of self-dual irreducible cuspidal representations of $\mathrm{GL}_N(F)$, and $\mathrm{Cusp}^*(F) = \bigcup_{N\geq 1} \mathrm{Cusp}^*_N(F)$.

We recall that a representation $\pi \in \mathrm{Cusp}(F)$ is of *depth zero* if there exist non-zero vectors which are fixed by the pro-p-radical of the maximal parahoric subgroup $\mathrm{GL}_N(\mathfrak{o}_F)$ of $\mathrm{GL}_N(F)$. The set of equivalence classes of irreducible cuspidal representations contained in $\mathrm{Cusp}(F)$ of depth zero is denoted $\mathrm{Cusp}_{[0]}(F)$. Similarly, we write $\mathrm{Cusp}^*_{[0]}(F)$ for the set of equivalence classes of depth zero self-dual irreducible cuspidal representations, which is contained in $\mathrm{Cusp}(F)$.

Any $\pi \in \mathrm{Cusp}_{[0]}(F)$ is of the form

$$\pi = \mathrm{ind}_{F^\times \mathrm{GL}_N(\mathfrak{o}_F)}^{\mathrm{GL}_N(F)} \omega_\pi \lambda_\pi$$

where λ_π is the inflation of an irreducible cuspidal representation τ_π from the finite reductive quotient $\mathrm{GL}_N(k_F)$. One then extends λ_π to $F^\times \mathrm{GL}_N(\mathfrak{o}_F)$ by the central character ω_π and compactly induces to π. Provided the representation λ_π is self-dual and ω_π is quadratic, then its compactly induced representation π is also self-dual [Adl97].

We write $\mathrm{Cusp}(G)$ for the set of equivalence classes of irreducible cuspidal representations of G, which contains the set of equivalence classes of depth zero cuspidal representations of the classical group G, which we denote $\mathrm{Cusp}_{[0]}(G)$. Any $\sigma \in \mathrm{Cusp}_{[0]}(G)$ can be written as

$$\sigma = \mathrm{ind}_{J_\sigma}^{G} \Lambda_\sigma$$

where J_σ is the normalizer of a maximal parahoric J_π° of G, itself a classical group. Moreover, $\Lambda_\sigma \mid_{J^\circ\pi} = \lambda_\pi$ is the inflation of $\tau_\sigma = \tau_\sigma^{(1)} \otimes \tau_\sigma^{(2)}$ an irreducible cuspidal representation of the finite reductive quotient $J_\sigma^\circ / J_\sigma^1 \simeq \mathcal{G}_{N_1} \times \mathcal{G}_{N_2}$. The integers N_i satisfy $N_1 + N_2 = \dim G$ and are wholly determined by σ.

4.2 Covers and Hecke Algebras

Let π be a depth-zero irreducible cuspidal representation of $\mathrm{GL}_{m_\pi}(F)$ and σ be a depth-zero irreducible cuspidal representation of a classical group G. We naturally view $\mathcal{M} \simeq \mathrm{GL}_{m_\pi}(F) \times G$ as a maximal Levi subgroup of G' a larger classical group of the same type as G. We now construct a cover in the sense of Bushnell–Kutzko using the local data describing π and σ, as given in the previous section.

Write $\mathcal{P}^+ = \mathcal{M}\mathcal{N}^+$ for a parabolic subgroup of G' with Levi factor $\mathcal{M}$ and denote by $\mathcal{P}^- = \mathcal{M}\mathcal{N}^-$ for the opposite parabolic subgroup to $\mathcal{P}^+$. Set $J_\mathcal{M} = \mathrm{GL}_{m_\pi}(\mathfrak{o}_F) \times J_\sigma$ a compact open subgroup of $\mathcal{M}$ and $\lambda_\mathcal{M} = \lambda_\pi \otimes \lambda_\sigma$ an irreducible representation of $J_\mathcal{M}$. The pair $(J_\mathcal{M}, \lambda_\pi)$ is a type for $\mathcal{M}$.

Recall from [BK98, (8.1)] that there exist a compact open subgroup J of G and a representation λ of J satisfying:

i) $J \cap \mathcal{M} = J_\mathcal{M}$;

ii) $\lambda\mid_{J_\mathcal{M}} = \lambda_\mathcal{M}$;

iii) $\lambda\mid_{J\cap\mathcal{N}^\pm}$ is trivial.

Whilst J is not itself a maximal compact subgroup of G', it is contained in the intersection of two maximal compact subgroups, namely $J_1 := J_{N_1+m_\pi,N_2}$ and $J_2 := J_{N_1,N_2+m_\pi}$. The reductive quotients J_1/J_1^1, J_2/J_2^1 are isomorphic to $\mathcal{G}_{(1)} := \mathcal{G}_{N_1+m_\pi,N_2}$, $\mathcal{G}_{(2)} := \mathcal{G}_{N_1,N_2+m_\pi}$ respectively. The maximal compacts J_i come equipped with Weyl group elements $s_i \in J_i$. These elements are denoted $s_1, s_2 = s_1^\varpi$ in [Ste08, Section 6.2] and interchange (up to scalars) the GL_{m_π} factors in $\mathcal{M}$ whilst stabilizing the block associated to G.

The embedding of J into the maximal compact subgroup J_i give rise to the following commutative diagram:

$$\begin{tikzcd} J \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & J_i \arrow[d, two heads] \\ \mathcal{P}^{(i)} \arrow[r, hook] & J_i/J_i^1 \end{tikzcd}.$$

Here $\mathcal{P}^{(i)}$ are parabolic subgroups: they are the image in J_i/J_i^1 of the parahoric J° associated to J. The parabolics have Levi factors $\mathcal{M}^{(i)} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathcal{G}_{N_1} \times \mathcal{G}_{N_2}$. We write $\mathcal{G}_{N_i}^{(m_\pi)} := \mathrm{GL}_{m_\pi}(k_F) \times \mathcal{G}_{N_i}$.

The embedding of J into the maximal compact J_i also gives rise to an embedding of spherical Hecke algebras

$$\mathrm{End}_{\mathcal{G}_{(i)}}(\mathrm{Ind}\,_{\mathcal{G}_{N_i}^{(m_\pi)}}^{\mathcal{G}_{(i)}} \tau^{(i)}) = \mathcal{H}(\mathcal{G}_{(i)}, \tau^{(i)}) \simeq \mathcal{H}(J_i, \lambda) \hookrightarrow \mathcal{H}(G, \lambda),$$

for $\tau^{(i)}$ a representation of the parabolic $\mathcal{P}^{(i)}$ which satisfies $\tau^{(i)} \mid_{\mathcal{G}_{N_i}^{(m_\pi)}} = \tau_\pi \otimes \tau_\sigma^{(i)}$. The endomorphism algebra $\mathrm{End}_{\mathcal{G}_{(i)}}(\mathrm{Ind}\,_{\mathcal{G}_{N_i}^{(m_\pi)}}^{\mathcal{G}_{(i)}} \tau^{(i)})$ is two-dimensional, so $\mathrm{Ind}\,_{\mathcal{G}_{N_i}^{(m_\pi)}}^{\mathcal{G}_{(i)}} \tau^{(i)} = \tau'_{(i)} \oplus \tau''_{(i)}$ with $\dim \tau'_{(i)} \geq \dim \tau''_{(i)}$ [HL80, 3.18,4.5]. We take $\overline{T}_i \in \mathcal{H}(\mathcal{G}_{(i)}, \tau')$ with support on the non-trivial double coset $\mathcal{P}^{(i)} s_i \mathcal{P}^{(i)}$ and which satisfies the quadratic relation $(\overline{T}_i + 1)(\overline{T}_i - q^{r_i}) = 0$, where (up to normalization)

$$q^{r_i} := \frac{\dim \tau'_{(i)}}{\dim \tau''_{(i)}}.$$

Through the embedding, this element corresponds to $T_i \in \mathcal{H}(G', \lambda)$ which is supported on the non-trivial double coset $J s_i J$, which also satisfies the quadratic relation $(T_i + 1)(T_i - q^{r_i}) = 0$.

We form the element $\phi = T_2 T_1$, which is invertible since each T_i is invertible, with support

$$\begin{aligned} \mathrm{supp}\,(\phi) &\subseteq J s_2 J s_1 J \\ &= J \underbrace{s_2 (J \cap \mathcal{N}^-) s_2^{-1}}_{\subseteq J \cap \mathcal{N}^+} \underbrace{s_2 (J \cap \mathcal{M}) s_2^{-1}}_{\subseteq J \cap \mathcal{M}} s_2 s_1 \underbrace{s_1^{-1} (J \cap \mathcal{N}^+) s_1}_{\subseteq J \cap \mathcal{N}^-} J \\ &= J s_2 s_1 J. \end{aligned}$$

Thus ϕ is an invertible element of $\mathcal{H}(G', \lambda)$ which is supported on the double coset Js_2s_1J, with s_2s_1 a strongly positive element of the centre of $\mathcal{M}$, showing that (J, λ) is a cover of $(J_{\mathcal{M}}, \lambda_{\mathcal{M}})$.

4.3 Reducibility of Parabolic Induction and the Jordan Set

We are motivated in this section to understand the nature of reducibility of parabolically induced representations. More precisely, we want to answer this question when we consider G as part of a maximal Levi subgroup $\mathcal{M} \simeq \mathrm{GL}_{m_\pi}(F) \times G$ of a larger classical group G' of the same type as G. This means we concern ourselves with the parabolically induced representation

$$I(\pi, s, \sigma) = \mathrm{Ind}\,^{G'}_{\mathcal{M},\mathcal{P}}\,\pi\ \mid \det \mid^{\,s} \otimes \sigma$$

for $s \in \mathbb{C}$. In each inertial equivalence class $[\pi] = \{\pi \mid \det \mid^{\,t} : t \in \mathbb{C}\}$ it is sufficient to fix one representation π and consider $I(\pi, s, \sigma)$, since $I(\pi \mid \det \mid^{\,t}, s, \sigma) = I(\pi, s + t, \sigma)$.

The following Theorems due to Silberger, the first coming from [Sil79, 5.4.2.2 − 3], and the second from [Sil80, Theorem 1.6], tells us the importance of self-dual representations in our situation. Note that the results of Silberger apply to arbitrary representations π, not just depth-zero representations of a classical group G.

Theorem 4.3.1. *(i) If there exists $s \in \mathbb{R}$ such that $I(\pi, s, \sigma)$ is reducible, then there exists $t \in \mathbb{R}$ such that $\pi|\det|^{\,t}$ is self-dual.*

(ii) Suppose $I(\pi, s, \sigma)$ is reducible for some $s \in \mathbb{R}$ and π self-dual. Then there exists a unique real number $s_\sigma(\pi) \in \mathbb{R}_{\geq 0}$ such that, for $s \in \mathbb{R}$, $I(\pi, s, \sigma)$ is reducible if and only if $s = \pm s_\sigma(\pi)$.

Remark 4.3.2. We note that while Silberger's result only tells us when we obtain *real* points of reducibility, we are able to extrapolate from it points of complex reducibility. This is because if π is a self-dual irreducible cuspidal representation of $\mathrm{GL}_{m_\pi}(F)$ of depth zero, then there are two unramified twists of π which are self-dual: namely π and $\pi' := \pi \mid \det \mid^{\frac{\pi i}{m \log q}}$.

Knowing the reducibility points of parabolically induced representations has the following impact for the local Langlands correspondence. In the same way as [LS15] and [BHS18], we define $\mathrm{Red}(\sigma)$ as the set of isomorphism classes of cuspidal representations π of some $\mathrm{GL}_{m_\pi}(F)$, with $m_\pi \geq 1$, such that $n := 2s_\sigma(\pi) - 1 \in \mathbb{Z}$ is non-negative. We then define the *Jordan set* $\mathrm{Jord}(\sigma)$ as the set of pairs (π, n) such that $n \geq 1$ and $(\pi, n+2k) \in \mathrm{Red}(\sigma)$.

Using the language of Jordan sets, Mœglin [Mg14] gives a criterion in which one is (hypothetically) able to determine the Langlands parameter ϕ for a given irreducible cuspidal representation σ of G. Explicitly, let $\mathcal{W}_F$ denote the Weil group of F and ${}^LG$ be the Langland's dual group of G of dimension $N_{{}^LG}$. Assume G is split. Let ϕ be the Langlands parameter

$$\phi : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$$

whose L-packet $\prod_\phi$ contains σ (as conjectured by the local Langlands correspondence). Let ι denote the natural injection from ${}^LG$ into $\mathrm{GL}_{N_{{}^LG}}(\mathbb{C}) \times \mathcal{W}_F$. If one has an explicit description of $\mathrm{Jord}(\sigma)$ then one expects ϕ to be of the form

$$\iota \circ \phi = \bigoplus_{(\pi,n) \in \mathrm{Jord}(\sigma)} \phi_\pi \otimes \mathrm{St}_n$$

where ϕ_π is the irreducible representation of $\mathcal{W}_F$ corresponding to π via the local Langlands correspondence for $\mathrm{GL}_{m_\pi}(F)$, and St_n is the unique irreducible n-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. This result implies the following equality:

$$\sum_{(\pi,n) \in \mathrm{Jord}(\sigma)} m_\pi n = N_{{}^LG},$$

which is equivalent to

$$\sum_{\pi \in \mathrm{Cusp}(F)} \left\lfloor s_\sigma(\pi)^2 \right\rfloor m_\pi = N_{{}^LG}$$

since all but finitely many $s_\sigma(\pi)$ are 0 or $\frac{1}{2}$ so $\lfloor s_\sigma(\pi) \rfloor^2 = 0$.

Remark 4.3.3. This sum includes both depth-zero and positive-depth irreducible cuspidal representations π of $\mathrm{GL}_{m_\pi}(F)$. Whilst in this thesis we only consider $(\pi, n) \in \mathrm{Jord}(\sigma)$ with π of depth-zero, we do not verify that there is no contribution from positive-depth cuspidal representations. However, we later see that, at least for certain groups G,

$$\sum_{\substack{(\pi,n)\in\mathrm{Jord}(\sigma)\\ \pi\in\mathrm{Cusp}(F)\text{ of depth zero}}} \left\lfloor s_\sigma(\pi)^2 \right\rfloor m_\pi = N_{{}^LG}$$

which, when combined with Mœglin's result, implies that the we have found all of $\mathrm{Jord}(\sigma)$.

4.4 A Result of Blondel

Our problem of finding reducibility points for parabolic induction reduces to finding the numbers $\pm s_\sigma(\pi)$ and $\pm s_\sigma(\pi')$. The following Proposition, due to Blondel [Blo12, 3.12], shows the connection between the points of reducibility for the parabolically induced representation $I(\pi, s, \sigma)$ and the eigenvalues of the generators for the spherical Hecke algebra $\mathcal{H}(G', \lambda)$ in Section 4.2.

We note that Blondel works under the hypothesis that the residue characteristic is odd. This is necessary since she considers positive-depth cuspidal representations of a classical group, which were classified by [Ste08] in the case of odd residue characteristic. Moreover, she uses the construction of covers given in [MS14] which only holds under this assumption. However, since the classification of depth-zero cuspidal representations of an arbitrary connected reductive algebraic group is known with no restrictions on residue characteristic, her result stands with only minor modifications. Namely we use the explicit construction of a cover for a maximal Levi given in Section 4.2.

Let $\mathcal{M}$ be a maximal Levi subgroup of G', so $\mathcal{M} \simeq \mathrm{GL}_m(F) \times G$. Take π an irreducible cuspidal representation of $\mathrm{GL}_m(F)$ and σ an irreducible cuspidal representation of G, both of depth-zero. We can therefore write $\pi \simeq \mathrm{ind}_{F^\times J_\pi}^{\mathrm{GL}_m(\mathfrak{o}_F)} \Lambda_\pi$ and $\sigma \simeq \mathrm{ind}_{J_\sigma}^{G} \lambda_\sigma$ as described in Section 4.1. The type (J, λ) is then a G'-cover of $(J_\mathcal{M}, \lambda_\mathcal{M}) = (J_\pi \times J_\sigma, \lambda_\pi \otimes \lambda_\sigma)$. Moreover, we saw that the spherical Hecke algebra $\mathcal{H}(G', \lambda) = \mathrm{End}_{G'}(\mathrm{ind}_J^{G'} \lambda)$ is two-dimensional with generators T_1, T_2 subject only to the quadratic relations

$$(T_i + 1)(T_i - q^{r_i}) = 0,$$

for $i = 1, 2$ and $r_i \in \mathbb{R}$ non-negative.

Proposition 4.4.1. *Let $\mathcal{M}, \pi, \sigma$ be as above. The real parts of the points of reducibility of the parabolically induced representation* $\mathrm{Ind}_{\mathcal{M},\mathcal{P}}^{G'} \pi|\det|^s \otimes \sigma$ *are*

$$\{\pm s_\sigma(\pi), \pm s_\sigma(\pi')\} = \left\{\pm\frac{(r_1 + r_2)}{2m}, \pm\frac{(r_1 - r_2)}{2m}\right\}.$$

Proof. Since $(J_{\mathcal{M}}, \lambda_{\mathcal{M}})$ is an $\mathcal{M}$-type, using the Bernstein Decomposition of $\mathcal{R}(\mathcal{M})$, the block $\mathcal{R}^{[\pi\otimes\sigma]}(\mathcal{M})$ is the full subcategory of $\mathcal{R}(\mathcal{M})$ consisting of elements whose irreducible subquotients are representations of $\mathcal{M}$ which are unramified twists of $\pi \otimes \sigma$. The functor $\mathbf{m}_{\mathcal{M}} : \mathcal{R}^{[\pi\otimes\sigma]}(\mathcal{M}) \to \text{Mod–}\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ which sends the representation τ to the module $\mathrm{Hom}_{J_{\mathcal{M}}}(\lambda_{\mathcal{M}}, \tau)$ is an equivalence of categories.

Similarly, using the Bernstein decomposition for $\mathcal{R}(G')$ we have the block $\mathcal{R}^{[\pi\otimes\sigma]}(G')$ corresponding to the type (J, λ) is the full subcategory of $\mathcal{R}(G')$ whose irreducible subquotients are representations of G' which have supercuspidal support an unramified twist of $\pi \otimes \sigma$. The functor $\mathbf{m}_G{}' : \mathcal{R}^{[\pi\otimes\sigma]}(G') \to \text{Mod-}\mathcal{H}(G', \lambda)$ which sends the representation τ to the module $\mathrm{Hom}_J(\lambda, \tau)$, which again gives an equivalence of categories.

As (J, λ) is a cover of $(J_{\mathcal{M}}, \lambda_{\mathcal{M}})$, we have a normalized embedding of spherical Hecke algebras $t : \mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}}) \hookrightarrow \mathcal{H}(G', \lambda)$ which gives the following commutative diagram:

$$\begin{tikzcd}
\mathcal{R}^{[\pi\otimes\sigma]}(G') \arrow[r, "\mathbf{m}_{G'}"] & \mathrm{Mod} - \mathcal{H}(G', \lambda) \\
\mathcal{R}^{[\pi\otimes\sigma]}(\mathcal{M}) \arrow[u, "\mathrm{Ind}_{\mathcal{P}}^{G'}"] \arrow[r, "\mathbf{m}_{\mathcal{M}}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Mod} - \mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}}) \arrow[u, "t_*"']
\end{tikzcd}.$$

Here $\mathrm{Ind}_{\mathcal{P}}^{G'}$ denotes the functor of parabolic induction and t_* is the functor mapping a module $X \in \text{Mod-}\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ to $\mathrm{Hom}_{\mathcal{H}(\mathcal{M},\lambda_{\mathcal{M}})}(\mathcal{H}(G', \lambda), X)$ with the module structure of $\mathcal{H}(G', \lambda)$ given by the embedding t.

Since the diagram is commutative, the representation $\mathrm{Ind}_{\mathcal{P}}^{G'} \pi|\det|^s \otimes \sigma$ is reducible if and only if the module $t_*\mathbf{m}_{\mathcal{M}}(\mathrm{Ind}_{\mathcal{M}}^{G'} \pi|\det|^s \otimes \sigma)$ is reducible. We therefore need to know when the module t_*X of an irreducible module $X \in \text{Mod-}\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ is reducible.

Since $(J_{\mathcal{M}}, \lambda_{\mathcal{M}})$ is a G'-cover, we know that the spherical Hecke algebra $\mathcal{H}(G', \lambda)$ has two generators T_1, T_2 subject to the quadratic relations

$$(T_1 + 1)(T_i - q^{r_i}) = 0$$

for $i = 1, 2$ (see section 4.2). Moreover, the element T_2T_1 is supported on the double coset $J\zeta J$ for ζ a strongly-positive element of the centre of $\mathcal{M}$. The spherical Hecke algebra $\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ is isomorphic to $\mathcal{H}(\mathrm{GL}_m(F), \lambda_\pi)$, which in turn is isomorphic to $\mathbb{C}[Z^{\pm 1}]$ by [BK93a, Section 5.5] with Z supported on $\zeta J_{\mathcal{M}}$. The irreducible representations of this algebra are characters defined by their value on Z. Since $t(Z)$ also has support on the double coset $J\zeta J$, we normalize Z so that $t(Z) = T_2T_1$.

The group $X_0(\mathrm{GL}_m(F))$ of unramified characters of $\mathrm{GL}_m(F)$ acts on $\mathcal{H}(\mathrm{GL}_m(F), \lambda_\pi)$ by

$$(\chi f)(x) = \chi(x) f(x)$$

where $\chi \in X_0(\mathrm{GL}_m(F)), f \in \mathcal{H}(\mathrm{GL}_m(F), \lambda_\pi)$ and $x \in \mathrm{GL}_m(F)$. If $\pi \in \mathrm{Irr}(\mathrm{GL}_m(F))$ and $\chi \in X_0(\mathrm{GL}_m(F))$ the image of $\pi \otimes \chi$ under $\mathbf{m}_{\mathcal{M}}$ is the character of $\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ defined by

$$\mathbf{m}_{\mathcal{M}}(\pi \otimes \chi)(Z) := \chi^{-1}(\varpi_F)\mathbf{m}_{\mathcal{M}}(\pi)(Z). \tag{$\circ$}$$

Since $\text{Mod-}\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ is a commutative ring, all simple modules are 1-dimensional. The embedding $t(\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}}))$ has index 2 in $\mathcal{H}(G', \lambda)$, so for any simple $\mathcal{H}(\mathcal{M}, \lambda_{\mathcal{M}})$ module M, the $\mathcal{H}(G', \lambda)$ module $t_*(M)$ is 2-dimensional. Such a module is reducible if and only if it contains a 1-dimensional submodule. Suppose V is a 1-dimensional $\mathcal{H}(G', \lambda)$-module. Then for any $v \in V$ we have

$$v \cdot T_i = \lambda_i v_i$$

for some $\lambda_i \in \mathbb{C}^\times$ and $i = 1, 2$. The quadratic relations for the T_i give the possible values for λ_i, namely

$$\lambda_i \in \{-1, q^{r_i}\}$$

for $i = 1, 2$. This gives at most 4 possible 1-dimensional modules V. If V were now a submodule of $t_*(M)$, then by adjunction we have that $V\mid_{\mathcal{H}(\mathcal{M},\lambda_{\mathcal{M}})} = M$. These four $\mathcal{H}(\mathcal{M},\lambda_{\mathcal{M}})$-modules M are precisely the modules for which $t_*(M)$ is reducible. On the modules, Z acts as T_2T_1, i.e. with eigenvalue in

$$\{1, -q^{r_1}, -q^{r_2}, q^{r_1+r_2}\} \tag{†}$$

Suppose now that π is chosen such that $\operatorname{Ind}_{\mathcal{M},\mathcal{P}}^{G'} \pi \mid \det\mid^s \otimes \sigma$ is reducible for some $s \in \mathbb{R}$. Then Theorem 4.3.1 tells us that $s = \pm s_1$ for some non-negative $s_1 \in \mathbb{R}$. Moreover, we know that given such a π there is a unique inequivalent unramified twist π' of π with π' is self-dual, namely $\pi' = \pi \mid \det \mid^{\frac{\pi i}{n \log q}}$. Again this is reducible for $s' = \pm s_2$ with $s_2 \in \mathbb{R}$ non-negative. This gives at most 4 points of reducibility, associated to the representations

$$\begin{aligned}\{\operatorname{Ind}_{\mathcal{M}}^{G'} \pi|\det|^{s_1} \otimes \sigma, \operatorname{Ind}_{\mathcal{M}}^{G'} \pi|\det|^{-s_1} \otimes \sigma,&\\ \operatorname{Ind}_{\mathcal{M}}^{G'} \pi|\det|^{s_2+\frac{\pi i}{n\log q}} \otimes \sigma, \operatorname{Ind}_{\mathcal{M}}^{G'} \pi|\det|^{-s_2+\frac{\pi i}{n\log q}} \otimes \sigma\}.&\end{aligned}$$

Using (∘) the representations $\pi|\det|^{\pm s_1}$ and $\pi|\det|^{\pm s_2+\frac{\pi i}{n\log q}}$ correspond to the simple modules in $\mathcal{H}(\mathcal{M},\lambda_{\mathcal{M}})$ on which Z acts by

$$\left\{q^{s_1 n}\mathbf{m}_{\mathcal{M}}(\pi)(Z), q^{-s_1 n}\mathbf{m}_{\mathcal{M}}(\pi)(Z), -q^{s_2 n}\mathbf{m}_{\mathcal{M}}(\pi)(Z), -q^{-s_2 n}\mathbf{m}_{\mathcal{M}}(\pi)(Z)\right\}. \tag{‡}$$

The sets (†) and (‡) must coincide. By taking quotients of pairs of elements of each set, and then looking at which pairs give a positive quotient, we find that

$$\{\pm s_1, \pm s_2\} = \left\{\pm\frac{(r_1+r_2)}{2m}, \pm\frac{(r_1-r_2)}{2m}\right\}.$$

□

4.5 Jordan Decomposition of Characters

Let G be a linear algebraic group. Its Jordan decomposition means that we can write every $g \in G$ uniquely as $g = su$ with s semisimple and u unipotent such that s, u commute. If G is abelian, then G is isomorphic to the group $\hat{G}$ of characters $\chi : G \to \mathbb{C}$. In this way we obtain a Jordan decomposition for $\hat{G}$. The idea behind the work of Lusztig [Lus77] is to do this for G non-abelian. To keep with normal notation for finite reductive groups on this matter, the notation used in this section is independent from the rest of the thesis.

In order to classify the unipotent cuspidal representations of our finite classical groups, we need to introduce the notion of self-dual polynomials. Let $k = \mathbb{F}_q$ be a finite field of characteristic 2. An irreducible polynomial $P \in k[X]$ is *self-dual* if $P(0)P(X) = X^{\deg P} P(X^{-1})$.

Suppose P is a self-dual irreducible polynomial of odd degree. We write $P(X) = a_{\deg P} X^{\deg P} + \cdots + a_1 X + a_0$ so $a_0 = P(0) \neq 0$. By definition, the coefficients of P satisfy $a_0 a_i = a_{\deg P - i}$ for all i, and so an even number of the a_i are non-zero. This implies $P(1) = 0$. By irreducibility of P it follows that the only self-dual irreducible polynomial of odd degree is precisely $X + 1$. For P self-dual irreducible of even degree, let k_P be a degree P extension of k and k_P° be the degree $(P/2)$ extension of k contained in k_P.

Let G be a classical group defined over $\bar{k}$. Denote by $\mathcal{F}$ the standard Frobenius map which raises each coefficient of g to the q^{th} power. The fixed points of G under $\mathcal{F}$ is the classical group $G^{\mathcal{F}}$ defined over the finite field k. By classical group we mean G is one of the following types:

(a) $G^{\mathcal{F}} = \mathrm{Sp}_{2n}(k)$ (for $n \geq 1$);

(b) $G^{\mathcal{F}} = \mathrm{SO}^{\pm}_{2n}(k)$ (for $n \geq 2$).

Remark 4.5.1. Recall from Proposition 2.5.2 that for finite fields of characteristic 2 the groups Sp_{2n} and O_{2n+1} are isomorphic. We therefore need only consider Special Orthogonal groups of even dimension.

The group G is defined by its root datum, and by taking the dual root datum, we obtain the dual group G^* to G. Writing $\mathcal{F}$ for the standard Frobenius map on the G^* we have that $G^{*\mathcal{F}}$ is a finite group dual to $G^{\mathcal{F}}$. In particular

(a) $G^{*\mathcal{F}} = \mathrm{SO}_{2n+1}(k)$ (for $n \geq 1$);

(b) $G^{*\mathcal{F}} = \mathrm{SO}^{\pm}_{2n}(k)$ (for $n \geq 2$).

Denote by $\mathcal{E}(G^{\mathcal{F}})$ the set of equivalence classes of complex irreducible representations of $G^{\mathcal{F}}$. This set has a partition into *geometric Lusztig series*

$$\mathcal{E}(G^{\mathcal{F}}) = \bigsqcup_{s} \mathcal{E}(G^{\mathcal{F}}, s)$$

where s runs over conjugacy classes of semisimple elements of $G^{*\mathcal{F}}$. We now describe this partition.

Let $\mathcal{T}$ be any $\mathcal{F}$-stable maximal torus in G^* containing s and $R^s_{\mathcal{T}}$ be the corresponding Deligne–Lusztig character [Car85, Proposition 7.2.3]. An irreducible representation $\rho \in \mathcal{E}(G^{\mathcal{F}})$ lies in $\mathcal{E}(G^{\mathcal{F}}, s)$ if and only if

$$\langle R^s_{\mathcal{T}}, \rho \rangle = \frac{1}{\mid G^{\mathcal{F}} \mid} \sum_{g \in G^{\mathcal{F}}} R^s_{\mathcal{T}}(g) \mathrm{tr}(\overline{\rho(g^{-1})}) \neq 0.$$

One can also obtain a criterion for checking whether a given representation $\rho \in G^{\mathcal{F}}$ is cuspidal. A representation ρ is cuspidal if and only if, for any pair $(\mathcal{T}, s)$ with $\mathcal{T}$ an $\mathcal{F}$-stable maximal torus contained in a proper $\mathcal{F}$-stable parabolic subgroup of $G^{*\mathcal{F}}$, we have $\langle R^s_{\mathcal{T}}, \rho \rangle = 0$.

We wish to be able to obtain information about cuspidal representations appearing in a particular $\mathcal{E}(G^{\mathcal{F}}, s)$, in particular we want to know the dimensions of these representations. This motivates the following definition. An irreducible representation ρ is *unipotent* if it appears in $\mathcal{E}(G^{\mathcal{F}}, 1)$.

Write $G^{*\mathcal{F}}_s$ for the centralizer of s in $G^{*\mathcal{F}}$. The Jordan decomposition of characters [Lus77, Section 7] gives a bijection of sets

$$\psi^G_s : \mathcal{E}(G^{\mathcal{F}}, s) \longrightarrow \mathcal{E}(G^{*,\mathcal{F}}_s, 1),$$

which satisfies the following properties:

(i) for any $\rho \in \mathcal{E}(G^{\mathcal{F}}, s)$ there exists a constant c_s such that

$$\dim \psi^G_s \rho = c_s \dim \rho; \qquad (\clubsuit)$$

(ii) if the identity components of the centres of G^* and G^*_s have the same k-rank, then ψ^G_s maps cuspidal representations to cuspidal representations.

For $m \in \mathbb{Z}$ let $m_{p'}$ be the maximal divisor of m prime to p. The constant c_s above is then

$$c_s = \mid G^{*\mathcal{F}} \mid_{p'}^{-1} \cdot \mid G_s^{*\mathcal{F}} \mid_{p'} .$$

We are therefore able to classify irreducible cuspidal representations of $G^{\mathcal{F}}$ for any G, providing we can classify the pairs (s, ρ) where s is a semisimple element of $G^{*\mathcal{F}}$ such that the identity components of $Z(G^{\mathcal{F}})$ and $Z(G^{*\mathcal{F}})$ have the same k-rank and ρ an irreducible cuspidal unipotent representation of $G_s^{*\mathcal{F}}$. In [Lus77, Section 8] Lusztig classified the irreducible cuspidal unipotent representations of finite classical groups. He showed that in any given geometric Lusztig series $\mathcal{E}(G^{\mathcal{F}}, s)$ there is at most one cuspidal representation. Moreover, the author proceeded to show that the equivalence classes of irreducible cuspidal representations of $G^{\mathcal{F}}$ are in bijection with the conjugacy classes of semisimple elements $s \in G^{*\mathcal{F}}$ which have characteristic polynomial

$$P_s(X) = \prod_P P(X)^{a_P}(X+1)^{a_+},$$

where P runs over all self-dual polynomials of even degree and the exponents satisfy

Case (a)
- $\sum_P a_P \deg\, P + a_+ = 2n + 1$;
- $a_P = \frac{1}{2}(m_P^2 + m_P)$ for some $m_P \in \mathbb{Z}$;
- $a_+ = 2(m_+^2 + m_+) + 1$ for some $m_+ \in \mathbb{Z}$.

Case (b)
- $\sum_P a_P \deg\, P + a_+ = 2n$;
- $a_P = \frac{1}{2}(m_P^2 + m_P)$ for some $m_P \in \mathbb{Z}$;
- $a_+ = 2m_+^2$ for some $m_+ \in \mathbb{Z}$ with the sign $\pm = (-1)^{m_+}$.

Remark 4.5.2. If $G^{*\mathcal{F}}$ is an arbitrary reductive group then it is no longer the case that a geometric Lusztig series contains at most one cuspidal representation. For example, if $G^{*\mathcal{F}}$ is an exceptional group then there are at least 2 unipotent cuspidal representations, and so it is possible for a geometric Lusztig series to contain more than one cuspidal representation.

In addition, Lusztig tells us precisely when the groups $\mathrm{Sp}_{2n}(k)$ and $\mathrm{SO}^{\pm}_{2n}(k)$ contain irreducible cuspidal unipotent representations. For $G = \mathrm{Sp}_{2n}(k)$ we require $n = t^2 + t$ for some $t \geq 1$, and this unique representation has dimension

$$\frac{|\mathrm{Sp}_{2n}(k)|_{p'} \cdot q^{\binom{2t}{2}+\binom{2t-2}{2}+\cdots}}{2^t(q+1)^{2t}(q^2+1)^{2t-1}\cdots(q^{2t}+1)}.$$

We note that if $n = 2$ then this is the representation θ_{10} introduced by Srinivasan [Sri68].

Similarly, the Special Orthogonal group $G^{\mathcal{F}} = \mathrm{SO}^{\pm}_{2n}(k)$ (with $n > 1$) has an irreducible cuspidal unipotent representation when $n = t^2$ for some $t \geq 2$, with $\pm = (-1)^t$. This representation has dimension

$$\frac{|\mathrm{SO}^{\pm}_{2n}(k)|_{p'} \cdot q^{\binom{2t-1}{2}+\binom{2t-3}{2}+\cdots}}{2^{t-1}(q+1)^{2t-1}(q^2+1)^{2t-2}\cdots(q^{2t-1}+1)}.$$

Now consider $s \in G^{*\mathcal{F}}$ semisimple and suppose $\mathcal{M}^*$ is an $\mathcal{F}$-stable Levi subgroup contained in an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}^*$ of G containing s. By dualizing, we have an $\mathcal{F}$-stable Levi subgroup $\mathcal{M}$ of an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}$ of G. Write $\mathcal{M}^*_s$ for the centralizer of s in $\mathcal{M}$, which is an $\mathcal{F}$-stable Levi subgroup of G^*_s. In this way, we obtain an analogous Jordan decomposition of characters for our Levi $\mathcal{M}^{\mathcal{F}}$

$$\psi_s^{\mathcal{M}^{\mathcal{F}}} : \mathcal{E}(\mathcal{M}^{\mathcal{F}}, s) \longrightarrow \mathcal{E}(\mathcal{M}_s^{*\mathcal{F}}, 1)$$

which has the same properties as ψ_s^G.

Every irreducible representation ρ of $G^{\mathcal{F}}$ appears as a component in the composition series of a representation parabolically induced from an irreducible cuspidal representation of a Levi subgroup $\mathcal{M}^{\mathcal{F}}$ to a parabolic subgroup $\mathcal{P}^{\mathcal{F}}$ of $G^{\mathcal{F}}$. This means that the study of irreducible representations of $G^{\mathcal{F}}$ reduces to understanding the irreducible cuspidal representations of Levi subgroups of $G^{\mathcal{F}}$. Any Levi subgroup $\mathcal{M}^{\mathcal{F}}$ of $G^{\mathcal{F}}$ is of the form

$$\mathcal{M}^{\mathcal{F}} \simeq \prod_{n_i} \mathrm{GL}_{n_i}^{\mathcal{F}} \times H^{\mathcal{F}},$$

with $H^{\mathcal{F}}$ a classical group the same type as $G^{\mathcal{F}}$. By dualizing, we have

$$\mathcal{M}^{*\mathcal{F}} \simeq \prod_{n_i} \mathrm{GL}_{n_i}^{\mathcal{F}} \times H^{*\mathcal{F}}$$

since finite general linear groups are self-dual. Therefore, we can write $s = (s_1, \ldots, s_m, s_H)$. In this way, any cuspidal representation ρ appearing in $\mathcal{E}(\mathcal{M}^{\mathcal{F}}, s)$ is of the form

$$\rho = \rho_1 \otimes \cdots \rho_m \otimes \rho_H,$$

with each $\rho_i \in \mathcal{E}(\mathrm{GL}_{n_i}^{\mathcal{F}}, s_i)$, $\rho_H \in \mathcal{E}(H^{\mathcal{F}}, s_H)$ cuspidal. Using the Jordan decomposition of characters, the unipotent cuspidal representation $\psi_s^{\mathcal{M}^{\mathcal{F}}}(\rho)$ has a similar decomposition.

This gives the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}\mathcal{E}(G^{\mathcal{F}}, s) \arrow[r, "\psi_s^G"] & \mathcal{E}(G_s^{*\mathcal{F}}, 1) \\
\mathbb{Z}\mathcal{E}(\mathcal{M}^{\mathcal{F}}, s) \arrow[u, "\mathrm{Ind}_{\mathcal{M}^{\mathcal{F}}, \mathcal{P}^{\mathcal{F}}}^{G^{\mathcal{F}}}"] \arrow[r, "\psi_s^{\mathcal{M}}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{Z}\mathcal{E}(\mathcal{M}_s^{*\mathcal{F}}, 1) \arrow[u, "\mathrm{Ind}_{\mathcal{M}_s^{*\mathcal{F}}, \mathcal{P}_s^{*,\mathcal{F}}}^{G_s^{*\mathcal{F}}}"']
\end{tikzcd},$$

with the vertical arrows corresponding to parabolic induction. On the left hand side, we have normal parabolic induction from our maximal Levi $\mathcal{M}^{\mathcal{F}}$ to $G^{\mathcal{F}}$. As discussed before, the induced representation has length two and the quotient of the dimensions of the representations is precisely the parameter q^{r_i}. However, on the right hand side, the nature of the induced representation is the same by (♣). Moreover, the quotient of the dimensions of the representations is again q^{r_i}. Since the right hand side consists of unipotent representations, we can use Table II from [Lus78] to find the parameter q^{r_i}, once we identify the groups $\mathcal{M}_s^{*\mathcal{F}}$ and $G_s^{*\mathcal{F}}$.

4.6 Calculation of Parameters

We now return to the notation used previously in this thesis. In this section G is either:

– $\mathrm{Sp}_{2n}(F)$ for F an arbitrary non-archimedean local field of even residue characteristic;

– a Split Special Orthogonal group defined over F a dyadic field.

We consider each case separately. We use the results of the previous section in order to verify the calculation of Mœglin [Mg14]. Explicitly, we prove the following (see Section 4.3 for notation).

Theorem 4.6.1. *Suppose G is as above. If G is a Symplectic group, let π be an arbitrary irreducible cuspidal depth-zero representation. If G is a Split Special Orthogonal group, let π be an irreducible cuspidal depth-zero representation arising from a maximal parahoric subgroup as considered in Corollary 2.10.7. Then*

$$\sum_{\substack{(\pi,n)\in\mathrm{Jord}(\sigma)\\ \pi\in\mathrm{Cusp}(F)\ \textit{depth-zero}}} \left\lfloor s_\sigma(\pi)^2\right\rfloor m_\pi = N_{{}^LG}.$$

4.6.1 Symplectic Group

Let $G = \mathrm{Sp}_{2n}(F)$ and σ be an irreducible cuspidal depth zero representation of G, so we write

$$\sigma = \mathrm{ind}_J^G \lambda_\sigma.$$

As in Section 2.9 J is the maximal parahoric associated to the almost self-dual lattice L_m with irreducible representation λ_σ which is the inflation of an irreducible cuspidal representation $\tau_\sigma = \tau_\sigma^{(1)} \otimes \tau_\sigma^{(2)}$ of the reductive quotient $J/J^1 \simeq \mathrm{Sp}_{2m}(k_F) \times \mathrm{Sp}_{2(n-m)}(k_F)$. Put $N_1 = m$ and $N_2 = n - m$.

For $i = 1, 2$, there exists a unique conjugacy class $s_\sigma^{(i)}$ in $\mathrm{SO}_{2N_i+1}(k_F)$ such that $\tau_\sigma^{(i)}$ is in the Lusztig series $\mathcal{E}(\mathrm{Sp}_{2N_i}(k_F), s_\sigma^{(i)})$. We denote by

$$\prod_P P(X)^{a_P^{(i)}}(X+1)^{a_+^{(i)}}$$

the characteristic polynomial of $s_\sigma^{(i)}$ where the product runs over self-dual irreducible monic polynomials $P \in k_F[X]$ of even degree. From the the previous section we know that the exponents $a_P^{(i)}$ satisfy the conditions

– $\sum_P a_P^{(i)} \deg P + a_+^{(i)} = 2N_i + 1$;

– $a_P^{(i)} = \frac{1}{2} m_P^{(i)}(m_P^{(i)} + 1)$ for some integer $m_P^{(i)}$;

– $a_+^{(i)} = 2m_+^{(i)}(m_+^{(i)} + 1) + 1$ for some integer $m_+^{(i)}$.

Let π be a cuspidal self-dual depth-zero irreducible representation of $\mathrm{GL}_{m_\pi}(F)$. Then we can write

$$\pi = \mathrm{ind}_{F^\times \mathrm{GL}_{m_\pi}(\mathfrak{o}_F)}^{\mathrm{GL}_{m_\pi}(F)} \Lambda_\pi$$

where $\Lambda_\pi \mid_{\mathrm{GL}_{m_\pi}(\mathfrak{o}_F)}$ is inflated from τ_π a cuspidal self-dual irreducible representation of $\mathrm{GL}_{m_\pi}(k_F)$. We now consider the group $\mathcal{M}_{k_F} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{Sp}_{2N_i}(k_F)$ with representation $\tau_\pi \otimes \tau_\sigma^{(i)}$ which naturally appears as a maximal Levi subgroup of $\mathcal{G}_{k_F} \simeq \mathrm{Sp}_{2(N_i+m_\pi)}(k_F)$. We are interested in the quadratic parameter q^{r_i} for the generators T_i of the spherical Hecke algebra $\mathcal{H}(\mathrm{Sp}_{2(N_i+m_\pi)}(F), \pi \otimes \sigma)$, which arises from the spherical Hecke algebra

$$\mathcal{H}(\mathcal{G}_{k_F}, \tau_\pi \otimes \tau_\sigma^{(i)}) = \mathrm{End}_{\mathcal{G}_{k_F}}\left(\mathrm{Ind}\,_{\mathcal{M}_{k_F}}^{\mathcal{G}_{k_F}} \tau_\pi \otimes \tau_\sigma^{(i)}\right)$$

over the residue field.

We require that the induced representation $\mathrm{Ind}_{\mathcal{M}_{k_F}}^{\mathcal{G}_{k_F}} \tau_\pi \otimes \tau_\sigma^{(i)}$ is reducible, which we know occurs if and only if the representation τ_π is self-dual, which implies $m_\pi = 1$ or m_π even. The representation τ_π is in the Lusztig series associated to some conjugacy class s_π in $\mathrm{GL}_{m_\pi}(k_F)$, with self-dual irreducible characteristic polynomial Q.

Suppose $m_\pi = 1$ so $Q(X) = X + 1$. Over the residue field, this gives a maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{Sp}_{2N_i}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{Sp}_{2N_i+2}(k_F)$. There exists a unique conjugacy class $s = (1, s_\sigma^{(i)})$ in $\mathcal{M}_{k_F}^*$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(i)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}_{k_F}^*, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}_{s,k_F}^* \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}_{a_+^{(i)}}(k_F) \times \prod_P \mathrm{U}_{a_P^{(i)}}(k_P/k_P^\circ).$$

The corresponding centralizer of s in $\mathcal{G}^*$ is

$$G_{s,k_F}^* \simeq \mathrm{SO}_{a_+^{(i)}+2}(k_F) \times \prod_P \mathrm{U}_{a_P^{(i)}}(k_P/k_P^\circ).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and $\mathcal{G}^*_{s,k_F}$ above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_F) \times \mathrm{SO}_{a_+^{(i)}}(k_F) \subseteq \mathrm{SO}_{a_+^{(i)}+2}(k_F)$. Since $\mathrm{SO}_{2t+1}(k_F)$ is of type B_t, we find from Table II in [Lus78] that

$$r_i = 2m_+^{(i)} + 1.$$

Using Proposition 4.4.1 we have

$$\{\pm s_\sigma(\pi), \pm s_\sigma(\pi')\} = \left\{ \pm\frac{((2m_+^{(1)}+1) + (2m_+^{(2)}+1))}{2}, \pm\frac{((2m_+^{(1)}+1) - (2m_+^{(2)}+1))}{2} \right\},$$

and so

$$\begin{aligned}\lfloor s_\sigma(\pi)\rfloor^2 + \lfloor s_\sigma(\pi')\rfloor^2 &= (m_+^{(1)} + m_+^{(2)} + 1)^2 + (m_+^{(1)} - m_+^{(2)})^2 \\ &= 2m_+^{(1)}(m_+^{(1)}+1) + 2m_+^{(2)}(m_+^{(2)}+1) + 1 \\ &= a_+^{(1)} + a_+^{(2)} - 1.\end{aligned}$$

Now suppose m_π is even. Over the residue field, this gives a maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{Sp}_{2N_i}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{Sp}_{2(N_i+m_\pi)}(k_F)$. There exists a unique conjugacy class $s = (s_\pi, s_\sigma^{(i)})$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(i)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_Q) \times U_{a_Q^{(i)}}(k_Q/k_Q^\circ) \times \prod_{P \neq Q} U_{a_P^{(i)}}(k_P/k_P^\circ) \times \mathrm{SO}_{a_+^{(i)}}(k_F).$$

The centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq U_{a_Q^{(i)}+2}(k_Q/k_Q^\circ) \times \prod_{P\neq Q} U_{a_P^{(i)}}(k_P/k_P^\circ) \times \mathrm{SO}_{a_+^{(i)}}(k_F).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and $\mathcal{G}^*_{s,k_F}$ above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times U_{a_Q^{(i)}}(k_Q/k_Q^\circ)$. Since U_t is of type ${}^2A_{t-1}$, we have from Table II in [Lus78] that

$$r_i = (2m_Q^{(i)} + 1)\frac{m_\pi}{2}.$$

Using Proposition 4.4.1 we have

$$\{\pm s_\sigma(\pi), \pm s_\sigma(\pi')\} = \left\{ \pm\frac{((2m_Q^{(1)}+1)\frac{m_\pi}{2} + (2m_Q^{(2)}+1)\frac{m_\pi}{2})}{2m_\pi}, \pm\frac{((2m_Q^{(1)}+1)\frac{m_\pi}{2} - (2m_+^{(2)}+1)\frac{m_\pi}{2})}{2m_\pi} \right\}.$$

Precisely one of these quantities is an integer, whilst the other is a half-integer. Taking this into account gives

$$\begin{aligned}\lfloor s_\sigma(\pi)\rfloor^2 + \lfloor s_\sigma(\pi')\rfloor^2 &= \frac{(m_Q^{(1)} + m_Q^{(2)} + 1)^2}{4} + \frac{(m_Q^{(1)} - m_Q^{(2)})^2}{4} - \frac{1}{4} \\ &= \frac{1}{2} m_Q^{(1)}(m_Q^{(1)} + 1) + \frac{1}{2} m_Q^{(2)}(m_Q^{(2)} + 1) \\ &= a_Q^{(1)} + a_Q^{(2)}.\end{aligned}$$

Therefore, summing over all self-dual irreducible cuspidal depth-zero representations π of $\mathrm{GL}_{m_\pi}(F)$ gives

$$\begin{aligned}\sum_{\pi \in \mathrm{Cusp}^*_{[0]}(F)} \lfloor s_\sigma(\pi)\rfloor^2 n_\pi &= \left(\sum_P (a_P^{(1)} + a_P^{(2)}) \deg P\right) + a_+^{(1)} + a_+^{(2)} - 1 \\ &= (2N_1 + 1) + (2N_2 + 1) - 1 \\ &= 2n + 1.\end{aligned}$$

The Langlands dual group of $G = \mathrm{Sp}_{2n}(F)$ is ${}^L G = \mathrm{SO}_{2n+1}(\mathbb{C})$, and since the the summation above gives $2n + 1 = N_{{}^L G}$, we have found all of $\mathrm{Jord}(\sigma)$ and so Theorem 4.6.1 is verified in this case.

4.6.2 Even Split Special Orthogonal Groups

We now consider the case $G = \mathrm{SO}^+_{2n}(F)$. As in Section 2.10.2, let J° be a maximal parahoric subgroup associated to the almost self-dual lattice L_m for $0 \leq m \leq n$. Recall that we impose $m \neq 1, 2, n-2, n-1$. Take σ an irreducible cuspidal depth-zero representation of G, so we can write

$$\sigma = \mathrm{ind}_J^G \Lambda_\sigma$$

for J the normalizer of J° in G. Here Λ_σ is the extension of the representation λ_σ of J which is the inflation of an irreducible cuspidal representation $\tau_\sigma = \tau_\sigma^{(1)} \times \tau_\sigma^{(2)}$ of the reductive quotient $J^\circ/J^1 \simeq \mathrm{SO}^+_{2m}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F)$. Put $N_1 = m$ and $N_2 = n - m$.

For $i = 1, 2$, there exists a unique conjugacy class $s_\sigma^{(i)}$ in $\mathrm{SO}_{2N_i}^+(k_F)$ such that $\tau_\sigma^{(i)}$ is in the Lusztig series $\mathcal{E}(\mathrm{SO}_{2N_i}^+(k_F), s_\sigma^{(i)})$. We denote by

$$\prod_P P(X)^{a_P^{(i)}}(X+1)^{a_+^{(i)}}$$

the characteristic polynomial of $s_\sigma^{(i)}$ where the product runs over self-dual irreducible polynomials P of even degree in $k_F[X]$. From the the previous section we know that the exponents $a_P^{(i)}$ satisfy the conditions

- $\sum_P a_P^{(i)} \deg P + a_+^{(i)} = 2N_i$;
- $a_P^{(i)} = \frac{1}{2} m_P^{(i)}(m_P^{(i)} + 1)$ for some integer $m_P^{(i)}$;
- $a_+^{(i)} = 2(m_+^{(i)})^2$ for some integer $m_+^{(i)}$ and $\epsilon^{(i)} = (-1)^{m_+^{(i)}}$.

As before let π be a cuspidal self-dual depth zero irreducible representation of $\mathrm{GL}_{m_\pi}(F)$ so

$$\pi = \mathrm{ind}_{F^\times \mathrm{GL}_{m_\pi}(\mathfrak{o}_F)}^{\mathrm{GL}_{m_\pi}(F)} \Lambda_\pi$$

where $\Lambda_\pi \mid_{\mathrm{GL}_{m_\pi}(\mathfrak{o}_F)}$ is inflated from τ_π a cuspidal self-dual irreducible representation of $\mathrm{GL}_{m_\pi}(k_F)$. The representation τ_π is in the Lusztig series associated to some conjugacy class s_π in $\mathrm{GL}_{m_\pi}(k_F)$ with self-dual irreducible characteristic polynomial Q.

Suppose $m_\pi = 1$ so $Q = X+1$. We consider the maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}_{2N_i}^+(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{SO}_{2N_i+2}^+(k_F)$. There exists a unique conjugacy class of $s = (1, s_\sigma^{(i)})$ in $\mathcal{M}_{k_F}^*$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(i)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}_{k_F}^*, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}_{s,k_F}^* \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}_{a_+^{(i)}}^{\epsilon^{(i)}}(k_F) \times \prod_P U_{a_P^{(i)}}(k_P/k_P^\circ),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}_{s,k_F}^* \simeq \mathrm{SO}_{a_+^{(i)}+2}^{\epsilon^{(i)}}(k_F) \times \prod_P U_{a_P^{(i)}}(k_P/k_P^\circ).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_F) \times \mathrm{SO}^{\epsilon^{(i)}}_{a_+^{(i)}}(k_F) \subseteq \mathrm{SO}^{\epsilon^{(i)}}_{a_+^{(i)}+2}(k_F)$. Since SO_{2t} is of type D_t, we have from Table II in [Lus78] that

$$r_i = 2m_+^{(i)}.$$

Using Proposition 4.4.1 gives

$$\{\pm s_\sigma(\pi), \pm s_\sigma(\pi')\} = \left\{ \pm \frac{\left(2m_+^{(1)} + 2m_+^{(2)}\right)^2}{2}, \pm \frac{\left(2m_+^{(1)} - 2m_+^{(2)}\right)^2}{2} \right\},$$

and so

$$\begin{aligned}
\lfloor s_\sigma(\pi) \rfloor^2 + \lfloor s_\sigma(\pi') \rfloor^2 &= (m_+^{(1)} + m_+^{(2)})^2 + (m_+^{(1)} - m_+^{(2)})^2 \\
&= 2(m_+^{(1)})^2 + 2(m_+^{(2)})^2 \\
&= a_+^{(1)} + a_+^{(2)}.
\end{aligned}$$

Now suppose m_π is even. We consider the maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{SO}^+_{2N_i}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{SO}^+_{2(N_i+m_\pi)}(k_F)$. There exists a unique conjugacy class of $s = (s_\pi, s_\sigma^{(i)})$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(i)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(i)}}(k_Q/k_Q^\circ) \times \prod_{P \neq Q} U_{a_P^{(i)}}(k_P/k_P^\circ) \times \mathrm{SO}^{\epsilon^{(i)}}_{a_+^{(i)}}(k_F),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq \mathrm{U}_{a_Q^{(i)}+2}(k_Q/k_Q^\circ) \times \prod_{P \neq Q} U_{a_P^{(i)}}(k_P/k_P^\circ) \times \mathrm{SO}^{\epsilon^{(i)}}_{a_+^{(i)}}(k_F).$$

We are interested in the quadratic parameter q^{a_i} for the generators T_i of the spherical Hecke algebra $\mathcal{H}(\mathrm{SO}^+_{2n+2}(F), \pi \otimes \sigma)$, which arises from the spherical Hecke algebra

$$\mathcal{H}(\mathcal{G}_{k_F}, \tau_\pi \otimes \tau_\sigma^{(i)}) = \mathrm{End}_{\mathcal{G}_{k_F}} \left(\mathrm{Ind}\,^{\mathcal{G}_{k_F}}_{\mathcal{M}_{k_F}} \tau_\pi \otimes \tau_\sigma^{(i)} \right)$$

over the residue field.

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(i)}}(k_Q/k_Q^\circ) \subseteq \mathrm{U}_{a_Q^{(i)}+2}(k_Q/k_Q^\circ)$. Since U_t is of type ${}^2A_{t-1}$, we have from Table II in [Lus78] that

$$r_i = (2m_Q^{(i)} + 1)\frac{m_\pi}{2}.$$

This is precisely the same as the case $m_\pi \in 2\mathbb{Z}$ for the Symplectic group and so we have

$$\begin{aligned}\lfloor s_\sigma(\pi)\rfloor^2 + \lfloor s_\sigma(\pi')\rfloor^2 &= \frac{\left(m_Q^{(1)} + m_Q^{(2)} + 1\right)^2}{4} + \frac{\left(m_Q^{(1)} - m_Q^{(2)}\right)^2}{4} - \frac{1}{4} \\ &= a_Q^{(1)} + a_Q^{(2)}.\end{aligned}$$

Therefore, summing over all self-dual irreducible cuspidal representations π gives

$$\begin{aligned}\sum_{\pi \in \mathrm{Cusp}^*_{[0]}(F)} \lfloor s_\sigma(\pi)\rfloor^2 m_\pi &= \left(\sum_P (a_P^{(1)} + a_P^{(2)}) \deg P\right) + a_+^{(1)} + a_+^{(1)} \\ &= 2N_1 + 2N_2 \\ &= 2m + 2(n-m) \\ &= 2n,\end{aligned}$$

and so Theorem 4.6.1 is verified in this case.

4.6.3 Odd Split Special Orthogonal Groups

We now consider the case $G = \mathrm{SO}_{2n+1}(F)$ the group isometries of a non-degenerate quadratic form Q with a 1-dimensional anisotropic subform. As in Section 2.10.2, let J° be a maximal parahoric subgroup associated to the almost self-dual lattice L_m for $0 \leq m \leq n$. Recall that we impose $m \neq n-2, n-1$ so that we do not have a factor of $\mathrm{SO}_2^+(k_F)$ or $\mathrm{SO}_4^+(k_F)$ appearing in the reductive quotient. Take σ an irreducible cuspidal depth-zero representation of G, so we can write

$$\sigma = \mathrm{ind}_J^G \Lambda_\sigma$$

for J the normalizer of J° in G. Here Λ_σ is the extension of the representation λ_σ of J which is the inflation of an irreducible cuspidal representation $\tau_\sigma = \tau_\sigma^{(i)} \times \tau_\sigma^{(2)}$ of the reductive quotient $J^\circ/J^1 \simeq \mathrm{O}_{2m+1}(k_F) \times \mathrm{SO}^+_{2(n-m)}(k_F)$. Put $N_1 = m$ and $N_2 = n - m$. In what follows we use the isomorphism $\mathrm{O}_{2N_1+1}(k_F) \simeq \mathrm{Sp}_{2N_1}(k_F)$ for finite fields of characteristic 2 so that we can use the work of Lusztig on the Jordan Decomposition of Characters for the odd Orthogonal group.

Since we have different classical groups arising in the reductive quotient, we cannot consider the cases $i = 1, 2$ concurrently. For $i = 1$ there exists a unique conjugacy class $s_\sigma^{(1)}$ in $\mathrm{O}^+_{2N_1+1}(k_F) \simeq \mathrm{Sp}_{2N_1}(k_F)$ such that $\tau_\sigma^{(1)}$ is in the Lusztig series $\mathcal{E}(\mathrm{Sp}_{2N_1}(k_F), s_\sigma^{(1)})$. We denote by

$$\prod_P P(X)^{a_P^{(1)}}(X+1)^{a_+^{(1)}}$$

the characteristic polynomial of $s_\sigma^{(1)}$ where the product runs over self-dual irreducible polynomials P of even degree in $k_F[X]$. From the the previous section we know that the exponents $a_P^{(1)}$ satisfy the conditions

- $\sum_P a_P^{(1)} \deg P + a_+^{(1)} = 2N_1 + 1$;
- $a_P^{(1)} = \frac{1}{2} m_P^{(1)}(m_P^{(1)} + 1)$ for some integer $m_P^{(1)}$;
- $a_+^{(1)} = 2m_+^{(1)}(m^{(1)} + 1) + 1$ for some integer $m_+^{(1)}$.

Similarly, for $i = 2$ there exists a unique conjugacy class $s_\sigma^{(2)}$ in $\mathrm{SO}^+_{2N_2}(k_F)$ such that $\tau_\sigma^{(2)}$ is in the Lusztig series $\mathcal{E}(\mathrm{SO}^+_{2N_2}(k_F), s_\sigma^{(2)})$. We denote by

$$\prod_P P(X)^{a_P^{(2)}}(X+1)^{a_+^{(2)}}$$

the characteristic polynomial of $s_\sigma^{(2)}$ where the product runs over self-dual irreducible polynomials P of even degree in $k_F[X]$. From the the previous section we know that the exponents $a_P^{(2)}$ satisfy the conditions

- $\sum_P a_P^{(2)} \deg P + a_+^{(2)} = 2N_2$;
- $a_P^{(2)} = \frac{1}{2} m_P^{(2)}(m_P^{(2)} + 1)$ for some integer $m_P^{(2)}$;
- $a_+^{(2)} = 2\left(m_+^{(2)}\right)^2$ for some integer $m_+^{(2)}$ and $\epsilon = (-1)^{m_+^{(2)}}$.

Let π be a cuspidal self-dual depth-zero irreducible representation of $\mathrm{GL}_{m_\pi}(F)$. We consider the group

$$\mathcal{M}^{(i)}_{k_F} \simeq \begin{cases} \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{Sp}_{2N_1}(k_F) & \text{if } i = 1, \\ \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{SO}^+_{2N_2}(k_F) & \text{if } i = 2, \end{cases}$$

with representation $\tau_\pi \otimes \tau^{(i)}_\sigma$ which naturally appears as a maximal Levi subgroup of

$$\mathcal{G}^{(i)}_{k_F} \simeq \begin{cases} \mathrm{Sp}_{2(N_1+m_\pi)}(k_F) & \text{if } i = 1, \\ \mathrm{SO}^+_{2(N_2+m_\pi)}(k_F) & \text{if } i = 2. \end{cases}$$

As before, we require that the induced representation $\mathrm{Ind}^{\mathcal{G}^{(i)}_{k_F}}_{\mathcal{M}^{(i)}_{k_F}} \tau_\pi \otimes \tau^{(i)}_\sigma$ be reducible, which we know occurs if and only if the representation τ_π is self-dual, which implies $m_\pi = 1$ or m_π even. The representation τ_π is in the Lusztig series associated to some conjugacy class s_π in $\mathrm{GL}_{m_\pi}(k_F)$, with self-dual irreducible characteristic polynomial Q.

Suppose $m_\pi = 1$ so $Q = X + 1$. First we consider $i = 1$ so we have the maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{Sp}_{2N_1}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{Sp}_{2N_1+2}(k_F)$. There exists a unique conjugacy class of $s = (1, s^{(1)}_\sigma)$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau^{(1)}_\sigma$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}_{a^{(1)}_+}(k_F) \times \prod_P U_{a^{(1)}_P}(k_P/k^\circ_P),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq \mathrm{SO}_{a^{(1)}_+ +2}(k_F) \times \prod_P U_{a^{(1)}_P}(k_P/k^\circ_P).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times \mathrm{SO}_{a^{(1)}_+}(k_F) \subseteq \mathrm{SO}_{a^{(1)}_+ +2}(k_F)$. Since SO_{2t+1} is of type B_t, we have from Table II in [Lus78] that

$$r_1 = 2m^{(1)}_+ + 1.$$

Now suppose $i = 2$ so $\mathcal{M}_{k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}^+_{2N_2}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{SO}^+_{2N_2+2}(k_F)$. There exists a unique conjugacy class of $s = (1, s^{(2)}_\sigma)$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau^{(2)}_\sigma$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_F) \times \mathrm{SO}^{\epsilon}_{a_+^{(2)}}(k_F) \times \prod_P U_{a_P^{(2)}}(k_P/k_P^\circ),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq \mathrm{SO}^{\epsilon}_{a_+^{(2)}+2}(k_F) \times \prod_P U_{a_P^{(2)}}(k_P/k_P^\circ).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times \mathrm{SO}^{\epsilon}_{a_+^{(2)}}(k_F) \subseteq \mathrm{SO}^{\epsilon}_{a_+^{(2)}+2}(k_F)$. Since SO_{2t} is of type D_t, we have from Table II in [Lus78] that

$$r_2 = 2m_+^{(2)}.$$

Proposition 4.4.1 yields

$$\{\pm s_\sigma(\pi), \pm s_\sigma(\pi')\} = \left\{ \pm \frac{(2m_+^{(1)} + 1 + 2m_+^{(2)})}{2}, \pm \frac{(2m_+^{(1)} + 1 - 2m_+^{(2)})}{2} \right\}.$$

Since both reducibility points are half-integers, we have

$$\begin{aligned}
\lfloor s_\sigma(\pi) \rfloor^2 + \lfloor s_\sigma(\pi') \rfloor^2 &= \left(m_+^{(1)} + \frac{1}{2} + m_+^{(2)} \right)^2 + \left(m_+^{(1)} + \frac{1}{2} - m_+^{(2)} \right)^2 - \frac{1}{2} \\
&= 2\left(m_+^{(1)}\right)^2 + 2m_+^{(2)}(m_+^{(2)} + 1) \\
&= a_+^{(1)} + a_+^{(2)} - 1.
\end{aligned}$$

Now suppose m_π is even and $i = 1$. We consider the maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{Sp}_{2N_1}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{Sp}_{2(N_1+m_\pi)}(k_F)$. There exists a unique conjugacy class of $s = (s_\pi, s_\sigma^{(1)})$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(1)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(1)}}(k_Q/k_Q^\circ) \times \prod_{P \neq Q} U_{a_P^{(1)}}(k_P/k_P^\circ) \times \mathrm{SO}_{a_+^{(1)}}(k_F),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq \mathrm{U}_{a_Q^{(1)}+2}(k_Q/k_Q^\circ) \times \prod_{P \neq Q} U_{a_P^{(1)}}(k_P/k_P^\circ) \times \mathrm{SO}_{a_+^{(1)}}(k_F).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(1)}}(k_Q/k_Q^\circ) \subseteq \mathrm{U}_{a_Q^{(1)}+2}(k_Q/k_Q^\circ)$. Since U_t is of type ${}^2A_{t-1}$, we have from Table II in [Lus78] that

$$r_1 = \left(2m_Q^{(1)} + 1\right)\frac{m_\pi}{2}.$$

For $i = 2$ we have the maximal Levi subgroup $\mathcal{M}_{k_F} \simeq \mathrm{GL}_{m_\pi}(k_F) \times \mathrm{SO}^+_{2N_2}(k_F)$ of $\mathcal{G}_{k_F} \simeq \mathrm{SO}^+_{2(N_2+m_\pi)}(k_F)$. There exists a unique conjugacy class of $s = (s_\pi, s_\sigma^{(2)})$ in $\mathcal{M}^*_{k_F}$ such that the representation $\tau_\pi \otimes \tau_\sigma^{(2)}$ lies in the Lusztig series $\mathcal{E}(\mathcal{M}^*_{k_F}, s)$. The corresponding centralizer of s in $\mathcal{M}^*$ is

$$\mathcal{M}^*_{s,k_F} \simeq \mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(2)}}(k_Q/k_Q^\circ) \times \prod_{P\neq Q} U_{a_P^{(2)}}(k_P/k_P^\circ) \times \mathrm{SO}^\epsilon_{a_+^{(2)}}(k_F),$$

whereas the centralizer of s in $\mathcal{G}^*$ is

$$\mathcal{G}^*_{s,k_F} \simeq \mathrm{U}_{a_Q^{(2)}+2}(k_Q/k_Q^\circ) \times \prod_{P\neq Q} U_{a_P^{(2)}}(k_P/k_P^\circ) \times \mathrm{SO}^\epsilon_{a_+^{(2)}}(k_F).$$

From the description of the groups $\mathcal{M}^*_{s,k_F}$ and G^*_{s,k_F} above, we see that parabolic induction is only occurring on the groups $\mathrm{GL}_1(k_Q) \times \mathrm{U}_{a_Q^{(2)}}(k_Q/k_Q^\circ) \subseteq \mathrm{U}_{a_Q^{(2)}+2}(k_Q/k_Q^\circ)$. Since U_t is of type ${}^2A_{t-1}$, we have from Table II in [Lus78] that

$$r_2 = \left(2m_Q^{(2)} + 1\right)\frac{m_\pi}{2}.$$

In the same way as for the Symplectic group and the Even Split Special Orthogonal group we have that

$$\begin{aligned}\lfloor s_\sigma(\pi)\rfloor^2 + \lfloor s_\sigma(\pi')\rfloor^2 &= \frac{\left(m_Q^{(1)} + m_Q^{(2)} + 1\right)^2}{4} + \frac{\left(m_Q^{(1)} - m_Q^{(2)}\right)^2}{4} - \frac{1}{4} \\ &= a_Q^{(1)} + a_Q^{(2)}.\end{aligned}$$

Therefore, summing over all self-dual irreducible cuspidal representations π gives

$$\begin{aligned}\sum_{\pi\in\mathrm{Cusp}^*_{[0]}(F)} \lfloor s_\sigma(\pi)\rfloor^2 m_\pi &= \left(\sum_P (a_P^{(1)} + a_P^{(2)})\deg P\right) + a_+^{(1)} + a_+^{(2)} - 1 \\ &= \left(\sum_P a_P^{(1)}\deg P + a_+^{(1)}\right) + \left(\sum_P a_P^{(2)}\deg P + a_+^{(2)}\right) - 1\end{aligned}$$

$$
\begin{aligned}
&= (2N_1 + 1) + (2N_2) - 1 \\
&= 2m + 1 + 2(n - m) - 1 \\
&= 2n.
\end{aligned}
$$

This completes the proof of Theorem 4.6.1.

Chapter 5

Positive Depth Representations of $\mathrm{Sp}_4(F)$

5.1 Notation

Let F be a dyadic field with $\mathfrak{o}_F$ its ring of integers and $\mathfrak{p}_F$ its unique maximal ideal so that the residue field $k_F \simeq \mathfrak{o}_F/\mathfrak{p}_F$ is finite. Fix a uniformizer ϖ of F.

Let V be a 4-dimensional F-vector space and write $A = \mathrm{End}_F(V)$. Let $h : V \times V \to F$ be a symplectic bilinear form with ordered Witt basis $\{e_{-2}, e_{-1}, e_1, e_2\}$ so that the Gram matrix associated to h is

$$A_h = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

With respect to this basis, we identify $\mathrm{Aut}_F(V)$ with $\widetilde{G} = \mathrm{GL}_4(F)$. The Symplectic group

$G = \mathrm{Sp}_4(F)$, the subgroup of $\widetilde{G}$ consisting of elements which preserve the symplectic form h, is then

$$\mathrm{Sp}_4(F) = \{g \in V \ | \ g^T A_h g = A_h\}.$$

5.2 Root System of $\mathrm{Sp}_4(F)$

Let $\mathfrak{g} = \{X \in M_4(F) : A_h X + X^T A_h = 0\}$ denote the Lie algebra of G. By this definition, a matrix $X \in A$ is in $\mathfrak{g}$ if and only if X is of the form

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{13} \\ x_{31} & x_{32} & -x_{22} & -x_{12} \\ x_{41} & x_{31} & -x_{21} & -x_{11} \end{pmatrix}.$$

Since G is a linear algebraic group it has the rational representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ with action given by conjugation. Let $T = \{\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) : t_i \in F^\times\}$ be a maximal F-split torus in G. If we consider the image of T under the adjoint representation $\mathrm{Ad}(T)$ we get a set of commuting semisimple elements, which can be diagonalized. We write $X(T) = \mathrm{Hom}(T, F^\times)$ for the set of rational characters of T. For $\chi \in X(T)$ the *weight space* associated to χ is the T-eigenspace

$$\mathfrak{g}_\chi = \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = \chi(t)X \text{ for all } t \in T\}.$$

We call χ the *weight* of $\mathfrak{g}_\chi$. The set Φ of non-zero weights with non-zero eigenspaces is called the *set of roots* of G. Let $\mathfrak{g}_0$ denote the 0-weight space, which is a self-normalizing nilpotent subalgebra of $\mathfrak{g}$ called the *Cartan* subalgebra. With respect to our basis, $\mathfrak{g}_0$ is the subalgebra of diagonal matrices. We obtain the *weight space decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\gamma \in \Phi} \mathfrak{g}_\gamma.$$

Let E_{ij} denote the monomial matrix with (i, j) coefficient 1 and all other coefficients 0. From our explicit description of the Lie algebra of G we have the following basis for $\mathfrak{g}$, where $\mathfrak{g}_\gamma$ is spanned by X_γ and, writing $\mathfrak{g}_0 = \mathfrak{g}_{T_1} + \mathfrak{g}_{T_2}$, $\mathfrak{g}_{T_i}$ is spanned by X_i $(i = 1, 2)$.

$X_\alpha = E_{12} - E_{34}$	$X_{-\alpha} = E_{21} - E_{43}$
$X_\beta = E_{23}$	$X_{-\beta} = E_{32}$
$X_{\alpha+\beta} = E_{13} + E_{24}$	$X_{-(\alpha+\beta)} = E_{31} + E_{42}$
$X_{2\alpha+\beta} = E_{14}$	$X_{-(2\alpha+\beta)} = E_{41}$
$X_1 = E_{11} - E_{44}$	$X_2 = E_{22} - E_{33}$

In order to describe the roots in our root system we consider the adjoint action of T on basis elements X_γ of $\mathfrak{g}_\gamma$. For example, with $t = \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1})$,

$$\mathrm{Ad}(t)X_\alpha = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} = t_1 t_2^{-1} X_\alpha,$$

which gives the root $\alpha(t) = t_1 t_2^{-1}$. A routine calculation gives the following set of roots of $\mathfrak{g}$:

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\},$$

where

$$\begin{array}{cccc} \alpha : T \to F^\times & \beta : T \to F^\times & \alpha+\beta : T \to F^\times & 2\alpha+\beta : T \to F^\times \\ t \mapsto t_1 t_2^{-1} & t \mapsto t_2^2 & t \mapsto t_1 t_2 & t \mapsto t_1^2 \end{array}$$

and the negative roots are the inverse of their positive counterparts. We call the one-dimensional subspace $\mathfrak{g}_\gamma$ of $\mathfrak{g}$ generated by X_γ the *root subspace* corresponding to $\gamma \in \Phi$.

We write $\Phi = \Phi_S \sqcup \Phi_L$ where $\Phi_S = \{\pm\alpha, \pm(\alpha+\beta)\}$ denotes the short roots and $\Phi_L = \{\pm\beta, \pm(2\alpha+\beta)\}$ denotes the long roots. We see that Φ is of type C_2 with base $\Delta = \{\alpha, \beta\}$.

Let $Y(T) = \mathrm{Hom}(F^\times, T)$ denote the set of rational cocharacters of T. There is a natural non-degenerate pairing $\langle , \rangle : X(T) \times Y(T) \to \mathbb{Z}$ given by evaluation: for $\delta \in X(T)$ and $\gamma^\vee \in Y(T)$ the pairing $\langle \delta, \gamma^\vee \rangle$ corresponds to the integer exponent

$$\delta \circ \gamma^\vee(x) = x^{\langle \delta, \gamma^\vee \rangle} \text{ for all } x \in F^\times.$$

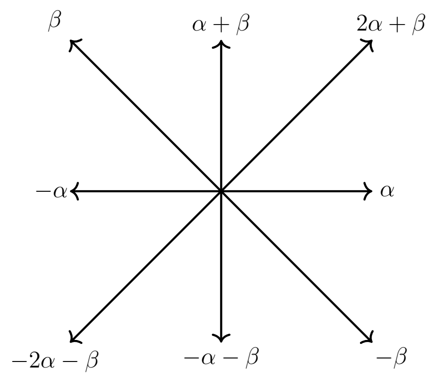

Figure 5.1: Root System of $\mathrm{Sp}_4(F)$

Let s_γ denote the reflection in the hyperplane perpendicular to the root $\gamma \in \Phi$ in the space $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Then there is a unique $\gamma^\vee \in Y(T)$ such that

$$s_\gamma(\delta) = \delta - \langle \delta, \gamma^\vee \rangle \gamma \text{ for all } \delta \in \Phi.$$

Moreover, $\gamma^\vee$ satisfies $\langle \gamma, \gamma^\vee \rangle = 2$. The set $\Phi^\vee = \{\gamma^\vee : \gamma \in \Phi\}$ is called the *set of coroots* of $\mathfrak{g}$. Thus

$$\Phi^\vee = \{\pm\alpha^\vee, \pm\beta^\vee, \pm(\alpha+\beta)^\vee, \pm(2\alpha+\beta)^\vee\},$$

where

$$\alpha^\vee : F^\times \to T \qquad\qquad \beta^\vee : F^\times \to F$$
$$\alpha^\vee(x) = \mathrm{diag}(x, x^{-1}, x, x^{-1}) \qquad\qquad \beta^\vee(x) = \mathrm{diag}(1, x, x^{-1}, 1)$$

and

$$(\alpha+\beta)^\vee : F^\times \to T \qquad\qquad (2\alpha+\beta)^\vee : F^\times \to F$$
$$\alpha^\vee(x) = \mathrm{diag}(x, x, x^{-1}, x^{-1}) \qquad\qquad \beta^\vee(x) = \mathrm{diag}(x, 1, 1, x^{-1}).$$

5.3 Parahoric Subgroups

There are three G-conjugacy classes of self-dual lattice chains in V, namely:

$$\Lambda_0 : \quad \ldots \supset \mathfrak{o}_F e_{-2} \oplus \mathfrak{o}_F e_{-1} \oplus \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2 \supset \mathfrak{p}_F e_{-2} \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_2 \supset \ldots$$

$$\begin{aligned}\Lambda_1: \quad & \ldots \supset \mathfrak{o}_F e_{-2} \oplus \mathfrak{o}_F e_{-1} \oplus \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_2 \supset \mathfrak{o}_F e_{-2} \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_2 \\ & \qquad \supset \mathfrak{p}_F e_{-2} \oplus \mathfrak{p}_F e_{-1} \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F^2 e_2 \supset \ldots \\ \Lambda_2: \quad & \ldots \supset \mathfrak{o}_F e_{-2} \oplus \mathfrak{o}_F e_{-1} \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F e_2 \supset \mathfrak{p}_F e_{-2} \oplus \mathfrak{p}_F e_1 \oplus \mathfrak{p}_F^2 e_1 \oplus \mathfrak{p}_F^2 e_2 \supset \ldots\end{aligned}$$

Let $\mathfrak{A}_i$ be the hereditary $\mathfrak{o}_F$-order corresponding to Λ_i with Jacobson radical $\mathfrak{P}_i$. The stabilizers of these almost self-dual lattice chains $\mathfrak{A}_i^\times \cap G$ are maximal parahoric subgroups of G. Each maximal parahoric $K_i := \mathfrak{A}_i^\times \cap G$ has a filtration by normal compact open subgroups $K_i^n := U^n(\mathfrak{A}_i) \cap G$. With respect to our chosen Witt basis these groups have the following description:

$$K_0 = \mathrm{Sp}_4(\mathfrak{o}_F) \qquad\qquad K_0^n = (1 + \mathrm{Mat}_4(\mathfrak{p}_F^n)) \cap G;$$

$$K_1 = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix} \cap G \quad K_1^n = \begin{pmatrix} 1+\mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor} & \mathfrak{p}_F^{\lceil \frac{n}{2} \rceil - 1} \\ \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor + 1} & 1+\mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} & \mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor} \\ \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor + 1} & \mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} & 1+\mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor} \\ \mathfrak{p}_F^{\lceil \frac{n}{2} \rceil + 1} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor + 1} & \mathfrak{p}_F^{\lfloor \frac{n}{2} \rfloor + 1} & 1+\mathfrak{p}_F^{\lceil \frac{n}{2} \rceil} \end{pmatrix} \cap G;$$

$$K_2 = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} & \mathfrak{p}_F^{-1} \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} & \mathfrak{p}_F^{-1} \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix} \cap G \quad K_2^n = \begin{pmatrix} 1+\mathfrak{p}_F^n & \mathfrak{p}_F^n & \mathfrak{p}_F^{n-1} & \mathfrak{p}_F^{n-1} \\ \mathfrak{p}_F^n & 1+\mathfrak{p}_F^n & \mathfrak{p}_F^{n-1} & \mathfrak{p}_F^{n-1} \\ \mathfrak{p}_F^{n+1} & \mathfrak{p}_F^{n+1} & 1+\mathfrak{p}_F^n & \mathfrak{p}_F^n \\ \mathfrak{p}_F^{n+1} & \mathfrak{p}_F^{n+1} & \mathfrak{p}_F^n & 1+\mathfrak{p}_F^n \end{pmatrix} \cap G.$$

The pro-p-radical K_i^1 is the maximal normal pro-p-subgroup of K_i. The maximal parahorics have reductive quotients $\mathcal{G}_0 = K_0/K_0^1 \simeq K_2/K_2^1 = \mathcal{G}_2 \simeq \mathrm{Sp}_4(k_F)$ and $K_1/K_1^1 = \mathcal{G}_1 \simeq \mathrm{Sp}_2(k_F) \times \mathrm{Sp}_2(k_F) = \mathrm{SL}_2(k_F) \times \mathrm{SL}_2(k_F)$.

Let $S = \{s_0, s_1, s_2\}$ be a set of fundamental reflections for the affine Weyl group W. We choose the following representatives for s_i in W:

$$s_0 = \begin{pmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ -\varpi & & & \end{pmatrix}, \qquad s_1 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

The fundamental reflections s_i satisfy the braid relations:

$$s_i^2 = \mathbb{1};$$

$$s_0 s_2 = s_2 s_0;$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0;$$

$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1.$$

where $\mathbb{1}$ denotes the trivial word.

Let $\mathfrak{I} = \bigcap_{i=0}^{2} K_i$ denote the standard Iwahori subgroup of G. For $S' \subset S$ let $W_{S'}$ denote the subgroup of W generated by S'. The standard parahoric subgroups of G correspond to proper subsets S' of S via the map

$$S' \mapsto G_{S'} = \mathfrak{I} N_{S'} \mathfrak{I}, \tag{$\heartsuit$}$$

where $N_{S'}$ is any set of representatives of $W_{S'}$ in G. In particular the maximal parahorics K_i correspond to the sets $S_i := S \backslash \{s_i\}$. In this case we write $W_i = W_{S_i}$.

5.4 Characters of Filtration Subgroups

We now turn the the question of describing characters of the abelian quotients K_i^n / K_i^{n+1}. For $1 \leq m \leq n \leq 2m$ we have

$$\mathfrak{P}_i^m / \mathfrak{P}_i^n \simeq U^m(\mathfrak{A}_i) / U^n(\mathfrak{A}_i)$$
$$\beta \mapsto 1 + \beta$$

Remark 5.4.1. In this Chapter we use β for both a root in the root system Φ and for an element of some power of the Jacobson radical $\mathfrak{P}^n$. We do not distinguish further since it is clear from the context which meaning is implied.

We fix ψ_F an additive character of F with conductor $\mathfrak{p}_F$. Set $\psi_A = \psi_F \circ \mathrm{tr}$ a character of $A = \mathrm{End}_F(V)$ where tr denotes the trace map. For S a subset of A let

$$S^* = \{a \in A : \psi_A(aS) = 1\}.$$

This gives the identification $(\mathfrak{P}_i^n)^* = \mathfrak{P}_i^{1-n}$ which in turns gives rise to the isomorphism

$$\mathfrak{P}_i^{-n}/\mathfrak{P}_i^{-m} \simeq \left(\mathfrak{P}_i^{m+1}/\mathfrak{P}_i^{n+1}\right)^{\wedge}.$$

Thus, if we impose $0 \leq m \leq n \leq 2m+1$, then we have an isomorphism between cosets $\beta + \mathfrak{P}_i^{1-n}$ and characters of the abelian quotient $U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)$:

$$\begin{aligned} \mathfrak{P}_i^{-n}/\mathfrak{P}_i^{1-n} &\simeq \left(U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)\right)^{\wedge} \\ \beta + \mathfrak{P}_i^{1-n} &\mapsto \left(\widetilde{\psi}_\beta : x \mapsto \psi_A(\beta(x-1)) \text{ for } x \in U^n(\mathfrak{A}_i)\right). \end{aligned}$$

Since G is a subgroup of $\widetilde{G}$ we can consider the restriction map

$$\begin{aligned} \operatorname{Res} : \left(U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)\right)^{\wedge} &\to \left(K_i^n/K_i^{n+1}\right)^{\wedge} \\ \widetilde{\psi}_\beta &\mapsto \psi_\beta \end{aligned}$$

which gives an induced map

$$\begin{aligned} \mathfrak{P}_i^{-n}/\mathfrak{P}_i^{1-n} &\to \left(K_i^n/K_i^{n+1}\right)^{\wedge} \\ \beta &\mapsto \psi_\beta. \end{aligned}$$

The induced map $\beta \mapsto \psi_\beta$ is a homomorphism of abelian groups since $\psi_{\beta+\beta'} = \operatorname{Res}\circ\widetilde{\psi}_{\beta+\beta'} = (\operatorname{Res}\circ\widetilde{\psi}_\beta)\cdot(\operatorname{Res}\circ\widetilde{\psi}_{\beta'}) = \psi_\beta \cdot \psi_{\beta'}$. Thus, in order to calculate the fibres of Res, it is enough to compute the kernel of Res. Once we do this, and show that it is surjective, we have the following commutative diagram

$$\begin{tikzcd} \mathfrak{P}^{-n}/\left(\mathfrak{P}^{1-n} + \operatorname{Ker}(\operatorname{Res})\right) \arrow[r, "\beta\mapsto\widetilde{\psi}"] \arrow[dr, dashed, "\sim"] & \left(U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)\right)^{\wedge} \arrow[d, two heads, "\operatorname{Res}:\widetilde{\psi}\mapsto\psi_\beta"] \\ & \left(K_i^n/K_i^{n+1}\right)^{\wedge} \end{tikzcd}.$$

We now show that the map Res is surjective. Let χ be a character of K_i^n trivial on K_i^{n+1} and set $L = U^{n+1}(\mathfrak{A}_i)K_i^n$ a subgroup of $U^n(\mathfrak{A}_i)$. Define a character χ_L of L by $\chi_L(hg) := \chi(g)$ for all $h \in U^{n+1}(\mathfrak{A}_i), g \in K_i^n$. Note that this is well-defined since χ is trivial on $U^{n+1}(\mathfrak{A}_i) \cap K_i^n = K_i^{n+1}$, and defines a character since K_i^n normalizes the trivial character of $U^{n+1}(\mathfrak{A}_i)$. Using Mackey Restriction–Induction:

$$\operatorname{Res}^{U^n(\mathfrak{A}_i)}_{U^{n+1}(\mathfrak{A}_i)}\left(\operatorname{Ind}^{U^n(\mathfrak{A}_i)}_{K}\chi_L\right) = \bigoplus_{U^{n+1}(\mathfrak{A}_i)\backslash U^n(\mathfrak{A}_i)/L} \operatorname{Ind}^{U^{n+1}(\mathfrak{A}_i)}_{{}^gK\cap U^{n+1}(\mathfrak{A}_i)} \operatorname{Res}^{{}^gK}_{{}^gK\cap U^{n+1}(\mathfrak{A}_i)} {}^g\chi_L$$

$$
\begin{aligned}
&= \bigoplus_{U^{n+1}(\mathfrak{A}_i)\backslash U^n(\mathfrak{A}_i)/L} \operatorname{Ind}_{U^{n+1}(\mathfrak{A}_i)}^{U^{n+1}(\mathfrak{A}_i)} \operatorname{Res}^{{}^gK}_{U^{n+1}(\mathfrak{A}_i)} {}^g\chi_L \\
&= \bigoplus_{U^{n+1}(\mathfrak{A}_i)\backslash U^n(\mathfrak{A}_i)/L} \operatorname{Res}^{{}^gK}_{U^{n+1}(\mathfrak{A}_i)} {}^g\chi_L \\
&= \bigoplus_{U^{n+1}(\mathfrak{A}_i)\backslash U^n(\mathfrak{A}_i)/L} \mathbb{1}_{U^{n+1}(\mathfrak{A}_i)}
\end{aligned}
$$

since ${}^gK \cap U^{n+1}(\mathfrak{A}_i) = U^{n+1}(\mathfrak{A}_i)$ and χ_L is trivial on $U^{n+1}(\mathfrak{A}_i)$. As $U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)$ is abelian, $\operatorname{Ind}_L^{U^n(\mathfrak{A}_i)} \chi_L \hookrightarrow \operatorname{Ind}_{U^{n+1}(\mathfrak{A}_i)}^{U^n(\mathfrak{A}_i)} \mathbb{1}_{U^{n+1}(\mathfrak{A}_i)}$ is a sum of characters of $U^n(\mathfrak{A}_i)$ trivial on $U^{n+1}(\mathfrak{A}_i)$. Using Frobenius reciprocity each of these restricts to χ_L. Thus we are able to extend characters of K_i^n/K_i^{n+1} to characters of $U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)$.

In order to find Ker(Res) we need to find all $\beta \in \mathfrak{P}_i^{-n}$ such that $\widetilde{\psi}_\beta$ is the trivial character of K_i^n. This is equivalent to finding conditions on $\beta = (\beta_{ij})$ such that $\operatorname{tr}(\beta x) \subseteq \mathfrak{p}_F$ for all $1 + x \in K_i^n$ since we fixed our additive character ψ_F to have conductor $\mathfrak{p}_F$. When calculating $\operatorname{tr}(\beta x) = \sum_{l,k} \beta_{lk} x_{kl}$ we reduce modulo $\mathfrak{p}_F$ to find that some $\beta_{lk} x_{kl}$ already lie inside $\mathfrak{p}_F$. Since the containment must hold for all x, on the remaining β_{ij} we may pick certain elements $x \in \mathfrak{P}^{-n}$ to find necessary conditions on β, and then check that these are in fact sufficient.

Example 5.4.2. Consider $K_0 = \operatorname{Sp}_4(\mathfrak{o}_F)$. We take $x \in \mathfrak{P}_0^n$ such that $1 + x \in G$ and $\beta \in \mathfrak{P}_0^{-n}$. We calculate

$$
\operatorname{tr}(\beta x) = \sum_{l,k=1}^{4} \beta_{lk} x_{kl} \bmod \mathfrak{p}_F.
$$

Consider the summands $\beta_{12}x_{21} + \beta_{43}x_{34}$, which corresponds to intersecting K_0^n with the root subgroup U_α of G. Choose

$$
1 + x = \begin{pmatrix} 1 & x_{12} & & \\ & 1 & & \\ & & 1 & x_{34} \\ & & & 1 \end{pmatrix} \in K_0^n \backslash K_0^{n+1}
$$

which forces $x_{34} = -x_{12} \in \mathfrak{p}_F^n \backslash \mathfrak{p}_F^{n+1}$ since $1 + x \in G$. Now

$$\begin{aligned} \operatorname{tr}(\beta x) &\equiv \beta_{21}x_{12} + \beta_{43}x_{34} \bmod \mathfrak{p}_F \\ &\equiv x_{12}(\beta_{21} - \beta_{43}) \bmod \mathfrak{p}_F \\ &\equiv 0 \bmod \mathfrak{p}_F \end{aligned}$$

implies that $\beta_{21} - \beta_{43} \in \mathfrak{p}_F^{1-n}$ i.e. $\beta_{21} \equiv \beta_{43} \bmod \mathfrak{p}_F^{1-n}$. Carrying out this calculation for all root subgroups and standard maximal parahorics we find the kernel of Res. This leads to the following Proposition.

Proposition 5.4.3. *Let $\beta \in \mathfrak{P}_i^{-n}/\mathfrak{P}_i^{1-n}$ correspond to the character $\widetilde{\psi}_\beta$ of the abelian group $U^n(\mathfrak{A}_i)/U^{n+1}(\mathfrak{A}_i)$. Let* Res : $\big(U(\mathfrak{A}_i)^n/U^{n+1}(\mathfrak{A}_i)\big)^\wedge \to \big(K_i^n/K_i^{n+1}\big)^\wedge$ *denote the restriction map on characters. The following table gives necessary and sufficient conditions on β such that $\widetilde{\psi}_\beta$ lies in the kernel of* Res.

$\mathfrak{A}_0$	$\mathfrak{A}_1$		$\mathfrak{A}_2$
	n even	n odd	
$\beta_{11} \equiv \beta_{44} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{11} \equiv \beta_{44} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{12} \equiv \beta_{34} \bmod \mathfrak{p}_F^{-n}$	$\beta_{11} \equiv \beta_{44} \bmod \mathfrak{p}_F^{1-n}$
$\beta_{22} \equiv \beta_{33} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{22} \equiv \beta_{33} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{13} \equiv -\beta_{24} \bmod \mathfrak{p}_F^{-n}$	$\beta_{22} \equiv \beta_{33} \bmod \mathfrak{p}_F^{1-n}$
$\beta_{12} \equiv \beta_{34} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{14} \in \mathfrak{p}_F^{-n}$	$\beta_{21} \equiv \beta_{43} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{12} \equiv \beta_{34} \bmod \mathfrak{p}_F^{1-n}$
$\beta_{21} \equiv \beta_{43} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{23}, \beta_{32} \in \mathfrak{p}_F^{1-n}$	$\beta_{31} \equiv -\beta_{42} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{21} \equiv \beta_{43} \bmod \mathfrak{p}_F^{1-n}$
$\beta_{13} \equiv -\beta_{24} \bmod \mathfrak{p}_F^{1-n}$	$\beta_{41} \in \mathfrak{p}_F^{2-n}$		$\beta_{13} \equiv -\beta_{24} \bmod \mathfrak{p}_F^{2-n}$
$\beta_{31} \equiv -\beta_{42} \bmod \mathfrak{p}_F^{1-n}$			$\beta_{31} \equiv -\beta_{42} \bmod \mathfrak{p}_F^{-n}$
$\beta_{14}, \beta_{23}, \beta_{32}, \beta_{41} \in \mathfrak{p}_F^{1-n}$			$\beta_{23},\ \beta_{41} \in \mathfrak{p}_F^{2-n}$
			$\beta_{14},\ \beta_{32} \in \mathfrak{p}_F^{-n}$

Remark 5.4.4. Since we are working in residue characteristic 2, where $1 \equiv -1 \,(\bmod\, \mathfrak{p}_F)$, we do not need to have minus signs in the table above. However, since our calculations do not depend on the characteristic, we retain them to allow for comparison to similar results in arbitrary residue characteristic.

Example 5.4.5. Let

$$\beta = \varpi^{-n} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \beta' = \varpi^{-n} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $1+x \in K_0^n$ and $x = \varpi^n(x_{ij})$. By definition

$$\begin{aligned} \widetilde{\psi}_\beta(1+x) = \psi_F \circ \mathrm{tr}(\beta x) &= \psi_F \circ \mathrm{tr}\left(\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \right) \\ &= \psi_F \circ \mathrm{tr}\left(\begin{pmatrix} x_{21}+x_{41} & x_{22}+x_{42} & x_{23}+x_{43} & x_{24}+x_{44} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11}+x_{41} & x_{12}+x_{42} & x_{13}+x_{43} & x_{14}+x_{44} \\ -x_{21} & -x_{22} & -x_{23} & -x_{24} \end{pmatrix} \right) \\ &= \psi_F(x_{13} - x_{24} + x_{21} + x_{43} + x_{32} + x_{41}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \widetilde{\psi}_{\beta'}(1+x) = \psi_F \circ \mathrm{tr}(\beta' x) &= \psi_F \circ \mathrm{tr}\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \right) \\ &= \psi_F \circ \mathrm{tr}\left(\begin{pmatrix} x_{41} & x_{42} & x_{43} & x_{44} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\ &= \psi_F(x_{32} + x_{41}). \end{aligned}$$

Thus $\widetilde{\psi}_\beta \neq \widetilde{\psi}_{\beta'}$. However, by appealing to Proposition 5.4.3, we find that $\psi_\beta = \psi_{\beta'}$.

This now raises the following important question: is the stratum $[\Lambda, n, n-1, \beta]$ with associated character ψ_β fundamental? Recall that a stratum is fundamental if the characteristic polynomial $\varphi_\beta(X) \neq X^4$. A quick calculation shows that

$$\varphi_\beta(X) = X^4 + 1, \qquad\qquad \varphi_{\beta'}(X) = X^4.$$

Thus the stratum $[\Lambda_0, n, n-1, \beta]$ with character $\widetilde{\psi}_\beta$ is fundamental whilst the stratum $[\Lambda_0, n, n-1, \beta']$ with character $\widetilde{\psi}_{\beta'}$ is non-fundamental, and yet they determine the same character of K_0^n. In order to answer the above question, we turn to the Moy–Prasad filtration [MP94, Section 3].

The phenomena exhibited above motivates the following definition.

Definition 5.4.6. Let $\mathfrak{A}$ be a hereditary order with Jacobson radical $\mathfrak{P}$ and let $\beta, \beta' \in \mathfrak{P}^{-n}$ for some $n \in \mathbb{N}$. We say β' is *anti-upper triangular* if β' is of the form

$$\beta' = \begin{pmatrix} * & * & * & * \\ * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, β' is obtained from β arbitrary by *anti-upper triangularization* if β' is anti-upper triangular and $\psi_\beta = \psi_{\beta'}$. We call a stratum $[\Lambda, n, n-1, \beta]$ with β upper-anti triangular *skew*.

5.5 Moy–Prasad Filtration

Given the Lie algebra $\mathfrak{g}$ and a point x in the Bruhat–Tits building of G, there exist two filtrations. One filtration is given by [MP94, Section 3], in which Moy and Prasad use the filtration $\mathfrak{g}_{x,r}$ (for $r \in \mathbb{R}$), and its dual $\mathfrak{g}^*_{x,-r}$, to define characters of abelian quotients of filtration subgroups of a parahoric subgroup of G associated to the point x. The second filtration is given by [BS09, Section 9], in which the authors again use the filtration $\mathfrak{g}_{x,r}$, but instead interpret x as a self-dual lattice function. Here a self-dual lattice function is a self-dual lattice sequence in which the domain is $\mathbb{R}$ instead of $\mathbb{Z}$, along with a necessary continuity condition [BL02, Section 2].

In [Lem09, Theorem 1.8], the author showed that these two filtrations coincide (up to normalization). Moreover, Lemaire proves that the filtration given by Broussous–Stevens extends to include the case that F is dyadic. We can therefore move between the lattice theoretic setting of Bushnell–Kutzko–Stevens and the filtration of the dual of the Lie algebra $\mathfrak{g}^*$ given by Moy–Prasad. We can then answer the question of whether the stratum given in Example 5.4.5 is fundamental by interpreting ψ_β in the language of Moy–Prasad.

Let Λ be a self-dual lattice sequence in A and K be the stabilizer of Λ, a parahoric subgroup of G. In the lattice theoretic setting, characters of the abelian quotients K^n/K^{n+1} are determined by an element $\beta \in \mathfrak{P}^{-n}/\mathfrak{P}^{1-n}$. In the Moy–Prasad setting, we turn to the dual of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, F)$ denote the dual of $\mathfrak{g}$, and x be the point in the Bruhat–Tits building associated to Λ. Given the filtration $\mathfrak{g}_{x,r}$ for $r \in \mathbb{R}$, there is an associated filtration of the dual $\mathfrak{g}^*$, given by

$$\mathfrak{g}^*_{x,-r} = \{X \in \mathfrak{g}^* \mid X(Y) \in \mathfrak{p}_F \text{ for all } Y \in \mathfrak{g}_{x,r^+}\},$$

where $\mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$. In this setting, characters of K^n/K^{n+1} above correspond to the coset $X + \mathfrak{g}^*_{x,-\frac{n}{e}+}$, where $e = e(\Lambda)$ [MP94, 3.7-3.8]. The character χ_X associated to $X + \mathfrak{g}^*_{x,-\frac{n}{e}+}$ is said to *non-degenerate* if the coset does not contain any nilpotent elements. If the character ψ_β is equivalent to a non-degenerate character χ_X then the stratum $[\Lambda, n, n-1, \beta]$ containing ψ_β is *fundamental*. Thus, given a non-fundamental stratum $[\Lambda, n, n-1, \beta]$, one can find an element $X \in \mathfrak{g}^*_{x,-\frac{n}{e}}$ such that the coset $X + \mathfrak{g}^*_{x,-\frac{n}{e}+}$ contains a nilpotent element. We therefore have the following definition.

Definition 5.5.1. Let Λ be a lattice sequence in A with stabilizer K and $[\Lambda, n, n-1, \beta]$ be a stratum. Let ψ_β be the character of K^n trivial on K^{n+1} associated to $[\Lambda, n, n-1, \beta]$. Let X be an element of the filtration of the dual Lie algebra $\mathfrak{g}^*_{x,-\frac{n}{e}}$ so that the character χ_X coincides with ψ_β. We say that the stratum $[\Lambda, n, n-1, \beta]$ is *G-fundamental* if the character χ_X is non-degenerate.

Now we must be able to translate our choice of $\beta \in \mathfrak{P}^{-n}$ to an $X \in \mathfrak{g}^*_{x,-\frac{n}{e}}$ so that $\psi_\beta = \chi_X$. For $\mathfrak{g}$ we have the weight space decomposition, which gives a decomposition of $\mathfrak{g}$ into one-dimensional subspaces associated to the root system Φ and the Cartan subalgebra $\mathfrak{g}_0$. We have a similar weight space decomposition for $\mathfrak{g}^*$, namely

$$\mathfrak{g}^* = \mathfrak{g}_0^* \oplus \bigoplus_{\gamma\in\Phi} \mathfrak{g}^*_\gamma$$

where $\mathfrak{g}_0^* = \mathrm{Hom}(\mathfrak{g}_0, F)$ and $\mathfrak{g}^*_\gamma = \{X \in \mathfrak{g}^* \mid \mathrm{Ad}^*(t)X = \gamma(t)X \text{ for all } t \in T\}$ for Ad the coadjoint action. Each $\mathfrak{g}^*_\gamma$ is a one-dimensional subspace of $\mathfrak{g}^*$ and is identified with the dual of $\mathfrak{g}_{-\gamma}$. Given $X_{-\gamma}$ the basis vector for $\mathfrak{g}_{-\gamma}$ defined previously, we denote by X^*_γ the unique vector in $\mathfrak{g}^*_\gamma$ such that $X^*_\gamma(X_{-\gamma}) = 1$. Thus any $X \in \mathfrak{g}^*$ can be uniquely written as

$$X = \sum_{i=1}^{2} a_i X_i^* + \sum_{\gamma\in\Phi} a_\gamma X^*_\gamma,$$

where $a_i, a_\gamma \in F$ and X_1^*, X_2^* is the standard basis for $\mathfrak{g}_0^*$.

Using [MP94, 4.2-4.3], if we can find a one-parameter subgroup $\lambda : \mathrm{GL}_1(k_F) \to K/K^1$ so that

$$\lim_{t\to 0} \mathrm{Ad}\lambda(t)X = 0 \qquad (\spadesuit)$$

for all $t \in T$, then the coset $X + \mathfrak{g}^*_{x,-\frac{n}{e}+}$ contains a nilpotent element, and so the stratum $[\Lambda, n, n-1, \beta]$ with character ψ_β is not fundamental. Let $\lambda(t) = \mathrm{diag}(t^a, t^b, t^{-b}, t^{-a}) \in T$. Then by translating from X to a β so that $\chi_X = \psi_\beta$, we can find conditions on $a, b \in \mathbb{Z}$ so that the one-parameter subgroup λ satisfies ($\spadesuit$). We note that if we wish to satisfy ($\spadesuit$) then we always require $a_i = 0$ for $i = 1, 2$.

Example 5.5.2. For example, consider $X = a_{-\alpha}X^*_\alpha$ with $a_{-\alpha} \neq 0$ and all other coefficients zero. After upper anti-triangularizing, the corresponding β is of the form

$$\beta = \begin{pmatrix} 0 & a_{-\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\lambda(t) = \mathrm{diag}(t^a, t^b, t^{-b}, t^{-a}) \in T$. Then

$$\begin{aligned}\lambda(t)\beta\lambda^{-1}(t) &= \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^{-b} & 0 \\ 0 & 0 & 0 & t^{-a} \end{pmatrix} \begin{pmatrix} 0 & a_{-\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^{-a} & 0 & 0 & 0 \\ 0 & t^{-b} & 0 & 0 \\ 0 & 0 & t^b & 0 \\ 0 & 0 & 0 & t^a \end{pmatrix} \\ &= \begin{pmatrix} 0 & t^{a-b}a_{-\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

By translating back into the Moy–Prasad language, we have $X = a_{-\alpha}X^*_{-\alpha}$ satisfies (♠) if and only if t^{a-b} tends to 0 as t tends to 0. This is true if and only if $a > b$.

Coefficient $a_\gamma \neq 0$	Condition on $\lambda(t)$
a_α	$b > a$
a_β	$0 > b$
$a_{\alpha+\beta}$	$0 > a + b$
$a_{2\alpha+\beta}$	$0 > a$
$a_{-\alpha}$	$a > b$
$a_{-\beta}$	$b > 0$
$a_{-(\alpha+\beta)}$	$a + b > 0$
$a_{-(2\alpha+\beta)}$	$a > 0$

Suppose now that Λ is a lattice chain of period $e = e(\Lambda)$, so that $K(\Lambda)$ is a standard parahoric. Fix a skew stratum $[\Lambda, n, n-1, \beta]$ with ψ_β of depth $\frac{n}{e}$. Write $n = ek - m$, with $m \in \{0, 1, \ldots, e-1\}$. For a fixed m, we have ψ_β is a character of K^{ek-m} trivial on K^{1+ek-m} with $\beta \in \mathfrak{a}_{m-ek} \setminus \mathfrak{a}_{1+m-ek}$. Since ψ_β on K^{ek-m} depends only on the coset of β in $\mathfrak{a}_{m-ek}/\mathfrak{a}_{1+m-ek}$, we can assume β is a matrix whose non-zero coefficients are contained in those where there is a jump in the filtration from K^{ek-m} to K^{1+ek-m}, or equivalently from $\mathfrak{g}^*_{x,-\frac{n}{e}}$ to $\mathfrak{g}^*_{x,-\frac{n}{e}+}$. In this quotient we let Ξ_m denote the subset of Φ such that $\mathfrak{g}^*_{x,-\frac{n}{e}} \cap \mathfrak{g}^*_\gamma \neq \mathfrak{g}^*_{x,-\frac{n}{e}+} \cap \mathfrak{g}^*_\gamma$ for all $\gamma \in \Xi_m$. One readily sees that as m varies $\Phi = \bigsqcup_m \Xi_m$.

If $\psi_\beta = \chi_X$ then we write $X = \sum_{\gamma \in \Xi_m} a_\gamma X^*_\gamma$ and set $\Xi(\beta) = \{\gamma \in \Xi_m : a_\gamma X^*_\gamma \notin \mathfrak{g}_{x,-\frac{n}{e}+}\}$ so that we can replace X by $\sum_{\gamma \in \Xi(\beta)} a_\gamma X^*_\gamma$ without changing the character χ_X on K^{ek-m}. We consider all the possibilities for $\Xi(\beta)$ for which there is a one-parameter subgroup λ satisfying ($\spadesuit$) as per the table above. For example, we cannot have both a_α and $a_{-\alpha}$ non-zero because if $\lambda(t) = \mathrm{diag}(t^a, t^b, t^{-b}, t^{-a})$ satisfied ($\spadesuit$) then we would need both $b > a$ and $a > b$, a contradiction. This immediately implies that we can only have at most four coefficients non-zero for any given m. Similarly, we see that we can not have all of $a_\alpha, a_\beta, a_{-(\alpha+\beta)}$ non-zero, since we have the conditions $b > a$, $0 > b$ and $a + b > 0$ which is absurd.

In each case considered for $\Xi(\beta)$ above, we can find a stratum $[\Lambda', n', n' - 1, \alpha]$ such that $\beta + \mathfrak{a}_{1+m-ek} \subseteq \mathfrak{a}'_{-n'}$ and $\frac{n'}{e'} < \frac{n}{e} = k - \frac{m}{e}$. This containment implies that the character ψ_β on K^{ek-m} restricts to the trivial character of ${K'}^{n'+1}$. Therefore, there exists a character ψ_α of ${K'}^{n'}$ trivial on ${K'}^{n'+1}$ with depth $\frac{n'}{e'}$. This means that given a character ψ_β of a prescribed depth, we can find another character ψ_α of a strictly smaller depth. The lattice chain Λ' which we move to need not be a standard parahoric. In fact, in most cases we must move to a conjugate of a standard parahoric. This gives the following result, which is a direct proof of [MP94, 6.3] in our case.

Theorem 5.5.3. *Let Λ be a self-dual lattice chain of period $e = e(\Lambda)$ and $[\Lambda, n, n-1, \beta]$ be a non-fundamental skew stratum. Then there exist a self-dual lattice chain Λ' of period $e' = e(\Lambda')$ and an integer n' such that*

$$\frac{n'}{e'} < \frac{n}{e} \qquad \textit{and} \qquad \beta + \mathfrak{a}_{1-n} \subseteq \mathfrak{a}'_{-n'}.$$

We now give an example of such a calculation outlined above.

Example 5.5.4. Consider the chain of period 1 with $\Lambda = \Lambda_0 = \mathfrak{o}e_{-2} \oplus \mathfrak{o}e_{-1} \oplus \mathfrak{o}e_1 \oplus \mathfrak{o}e_2$.

Then $K = K_0$, $m = 0$ and $\Xi_m = \Xi_0 = \Phi$. Here

$$\mathfrak{a}_0 = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

and $\mathfrak{a}_n = \varpi^n \mathfrak{a}_o$ for $n \in \mathbb{Z}$. Since $e(\Lambda) = 1$, we need only consider the stratum $[\Lambda, k, k-1, \beta]$ with $\beta \in \mathfrak{a}_{-k}$. Here ψ_β has depth k, $\varpi^k \beta \in \mathfrak{a}_0$ and we identify $\mathfrak{a}_0/\mathfrak{a}_1$ with

$$\begin{pmatrix} * & * & * & * \\ * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Since all short roots (resp. long roots) are conjugate by the Weyl group and W normalizes T by definition, by conjugating if necessary, we need only consider the cases $X = \sum_\gamma a_\gamma X_\gamma^*$ with γ negative. Therefore, we may choose Ξ_0 a subset of $\{-\alpha, -\beta, -(\alpha+\beta), -(2\alpha+\beta)\}$, and we need only consider 8 possible cases for $X = \sum_{\delta \in \Xi} a_\delta X_\delta^*$. In what follows we write L for a "long root" and S for a "short root".

Remark 5.5.5. The lattice chain Λ' which we move to need not be unique. In fact, in all the cases above, we could also move to the lattice chain Λ' associated to the standard Iwahori with $n' = 4k - 1$ and $\frac{n'}{e'} = k - \frac{1}{4} < k$.

Roots in Ξ	X	Λ'	n'
$1S$	$a_{-\alpha}$	$\mathfrak{oooo} \supset \mathfrak{opop}$	$2k-1$
$1L$	$a_{-\beta}$	$\mathfrak{oooo} \supset \mathfrak{oopp}$	$2k-1$
$1S+1L$	$a_{-\alpha}, a_{-\beta}$	$\mathfrak{oooo} \supset \mathfrak{ooop} \supset$ $\mathfrak{oopp} \supset \mathfrak{oppp}$	$4k-1$
$2S$	$a_{-\alpha}, a_{-(\alpha+\beta)}$	$\mathfrak{oooo} \supset$ $\mathfrak{ooop} \supset \mathfrak{oppp}$	$3k-1$
$2L$	$a_{-\beta}, a_{-(2\alpha+\beta)}$	$\mathfrak{oooo} \supset \mathfrak{oopp}$	$2k-1$
$2S+1L$	$a_{-\alpha}, a_{-(\alpha+\beta)}, a_{-(2\alpha+\beta)}$	$\mathfrak{oooo} \supset$ $\mathfrak{ooop} \supset \mathfrak{oppp}$	$3k-1$
$1S+2L$	$a_{-(\alpha+\beta)}, a_{-\beta}, a_{-(2\alpha+\beta)}$	$\mathfrak{oooo} \supset \mathfrak{oopp}$	$2k-1$
$2S+2L$	$a_{-\alpha}, a_{-\beta}, a_{-(\alpha+\beta)}, a_{-(2\alpha+\beta)}$	$\mathfrak{oooo} \supset \mathfrak{ooop} \supset$ $\mathfrak{oopp} \supset \mathfrak{oppp}$	$4k-1$

Theorem 5.5.6. *Let π be a smooth irreducible representation of G of positive depth. Then π contains some G-fundamental skew stratum $[\Lambda, n, n-1, \beta]$.*

Proof. Let $\mathcal{S}$ denote the set of pairs (Λ, n) with Λ a lattice chain in A and $n \in \mathbb{N}$ such that π contains the trivial character of $K^{n+1}(\Lambda)$. This is non-empty by smoothness of π. We choose $(\Lambda, n) \in \mathcal{S}$ with $\frac{n}{e(\Lambda)}$ minimal, which is possible since $e(\Lambda)$ is bounded. Since π contains the trivial character of $K^{n+1}(\Lambda)$, it contains some character ψ_β of $K^n(\Lambda)$ trivial on $K^{n+1}(\Lambda)$ i.e. π contains the stratum $[\Lambda, n, n-1, \beta]$. Suppose $[\Lambda, n, n-1, \beta]$ is not fundamental. By Theorem 5.5.3 there exist a self-dual lattice chain Λ' of period e' and an integer n' with $\beta + \mathfrak{a}_{1-n} \subseteq \mathfrak{a}'_{-n'}$ and $\frac{n'}{e'} < \frac{n}{e}$. This means that ψ_β restricts to the trivial character of $K^{n'+1}(\Lambda')$, and so π contains the trivial character of $K^{n'+1}(\Lambda')$. Moreover, $n' > 0$ since π has positive depth. Therefore $(\Lambda', n') \in \mathcal{S}$ with $\frac{n'}{e'} < \frac{n}{e}$, contradicting the minimality of n/e. $\square$

5.6 Future Work

It was hoped that once we had verified that a smooth irreducible representation π of G of positive-depth contains some G-fundamental stratum, we would then move on to obtain intertwining results akin to [Ste01]. This relies on having a nice set of double coset representatives, and since we have been working explicitly with the example of $\mathrm{Sp}_4(F)$, we would also need explicit descriptions of such sets. In the work that follows, we give an explicit description of the double coset spaces $K\backslash G/K$ for K a maximal parahoric subgroup of G. This was intended to be the basis for obtaining results on the intertwining of G-fundamental strata, but time constraints prohibited this.

5.6.1 The Geometric Representation

We now recall the relative theory of Coxeter groups which will be of use to us. We will apply the following with W the affine Weyl group and S the set of fundamental reflections, although makes sense in greater generality. For more information, see [Hum90, Chapter 5].

A *Coxeter system* is a pair (W, S) consisting of a group W and a subset S of generators subject to relations of the form

$$(ss')^{m(s,s')} = 1, \qquad \text{for } s, s' \in S,$$

where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$. If there is no relation between s and s' in W then we set $m(s,s') = \infty$. Any $w \in W\backslash\{\mathbb{1}\}$ can we written in the form $w = s_1 s_2 \cdots s_r$ for some $s_i \in S$, but by virtue of the braid relations above, this need not be unique. If w has such a presentation with r minimal, then we say that the presentation is *reduced*; all reduced presentations of w have the same *length* r, which we call the *length* $l(w)$ of w. We interpret the trivial element $\mathbb{1}$ as having length zero.

Let $\mathcal{V}$ be a real vector space with basis $\{\alpha_s : s \in S\}$. Define a symmetric bilinear form Υ on $\mathcal{V}$ by

$$\Upsilon(\alpha_s, \alpha_{s'}) := -\cos\left(\frac{\pi}{m(s,s')}\right),$$

which, in the case that $m(s,s') = \infty$, we interpret as $\Upsilon(\alpha_s, \alpha_{s'}) = -1$. For each $s \in S$, we define the reflection $\sigma_s : \mathcal{V} \to \mathcal{V}$ by

$$\sigma_s(v) := v - 2\Upsilon(\alpha_s, v)\alpha_s \text{ for all } v \in \mathcal{V}.$$

The reflection σ_s sends α_s to $-\alpha_s$ and fixes the hyperplane $H_s = \{v \in V : \Upsilon(\alpha_s, v) = 0\}$. The symmetric bilinear form Υ is preserved by the action of σ_s i.e. for all $v, v' \in \mathcal{V}$

$$\begin{aligned}
\Upsilon(\sigma_s(v), \sigma_s(v')) &= \Upsilon(v - 2\Upsilon(\alpha_s, v)\alpha_s, v' - 2\Upsilon(\alpha_s, v')\alpha_s) \\
&= \Upsilon(v, v') - 2\Upsilon(\alpha_s, v')\Upsilon(v, \alpha_s) - 2\Upsilon(\alpha_s, v)\Upsilon(\alpha_s, v') \\
&\qquad + 4\Upsilon(\alpha_s, v)\Upsilon(\alpha_s, v')\Upsilon(\alpha_s, \alpha_s) \\
&= \Upsilon(v, v')
\end{aligned}$$

since Υ is symmetric and $\Upsilon(\alpha_s, \alpha_s) = 1$. One would hope that $s \mapsto \sigma_s$ extends to a homomorphism from W to the subgroup of $GL(\mathcal{V})$ generated by the reflections σ_s. This turns out to be true and is summarised in the following Proposition.

Proposition 5.6.1. *There is a unique homomorphism $\sigma : W \to \mathrm{GL}(\mathcal{V})$ which sends $s \in S$ to $\sigma_s \in \mathrm{GL}(\mathcal{V})$. Moreover, $\sigma(W)$ preserves the bilinear form Υ on $\mathcal{V}$.*

We call σ the *geometric representation* of W.

Now let $\Phi_W = \{\sigma(w)(\alpha_s) : w \in W, s \in S\}$ be the *root system of* W. We can write any root $\alpha \in \Phi_W$ (uniquely) as

$$\alpha = \sum_{s \in S} \lambda_s \alpha_s,$$

with $\lambda_s \in \mathbb{R}$ all of the same sign. We say that α is *positive*, and write $\alpha > 0$, if $\lambda_s \geq 0$ for all s. We have the analogous definition for α being *negative*. The following theorem highlights the interplay between the geometric representation, positive/negative roots and the length function.

Theorem 5.6.2. [Hum90, Chapter 5.4] *Let $w \in W$ and $s \in S$. Let $l : W \to \mathbb{N}$ denote the length function. Then $l(ws) > l(w)$ if and only if $\sigma(w)(\alpha_s) > 0$. Moreover, $l(ws) < l(w)$ if and only if $\sigma(w)(\alpha_s) < 0$.*

5.6.2 Distinguished Double Coset Representatives

We now turn to the question of finding a set of representatives for the double coset space $K_i\backslash G/K_i$ with particular properties, which by ($\heartsuit$) in section 5.3 is equivalent to finding a set of representatives for the double coset space $W_i\backslash W/W_i$. For $S', S'' \subset S$, let

$$
\begin{aligned}
{}_{S'}\overline{D} &= \{w \in W : l(s'w) > l(w) \text{ for all } s' \in S'\},\\
\overline{D}_{S''} &= \{w \in W : l(ws'') > l(w) \text{ for all } s'' \in S''\},
\end{aligned}
$$

denote the unique sets of coset representatives of minimal length for the right coset space $W_{S'}\backslash W$ and left coset space $W/W_{S''}$ respectively. We call ${}_{S'}\overline{D}_{S''}$ a set of *distinguished (double coset) representatives* for $W_{S'}\backslash W/W_{S''}$ if

$$W = \bigsqcup_{d \in {}_{S'}\overline{D}_{S''}{}''} W_{S'} d W_{S''}$$

and each $d \in {}_{S'}\overline{D}_{S''}$ has minimal length in its double coset [Mor93, Section 3]. We say that a set ${}_{S'}D_{S''}$ of double coset representatives for $G_{S'}\backslash G/G_{S''}$ is *distinguished* if the projection ${}_{S'}\overline{D}_{S''} \subset N_G(T)$ of ${}_{S'}D_{S''}$ to W is distinguished. Distinguished coset representatives satisfy

$$l(s'ds'') = l(s') + l(d) + l(s'')$$

for all $s' \in S'$, $s'' \in S''$.

We now construct a set of distinguished double coset representatives for the space $W_i\backslash W/W_i$, $i = 0, 1, 2$. While the method we use can be generalised to any symplectic group, it is not feasible for larger groups for reasons which will become evident.

Theorem 5.6.3. *Let $S = \{s_0, s_1, s_2\}$ be a set of fundamental reflections in G. Let $S_i := S\backslash\{s_i\}$ and $W_i := W_{S_i}$ denote the subgroup of the affine Weyl group W generated by reflections in S_i. Let*

- $DCR_{S_0} = \{\mathbb{1},\, s_0,\, s_0s_1s_0,\, s_0A^r,\, s_0s_1s_0B^s,\, s_0s_1s_0B^tA^u : r, s, t, u \in \mathbb{N}$ *and t odd*$\}$

  where $A = s_1s_2s_1s_0$ and $B = s_2s_1s_0$;

- $DCR_{S_1} = \{\mathbb{1},\, s_1,\, s_1C^r,\, s_1A^{-s},\, s_1A^{-t}s_0s_1,\, A^us_1,\, A^vs_1s_2s_1, s_1A^{-w}C^x,$

  $A^ys_1C^z : r, s, t, u, v, w, x, y, z \in \mathbb{N}\}$ *where $C = s_0s_2s_1$;*

– $DCR_{S_2} = \{\mathbb{1},\, s_2,\, s_2s_1s_2,\, s_2D^r,\, s_2s_1s_2E^s,\, s_2s_1s_2E^tD^u : r,s,t,u \in \mathbb{N} \text{ and } t \text{ odd}\}$

$$\text{where } D = s_1s_0s_1s_2 \text{ and } E = s_0s_1s_2.$$

Then DCR_{S_i} is the set $s_i\bar{D}s_i$ of distinguished double coset representatives for $W_i\backslash W/W_i$ for each i.

Proof. The proof can be split into two parts. The first is to show that the conjectured set of representatives have minimal length in their double cosets. This shows that $DCR_{S_i} \subset {}_{S_i}\overline{D}_{S_i}$. The second is an inductive argument to show that DCR_{S_i} exhausts all distinguished representatives, which forces equality above.

For the first part, since distinguished representatives have minimal length in their cosets, we need to show that our conjectured list consists of distinguished elements. By definition, we have that $d \in {}_{S_i}\overline{D}_{S_i}$ if and only if $d \in {}_{S_i}\overline{D}$ and $d \in \overline{D}_{S_i}$. These sets of distinguished right and left cosets representatives are in bijection by the anti-automorphism $w \mapsto w^{-1}$. This means that d is a distinguished left coset representative if and only if d^{-1} is a distinguished right coset representative. Thus d is a distinguished double coset representative if and only if both d and d^{-1} are distinguished right coset representatives.

Remark 5.6.4. If w is a distinguished word in the double coset $W_i\backslash W/W_i$ then the number of s_i appearing in the presentation for any element of that coset is determined.

In order to motivate the inductive nature of our exhaustion argument we have the following result.

Lemma 5.6.5. *Let $i = 0,1,2$ and W_i, S_i be as above. Let $w \in W$ be a distinguished word with $n+1$ occurrences of s_i appearing in its reduced presentation. Then there exists a distinguished word d with n occurrences of s_i in a reduced presentation and $w_i \in W_i$ such that $w = dw_is_i$.*

Proof. Suppose $w \in W$ is distinguished with $n+1$ occurrences of s_i in its reduced presentation. Write a reduced presentation

$$w = us_ivs_i$$

with $v \in W_i$. The word us_i is reduced and has n occurrences of s_i in its reduced presentation. This means we can write

$$us_i = xdy$$

for some $x, y \in W_i$ and d distinguished with n occurrences of s_i. Then

$$l(x^{-1}us_i) = l(dy) = l(xdy) - l(x) = l(us_i) - l(x)$$

implies

$$l(x^{-1}w) \leq l(x^{-1}us_i) + l(vs_i) = l(us_i) + l(vs_i) - l(x) = l(w) - l(x).$$

Since w is distinguished, we conclude that $l(x) = 0$ and so $x = \mathbb{1}$. Thus

$$w = us_ivs_i = d(yv)s_i = dw_is_i$$

with $w_i = yv \in W_i$ as required. □

We now proceed to show that DCR_{S_i} is contained in $s_i\bar{D}s_i$.

Lemma 5.6.6. *Let DCR_{S_i} be as in Theorem 5.6.3. Then every element of DCR_{S_i} as an element of W is distinguished. Moreover, the expressions given for the elements of DCR_{S_i} are reduced.*

Proof. With respect to the ordered basis $\{\alpha_{s_0}, \alpha_{s_1}, \alpha_{s_2}\}$ of $\mathcal{V}$ we have

$$\sigma_{s_0} = \begin{pmatrix} -1 & \sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{s_1} = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2} & -1 & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_{s_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{2} & -1 \end{pmatrix}.$$

For each $d \in DCR_{S_i}$ we compute $\sigma(d)(\alpha_s)$ and $\sigma(d^{-1})(\alpha_s)$ for $s \in S_i$. In all cases, the resulting vectors are positive i.e. all coefficients are nonnegative. Theorem 5.6.2 implies that $l(ds) > l(d)$ and $l(d^{-1}s) > l(d^{-1})$, so every element is distinguished.

To show each element d is reduced we compute $l(d)$ by building it up as a product of s_i (from left to right) and verifying (using Theorem 5.6.2) that the length increases at each step. This is done by induction and a direct calculation. We note that the cases $i = 0, 2$ are dual to each other by swapping s_0 with s_2. □

Example 5.6.7. We give an example of the calculations needed in the Lemma above for the case $i = 0$. Consider the element s_0A^n in the double coset space $W_0\backslash W/W_0$, where $A = s_1s_2s_1s_0$ and $n \geq 1$. Then

$$\begin{aligned}\sigma(s_0A^n) = \sigma(s_0)\sigma(A)^n &= \begin{pmatrix}-1 & \sqrt{2} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-2n+1 & n\sqrt{2} & 0\\ -2n\sqrt{2} & 2n+1 & 0\\ -2n & n\sqrt{2} & 1\end{pmatrix}\\ &= \begin{pmatrix}-2n-1 & (n+1)\sqrt{2} & 0\\ -2n\sqrt{2} & 2n+1 & 0\\ -2n & n\sqrt{2} & 1\end{pmatrix}.\end{aligned}$$

Now

$$\sigma(s_0A^n)(\alpha_{s_1}) = \begin{pmatrix}-2n-1 & (n+1)\sqrt{2} & 0\\ -2n\sqrt{2} & 2n+1 & 0\\ -2n & n\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}(n+1)\sqrt{2}\\ 2n+1\\ n\sqrt{2}\end{pmatrix},$$

and

$$\sigma(s_0A^n)(\alpha_{s_2}) = \begin{pmatrix}-2n-1 & (n+1)\sqrt{2} & 0\\ -2n\sqrt{2} & 2n+1 & 0\\ -2n & n\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

This shows that $l(s_0A^ns_1) > l(s_0A^n)$ and $l(s_0A^ns_2) > l(s_0A^n)$ by Theorem 5.6.2. Thus $s_0A^n \in \overline{D}_{S_0}$ and, since $(s_0A^n)^{-1} = s_0A^n$, we also have $(s_0A^n)^{-1} \in \overline{D}_{S_0}$ so $s_0A^n \in {}_{S_0}\overline{D}$. Similarly,

$$\begin{aligned}\sigma(s_0s_1s_0B^n) &= \sigma(s_0s_1s_0)\sigma(B)^n\\ &= \begin{pmatrix}-1 & 0 & 2\\ -\sqrt{2} & 1 & \sqrt{2}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-n+\frac{1}{2}(1+(-1)^n) & \frac{\sqrt{2}}{2}(1-(-1)^n) & n-\frac{1}{2}(1-(-1)^n)\\ -n\sqrt{2} & 1 & n\sqrt{2}\\ -n-\frac{1}{2}(1-(-1)^n) & \frac{\sqrt{2}}{2}(1-(-1)^n) & n+1-\frac{1}{2}(1-(-1)^n)\end{pmatrix}\\ &= \begin{pmatrix}-(n+1)+\frac{1}{2}(1-(-1)^n) & \frac{\sqrt{2}}{2}(1-(-1)^n) & (n+2)-\frac{1}{2}(1-(-1)^n)\\ -(n+1)\sqrt{2} & 1 & (n+1)\sqrt{2}\\ -(n+1)-\frac{1}{2}(1-(-1)^n) & \frac{\sqrt{2}}{2}(1-(-1)^n) & (n+1)-\frac{1}{2}(1-(-1)^n)\end{pmatrix}.\end{aligned}$$

We see that since both $\sigma(s_0s_1s_0B^n)(\alpha_{s_1})$ and $\sigma(s_0s_1s_0B^n)(\alpha_{s_2})$ are positive, $s_0s_1s_0B^n$ is positive.

It remains to show that each word is reduced. We inductively assume $d = s_0A^r$ is reduced, with the base case s_0 trivially satisfied.

$$\sigma(d) = \begin{pmatrix} -2r-1 & (r+1)\sqrt{2} & 0 \\ -2r\sqrt{2} & 2r+1 & 0 \\ -2r & r\sqrt{2} & 1 \end{pmatrix} \text{ and } \sigma(d)(\alpha_{s_1}) = \begin{pmatrix} (r+1)\sqrt{2} \\ 2r+1 \\ r\sqrt{2} \end{pmatrix}$$

is positive so $l(ds_1) = l(d)+1$,

$$\sigma(ds_1) = \begin{pmatrix} 1 & -(r+1)\sqrt{2} & 2r+2 \\ \sqrt{2} & -2r-1 & (2r+1)\sqrt{2} \\ 0 & -r\sqrt{2} & 2r+1 \end{pmatrix} \text{ and } \sigma(ds_1)(\alpha_{s_2}) = \begin{pmatrix} 2r+2 \\ (2r+1)\sqrt{2} \\ 2r+1 \end{pmatrix}$$

is positive so $l(ds_1s_2) = l(d)+2$,

$$\sigma(ds_1s_2) = \begin{pmatrix} 1 & (r+1)\sqrt{2} & 0 \\ \sqrt{2} & 2r+1 & -(2r+1)\sqrt{2} \\ 0 & (r+1)\sqrt{2} & -(2r+1) \end{pmatrix} \text{ and } \sigma(ds_1s_2)(\alpha_{s_1}) = \begin{pmatrix} (r+1)\sqrt{2} \\ 2r+1 \\ (r+1)\sqrt{2} \end{pmatrix}$$

is positive so $l(ds_1s_2s_1) = l(d)+3$,

$$\sigma(ds_1s_2s_1) = \begin{pmatrix} 2r+3 & -(r+1)\sqrt{2} & 0 \\ (2r+2)\sqrt{2} & -(2r+1)\sqrt{2} & 0 \\ 2r+2 & -(r+1)\sqrt{2} & 1 \end{pmatrix} \text{ and } \sigma(ds_1s_2s_1)(\alpha_{s_0}) = \begin{pmatrix} 2r+3 \\ (2r+2)\sqrt{2} \\ 2r+1 \end{pmatrix}$$

is positive so $l(ds_1s_2s_1s_0) = l(dA) = l(d)+4$.

This shows that dA is reduced. Next consider $d = s_0s_1s_0B^s$. The base case is $s_0s_1s_0$ which is certainly reduced. We inductively assume that $d = s_0s_1s_0B^s$ is reduced.

$$\sigma(d) = \begin{pmatrix} -(s+1)+\frac{1}{2}(1-(-1)^s) & \frac{\sqrt{2}}{2}(1-(-1)^s) & (s+2)-\frac{1}{2}(1-(-1)^s) \\ -(s+1)\sqrt{2} & 1 & (s+1)\sqrt{2} \\ -(s+1)-\frac{1}{2}(1-(-1)^s) & \frac{\sqrt{2}}{2}(1-(-1)^s) & (s+1)-\frac{1}{2}(1-(-1)^s) \end{pmatrix}$$

$$\text{and } \sigma(d)(\alpha_{s_2}) = \begin{pmatrix} (s+2) - \frac{1}{2}\left(1-(-1)^s\right) \\ (s+1)\sqrt{2} \\ (s+1) - \frac{1}{2}\left(1-(-1)^s\right) \end{pmatrix} \text{ is positive so } l(ds_2) = l(d)+1,$$

$$\sigma(ds_2) = \begin{pmatrix} -(s+1) - \frac{1}{2}(1-(-1)^s) & (s+2)\sqrt{2} & -(s+2) + \frac{1}{2}(1-(-1)^s) \\ -(s+1)\sqrt{2} & 2s+3 & -(s+1)\sqrt{2} \\ -s - \frac{1}{2}(1-(-1)^s) & (s+1)\sqrt{2} & -(s+1) + \frac{1}{2}(1-(-1)^s) \end{pmatrix}$$

$$\text{and } \sigma(ds_2)(\alpha_{s_1}) = \begin{pmatrix} (s+2)\sqrt{2} \\ 2s+2 \\ (s+1)\sqrt{2} \end{pmatrix} \text{ is positive so } l(ds_2s_1) = l(d)+2,$$

$$\sigma(ds_2s_1) = \begin{pmatrix} (s+3) - \frac{1}{2}(1-(-1)^s) & -(s+2)\sqrt{2} & (s+2) + \frac{1}{2}(1-(-1)^s) \\ (s+2)\sqrt{2} & -2s-3 & (s+2)\sqrt{2} \\ (s+1) - \frac{1}{2}(1-(-1)^s) & -(s+1)\sqrt{2} & (s+1) + \frac{1}{2}(1-(-1)^s) \end{pmatrix}$$

$$\text{and } \sigma(ds_2s_1)(\alpha_{s_0}) = \begin{pmatrix} (s+3) - \frac{1}{2}\left(1-(-1)^s\right) \\ (s+2)\sqrt{2} \\ (s+1) - \frac{1}{2}\left(1-(-1)^s\right) \end{pmatrix} \text{ is positive so}$$

$$l(ds_1s_2s_0) = l(dB) = l(d)+3.$$

Thus dB is reduced. We finally consider $d = s_0s_1s_0B^tA^u$ with t odd. We have

$$\sigma(s_0s_1s_0B^t) = \begin{pmatrix} -(t+2) & \sqrt{2} & t+1 \\ -(t+1)\sqrt{2} & 1 & (t+1)\sqrt{2} \\ -(t+1) & \sqrt{2} & t \end{pmatrix}$$

is reduced by the previous case which provides the base case of an induction on u. Then

$$\sigma(d) = \begin{pmatrix} -(t+2u+2) & (u+1)\sqrt{2} & (t+1) \\ -(t+2u+1)\sqrt{2} & 2u+1 & (t+1)\sqrt{2} \\ -(t+2u+1) & (u+1)\sqrt{2} & t \end{pmatrix} \text{ and } \sigma(d)(\alpha_{s_1}) = \begin{pmatrix} (u+1)\sqrt{2} \\ 2u+1 \\ (u+1)\sqrt{2} \end{pmatrix}$$

is positive so $l(ds_1) = l(d)+1$,

$$\sigma(ds_1) = \begin{pmatrix} -t & -(u+1)\sqrt{2} & (t+2u+3) \\ -t\sqrt{2} & -(2u+1) & (t+2u+2)\sqrt{2} \\ -(t-1) & -(u+1)\sqrt{2} & (t+2u+2) \end{pmatrix} \text{ and } \sigma(ds_1)(\alpha_{s_2}) = \begin{pmatrix} t+2u-3 \\ (t+2u+2)\sqrt{2} \\ t+2u+2 \end{pmatrix}$$

is positive so $l(ds_1s_2) = l(d) + 2$,

$$\sigma(ds_1s_2) = \begin{pmatrix} -t & (t+u+2)\sqrt{2} & -(t+2u+3) \\ -t\sqrt{2} & 2t+2u+3 & -(t+2u+2)\sqrt{2} \\ -(t-1) & (t+u+1)\sqrt{2} & -(t+2u+3) \end{pmatrix} \text{ and } \sigma(ds_1s_2)(\alpha_{s_1}) = \begin{pmatrix} (t+u+2)\sqrt{2} \\ 2t+2u+3 \\ (t+u+1)\sqrt{2} \end{pmatrix}$$

is positive so $l(ds_1s_2s_1) = l(d) + 3$,

$$\sigma(ds_1s_2s_1) = \begin{pmatrix} t+2u+4 & -(t+u+2)\sqrt{2} & t+1 \\ (t+2u+3)\sqrt{2} & -(2t+2u+3) & (t+1)\sqrt{2} \\ t+2u+3 & -(t+u+1)\sqrt{2} & t \end{pmatrix} \text{ and } \sigma(ds_1s_2s_1)(\alpha_{s_0}) = \begin{pmatrix} t+2u+4 \\ (t+2u+3)\sqrt{2} \\ t+2u+3 \end{pmatrix}$$

is positive so $l(ds_1s_2s_1s_0) = l(dA) = l(d) + 4$.

It now remains to show that the sets DCR_{S_i} exhaust all distinguished double coset representatives. Let $(DCR_{S_i})_n$ denote the subset of DCR_{S_i} consisting of elements with n lots of s_i occurring in its reduced presentation. Then $DCR_{S_i} = \bigsqcup_{n\geq 0}(DCR_{S_i})_n$, with the understanding that $w \in DCR_{S_i}$ having no occurrences of s_i implies $w = \mathbb{1}$. The details for the case $i = 0$ are given below.

Lemma 5.6.8. *Let $A = s_1s_2s_1s_0$ and $B = s_2s_1s_0$. Let DCR_{S_0} be as in Theorem 5.6.3 and $(DCR_{S_0})_n$ be as above. Assume that $d \in W$ is distinguished for $W_0\backslash W/W_0$ and has $n \in \mathbb{N} \cup \{0\}$ lots of s_0 appearing in its presentation (with the understanding that $n = 0$ corresponds to the trivial word.) Then*

$$d \in \bigsqcup_{n=0}^{3}(DCR_{S_0})_n = \left\{\mathbb{1}, s_0,\ s_0s_1s_0,\ s_0A, s_0s_1s_0B, s_0A^2\right\}$$

if $n < 4$, and

$$d \in (DCR_{S_0})_n = \left\{s_0\, A^{n-1}, s_0\, s_1\, s_0\, B^{n-2}, s_0\, s_1\, s_0\, B^x A^y : x + y = n-2 \text{ and } x \text{ odd } \right\}$$

if $n \geq 4$.

Proof. We proceed by induction on n, with the base case ($n = 0$) trivial. The inductive hypothesis and Lemma 5.6.5 tells us that any distinguished representative with $n + 1$ occurrences of s_0 is of the form dw_0s_0 with $d \in (DCR_{S_0})_n$ and $w_0 \in W_0$. There are only a finite number of possible elements of this form, which is bounded by $\mid W_0 \mid \cdot \mid (DCR_{S_i})_n \mid$.

The set W_0 consists of 8 elements, namely

$$W_0 = \{\mathbb{1}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}.$$

Using the braid relations, we see that the elements $s_1s_2s_1$ and $s_2s_1s_2s_1s_2$ represent the same word, but have differing lengths. Since distinguished words have minimal length in their double cosets, we choose elements of minimal length which represent a word. If $w_0 \in W_0$ ends in s_2 then $(dw_0s_0)s_2 = d(w_0s_2)s_0$ and w_0s_2 is shorter and ends in s_1. On the other hand, ds_0 ends in s_0s_0 and so has reduced expression with fewer than n lots of s_0. Thus we need only consider $ds_1s_0, ds_2s_1s_0 = dB$ and $ds_1s_2s_1s_0 = dA$.

We write s_i, s_j to indicate that we have considered the element $s_idw_0s_0s_j$ for $s_i, s_j \in W_0$, which we are permitted to do since we are in $W_0\backslash W/W_0$. To ease notation we abbreviate $i := s_i$.

Number of s_0's	Distinguished Representatives
1	0
2	010, $0A$
3	$010B$, $0A^2$
4	$010B^2$, $010BA$, $0A^3$
5	$010B^3$, $010BA^2$, $010A^4$
6	$010B^4$, $010BA^3$, $010B^3A$, $0A^5$
7	$010B^5$, $010BA^4$, $010B^3A^2$, $0A^6$
8	$010B^6$, $010BA^5$, $010B^3A^3$, $010B^5A$, $0A^7$

Table 5.1: Distinguished Reps of $W_0\backslash W/W_0$ with up to 8 occurrences of s_0.

We consider Table 5.1 as our base case in our induction. In what follows, we use "=" to mean that two elements reside in the same double coset. We readily make use of the following relations, which can be derived directly from the braid relations:

$$B^2A = AB^2; \tag{$\circ$}$$

$$A(10B) = (10B)A; \qquad (\square)$$

$$A2 = 2A. \qquad (\triangle)$$

Case 1: Let $d = 0A^{n-1}$:

(i) $d10 = 0A^{n-1}10 = 0A^{n-2}12\overline{1010}1 = 0A^{n-2}1\overline{2}010 = 0A^{n-2}10B$
$\overset{\square}{=} 010BA^{n-2} \in (DCR_{S_0})_{n+1}$.

(ii) $dB = 0A^{n-1}B = 0A^{n-1}210 \overset{\triangle}{=} \overline{2}02A^{n-2}12\overline{1010}1 = 0A^{n-2}1\overline{2}010$
$= 0A^{n-2}10B \overset{\square}{=} 010BA^{n-2} \in (DCR_{S_0})_{n+1}$.

(iii) $dA = 0A^{n-1}A = 0A^n \in (DCR_{S_0})_{n+1}$.

Case 2: Let $d = 010B^{n-2}$:

(i) $d10 = 010B^{n-2}10 = 010B^{n-4}2102\overline{1010}1 = 010B^{n-4}210\overline{2}010 = 010B^{n-4}\overline{21210}$
$= 010B^{n-4}121\overline{20}2 = 010B^{n-4}A$ has fewer than n lots of 0 in its reduced expression.

(ii) $dB = 010B^{n-2}B = 010B^{n-1} \in (DCR_{S_0})_{n+1}$.

(iii) If n is odd then $dA = 010B^{n-2}A \in (DCR_{S_0})_{n+1}$,
If n is even then $dA = 010B^{n-2}A \overset{\circ}{=} 010AB^{n-2} = \overline{1010}1210B^{n-2}$
$= 010B^{n-1} \in (DCR_{S_0})_{n+1}$.

Case 3: Let $d = 010B^xA^y$ with x odd and $x + y = n - 2$:

(i) If $x = 1$ then $d10 = 010BA^y10 \overset{\square}{=} 0A^y10B10 = 0A^y102\overline{1010}1$
$= 0A^y10\overline{2}010 = 0A^{y+1}$ has fewer than
n lots of 0 in its reduced expression,
If $x > 1$ then $d10 = 010B^xA^y10 = 010B^xA^{y-1}12\overline{1010}1$
$= 010B^xA^{y-1}1\overline{2}010 = 010B^xA^{y-1}10B$
$\overset{\square}{=} 010B^x10BA^{y-1} = 010B^{x-2}2102\overline{1010}BA^{y-1} =$
$010B^{x-1}210\overline{20}101BA^{y-1}$ has fewer than n lots of 0
in its reduced expression.

(ii) $dB = 010B^xA^yB = 010B^xA^{y-1}AB = 010B^xA^{y-1}B^2$
$= 010B^{x+2}A^{y-1} \in (DCR_{S_0})_{n+1}$.

(iii) $dA = 010B^xA^yA = 010B^xA^{y+1} \in (DCR_{S_0})_{n+1}$

Thus every distinguished element with $n+1$ lots of s_0 in its reduced expression lies in $(DCR_{S_0})_{n+1}$. $\square$

Theorem 5.6.3 now follows immediately from Lemma 5.6.6 and Lemma 5.6.8 (and its analogous statement for $i = 1, 2$). $\square$

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