Problematizing mathematics and its pedagogy through teacher engagement with history-focused and classroom situation-specific tasks

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We explore the conjecture that engaging teachers with activities which feature mathematical practices from the past (*history-focused* tasks) and in today's mathematics classrooms (*mathtasks*) can promote teachers' problematizing of mathematics and its pedagogy. Here, we sample evidence of discursive shifts observed as twelve mathematics teachers engage with a set of *problematizing activities* (PA) – three rounds of *history-focused* and *mathtask* combinations – during a four–month postgraduate course. We trace how the commognitive conflicts orchestrated in the PA triggered changes in the teachers' narratives about: mathematical objects (such as what a function is); how mathematical objects come to be (such as what led to the emergence of the function object); and, pedagogy (such as what value may lie in listening to students or in trialing innovative assessment practices). Our study explores a hitherto under-researched capacity of the commognitive framework to steer the design, evidence identification and impact evaluation of pedagogical interventions.

Teacher mathematical and pedagogical discourses; problematizing; theory of commognition; history of mathematics; MathTASK; mathematics teacher education

1. Means for problematizing teacher mathematical and pedagogical discourses

In engaging with discussions on history of mathematics and on situations they are likely to face in the classroom, teachers may problematize what mathematics is, how it changes, as well as how it is learned and taught (e.g. Bernardes & Roque, 2018; Biza, Nardi & Zachariades, 2007). The study we draw on in this paper (Moustapha-Corrêa, 2020) investigates how such problematizations may be triggered by a combination of *history-focused tasks* (inspired by Arcavi, & Isoda, 2007 and Kjeldsen, & Blomhøj, 2012) and *situation-specific tasks* (thereafter *mathtasks*, as per Biza et al., 2007; 2018). We call this combination of tasks – as well as the set of specifically designed assignments (reflective diaries, portfolios and teaching plans) that teachers engage with in teacher education settings – *problematizing activities* (PA).

The mathematics teachers who engaged with the PA were enrolled in an in-service professional master's program. The PA were designed to provide them with multiple opportunities for reflection on learning, on teaching and on mathematics itself. We conjecture that problematizing teachers' discourses on mathematics and its teaching, may lead to shifts in these discourses. We note that, in the study, while we did not collect data from teachers' actual practice, we did observe shifts in their utterances on intended practice. Therefore, we analyze evidence of problematization, triggered by the tasks, and we examine, in particular, cases where engagement with the PA results in discursive shifts.

In our work, *problematizing* means to put into question conventional views – especially those usually taken for granted – on mathematics as well as on its teaching and learning, and to consider different perspectives. Our perspective reverberates Freire's (2016) problematizing education as a counterpoint to a *banking pedagogy*, a metaphor for a paradigm which considers teachers as sole owners of knowledge, whose role is to *deposit* it in learners, seen as empty vessels. Freire conceives education as a “praxis which implies the action and reflection of people over the world to transform it” (Freire, 2016, p. 118). Such reflection must be pervaded of intentionality; this requires *problematizing* relations of people with the world.

In our study, a problematizing perspective on the teaching and learning of mathematics is intertwined with a problematizing view of mathematics itself. This involves challenging a view according to which mathematical knowledge has always been the way it is in the present, or develops through a linear and universal process setting out from a more primitive to a more advanced state, carried out by the isolated inspiration of persons with inborn giftedness (Giraldo, 2018). If that were the case, mathematics teaching would be constrained to the exposition of facts and procedures. In contrast, a problematizing view of mathematics (and its teaching) focuses on socially and culturally situated contexts in which mathematical knowledge is produced, both from historical and subjective perspectives.

Our take on history of mathematics is aligned with Grattan-Guinness’ (2004) distinction between two ways of interpreting the mathematics of the past, *history* and *heritage*. Those who endorse the history perspective are not interested only in heroic successes, but also in what *did not* work out. They consider not only mathematical aspects, but also social and cultural contexts in which these are situated. Those who endorse the heritage perspective, in contrast, address the question “how did we get here?”, as if the present was “*photocopied* onto the past” (Grattan-Guinness, 2004, p. 165, emphasis in the original), that is, as if the meanings and uses of mathematical objects were the same in the past as they are today. Our PA aim to promote a problematized understanding of mathematics, in opposition to a heritage perspective of mathematical knowledge production as a linear, cumulative and progressive development. We conjecture that presenting discontinuities and difficulties that emerged over time and across different cultures opens up possibilities for learning environments which nurture confidence and authority to do mathematics.

Aligned with our take on history of mathematics and its pedagogy is our discursive approach, the theory of commognition (Sfard, 2008), to designing the PA and to tracing and analyzing evidence of our participants’ engagement with the PA. In commognitive theory, learning occurs through, and is evidenced in, participation in a discourse. Discursive actions are shaped by metarules, namely rules about what makes such actions legitimate within a given discourse (for example, what actions are seen as legitimate ways to validate a mathematical argument). Thus, learning involves changes in the metarules used and/or the narratives endorsed by learners. Discursive patterns (routines) result from such rule-governed processes and the endorsed narratives. In this vein, the development of mathematics can be understood as the changing of metarules, and subsequent changes of endorsed narratives and routines, across time and contexts.

Drawing on commognitive theory, Kjeldsen and Blomhøj (2012) argue that “history can be used in mathematics education to reveal metadiscursive rules and make them explicit objects of reflection and – ultimately – to provoke commognitive conflicts” (p. 330). At the core of Kjeldsen and colleagues’ approach is the use of historical sources to showcase situations in which mathematical discourse is governed by metarules that are quite different to the ones that govern mathematics today. Engaging with such situations, “in which different discursants are acting according to different metarules” (commognitive conflicts, per Sfard, 2008, p. 256), may instigate meta-level learning, namely learning about changes in the rules that govern a discourse.

We consider that the heritage perspective may inhibit the opportunity to experience commognitive conflicts, namely to experience how the metarules, endorsed narratives and routines governing mathematics have been changing over time. We therefore align our study with the “genuine approach to history” (Kjeldsen & Blomhøj, 2012, p. 330) in which primary sources are studied and past episodes are placed in their contexts, enabling differences to surface.

Further, our PA draw inspiration from Arcavi and Isoda's (2007) listening approach, according to which engaging in tasks based on primary historical sources that may be seen as not straightforward may help teachers engage with more tolerance and sensitivity with unexpected or unusual student productions. In our PA, we adapt the Arcavi and Isoda listening approach by intentionally giving careful attention to teachers' mathematical and pedagogical discourses. We do so by creating several moments during the course when the teachers freely share their teaching ideas and practices, integrating them into the tasks, and also by allowing teachers to determine parts of the course. By doing so, the teacher is placed at the center of (her/his) education process. We conjecture that through the experience of the PA, teachers' listening to their students may develop as well.

The design of the PA also draws upon perspectives that see teachers' views on mathematics and its teaching as complex and dynamic (e.g. Davis & Renert, 2013). Our take on teachers' discourses is consonant with Davis and Renert's work (ibid) in: (1) considering that teachers need to articulate understandings on present established mathematics and on how mathematical knowledge is produced; (2) acknowledging teachers as key agents on the production of a range of mathematical possibilities situated in different social contexts; (3) focusing teachers' professional development from the perspective of relationships between individual and collective aspects, rather than on what an individual knows or does not know. In this sense, we concur with Cochran-Smith and Lytle (1999) who highlight the importance of communities of inquiry, in which teachers take their own practice as intentional objects of inquiry – a positioning they call *inquiry as stance* – so that theory and practice cannot be dichotomized.

In our PA, collectivity is paramount, since the PA are designed to encourage teachers to collectively engage in reflections upon their practices to develop problematized views of mathematics, its teaching and learning. Biza et al.'s (2007, 2018) mathtasks engage teachers with situations that are likely to occur in mathematics classrooms and are informed by issues that research has highlighted as seminal. We see the design and application of mathtasks as a tool akin to our perspective on problematizing the teaching and learning of mathematics: through reflecting on such situations, mathtasks are a means to bring teachers' own practices into the spotlight as intentional objects of inquiry. We note that this is particularly pertinent in the Brazilian context of mathematics teacher education where it is often the case that teachers in in-service training courses are regarded as mere recipients of knowledge from an external authority, and thus, the opportunity to express themselves sometimes is neglected. The mathtasks are, therefore, an authentic way to invite teachers into discussing situations that are highly likely to arise in the classroom but also that the teachers are likely to have experienced and have much to say about.

In a nutshell, our study explores the conjecture that engaging teachers with history-focused tasks and mathtasks can promote object-level and meta-level learning about mathematics and its pedagogy. We trace this learning in evidence of discursive shifts, namely in changes in the narratives and routines about mathematics and its pedagogy that our participants endorse. These shifts might be at object-level or at meta-level and may concern changes in the teachers' narratives about: mathematical objects (such as: what is a function?); how mathematical objects come to be (such as: what led to the emergence of the object of function?); and, pedagogy (such as: the value that lies in listening to students or trialing innovative assessment practices). We now introduce the PA, the participants and the study's methods of data collection and analysis.

2. Problematizing Activities: Participants, design, data collection and analysis

The PA were implemented in a course on History of Mathematics in an in-service professional master's program, in Rio de Janeiro, Brazil, with 12 mathematics teachers (Table 1). The application of the PA spanned a total of 13 three-hour meetings (Table 2), was organized in three *task rounds*, was conducted by the first two authors, and the third author participated in the first and the last meeting. Throughout the process, participants wrote *reflective diaries*. Each meeting had a participant responsible for reporting what happened in the previous one. Meetings were audio and video recorded. Individual written responses to the tasks were collected. After the three task rounds, the participants shared: a *teaching plan*, in which they presented in pairs an approach to teaching a topic from the mathematics curriculum; and, *portfolios*, in which they reported their experiences of the PA. After the completion of the meetings, there were individual responses to an online questionnaire, with 42 open and closed questions, and individual semi-structured interviews lasting 40 to 60 minutes. Our analyses draw on this rich dataset collected during and after the application of the PA and evidence the problematization of mathematics and its pedagogy generated as a result of the teachers' participation as well as their discursive shifts.

Participant	Previous years of teaching experience	Lower secondary school teaching experience	Upper secondary school teaching experience	University teaching experience	Mathematics	Physics
		At the time of study			Degree	
Apolo	10	✓	✓ (also primary school)		✓	
Cleber	12	✓	✓		✓	
Gabiangela	11	✓			✓	
Gladson	6	✓	✓		✓	
Guilherme	50	✓	✓	✓ (retired)	✓	
Leticia	5	✓	✓		✓	
Luciane	7	✓ (no longer teaches)			✓	
Michel	8	✓	✓		✓	
Naylor	No response	✓	✓		✓	
Silvio	7	✓	✓		✓	
Ulisses	40	✓				✓
Vinicius	3	✓	✓		✓	

Table 1. The participants (names are pseudonyms).

We use the following codes to refer to the data:

- numbers for the meetings (1-13), as in M7 or 7 for the seventh meeting;
- number for the rounds (1, 2, 3), as in R2 for the second round and 6R2 for the sixth meeting, which was part of the second round;

- PP: portfolio presentation;
- TP: teaching plan;
- I[first 2 letters of participant name] for interview excerpt (e.g. IVI designates excerpt from **Vi**nicius' interview);
- RD[Meeting Number] [first 2 letters of participant name] for reflective diaries (e.g. RD2AP designates excerpt from **Ap**olo' reflective diary of the second meeting).

M1 to M13 took place almost every week between mid-August and mid-December. Each round of the PA consisted of three phases: *triggering discussion*, *historical zoom in and zoom out (ZIZO)*, and *reflection-on-teaching*.

<table>
<tr><th>Meeting</th><th colspan="3">Content</th></tr>
<tr><td>1</td><td colspan="2">Introduction and participant consent</td><td>R1 – Trigger</td></tr>
<tr><td>2</td><td colspan="3">R1 – ZIZO</td></tr>
<tr><td>3</td><td>R1 – ZIZO</td><td>R1 – Reflection</td><td>Planning of R3</td></tr>
<tr><td>4</td><td colspan="3">R2 – Trigger</td></tr>
<tr><td>5</td><td colspan="3">R2 – ZIZO</td></tr>
<tr><td>6</td><td colspan="3">R2 – ZIZO</td></tr>
<tr><td>7</td><td colspan="3">R2 – Reflection</td></tr>
<tr><td>8</td><td colspan="2">R2 – Reflection</td><td>R3 – Trigger</td></tr>
<tr><td>9</td><td>R3 – Trigger</td><td colspan="2">R3 – ZIZO</td></tr>
<tr><td>10</td><td colspan="3">R3 – ZIZO</td></tr>
<tr><td>11</td><td>R3 – ZIZO</td><td colspan="2">R3 – Reflection</td></tr>
<tr><td>12</td><td colspan="3">Teaching Plan</td></tr>
<tr><td>13</td><td colspan="3">Portfolio Presentation</td></tr>
</table>

Table 2. The 13 PA meetings (in bold: the parts we draw on in this paper).

In the mathtasks (Biza et al., 2007, 2018) used in the triggering and reflection-on-teaching phases, teachers are presented with hypothetical situations from the mathematics classroom and invited to describe how they would react. These situations feature, for example, commognitive conflicts about words or phrases used commonly but typically not debated in lessons. We thus problematize how teachers may deal with ordinary teaching situations and how they reflect on these. We ask participants first to engage in small groups, then write down their response to the task and then reconvene for a plenary discussion of their responses.

Inspired by Kjeldsen and colleagues' (Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014) approach, we design the historical ZIZO phase comprising three key moments: an *immersion* in primary sources, an *historical overview*, and *unveiling* the source. While immersing in a source, teachers are invited to consider excerpts from history that are as similar to the primary source as possible. We do so intentionally to orchestrate commognitive conflicts emerging from the difference in metarules governing the mathematical practices in the primary sources and those of today. We also do so to encourage teachers to develop intentional listening capabilities as a benefit of "the linkage between interpreting texts and interpreting students" (Arcavi & Isoda 2007, p. 125). The historical overview encompasses a social, cultural and mathematical contextualization. We discuss the type of mathematics that was produced in that specific period, aiming to relativize the sovereignty of the contemporary mathematics over mathematics of the past. The overview is done through the reading of a textbook (e.g. Roque, 2012), culminating with a discussion in class. Finally, we return to the primary source to unveil it, namely appreciate past mathematical practices, and, emphatically, not judge them in the light of those of the present. Our aim is that the teachers notice the diversity of ways in which mathematics can be produced. We end up also problematizing what mathematics is, since the teachers may realize that the growth of mathematics and the directions this growth may take is socially and culturally contingent.

The first round (Moustapha-Corrêa, Bernardes & Giraldo, 2019) focused on the Ahmes Papyrus, written in hieroglyphic: specifically, the Aha problems and the different approaches to solving linear equations (Arcavi & Isoda, 2007). In the second round, we focused on approaches to teaching about area and Pythagoras' Theorem, triggered by the study of the first two books of Euclid's *Elements* as in Bicudo (2009). Finally, in the third round, we focused on a proposition on uniformly accelerated motion proposed by Galilei (1638), on definitions of function proposed by Euler (1748; 1755) and Dirichlet (Lejeune-Dirichlet, 1837), and provided opportunities to problematize how a mathematical object changes over time. The sources for each round can be found in Moustapha-Corrêa (2020).

To conclude the round, in the reflection-on-teaching mathtask, we discuss with the participants pedagogical and mathematical aspects of the topic also drawing on historical facts. Due to limitations of space, we refrain from detailing these here and refer the interested reader to Appendix 3a and 3b.

For the choice of topic in the tasks of each round we consider: availability of primary and secondary historical sources; potential to foreground diverse and alternative perspectives; and, relevance to the participants' practice. The themes of the first and second rounds were selected by the course leaders, the theme of the third round was chosen by the class.

3. The third round of the PA

We exemplify the PA by presenting details from the third round, which revolved around: (at object and meta-level) what is a function, what is the difference between a variable and an unknown, functions and variability; and (at meta-level) mathematics as an ever-changing discipline.

In the triggering mathtask (Appendix 1), the classroom situation concerns an incident in which the use of the word "solve" evokes a "widely enacted metarule" (Sfard, 2008, p. 211) that is regularly present in Brazilian lessons: that of solving a second-degree equation when second degree polynomials appear in the formulation of a problem, regardless of what is requested in the exercise. This mathtask was designed to:

- highlight different uses of letters in school algebra;
- problematize teachers' tendency to blame students' perceived inability to cope with problems in algebra and calculus; and,
- create opportunities for the participants to share their teaching practices on functions.

We chose to design this mathtask around the issue of "solving a function" in order to bring attention to a strong "precedent event" (Lavie, Steiner & Sfard, 2019) in the students' experience: an expression that evokes solving equations, while studying functions. A key point in the intended discussion (Kieran, 2006) is that, in both quadratic equations and functions, the same visual mediator is used to realize two different things: in $ax^2 + bx + c = 0$, x indicates an unknown, while in $f(x) = ax^2 + bx + c$, x indicates a variable. For the teacher, seeing $f(x)$ might be sufficient to evoke the meaning of x as a variable; for the students, who are newcomers to functions, this might not be so evident. What is behind the situation is a commognitive conflict between two incommensurable narratives (on functions and on equations), regarding two meanings of the same visual mediator (x).

In the immersion moment of the ZIZO part (Appendix 2), we engaged the teachers with four excerpts:

- Galilei's (1638) proposition on uniformly accelerated motion which illustrates a mathematical practice on relations between magnitudes that was common before the emergence of the notion of function and its first definitions;
- two definitions proposed by Euler: one in which function appears as an analytical expression (Euler, 1748); and, a comprehensive and updated version of this, influenced by D'Alembert's solution of the vibrating strings problem (Euler, 1755);
- Dirichlet's (Lejeune-Dirichlet, 1837) definition in which the scope of what is meant by "function" is broadened to include relations not expressed algebraically.

Our intention in studying these excerpts was to encourage participants to: i) become acquainted with a practice which dealt with variation before the object of function as we know it today emerged; ii) observe different definitions and identify similarities and differences between them; and, iii) realize that there was more than one definition over time of what eventually became the object of function – even for the same mathematician. In this sense, the role of the historical source is to challenge the participants' comfort zones, by presenting them with different ways of defining a function.

Our history-focused tasks feature commognitive conflicts through contrasting different metarules governing mathematics then and nowadays. For instance, Euler – and several eighteenth-century analysts – was influenced by the "generality of the variable" metarule, according to which "a variable in a function could take on all values and could not be restricted to an interval" (Kjeldsen & Petersen, 2014, p. 34). Dirichlet, in contrast, was not guided by this metarule. So, in realizing that there is no domain in Euler's definitions of function, we may contrast with today's definition of function. In this way, we orchestrated a commognitive conflict concerning the different narratives about function endorsed in the sources, and in contemporary mathematics. Following the study of these excerpts, an animated discussion about the historical contexts of the excerpts ensued, guided by Roque (2012), and we concluded by sharing the set-theoretic Bourbaki definition of function.

Finally, in the two reflection-on-teaching mathtasks, the first (Appendix 3a) problematizes the teaching of functions and to which extent teachers are open to different ways of teaching, including being informed by historical sources. The second (Appendix 3b) involves a debate on whether mathematics is discovered or invented.

Across the PA, we aimed to challenge participants' comfort zones and to propel them towards problematizing often long-held narratives about mathematics and its pedagogy. We did so through orchestrating potentially productive commognitive conflicts. Specifically, in the third round of the PA, the design of the mathtask aimed to orchestrate a commognitive conflict between narratives on equations and on functions, which are incommensurable in the ways they refer to x as an unknown and as a variable, and in relation to routines associated with the word "solve". Through exposing the different uses of the words "function" and "variable" by Euler and by Dirichlet, the history-focused task aimed to demonstrate the different metarules governing the mathematics of Euler and Dirichlet. Furthermore, by shedding light on the problems that motivated the emergence and changing of mathematical objects – such as function –, we also aimed to problematize teacher narratives on how mathematics grows. So, the combination of the mathtasks and the history-focused tasks became a fruitful way to connect and problematize deeply rooted teaching routines and historical practices. We aimed that such problematizing would generate discursive shifts and we sample evidence of these in what follows.

4. Evidence of problematizing mathematics and its pedagogy in the teachers' PA work

The problematizations, and resulting discursive shifts, experienced by the teachers throughout their participation in the PA span nuanced reflections, at object and meta-level, on mathematics and its pedagogy. We sample evidence of said problematizations and discursive shifts in four episodes that concern: the object of function (object level, 4.1); the mutability of mathematics (object/meta-level, 4.2); listening to, and debating with, students (object/meta-level, 4.3); and, assessment (meta-level, 4.4). Two episodes are directly related to the third round, and two refer also to the PA as a whole. Across the four episodes, we discuss commognitive conflicts, both planned and unanticipated, that emerged from the participants' engagement with the PA.

To analyze the data, we first constructed a factual narrative account summarizing each meeting. While doing so, we highlighted episodes which substantiate our participants' problematizing utterances and instantiate evidence of discursive shifts. We then sought further evidence in other data sources (e.g. interviews, written responses to the mathtasks, reflective diaries). We count self-reporting from participants and our own observations in the participants' utterances and behavior through the PA as evidence. Key participant utterances were then transcribed verbatim for inclusion in the accounts of the episodes. We selected four characteristic episodes for presentation in this paper, with a particular focus on episodes that feature the connection between the data and the intentions of the PA. Data have been translated from Portuguese. We note that we have maintained the pronoun "he" when the interviewees say "ele", even though technically they mean "any student in class" (male, female or other).

4.1. What does it mean "to solve a function"? Problematizing the role of visual mediator x in the teaching about functions and equations

The participants related to the situation that is at the heart of the triggering mathtask (Appendix 1) at once, and often recognized themselves in the mathtask's teacher (e.g. Cleber in the portfolio and interview). They discussed students' references to equations while studying functions and conjectured about what students may mean by "to solve a function". Michel and Luciane, for example, suggested that students may evoke routines related to solving second-degree equations in the context of quadratic functions because of the regularly recurrent precedent task situations in which students are expected to solve such equations: "… it has to do with the previous year, he learns a lot about working, solving second-degree equations. He's thinking it's a second-degree equation." (Michel, 8R3); "x squared, [lowers her head and starts writing right away], it's already on automatic" (Luciane, 8R3). Luciane means that, whenever the students encounter x^2, they automatically start solving an equation, regardless of whether the task asks them to do so or not. As intended by the mathtask, Michel's and Luciane's utterances indicate that they see students as not engaging with the new meaning of "x" as a variable.

For Gabiangela and Ulisses, however, there are situations in which the expression "to solve a function" may conjure different meanings to students. For Gabiangela, one such situation is when we "find the roots to sketch the graph": "We equal to zero and calculate the roots to draw in the Cartesian plane. So, I think we think we're solving the function!" (Gabiangela, 8R3). For Ulisses, by "solving a function", the student in the mathtask means to calculate the value of $f(x)$ for certain values of x. He stresses that he does not see the connection to second-degree equation because the exercise "doesn't say anywhere it's a second-degree equation". He stresses that the exercise is

> asking to find the values, to find the image. .... [the teacher] asks to determine the coefficients. [Mentioning an] equation of the second-degree or not, you can find the coefficients, and solve the function, according to the values [the teacher] gave there (Ulisses, 9R3).

Thus, both Gabiangela and Ulisses were looking for alternative meanings for the expression "to solve a function" that the students may have constructed. Triggered by the situation in the mathtask, they therefore problematized uses of this particular phrase, and of associated visual mediators. Their classroom experience was brought into, and enriched, the debate. They moved beyond the situation presented in the mathtask and reflected on the nuances of phrases commonly present in lessons, and which have multiple potential meanings that are often not explicitly negotiated. We aimed the engagement with this part of the PA to achieve awareness of the need to pursue such nuanced negotiation in lessons. In particular, the commognitive conflict set up for the participants was experienced as planned, as we can see through the different narratives of Luciane, Michel and Ulisses. Furthermore, the emergence of participant narratives that we had not foreseen evidences the potential of the PA to instigate problematization.

For Silvio, the situation seemed more centered on that "[the student] did not understand the $f(x)$ that is there" (8R3). When asked if it was just $f(x)$ that the student "did not understand", he replied: "x squared, from the equal sign, she knows what it is. I assure you." (8R3). Silvio's utterance obfuscates the fact that, in the expressions for an equation and for a function, x mediates different meanings. To him, the only difference is the presence of $f(x)$. This may be suggestive of his own realizations of unknowns and variables, and of his classroom practices in analogous situations. Since he did not seem explicitly aware of the fact that the same visual mediator may carry different meanings in different situations, he may not address this issue in his lessons.

The mathtask spurred at least two commognitive conflicts experienced by the teachers, one regarding the meaning of "solving a function" and another one regarding the meaning of the visual mediator x. Regarding both situations, we described participants' discursive activities. We credit the teachers' engagement with the mathtask for allowing pedagogical and mathematical reflections on functions at object level, which in turn, unfolded differences in their endorsed narratives to surface and be debated.

We are heartened by Cleber's, Michel's, Luciane's, Gabiangela's and Ulisses' utterances that the situation in the mathtask triggered potential for a shift in their teaching practice. We see problematizing the multiple and often tacit meanings that certain phrases and visual mediators may have in mathematics lessons as a potential discursive shift in the way the teachers speak about the mathematical objects at stake (x, equation, function). To negotiate the different shades of meaning they saw, their phrasing became more elaborate. We aim that this shift may translate into analogous action in lessons: for instance, the teachers may choose to address how the same visual mediators can have different meanings – and anticipate classroom situations in which the students' utterances are suggestive of limited awareness of these meanings.

4.2 How does mathematics change over time? Problematizing narratives on the immutability of mathematics

In the first meeting, we introduced history and heritage perspectives (Grattan-Guinness, 2004), and declared that the design of the PA is aligned with the former. All participants were receptive to a proposition that seemed new to them, except Guilherme, who, during the first and second rounds, kept asking when we were going to start the history lessons, and if we could "take an *en passant* tour of the history of mathematics" (4R2). His own take on history seemed to align with a heritage approach as evident also in the historical anecdotes he often shared during the meetings (e.g., on how, during the Siege of Syracuse, Archimedes destroyed enemy ships with fire, using parabolic mirrors, 6R2). Guilherme seemed also uncomfortable with the teaching approach in the PA as a whole: he was the only one who did not vote for another task round, when, in the third meeting, we asked the class to decide. He attended all meetings nonetheless, and engaged with what he and other participants saw initially as unusual writing assignments. His *agent provocateur* presence turned helpful in one significant way: his provocations led us to explicitly highlight meta-level issues, such as the mutability of mathematics. When asked whether he thinks that mathematics changes over time, he responded with what we see as a cumulative account of how mathematics grows but does not necessarily change:

> ... So this is the deal ... mathematics is a science where, with each new proof discovered, a brick is laid, and you don't remove a brick to put a new one, as it is in Physics. You use what's there, because it ... you can't really take it out. And, you build one more, one more and grow on top of the other. So, [that's it] in my modest point of view [he laughs]. (9R3)

When we then asked the class whether mathematics is discovered or invented, Guilherme had a firm view: "[it] was fully discovered by us, because it is already there." (9R3). When we brought up the example of Euclid's fifth postulate and the different notions of parallelism implied by accepting or rejecting it, Guilherme seemed skeptical. Later in the meeting, while discussing the Dirichlet excerpt in the history-focused task, Guilherme articulated a revised narrative about how mathematics changes and becomes better as it grows:

> I mean, it increases [...]. You can see it better, right? And it expands what you already knew. It's just a better way to say ... that's why you said it changes. I just don't know how to thank you guys for the unique opportunity [to think about this in this way]. (9R3)

While Guilherme seems to still hold a relatively narrow developmental view of how mathematics changes over time (namely that mathematics becomes somehow better with time), what we see as significant is how the discussion of excerpts from primary sources has generated reflections and unprompted utterances that evidence rethinking about mathematics and its history. In general, the commognitive conflict orchestrated in the history-focused task was detected by the participants through the different narratives regarding the definition of a function endorsed by Euler, Dirichlet and the one accepted in today's mathematics. Such differentiation surprised the participants. During the presentation of his portfolio, Cleber, for example, explicitly stated the difference in his views before and after the PA:

> We had an image of ... linear knowledge, right? As if this knowledge just came about, then someone took this book and walked a little bit more, someone else walked two more steps. […] I think most [of the class] had a linear view on history. Then we began to see how a concept evolved. Sometimes in a nonlinear way, […]. (13PP)

Both Guilherme's and Cleber's statements exemplify how the PA, particularly the combination of ZIZO moments for studying historical facts, facilitated the problematization of long-held narratives on how mathematics grows. Participants often moved from linear and static to nonlinear and dynamic views of this growth. In this episode, we see in Guilherme and Cleber's utterances evidence that they problematized how mathematics changes over time. We credit the engagement with the ZIZO parts of the PA for allowing new narratives on mathematics to emerge. Euler and Dirichlet's excerpts, and the following contextualization, triggered the debate and endorsement of elaborate narratives about the object of function. The narrative of the mutability of mathematics was formulated, debated and eventually endorsed. Therefore, in this episode, we illustrate how the commognitive conflicts orchestrated in the PA has triggered problematizations, which in turn instigated changes in some participants' endorsed narratives on the mutability of mathematics.

The following episodes describe how the participation in the PA may have shifted the way teachers deal with – and, more especially, listen to and assess – their own students. Differently from the two episodes so far, the following two evidence problematizations and discursive shifts that emerged from engagement with the PA as a whole.

4.3 Do we listen? Problematizing monologic classroom discourse

Here, we report evidence of how our practice of listening to the participants fostered appreciation for this practice. For instance, the teachers were invited to consider whether they wish to have another round of the PA, and, if so, on what theme. Our listening stance inspired some participants to start listening to their own students. Gladson, for example, reported that he now invited his students to point out weaknesses in his lessons. To his bemusement, his students pointed out nothing – apart from a comment on his whiteboard organization – even after he pleaded with them further. When asked if that was the first time he asked such a question, he replied "[first time] in my whole life". By problematizing his own teaching routines, Gladson revised and tried to reform them. His testimony suggests that he was trying – for the first time – to adopt a dialogic approach with his students. Our interpretation of his observing the students' unresponsiveness is that, in the absence of precedent events to draw on – in which students were invited to engage with a critique of his teaching – this reaction is far from surprising.

Guilherme, who has been teaching for fifty years, and set out from a position of weariness towards the PA (see 4.2), offered, during the interview, ample evidence of the PA's impact on his teaching practices. In his interview, he declared that, instead of merely having "students listen and [...] the teacher, the holder of the power in quotation marks, speaks", he started to acknowledge how important is to "interact with the class". He stated that this was "because we got into those debates [in the PA], in the discussions of the things we experienced in the classroom." Guilherme called the PA a "lesson model" that

> was so good for me that from the third, fourth meeting, I was changing the conception of how to behave with my students. So, I started to want to hear more from my students. I want to listen. … [because when you listen to students], you see their difficulties, you realize where they are, where they are failing. What is missing in their understanding of what they ... are studying. And when only you speak, you think everyone is on the same level [and only] in the end, on the exams, you see the failure. (IGU)

According to Guilherme, with a more dialogic teaching approach, it is possible to follow students' steps as they learn.

In Gladson and Guilherme's utterances, we see problematizations of monologic teaching routines and self-reported actions towards reforming those. This is in accordance with the design of the PA, which acknowledges teachers as authors and protagonists of their own education. The tasks were designed to create opportunities for the teachers to share narratives about teaching and teaching practice. Throughout the PA, sharing was explicitly encouraged and valued, particularly in the parts involving mathtasks, which were designed to trigger reflection on the participants' professional experiences and foster problematization of long-established pedagogical routines. In this episode, the problematizations resulted in discursive shifts regarding the narratives that underpin these routines. The teachers seem to endorse a narrative that steers them away from a passive pedagogy and favors active participation and critical thinking from their students. They tried out our listening practice in their own practice, starting to listen and/or interacting more with their students.

Intended in-lesson routines were not the only ones to shift significantly as a result of the PA though. We now sample an episode evidencing discursive shifts regarding how mathematical learning may be assessed.

4.4. How do we know learning happened? Problematizing deeply rooted ways of assessment

The engagement with the PA made some participants problematize the way they assess their students' learning. Assignments such as the portfolio and the reflective diaries, deployed during the PA, came across as a stark contrast to the testing routines that dominate school mathematics and typically involve narrow, closed questioning. Gladson, for example, intimated in M9 that he was considering a review of how he assesses his students. Three meetings later, during the presentation of the portfolios, he reported that, in his school's final exam, he asked the students an open-ended question that resembled the openness of those deployed in the PA: "write everything that you have studied and is not in this exam […] anything that comes to mind". He also reported that he is considering making reflective diaries part of his assessment repertoire.

> Even if [students] don't study anything, at least to pay attention in the class, they have to write things down, write to me. This even helps us ... imagine when preparing an assessment? To see what you gave, how much students understood and so, to go there and prepare the assessment. Even to get that guy who is bad, I'll try to help him, I'll see what he wrote down, what he learned, I'll put there. (Gladson, 13PP)

In these utterances, Gladson shared his reflections on the benefits of reflective diaries as assessment, specifically in encouraging students' attentiveness in lessons and in informing teacher decisions about the students' learning progress and where his help may be needed. When reporting his experience with a more open question in an exam, Gladson said that he wishes to prioritize activities that convey mathematics as less objective and activities that go beyond finding right and wrong answers: "[The PA] are making us think, especially about assessment. Sometimes we overwhelm the students with assessments [that are] too objective. And, we see that in fact we can have assessment [that is] a little bit more subjective. And I have changed somewhat about this." (M10) According to newly endorsed narratives, Gladson has problematized his views on how mathematical assessment should be and reports new assessment routines.

Vinicius, who was presenting the portfolio with Gladson, reported that he had already tried the reflective diary with his students, asking them to write an essay about logarithms.

> My first attempt wasn't so cool … I said I wish you guys … to write about it [logarithms], an essay. I said informally, I'm not worried here about hits and misses, I want you to talk, I want you to express yourself through the text. … The vast majority copied the definition of logarithm and put it on paper. (Vinicius, 13PP)

Both Gladson and Vinicius shifted their way of speaking about assessment, as an (in)formative process rather than mere testing. Vinicius even shared stories of shifts in his practice. Underlying his students' resistance to engage with his innovative approach to assessment may be unfamiliarity. Similarly to the unresponsiveness of Gladson's students in 4.3, Vinicius' students may have simply copied and pasted the definition of logarithms because, for them, completing an assignment was governed by metarules – regarding the pedagogical discourse – that see performing tasks that require minimum levels of "agentivity" (Lavie, Steiner & Sfard, 2019, p. 170). New metarules were then at stake. Being invited to write a reflective diary was asking students to engage with more than just answering a question with only right-and-wrong answers, and there were hardly any precedent events that would steer the students towards embracing this invitation.

Vinicius seemed aware of the novelty he is introducing in his classroom's discursive practice and was willing to explicitly negotiate the new rules that govern the routine of composing a reflective diary – and to do so promptly: "for sure, next year, I'll implement this at the beginning ... of course I'll have time to fix it" (Vinicius, 13PP).

In this episode, we illustrate how the engagement with the PA generated Gladson and Vinicius' problematizations of – and self-reported efforts to reform – their assessment routines. In both teachers' utterances, we see evidence of how the problematizations spurred by the PA resulted in discursive shifts, in the narratives regarding assessment routines they endorsed, and in the novelties they introduced in their assessment practices. Even though our aim with the PA was to trigger teachers' problematizing of the ways they teach, our design did not specifically target the contrasting of different ways of assessment. The participants' problematizations –

and ensuing discursive shifts – regarding alternative ways of assessing mathematical learning were a surprising and welcome addition to the PA's impact.

5. Problematizing activities as a vehicle for discursive shifts on mathematics and its pedagogy

Our paper reports empirical research in a novel teacher professional development setting (the *problematizing activities*, PA) that encompasses contemporary trends in historiography of mathematics (e.g. Grattan-Guinness, 2004), and deploys the commognitive lens (Sfard, 2008) to trace in-service mathematics teachers' problematizations of – and discursive shifts regarding – mathematics and its pedagogy. The PA aim to provide opportunities for such problematization with a suite of activities that include a combination of history-focused tasks (inspired by Arcavi, & Isoda, 2007; Kjeldsen, & Blomhøj, 2012) and mathtasks (Biza, et al., 2007; 2018). We see this amalgamated use of tasks as a potent cocktail that can facilitate foregrounding and challenging longstanding narratives about mathematics and its pedagogy.

During the PA meetings, history-focused tasks played a central role in highlighting how the growing of mathematics over time is socially and culturally situated, and in shaking a widespread view of mathematics as a body of facts that has always been the way it is today. Our study builds on prior works (e.g. Bernardes & Roque, 2016; Kjeldsen & Petersen, 2014) which evidence how historical sources can be used in designing teaching and learning situations that engage participants with mathematical discourse – at object and meta-level – in ways they rarely have the opportunity to consider. The history-focused tasks and the mathtasks in our study provide a setting that allows contrasting past and current mathematical metarules and instigates productive commognitive conflicts. In this paper, we showcased how this occurred during the participants' engagement with the PA's third round. We presented milestones of how one mathematical object – eventually known as function – changed over time, we highlighted the mutability of mathematics, and we exposed the participants to a take on history of mathematics that is distinctly different from the one they have been exposed to over many years of prior education. Compellingly and explicitly, the participants reflected, for example, on how the metarules that govern mathematical discourse change over time (see Guilherme's utterances, in 4.2, regarding mathematics as an ever-changing discipline), leading to object and meta-level mathematical learning.

In tandem with the history-focused tasks, mathtasks played a central role in engaging teachers with situations they are likely to face in the classroom and, through the discussion of said situations, provided opportunities to identify evidence of shifts in the teachers' mathematical and pedagogical discourses. Mathtasks allowed access to aspects of teachers' discursive activities that are often not accessible to researchers. The specificity of these aspects could not have emerged through direct questions about teachers' views on mathematics and its teaching at large. See, for example, how Silvio's utterances in 4.1 hint at possible repercussions of not highlighting in lessons the multiple meanings that a single visual mediator may carry in different task situations. We credit mathtasks with the capacity to trigger highly situated concurrent discussions of mathematics and its pedagogy, and, in this sense, we observe evidence of object level and meta-level learning.

As the PA were drawing to the end of the third round, participants' discursive shifts, including the confidence with which they provided articulate accounts of these shifts, became more evident. In 4.1, we see the teachers discussing equation-solving routines and sharing different interpretations of what "to solve a function" might mean in different task situations during lessons – and, as we were not aware of some of these interpretations, we were intrigued. This fueled our confidence in the capacity of the PA to reveal hidden elements of the teachers' expertise and experience further. In 4.1, we also see the participants reflecting on studentblaming dominant narratives about learning in school mathematics – and we see them problematizing teaching routines that emerge from these narratives. In 4.2, we see participants' narratives on how mathematics changes over time moving from linear and static to nonlinear and dynamic. We credit the engagement with the history-focused tasks for allowing new narratives on mathematics to emerge. In 4.3 and 4.4, we see how the participants intend to work on their capacity to listen in the classroom and how they become more intrigued towards open-ended ways to assess their students' learning. In sum, across the four episodes, we observe evidence of object level and meta-level learning concerning mathematics and its pedagogy.

The openness of the PA also allowed participants to share their thoughts often regardless of, and beyond, what we had expected. As we take stock of the outcomes of this application of the PA, we note that it was not our intention to analyze the participants individually. Rather, our aim was to trace the reflexive potential of the PA, how individual experiences of the PA are informed by, and, in turn, influence its components with mutuality and reciprocity – and how participants alike benefited collectively from this experience.

The extent and quality of problematizing, as well as the evidence of discursive shifts observed as the participants engaged with the PA illustrate the potential of this approach to in-service teacher professional development (notably in the context of a part-time masters' program for mathematics teachers). In this sense, the PA constituted an instance of trialing and evaluating how a more critical study of history of mathematics – particularly one that endorses a history, not a heritage, perspective (Grattan-Guinness, 2004) and which encompasses the study of primary sources – may become an integral and productive component of teacher education. Moreover, the PA emerged as a means of putting teachers in the role of authors of their own in-service education processes. The mathtasks were designed to create opportunities for the teachers to draw extensively on their experiences, rather than rely on alienated forms of learning from academic experts who have little involvement with the classroom. We note that this tenet resonates with the participationist perspective in the commognitive lens. It also challenges models of teacher education in which teachers are merely fed with readymade prescriptions. Furthermore, the PA also encompass assignments that often appeared unusual to the teachers (reflective diaries, portfolios and teaching plans). The reflective elements of the meetings and the assignments also resonate with the overall aim of the PA, namely, to problematize mathematics and its pedagogy, and to challenge common and deeply-rooted ways of teaching and assessing mathematical learning.

We fully recognize that a one-off application of the PA may have modest, short-term influence on teachers' mathematical and pedagogical discourses, and we stress that more applications would certainly generate more insight into its impact. We are nonetheless heartened by the robustness of the evidence generated in this application, and we advocate the inclusion of suitably adjusted variations of the PA as a curricular component in mathematics teacher education and professional development programs at large. Through orchestrating commognitive conflicts comes problematization – and then shaking and re-shaping – of teacher discourses on mathematics and its pedagogy. Our PA evidence how this is a feasible, replicable and worthy enterprise.

With the steer and support of the commognitive lens – especially in designing the PA, analyzing the evidence of its implementation and evaluating its impact – we orchestrated engagement with the PA that resulted in concrete evidence of participants' problematization and discursive shifts. Our participants problematized what they have experienced so far as perennial narratives and metarules that govern mathematics (such as that mathematics is immutable and produced exclusively by isolated, gifted individuals) as well as narratives and routines that govern its pedagogy (such as that the teacher has an unshakeable, and overbearing, role of a classroom's "ultimate substantiator" (Sfard, 2008, p. 234) compliantly listened to by students). The commognitive language allowed us to trace these problematizations and discursive shifts over the relatively long period of the PA's four months. We see the unveiling of these affordances of the commognitive framework as suggestive of "a shift in the grain size of commognitive research" (Nardi et al., in press, p. tbc) and as a potentially fruitful expansion of the framework's utility. Pedagogical reform needs to be grounded in robust theoretical foundations – as proponents of sustainable change research emphasize (Reinholz, Rasmussen & Nardi, 2020). With our work, we make the case that the commognitive framework has the capacity to be exactly this.

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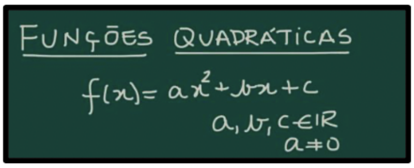
Appendices (or Supplementary Materials subject to limitations of space)

Appendix 1. The triggering *mathtask* in Round 3

In a High School Year 1 class, the teacher, Mr V, introduces the study of quadratic functions. He writes on the blackboard as below and stresses that the coefficients a, b and c take specific values.



He then gives the following exercises to the students:

Exercises:
1. Name the coefficients of the following functions.
a) $f(x) = 5x^2 + x - 7$;
b) $f(x) = 4 - x^2$;
c) $f(x) = x^2 - x$;
d) $f(x) = -4 + 2x^2 - 9x$

2. Consider $f(x) = x^2 - 3x + 4$. Determine.

x	$f(x)$
1	
2	
0	
-1	
	4
	2
	0

Students A and B work together on the exercises.

B: How are we going to solve this function?
A: This question here [1] is just to indicate the coefficients.
B: Ok. But what about the other one [2]? Don't we have to solve?

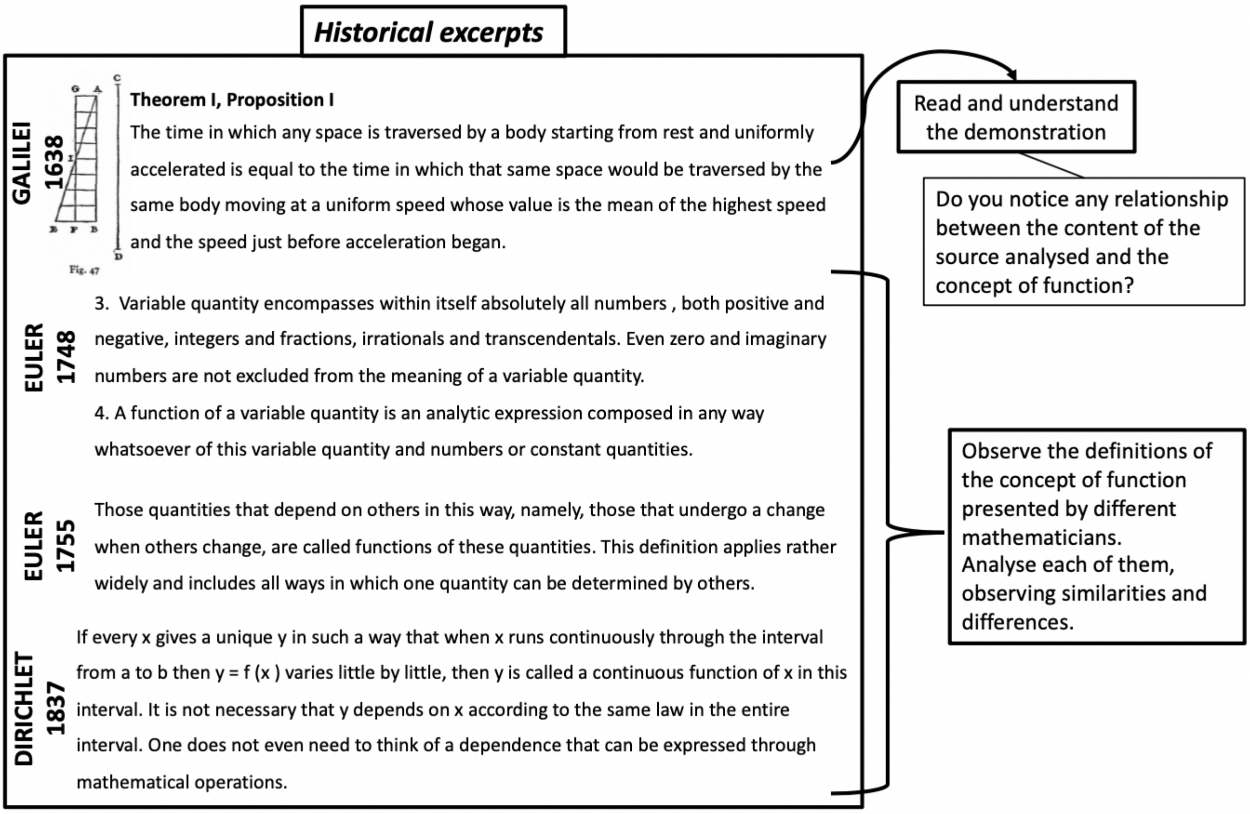
Mr V was intrigued with by Student B's question about "solving functions". After the class, he meets a colleague and says the following:

Mr V: I teach for years and until now, I cannot understand why the students keep saying that they need to "solve the function".

Questions:

a. What lies behind B's request to "solve the function" and Mr V's observation?
b. As a teacher, how would you approach a student who wants to "solve the function"?

Appendix 2. The *history-focused* task in Round 3



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Appendix 3a. The *reflection-on-teaching mathtask* in Round 3 *(object level)*

Pedro and Levi are two teachers who have been reviewing the examples they bring to their students. They have been studying new ways of teaching function for some time. Ronaldo, a newcomer to the school, found a piece of paper on the mathematics classroom desk and was sceptical about the exercises described.

The three exercises involve functions.

- The first exercise asks the students to relate graphs to tables.
- The second exercise presents six graphs, organized in two rows and three columns. The students are asked to indicate similarities and differences between the graphs in each row and each column.
- The third exercise is a practical activity for solving optimization problems in order to construct a box.

Questions

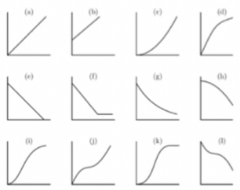
a. For each one of the exercises, indicate

i. goals;

ii. content that can be addressed.

b. Would you use these exercises? Why? In which grade?

c. What difficulties do you think students may face if they are asked to do these exercises? How would you address these difficulties? (answer for each exercise)



I. Coffee cooling

Time (minutes)	0	5	10	15	20	25	30
Temperature (ºC)	90	79	70	62	55	49	44

II. Preparing the supper

Weight (kilos)	3	4	5	6	7	8	9
Time (hours)	2,5	3	3,5	4	4,5	5	5,5

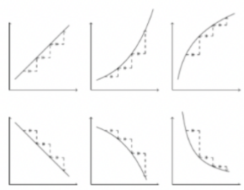
III. After three beers...

Time (hours)	1	2	3	4	5	6	7
Blood alcohol (mg/100ml)	90	75	60	45	30	15	0

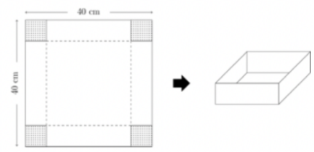
IV. How a baby grows before birth

Gestation time (months)	2	3	4	5	6	7	8	9
Baby's length (cm)	4	9	16	24	30	34	38	42

Exercise 1



Exercise 2



Exercise 3

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Appendix 3b. The *reflection-on-teaching mathtask* in Round 3 *(meta-level)*

Cassiano and Raphael are two undergraduate students in Mathematics.

Impressed with the International Congress of Mathematics (ICM) that took place in Rio de Janeiro in 2018, they talk about what they had seen at the biggest mathematics event in the world.

Cassiano: Did you see? *Matemaníaca* was at ICM, she recorded a video with some Fields Medal winners. One of the interviewees said it was the best question he has been asked, ever!
Raphael: Yeah ... I always wonder if mathematics was invented or discovered ...
Cassiano: What are you talking about?!?!
Raphael: Wow ... that was the question she asked them!
Cassiano: No, the best question was whether zero is natural or not!

Questions
a. In your opinion, is mathematics invented or discovered? Justify.

b. In the video, the arguments used by the two interviewed mathematicians are:

Alessio Figalli: Ah… This is just a tricky question, you know…. Everyone has its own opinion. You know… Mathematics was born to model the world, was born to understand whether the earth was round or not, then the diameter, how big was the earth and the numbers in general and then mathematics after. Always with this ideal of transforming nature and the physics into numbers and transforming our world into a description that we can do with numbers. So ... to me, I think it was invented by humans to describe the world.

Akshay Venkates: I think that depends a lot on which part of the math, but in the parts of math where I've worked, it's… I certainly feel , think of it as been discovered.

i. What is your opinion about the response of each of the medal laureates?
ii. Have you ever considered these arguments before? Or rather, have you ever asked yourself this question?

a. As a teacher, do you think it is important to consider this question? If not, why? If so, how?

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