FROM MATHEMATICS TO MATHEMATICS EDUCATION: TRIGGERING AND ASSESSING MATHEMATICS STUDENTS' MATHEMATICAL AND PEDAGOGICAL DISCOURSES

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In this paper, we present examples of activities and their assessment frame for mathematics undergraduate students' introduction to mathematics education research. The activities are inspired by studies that have identified and addressed differences between discursive practices in mathematics and in mathematics education. The proposed set of activities uses task design principles that contextualise mathematical content and the use of mathematics education theory to specific learning situations. Students' responses to these activities are assessed in relation to: clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory. We exemplify the application of these activities through responses from one student.

Keywords: Novel approaches to teaching, teachers' and students' practices at university level, mathematical discourse, mathematics education discourse, MathTASK.

INTRODUCTION

Some institutions have introduced courses on mathematics education in mathematics undergraduate programmes. The motivation for such courses is to introduce mathematics students to the field of mathematics education research or/and to prepare them for mathematics teaching. Very often, these courses familiarise students not only with the new content of the social science of education but also with the new, to them, practices of educational research, which is a very different enterprise from research in mathematics (Schoenfeld, 2000). For example, in mathematics education, in comparison to mathematics, the perspective is less absolutist, more contextually bounded and more focus on the reasons behind a student's error. Approaches are more relativist on what constitutes knowledge (Nardi, 2015) and evidence is not in the form of proof, but rather more "cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt" (Schoenfeld, 2000, p. 649). Thus, findings are rarely definitive and are more suggestive. Such epistemological differences affect the experiences of those who, although familiar with mathematics research and practices, are newcomers to mathematics education. Boaler, Ball and Even (2003) analysed the challenges of mathematics graduates when they embark on postgraduate studies in mathematics education. They describe the epistemological shift these students experience in their transition from systematic enquiry in mathematics to systematic enquiry in mathematics education. Nardi (2015) addresses challenges with such epistemological shifts in the context of a postgraduate programme in mathematics education that enrols mathematics graduates and with a focus on the programme's activities "designed to facilitate incoming students' engagement with the mathematics education research literature" (ibid, p. 135).

In this paper, we draw on studies that have observed and addressed such shifts at a postgraduate level to discuss a course that introduces mathematics education to undergraduate mathematics students. Specifically, we propose course activities and an assessment frame for students' engagement with both mathematics and mathematics education discourses. Mathematical discourse is related to the mathematical content seen at upper secondary and first year university level, whereas mathematics education discourse is related to theories on the teaching and learning of mathematics and key findings from mathematics education research.

In the next sections, we describe the theoretical underpinnings of this proposal and the teaching context in which these activities are implemented. Then, we offer an outline of the course and its learning objectives before presenting the assessment and the marking criteria with examples of activities. Finally, we exemplify data collected from one student, Emily, as well as analysis of this data in which we apply the proposed assessment frame to evaluate her responses. Our goal is to investigate whether and how the proposed activities and their assessment frame can generate insight into mathematics students' engagement with both mathematical content and mathematics education theory. We conclude with a brief discussion of the potentialities of such activities in undergraduate students' introduction to mathematics education research.

CONTEXTUALISING MATHEMATICS EDUCATION DISCOURSE

The theoretical perspective of this work is discursive and is inspired by the commognitive framework proposed by Sfard (2008) that sees mathematics and mathematics education as distinctive discourses and learning of mathematics and mathematics education as a communication act within these discourses. We are interested in discursive differences – and potential conflicts – between mathematics and mathematics education and we aim towards a balanced engagement with both. Specifically, we are interested in how students transform what they know about mathematics from their mathematical studies and about mathematics education theory they are introduced to during aforementioned courses into discursive objects that can be used to describe teaching and learning. This transformation is the productive discursive activity of *reification* proposed by Sfard (2008, p. 118). For example, the reification of the theoretical construct of *sociomathematical norms* (Cobb & Yackel, 1996) can describe a situation in which students negotiate different approaches in solving a problem with integrals, while the reification of integration processes can describe the mathematical choices, and the accuracy of such choices.

Nardi (2015) proposed a set of activities for Masters and doctoral level students for their introduction to mathematics education research. In these activities, students are asked to engage with literature from mathematics education research and to produce accounts of their readings. In addition, students are asked to produce accounts of instances in "their personal and professional experiences that can be narrated in the language of the theoretical perspective" (ibid, p. 151) featured in those readings. These accounts of students' experiences are called *Data Samples*. Engagement with literature together with the production of Data Samples has supported students situating their readings in their own experiences and their engagement with the discourse of mathematics education research. From the analysis of student interviews and written productions, emerged four themes regarding students' transition from studies in mathematics to studies in mathematics education: *learning how to identify appropriate mathematics education literature*; *reading increasingly more complex writings in mathematics education*; *coping with the complexity of literate mathematics education discourse*; and, *working towards a contextualised understanding of literate mathematics education discourse* (ibid). The contextualisation of the mathematics education discourse triggered by the Data Samples and described by the fourth theme are the inspiration for the activities we outline in this paper.

Another inspiration was from our work with pre- and in- service mathematics teachers in the *MathTASK*[1] programme in which we engage teachers with fictional but realistic classroom situations, which we call *mathtasks* (Biza, Nardi & Zachariades, 2007). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students. The mathematical problem, the student responses and the teacher reactions are all inspired by the vast array of issues that typically emerge in the complexity of the mathematics classroom and what prior research has highlighted as seminal. Teachers are invited to engage with these tasks through reflecting, responding in writing and discussing. At the heart of MathTASK is the claim that, theoretical discussion related to the teaching and learning of mathematics is not productive unless it becomes focused on particular elements of mathematics and its teaching embedded in classroom situations that are likely to occur in actual practice (Speer, 2005). The MathTASK design was followed in the activities we outline in this paper.

Recently, we analysed the responses to mathtasks of mathematics teachers who attended a master's level course in mathematics education (Biza, Nardi & Zachariades, 2018). Our analysis focused on teachers' engagement with mathematics and mathematics education research discourses – particularly in relation to mathematics education theories they had been introduced to during the course. A typology of four interrelated characteristics emerged from this analysis of the teachers' responses and used later in the analysis of trainee teachers' engagement with mathtasks (Biza & Nardi, 2019). An adaptation of this typology became the frame we deployed to assess students' engagement with the course activities:

[1] We use MathTASK (https://www.uea.ac.uk/groups-and-centres/a-z/mathtask) when we refer to the programme and its principles, whereas we use *mathtask* to refer to specific tasks designed with the principles of the MathTASK.

Consistency: how consistent is a response in the way it conveys the link between the respondent's stated pedagogical priorities and their intended practice? For example, do those who prioritise student participation in class propose a response to a classroom situation that involves such participation of students? Or, does their proposed response involve only telling students the expected answer to a mathematical problem?

Specificity: how contextualised and specific is a response to the teaching situation under consideration? For example, do those who write generally about valuing the use of vivid, visual imagery in mathematics teaching, propose a response to a classroom situation that involves specific examples of such imagery? Or, does the response include only a general statement of their preference?

Reification of pedagogical discourse: how reified is the pedagogical discourse that respondents have become familiar with during the course? For example, how productively are terms such as "relational understanding" (Skemp, 1976) or "sociomathematical norms" (Cobb and Yackel, 1996) used in the responses?

Reification of mathematical discourse: how reified is the mathematical discourse that respondents have become familiar with during prior mathematical studies? For example, how productively does prior familiarity with natural, integer, rational and real numbers inform a respondent's discussion about fractions in a primary classroom situation?

Before presenting how the typology was used in the assessment of students' responses to the activities, we first describe the context of the course and its learning objectives.

THE COURSE: CONTEXT, OBJECTIVES, STRUCTURE

The mathematics education course we discuss in this paper is offered as optional to final year mathematics undergraduate students in a research-intensive university in the UK. The aim of the course is to introduce students to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum. The learning objectives of the course include: to become familiar with learning theories in mathematics education; to be able to critically appraise research papers in mathematics education; to be able to compose arguments regarding the learning and teaching of mathematics by appraising and synthesising recent literature; to become familiar with the requirements of teaching mathematics – mathematical knowledge for teaching; to become familiar with key findings in research into the learning and teaching of mathematics; and, to practise reading, writing, problem solving and presentation skills with a particular focus on texts of theoretical content, yet embedded in key issues in mathematics education research.

Teaching activities include four hours per week (two for lectures and two for seminars). In the lectures, led by the first author, the theoretical content is introduced while in the seminars, led by the first author and teaching assistants, students present and discuss their work that involves preparing presentations of papers they have read, identifying examples from their experience (data samples, as per Nardi, 2015), solving problems and reflecting on their solution; and, responding to mathtasks (Biza et al., 2007).

Opportunities for feedback are offered during the seminars and in formative and summative pieces of writing. We now exemplify how mathtasks are used in the course and how the typology of the four characteristics (Biza et al., 2018) shaped the frame we deployed to assess student engagement with said tasks.

SUPPORTING AND ASSESSING STUDENTS' ENGAGEMENT WITH MATHEMATICS EDUCATION AND MATHEMATICS DISCOURSES

We now present an example from the summative assessment that was taken by the students in the middle of the term. This assessment had two parts. In Part I, students were asked to solve a mathematical problem and reflect on their solution by using the mathematics education terms they had been introduced up to that point. In Part II (Figure 1), which is our focus in this paper and was inspired by the MathTASK design, students are asked to choose and discuss *one set* of mathematics education theoretical constructs from a list of four that had been discussed in the sessions up to that point and, then, to use these constructs to respond to *one* of two proposed mathtasks.

In discussing the theoretical constructs, the students were also expected to give examples of (1) how these constructs have been used in research, and, (2) how these constructs can be used to describe their own experiences. (1) was aiming to assess students' skills to identify relevant literature and (2) to contextualise the use of these theoretical constructs in their own experiences (as in Nardi's (2015) Data Samples).

Mathtask A (Differential Equation) is in Figure 1 (left) and mathtask B (Reasoning) is in Figure 1 (right). Students' use of the theoretical constructs in their responses to these mathtasks, together with their aforementioned Data Samples, provide evidence of how mathematics education and mathematics discourses have been reified in the students' communication about teaching and learning issues.

For the purpose of this paper, we analysed students' written responses according to the marking criteria: *clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory* (Figure 2) based on the four characteristics proposed by Biza et al. (2018): *consistency, specificity*, *reification of pedagogical discourse* and *reification of mathematical discourse*, where "reification of the pedagogical and the mathematical discourses" have been replaced by the "use of terms and processes from mathematical theory" and "use of terms and processes from mathematical theory", respectively.

Our aim is to investigate mathematics students' engagement with both mathematical content (mathematical discourse) and mathematics education theory (mathematics education research). We now present excerpts from the responses of Emily (pseudonym), one of the students who attended the course and consented to the use of her responses as data for our study. Emily's responses were chosen for presentation in this paper as their articulation and subtlety allows us to illustrate how we used the assessment frame consisting of the aforementioned six marking criteria.

In Part II (2,000 words), you will discuss mathematics education theoretical constructs we have seen thus far and use these constructs to discuss learning incidents. Specifically, for this part of your assignment, you will choose **one** of the options below:

- Relational and instrumental understanding (Skemp, 1976)
- Procepts and reification (Gray & Tall, 1994)
- Social and sociomathematical norms (Cobb & Yackel, 1996)
- Semantic and syntactic proof (Weber & Alcock, 2004)

and **one** of the learning incidents below:

<table>
<tr><th>A: Differential Equation</th><th>B: Reasoning</th></tr>
<tr>
<td>
In a Year 13 class, students are asked to find the general solution of the differential equation $\frac{dy}{dx}=y^2$. One student proposes the following.

Student: This is a separable equation, so, I need to separate the variables:
$\frac{1}{y^2}dy=dx$
Now, I integrate both sides:
$\int y^{-2}dy=\int dx$
Which gives:
$-y^{-1}=x$
$y=-\frac{1}{x}$
The problem asks for the general solution, I need to add the constant. So, the general solution should be:
$y=-\frac{1}{x}+C$
Let me check what the answer says at the back of the book … [checks] Mmmm, it says:
$y=-\frac{1}{x+C}$
No, this cannot be correct, it must be a typo and it won't be the first time!
</td>
<td>
In a Year 7 lesson, students are asked to solve the following problem:

"Can you make the two columns of numbers below add up to the same total by swapping just two numbers between the columns? Explain why or why not."

<table>
<tr><td>1</td><td>7</td></tr>
<tr><td>3</td><td>2</td></tr>
<tr><td>8</td><td>4</td></tr>
<tr><td>5</td><td>9</td></tr>
</table>
The following conversation between students A and B takes place:
Student A: If I add up the columns at the moment, I get totals of, umm…17 and 22. So we need to make these the same.
Student B: How about we just try swapping some numbers and see what happens!
Student A: Okay, let's try the top two numbers first…If we swap 1 and 7, we get new totals of 23 and 16. That's worse than before!
Student B: Let's try some others…what about swapping 5 and 7?
Student A: No, that gives 19 and 20.
Student B: We're getting closer, though!
Student A: What about if we swap two numbers that are close together, like 2 and 3?
Student B: Ummm…that gives 16 and 23, that can't be right.
Student A: We could be here doing this forever!
Student C joins the conversation.
Student C: Maybe it can't be done and we have to show why not.
Student A: How would we do that then? We can't try every single possible swap…that would take too long!
</td>
</tr>
</table>

You will structure your work on Part II as follows:

Discussion of theoretical constructs [1000 words, 25 marks]: You will present the theoretical constructs of your choice through: discussing their meaning; describing their relationship with learning theories we have seen so far; giving examples (from research papers) on how these constructs have been used to analyse students' responses or behaviour in the classroom; and, giving an example from your own experience.

Discussion of the learning incident [1000 words, 25 marks]: You will discuss the incident of your choice by using the language of the theoretical constructs you have chosen in the first section. It will help you to choose a theoretical construct that can explain the issues you have identified in the incident of your choice. In this section: you will solve the mathematical problem of the incident; you will identify what the issues are in students' responses; and, you will describe your interpretation of why the student(s) have responded in such way.

Figure 1: Assessment activity inspired by the MathTASK design

Clarity: How clear, justified and transparent the arguments are. **Coherence:** How logically connected the arguments are. **Consistency:** How consistent the arguments are across the text. **Specificity:** How contextualised and specific the arguments are in the used examples and the discussed situation. **Use of terms and constructs from mathematics education theory:** How precise and accurate the arguments are in relation to the used mathematics education constructs and terms. **Use of terms and processes from mathematical theory:** How precise and accurate the arguments are in relation to the used mathematical concepts and process, such as definitions and proof.
Figure 2: The six marking criteria.

EMILY'S ENGAGEMENT WITH THE ACTIVITY

Emily chose the theoretical constructs of *instrumental and relational understanding* (Skemp, 1976) and mathtask A (Differential Equation). In her response, Emily summarises the constructs well (*use of terms and constructs from mathematics education theory*) and draws on a range of research literature that uses these constructs. Also, she reflects on her experiences with high *specificity*, by attributing students' approaches to their schooling experience (e.g., teaching practices, assessment, etc.) and by recognising that relational understanding "has never been required":

> It is clear that achieving a relational understanding is ideal, however, it does have its drawbacks and isn't always necessarily the optimal form of understanding. In lower levels of a student's mathematical education, topics do not need to be understood at a relational level [Skemp, 1976]. Throughout our schooling, when certain topics are met, pupils are often told that they do not need to understand how something works and just simply how to apply it. In my experience of first dealing with quadratic equations at GCSE, I did not know how the formula found the roots of the equation and was told that I did not need to know at that level. As I have progressed throughout my mathematical education there has never been a stage where it is thought necessary to gain a relational understanding as it is not required and is unknown by the majority of people. This lack of relational understanding is not due to a lack of disinterest or ability to understand but is purely due to the fact that such knowledge has never been required.

Later in her response to mathtask A, her approach takes a distance from the school influence and attributes students' approaches to their idiosyncratic characteristic as "instrumental" and "relational learners".

> In the classroom, pupils that understand in an instrumental way exhibit different characteristics to those who relationally understand. One of the main differences between the two types of pupil is not only how they answer questions they are asked, but also in the questions they ask and the answers they expect. A pupil who desires to achieve a relational understanding will eventually come up with an answer to a variety of questions even if it takes an extended period of time, whereas an instrumental learner can only answer an immediate answer to particular questions. […] This leads to the relational learner continuing to try until they gain an answer, unlike the instrumental learner who when they can no longer make any progress, often give up.

This characterisation of learners (as instrumental or relational) contradicts (*consistency*) her earlier view of approaches embedded in institutional practices. Although subtle, this *inconsistency* in Emily's response is a great opportunity for discussion around the simplistic lens of individual learning styles versus the actual complexity of institutional influences on learning processes.

Later in her response, she attempts to combine instrumental and relational understanding:

> Perhaps instrumental understanding should be viewed as a stage within the relational understanding and so students should be taught the skills required for both understanding. Merging the two states of understanding could result in being more powerful than either one alone thanks to the speed and ease of instrumental understanding alongside the profound knowledge gained through relational. Undoubtedly both understandings create a foundation on which new knowledge can develop which is key in mathematical education.

We note that, during class discussions, avoiding the dichotomy between instrumental and relational understanding had been repeatedly emphasised (*use of mathematics education terms and constructs*). This discussion has been assimilated in Emily's attempt to describe instrumental understanding as a "stage within" relational understanding.

In her response to mathtask A, Emily solved the problem correctly and spotted the mathematical error of the student in the incident (*use of terms and processes from mathematical theory*). In her explanation, she uses the relational/instrumental understanding language with precision:

> In the learning incident, it can be argued that the child in focus has an instrumental understanding of integration. Upon first reading the incident, this becomes evident due to the misunderstanding of where to place the constant of integration, c, as the pupil shows that they know they must include a constant when solving an indefinite integral. The student has displayed a common mistake of adding the constant once the equation had been rearranged to make y the subject.

However, her response does not explain the purpose of using the constant "c" in the integration. She thus misses the opportunity to demonstrate the mathematical explanation of *why* this is the correct integration (*specificity*, *use of terms and processes from mathematical theory*).

Overall, Emily's response demonstrates high *specificity* in the examples she provides and in her discussion of the incident. Her arguments are *clear* and *coherent*, although they are not always *consistent*, especially in relation to her views on institutional vs individual factors influencing students' approaches to learning. The use of mathematics education terms and constructs is precise and accurate (*use of terms and constructs from mathematics education theory*), while the *use of terms and processes from mathematical theory*, although without errors, does not demonstrate the precision and the mathematical detail we expect in the discussion about integration.

CONCLUSIONS

In this paper, we presented examples of mathtasks and their assessment frame used in a mathematics education course for mathematics undergraduates. The course activities are inspired by studies that have identified the epistemological differences between practices in mathematics and mathematics education (Boaler et al. 2003; Nardi, 2015; Schoenfeld, 2000) and have addressed these differences in the learning of postgraduate students (Nardi, 2015). The outlined set of activities uses task design principles that contextualise the use of mathematics education theory and mathematical content in specific learning situations (MathTASK design, Biza et al. 2007). Students' responses to these activities are assessed in relation to: clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory inspired by the four characteristics proposed by Biza et al. (2018). We see the potency of these activities in the introduction of mathematics students to mathematics education research as they invite students to engage both with mathematics and mathematics education discourses and to contextualise learning about mathematics education theories in their own learning experiences. Finally, we see these activities as affording opportunities for nuanced and concrete formative feedback.

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