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Mathematics teachers' work with resources

Four cases of secondary teachers using technology

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List of Acronyms

Information and communication technologies	ICT
Personal computer	PC
Documentational Approach	DA
Knowledge Quartet	KQ
Subject matter knowledge	SMK
Pedagogical content knowledge	PCK
Curricular knowledge (Shulman, 1986)	CK[1]
Technological pedagogical content knowledge	TPCK
Technological content knowledge	TCK
Technological pedagogical knowledge	TPK
Content knowledge	CK[2]
pedagogical knowledge	PK
Technological knowledge	TK
Mathematical Knowledge for teaching	MKT
Common content knowledge	CCK
Specialized content knowledge	SCK
Knowledge of content and students	KCS
Knowledge of content and teaching	KCT
Mathematics teachers' specialised knowledge	MTSK
Knowledge of Mathematics Teaching	KMT
Knowledge of Features of Learning Mathematics	KFLM
Knowledge of Mathematics Learning Standards	KMLS
Critical incidents	CI

[1] CK here is Shulman's (1986)
[2] CK here is based on Mishra and Koehler (2006)

Abstract

This study examines teachers' work with paper-based, technology and social resources with the use of two theoretical frameworks: the Documentational approach and the Knowledge Quartet. The former affords looking at teachers' resources and resource systems and how these are utilized under schemes of work. The latter affords a closer look at teachers' work during lessons and at their knowledge-in-action. Specifically, the study investigates how four upper secondary teachers use, re-use and balance their resources by looking at their schemes of work in class, through lesson observations; and, by reflecting on the details of their work and knowledge-in-action in pre- and post-observation interviews. Analysis examines five themes in relation to teachers' work. First, teachers use students' contributions as a resource during lessons. Second, teachers connect (or not) different resources. Third, institutional factors, such as examinations requirements and school policy, have impact on teachers' decisions and on how they balance their resource use. Fourth, when mathematics-education software is used, teacher knowledge of the software comes into play. Fifth, there is ambiguity in the identification of contingency moments, particularly regarding whether these moments were anticipated (or not) or provoked by the teacher. These five themes also suggest theoretical findings. In relation to the Knowledge Quartet, the findings indicate the potency of adding a few new codes or extending existing codes. This is especially pertinent in the context of teaching upper secondary mathematics with technology resources. In relation to the Documentational approach, this study introduces two constructs: *scheme-in-action* and *re-scheming*. A scheme-in-action is the scheme followed in class and documented from the classroom. Re-scheming is scheming again or differently from one lesson to another. Finally, the study discusses implications for practice and proposes the use of key incidents extracted from classroom observations towards the development of teacher education resources (e.g. for the MathTASK programme).

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1 Introduction

Teaching is a complex endeavour in which teachers interact with a range of tools and factors, and are expected to act promptly in response to expected and unexpected incidents in their classrooms by drawing on their knowledge and beliefs (Mishra & Koehler, 2006; Nardi, Biza, & Zachariades, 2012). "As a rule, teachers do not have full control of their working conditions. Instead, they have only limited control in relation to education, the context, the curriculum, being subject to the decisions and rules established by others" (Cyrino, 2018, p. 271). The demands the different factors and social interactions put on teachers implies that teachers' practices are highly contextual and not a mere replication of their lesson plans, knowledge or beliefs (Biza, Joel, & Nardi, 2015; Bretscher, 2014). In addition, such factors require mathematics teachers to adapt and re-adapt to the different and changing factors in their working environment (Gueudet & Trouche, 2009). For these reasons, it is essential that such factors are taken into account when studying teachers' practices (Herbst & Chazan, 2003). Factors that come into play in teaching include teachers' and students' roles and approaches, institutional policies, time pressure, unexpected incidents, and resources (Nardi, Biza, & Zachariades, 2012). To consider teachers' roles and approaches, the researcher can look at teachers' knowledge, beliefs or practices (Bretscher, 2014) or even at how these interplay through looking at teachers' identity (Goos, 2013). In turn, students influence the teaching/learning process through their interactions with the teacher (Jaworski & Potari, 2009), including the contributions they offer voluntarily or when invited (Rowland, Turner, Thwaites, & Huckstep, 2009). Institutional policies and restrictions including curricula and time pressure also have an impact on the teaching/learning process (Bretscher, 2014; Cyrino, 2018; Ruthven, Hennessy, & Deaney, 2008). Unexpected incidents that the teacher has not planned for push the teacher to make instantaneous decisions and impact on lessons' agendas (Clark-Wilson & Noss, 2015; Rowland, Thwaites, & Jared, 2015). In addition, resources play an important role in teaching. For example, for some decades, it was thought that having access to more teaching resources (e.g., textbooks, technology resources) meant better teaching practice (Cohen, Raudenbush, & Ball, 2003). However, such access creates opportunities to use resources, but "does not cause learning. Researchers report that schools and teachers with the same resources do different things, with different results for learning" (Cohen et al., 2003, p. 119). This implies that "more resources do not necessarily lead to better practice" (Adler, 2000, p. 206), and attention should be given to how these resources are employed in the classroom (Adler, 2000; Cohen et al., 2003; Mishra & Koehler, 2006). Inspired by this latter argument, this study takes a focus on resources (including technology resources) and their use as a starting point to investigate teachers' work and the different factors at play with it. It looks at how teachers balance the different factors in play and use resources for the purpose of accomplishing their teaching objectives.

This study adopts a broader view on resources in which resources can be material resources (e.g. textbooks) or social interactions (e.g. discussion with colleagues) (Gueudet & Trouche, 2009). Teachers interact with these resources, and "these interactions play a central role in the teacher's professional activity" (Gueudet, Buteau, Mesa, & Misfeldt, 2014, p. 141) and have a great impact on students' learning (Carrillo, 2011). Teachers choose their resources and manage them (Gueudet et al., 2014), and their professional knowledge influences their choices and use of resources. This use of resources in turn shapes teachers' professional knowledge. Hence, a close look at the interactions between the teachers and resources help identify opportunities to develop both (Rowland, 2013). Such a close look is particularly important when technology resources are used due to: the inspirations and aspirations that accompanied the development of technology resources (e.g. (Papert, 1993)), the arguments on how these add to the complexity of the teaching profession (Clark-Wilson & Noss, 2015; Mishra & Koehler, 2006); the overwhelming number of resources currently available online; and arguments on the use of technology resources for mathematics teaching being not well-established in practice yet (Bretscher, 2014; Hoyles & Lagrange, 2009; Mishra & Koehler, 2006; OECD, 2015; Trouche & Drijvers, 2014). Bretscher (2014) reports on a "widely perceived quantitative gap and qualitative gap between the reality of teachers' use of ICT and the potential for ICT suggested by research and policy" (p.43) at secondary schools. She suggests:

> "that given the right conditions, at least those currently existing in England, ICT might contribute as a lever for change; however, the direction of this change might be construed as an incremental shift towards more teacher-centred practices rather than encouraging more student-centred practices" (Bretscher, 2014, p. 43).

She adds that ICT is not often used in mathematics classes, and specifically mathematical software (e.g. GeoGebra) is even less frequently used. Ruthven et al. (2008) comment on the quality of technology (particularly mathematical software) use in "older classes" that are close to national examinations. They think that activities implementing such use "were more sharply focused on the assessed curriculum" (p.313). They add that in these classes "the emphasis was on avoiding student work with the software or making it manageable, and on structuring dynamic geometry use more directly towards standard mathematical tasks" (p.315). These arguments on technology use in secondary classes, especially older classes, motivated me to focus on mathematical software use in secondary mathematics teaching with attention to older classes (i.e. upper secondary classes).

Therefore, in this study, I investigate secondary mathematics teachers' ways of working with resources, including technology resources by looking at two aspects. One aspect is focused on looking at how teachers use the available resources and manage that use to achieve their teaching aims, and in line with their views about mathematics and of its teaching and learning (Gueudet & Trouche, 2009). Another aspect is focused on looking at how they enact their lessons and balance the different factors in their working environment (Jaworski, 1994; Rowland, Huckstep, & Thwaites, 2005) in light of the available resources. To investigate these aspects, I draw on two theoretical frameworks: the Documentational Approach (Gueudet & Trouche, 2009) to address the first aspect; and the Knowledge Quartet (Rowland et al., 2005) to address the second aspect. The two theories are used to analyse data from four secondary mathematics teachers that was collected in three stages: pre-observation interviews, lesson observations and post-observation interviews. The lesson observations were audio- or video-recorded. They covered a range of mathematical topics including volume of revolution, trigonometry, iteration, integration, modulus equations, polynomials and transformations. The interviews were audio-recorded and were based on inviting participants' reflections on critical incidents, where a critical incident is considered as an everyday instance of when a teacher makes a decision that shapes classroom interactions; a decision that reveals the teacher's approach and purpose. While the pre-observation interview was designed to allow the teacher to reflect on specific scenarios that can happen in classroom; the post-observation interviews were based on reflections on critical incidents from the observation. The three stages of data collection help answer my main research question:

RQ: How do secondary mathematics teachers work with resources including technology in their classroom?

Inspired by this research question, I include in the chapters that follow a review of relevant literature and the adopted framework, the methodology, data and analysis, and findings and conclusions. The second chapter, the literature review, includes seven sections covering discussion of: resources; technology in mathematics education; teachers' knowledge; teachers' beliefs, practices and identity; the unexpected in the classroom; four theories on observing teaching (the teaching triad, the instrumental approach, the Documentational Approach and the Knowledge Quartet); and finally the theoretical framework of this study. The third chapter sets the methodology of this work including the methodological approach, the definition of critical incidents, this study's context and design, ethics, data collection overview and how the data is analysed. The fourth chapter on data and analysis overviews the data sets and their analysis for the four participants George, Martin, Adam and Charlie respectively. The fifth chapter draws on the analysis in the fourth chapter and offers a summary of findings, answers to the research question, study limitations, future research, implications for practice and my personal reflections.

2 Literature Review

As the research question of this study is "How do secondary mathematics teachers work with resources including technology in their classroom?", the literature discussed will be related to the key aspects: teachers' work (i.e. teaching), resources, and technology. To look at teachers' work and teaching approaches, I review some theoretical constructs and lenses in relation to: the definition of resources; technology in mathematics education; teachers' knowledge; beliefs, practices and identities; the unexpected in the classroom; and four theoretical lenses for observing teaching the teaching triad, the instrumental approach, the Documentational Approach, and the Knowledge Quartet. To start with, I review how resources and technology resources are seen in mathematics education and what they present in this study.

2.1 Resources

Resources are artefacts, teaching materials, and social and cultural interactions that are available for a teacher in her teaching and preparation for teaching (Gueudet & Trouche, 2009). An artefact is defined as an object that is designed to be used for specific purposes (Gueudet & Trouche, 2009). It can be a tool (e.g. a ruler), a mathematical technique (e.g. the quadratic formula), or a piece of software (e.g. Excel). Teaching materials are materials that are designed for mathematics teaching and learning including textbooks and calculators, for example (Adler, 2000). Social and cultural interactions refer to interactions with the environment, students and colleagues, for example a conversation with a colleague or a student's feedback on an activity (Gueudet & Trouche, 2009). The term "resource" can also be "the verb re-source, to source again or differently" (Adler, 2000, p. 207).This implies that teachers interact with resources, manage them and reuse them to achieve their teaching aims (Gueudet, 2017), and that "the effectiveness of resources for mathematical learning lies in their use, that is, in the classroom teaching and learning context" (Adler, 2000, p. 205). It is noted that "different teachers enact the same curriculum materials differently (e.g., Chavez, 2003), and that the same teacher may enact the same curriculum materials differently in different classes (Eisenmann & Even, 2011)" (Kieran, Tanguay, & Solares, 2011, p. 190). In this study, I examine four teachers' work and interactions with resources, in the wider range defined here, but with a focus on cases where technology resources are integrated as well as other resources. Discussion about the rationale behind the focus on technology is in the next section.

2.2 Technology in mathematics teaching

Interpretations of the term technology in mathematics education can vary (Clark-Wilson & Noss, 2015; Mishra & Koehler, 2006). It can be used to refer to artefacts including for example textbooks, pens, projectors, calculators and posters (Mishra & Koehler, 2006). However, its more commonly used nowadays to refer to digital technologies including computers, educational software, digital resources (like e-textbooks), and other new resources that are “not yet part of the mainstream” (Mishra & Koehler, 2006, p.1023). In this study, the latter interpretation is used, where the technology refers to digital technologies that transform the classroom environment, shape teacher-student-resources interactions, and facilitate new ways of expressing mathematical meanings (Clark-Wilson & Noss, 2015; Pepin, Choppin, Ruthven, & Sinclair, 2017). In the following three subsection I review the inspirations and aspirations around technology, practice with technology, and define mathematics education software (which is the focus of this study).

2.2.1 Inspirations and aspirations around technology

Papert’s (1993) inspirational Mindstorms anticipated that computers “can be carriers of powerful ideas … they can help people form new relationships with knowledge… and [establish] an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building” (Papert, 1993, pp. 4-5). Since then, many developments in the field of mathematics-teaching technology have taken place, and these were accompanied by a great number of research studies. To start with, research studies on the use of technology were mainly based on a constructivist perspective implying that knowledge is constructed by the learner (Drijvers, Kieran, Mariotti, Ainley, Andresen, Chan, Dana-Picard, Gueudet, Kidron, Leung, & Meagher, 2009; Guin & Trouche, 1998). After the rise of Vygotsky’s sociocultural perspective (Vygotskiĭ, 1978), some theories were influenced by this as well as the constructivist approach (Drijvers et al., 2009). For example, Noss and Hoyles (1996) suggested that the computers can act as a "window" on mathematics education as they:

> "… afford us insights into people's mathematical meanings […]. It will provide a setting to help us make sense of the psychology of mathematical learning, and at the same time, afford us an appreciation of mathematical meaning-making which takes us well beyond the psychological realm. Above all, the computer will open new windows on the construction of meanings forged at the intersection of learners' activities, teachers' practices, and the permeable boundaries of mathematical knowledge" (p.2)

They used the window metaphor: computers can be like windows on mathematical knowledge for learners, and windows on student's ideas and progress for educators. However, they added that:

> "windows are for looking through, not looking at. It is true that windows mediate what we see and how we see it. Equally, windows can, at times, be objects for design and study. But in the end, what counts is whether we can see clearly beyond the window itself onto the view beyond." (Noss & Hoyles, 1996, p. 10)

Similarly, Falbel (1991) thinks that computers are neutral, and that the way we use computers in education is what makes them "good or bad" (p.35). As well, Papert (1993) criticized "technocentrism" - "the fallacy of referring all questions to the technology" - which only studies the computers' role in education in a superficial way. He added that computers had no control on education; they should not be presented as providers of facts and information, nor as directors of learners' work, and not even as instructors to learners (Papert, 1993). If computers are to be employed in education, they should be used to "foster individual development" and give learners the chance to lead their work and expand their choices (Papert, 1993). Criticism about computers’ use was also built on what Papert (1990) called "scientism", which only tested the effects of technology on education by conducting "small" experiments. Such experiments, he argued, did not change enough educational factors to reveal valid results; they did not allow computers to act as devices for a fundamental change in mathematics education. Therefore, Papert (1990) thinks that we should appropriate technology to our needs in order to make use of it - teachers and children are still the leaders in the educational process. This argument refutes any criticism based on dehumanizing the educational process by using technology by emphasizing that the use of computers for education should be an interactive process.

> "Not the imparting of 'information' from machine to person, but an enhanced communication between people; not the transmission of A's understanding to B, but an arena in which A and B's understandings can be externalized; not a means of displaying A's knowledge for B to see, but a setting in which emerging knowledge of both can be expressed." (Noss & Hoyles, 1996, p.6)

The main point here is that computers should mediate between teachers and learners (Hoyles & Noss, 1992). "The computer, as we shall see, not only affords us a particularly sharp picture of mathematical meaning-making; it can also shape and remould the mathematical knowledge and activity on view" (Noss & Hoyles, 1996, p.5), pointing out the "construction of meaning is a human quality" (p.2). Papert (1991) argues that "the thrust of constructionism is to create a learning environment in which learning will come about in activities driven by enterprise and initiative" (Papert, 1991, p.22). Theories adopting a sociocultural view on learning with computers appeared after the rise of Vygotsky's sociocultural perspective (Vygotskiĭ, 1978); one example of these is the instrumental approach (Guin & Trouche, 1998) which will be discussed in section 2.6.2 (Drijvers et al., 2009).

Papert & Turkle (1991) argue that:

> "… the computer stands betwixt and between the world of formal systems and physical settings; it has the ability to make the abstract concrete. In the simplest case, an object moving on a computer screen might be defined by the most formal of rules and so be like a construct in pure mathematics; but at the same time it is visible, almost tangible and allows a sense of direct manipulation that only the encultured mathematician can feel in traditional systems" (Papert & Turkle, 1991, p. 162)

They add that this makes learning in an environment suitable for different learning styles by affording "a degree of closeness to objects" (Papert & Turkle, 1991, p.167). So, students are prepared to engage with computers and computers afford both soft and hard thinking styles. Computers also afford the integration of the informal idea and formal mathematics where the informal is students' language and perceptions and the formal is the language the technology understands (Hoyles & Noss, 2003). They afford the externalization of knowledge when students argue and generate their own explanations that are shaped by the tools in use (Harel & Papert, 1991; Hoyles & Noss, 2003). This can happen through verbalizing thoughts which helps examine and organize knowledge. Verbalised arguments, explanations and thoughts can be shared with the observer/teacher, peer(s), afford a closer look at the learners' progress and facilitates learning not only by doing, but also by "thinking and talking about what you do" (Harel & Papert, 1991, 42). And, the computer affords a qualitative feedback that allows self-evaluation and correction skills to be developed (Goldstein & Pratt, 2003; Harel & Kafai, 1991).

Research exposed aspects to consider when working in a computer environment. One aspect is that the environment should afford learners' expression of meaning and creativity and encourage discovery. It should afford what Falbel (1991) describes as "active learning" instead of a "passive education" that simply teaches the learners a series of instructions. In other words, computers should be used to promote "learning by doing", not "learning by being told" (Harel & Papert, 1991, p.41). Another aspect is motivation, activities should be motivating for learners to engage with the work and use the computer to explore. To achieve that, an activity should have a clear target the learners work to achieve (Ainley & Pratt, 2005; Goldstein & Pratt, 2003). Finally, the activities should be designed to encourage students to utilize mathematical ideas and concepts. It is argued that one of the reasons that keep learners away from mathematics is that they do not see the usefulness of its formal concepts. Computer-based activities can be designed to offer this usefulness and some instrumental learning (Ainley & Pratt, 1996; Goldstein & Pratt, 2003). Computers offer semi-concrete representations that are easier to students to understand than abstract knowledge (Noss & Hoyles, 1996).

2.2.2 Practice with technology

Despite all the inspirations discussed in the previous section, it is perceived that technology use is not yet well established in mathematics education. Trouche and Drijvers (2014) argue that:

> "… the integration of digital tools is considered as problematic in the sense that their availability questions the goals of mathematics education as well as current teaching practices (Lagrange et al., 2003). The latter type of question has not yet been answered in a satisfying way, which may explain the limited integration of digital tools in mathematics education in spite of earlier high expectations". (p.193)

Also, according to the OECD report (OECD, 2015), computers are not used frequently in mathematics classes. This in some cases is due to the lack of ICT resources, or a reliable internet connection, while in others it is due to the school's preference (OECD, 2015). The report emphasized the role of policies such as the national curriculum, as well as the efficacy of the training provided, as drivers in the use of technology, but it also reported that the tendency to employ technology in mathematics classes is dependent on teachers rather than school policies (OECD, 2015). Furthermore, a survey conducted by Bretscher (2014, p. 43) explored "the widely perceived quantitative gap and qualitative gap between the reality of teachers' use of ICT and the potential for ICT suggested by research and policy" at secondary schools in England. The study showed that although teachers generally appreciated the role of technology in supporting students learning of mathematics, the quantitative gap was evident in students' direct access to technology, which took place no more than twice per term in 75% of the studied cases, with very rare access to mathematical software (Bretscher, 2014). Whereas, the qualitative gap reflected teachers' tendencies to use technology in whole-class contexts and in ways that did not promote student-centred work (Bretscher, 2014). The study also commented on computer suites access as an obstacle that could have influenced teachers' use of technology in mathematics lessons (Bretscher, 2014). Though, even when such suites were available, teachers were using technology in ways that did not disturb their classroom management and leadership and with main reliance on presentational forms of technology like interactive whiteboards and projectors (Bretscher, 2014). Ruthven et al. (2008) also show teacher's tendencies to follow a teacher-centred approach when using technology. They add that in higher secondary classes, where students are close to taking national examinations "the emphasis was on avoiding student work with the software or making it manageable, and on structuring dynamic geometry use more directly towards standard mathematical tasks" (Ruthven et al., 2008, p. 315).

The issues behind specific use/underuse of technology in mathematics education might hide some underlined criticism of such use. For example, the OECD (2015, p.146) concluded that:

> "Resources invested in ICT for education are not linked to improved student achievement … [L]imited use of computers at schools may be better that no use at all, but levels of computer use above the current OECD average are associated with significantly poorer results. (OECD, 2015, p.146)

The OECD (2015) report considered the quantity of ICT use in classrooms but failed to look at the quality of such use, which is an equally important factor. Also, the report considered using technology to support "(traditional) learning experiences… and perhaps [act] as a catalyst of wider change" (OECD, 2015, p.50). The first scenario (i.e. supporting traditional learning approaches) is not a strong case for the use of technology. The main point of using technology is to revolutionize the teaching of mathematics, and achieve a major change in the way students learn mathematics. The report recognized that such drop in students' performance might be associated with the distractions the computer offers like "chatting on line… and practicing and drilling" (OECD, 2015, p.154). It is unavoidable that students will find distractions in all sorts of learning environments (traditional and computer-based), and here the agency of the teacher should come to play its role.

Although digital resources can be adapted to provide access to formal mathematical knowledge, the overwhelming availability of such resources “yields a deep change in teachers’ professional knowledge and development” (Gueudet & Trouche, 2009, p. 199) as it adds to the complexity of the job (Fuglestad, Healy, Kynigos, & Monaghan, 2010). It is also argued that digital resources act best when designed by teachers as they “challenge teachers’ knowing with respect to teaching and learning mathematics, [and] also regarding their view of the nature of mathematics itself”. The interactions with such challenging-to-design digital resources develop teachers’ identities, and hence develops the ways teachers orchestrate their classroom instructions. And when looking at teachers’ work with digital resources, it is important to consider teachers’ “hiccups” which Clark-Wilson (2013, p. 34) defines as “the perturbation experienced by the teacher, triggered by the use of the technology that prompted him or her to rethink their personal mathematical, pedagogical or technological knowledge”. These should be evident moments in which the teacher reconsiders her/his choice of instructions because of an unexpected experience during the use of technology to teach mathematics (Clark-Wilson, 2013, p. 34). Clark-Wilson and Noss (2015, p. 97) categorised hiccups in the following seven categories:

- Aspects of the initial activity design
- Interpreting the mathematical generality under scrutiny
- Unanticipated student responses as a result of using the technology
- Perturbations experienced by students as a result of the representational outputs of the technology
- Instrumentation issues experienced by students when making inputs to, and actively engaging with the technology
- Instrumentation issues experienced by teachers whilst actively engaging with the technology
- Unavoidable technical issues

The hiccups framework is used to reflect on unexpected events in classroom. When employing technology, the chances of hiccups are higher because resources become more complex: students’ needs become more evident (e.g. if a student knows more about technology than his/her teacher); tasks can be more challenging for teachers to design; and the management of learning becomes more complicated with the higher chances of distraction. In addition, there are some factors to consider when technology is used, and these are based on a number of premises: first, the ICT use dependency on the teacher training provided (Gueudet et al., 2014; OECD, 2015); secondly, the availability of ICT devices and internet connection (Bretscher, 2014; OECD, 2015); thirdly, the national curriculum obligations (OECD, 2015); fourthly, and most importantly, the "[e]ducation policies that aim to embed ICT" (OECD, 2015). To reflect on the complexity of teaching with technology, this study will look at teachers' work when using technology tools for mathematics teaching and their ways of orchestrating lessons. The next section explains the technology tools and resources of interest to this study.

2.2.3 Mathematics-education software

The terms used to refer to technology resources in the previous two sections (i.e. technology, computer and ICT) are general and inclusive of both hardware devices and software pieces and digital resources. Among software pieces there are some that have been adapted and used for teaching mathematics like spreadsheets; and others that have been designed specifically for mathematics education like Cabri (Drijvers et al., 2009). In this study, the focus is on software that is specifically designed for mathematics teaching; in this work I will call it mathematics-education software. Mathematics education software started in forms of algebra software packages (e.g. Maple, Mathematica), and then in forms of tools for geometry (e.g. Logo, Cabri) (Jones, Mackrell, & Stevenson, 2010). The latter have become more widespread and commonly used in mathematics classrooms (Jones et al., 2010). Some software combines algebra and geometric packages (e.g. GeoGebra) (Hall & Chamblee, 2013). One main feature in Geometry software that they afford "a way of building, and developing, our visual intuition across a range of geometries" (Jones et al., 2010, p. 50). They also facilitate "watching and controlling animations on the screen" (Butler, 2012, p. 44). Some software offer working in two-dimensional or three-dimensional environments (e.g. Cabri, Autograph and GeoGebra). The dragging feature in software like Cabri allows the user to watch continuous variation of the object whose element is being dragged, while maintaining the mathematical relationships of the constructed object (Jones et al., 2010; Sinclair & Yurita, 2008). Software like Cabri, Autograph, GeoGebra and Desmos allow the user to see multiple representations that are linked (Drijvers et al., 2009; Hall & Chamblee, 2013; Sinclair & Yurita, 2008). This includes graphical, algebraic representations as well as computer based and paper-based ones. Another feature is in the option of using sliders to control and continuously change values of some variables; and watching the effect of this change on the graphical representation (Hall & Chamblee, 2013). Mathematics-education software provides limited feedback to the students by validating or not validating (e.g. not reserving one object's intended mathematical properties when one of its elements is dragged) their answer, and leave the prospect open for students' exploration and investigation (Drijvers et al., 2009). All these affordances come with limitations. These include the flatness of screens and shortcomings in relation to the dragging feature (Jones et al., 2010). Also, there are concerns about cases when users do not question what they are seeing on the screen, leading them to non-reflective “fishing behaviour” (Artigue, 1995) in (Guin & Trouche, 1998).

This study aims to address issues of mathematics-education software use for teaching, in relation to their affordances and limitations of software, but also in relation to how they are used and balanced when integrated with other resources. For that purpose, the study needs to use theoretical constructs and lenses. The next sections in this chapter will address some theoretical constructs and lenses that are used to look at teaching, the section after identifies which of those form the theoretical framework of this study. First, I review arguments and discussion in relation to teachers’ knowledge, beliefs, practices and identities, and the unexpected in the classroom. Then, I review four theoretical lenses that are used for observing teaching: the teaching triad, the instrumental approach, the Documentational Approach and the Knowledge Quartet respectively.

2.3 Teachers’ knowledge

Teachers’ knowledge is one of the factors that affect the teaching process, as it has implications for what teachers teach, how they teach it and why they choose to teach it in specific way (Mishra & Koehler, 2006; Shulman, 1986). Shulman’s (1986) work proposed that teachers’ knowledge includes subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK). SMK represents teacher’s knowledge of the meanings and concepts of the taught subject, as well as her knowledge of “why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice” (Shulman, 1986, p. 9). It also includes knowledge about how the subject’s facts are based on specific organisation of perceptions and principles, and how the validity or truth of arguments is determined (Shulman, 1986). PCK concerns knowing how to present the subject’s concepts and make it accessible to learners; this requires ability to explain and articulate the subject’s concepts using different forms of representation where appropriate and taking into account the complexity of the taught topics and how to make them more comprehensible to learners (Shulman, 1986). Curricular knowledge (CK) represents the knowledge about the curriculum, framework and teaching materials designed to teach a specific subject at a specific level. It also includes lateral knowledge of the curriculum of other subjects taught at the same level (Shulman, 1986). Furthermore, Shulman (1987) addressed the knowledge base which is “a codified or codifiable aggregation of knowledge, skill, understanding, and technology, of ethics and disposition, of collective responsibility — as well as a means for representing and communicating it" (p.4). He described the knowledge base seven categories (p.8):

- Content knowledge
- General pedagogical knowledge including classroom management strategies
- Curriculum knowledge including knowledge about "materials and programs that serve as "tools of the trade" for teachers"
- Pedagogical content knowledge
- Knowledge of learners and their characteristics
- Knowledge of educational contexts including knowledge about the classroom context, school governance and financing and the culture of the community
- Knowledge of educational aims and values, along with their philosophical and historical backgrounds.

Mishra and Koehler (2006) suggested the inclusion of knowledge about technology in teachers' knowledge, they argued for extending Shulman's pedagogical content knowledge PCK to technological pedagogical content knowledge TPCK. The latter, they think, includes content knowledge (CK), pedagogical knowledge (PK), technological knowledge (TK), the intersection between each pair of these (pedagogical content knowledge (PCK), technological pedagogical knowledge (TPK), technological content knowledge (TCK), and the intersection between the three of them (TPCK). To explain what Mishra and Koehler (2006) have added in relation to technology, I summarize their definitions of the technological knowledge as well as the last three categories (TPK, TCK and TPCK). While PK and PCK definitions are based on those mentioned earlier from Shulman (1987).

- Technological knowledge (TK) is the knowledge of how to use and manage standard (e.g. books) and advanced technologies (e.g. digital Autograph software), and the "ability to learn and adapt to new technologies" (Mishra & Koehler, 2006, p. 1028).
- Technological pedagogical knowledge (TPK) is about knowing how to use technology for teaching and learning, for example for creating discussions and keeping progress records (Mishra & Koehler, 2006).
- Technological content knowledge (TCK) is about the interplay between the content and the technology, and how the content can be viewed and taught using a specific technology (Mishra & Koehler, 2006). For example, how one can use GeoGebra to construct geometrical shapes.
- Technological pedagogical content knowledge (TPCK):

> “TPCK is the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones”. (Mishra & Koehler, 2006, p. 1029)

Also in relation to Shulman’s categories, Ball, Thames, and Phelps (2008, p. 402) suggested what they described as a “refinement” of Shulman’s categories of knowledge. Their refinement includes six categories defined as follows:

- Common content knowledge (CCK): the mathematical skills used in teaching as well as other settings; by teachers as well as others. In brief, this knowledge is not exclusive to teachers and teaching.
- Specialized content knowledge (SCK): the mathematical skills that are exclusive to teaching, like “looking for patterns in students errors” (Ball et al., 2008, p. 400). It goes beyond knowing what is being taught to students and requires higher awareness of mathematical reasoning and interpretations. It also includes knowledge about mathematical terminology and representations.
- Horizon knowledge: the knowledge about “how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403).
- Knowledge of content and students (KCS): includes teachers’ knowledge about students as well as about mathematics. In other words, it looks at teachers’ appreciation of students’ needs, how they think and how they express their thoughts.
- Knowledge of content and teaching (KCT): includes teachers’ knowledge about teaching as well as about mathematics, i.e. knowing how to design tasks and instructions, how to sequence them, what representations and methods to use, and how to respond to students’ ideas and contributions.
- Knowledge of content and curriculum: is the same as Shulman’s curricular knowledge.

The first three categories (common content knowledge, specialized content knowledge, horizon knowledge) correspond subject matter knowledge and the last three (knowledge of content and students, knowledge of content and teaching, knowledge of content and curriculum) correspond pedagogical content knowledge and include curricular knowledge (Figure 1). All of these categories come under what they called mathematical knowledge for teaching (MKT) which is defined as the knowledge “entailed by teaching” referring to the “mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” (Ball et al., 2008, p. 399).

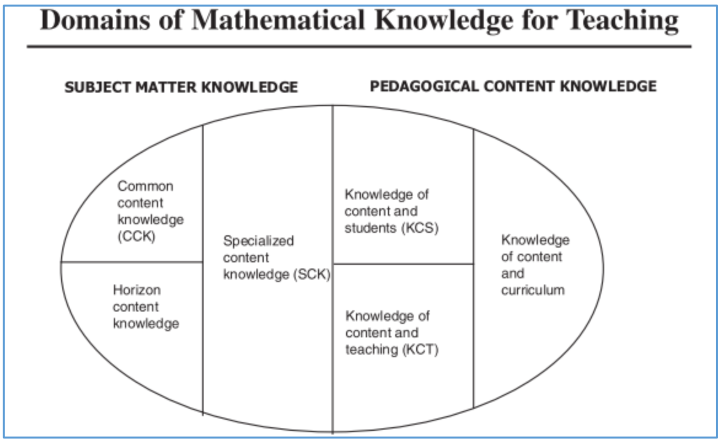

Figure 1: MKT domains as in Ball et al. (2008, p. 403)

Carrillo-Yañez et al. (2018) argue that the MKT categories focus “on practice as carried out in class, ignoring the knowledge that teachers might bring into play when carrying out any other kind of activity as a teacher” (p. 238). They also argue that the overlay between categories makes using the model for analysis difficult. Instead, they suggested a different model, mathematics teachers’ specialised knowledge (MTSK), that in their opinion offers a clearer view of the specialised knowledge teachers need for teaching. One that links mathematical knowledge to the knowledge of its teaching and learning. Their MTSK includes two main categories: Mathematical Knowledge and Pedagogical content knowledge, defined as follows:

- Mathematical Knowledge: Similar to Shulman’s definition of mathematical content knowledge with broader view that considers the “characteristics of mathematics as a scientific discipline, and at the same time recognise a differentiation between Mathematics per se and School Mathematics” (Carrillo-Yañez et al., 2018, p. 240). It includes three sub-domains: Knowledge of Topics which is about knowing the mathematical content itself; Knowledge of the Structure of Mathematics which is about the interlinks and connections between mathematical concepts; and Knowledge of Practices in Mathematics which is about how one works systematically in mathematics (Carrillo-Yañez et al., 2018).

- Pedagogical Content knowledge: It includes pedagogical knowledge that arises mainly from mathematics and its teaching and learning; and excludes general pedagogical knowledge even if it is applicable to mathematical contents. Its subdomains are: Knowledge of Mathematics Teaching (KMT) which can be developed from familiarizing self with research or from personal experiences, Knowledge of Features of Learning Mathematics (KFLM) which focuses on mathematical content and how students construct meanings, Knowledge of Mathematics Learning Standards (KMLS) which includes ways of assessing students' levels such as curriculum specifications.

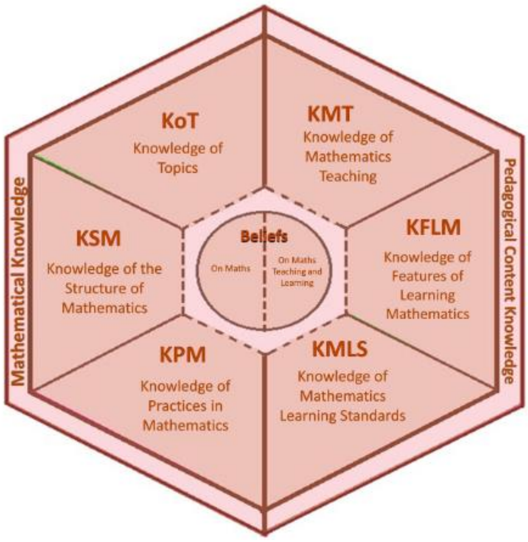

Figure 2: MTSK diagram as in Carrillo-Yañez et al. (2018, p. 241)

MTSK (Figure 2) has teachers' beliefs about mathematics and its teaching and learning at its centre, and as part of teachers' knowledge. On the other hand, knowledge can be defined as "beliefs held with certainty or justified true belief. What is knowledge for one person may be belief for another, depending upon whether one holds the conception as beyond question" (Philipp, 2007, p. 259). Therefore, beliefs and knowledge are interconnected. The next section discusses teachers' beliefs in more details, along with teachers' practices and identities.

2.4 Teacher beliefs, practices and identity

Beliefs play an essential role in influencing practices and actions in education mainly because teachers usually try to translate their beliefs into "pedagogical practices" (Biza, Nardi, & Joel, 2015). McLeod and McLeod (2002) acknowledged the "difficulty" of reaching a general agreement on one single definition. However, they emphasized that the definition of beliefs is audience and focus dependent (McLeod & McLeod, 2002). The definition adopted in this study is:

> Beliefs—Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (Philipp, 2007, p. 259).

Research looked at teachers' beliefs in relation to different areas including:

- Pedagogical beliefs: looking teachers' preferences of a student-cantered approach or a teacher-led one; how investigative and open-ended the tasks they choose are; and their tendency to teach through playing, stories, or through the use of technology etc (Speer, 2005).
- Curricular beliefs (Philipp, 2007): These are about teachers' instructional targets and mathematical content, like consideration of specific solutions as valid/not valid alternative methods (e.g. visual and algebraic solutions) and addressing a mathematical skill as essential or advantage (e.g. factoring of linear equation (Speer, 2005)).
- Epistemological beliefs (Goos, 2013): These concern teachers' beliefs about mathematics, its nature, and methods.
- Technology-related beliefs (Philipp, 2007): These concern teachers' individual preferences of using or not using technology, and what types of technology use are useful if any.
- Beliefs in relation to students (McLeod & McLeod, 2002; Philipp, 2007; Speer, 2005): These concern how students think and learn, what they wish to achieve, and what their behaviour is like.
- Beliefs about the social context: this was used by McLeod and McLeod (2002) to express the students' beliefs about the social context. However, the same category can be helpful to address teachers' beliefs about schools, policies, examinations, parents' expectations, and social environment. This category enables one to extend focus and address the social environment surrounding teachers within and outside schools.

Beliefs were categorized as "professed" -in reference to the verbally claimed ones- or "attributed" -in reference to the ones reflected by practice- (Speer, 2005, p.361). They were categorized also as espoused (i.e. stated) and enacted (i.e. refined by constraints and contextual conditions, in order to be used in practice) (Ernest, 1989). Also in consideration of contextual conditions, Hoyles (1992) did not approve the decontextualization of beliefs; and argued that instead of talking about beliefs and beliefs-in-practice or about espoused and enacted beliefs one should consider the "situatedness of beliefs" (p.40). Hoyles definition of situated beliefs implied that they are dependent on the contexts in which they are shaped and expressed like the classroom context. That in turn means that "situations are co-producers of beliefs, and as situations differ, so do beliefs" (Skott, 2001, p. 5).

In addition, the complexity of the teaching situations imposes several contextual factors that have impact on teachers' classroom actions (Nardi et al., 2012; Speer, 2005). Therefore, although teachers' practices are performed by them and to some extent informed by their beliefs, there are more factors in play and practices are not a perfect reflection of beliefs (Nardi et al., 2012; Speer, 2005). Cyrino (2018) argues that teachers are subject to political commitments and relationships including their school contexts and policies, educational organizations, and public policies in relation to education. These institutional commitments limit teachers' control over their teaching and make them subject to rules they did not establish. Steele (2005, p. 296) argues that teaching "is by nature conditional: conditional on the students, the precise moment in time, the content at hand, the teachers' beliefs, and so forth". This implies that any teaching act is highly contextualised and consequently unique due to: institutional and contextual conditions that can play different roles even within one classroom (Skott, 2001); and teachers' beliefs are "mediated by the social contexts of education" (Skott, 2001, p. 3) or even situated (Hoyles, 1992). To shed light on this, one can consider the "practical rationality of teaching" which is "a network of dispositions activated in specific situations" (Herbst & Chazan, 2003, p. 13). "Dispositions" here are contextual mutual pledges and insights held by individuals sharing a profession, and are developed throughout the course of work practice (Herbst & Chazan, 2003, p. 12). Using the practical rationality of mathematics teaching one can study practices while taking into account the different contextual conditions in relation to the actors' personalities, institutions, circumstances, epistemology, time issues and materials (Herbst & Chazan, 2003). Though, teachers' agency in the educational process should not be underestimated, as teachers act on these contexts in the light of their beliefs and by reflecting on previous practices and experiences (Goos, 2013;

Hoyles, 1992; McLeod & McLeod, 2002). This latter combination of beliefs and practices is addressed as teachers' identities.

As beliefs and practices seem to continuously interact and affect the development of each other, the concept of teachers' identities was introduced to combine these two interrelated concepts and offer a more productive tool of analysis (Lerman, 2001). By definition, "identity is at the core of teachers' ways of being and acting" (Thames & van Zoest, 2013, p. 583). Therefore, Goos (2013) suggests a sociocultural analysis of "person-in-practice-in-person" (p.522) that considers the individual's contribution to learning through social interactions in its first part, and those interactions influence on developing the person's identity in its second is appropriate to study teachers' identities . According to Battey and Franke (2008), a focus on identity facilitates seeing "teacher learning as both situated in practice and as an integrated, complex system embedded in the structures, histories, and cultures of schools" (p.127). They argue that identity can be used to examine and distinguish between how teachers perceive professional development and how they put it into practice. "The construct of identity allows us to begin to understand why professional development can look very different as teachers take new ideas and put them into classroom practice" (Battey & Franke, 2008, pp. 127-128).

Studying identity affords "a practice-based approach", and means of developing strategical use of identity as a source of practice and practice development (Thames & van Zoest, 2013). Thus, in this study, the attention is on identity, where identity includes the individual's ways of thinking, acting, and interacting (Philipp, 2007). This is because the sociocultural approach (like in the case of teachers' identity) focuses on actions, where actions "have a psychological moment or dimension" without limiting that to individualistic psychological analysis (Wertsch, Río, & Alvarez, 1995, p. 10). In other words, psychological approaches lie within the sociocultural approach (Kieran, Forman, & Sfard, 2002; Wertsch et al., 1995). This implies that identity embraces psychological theoretical constructs such as the interrelated beliefs and knowledge about subject matter, its teaching and learning, and teacher's self-efficacy and ways of work (Collopy, 2003; Cyrino, 2018). Cyrino (2018) adds to these teachers' vulnerability and sense of agency, where vulnerability is not seen as a weakening aspect but one that allows teachers to identify their difficulties and limitations (for example, in case of a critical incident) and deal with them. This in turn helps them develop a sense of agency in relation to where they position themselves within a specific context of practice (Cyrino, 2018). In turn, "agency is always mediated by the interaction between the individual (attributes and inclinations), and the tools and structures of a social setting" (Lasky, 2005, p. 900). In general, according to

Wenger (1998) “identity in practice is defined socially not merely because it is reified in a social discourse of the self and of social categories, but also because it is produced as a lived experience of participation in specific communities” (p. 151). Hence, identity is developed and negotiated through engagement with sociocultural contexts. Sfard and Prusak (2005) suggest that identity is used in research that aims to investigate “human beings in action and on the mechanisms underlying human action” (p. 14). They define identities as “collections of stories about persons” (p.15); argue that “they are collectively shaped even if individually told, and they can change according to the authors' and recipients' perceptions and needs” (p.17); and stress the dynamic nature of identities as the stories that constitute them evolve. Due to “the dynamic nature of identity as something that is constantly negotiated through the interplay of one’s lived experience in the world and how [s/he] and others discursively interpret that experience” Goos (2013, p. 522), Teachers, while planning and teaching, are surrounded by many factors to consider. Those factors in turn help form and re-form the teacher’s identity including their knowledge, beliefs and practices. Among these factors, there are some unexpected ones and this is what the next section discusses.

2.5 The unexpected in the classroom

One face of the complexity of teaching is that although teachers can plan their intended classroom actions, they cannot predict their students’ responses and contributions (Rowland, Huckstep, & Thwaites, 2005). This implies that teachers need to act spontaneously in many situations. “Split-second decision making is a crucial aspect of teaching. Given the daily madness of life in a classroom, considering all the options and consequences is difficult” (Hole & McEntee, 1999, p. 35). Hence, Lampert and Ball (1999) argue that teaching requires more than applying pre-learnt teaching knowledge; it actually “depends fundamentally on being able to know things in the situation” (p.37). That is to know about the things as they are happening, things the teacher has not learnt about and cannot predict (e.g. student engagement in the lesson). And, in light of that momentary “uncertain, provisional, evolving knowledge”, the teacher must act (Lampert & Ball, 1999, p. 38). That is “to be prepared for the unpredictable” as well as the predictable (Lampert & Ball, 1999, p. 39). The unpredictable in turn, makes decision making complex and dependent on the individual’s momentary judgement rather than professionals’ advice (Rowland et al., 2015).

Any unpredictable moment or event in teaching is described by Thames and van Zoest (2013) as “pivotal teaching moment” and defined as “an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p.127) . While the term "teachable moments" is used by Arafeh, Smerdon, and Snow (2001) and defined as "the set of behaviors within a lesson that indicated students are ripe for, or receptive to, learning because they express confusion, misunderstanding, uncertainty, struggle, or difficulty with a mathematical problem, concept, or procedure" (pp. 2–3). Similar moments were looked at using the Knowledge Quartet framework as contingent moments (Rowland et al., 2005). Also, within technology settings such moments were addressed using the hiccups framework (Clark-Wilson & Noss, 2015), as discussed in the technology in mathematics education section (section 2.2.2). Although the four terms look at unexpected moments, one main difference is that teachable moments, contingent moments and pivotal teaching moments are identifiable by researchers while hiccups must be recognised and identified by teachers. The second main difference is that hiccups occur when the "technology informs and underpins the contributions made by students whilst also impinging directly on classroom practices when it is used as a mathematical authority within whole-class teaching episodes – to which the teacher is expected to respond" (Clark-Wilson & Noss, 2015, p. 100). Contingent moments are discussed in detail in the following section.

2.6 Observing Teaching

The teaching and learning process is the result of a complex set of interactions in and outside classrooms (Ball, 1988). Teachers, while planning and teaching, are surrounded by many factors that form and re-form the teacher's identity. In the classroom, interactions emerge as interrelated sets of contributions from the teacher and the students (Skott, 2001). Such interactions were studied using theories like the Teaching Triad (Jaworski, 1994), Knowledge Quartet (Rowland et al., 2005), Instrumental Approach (Trouche, 2004), and the Documentational Approach (Gueudet et al., 2014; Gueudet & Trouche, 2009). A brief review of these theories is presented in this section, while a more detailed review of the Knowledge Quartet and the Documentational Approach, is offered in the next section.

2.6.1 The Teaching Triad

The Teaching Triad (TT) looks at teaching as an act of harmony between three factors: student sensitivity (SS), mathematical challenge (MC) and management of learning (ML) (Jaworski, 1994). These factors are evident when a teacher plans a lesson and starts to think of how to consider teaching a specific mathematical idea (MC), his/her particular students' needs (SS), the best way to work on the task with the students (ML) (group work, individual work or classroom discussion). The same factors will be in play during lessons, but within a different context as this time the interactions with the students are happening and the teacher has to respond on demand, in many cases not according to what was planned. So, the TT offers a simple representation for the rich social interactions mathematics teachers have to balance, from a constructivist point of view. It can be "used as an analytical device (by the researchers) and as a reflective agent for teaching development (by the teachers) " (Potari & Jaworski, 2002, p. 351). As a complement to the Teaching Triad, there is a fourth dimension that is strongly evident and affects teachers' plans and practices, that is the social factor which is the social dimension. (Potari & Psycharis, 2018) argue that sensitivity to students' needs (SS) has a social dimension (SSS) which involves sensitivity to students' social background, as well as an affective dimension (e.g., praising students) (SSA) and a cognitive dimension (e.g., inviting students' contributions) (SSC). In fact, social factors include some that Jaworski (1994) already addressed in her book, like time pressure having to go through a set syllabus, the requirement that students know specific things for examinations purposes, expectations from the teacher, school ethos, and the training provided for teachers (Jaworski, 1994; Potari & Jaworski, 2002). Social factors were considered in Jaworski (1994) and Potari and Jaworski (2002) as part of their "macro analysis" of classroom interactions, but not in their "micro analysis" of dialogues, but were also included in their presentation of classroom interactions in light of the teaching triad. The consideration of social factors is also supported by the way Goos (2013) recognised the activities "permitted"/ "offered" to teachers within a specific environment (Goos, 2013, p.523), she argued that "teachers re-organize aspects of their professional environment to better align with goals emerging from new knowledge or changed beliefs". Social factors include students' social culture, teaching resources and materials, syllabus, assessment schemes, time restrictions, room constraints, and cultural considerations of what constitutes good teaching practices (Goos, 2013). In relation to the use of technology in mathematics classes, social factors also come from school or government policies regarding such use and the facilities offered to promote it. So, social factors are at the centre of teachers' work, and the teaching triad should be extended to include social factors in order to reflect the complexity of the factors teachers face when trying to put their knowledge into practice.

2.6.2 Instrumental Approach

Influenced by Vygotsky's sociocultural perspective (Vygotskiĭ, 1978), the instrumental approach considers tools as active carriers of social experiences and an important part of that learning environment (Trouche, 2004). Hence, "emphasis is put on the fact that, due to their characteristics and also due to the way they shape and constrain the possibilities of interaction with mathematical objects, they deeply condition the mathematics which can be produced and learnt” (Artigue, 1997, p. 2) in (Guin & Trouche, 1998, p. 199). (Trouche & Drijvers, 2014) argue that the instrumental approach “highlights the importance of a meaningful relationship between a tool and its user for carrying out a specific task” (p.194).

The instrumental approach (Guin & Trouche, 1998; Trouche, 2004) started by looking at students’ interactions with artefacts and how they transform them into instruments. Here, an artefact is an object designed and used for a specific purpose (Gueudet et al., 2014; Gueudet & Trouche, 2009; Trouche & Drijvers, 2014), it could be a mathematical technique for solving a specific problem, or a tool like a pen. An instrument is developed when the artefact is used by the subject according to a specific scheme to achieve a specific purpose (Gueudet et al., 2014; Gueudet & Trouche, 2009; Trouche & Drijvers, 2014). A scheme here is defined as a set of organized procedures carried out on an artefact; it consists of aims of use and invariant utilization scheme (Gueudet & Trouche, 2009). The utilization scheme includes the subject’s aims of the activity, operational invariants (i.e. perceptions established throughout the activity to be used in comparable situations), and inferences or conclusions the subject reaches due to specific experiences. The complex process of creating an instrument is called instrumental genesis, it includes both instrumentation and instrumentalization: instrumentation refers to the artefact affecting the subject’s actions and knowledge; and instrumentalization refers to the subject’s perceptions affecting the way the artefact is used (Gueudet et al., 2014; Gueudet & Trouche, 2009; Trouche & Drijvers, 2014). Instrumentation is influenced by the artefact’s affordances (e.g. commands in a software) and limitations; while instrumentalization is dependent on the user (e.g. discovery approach); and the two processes are interlinked (Trouche, 2004). “In the instrumental approach, developing techniques is creating reasons”, and is entwined with “conceptualisation” (Trouche & Drijvers, 2014, p. 196). Schemes carry individual and social aspects, and so do instrumental genesis (Trouche, 2004). The balance between the individual and the social is influenced by: whether the artefact facilitates group or individual work; the availability of the artefact; and, the way the teacher integrates the artefact in class (Trouche, 2004).

Initially, the teacher’s role in the classroom was addressed under instrumental orchestration which is the act of organising the learning environment (space, time, dialogs…) by the teacher, whose responsibility is to manage the students’ instrumental genesis according to the requisites of the task (Trouche, 2004). Attributing the role of orchestra conductor to the teacher “allows some variation, playing on the kind of orchestra—the jazz band offers room for student improvisation, the choice of the score, the choice of the didactical configuration. But in each of these cases, the teacher is always central” (Trouche & Drijvers, 2014, p. 199). Trouche and

Drijvers (2014) note that the initial idea of instrumental orchestration “included didactical configuration and exploitation modes […] and as such did not focus on the dynamic adaptions made on the fly” (p.201). Later on, Gueudet and Trouche (2009) and Trouche and Drijvers (2014) argued that teachers’ activities also involve a process of geneses in order to dynamically integrate different resources, hence the Documentational Approach was developed as explained in the next section.

2.6.3 Documentational Approach

The Documentational Approach was developed to look at teachers’ work when they integrate different resources (Gueudet & Trouche, 2009). Hence, the above Instrumental Approach theoretical constructs were expanded to establish the *Documentational Approach* (Gueudet et al., 2014; Gueudet & Trouche, 2009) theoretical constructs. So, resources were defined as “anything that can possibly intervene in [a teacher’s] activity”, to include artefacts, teaching materials, and even social and cultural interactions that affect a teacher’s teaching or preparation work (Gueudet et al., 2014, p. 142). An artefact is defined as mentioned in the previous section, as an object designed for a specific purpose. Teaching materials are textbooks, calculators, and any other materials designed for teaching mathematics (Adler, 2000). Social and cultural interactions are the interactions with the environment, students and colleagues, for example a student’s feedback on an activity (Gueudet & Trouche, 2009). Both Gueudet and Trouche (2009, p. 205) and Gueudet et al. (2014, p. 141) referred to Adler’s (2000) definition of a resource “as the verb re-source, to source again or differently” (Adler, 2000, p. 207). This implies that teachers interact with resources, manage them and reuse them to achieve their teaching aims (Gueudet, 2017).

> "Teachers look for resources, though sometimes they meet resources that they were not looking for (discussions with a colleague at the coffee machine, for example). They associate these resources, modify them, conceive their own resources and use them with students." (Gueudet, 2017, p. 201)

Because “the effectiveness of resources for mathematical learning lies in their use, that is, in the classroom teaching and learning context” (Adler, 2000, p. 205), the Documentational Approach was developed to look at teachers’ use of resources. It examines how a teacher develops documents, where a document is a set of resources used according to a specific scheme for a specific goal (Gueudet et al., 2014). The scheme was addressed (Gueudet & Trouche, 2009) as a utilization scheme with a specific goal, operational invariants and usage. While in later work a scheme was defined as a set of organized procedures carried out on an artefact; consisting of “the *goal* of the activity; *rules of action*; *operational invariants*; and *inferences*” (Gueudet et al., 2014, p. 140, italics in original):

- The aim of the teaching activity represents the teacher’s goal, for example revising for the examinations. González-Martín, Nardi, and Biza (2018) suggest that aims can be general (e.g. to teach a specific topic) or specific (e.g. to solve a specific textbook question).
- The rules of action are the actions the teacher follows to achieve her goals like solving past-examination questions on a specific topic (Gueudet, 2017). Like in the case of aims, rules of actions can be split into general (e.g. to do a specific activity on Autograph) and specific (e.g. to start the activity by asking the students a specific question).
- Operational invariants are the perceptions established throughout the activity to be used in comparable situations (Gueudet et al., 2014; Gueudet & Trouche, 2009), for example past examination papers are a good resource for examination revision as they familiarize students with examination-style questions. For teachers, this is “professional knowledge” (Gueudet et al., 2014, p. 142), they are the reasons adopted by a teacher to justify her stable actions in a range of similar situations. More recently, they have been addressed as “beliefs and knowledge in action” (Trouche, Gitirana, Miyakawa, Pepin, & Wang, 2019, p. 54).
- Inferences are the interpretations and conclusions a teacher reaches due to specific teaching experiences.

Hence, a documented action scheme is the set of procedures a teacher develops to make use of resources (Gueudet et al., 2014). And a teacher’s documentational work is the set of resources encountered, collected, amended or developed by a teacher for a specific goal (Gueudet et al., 2014). A resource system is “the set of resources accumulated and organised (over time) by a teacher in line with his/her regular teaching activity” (Trouche et al., 2019, p. 54). The process of developing schemes for adapting different sets of resources to achieve a specific target is called the *Documentational Genesis* (Gueudet et al., 2014; Gueudet & Trouche, 2009). The *Documentational Approach* studies the development of “structured documentation system[s]” that represent teachers’ work and progress as a result of influencing and being influenced by different resources (Gueudet et al., 2014). The Documentational Approach offers an overview of a teacher’s work including the resources, teacher’s goals, rules of actions, operational invariants and inferences. It acknowledges that “resources are essentially social, as they take place, for example, in schools, or via Internet, and often in collectives” (Trouche et al., 2019, p. 54). Moreover, it facilitates viewing “digital technologies simply as a particular type of resource amongst a wider range of curriculum resources and not as something special or unique" (Bretscher, 2014, p. 46).

2.6.4 The Knowledge Quartet

The Knowledge Quartet (Rowland et al., 2005) was initially developed as a tool for analysing and reflecting on teachers' knowledge and beliefs with the aim of developing mathematics teaching. It was empirically developed as a conceptual framework to reflect on elementary trainees' mathematical knowledge and initiate discussions between teacher educators, trainees and teacher-mentors during school-based placements (Rowland et al., 2005). Its scope was then widened to include elementary and secondary in service teachers and trainees (e.g. Rowland (2010), Turner (2008) and (Carrillo, 2011)). It is argued that "the quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom" (Rowland et al., 2005, p. 259). According to Rowland et al. (2005) subject knowledge in this case includes both Shulman's (1986) subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Based on SMK and PCK, the Quartet was developed as "a framework for the observation, analysis and development of mathematics teaching, with a focus on the teacher's mathematical content knowledge" (Thwaites, Jared, & Rowland, 2011, p. 227). The Quartet is defined by its four dimensions: *Foundation*, *Transformation*, *Connection* and *Contingency* (Rowland et al., 2009, p. 29), under each of these dimensions Rowland et al. (2005) and Thwaites et al. (2011) used codes that emerged empirically. Table 1 which is taken from Rowland and Turner (2017, p. 106) summarizes the Quartet's dimensions and their contributory codes. I include in the following subsection the definitions of each of the four dimensions and their constituent codes.

The Knowledge Quartet's dimensions

The Quartet's dimensions and their constituent codes are explained in the list below. In this list, and also for the rest of the thesis, the Quartet's dimensions the Quartet's dimensions are capitalized and italicized (e.g. *Foundation*) and their contributory codes are italicized (e.g. *identifying errors*).

- *Foundation*

The first dimension "*Foundation*" represents teachers' knowledge and beliefs that they acquire during academic studies or practice. Knowledge here includes their knowledge about mathematics as well as about the teaching and learning of mathematics. The knowledge about mathematics goes beyond instrumental understanding of mathematics to involve "knowing why"; it also includes "careful and deliberate" use of mathematical terminology (Rowland et al., 2005, p. 260). The knowledge about mathematics teaching concerns awareness of the basic and important principles that inform teachers' lesson plans and choices for teaching, "whilst leaving open to deliberation the details of exposition and task design" (Rowland et al., 2005, p. 261). Teachers' beliefs in the *Foundation* dimension are those about mathematics and its teaching and learning, and they cover three aspects. The first aspect includes "beliefs about

Dimension	***Contributory codes***
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy; beliefs concerning effective mathematics instruction, the nature of mathematics, and the purposes of mathematics education.	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	choice of examples; choice of representation; (mis)use of instructional materials; teacher demonstration (to explain a procedure)
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness; making connections between procedures; making connections between concepts; making connections between representations
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction; responding to the (un)availability of tools and resources

Table 1: The Knowledge Quartet dimensions and their contributory codes as listed in (Rowland & Turner, 2017, p. 106)

the nature of mathematics itself and different philosophical positions regarding the nature of mathematical knowledge" (Rowland et al., 2005, p. 261). The second aspect includes beliefs about why we learn mathematics and why specific topics are to be taught in school. The third aspect includes beliefs about when students learn best in mathematics and what teaching practices work well (Rowland et al., 2005). Rowland et al. (2005) argue that the knowledge and beliefs of teachers have great influence on their teaching practices and lesson plans.

Rowland (2010) compares the foundation knowledge needed for teaching elementary school mathematics and that needed for teaching secondary school mathematics. He suggests that establishing the foundations of learners' mathematical knowledge during elementary school years carries its own difficulty and requires considerable PCK knowledge in relevance to the basics of mathematics. This is because the teacher is working on making basic concepts (e.g. subtraction), that are very simple and self-evident to educated people, accessible to young learners. To the teacher, such concept is "invisible until it becomes necessary to know it", and this makes teaching it challenging (Rowland, 2010, p. 1841). While, teaching at secondary level requires more advanced MCK which includes appreciation of the complexity of the taught concepts, the required prerequisite knowledge, and the representation systems in school mathematics education.

Under the *Foundation* dimension, we find codes addressing teachers' knowledge and beliefs: *adheres to textbook*, *awareness of purpose*, *concentration on procedures*, *identifying errors*, *overt display of subject knowledge*, *theoretical underpinning of pedagogy* and *use of mathematical terminology* (Rowland & Turner, 2017; Turner & Rowland, 2011).

While the *Foundation* dimension focuses on teachers' mathematical knowledge, the rest of the Quartet dimensions discussed in the following "focus on knowledge-in-action" by looking at how the teacher's knowledge is employed during teaching and lesson planning (Rowland et al., 2005, p. 261).

- *Transformation*

The *Transformation* dimension "concerns the ways that teachers make what they know accessible to learners, and focuses in particular on their choice and use of representations and examples" (Thwaites et al., 2011, p. 227). It examines how teachers illustrate mathematical concepts and ideas to students by looking at: the examples they use (e.g. a selected exercise for students' activity); the representations they employ (e.g. graph or table); the instructional materials they opt for (e.g. textbook or worksheet); and the demonstrations they give to convey the mathematical ideas to the learners. These aspects are all informed by teachers' knowledge and also by the literature resources available to them for example through their institutions, professional development training or even on the internet (Rowland et al., 2005). The codes for the transformation dimension are about the teacher's *choice of examples*, *choice of representation*, *demonstration,* and (mis)*use of instructional materials* (Rowland & Turner, 2017; Turner & Rowland, 2011). In this dimension, the focus on choices of representations and examples led to recommendations in relation to these two aspects. Rowland et al. (2009)

recommend that teachers should "consider how different representations act as 'scaffolds' between concrete (enactive) experience and the abstract (symbolic) recording used in mathematics" (p.65). They also suggest that "randomly generated examples" (p.99) should be avoided, and that examples should be chosen after careful consideration of their educational aims.

- *Connection*

The *Connection* dimension (Rowland et al., 2005) focuses on teacher's choices in terms of the plan of lesson, the sequence of activities, connecting ideas and concepts and doing so in a coherent way. These choices are related to the mathematical content and to "the coherence of the planning or teaching displayed across an episode, lesson or series of lessons" (Rowland et al., 2005, p. 262). This coherence is examined in terms of: the teacher's awareness of the mathematical content's integrity; her/his organization of the mathematical discourse during lessons; and the ordering of topics and instructions s/he chooses to follow within and between lessons (e.g. ordering of exercises) (Rowland et al., 2005). These "reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks" (Rowland et al., 2005, p. 263). The codes under the connection dimension are *anticipation of complexity*, *decisions about sequencing*, *making connections between procedures*, *making connections between concepts*, *making connections between representations*, and *recognition of conceptual appropriateness* (Rowland & Turner, 2017; Turner & Rowland, 2011).

- *Contingency*

The *Contingency* dimension "concerns responses to unanticipated and unplanned events: until now, these have centred on responses to unexpected pupil contributions, and also on notable 'in-flight' teacher insights" (Thwaites et al., 2011, p. 227). Rowland et al. (2015) recognised three triggers of contingency: the teacher reassessing the lesson plan; students giving an unexpected contribution; more/less pedagogical tools and artefacts available. Thus, this dimension includes the codes: *deviation from agenda*, *responding to students' ideas*, *use of opportunities*, *responding to the (un)availability of tools and resources, and teacher insight during instruction* (Thwaites et al., 2011; Turner & Rowland, 2011). The first and third codes are triggered by the teacher, the second code is triggered by students and the fourth code is triggered by tools and resources (Rowland et al., 2015). The fifth code is also triggered by the teacher (Rowland & Turner, 2017). In response to unexpected students' ideas, a teacher can choose "to ignore, to acknowledge but put aside, and to acknowledge and incorporate"

(Rowland et al., 2015, p. 79). One should also distinguish “between the contingent action of the teacher and the contingent potential of the situation” (Rowland et al., 2015, p. 88). The first refers to the instantaneous teacher’s response to a trigger of contingency. The latter is about the teachers’ potential development which can result from reflecting on the contingent moments along with teachers’ corresponding actions. This development arises either when a teacher realizes that her/his contingent action is inadequate and reflects on it; or when the teacher does not realize such inadequacy but a more experienced observer brings it to the teacher’s attention for reflection (Rowland et al., 2015).

Rowland et al. (2015, p. 88) recognised that the data used to give rise to the *Knowledge Quartet’s Contingency* did not include:

> “… technology-related trigger of contingency, in which the technology functions as intended but with unexpected consequences. This might be expected to occur especially when a technology resource (such as a graph plotter, spreadsheet, dynamic geometry environment) is used in an exploratory way in the course of teaching” (p.88).

Thus, technology-related moments that are similar to Rowland’s contingent moments were described by (Clark-Wilson, 2013) as “hiccups” where the latter are also “moments that are almost impossible to plan for” (Rowland et al., 2005, p. 263). Although hiccups are similar to the contingent moments suggested by Rowland et al. (2005), the main difference between the two is the technology factor. Hiccups are the result of working with technology in classrooms (Clark-Wilson & Noss, 2015). Another difference is teacher’s awareness of the hiccups when they happen, which is not always the case with *Contingency*. In summary, the hiccups framework is used to reflect on unexpected events in classroom and not on teachers’ actions; it is technology specific and includes only events noticed by the teacher. For this study, we find contingent moment a more suitable tool that helps reflect on teachers’ practices with and without the use of technology, regardless of whether the teacher notices the contingency or not. Further aspects that lead to this choice of the Knowledge Quartet are discussed in the following section.

2.7 Theoretical framework

To look at teachers work with resources; I use two theories: the Documentational Approach and the Knowledge Quartet. Rationale behind these choices are explained in this section.

2.7.1 Why the Knowledge Quartet?

The Knowledge Quartet (KQ) can be used to reflect on teachers' actions in class and not only their knowledge (Neubrand, 2018). It offers a tool to explore teachers' work in detail focusing on teachers' practices in classroom. According to Carrillo (2011, p. 274) the Quartet affords analysis that does not only focus on "management issues" in teaching, but goes beyond that by placing teachers' mathematical knowledge at the centre of the analysis of mathematics teaching. It even goes beyond looking at teachers' professional knowledge to looking at teachers' practices and their knowledge "in action" (Carrillo, 2011, p. 275; Rowland, Turner, & Thwaites, 2014, p. 319), which is "the knowledge that can be observed in the actual practice of teaching" (Carrillo, 2011, p. 275). The importance of knowledge in action is based on the argument that this knowledge impacts on students' learning and developing it develops learning outcomes (Carrillo, 2011). In other words "practice is both the source of problems and the space for implementing possible solutions and approaches"(Carrillo, 2011, p. 275), thus any study using this framework should start from the classroom.

The Knowledge Quartet is a tool that is "manageable, and not overburdened with structural complexity" (Rowland et al., 2005, p. 256). Corcoran and Pepperell (2011) used the Quartet and noted how the four dimensions should not be perceived as linear. In their study, and with focus on students' learning, participants looked at the *Contingency* dimension first and then moved to *Connection*, *Transformation* and *Foundation* dimensions respectively. Thus, the four dimensions are manageable and can be reordered in order to promote a specific focus that might seem more relevant to a chosen 'first' dimension (Carrillo, 2011).

> "The four dimensions of the KQ apparently seem to react to the broad band of possible actions a teacher is exposed to before and during the giving of a lesson. Indeed, the KQ works quite well in interpreting what can be seen in a mathematics lesson from outside" (Neubrand, 2018, p. 608)

However, considering that the data the Knowledge Quartet originally used did not include lessons where digital resources were used (Rowland et al., 2015), more codes might be needed to address the use of such resources. Oates, Callingham and Hay (2019) support the use of the Knowledge Quartet to explore teachers' knowledge when using technology, and suggest that such a use may impact on the contributory codes and lead to extending or modifying them. Also, Rowland et al. (2015) do not eliminate the possibility of finding more triggers of contingency when looking at different data sets.

> Although our experience to date indicates that the fundamental anatomy of the Knowledge Quartet is complete, we take the view that the details of its component codes, and the conceptualisation of each of its dimensions, are perpetually open to revision. This fallibilist position seems to us to be as appropriate for a theory of knowledge-in-mathematics-teaching as it is for mathematics itself. (Rowland & Turner, 2017, p. 108)

2.7.2 Why the Documentational Approach (DA)?

The Documentational Approach offers an overview of a teacher's work including the resources, teacher's goals, rules of actions, operational invariants and inferences. Most of these will not be covered if the Knowledge Quartet is used on its own. As well, "the documentational approach aims at presenting a more holistic view of the teachers' activity. It can naturally be used to study technology integration phenomena and more generally to understand the professional evolutions resulting from the generalised availability of digital resources" (Gueudet & Trouche, 2011, p. 38). It facilitates a wider view of resources, inclusive of digital resources but not singling them out (Bretscher, 2014). It even embraces considerations of institutional factors, such as time pressure and the working environment (Bretscher, 2014). Furthermore, the instrumental orchestration offers a descriptive classification of technology use, so another tool is needed to cover the combination of resources and the details of teacher's actions. González-Martín et al. (2018) argue that "[w]hile DA says much about the use of resources but not about the user, we believe that individual agency needs to come into consideration in teachers' documentation work" (González-Martín et al., 2018, p. 251). In order to consider teachers' agency, González-Martín et al. (2018) take into account what Chevallard (2003) termed as teachers' "personal relationship" with the topic to be taught, in other words "their overall experiences as students and teachers of mathematics and its teaching" (González-Martín et al., 2018, p. 233). In order to say more about the user (teacher), this study uses the Knowledge Quartet to include and consider teachers' beliefs-, practices- and knowledge-in-action where appropriate.

2.7.3 The Documentational Approach in tandem with the Knowledge Quartet

As explained in the previous two sections the Documentational Approach helped me identify the more general elements in a lesson observation; while the Knowledge Quartet dimensions afford looking deeper into the details. Considering my interest in looking at how secondary mathematics teachers work with resources including technology in their classroom, and having in mind the Documentational Approach and Knowledge Quartet lenses, my research question can be supported by two sub-questions as follows:

Research question (RQ): How do secondary mathematics teachers work with resources including technology in their classroom?

I. What are their resource systems and their scheme of work?
II. How do they enact their schemes of work in class?

The first sub-question is answered using the Documentational Approach. The second sub-question is answered using both the Documentational Approach and the Knowledge Quartet. This allows me to move from the overview using the scheme of work from the former, to the details using the dimensions of the latter: *Foundation*, *Transformation*, *Connection*, *Contingency*. For the purpose of this study, the documentational genesis is not discussed because it is beyond the focus of this work. Hence, there was no long-term observations and detailed documentation by the teacher as in some DA studies. The elements that are used here from the Documentational Approach are the ones related to teacher's document: resources and schemes of work including aims, rules of action, operational invariants and inferences. The two lenses Documentational Approach and Knowledge Quartet, are used to explore how teachers' work with technology by addressing the two sub-questions I and II; and by using the methodological approach and study design that are explained in detail in the next chapter (methodology).

3 Methodology

3.1 Methodological Approach

This study focuses on teachers' work with resources, when they employ mathematics-education software along with other resources, in secondary mathematics lesson. According to Adler et al. (2005, p.369) "[h]aving teachers as the focus of research leads to high complexity. This increases the tendency to keep the sample small in order to reduce complexity" (Roesken-Winter, Hoyles, & Blomke, 2015, p. 5). So, qualitative methods will be used in order to provide rich qualitative findings with better understanding and higher validity (May, 2001; Merriam & Tisdell, 2016). A "qualitative research is a situated activity that locates the observer in the world. Qualitative research consists of a set of interpretive, material practices that make the world visible" (Denzin and Lincoln, 2013, p.6). A qualitative research methodology will help me understand how teachers "interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences" (Merriam & Tisdell, 2016, p. 6). It will facilitate listening to several views on lived experiences and social situations (Hesse-Biber, 2010). In this approach, the participants' views are sought by the researcher who considers the social reality as subjective and diverse; in other words "there is not just one story but multiple stories of lived experience" (Hesse-Biber, 2010, p. 455). This implies that the qualitative approach values participants' reflections and empower them. This however does not eliminate the researcher's subjectivity and the impact of the theoretical framework on the analysis (Merriam & Tisdell, 2016). As in other qualitative studies, in this study "the focus is on process, understanding, and meaning; the researcher is the primary instrument of data collection and analysis; the process is inductive; and the product is richly descriptive" (Merriam & Tisdell, 2016, p. 15). And, the researcher's role is to be as "responsive and adaptive" as possible (Merriam & Tisdell, 2016, p. 16). While being informed by a theoretical framework, she collects different pieces of information and inductively derives findings including, for example, themes and hypotheses (Merriam & Tisdell, 2016). According to Merriam and Tisdell (2016), a qualitative researcher should have a questioning eye and an observer stance; tolerate ambiguity during data collection and trust the inductive process in qualitative approaches; ask good questions and think inductively during data analysis.

It is thought that the most common approach to qualitative research is the interpretive approach (Merriam & Tisdell, 2016) which values the participants' interpretations and reflections and explores meanings within the participants' environments where the interpretations and meanings are the participants' not the researcher's (Merriam & Tisdell, 2016; Stake, 2010). These interpretations and meanings are socially constructed through interactions with others (Merriam & Tisdell, 2016). Such methodology “relies heavily on the observers defining and redefining the meanings of what they see and hear” (Stake, 2010, p. 36).

One way to conduct qualitative research is through case studies. A case study offers a comprehensive account and investigation of a unit of study within its real-life settings, where the boundaries of the unit of study or case are chosen and set by the researcher (Merriam & Tisdell, 2016). For example, a case can be a person or an institution; a teacher or a school. However, the boundaries should be set in a way that gives a finite data (e.g. finite number of participants and finite number of observations).

This study adopts an interpretive qualitative approach that aims to understand a situation based on the teachers’ interpretations as well as the researcher’s; therefore the experiences shared between the teacher and the researcher become more meaningful and expressive (Stake, 2010). The research design will include asking the participants’ questions, aiming to seek their own interpretations where needed and where possible. With the theoretical framework in mind, I will seek an inductive approach in my analysis. And I will focus on case study to achieve in-depth analysis. With these outlines in mind, the study design was set to include seeking participants’ interpretations and reflections on critical incidents. Discussion of what critical incidents are is in the next section.

3.2 Critical incidents

According to Skott (2001), “critical incidents of practice” are instances when a teacher makes classroom decisions taking into account several motives some of which can be conflicting, vital to the teacher’s school mathematics priorities, and crucial for the development of classroom interactions and students’ learning. Tripp (2012) thinks that a critical incident is an ordinary event or routine that tells the trends, purposes, and routines of a teacher’s practice; it becomes critical when someone chooses to see it as such which in his opinion is “problematic” because it is dependent on one’s interpretation (p.28). Goodell (2006) argues that “a critical incident can be thought of as an everyday event encountered by a teacher in his or her practice that makes the teacher question the decisions that were made, and provides an entry to improving teaching” (p.224). It is believed that reflections on critical incidents can play an important role for teachers’ learning and professional development (Goodell, 2006; Hole & McEntee, 1999; Potari & Psycharis, 2018; Skott, 2001; Tripp, 2012). Skott (2001) argues that critical incidents of practice are useful in two aspects.

> "First, they provide a window on the role of teachers' school mathematical priorities when these are challenged as informants of teaching practice by the emergence of multiple motives of their activities. Second, CIPs may prove significant for the long-term development of a teacher's school mathematical priorities" (Skott, 2001, p. 19)

Thus, identifying critical incidents and having teachers reflecting on them "may turn the classroom into a learning environment for teachers as well as for students" (Skott, 2001, p. 4), and consequently for researchers. Deep reflection on critical incidents inspires teachers to think of what happened, why it happened, what it could mean and what its implications are (Hole & McEntee, 1999).

Inspired by the arguments and discussion above, in this study, a critical incident is defined as an everyday event or an instance of when a teacher takes a classroom decision that shapes classroom interactions and students' learning; such decisions are not necessarily questioned by the teacher (like in (Skott, 2001)) and are mainly related to his/her choices of resources and schemes of use of these resources. In other words, they are the decisions that reveal the teacher's routines, purposes, approaches and tendencies. Therefore, and to promote teachers' reflections during interviews, critical incidents are used in this study in two phases.

The first phase was used for pre-observation interviews, and was inspired by the task methodology suggested by Biza, Nardi, and Zachariades (2007) where tasks are classroom scenarios based on critical incidents. They are defined as:

> "classroom scenarios (Tasks) which: are hypothetical but grounded on learning and teaching issues that previous research and experience have highlighted as seminal; are likely to occur in actual practice; have purpose and utility; and, can be used both in (pre- and in-service) teacher education and research through generating access to teachers' views and intended practices. Tasks initially had the following structure: reflecting upon the learning objectives within a mathematical problem (and solving it); examining a flawed (fictional) student solution; and, describing, in writing, feedback to the student." (Biza et al., 2007, p. 301).

It is believed that:

> "– in contrast to posing questions at a theoretical, decontextualised level – inviting teachers to respond to highly focused mathematically and pedagogically specific situations that are likely to

occur in the mathematics classrooms they are (or will be) operating in can generate significant access to teachers' views and intended practices" (Biza et al., 2007, p. 302).

The task methodology suggests that responses to tasks afford windows on teachers' mathematical knowledge; their pedagogical approach; their didactical practices; and on the interactions between the these (Biza, Kayali, Moustapha-Corrêa, Nardi, & Thoma, in press; Biza, Nardi, & Zachariades, 2009). Biza et al. (2009) recognize that as the teachers are not in the classroom when they reflect on these tasks they would, for example, have "some time to think about their reaction" (p.303). This would give the teachers chances to be more reflective; which would lead to their responses being "more representative of their intentions" (Biza et al., 2009, p. 304). Reflections on these tasks "help teachers develop their capacity to transform theoretical knowledge into theoretically-informed practice" (Biza et al., 2009). They also provide researchers with insights into Ball et al.'s (2008) mathematical knowledge for teaching and Shulman's (1987) pedagogical content knowledge (Biza et al., 2009); and with opportunities "to trigger teachers' reflection on their own considerations on the teaching of mathematics and their role as a teacher" (Biza, Nardi, & Joel, 2015, p. 184). Hence, engagement with tasks is through the teachers' giving written responses to the task first, and then through discussion of these responses. Further details on how this methodology is used in this study is in the next section "Research context and design".

In the second way, critical incidents were identified from lesson observations based on the definition adapted for this study. The incidents that were identified and used in the post-observations interview were selected because they were "crucial and exemplary incidents of the teacher's interaction with the students […] especially with regard to how the teacher organised and orchestrated the classroom communication and conceived his or her own role in it" (Skott, 2001, p. 8). So, for the post-observation interview, questions were based on these critical incidents and aimed to invite teachers to interpret and reflect on their choices and decisions during lessons. It is believed that teachers' interpretations of such critical incidents are based on their knowledge and experiences and relating these to the general teaching and learning ideologies (Potari & Psycharis, 2018). So, a close look at these interpretations provides insights on teachers' pedagogical argumentations and thus is "a means of understanding the resources upon which teachers base their interpretations" (Potari & Psycharis, 2018, p. 2). Therefore, the interviews work as an interpretative device that helps understand how teachers view their choices and orchestration of classroom interactions. Critical incidents is a tool that I use in the design of this study. The full study design is in the next section.

3.3 Research Context and Design

Having the interpretive research methodology in mind, the data collection for this study was designed to explore a number of case studies where each case is the case of one mathematics teacher at a secondary school setting. In this section, I first include information, relevant to this study, about: the context of secondary schools in England, secondary mathematics teacher training in England and the participants of the study. I then explain the research design which aims to give space for the participants to reflect on critical incidents and elaborate on their teaching approaches.

3.3.1 Secondary school context in England

In England, schooling is divided into stages called key stages (KS) (Table 2) ("The national curriculum," 2019). Primary school includes Key stages 1 and 2; 5-10 years old. And, secondary schools include Key stages 3, 4 and 5. Key stage 3 (KS3) covers year 7 to year 9; where students' ages are 11 to 14 ("The national curriculum," 2019). Key stage 4 (KS4) covers year 10 to year 11; where students' ages are 14 to 16 ("The national curriculum," 2019). Key stage 5 (KS5) covers years 12 and 13 with students' ages between 16 and 18. At the end of KS4 students take examinations for the General Certificate of Secondary Education (GCSE). GCSE grades used to be A*- U (i.e. A*, A, B, C*, C , D, E, F, G, U); with A* considered as a good pass and U considered as unsatisfactory (Table 3) (Jadhav, 2018). Since 2017, grades A* to G were replaced by numerical grades (9-1) with 9 being equivalent to A* and grade U was kept as it is to represent an unsatisfactory result (Jadhav, 2018). KS5 is also called upper secondary level, or A level as short for Advanced Level ("AS and A levels," 2019). At KS5, students can choose three to four subjects to study. Students who complete only their first year of KS5 get Advanced Subsidiary qualification (AS level) ("AS and A levels," 2019). At the end of the second year of KS5, students take their A level examinations; and those who wish to go to university are offered places based on their A level grades. For AS and A level examinations A* to E are the pass grades (Table 3) ("Grading and Marking of A-levels," 2019).

Key stages	School years	Age range in years
KS1	Reception, year 1 and year 2	5 to 7
KS2	Year 3 to year 6	8 to 10
KS3	Year 7 to year 9	11 to 14
KS4	Year 10 to year 11	14 to 16
KS5	Year 12 to year 13	16 to 18

Table 2: Key stages and their corresponding school years and students' ages

Examination	Year taken	Pass grades
GCSE	Year 11	A* to G or 9 to 1
AS level	Year 12	A* to E
A level	Year 13	A* to E

Table 3: Secondary school examinations and grades

Generally, schools in England have to follow the national curriculum guidelines which set up what subjects are taught and what the students' learning standards should be, without imposing a specific textbook on schools ("The national curriculum," 2019). Some schools are exempted from following the national curriculum ("The national curriculum," 2019). For example, private schools are not funded by the government; they charge students registration fees. They do not have to follow the national curriculum. They are also called independent schools.

3.3.2 Secondary mathematics teacher training in England

To qualify as a secondary mathematics teacher in England, one should have a degree and a teaching qualification (https://nationalcareers.service.gov.uk/job-profiles/secondary-school-teacher). An undergraduate degree in England is classified as a foundation degree or honours where the latter implies greater depth and breadth of materials and study than a foundation degree. Honours degrees are three-year long in England, they are abbreviated to "Hons", for example BSc (Hons) stand for a Bachelor of Science (QAA, 2014). An honours degree is classed as first, upper second, lower second or third based on the percentage the learner scores as detailed in Table 4 ("Study in the UK- UK Grading System," 2019).

Percentage score	Grade	Honours degree class
70%-100%	Excellent to outstanding	First
60%-69%	Good to very good	Upper second 2:1
50%-59%	Satisfying	Lower second 2:2
40%-49%	Sufficient	Third 3
< 40%	Unsatisfactory	Fail

Table 4: University grades and class for honours degrees ("Study in the UK- UK Grading System," 2019)

A qualified teacher in England is a teacher who holds qualified teacher status (QTS) which can be earned through undergraduate (e.g. Bachelor of Education (BEd)) or postgraduate university courses ("National Careers Service- Secondary school teacher," 2019). To achieve QTS in secondary mathematics one should enroll in a postgraduate teacher training course, such as postgraduate certificate in education (PGCE), postgraduate diploma in education (PGDE), postgraduate certificate in teaching ("Teaching- What is a PGCE?," 2019). These courses require the applicant to: hold a GCSE at grade C (i.e. grade 4) or above in mathematics and English; and have an undergraduate degree in mathematics or another closely related subject ("Teacing- Eligibility for teacher training," 2019).. If the applicant's degree is not in mathematics, s/he can enroll in a fully-funded subject knowledge course ("Teaching- Subject knowledge enhancement (SKE) courses," 2019).

The postgraduate certificate in education (PGCE) is one- or two-year course and is a common course for teacher training ("Teaching- What is a PGCE?," 2019). It offers the trainee "increased confidence" and it is "an internationally recognized qualification" ("Teaching- What is a PGCE?," 2019). The course, like other teacher training courses in England, offers:

- ✓ "plenty of classroom experience in at least two schools – a minimum of 24 weeks of school experience
- ✓ training to meet the Teachers' Standards, which will include classroom management and making your subject accessible to your pupils
- ✓ expert academic and practical guidance from mentors and tutors who are there to help you succeed" ("Teaching- Teacher training courses," 2019)

Once a teacher training course is completed successfully, newly qualified teacher (NQT) status is achieved. This implies that the NQT will need to go through a period of induction, which usually lasts for one academic year (three academic terms), during which s/he has a reduced teaching timetable and a mentor who holds a qualified teacher status (*Induction for newly qualified teachers (England) - Statutory guidance for appropriate bodies, headteachers, school staff and governing bodies*, 2018). This period is meant to be:

> "the bridge between initial teacher training and a career in teaching. It combines a personalised programme of development, support and professional dialogue with monitoring and an assessment of performance against the relevant standards […]. The programme should support

the newly qualified teacher (NQT) in demonstrating that their performance against the relevant standards is satisfactory by the end of the period and equip them with the tools to be an effective and successful teacher" (*Induction for newly qualified teachers (England) - Statutory guidance for appropriate bodies, headteachers, school staff and governing bodies*, 2018, p. 6).

On completion of the induction period the person is considered a qualified teacher.

3.3.3 Participants

The participants in this study were seven secondary school mathematics teachers, with different levels of experience and training, from four different schools in East England. Here is a list including information about each of the schools and the participants:

School 1: A mixed upper-secondary state school that enjoyed autonomy and flexibility in the curricular design, choices and resources as well as timetabling and terms' dates. From this school, three participants were willing to be interviewed (2 males and 1 female). I name the participants: George, Martin, and Hana.

School 2: A mixed secondary state school that enjoyed autonomy and flexibility in the curricular design, choices and resources as well as timetabling and terms' dates. From this school, one male participant was willing to be interviewed, the participant will be called Jude.

School 3: A mixed secondary state school that followed the national curriculum and local authorities' guidelines, but had the choice over which textbook to follow. From this school, two male participants were willing to be interviewed. I name the two participants Adam and Sam.

School 4: A secondary independent school that did not have to follow the national curriculum; and the textbook to follow was chosen by the school. From this school, one male participant was interviewed. I name him here Charlie.

The schools were chosen from diverse contexts. So, the first school was the only upper-secondary school, so all students were aged 16-18 and were preparing for their school leaving examination (A-level). The rest of the schools were secondary, so they were for students aged 11-18. One of the schools (school 4) was an independent school. School 1 and school 2 were not independent, yet they enjoyed more control over what they do than school 3 but less control than school 4. Further details about each school's profile are not mentioned here to maintain the anonymity of the schools. The participants from these school also had diverse profiles: two of them were the head of mathematics department; one was a newly qualified teacher; years of experiences ranged from one to twenty years. Due to the dependency on teachers' volunteering to participate, all participants were white European, and all the main participants were white British males.

The following table (Table 5) summarizes the teachers' profiles at the start of the data collection:

School	Teacher	Profile outline
1	George	He had a BSc in mathematics and PGCE in secondary mathematics teaching. He had fifteen years of teaching experience mostly in upper secondary education; and was the head of mathematics at his school during the data collection.
1	Martin	He had a BSc in mathematics and PGCE in secondary mathematics teaching. He was a newly qualified teacher and close to completing his first year of teaching experience at upper secondary level.
1	Hana	She had a BSc in mathematics and QTS in secondary mathematics teaching. She had seven-years teaching experience at upper secondary level.
2	Jude	He had a BSc in business and a PGCE in secondary mathematics teaching. He had three years' experience of teaching mathematics at lower secondary level.
3	Adam	He had four years' experience, during which he taught students aged 12-18 years. He held a degree in economics, a PGCE in secondary mathematics teaching and was about to finish an MA in education with his dissertation focusing on mathematics education.
3	Sam	He held a PhD in computer science and PGCE in secondary mathematics teaching. He had 10 years of teaching experience at secondary level.
4	Charlie	He had a BSc in mathematics and a PGCE in secondary mathematics teaching. He had twenty-year teaching experience at secondary level and was the head of mathematics at his school during the data collection.

Table 5: Participants profiles at the start of the data collection

These are all the participants who took part in part or all of this study. The following section describes the different phases of the study and ends with a table (Table 6) summarizing the participants' contribution or non-contribution to each phase.

3.3.4 Research Design

Having the interpretive research methodology in mind, I designed the phases of data collection in a way that allows me to look at their work with resources within context; and give the participants the space to reflect and express their interpretations around the different aspects of their work. Hence, the data collection for this study was done in five phases as detailed in the next sections: designing pre-observation interviews, pre-observation interviews, pilot observation, lesson observations, and post-observation interviews. The rationale behind each phase is explained in the section relevant to this phase.

The first phase- Designing pre-observation interviews

The purpose of the pre-observation interviews was to listen to the participants views and approaches about their work with technology and other resources. Having the qualitative approach in mind, I aimed to investigate the participants views and reflection within context. As this context was not yet available through actual teaching, I decided to follow the task methodology (Biza et al., 2007). This methodology involves asking the participants to provide written responses to *tasks*, and then conducting follow up interviews with them. Tasks were recommended by Biza et al. (2007) as a methodological tool. They are given classroom scenario that are designed as semi-structured interviews and aim to create and encourage discussions of specific classroom scenarios. A task provides an opportunity for researchers and teachers to put on the table the challenges faced, and decisions made during mathematics lessons. Tasks offer lenses to look into teachers' practices and beliefs and afford combining multiple factors from practice to be addressed. They encourage teachers to "*stop* and *reflect*", while taking into consideration "classroom dynamics" (Biza et al., 2015, p.36).

Inspired by the task methodology, and to look at my participants' use of resources, I created two scenarios: The first was on 3D geometry (in Figure 3) and the second was square and its properties (in Figure 4). These scenarios and their associated questions were validated through three stages:

- Discussion with my supervisor about the first draft of these tasks and what they address.
- Discussion at UEA's Research in Mathematics Education group meeting. Feedback on the first draft of these tasks was collected from the group; discussed in the group's meeting and considered to make the final version in Figures 3 and 4.
- The tasks were tested by having five practising teachers (not the participants) solve them for the purpose of giving me feedback on their content and format.

On 3D

A group of Year 11 students are asked the following question:

Design a milk container with a capacity of 1L. What dimensions and which design uses less materials? Why?

- What are the mathematical ideas and activities addressed in this question?
- Would you use this question in class? Why or why not? What are the learning objectives for which you would use this question? Would you modify it?
- Would you use technology with this question? If yes, what type of technology? If no, why?
- If you were to use technology, how would you use it?
- What teaching approaches and resources would you suggest for this question?

Do you anticipate any problems or challenges (either with students or resources)?

Figure 3: The 3D scenario

On the square and its properties

You have just introduced the properties of the square to Year 7 students. You are in a classroom in which you have computer access with internet connection and dynamic geometry software.

- How would you plan the next activities, and what kind of questions would you ask?
- What problems would you expect?
- In one of the activities, you asked your students to construct a square using dynamic geometry software. One student used the line segment command to draw a 4-sided figure that looked like a square, and then started chatting online.
- What are the issues in this case?
- How likely is this to happen?
- How would you react?

Figure 4: The scenario on square and its properties

The two task-inspired scenarios aim to give specific context of teaching situations and ask about the teachers' decisions, ideas, and reflections. The questions in the task were open-ended. In addition, as I would like to discover aspects of the teachers work with digital resources, I designed the first scenario and its questions without mentioning or suggesting any use of technology while I asked about this use directly in the second scenario. The aim is to leave the options fully open to the teacher in the first part of the interview; and, in the second part, try to clarify whether they are comfortable with using technology which is an important focus in this study.

My aim was to focus on 3D Geometry in one of the interview tasks, this choice of topic was inspired by arguments around the lack of research on technology use when teaching 3D geometry on one hand (Jones, Mackrell, & Stevenson, 2010, p.56), and the fact that "students have problems understanding three-dimensional geometry" (Dolonen & Ludvigsen, 2012, p. 167) on the other. Research suggested that students find it challenging to represent 3D figures and visualize the complex relationships between their components (Pittalis & Christou, 2010), "especially the relationship between surface area and volume" (Dolonen & Ludvigsen, 2012, p. 167). Also, the mental rotation of 3D figures "is a particularly undeveloped area of current mathematics curricula… [although it] can be used as a reliable strategy for understanding area measurement tasks… [and] composing and decomposing 2D and 3D figures…" (Bruce & Hawes, 2014, pp. 331-332). Pittalis and Christou (2010, p. 208), emphasized the importance of teachers' adopting explorative 3D geometry tasks that help students make connections between the visual perception of a figure and its properties, and encourage them to move away from memorizing formulas. To achieve that:

> "Battista (2003) suggested that students should develop two necessary skills to calculate conceptually the volume and surface area of a solid: (a) the conceptualisation of the numerical operations and the link of the formulas with the structure of the solid and (b) the understanding and visualisation of the internal structure of the solid" (Pittalis & Christou, 2010, p. 194).

The first scenario was designed with secondary school mathematical topics in mind, including: surface area, volume, unit conversion, optimization, cube, prisms, pyramids, cylinders, cones, problem solving or anything else the teacher would like to add. The aim from this task was to address any issues about time with open problems like the one in this task and the chances for different approaches and different designs: What designs would the teachers try and why? Will they consider all designs? Will they consider optimization? Will the teacher narrow the problem? If yes, how? For example, will they consider specific designs, or would they put specific conditions on dimensions e.g. cuboid. Also, aiming to hear teachers' preferences in this case: Will they use technology or not? Why? All these questions would help me look at the aspects of teachers' knowledge and beliefs in relation to the 3D task. The second scenario aimed also to looking at teachers' ideas, knowledge and beliefs in relation to the mathematical concept (square). But it was more directed to looking at the use of technology: whether the teachers would use technology or not; and whether they would consider behaviour issues and class management with technology. A teacher would answer the questions on these scenarios on paper; then there would be a discussion between him/her and I about their answers: to clarify or elaborate on these. A sample of the questions I used to manage the discussion on the written responses is included in Appendix E (from Adam's pre-observation interview). Teachers' answers to these scenarios would help the participants and me in making decisions about who would continue to the next phase. The decision would be based on the participants use or not use of technology and whether they see themselves as contributors to this study. This interview was not only designed to be informative about the participants' approaches, but also to be an ice-breaker between the participants and me before moving to the observation phase, which is the third phase.

The second phase- Pre-observation interviews

This phase is focused on recruiting and interviewing secondary school mathematics teachers and conducting pre-observation interviews with them. It required me to use my questioning skills to invite teachers to elaborate on their written answers. The interviewed teachers provided answers to the written questions related to the scenarios given to them, they were given the chance to further elaborate on the answers they provided and justify them. A sample of the questions I used to manage invite their elaboration on the written responses is included in 'Appendix E: Sample of Pre-observation interview questions' (where I include questions from Adam's pre-observation interview). The questions I asked were based on the written answers the teachers provided, so they had to be created during the interview. This meant that, there was a room for development by adding more questions or rephrasing them. However, as this part of the data is only used as an indication to who would be in a participant in the next phases, such shortcomings did not impact this study severely. At the end of this phase, and based on the pre-observation interviews and on the study context, some of the teachers would continue to the next phase. The choice was based on whether they used technology in their lessons, and whether they were comfortable with being observed during lessons. Out of the seven participants, two participants (Jude and Sam) did not wish to carry on to the next phase. Sam decided that he would not be comfortable with lesson observations. Jude claimed that he did not use technology, specifically mathematics-education software, because it took a lot of time to prepare.

The third phase- Recorded pilot observations

These are one-lesson observations done with all the teachers who were interviewed and willing to take part in the observation stage. These observations were audio recorded. The pilot observations were helpful in choosing the teachers who are going to participate in the following stage, based on the teachers' class work, as well as their responses to the previous interviews. Out of the five teachers who took part in this phase, four continued to the next phase. Hana did not seem comfortable using technology in her pilot observation, and after that observation she stated that she did not use mathematics-education software.

The fourth phase- Recorded lesson observations

In this phase, teachers were observed for up to ten lessons, depending on their use of mathematics education software (both in terms of frequency and diversity of use). These observations were audio or video recorded, based on the teacher's preference. Data was collected from the teachers' comments, actions, worksheets, as well as students' interactions. At this phase, I had four participants to work with. The number of observations was set up to ten to make sure that there is a set boundary to each case study. But the minimum number depended on the diversity of the individual teacher's approach to use technology. Ten observations were recorded with George, nine with Charlie, five with Adam and five with Martin.

The fifth phase- post-observation interviews

In this phase, the four teachers who were observed were interviewed. The interviews' questions were semi-structured and were about the teachers' planning, expectations, outcomes and evaluations of the lessons. The semi-structured interviews allowed the participants to answer using their own terms but within the set structure (May, 2010). The structure included reflections on critical incidents observed during lessons. The discussion about critical incidents in section 3.2 explains that a critical incident is defined in this study as an everyday event or an instance of when a teacher takes a classroom decision that shapes classroom interactions and students' learning; such decisions are not necessarily questioned by the teacher and are mainly related to his/her choices of resources and schemes of use of these resources. Based on this definition, I identified critical incidents from all observed lessons. Some of these incidents were more general (for example an incident related to how to prepare students for the examination), while other incidents were focused on specific topics (for example related to the teaching integration). Examples from the critical incidents were shared and discussed with my supervisors; and some were shared with the Research in

Mathematics Education group (RME group) at the university of East Anglia for discussion in the group meeting. For example, I shared an incident from Charlie's first lesson observation with the group. The feedback from my supervisors and the RME group was supportive and helpful in validating my approach of identifying critical incidents. Interpretations of both general and specific critical incidents were sought from each teacher based on their data, to enrich the analysis by giving each teacher a voice. A sample of post-observation questions is included in 'Appendix F: Sample of the post-observation interview', where I include the questions I asked Charlie based on the critical incidents identified in his lessons.

To summarize on all phases, the interviews and the observations were based on contextual data and participants' interpretations were sought in both interviews. The seven participants from the four schools were aware that they could take part in part of or all of the study. Table 6 summarizes the phases the participants from each school took part in.

3.4 Ethics

The ethical aspects of this study were addressed in an application to the Research Ethics Committee School of Education and Lifelong Learning (Appendix H) and the application was approved (Appendix I). Before taking any steps in the empirical work, I aimed to seek all the required adult and parental consents (where needed). In order to recruit participants for this study, a poster (included in 'Appendix A: Study poster') about the study was handed to the head of mathematics or to one mathematics teacher to circulate to the mathematics teachers in the school. The poster had information about the study including research questions and study aims, as recommended by Stake (2010). By trying to approach teachers directly, or using a poster in a school notice board, the chances of those teachers feeling obliged to participate because their head teachers want them to was reduced. Also, letters of invite, information sheet and consent forms (Appendices B: Interview opt in form, and Appendix C: Lesson-observations opt in form) were posted or handed in to secondary mathematics teachers in different schools. Before I started lesson observations, letters of invitations were prepared and given to students (at KS5) for adult consent (Appendix D: Student consent form for lesson observations). Although this study is about teachers and there is no direct data collection about students, volunteers from students were once sought when the progress of the data collection implied a need for students' feedback on their mathematics lessons; their input in this case was voluntary and it was audio recorded. For the teachers participating in this study, and also for

School	Participant	Pre-observation interview participant	Pilot observation	Observations/ Audio or video recorded	Number of observations if any	Post-observation interview participant
1	George	Yes	Yes	Yes/ video recorded	10	Yes
1	Martin	Yes	Yes	Yes/ video recorded	5	Yes
1	Hana	Yes	Yes	No	1 (pilot only)	Yes
2	Jude	Yes	No	No	0	No
3	Adam	Yes	Yes	Yes/ video recorded	5	Yes
3	Sam	Yes	No	No	0	No
4	Charlie	Yes	Yes	Yes/ audio recorded	9	Yes

Table 6:Participation in the different stages of data collection

their students, the consent was not only through signing the consent letter. But also by continuing to show willingness to take part in the study (Stake, 2010). The option to opt out was made clear to the participants through the consent letter and also through seeking their permission to observe lessons before each observation. This, for example, lead Martin to cancel one lesson observation within short notice. The consent letter explained that I would be the only person to access raw data.

I chose to audio- or video-record the lesson observations to be able to analyse in detail. Videos, audio-recordings and transcripts have been securely stored, and will be destroyed 10 years after the end of the work. All the names mentioned in the thesis are substitutes for the originals. All audio- and video-recordings have been reported in detail then analysed and discussed in relation to the theoretical framework. For audio-recording, a discrete microphone was provided to the teachers. Supporting pictures of the different steps on the board and/or worksheets were taken. In case of video-recording, students who did not wish to participate were seated in a way so that the camera did not capture them, and their data was not recorded or used at all in this study. The video-recording was mainly directed at the board. Pictures of worksheets or textbook activities were taken. Generally, following a recommendations in Stake (2010), I tried to capture the details through recordings and interviews and to get as close picture of the teaching as possible without intruding the teachers' private spaces.

3.5 Data Collection overview

As described in the research design section, data was collected using different methods: interviews and informal questioning, and lesson observations. The pre-observation interviews and pilot observations were used to establish the teachers' profiles for this study, and to decide which participants were going to be observed. Seven participants took part in the pre-observation interviews phase (Table 6): three from the first school (George, Martin, and Hana), one from the second school (Jude), two from the third school (Adam and Sam) and one from the fourth school (Charlie). Out of the seven participants, five participants were willing to take part in the observation phase, both Sam and Jude did not wish to continue with lesson observations. All other five participants were observed for one audio-recorded pilot observation. Hana tried to use technology in that lesson but for some reason what she prepared did not work during the lesson. So, after the pilot observation she told me that she rarely used technology, and that she did not know a lot about its use. So, she was not observed for the next phase. This meant that four teachers (George, Martin, Adam and Charlie) took part in the lesson observations and post-observation interview phases. All lesson observations (under the third phase) were video-recorded apart from Charlie's whose observations were audio recorded because he was not comfortable with video-recording. The four observed teachers used mathematics education software in most, but not all their observed lessons. And, although the plan was to observe up to 10 lesson for each teacher, only 5 and 6 were observed for Adam and Martin respectively. This was because the two teachers felt they showed me all the ways in which they usually used mathematics education software in their teaching and that more observations would not reveal new methods in relation to technology use.

The data collected from the observations included teachers' comments, actions, demonstration, worksheets, as well as interactions with students. The plan was to also include lesson plans, but none of the teachers was required to write lesson plans by their school and they all choose not to write lesson plans. Based on the data collected from the lesson observations, the teachers were asked to comment and reflect on critical incidents identified by the researcher; the incidents were chosen to reflect the choices teachers made during the observed lessons and their interactions with the students where relevant, as detailed in the critical incidents section.

3.6 Data analysis

The pre-observation interview and the pilot observation were used to collect contextual and background information about the participants and their suitability and willingness to take part in the next phases of the study. For the four participants who took part in the observation and post-observation phases, the analysis of the data collected in these phases happened in two stages. First, a preliminary analysis of the observations was done to determine the critical incidents in each lesson observation. This included identifying the teacher's main steps and choices. In the interview, each participant was invited to reflect on these choices and on the flow of the lesson (e.g. aims of using Autograph). Second, each participant's responses in the interview and actions during the lesson were analysed using the Documentational Approach and the Knowledge Quartet. The analysis using the Documentational Approach identified the resources used and their schemes of use: aims of the teaching activity, rules of actions, operational invariants and inferences in the context of the observed lesson. Each set of resources and their corresponding scheme of use were summarised in a documentational work table, similar to the one used by Gueudet (2017) in her analysis of university teachers' work. And, in the analysis using the Knowledge Quartet lens, the four dimensions of the Quartet were used to explore the details of specific situations in which the scheme of use is developed and applied. A sample of the analysis using the Knowledge Quartet lens in included in 'Appendix G: Sample of the analysis using the Knowledge Quartet'.

Early analysis led to identification of four themes or points: first, teachers use students' contributions as a resource during lessons; second, institutional factors like examination requirements and school policy have impact on teachers' decisions and on how they balance their resources; third, when mathematics-education software is used the teacher's knowledge of the software intertwines with this use; fourth, contingent moments can result from the lack of mathematics-education software knowledge (e.g. leading to unexpected outcome). Also, in some cases it is difficult for the researcher to judge whether a moment is a contingent moment or not if the element of teacher's expectancy (or lack of it) of that moment is not clear. The following table (Table 7) aims to introduce the reader to how these themes appear in the data.

These themes will appear in the next section (Data and analysis) frequently, in the forms outlined in the table above. The next chapter starts with analysis of data from George, Martin, Adam and Charlie respectively. For each participant, I aim to include one full lesson overview and analysis, and then focus on lesson episodes. The full lesson analysis includes analysis based on the Knowledge Quartet and the Documentational Approach where a document table similar to the one in Gueudet (2017) is included. The lesson episodes are part of lessons that include critical incidents relevant to this study and supportive of the main themes summarized in Table 7. For the lesson episodes, the document table I include is for the full lesson, not only for the chosen episodes.

Theme	***How it appears in the data***
Students' contributions as a resource	1. Contributions invited by the teacher to help solve questions on the board 2. Contributions invited by the teacher to support demonstration of new ideas 3. Students' questions on a new concept that would lead the teacher to demonstrate ideas to one student or to the class.
Institutional factors like examination requirements and school policy	1. Knowledge of examination requirements and school policy 2. Responding to contingent moments by referring exam requirements
Knowledge of the software	1. In planned use 2. In contingent use
Contingent moments	1. As a result of adhering to examination requirements 2. Moments that could be contingent, but the expectancy factor is not evident

Table 7: The main themes that emerged from the data analysis

4 Data and Analysis

4.1 First teacher data - George

George was a mathematics teacher with fifteen years of secondary teaching experience, mostly in upper secondary education (students aged 16-18 years). He held a BSc in mathematics and PGCE in secondary mathematics teaching. He was the head of mathematics at his school during the data collection. The school he worked for provided interactive whiteboards, iPads for students' use, and Autograph software (www.autograph-math.com).

From a set of eleven lessons in which George was observed, episodes from seven lessons are analysed and discussed in detail. These lessons are:

- Lesson 1- Trigonometry: the whole lesson is discussed as an example of George's teaching.
- Lessons 2, 4 and 7- lessons on the volume of revolution: episodes are chosen to reflect George's introduction to the volume of revolution in the three lessons, his comments on the use of the school website, and some questions from the students.
- Lesson 6, 9, 12 and 13- differential equations: the chosen episodes look at George's demonstration about differential equations in two different lessons with the same group of students and his answers to students' questions in the second of these lessons.
- Lesson 8- iteration: the episode looks at George's introduction to the iterative method.

The other four lessons are not discussed in detail because they were either revision lessons with students working on their iPads (which is the case in observations 5, 10 and 11) or technology software was not used (which is the case in observation 3). However, data from these lessons will be used where appropriate to provide a general view about how George worked with his students. To start with, lesson 1 is discussed in section 4.1.1 in order to give the reader an idea about George's teaching approach and its characteristics. Sections 4.1.2 – 4.1.6 review episodes from lessons 2 - 13 and section 4.1.7 summarizes the findings about George's teaching.

4.1.1 A characteristic lesson- Trigonometry

To give the reader a characteristic example of George's teaching, I include his lesson on trigonometry. This lesson contains elements and characteristic of his teaching that were recurrent in other lessons. In this section, I overview the trigonometry lesson, analyse it using first the Documentational Approach lens and second the Knowledge Quartet lens, then I offer a discussion of possible findings based on the analysis.

Lesson 1 Overview

This was the first lesson I observed for George. It was with a group of seventeen year 13 students. In this lesson, George mentioned that he was going to teach the following trigonometry formulae (Figure 5):

$$\mathrm{a}\sin x \pm b\cos x = R\sin(x \pm \alpha)$$

$$\mathrm{a}\ \cos x \pm b\sin x = R\cos(x \mp \alpha)$$

$$R = \sqrt{\mathrm{a}^2 + b^2}$$

$$R\cos\alpha = \mathrm{a} \quad \text{and} \quad R\sin\alpha = b$$

$$\cos\alpha = \frac{\mathrm{a}}{R}\ ,\ \sin\alpha = \frac{b}{R}\ ,\ \tan\alpha = \frac{b}{\mathrm{a}}$$

Figure 5: The trigonometry formulae taught in lesson 1

He introduced the formulae in Figure 5; solved some exercises on the board; answered students' questions and allowed some time for students to work independently.

In his introduction to the formulae, George started by displaying on the board a function (the same as the one I redrew in Figure 6) that was done on Autograph:

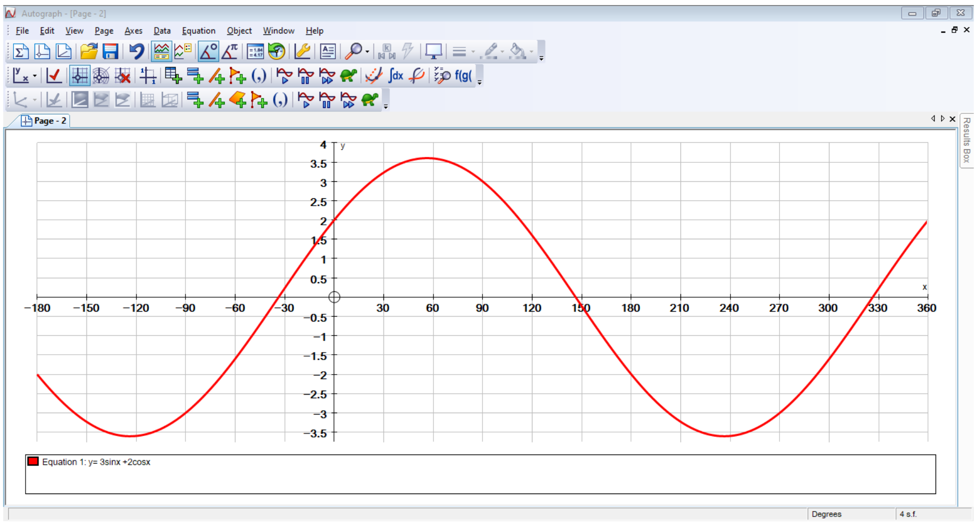

Figure 6: The graph of $y = 3\sin x + 2\cos x$ On Autograph (reproduction of George's work)

He then asked his students to estimate the graph's equation. The students gave different estimates and then agreed that $y = 3.6\sin(x + 35)$ was the closest. At that point, George revealed the equation of the graph on Autograph: "Autograph says it is $y = 3\sin x + 2\cos x$", and concluded that "the computer must be right, but we also know that this is sine translated and stretched, so the two must be equal". Based on that argument, George concluded that

$R\sin(x + \Theta) = 3\sin x + 2\cos x$. A student then suggested the use of compound angles rule, George agreed and wrote on the board:

$$R\sin(x + \Theta) = R\sin x\cos\Theta + R\sin\Theta\cos x$$
$$= 3\sin x + 2\cos x$$
$$R\cos\Theta = 3 \quad \text{and} \quad R\sin\Theta = 2$$
$$\cos\Theta = 3/R \quad \text{and} \quad sin\ \Theta = 2/R$$

Another student suggested Pythagoras, but George did not react and continued to find $\tan\Theta$ as follows:

$$\frac{\sin\Theta}{\cos\Theta} = \tan\Theta = \frac{2}{3}$$

Using their calculators, the students found that Θ = 33.7°. George was about to rearrange the equation, in order to find R, when the same student who mentioned Pythagoras before said Pythagoras again. George this time commented "oh yes, Pythagoras even better"; drew the

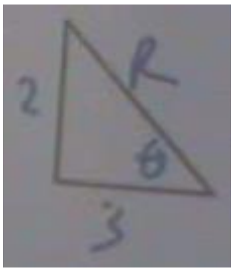

Figure 7: Right- triangle sketched by George

right triangle shown in Figure 7; and concluded that $R = \sqrt{3^2 + 2^2} = 3.61$.

After that, George started solving exercises from the school website, he started with solving a past-examination question. He mentioned that he chose a problem that had $R\cos(\Theta + x)$, unlike the one before which had $R\sin(x + \Theta)$. George proposed a question about how one would know if the tangent is 3/2 or 2/3; and he answered:

> "The long way is to work it out, to crunch these through and see what you get. The short way, now look at this… if it starts with a cos, they always ask you to do it in terms of cos here and if it starts with sin, its sin… Now let's double check".

To confirm his answer, George started checking if that was the case by looking at past-examination questions, which all followed that rule. But, one student questioned if that would always be the case. In response, George wondered what if the question had "$3\sin x + 4\cos x \qquad R\cos(\Theta + x)$", a student suggested to swap these around, and George approved:

"Exactly just swap them around". George then said: "I remember doing this few years ago, and in your textbook page number, can you go to page 198 please?". George asked his students to look at question 1, a to d in their textbooks, and notice how in question d there was cosine and yet they were asked for sine: "in that case you do need to swap it around". A student asked about what if they were asked about tangent. George explained that as "$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, so I guess in this case it's 3/4 instead of 4/3. Yea, but if we were to swap them around… Ah if it matches up then the short version is the second number over the first number, yeah…".

George then showed on the board the question in Figure 8 which he described as a "classic" question. Commenting on its mark scheme, George said that 3 marks were assigned for the last part of the question because it asked for the maximum value of the function and the angle. He added that the last part would have been given only 1 mark if it was asking only about the maximum value of the function, without asking about the angle.

It is given that $3\cos\theta - 2\sin\theta = R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(a) Find the value of R. *(1 mark)*

(b) Show that $\alpha \approx 33.7°$. *(2 marks)*

(c) Hence write down the maximum value of $3\cos\theta - 2\sin\theta$ and find a positive value of θ at which this maximum value occurs. *(3 marks)*

Figure 8: Past-examination question

George said he was going to look at his notes to check if he forgot something, and he again showed the formulae in Figure 5. George ended his part of demonstrating at this point, and kept the rule up on the board in response to his students' request. He set up homework from the exercises in the textbook (page 198, exercise 9E questions 1, 2 and 4), and gave his students a week to complete it. He then put a question on the board for the students to solve independently, in pairs or in groups, as they wished.

While students were solving questions, one student inquired about the signs in the second rule which was:

$$a\ \cos x\ \pm b\sin x = R\cos(x \mp a)$$

George started explaining by writing:

$$a\ \cos x + b\sin x = R\cos^2 \quad + \sin^2$$

$$R\cos^2 \quad = a\cos x$$

$$R\cos\Theta =$$

In these statements, George used $\cos^2$ instead of $\cos x \cos \Theta$, and $\sin^2$ instead of $sin\ x\ sin\ \Theta$. Although the student who asked the questions understood the idea and said: "Sir, you're right. So, when there is a positive sign between the two it is positive, when it is negative between the two it is negative". Another student was confused by what George wrote and asked: "why is it $\cos^2$?". George started amending what he wrote to clarify what he meant. So, he changed $R\cos^2 \quad = a\cos x$ to $R\cos x \cos \Theta = a \cos x$.

Then, George re-demonstrated how to find R by going back to the interactive whiteboard and what he wrote before, and telling the students that instead of the Pythagoras use to find R they could square R cos Θ =3 and R sin Θ =2 (i.e. $R^2 \cos \Theta = 9$ and $R^2 \sin \Theta = 4$). As $\cos^2 \Theta + \sin^2 \Theta = 1$, he explained that adding the two equations ($R^2 \cos \Theta = 9$ and $R^2 \sin \Theta = 4$) will give $R^2 (\sin^2 \Theta + \cos^2 \Theta) = 13$. A student asked why not substitute the values of sine and cosine, George advised him that it would be better not to because by using a whole number (i.e. $\sin^2 \Theta + \cos^2 \Theta = 1$) he would avoid the "tiny roundings" his calculator would do. George added that as $\sin^2 \Theta + \cos^2 \Theta = 1$ then $R^2 = 13$ and $R = \sqrt{13}$. After that, the students continued working on textbook questions, and George moved around the class answering students' questions until the lesson ended.

In the post-observations interview, George commented on his choice of resources for this lesson and on his aim from using Autograph. For example, in the interview he commented on the choice of the example and its demonstration in Autograph during the lesson:

> "So, the aim of Autograph at that point I was just using Autograph just to show them the graph. I deliberately wanted them to think that it's, I wanted them to tell me it's a sine graph that has been transformed. So, I wanted them to tell me it was like whatever it was 3.5 sin (x + 35), so I wanted that answer and then I wanted to show them. So, I was then using autograph to show them that it was something else, 3 whatever it was sin + 2 cos or whatever it was. So, to deliberately make them think hang on a second, we are right but we can't be right, and so to force that idea that these two things must be the same and we must be able to work things out between them".

Lesson 1 Analysis

In this section, I look at this lesson through the Documentational Approach and the Knowledge Quartet lenses respectively. This analysis is followed by a discussion of some themes that have been identified in George's teaching.

Using the Documentational Approach- Resources and Scheme of use

The resources George used included: interactive white board, board, curriculum of year 13, textbook, past examination questions and mark schemes, past teaching-experience, students' contributions (including students' previous knowledge), calculators, notebooks, Autograph, formulae sheet (hard and soft copies). His scheme of use (i.e. his teaching aims, rules of actions, operational invariants and inferences) is summarized in Table 8, and draws an overview of his 'documentational work towards the introduction of some trigonometry formulae (Figure 5). Beside the elements in the scheme of use that are related to this specific lesson, more general elements have been identified specifically towards students' preparation for the examinations. So, George had a specific aim (A1) which was to introduce the trigonometry formulae in Figure 5. He also had a general aim (A2) which was to prepare students for the examinations. George's rules of action (numbered R1-R10 in Table 8) were included demonstrating a range of examples for the students (R5 and R8); inviting students' contributions (R1 and R9); allowing some time for students' independent work and questions (R10); making connections with students' previous knowledge (R4); presenting some examination-style questions (R6); assigning homework (R7); using a typed trigonometry formulae sheet (R3); and using graphs and Autograph to illustrate ideas (R1-R2). The operational invariants (numbered O1-O5 in Table 8) were identified from George's teaching approach during the observation and his reflection during the interview. For example, George said in the interview that he had typed up the algebra and the trigonometry work (R3 & O2), so he can go back to it during the lesson to check if a mistake was made:

> "If we make a mistake in the algebra then they can get a bit confused […] And, so it's important for me to try and check and I'm checking all the way through 'is that correct? is that correct? is that correct?'. So, if I forget a step then it, kind of, they get a bit misled. But, it usually goes ok. If it does go wrong then we are usually able to go back and go 'okay, look I know what I'm expecting so was it this step, this step, this step, this step?' And, at, it's at that point that I can then go back I've got the trigonometry typed up. […] if I need to as backup, when I can show them it's supposed to be like this so they understand it that way"

In terms of inferences (I1 in Table 8), George expressed in the interview that the activity he used on Autograph for this lesson worked well, so he would use it again in the future: “as long as we need to teach that, I’ll probably carry on that way because it seems to work fairly well”.

Using the Knowledge Quartet

In relation to the foundation dimension, George’s mathematical and pedagogical knowledge was noticed throughout the lesson and the interview. *Awareness of purpose* was not explicitly expressed by George. His aims of teaching were: teaching the trigonometry formulae and preparing students for the examinations. He sometimes concentrated on procedures (*concentration on procedures*); for example, when he was solving exercises on finding R he moved away from using Pythagoras (as one student suggested) in his *demonstration* about the formulae, and followed the method he initially had in mind. Another example is “in his suggestion of mnemonic rules in the identification of the right formula” (Kayali & Biza, 2018b, p. 116). As well, George followed *examination requirements*. So, he solved past-examination questions (R6 & O5) during the lesson; and commented during the interview that this is important to prepare students for their examination: “Once you are ready you just practise past papers because they are the best way to get you the most experience of examination-style questions”. George also followed the *school policy* in relation to homework (R7), so he set up one homework assignment every week which was due in one week. For example, the homework for this lesson was chosen from the textbook and was due in one week. Hence, it seems that “*examination requirements*” and “*school policy*” are part of George’s foundation and they affected his choices in mathematics teaching.

In terms of the transformation dimension, George used Autograph to show a sine function that had been translated and stretched in order to *demonstrate* how trigonometry formulae could be related (R1, R2 & R4, O1 & O3). His *instructional materials* included both Autograph and the textbook, which was his main source for the questions to work on in class and for homework. He also relied on the *students’ contributions*, by asking for their estimate of the equation of the graph he prepared. Those *contributions* were a *resource* that George used along with other *instructional materials*. In terms of the connection dimension, George connected students’ contributions with the representation on Autograph and used examples and representations in order to connect concepts and procedures (*making connections between procedures*, *making connections between concepts*), he said: “So, I was using it to make them think one thing and then force them to see it in a different way and then make them make the connection between”. Also, George tried to connect the new concepts and procedures to the textbook and examination requirements by using textbook exercises and past-examination questions (R5 &

R6, O4 and O5). Thus, in this lesson, "*connection between resources*" and the mathematical meaning they bring was an important feature of George's teaching approach.

In terms of the *Contingency* dimension, I noticed that George did not react to one student's suggestion about using Pythagoras to find tan θ and then to find R. There is no evidence whether George heard the student or not. However, when the same student repeated his suggestion a bit later in the lesson, George followed his suggestion. But, when George moved to solving past-examination questions, he returned to using the method he initially had in mind (i.e. squaring the equations R cos θ = a and R sin θ = b and adding them) to find R, which is implicitly connected to Pythagoras. Another contingent moment was the moment a student inquired about George intentionally (maybe to write quickly and save time) writing $\cos^2$ instead of $\cos x \cos \Theta$, and $\sin^2$ instead of $sin\ x\ sin\ \Theta$. George was *responding to student's ideas* by correcting his writing (i.e. he changed $\cos^2$ to $\cos x \cos \Theta$, and $\sin^2$ to $sin\ x\ sin\ \Theta$). Also, in relation to the *Contingency* dimension, it was not clear from the data whether the student query about having a tangent in the trigonometry questions was expected or unexpected by George. George responded to the question, but there is no evidence whether that was a contingent moment or not. Elements of the Knowledge Quartet dimensions (foundation (*overt display of subject knowledge*), transformation (*use of instructional materials*: non-use of calculator in this case) and connection (*anticipation of complexity*)) were evident in George's response to the student who questioned if he could use his calculator to find R by substituting the values of sin θ and cos θ in the equation R^2 ($\sin^2$ θ+$\cos^2$ θ) =13 (instead of using $\sin^2$ θ+$\cos^2$ θ =1). George did not endorse that method and explained that by using $\sin^2$ θ+$\cos^2$ θ =1 one would "avoid the tiny roundings" a calculator would do if the values of sin θ and cos θ were substituted as the student suggested. He added that using $\sin^2$ θ+$\cos^2$ θ =1, a whole number, the answer would be more accurate.

Discussion

The analysis through the two lenses offers an overview of the teacher's documentational work towards the introduction of some trigonometry formulae (Figure 5) as well as preparing students for the examinations; and a look at the details of his actions in class using the Knowledge Quartet. The use of the Documentational Approach together with the Knowledge Quartet offers insights into George's work and capturing the dynamic nature of his teaching. For example, his comment on his work on Autograph during the post-observations interview reflected how he aimed to connect ideas, this is evident in his scheme of use (see Table 8) and is related to the Knowledge Quartet dimensions including his mathematical foundation, connection of mathematical ideas and use of examples and representations towards making these mathematical ideas accessible to the students. Thus, in this combined analysis, the Documentational Approach

<table>
<tr><th colspan="4">Scheme</th></tr>
<tr><th>Aims</th><th>Rules of action</th><th>Operational invariants</th><th>Inferences</th></tr>
<tr>
<td>Specific:
A1. To introduce the following trigonometry formulae:
$a\ \sin x\ \pm b\cos x = R\sin(x \pm a)$
$a\ \cos x\ \pm b\sin x = R\cos(x \mp a)$
$R = \sqrt{a^2 + b^2}$
$R\cos\alpha = a$ and $R\sin\alpha = b$
$\cos\alpha = \frac{a}{R}, \sin\alpha = \frac{b}{R}, \tan\alpha = \frac{b}{a}$
General:
A2. To prepare students for exam.</td>
<td>Specific:
R1. Use Autograph to show the graphs of a trigonometric function on the main whiteboard, where the teacher does the work and students contribute when appropriate.
R2. The trigonometric function was prepared before the lesson on Autograph.
R3. Use typed trigonometry formulae.
General:
R4. Connect the new ideas to students' previous knowledge.
R5. Choose exercises from the textbook, these exercises are at variety of difficulty levels.
R6. Use past exam questions to show styles of questions and mark schemes and offer some practice and preparation for exam.
R7. Set up homework with due date in one week.
R8. Find patterns and similarities/ differences between exercises.
R9. Answer students' questions.
R10. Students are given the time to solve questions and practice independently during class time.</td>
<td>Specific:
O1. It is easier to use Autograph to connect trigonometric ideas in this case.
O2. It helps to have the trigonometry typed up so at any point he can go back to that if needed, and show the students the correct steps.
General:
O3. It is useful to connect new ideas to previous knowledge.
O4. Choosing exercises from the textbook at variety of difficulty levels, helps to show students how to answer the question at each of these levels.
O5. The use of past-exam questions helps students practice and experience exam-style questions, and is something requested by students.</td>
<td>General:
I1. The activity on Autograph went well, so the teacher will use it whenever he is teaching these formulae.</td>
</tr>
</table>

Table 8: George's scheme of use in his first lesson

offers an overview of teachers' work in relation to the used resources, while the Knowledge Quartet offers a potent view of how the teacher's scheme of use "is activated and applied" in actual teaching (Kayali & Biza, 2018b). The Documentational Approach is used in tandem with the Knowledge Quartet in more teaching episodes with George aiming to address themes that came up from the analysis of lesson 1, these themes are:

1. The use of the Documentational Approach proposed looking at students' contributions (including their previous knowledge) as resources on which the teacher acts and builds the lesson.
2. The Knowledge Quartet analysis suggested paying attention to how the teacher created connections between resources (including students' contributions). This promoted the idea of the need to add more codes to the Knowledge Quartet in order to reflect such key details in the context of George's teaching.
3. Regarding the Knowledge Quartet, the data analysis suggested paying attention to how the teacher dealt with examination requirements and school policy (including school policy about homework).
4. Also, regarding the Knowledge Quartet, the analysis indicated towards considering the teacher's knowledge about mathematics and the software.
5. Another aspect in relation to the Knowledge Quartet is about contingent moments and how they can be identified. Identifying them is especially challenging in the cases when the teacher responds confidently to unexpected questions; one example is the case highlighted in the last paragraph of the analysis (about the student's questioning what to do when asked about tangents).

Having these points in mind, and to further examine what emerged from the analysis of the first lesson, I chose some teaching episodes for analysis in the next sections instead of whole lessons. The next sections offer a discussion of seven episodes to help reflect on the above five points.

4.1.2 Episode 1: A tour around the school website

George started one lesson by giving the students a "tour around" the school's website. He mentioned the parts of the school's website and classified them as "lots of handy documents"; these included worksheets, mathematics homework coversheet, link to the end of chapter questions, a link to the past mathematics examination papers, formula book, etc. According to George, all of these were linked to the mathematics scheme of work. He added:

> “The reason I’m showing you this is because we’ve added new column which is your homeworks. […] Sometimes the homeworks are the end of chapter questions, like this one… Other times it’s this one, we call it online test, we should call it online homework”

He then showed the link to integral maths (https://integralmaths.org/):

> “Have you used MyMaths [https://www.mymaths.co.uk/] before? Yep, so it’s a little bit like that. But this is the industry standard, this is the one that if you’re gonna use one in school, this is the one to use. […] Have you heard of advanced maths support program or people like that? They are the people and organisations who lead on A level maths teaching, it’s all linked together and if they were to recommend something they’ll recommend this”.

After that George asked his students to log into the website, but not all the students in this group were seeing the same content. Some students could only see core 2, the teacher promised to fix that. The teacher then clicked on core 3 and explained about its content. He asked his students how the pages appeared on mobile devices. A student answered: “it’s not the same”. George then started setting up homework for the students and explained that the website will enable him to see the homework assignments completed by the students and track students’ work and progress. The teacher then gave the students ten minutes to familiarize themselves with the site and its “resources”, in the meantime he was trying to sort out the issues some students had with course accessibility on the website. After that, he set up the homework of the week, which had due date next day on the website, but he asked the students to ignore the due date and take a week to complete homework. He emailed his students the link to the homework. The tour lasted about 25 minutes, after which George said they should now do some “new learning”.

Analysis

This episode is part of lesson 2, the scheme of work for this full lesson is summarized in Table 9. Episode 1 focuses on George’s resources, and specifically the school website. The website had links to the end of chapter questions, these were from the textbook - in this case Wiseman & Searle (2005). The website also linked to past-examination papers, which reflected an emphasis on examination requirements (O5). One link led to integral maths (https://integralmaths.org/) and George showed appreciation of the website (O3), as it was recommended by the people who led on A level. The part George aimed to show in this episode was the homework part. This part was George’s tool to see who did the homework;

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A3. To familiarize students with the school website. A4. To set up homework A5. To introduce the volume of revolution *General:* A6. To prepare students for exam	*Specific:* R11. Explain to the students about what they find on the school website R12. Check what students see on the screen once they log into integral maths R13. Set up homework on the website R14. Start introducing the volume of revolution R15. Remind the students of the formula used to find the area of a circle R16. Use Autograph to show the students a graph drawn in 3D R17. Show students the graph is for a function they previously learnt R18. Use trapezium rule and Simpson's rule on Autograph to shade the area between the graph and the x-axis R19. Explain that Simpson's rule is more accurate R20. Rotate the shaded area around the x-axis R21. Show the students different positions and rotations of the shaded area R22. Show another example prepared previously for the graph of $y = \sqrt{x}$ R23. Introduce the formula for volume of revolution using Autograph R24. Give some idea about the history of the integration notation R25. Explain why integration is used to find the volume of revolution R26. Use a textbook image to explain the formula of volume of revolution R27. Solve an example from the textbook, starting with an easy question R28. Use the formula sheet saved on the computer and give hard copies R29. Explain the formula for rotations around the y-axis *General:* R30. To give the students a tour around the school website R31. Give the students 10 minutes to familiarize themselves with the website R32. Introduce volume of revolution and connect it to students' previous knowledge R33. Give tips to the students and answer their questions R34. Use exam questions to familiarize students with questions on volume of revolution R35. Use the formula sheet	*Specific:* O1. Graphs on Autograph help students understand the volume of revolution O2. With Autograph we can make all these weird different shapes and have a bit of fun O3. Using a familiar graph helps reinforcing students' previous knowledge O4. Using graphs drawn before helps in focusing on the new topic during the lesson O5. The textbook image "shows it cut up a little bit easier… better than anything I could do on Autograph" *General:* O6. The school website is useful O7. Students need to see the website because of the addition of homework column O8. Integral maths is recommended to students, it is designed by those who lead on A level maths teaching and has lots of helpful resources O9. Setting up homework on the website enables the teacher to see how long students spends on the it, their scores, and attempts O10. Autograph helps students visualise the rotation and volume of revolution in 3D O11. The use of exam-style questions is in response to students' needs	*Specific:* I2. The textbook image is easier than anything done on Autograph I3. It is useful to have Autograph and textbook I4. "We can make all these weird different shapes and have a bit of fun with it" I5. Using the sine function is more interesting than using polynomials

Table 9: George's scheme for lesson 2 on volume of revolution

how much each student progressed and how much time s/he spend on the homework. All of these reflected George's appreciation of institutional factors: the school and its policies (e.g. homework policy), the textbook, and examination requirements (O1 and O2). One important resource here is the homework tool on the school website; which would help George monitor students' progress on homework, both in terms of the time spent on it and its quality.

4.1.3 Three episodes on the volume of revolution:

The three episodes about the volume of revolution were taken from the second, fourth and seventh lessons I observed for George. They aimed to teach the same topic, hence I grouped them together in this section. Each lesson lasted for fifty minutes. The first and third of these lessons were taught to one group (G1) of Year 13 students, in the first lesson George introduced the idea of volume of revolution and in the third one he reviewed the same idea to his students. The second lesson was taught to another group (G2) of the same year, and this was a lesson to introduce the volume of revolution to the students. A detailed overview of the episode from the first lesson (lesson 2) is offered in the next sections, after that two sections are devoted to show the changes noticed in lesson observations 4 and then 7.

4.1.3.1 *Episode 2- from Lesson 2*

This lesson included George familiarizing a group of year 13 students (G1) with the different components of school website including the one for homework (episode 1); and introducing the volume of revolution to them. The following three sections will look specifically at George's introduction to the volume of revolution through Autograph.

Introduction to "new learning" through Autograph

George started what he described as "new learning" in this lesson by stating the lesson topic, which was the volume of revolution. George asked a student he chose about the area of a circle. The student said $2\pi\ r$ first, then $2\pi r^2$, then he said that he did not remember. Another student said the correct formula (πr^2). The teacher then commented that these "are the two formulae" for a circle and that "something squared that's gonna give you the area, and 2 π r that's gonna give you the circumference".

George then started using the interactive white board. He showed the students a graph of y=x(3-x) and commented that although the graph might look like a "normal" graph to the students, he had actually "drawn it in 3D". He rotated the graph to show the students that it was 3D. He added that he wanted to find the area underneath the graph and that he would use the trapezium[3] rule, about which students learnt in core 2. The teacher tried to use Autograph to apply the trapezium rule, but he was not sure how to do it. His first attempt did not work, and he reached an unexpected result. His second attempt worked, and he applied the trapezium rule with five divisions (Figure 9). He looked for the zooming in option, a student pointed out that it can be done using the “magnifying glass on the left side”. George zoomed in and edited the image to get the graph in Figure 10.

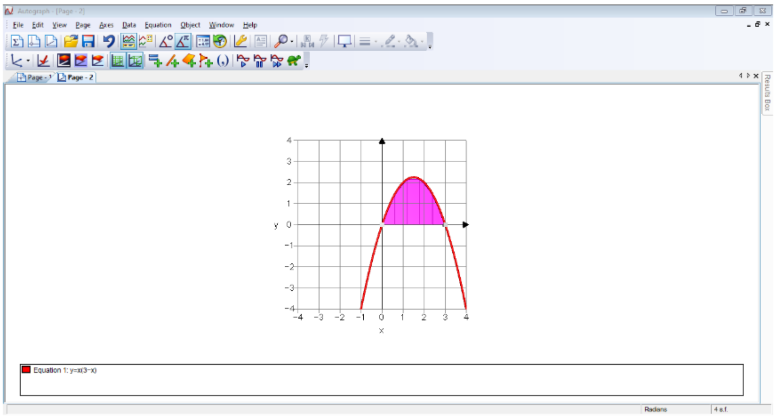

Figure 9: Trapezium rule for y=x(3-x) *with 5 divisions (reproduction of George’s work)*

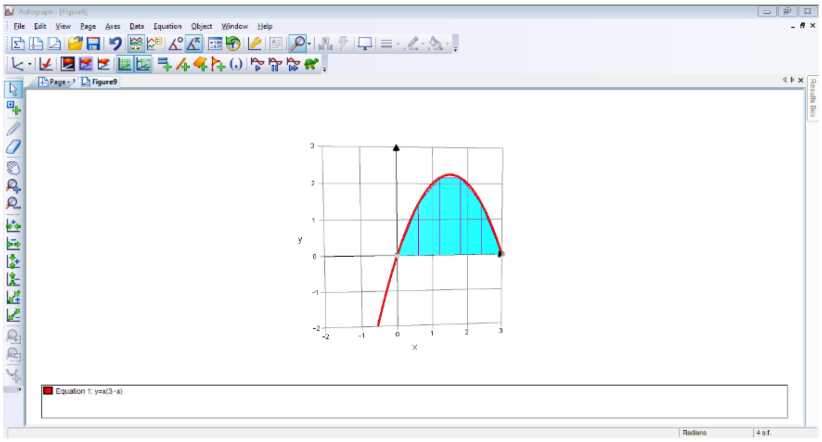

Figure 10: Trapezium rule for y=x(3-x) *with 5 divisions as edited by George (reproduction of George’s work)*

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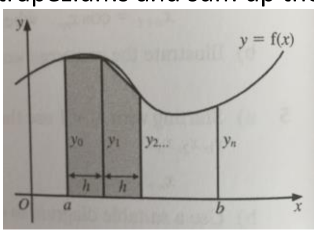
[3] Trapezium rule: to estimate the area under a curve $y = f(x)$, split the required area into n trapeziums and sum up these trapeziums’ area. This leads to the formula:



$\int_a^b y dx \approx \frac{h}{2}[\,(y_0 + y_n) + 2\,(y_1 + y_2 + \cdots + y_{n-1})]$ where $h = \frac{b-a}{n}$

(Wiseman & Searle, 2005, pp. 128-129)

George reminded the students that with fewer divisions the answer would be less accurate and showed them how the answer and the covered area would change when moving from 3 divisions to 12 divisions.

He then moved to using Simpson's[4] rule, which he said he was going to explain later: "I'm gonna use Simpsons rule because it filled it up a lot nicer. Even if you've got two divisions, it's filling in quite nicely", so he showed his student the graph of *y=x*(3-*x*) (Figure 11) with two divisions using Simpson's rule. George rotated the shaded area he got using Simpson's rule around the *x*-axis, and he got a shape (Figure 12) that looked like a "pointy sphere", a "Pacman[5]", or a "smarty" as he described it.

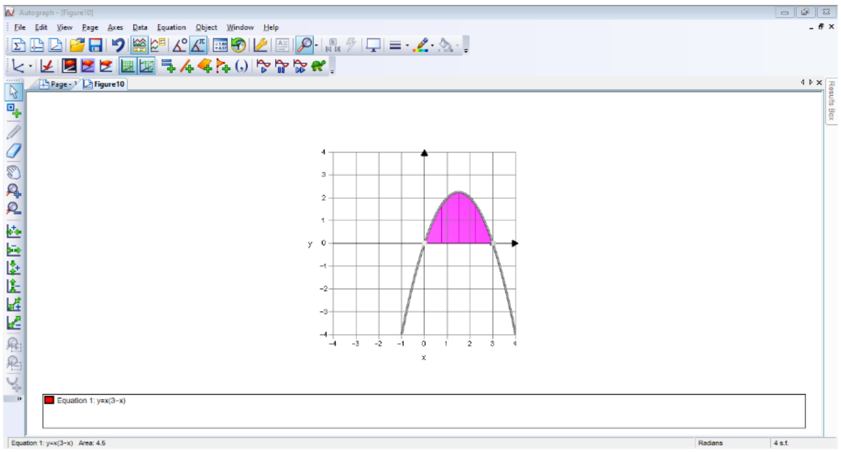


Figure 11: Simpson's rule for y=x(3-x) *with 2 divisions (reproduction of George's work)*

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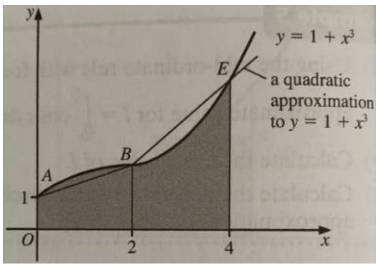
[4]Simpson's rule: to estimate the area under a curve, this rule fits a quadratic function to the curve where the quadratic passes through the mid-point and each of the two end points, leading to the formula:



$\int_a^b y dx \approx \frac{h}{3}[\,(y_0 + y_n) + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})]$, where $h = \frac{b-a}{n}$ and n is even

(Wiseman & Searle, 2005, p. 132)

[5] In reference to Pacman who was the hero character in the videogame Pacman that was released in 1980. The game can be accessed on www.webpacman.com

George opened another pre-prepared graph on Autograph, this time for $y = \sqrt{x}$, and showed the students the rotation of the area between this graph and the *x*-axis, around the *x*-axis (Figure 13). George added that the idea is not just to admire how "pretty" these shapes are, but also to work out their volume and for that the area of a circle is needed. So, he tried to explain to the students how to work out the volume of revolution using the software. To do that, he rotated the graph with a small angle on the software in order to show a "thin slice" of the shape for which he wanted to find the volume. He was dragging the end of the arc further from the axes and make the shaded area smaller and then bigger again. After that, he dragged a point again to show a thin slice of the shaded area (Figure 14). It seemed that George did not feel that the software showed what he was planning to show, and he was not satisfied with the *demonstration* he offered. He again tried to demonstrate how the formula can be concluded, emphasising that the formula will include integrating πr^2 between the limits the students are given in the question. After that, he moved to talking about who came up with the integration notations and explained that integration is like "sum", that was why the symbol for integration (∫) is like an (s) shape.

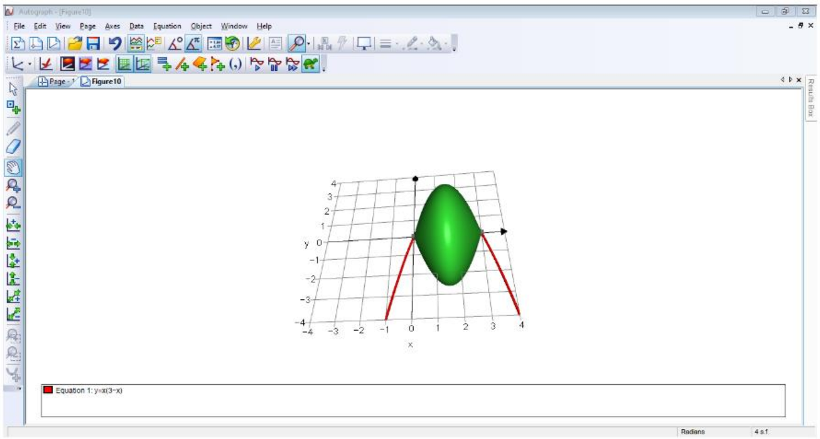


Figure 12: Rotating the shaded area in Figure 13 around the x-axis (reproduction of George's work)

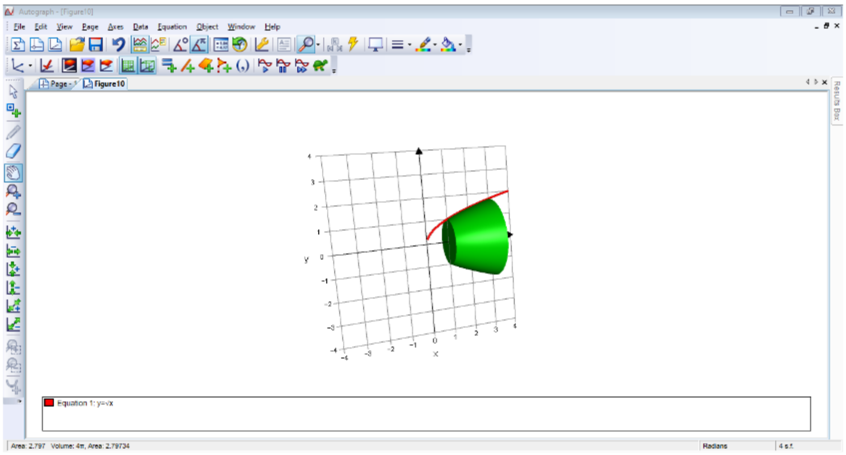


Figure 13: The rotation of the area between $y = \sqrt{x}$ and the x-axis, around the x-axis (reproduction of George's work)

Figure 14: George's attempt to show a thin slice of the shaded area on Autograph

Moving to the textbook and past-examination questions

His next step was to invite students to practice some textbook questions when he noticed a figure (Figure 15) on page 108 of the textbook (Wiseman & Searle, 2005) illustrating the formula. George used this Figure to re-explain the formula, and after that he started solving textbook questions (example 15) on the board, explaining that he was starting with an "easy example".

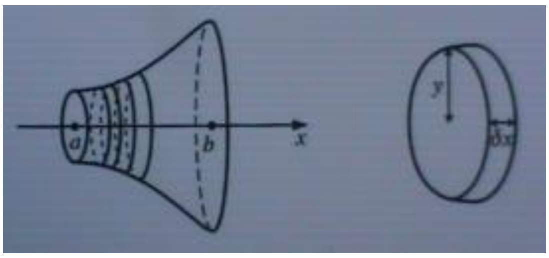


Figure 15: Diagram illustrating the formula of volume of revolution in the textbook by Wiseman and Searle (2005, p. 108)

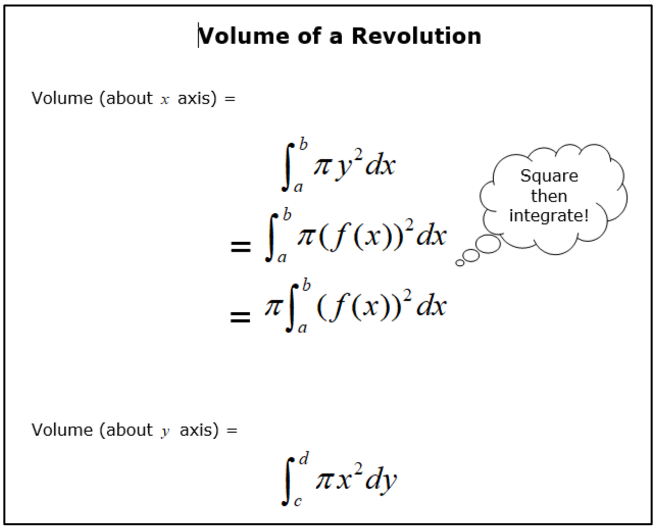


Figure 16: Formula sheet for volume of revolution, prepared by George

Afterwards, George displayed the formula sheet (Figure 16) on the interactive whiteboard, and remembered at that point that he had not explained the formula of the volume of revolution for rotations around the *y*-axis. So, he went quickly through this formula by advising the students to replace *y* by *x* and the d*x* by d*y* in the initial formula.

He then went back to talk about Newton and Leibniz and their notations for differentiation and integration:

> "So, Newton was busy doing y dot and y double dot which actually ties on with you, you know if you do f(x) the dot slowly became a dash and so we get the double dash. Have you seen that before, that notation? And then you have Leibniz doing the dy by dx and d2y by dx^2. Someone asked me the other day, so I'm not sure why do we say d2y instead of d^2y? I'm not sure about that. So, this is a little bit of history about where it came from. So, it was Leibniz and Newton both worked and independently to do Calculus…".

After that, George gave the students some guidance on how to solve example 15 from Wiseman and Searle (2005, p. 108). He also commented on another example (Figure 17) that contained "horrible square root", saying that because the students will get rid of the square root as soon as they apply the formula which includes squaring y. A student then asked why it was y^2, so George replied by explaining again about the thin disks that form the shape and how integrating the formula for their area, which is the formula for circle area, will give the volume required. Then, a student asked how the questions came in the national examination and whether there could be a need to do "integration by parts" in such questions or whether it was "more likely to be an easier way". The teacher answered "they could … I think so". George chose to do a past-examination question on the board next. He opened the school website on the interactive whiteboard and started checking for past examination questions. He chose the question in Figure 18 to solve on the board. The equation was $y = x^2 - 9$. George said the students needed x^2 for this question and he wrote $x^2 = y + 9$. So, $v = \cdots$. A student (student A) asked him about how to find the volume of revolution for the shaded area shown in Figure 18 when the rotation is done around the x-axis. Further details on students' questions and George's response are discussed in episode 7.

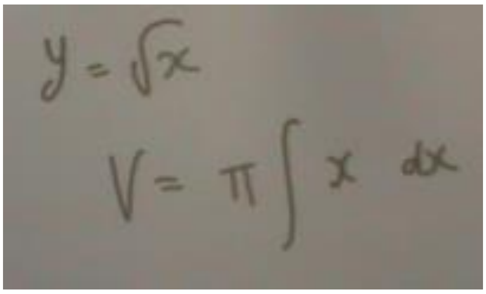
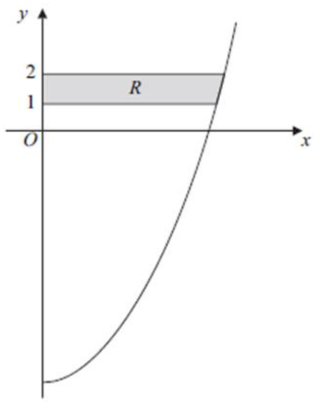


Figure 17: An example suggested by George

4 (a) Use integration by parts to find $\int x \sin x \, dx$. *(4 marks)*

(b) Using the substitution $u = x^2 + 5$, or otherwise, find $\int x\sqrt{x^2+5} \, dx$. *(4 marks)*

(c) The diagram shows the curve $y = x^2 - 9$ for $x \geqslant 0$.



The shaded region R is bounded by the curve, the lines $y = 1$ and $y = 2$, and the y-axis.

Find the exact value of the volume of the solid generated when the region R is rotated through 360° **about the y-axis**. *(4 marks)*

Figure 18: A past-examination question chosen by George to solve on the board

4.1.3.2 *Episode 3 - from Lesson 4*

During the second lesson on the volume of revolution, George was introducing the topic to another group of Year 13 students (G2) and he followed similar steps to the ones he did in lesson observation 2. So, in this section the focus is on addressing the similarities and differences with special attention to the latter.

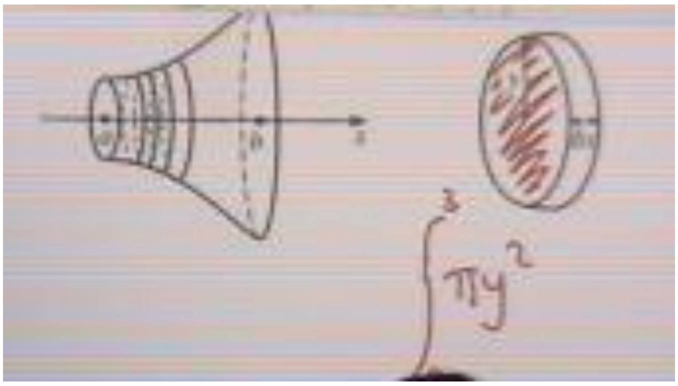


Figure 19: George's work on the diagram illustrating the formula of volume of revolution in the textbook by Wiseman and Searle (2005, p. 108)

In this lesson, the first difference was that George started the lesson by doing "some flicking through text book" starting from page 95 in Wiseman and Searle (2005). During that time, he was commenting on the topic of integration (integration by parts, integration by substitution, definite integrals…). In the meantime, he had the slide with the volume of revolution title and its formulae (Figure 16) displayed on the interactive whiteboard. Instead of using Autograph to explain the formula for the volume of revolution, George displayed on the board a pre-scanned copy of the textbook illustration (Figure 15). To explain why the diagram was displayed, he showed the graph of $y = x(3 - x)$ on Autograph in 3D. A student asked why that was done in 3D. George started explaining that it was because he needed to find the area between the curve and the x axis. And, he continued working on Autograph like he did in the lesson before. He would start with using the trapezium rule, which left some gaps uncovered and gave underestimate. Then he moved to using Simpson rule, which he was planning to teach the students in the next chapter. He was dragged the plane of the graph and showing it in different positions. He explained that the volume of the shape, resulting from rotating the area shaded when using Simpson rule around the x axis, was called the volume of revolution because it was obtained from "revolving something around". Students then started guessing what the shape would look like and one of them guessed that it would look like a "pointy sphere", others called it "smarties" or "Pacman". George showed the students different positions and rotations of the shaded area. To explain how to find the volume of revolution, George went back to the diagram he initially showed on the interactive whiteboard, which was scanned from the textbook

(Figure 15). He explained the formula of the volume of revolution in a way similar to the way he used in the previous lesson with this image, suggesting that students should think of the volume as “lots of slices” being added up using integration. He asked the students to look at one “slice” and think of what the area of its face was, it was a circle (πr^2). He explained that instead of r it is y in this case (Figure 19). Then, the explanation became more specific as he used the equation he entered on Autograph to find the volume of revolution and to show the students the limits of the integration for this example:

$$y = x(3 - x)$$

$$y = 3x - x^2$$

$$V = \int_0^3 \pi y^2 \, dx$$

$$V = \pi \int_0^3 (3x - x^2)^2 \, dx$$

$$V = \pi \int_0^3 9x^2 - 6x^3 + x^4 \, dx$$

A second difference was that George talked about the following example that contained a square root:

$$y = \sqrt{3x^2 - 7x + 2}$$

$$y^2 = 3x^2 - 7x + 2$$

He used it to explain that because students needed to find y^2 to find the volume of revolution, they would get rid of the “horrible” square root. He also reminded the students that they need to multiply the integral of y^2 by π. After that, George started explaining that to find the volume of revolution when the rotation is done around the y axis, students only need to change y^2 into x^2 and dx into dy. This was done while the formula sheet was displayed on the interactive whiteboard.

A third difference is that, in this lesson, George went back to Autograph to show two more examples not just one. He showed the students the rotation of $y = \sqrt{x}$, which was pre-prepared and used in the previous lesson. As well, he used a new pre-prepared example which was the rotation of the sine function: $y = sin(ax + b) + c$. With this example, he used a, b and c to transform the graph; the rotation of which gave different shapes (Like the one in Figure 20 for example). The students were very impressed with the images they saw, and George commented “Some fun… moving the sine curve along. Sadly, the exam questions don’t get this much fun”.

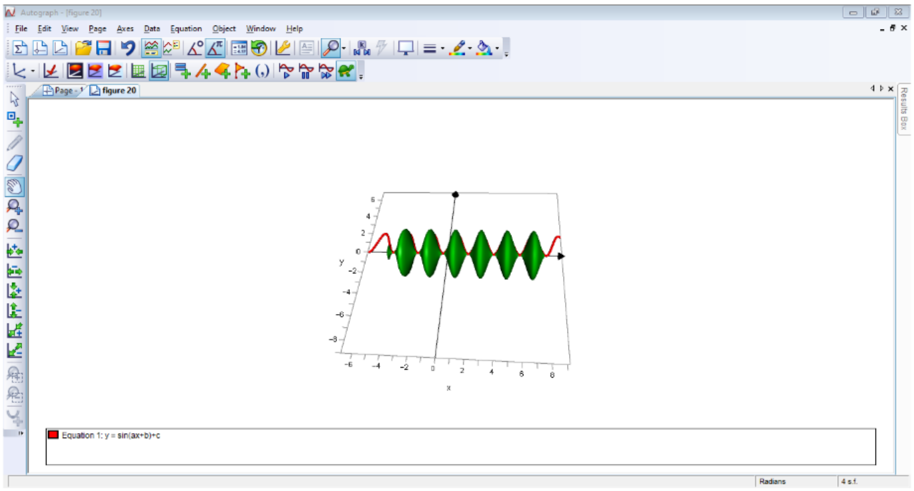
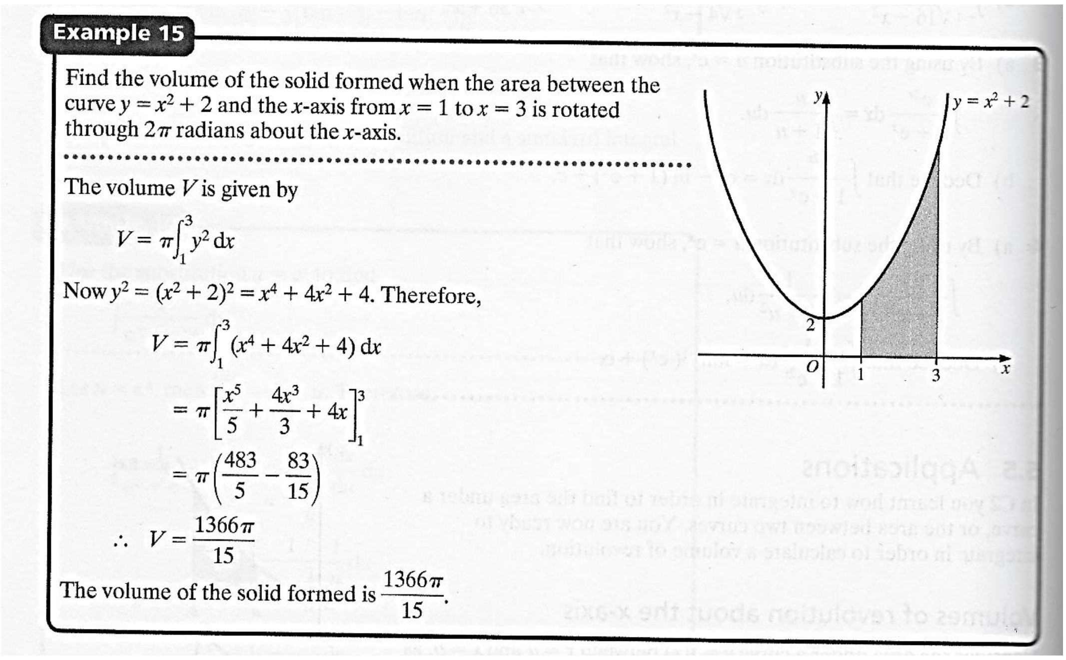


Figure 20: One shape created from rotating $y = sin(ax + b) + c$ on Autograph (reproduction of George’s work)

The fourth difference was that George solved two textbook examples in this lesson. So, beside examples 15 which he solved in the previous lesson (Figure 21), he also solved example 16 (Figure 232) in Wiseman and Searle (2005, p. 109).



Example 15

Find the volume of the solid formed when the area between the curve $y = x^2 + 2$ and the x-axis from $x = 1$ to $x = 3$ is rotated through 2π radians about the x-axis.

The volume V is given by

$$V = \pi \int_1^3 y^2 \, dx$$

Now $y^2 = (x^2 + 2)^2 = x^4 + 4x^2 + 4$. Therefore,

$$V = \pi \int_1^3 (x^4 + 4x^2 + 4) \, dx$$

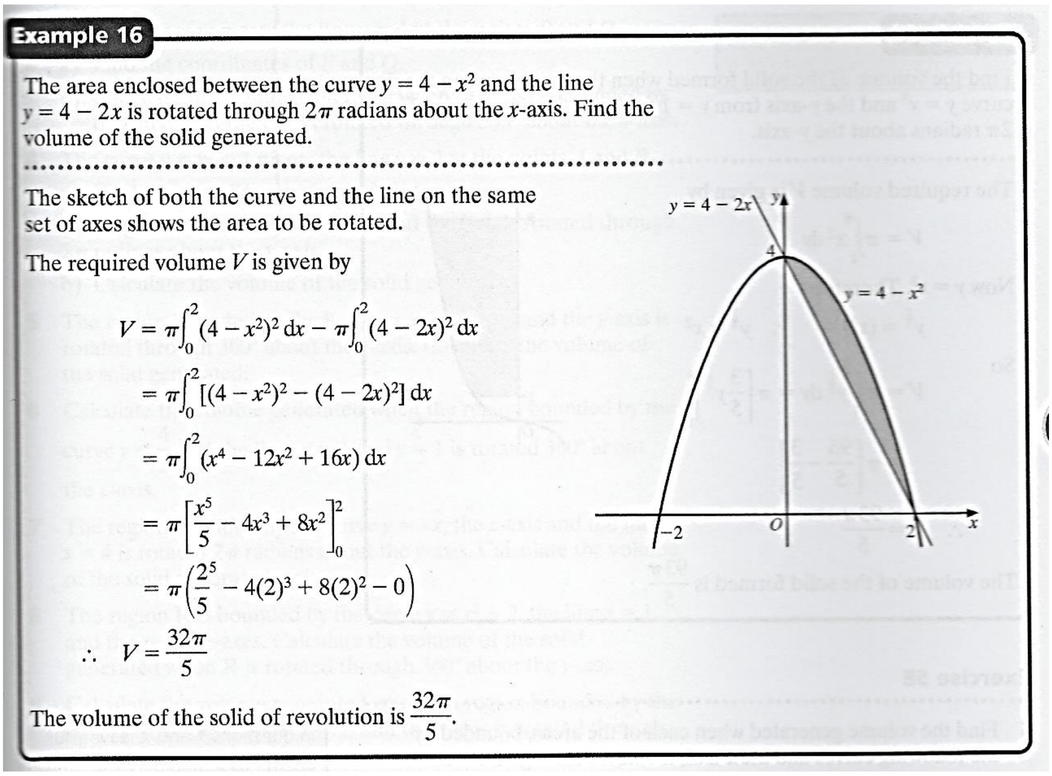
$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_1^3$$

$$= \pi \left(\frac{483}{5} - \frac{83}{15} \right)$$

$$\therefore \quad V = \frac{1366\pi}{15}$$

The volume of the solid formed is $\frac{1366\pi}{15}$.

Figure 21: Example 15 in Wiseman and Searle (2005, p. 109)



Example 16

The area enclosed between the curve $y = 4 - x^2$ and the line $y = 4 - 2x$ is rotated through 2π radians about the x-axis. Find the volume of the solid generated.

The sketch of both the curve and the line on the same set of axes shows the area to be rotated.

The required volume V is given by

$$V = \pi\int_0^2 (4 - x^2)^2\,dx - \pi\int_0^2 (4 - 2x)^2\,dx$$

$$= \pi\int_0^2 [(4 - x^2)^2 - (4 - 2x)^2]\,dx$$

$$= \pi\int_0^2 (x^4 - 12x^2 + 16x)\,dx$$

$$= \pi\left[\frac{x^5}{5} - 4x^3 + 8x^2\right]_0^2$$

$$= \pi\left(\frac{2^5}{5} - 4(2)^3 + 8(2)^2 - 0\right)$$

$$\therefore \quad V = \frac{32\pi}{5}$$

The volume of the solid of revolution is $\frac{32\pi}{5}$.

Figure 22: Example 16 in Wiseman and Searle (2005, p. 109)

The fifth difference during this lesson was when George recalled two questions asked by students during lesson 2 with G1. He talked about when a student (student A) asked him about how to find the volume of revolution for the shaded area shown in Figure 18 when the rotation is done around the x-axis. But, instead of using the original

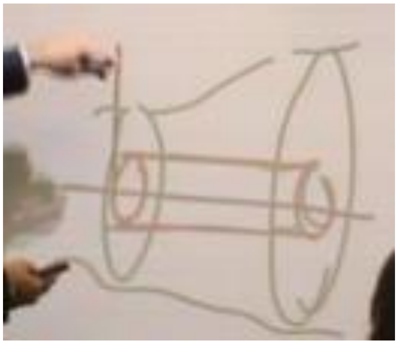


Figure 23: A shape sketched by George during his second lesson on volume of revolution

image in Figure 18, he used a shape he drew quickly (Figure 23) and stated that the student asked him about the volume of revolution in this case. One student voluntarily answered "cylinder". The teacher responded "Exactly, exactly, exactly, you do your volume of revolution as normal take away that cylinder…".

4.1.3.3 *Episode 4- from Lesson 7*

This was lesson was devoted to further work on the volume of revolution with G1. George started by quickly reviewing the idea and formula of volume of revolution. He showed the same example he used in the previous two lessons y=x(3-x), and, then, used the textbook illustration to explain how the formula was deduced (figure 17). He also used cards to remind the students of the formulae of volume of revolution. He mentioned that there were two types of questions: "easy ones" (Figure 24) and "more difficult" ones (Figure 25). Then, he proceeded with a past-examination question solution and a presentation of its mark scheme. After that, he gave the students some time to solve questions independently until the end of the lesson.

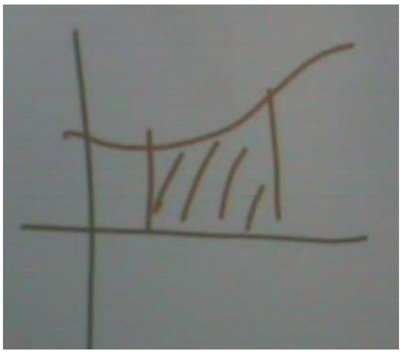


Figure 24: An "easier" question on volume of revolution

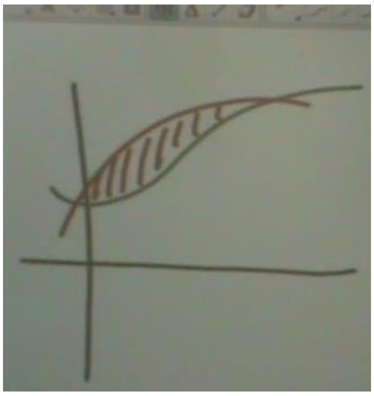


Figure 25: a "more difficult" question on volume of revolution

4.1.3.4 *Analysis of the three episodes on the volume of revolution*

Using the Documentational Approach

The resources George used in these three lessons were: interactive white board; board; curriculum of year 13; textbooks; past examination questions and mark schemes; past teaching-experience; students' contributions; calculators; notebooks; Autograph and pre-prepared graphs; formulae sheet; personal website; school website; and formulae cards. Although the formula cards were on display next to the board all the time and shown to the students only in the second and third lessons while the formulae sheet was used in the first and second lessons, the resources stayed almost the same across the three lessons. One more resource is added in the second and third episodes; that is George's experience teaching the first episode (specifically his experience in explaining the formula using Autograph and then the textbook diagram in Figure 15).

The schemes of use, George followed in lessons 2, 4 and 7 are summarized in Table 9, Table 10 and Table 11. The three schemes had the same aims: a specific aim "teach students about the volume of revolution", and a more general one "to prepare students for the exams". In terms of the other elements of George's scheme of use, I notice that George's rules of action in the first lesson included specific rules of actions that were under general ones. Under the general rule of action "to introduce the volume of revolution using Autograph" it is noticed that George went through specific rules of action including: showing the students the graph of $y = x(x - 3)$ drawn in 3D on Autograph; the use of the trapezium rule and Simpson's rule on Autograph; explaining that Simpson's rule is more accurate than the trapezium rule in this case; rotating the shaded area around the x-axis; showing the students different positions and rotations of the shaded area; showing another example prepared previously for the graph of $y = \sqrt{x}$; introducing the formula for volume of revolution around the x-axis using Autograph. While under the general rule of action "to connect new ideas with students' previous knowledge", rules of action included: reminding students of the formula of circle's area; explaining why integration is used to find the volume of revolution. Under the general rule of action "to use the textbook", specific rules of actions included: using the textbook diagram to explain the formula; solving an example from the textbook; and starting with an "easy" example. Other rules of action were: to talk about the history of integration notation; to use the formulae sheet to show the formulae; to explain the formula for rotations around the y-axis; to give tips to the students; to answer students' questions; to use past examination questions to give students idea about how they are tested on the volume of revolution; to give hard copies of the formulae sheet. The operational invariants included a general one like "George's appreciation of Autograph" under which there are specific ones including: Autograph helps students visualise the volume of revolution in 3D; with Autograph he "can make weird shapes and have fun"; his use of familiar graph on Autograph in order to "reinforce previous knowledge"; his use of pre-prepared graphs in order to help helps in focusing on the new topic and saves time. Other operational invariants included statements from George in relation to finding that for the purpose of explain the formulae the textbook diagram worked better than Autograph; and that the use of past-examination questions is in response to students' needs and to give them some practice for the examination.

Scheme			
Aims	**Rules of action**	**Operational invariants**	**Inferences**
Specific: A1. To introduce the volume of revolution *General:* A2. To prepare students for exam	*Specific:* R1. To flick through the textbook starting from page 9, talking about integration R2. Display the volume of revolution formulae on the board R3. Start introducing the volume of revolution R4. Show the textbook diagram that illustrates the volume of revolution formula R5. Use Autograph to show the students the graph of $y = x(3 - x)$ in 3D R6. Use trapezium rule and Simpson's rule on Autograph, and explain which is more accurate R7. Rotate the shaded area around the x-axis, and show its different positions and rotations R8. Use a textbook image to explain the formula of volume of revolution R9. Explain the formula specifically for $y = x(3 - x)$, rotated around the x-axis R10. Solve an example that has a square root R11. Explain the formula for rotations around the y-axis R12. Use Autograph to show another example prepared previously for the graph of $y = \sqrt{x}$ rotated around the x-axis R13. Use Autograph to show another example prepared previously for the graph of $y = \sin x$ rotated around the x-axis R14. Show the graphs are for functions the students had seen before R15. Stretch and shift sine graph on Autograph to show interesting shapes formed by rotation R16. Mention a question asked by a student from a different group, in a previous lesson on the volume of revolution and invite students to answer it R17. Start with an easy question (from a previous lesson) R18. Show students the difficult textbook questions that exceed exam requirements R19. Use the formula sheet saved on the computer to show the formula to the students *General:* R20. Connect new ideas about volume of revolution with students' previous knowledge R21. Solve examples from the textbook R22. Give tips to the students and answer students' questions R23. Use exam questions to show students how they are tested on volume of revolution	*Specific:* O1. The textbook image "shows it cut up a little bit easier… and is better than anything I could do on Autograph" O2. The graphs on Autograph help the students understand the volume of revolution and is a "kind of hook that interesting thing so they can see exactly what we are going on here" O3. With Autograph "we can make all these weird different shapes and have a bit of fun" O4. Using a familiar graph helps reinforcing previous knowledge about transformations O5. Using graphs drawn before helps in focusing on the new topic O6. Using an example that contains a square root helps the students see how they can get rid of the having a horrible square root in the formula *General:* O7. Autograph helps students visualise the volume of revolution in 3D O8. The use of exam-style questions is in response to students' needs, and to give practice for the exam	*Specific:* I1. The textbook image "shows it cut up a little bit easier… better than anything I could do on Autograph" I2. It is useful to have both Autograph and textbook I3. "We can make all these weird different shapes and have a bit of fun" I4. Using the sine function is more interesting than using polynomials

Table 10: George's scheme during the fourth lesson on volume of revolution

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A1. To continue working on the volume of revolution *General:* A1. To prepare students for exam	*Specific:* R1. Check with the students what they learnt in previous lessons R2. Re- introduce the volume of revolution R3. Use the formula card to remind students of the formulae R4. Use Autograph to show the students the graph of $y = x(3-x)$ in 3D R5. Connect the new ideas about volume of revolution with the students' previous knowledge R6. Use trapezium rule and Simpson rule on Autograph, explain that Simpson's rule is more accurate, and rotate the shaded area around the x axis R7. Show different positions and rotations of the shaded area and comment on colour/shade R8. Show the textbook diagram that illustrates the formula for the volume of revolution R9. Use a textbook image to explain the formula of volume of revolution R10. Answer a student's question about why they should learn about volume of revolution R11. Explain the formula specifically for the case of $= x(3-x)$, rotated around the x axis R12. Use an example that contains a square root to explain how to use the formula for the volume of revolution in similar situations R13. Explain that students do not need to learn how to find the formula, but how to use it R14. Explain the formula for rotations around the y-axis R15. Mention that there are two types of questions, easy ones asking about the area between a graph and the x/y-axis, and difficult ones asking about the area between two graphs R16. Show past-exam questions to give students idea how they are tested on the volume of revolution, and solve one question and review its mark scheme in detail on the board *General:* R17. Review how to find volume of revolution R18. Respond to students' contributions and questions R19. Give tips to the students R20. Solve past exam questions on board and let students practice solving some independently	*Specific:* O1. Graphs on Autograph help students understand the volume of revolution as is a "kind of hook that interesting thing so they can see exactly what we are going on here" O2. Autograph helps the students visualise the volume of revolution in 3D and shows the rotation O3. With Autograph we can make all these weird different shapes and have a bit of fun with it O4. Using graphs drawn before helps in focusing on the new topic O5. The textbook image "shows it cut up a little bit easier… better than anything I could do on Autograph" *General:* O6. The revision of the topic is useful for students who were not able to attend the previous lesson O7. The use of exam-style questions is in response to students' needs, and to give them practice for exams	*Specific:* I1. The textbook image "shows it cut up a little bit easier… better than anything I could do on Autograph" I2. It is useful to have both Autograph and textbook I3. "We can make all these weird different shapes and have a bit of fun"

Table 11: George's scheme during the seventh lesson on volume of revolution

In the second and third lessons, during which George introduced the volume of revolution to G2 and continued working on the topic with G1, the operational invariants stayed the same. And although, most of the rules of actions remained the same during the three lessons, there were differences in their appearance in George's teaching and in their order. In the rest of the analysis, the focus is on some of these differences in the rules of actions. Also, the inferences in George's scheme of use will be discussed in the three observations and the follow-up interview.

One main difference we see is in the use of Autograph. In the second lesson, did not attempt to use Autograph to introduce the formulae for volume of revolution. He used only the textbook diagram (Figure 19) for that purpose. He also showed one more example on Autograph in this lesson, then moved to working on textbook questions as he did in the previous lesson. In the third lesson, with G1, George continued working on the volume of revolution topic by quickly showing graph of function $y = x(x-3)$ and rotations of areas on Autograph, in a way similar to the way he followed in the first lesson and second lessons. Then, moved to solving textbook and past-examination questions.

In summary, in the last two lessons, "introduce the formula for volume of revolution around the x-axis using Autograph" was not a rule of action. And, in the interview, George commented on this by saying that the textbook diagram "show[ed] it cut up a little bit easier [...] and [was] better than anything [he] could do on Autograph". So, George's first inference from the three lessons is that he found the textbook diagram more helpful in explaining the formulae. He added that he found it useful to have "both Autograph and the textbook". This leads us to his other inference: it is useful to use both Autograph and the textbook as resources. Another inference is related to the functions entered on Autograph and how these were chosen to expand and build on students' previous knowledge by using familiar graphs. George commented during the interview on his choice of functions to graph on Autograph, and specifically his use of the sine function during the second lesson:

> "Partly from using that in previous lessons. So, knowing that that is going to give an interesting shape, and from playing around with sine graphs and things like that in previous lessons. So, using functions that they were aware of [...] So, a transformation of the sine curve I think, we were doing that with it. What I'm also doing there, I am also reinforcing or going back over making sure that they know about their transformations. So, I'm kind of teaching two topics at once. So, although we are doing this volume of revolution, I am also reminding them of what

they do when they do their transformations because I know they are going to get asked about that one"

George also thought that using the sine function was "more interesting than using polynomials". But he showed the sine function in the second episode (with G2) only. It was not clear from the data collected whether he showed the sine function to G1 in a different lesson, or whether he chose not to show it to them. However, in the interview he claimed that the use of the sine function on Autograph was a good choice for this topic. In terms of the use of past-examination questions, George used these after solving one example from the textbook in the first lesson. In the second lesson, he mentioned he was going to solve past-examination questions, but the lesson finished before he did. In the third lesson, he solved a past-examination question and explained its mark-scheme on the board. When asked about these choices, George mentioned that it was in response to students' needs that he now used past-examination questions frequently (one of his operational invariants). He added that students felt that not all textbook questions were examination-style questions, and some were even "more difficult" than examination questions (which something is he pointed to in the second lesson). It was also because he wanted to give his students some practice for the examination. As a result, he chose to use past-examination questions for every topic he taught. Finally, we noted that George did not have the time to solve past examination questions in the second lesson, maybe because he chose to solve two textbook examples. This was not evident in the data, which do not indicate the warrant of this choice.

Using the Knowledge Quartet

Most of the details of George's work stayed the same. In the first episode, George used Autograph (*use of instructional materials*) and was *making connections between concepts* (graphs, area under graph, trapezium rule, Simpson's rule, and volume of revolution). He was also *making connections between procedures* (area under graph, integration and volume of revolution), but these did not seem to work. When he decided to use the textbook to solve a question, he spotted Figure 15 in the textbook and decided to use it to explain the formulae of volume of revolution (*deviation from agenda*), this deviation led to change in the agenda in the next episodes by relying on Figure 15 only to explain the formula and using Autograph just for visualisation. This in turn involved connection between resources (Autograph, textbook, experience of teaching the first episode, formulae). His use of past-examination questions matches his aim in relation to preparing the students for the examination, so these along with the textbook are part of his instructional materials (*use of instructional materials)*. His reference to the history of mathematics, specifically in relation to the symbol of integration, does not come under any of the current Knowledge Quartet codes. One possible way to include it is by expanding on the foundation dimension to include knowledge about the history of mathematics.

Contingency was evident during the use of Autograph in the first episode, when George was trying different commands, in other words checking the availability of tools (e.g. to show "slices" of shape he got on Autograph). Having spotted the textbook diagram, he then responded to the availability of resources (i.e. the textbook diagram) (*responding to the (un)availability of tools and resources)*, and offered *demonstration* accordingly.

In summary, the differences between the three episodes on volume of revolution were related to the scheme of work in terms of order or rules of actions, but this change of scheme of work was due to the contingent moment George had when using Autograph in the first episode. This demonstrates the potencies of using Documentational Approach and the Knowledge Quartet to look at the dynamic nature of teachers' 'live' work and "explore the micro-evolution, namely the small changes and the rationale behind these changes, of teachers' documents from one lesson to another" (Kayali & Biza, 2018a, p. 202). Kayali and Biza (2018a) defined this micro-evolution as re-scheming, meaning scheming "again or differently" (Adler, 2000, p. 207), from one lesson to another. Re-scheming is further addressed in the Chapter 5 "Findings",

4.1.4 Episode 5 - from lesson 9: on differential equations

Lesson 9 was taught to a group of seventeen year 13 students. It started by George solving some textbook questions on numerical methods, specifically iteration. Then, he reminded the students that they learned about differential equations the day before and that the example they solved was a "nice" one. George tried to remember what the example was and thought it was $\frac{dy}{dx} = \frac{x+1}{y}$, which was question 2(b) in Wiseman and Searle (2005, p. 233). He asked his students to tell him how they solved it and he started writing:

$$\frac{dy}{dx} = \frac{x+1}{y}$$

$\int y.dy = \int (x+1).dx$ [the dots (.) here meant multiplied by]

$$\frac{y^2}{2} + c = \frac{(x\ +\ 1)^2}{2} + c$$

$$\frac{y^2}{2} + c = \frac{x^{\ 2}}{2} + x + c$$

Although the c on the first side on the last two equations was different from the one on the second side, George still used c on both sides instead of using c_1 and c_2 for example. He even continued writing:

$$\frac{y^2}{2} = \frac{x^{\,2}}{2} + x + c$$

And explained that in his last step he combined the c's in the previous equations and got "a big c", and that his next step is "rearranging".

$$y^2 = x^{\,2} + 2x + c$$

George added: "and then just square root both sides, ok. At that point you've got the general solution, and that's where we got the graphs like this. Because it's all different solutions, because we don't know what the + c is" (referring to Figure 26, which is for the equation $x(x-1)\frac{dy}{dx} = y$ on Autograph).

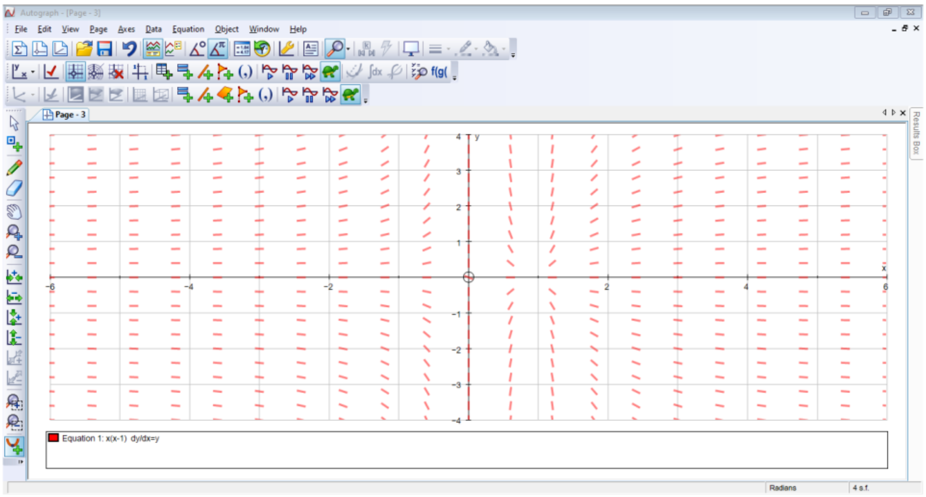


Figure 26: $x(x-1)\frac{dy}{dx} = y$ on Autograph (reproduction of George's work)

Then, the following conversation took place:

Students A: So, how did you do, you didn't see this?

George: okay, alright, let's. I've typed in a different equation (pointing to the equation line on Autograph $x(x-1)\frac{dy}{dx} = y$) that's the one we are about to do, okay. So, we are going to have a go at doing this question at the moment. All the computer understands is that there was a function that when it was differentiated it gave this answer here, okay. But the computer doesn't know what that value of c is so, it knows what shape it is. So, imagine this graph here, but it doesn't know, was it here? was it here? was it here? was it here was it all the way down here? So, there's all these different levels.

Student B: Can we just have the explicit one?

George: Okay, then take a step back

Student A: No, I get it I just can't visualise it properly. I can't see where, are they coming across the axis?

George: I'm not sure to be honest

Student B: Just imagine there is function where c is unknown

Student A: Yeah, I understand the concept of why there are so many

George: But, it's difficult to see from that

Student A: Yeah

George: I completely agree. I would struggle seeing that as well. I can see whatever it is, it is something coming along here.

Student B: And then it will go down

George: Goes down

Student B: I don't know, it contradicts

Student A: Looks like it goes round in a circle

Student B: Yeah

Student A: And the other one looks like it goes along and down and then along and up

George: I'll tell you what, let's find out by doing the question

Student A: I'm just confused how does it, like, go anywhere?

George: I don't know, so let's find out, here we go, I've chosen 11e question 2 part i.

The question was about finding the particular solution of the equation $x(x-1)\frac{dy}{dx} = y$ when $y = 1$ at $x = 2$ (Wiseman & Searle, 2005, p. 233).

Student B: Is this the same one?

George: It's the same one in the book, different to the one we did yesterday. In this case, we're going to go all the way to find a particular solution. We can do that because we can get a general solution from this, but then we are going to pin that solution. We know that it is the one that goes through this point, so that's going to help us out, okay. So, step number 1 rearrange…

George started to talk about the steps, but then he noticed that student C did not follow. So, he decided to write the solution down:

$$x(x-1)\frac{dy}{dx} = y$$

$$\int \frac{1}{y} dy = \int \frac{1}{x(x-1)} dx$$

By doing that, George said that he "prepared" both sides for integration and that the first side was "quite nice" as it was going to be "$\ln y$", while for the other side it would be "partial fractions". So, George displayed the work shown in Figure 27.

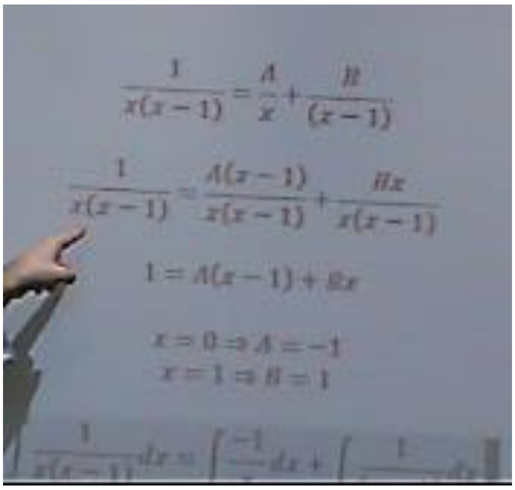


Figure 27: George's display of the "subroutine" for $\int \frac{1}{y} dy = \int \frac{1}{x(x-1)} dx$

He went through the "subroutine" shown in the image and then wrote:

$$\int \frac{1}{y} dy = \int \frac{-1}{x} + \frac{1}{x-1} dx$$

$$\ln y = -\ln x + \ln(x-1) + c$$

Then George commented on having the constant c on the second side only: "So, looks like the c went, we didn't bother writing this one, we combined it with that one". Then, he added that they were looking for y not $\ln y$, so he showed his students the following steps of the solution:

$$y = e^{-\ln x + \ln(x-1) + c}$$

$$y = A.e^{\ln x^{-1}} e^{+\ln(x-1)}$$

$$y = A.e^{\ln\frac{1}{x}} e^{\ln(x-1)}$$

$$y = A.\frac{1}{x}.(x-1)$$

$$y = A\frac{x-1}{x}$$

George showed the students how they could use the coordinates to (2, 1) to reach the particular solution $y = 2\frac{x-1}{x}$. He then went back to answer student A's question about the presentation on Autograph by saying that the "particular solution 2 – 2 – x sits in like that" and showed the graph (Figure 28) on the screen.

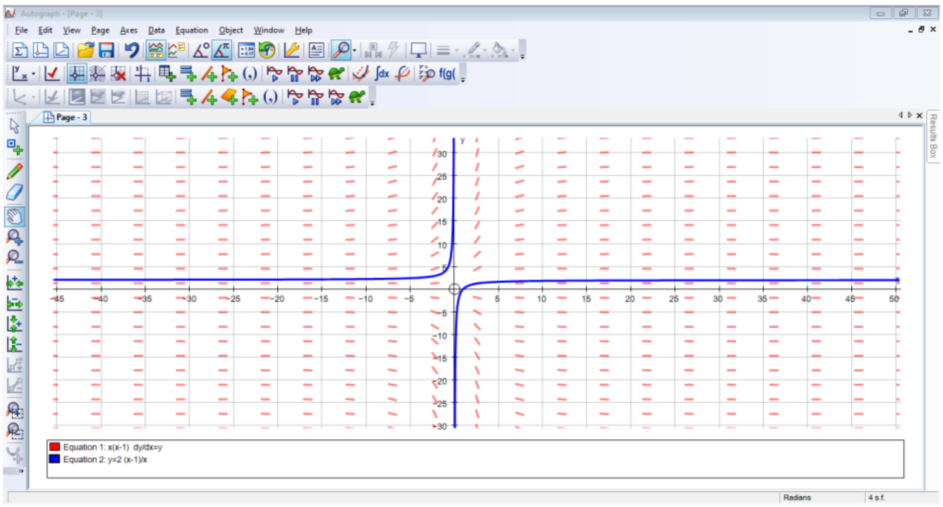


Figure 28: The particular solution $y = 2\frac{x-1}{x}$ *on Autograph (reproduction of George's work)*

Student B: So, it's a reciprocal. Oh, that makes sense

Student A: Look at the red line

Student B: It's a negative reciprocal.

Student A: It points down like and then we've got graph just drops horrifically between 1 and 0. The red lines don't show it accurately really

Student C: And then you've got the upside down one on the other side

George: These ones, yeah, what are they doing?

Student B: I think it's when it's minus (Figure 29)

George: Okay, so if we minus the whole lot?

Student B: Yeah

George: Because, can we have that

Student B: They seem to fit there as well

Student A: It worked better for a cubic graph. Do it just for a normal cubic graph because then it works a bit better.

George tried a function that integrates to a cubic one. He used $3x^2+2x$ first (Figure 30), which gave him "lots of vertical lines". So he decided to zoom in to show it "a bit better".

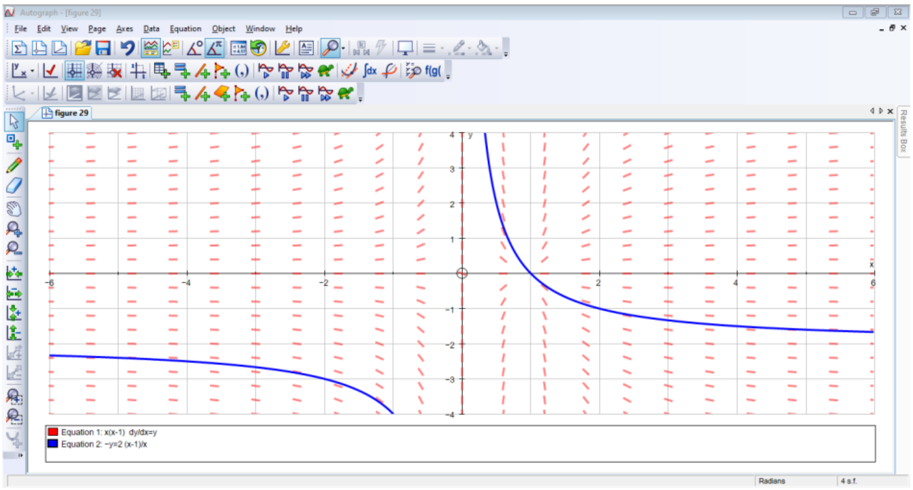


Figure 29: The particular solution $y = -2\frac{x-1}{x}$ on Autograph *(reproduction of George's work)*

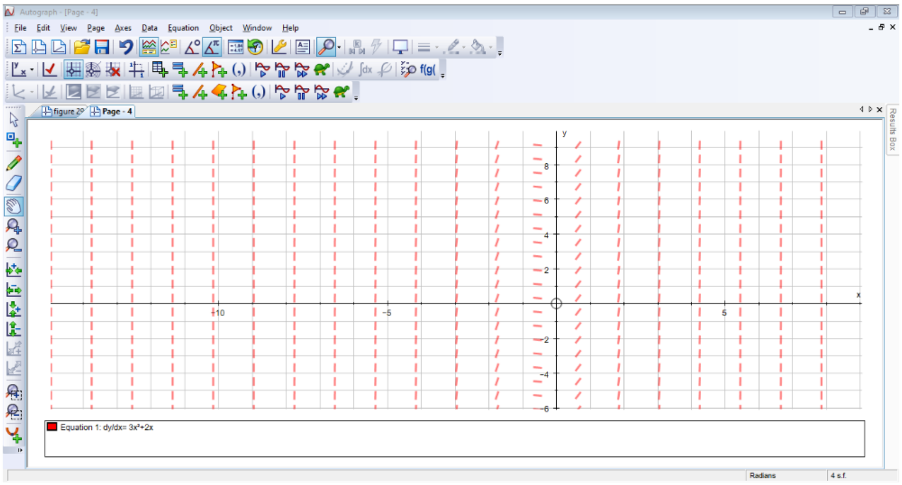


Figure 30: dy/dx= 3x^2+2x on Autograph showing "vertical lines", before zooming in (reproduction of George's work)

George then entered $y = x^3 + x^2 + 1$ to show one particular solution (Figure 31).

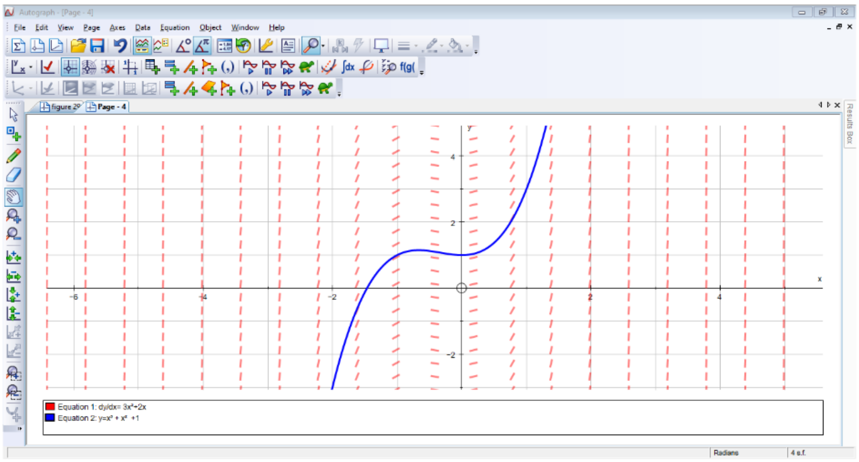


Figure 31: The graph of $y = x^3 + x^2 + 1$ on Autograph (reproduction of George's work)

Student B suggested "If you put $+ a$ and a slider, it's just to prove a point". George did that as shown in Figure 32, he said "actually those ones do fit on quite nice".

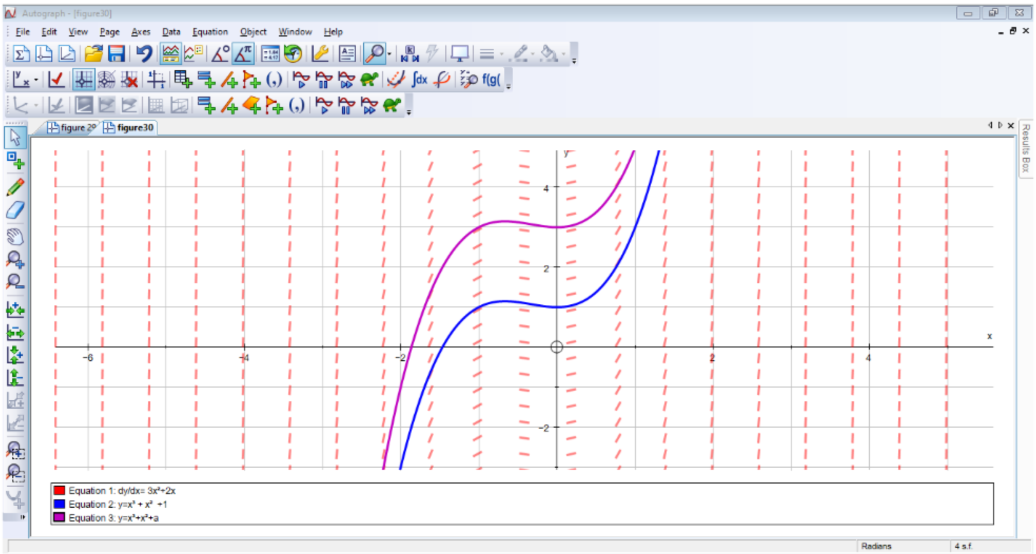


Figure 32: The graph of $y = x^3 + x^2 + a$ *on Autograph (reproduction of George's work)*

Then, student B suggested "Could you do animation between 5 and -5?". George did the animation and Student B commented: "There you go, that's what the computer is trying to say all along". George responded "Nicely, nicely. Thank you class, really happy we put that one in".

George ended the work on differential equations at that point and did one activity on a different topic (volume of revolution), until the end of the lesson.

4.1.4.1 *Analysis*

Using the Documentational Approach

The resources George used in this episode were: interactive white board; board; curriculum of year 13; textbooks; past examination questions and mark schemes; past teaching experience; students' contributions; calculators; notebooks; Autograph and pre-prepared graphs; formulae sheet; personal website; school website; and a worksheet.

George's scheme of work for lesson 9 is summarized in Table 12. His specific aims were: to introduce differential equations; and to use partial fractions in solving differential equations. His general aim: to prepare students for the examination. His rules of actions included general rules of actions: to answer students' questions; to start introducing chapter 11-differential equations and leave implicit differentiation and parametric equations to another lesson. Under the latter, there are specific rules of actions: to teach students how to find a general solution and then a particular solution using an example; to use partial fractions in solving differential equations; to use Autograph to show the students what they would see if they enter a differential equation on Autograph and use that to explain why they need to put the integration constant c. His operational invariants were all common with the operational invariants in previous lesson and were related to: connecting new ideas with concepts students learnt before; Autograph helps with visualisation; the textbook is a source of exercises and questions; the use of examination-style questions is in response to students' needs and to give them some practice for the examination.

Using the Knowledge Quartet

In this lesson, George's aims of teaching were: teaching differential equations and preparing students for the examination. In his quick introduction to differential equations, he used Autograph to demonstrate why the constant c should be included in the general solution. This showed his knowledge about the software affordances, and how he employed that to connect resources including the representations on Autograph, the algebraic solution, and students' contributions (in relation to steps to do on Autograph).

George used Autograph and the textbook as instructional materials (*use of instructional materials*), beside the other resources listed under the Documentational Approach analysis in the previous section. He demonstrated an example from the textbook and used Autograph to demonstrate the idea behind c the constant of integration. He also used the zooming in feature in Autograph to show the students that not all the line segments on the screen were vertical, as they initially saw. There is no evidence whether the segments appearing to be all vertical was something George anticipated or not. So, this might be a contingent moment to which George responded with confidence, but it could be something he anticipated and was ready for.

George was *making connections between procedures* and *making connections between concepts* (partial fractions, integration, and logarithms). He also connected between different resources including students' contributions, representation on Autograph and textbook questions. However, the connection he tried to create when he was solving question 2(b) on page 233 on the board, confused some students. He thought that the confusion was because the equation he initially inserted on Autograph was not the same as the one in question 2(b). So, he decided to solve the question 2 (i) which was about the equation in Figure 26 (i.e. equation $x(x-1)\frac{dy}{dx} = y$) in order to resolve the confusion. However, the students' question was about the directions of the line segments, which were going sometimes "up" and sometimes "down". In his response to the students' question about this, although George tried

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A1. To check students' work on iteration. A2. To introduce differential equations. A3. To use partial fractions in solving differential equations *General:* A4. To prepare students for exam	*Specific:* R1. Answer students' questions on iteration R2. Show the students the answer for question 3 p.237 and explain that it is a "meatier" question R3. Ask students to solve questions 2 (a and f) and 3 (a and c) from page 226 R4. Teach students how to find a general solution and then a particular solution of differential equations using an example R5. Use partial fractions in solving differential equations R6. Use Autograph to show the students what they would see if they enter a differential equation on Autograph and use that to explain why they need to put the integration constant c R7. Show students volume of revolution document on Autograph, using the sine function transformed $y = d\ sin(bx + a) + c$ *General:* R8. Answer students' questions R9. Start introducing chapter 11/differential equations and leave implicit differentiation and parametric equations to another lesson	*General:* O1. It is useful to connect new ideas with concepts students learnt before O2. Autograph helps with visualisation O3. The textbook is a source of exercises and questions O4. The use of exam-style questions is in response to students' needs, and to give them some practice for the exam	

Table 12: George's scheme of work during the ninth lesson on differential equations

to connect resources (i.e. the representation on Autograph and the particular solution he found), he did not connect the line segments on Autograph with the concept of slope and/or tangent. So, the students' reached the conclusion that those segments were not "accurate"; they "worked better for a cubic function"; and they were related to "reciprocal" or "the negative reciprocal". George used a cubic function with a slider on Autograph, in response to the students' request, and he was happy when a student concluded that this is what "the computer is trying to say all along". This was a contingent moment in which the teacher responded to the students' question (responding to students' ideas), but his response was limited as he could have created more connections with concepts the students already learnt about, to better clarify the display on the software. George did say that he did not know why the segments were going in different directions, so it could be that he did not know about the software features to make the latter connection.

4.1.5 Episode 6- from lesson 8 on Iteration

4.1.5.1 *Introduction to "Numerical methods"*

In his introduction to the chapter of "Numerical Methods" and its four methods (Change of Sign method, Iterative method, Mid-ordinate rule, and Simpson's rule) George justified its importance as follows:

> The whole chapter is about things that you can't solve by doing the algebra to them, or just by integrating them, okay. And, it turns out that there are way more functions and things that you can't solve algebraically than there are ones that you can, ok. There are loads of horrible things to integrate that are too difficult to integrate, that it's just easier to do a numerical thing and kind of approximately, a bit like a trapezium rule.

George reminded the students that they saw the trapezium rule and Simpson's rule in a previous lesson. He explained that Simpson's rule and the mid-ordinate rule were about finding the area and extension of the trapezium rule, while the change of sign method and the Iterative method were about solving equations. Then, he briefly introduced the four methods, and mentioned that he was going to focus on the "Iterative method" and when these methods are most useful.

4.1.5.2 *Rearranging an equation – Why $\boldsymbol{x^2+4x+1=0}$?*

After the introduction, George started explaining the iterative method, he mentioned that he needed to use an equation that cannot be factorised but can be solved using the quadratic formula. He spontaneously chose the quadratic equation $x^2+4x+1=0$, and asked his students to suggest its rearrangements of the form $x =\cdots$ (e.g. $x = \frac{-x^2}{4} - \frac{1}{4}$). George was commenting on the students' suggestions of rearrangement (i.e. when two rearrangements were the same and just written differently and when a student completed the square instead of rearranging the equation). He also asked them to find the equation roots using their previous knowledge on how to solve quadratic equations or using calculators. Students used calculators and gave the approximate answers -0.267 and -3.732.

4.1.5.3 *The use of Autograph*

George's next step was to use Autograph in order to further explain about the Iterative method and show how the different rearrangements would work. He used Autograph to graph the functions $y = x$ and $y =$ the other side of the rearrangement (e.g. for $x = \frac{-x^2}{4} - \frac{1}{4}$ he graphed $y = x$ and $y = \frac{-x^2}{4} - \frac{1}{4}$). The first rearrangement he graphed on Autograph was $x = \sqrt{-4x - 1}$ and it did not "work", so George said while pointing at the graphs (Figure 33) on Autograph:

> "Now, I can see that the blue one [$x = \sqrt{-4x - 1}$] does not quite, I think it almost touches, but not quite touches the red one [$y = x$], okay. So, they in fact don't cross. So, for that particular rearrangement sadly it's not going to work out. So, let's not do that one, let's pick a different one. Which one do you want?"

Then, one student picked $x = \frac{-1}{x+4}$, so George graphed $y = x$ and $y = \frac{-1}{x+4}$ on Autograph (Figure 34) and commented that the two graphs crossed this time at two points. Having noted that the two points are close to $x = -0.2$ and $x = -3$, George explained that the Iterative method is based on replacing the x in the denominator of this rearrangement by x_n and replacing the isolated x by x_{n+1}. Thus, he suggested the formula $x_{n+1} = \frac{-1}{x_n+4}$ to be used to complete the table in Figure 35 with a starting value $x_0 = 1$.

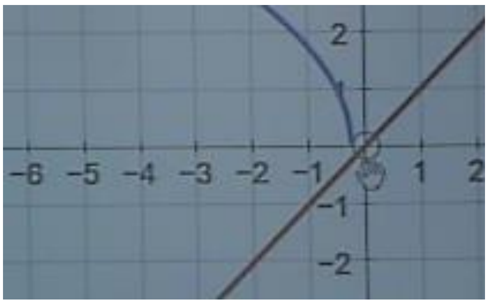


Figure 33: $y = x$ and $y = \sqrt{-4x - 1}$ on Autograph

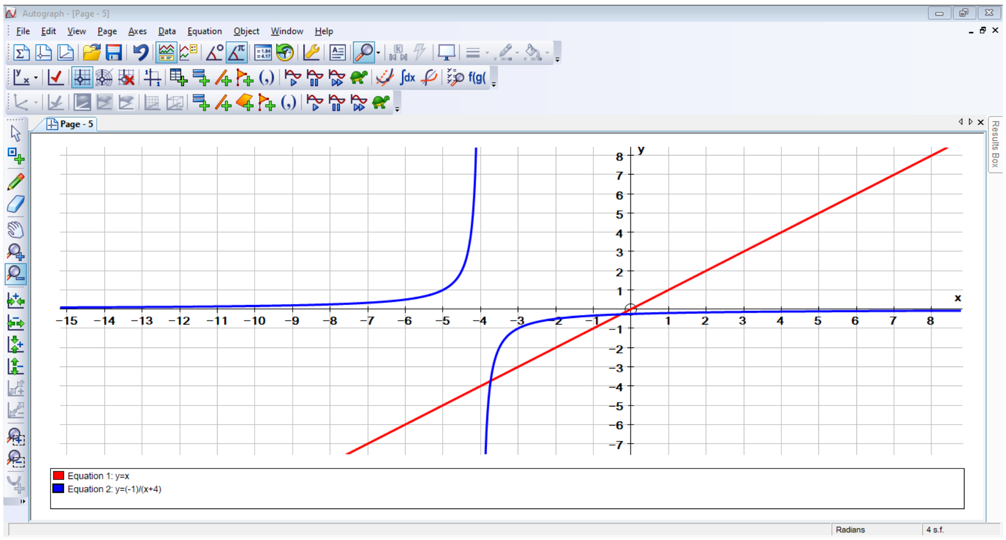


Figure 34: $\boldsymbol{y = x}$ *and* $\boldsymbol{y = \frac{-1}{x+4}}$ *on Autograph (reproduction of George's work)*

x_n	x_{n+1}
1	$-\frac{1}{5} = -0.2$
$-\frac{1}{5}$	$-\frac{5}{19} = -0.26315$
$-\frac{5}{19}$	$-\frac{19}{71} = -0.2676$
$-\frac{19}{71}$	-0.26769

Figure 35: Table completed using $\boldsymbol{x_{n+1} = \frac{-1}{x_n+4}}$ *(reproduction of George's work)*

After showing the Iterative method procedure on the board, George decided to show how the Iterative method works on the graph by using Autograph (Figure 36):

> "Look at this, you started with number 1, okay. The number 1 went, which way did it go? It's gone down to the curve, across to the line and that was our 0.2 [sic]. And then, here look at that down to the curve across the line, down to the curve if I zoom in further, oh missed it! Down to the curve and across to the line. Zooming in further, look at that! And we could keep on going down to the curve, across to the line zoom in, zoom in, zoom in, there it is, look at that, and the thing if I zoom out again looks like a staircase, kind of. So, this is staircase diagram".

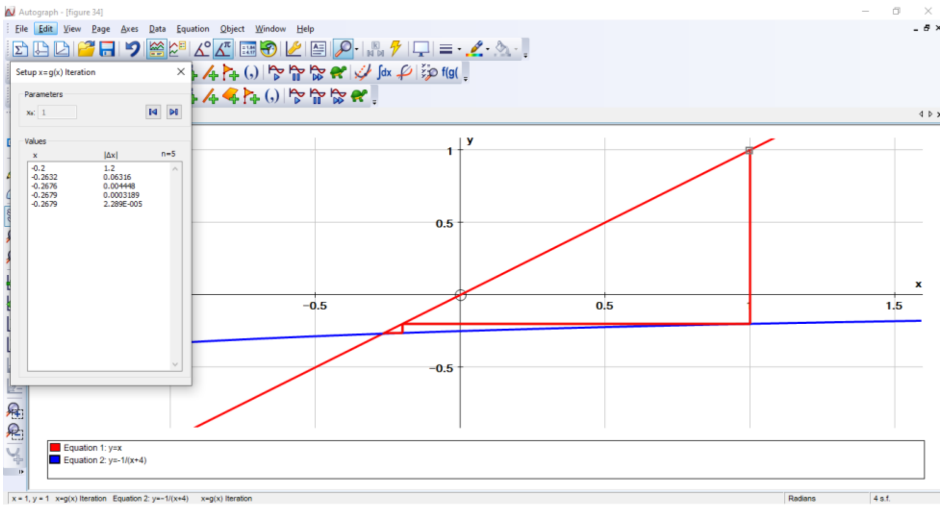


Figure 36: Autograph's Staircase diagram for $\boldsymbol{x = \frac{-1}{x+4}}$ (reproduction of George's work)

George then used $x_{n+1} = \frac{-1}{x_n+4}$ with different starting points for x_n on Autograph. When he used $x_0 = -10$ as a starting point, the software returned "overflow. George zoomed out; tried again $x_0 = -10$ and said "it worked this time" without commenting on the two different outcomes.

One student asked what would happen if $x_n + 4 = 0$ in $x_{n+1} = \frac{-1}{x_n+4}$. At that point George used the graph on Autograph to show that this would not work, as he commented:

> "You know like when you know your tan graph at 90 degrees, that 90 degrees on a tan graph is effectively saying we've got an infinitely tall triangle which wouldn't be a triangle at all, because it would be at 90 degrees. You're talking two parallel lines that's why tan of 90 doesn't work".

4.1.5.4 *Analysis*

Using the Documentational Approach

George's resources included: interactive white board, board, curriculum of year 13, textbooks, past examination questions and mark schemes, past teaching-experience, students' previous knowledge, students' contributions, calculators, notebooks, Autograph, handout, personal website, and the school website.

His scheme of work for lesson 8 is summarized in Table 13. His aims included the specific aims: to introduce the numerical methods chapter; to give a brief introduction to three of its four subsections (Simpson rule, change of sign method and mid-ordinate rule) and give detailed explanation of the fourth method (iterative method). Like previous lessons, his general aim was to prepare students for examination. His general rules of action were based on solving a range of examples for the students; asking for their contributions; giving them the time to practice independently and ask questions; making connections with what they already learned; showing some examination-style questions; and using graphs and Autograph to demonstrate how things work. The operational invariants were identified in his teaching approach (through the observation) and his reflection in the interview. For example, during the interview George reflected on his spontaneous choice of a range of examples and their *demonstration* in Autograph in the lesson:

> "So, it's not that I've chosen a specific and special example. So, I wanted to do it kind of live, but also then this thing about sometimes the live ones if I haven't tried them out before sometimes it doesn't work out. But, sometimes it's good to show that anyway. So, they see that different rearrangements work, different rearrangements don't work. […] I think it's worth showing the ones that don't work out anyway, so they understand that because otherwise they are going to assume that oh it always works out we can always get everything"

He suggested that *solving different examples would help show students that some rearrangements work, and some do not*. The *live demonstration* of a *range* of examples would appear to be a pattern of George's schemes of use.

Using the Knowledge Quartet

In his teaching about "numerical methods", George showed *awareness of purpose* when he stated the lesson topic, what he was going to focus on in that lesson "iterative method" and explained when this lesson's methods are useful to use. He was also trying to address *connections* (e.g. numerical methods and trapezium rule). And throughout the lesson, he frequently tried *making connections between concepts/procedures*. For example, he connected Simpson's rule which he used in previous lessons on Autograph to the methods he was teaching this lesson. He also said that the Simpson's rule and mid-ordinate point rule were about finding the area and extension of the trapezium rule, while the change of sign method and the iterative method were about solving equations.

In his *demonstration* about the iterative method, George was spontaneous in his *choice of examples* (the equation $x^2 + 4x + 1 = 0$ in this case). This spontaneity was visible during the lesson observation and was confirmed by George during the interview. Within the lesson, he created an equation that could not be factorized but can be solved using the quadratic formula. This choice was based on *awareness purpose* which is to show the students that the iterative method works and gives them an approximation of the equation's solutions. Then, he asked

Scheme		
Aims	***Rules of action***	***Operational invariants***
Specific: A1. To check and set up homework A2. To introduce numerical methods chapter and its four sections and explain the iterative method. *General:* A3. To prepare students for the exams	*Specific:* R1. Explain when the methods in this chapter are useful to use R2. Connect Simpson's rule and mid-ordinate rule with trapezium rule R3. Explain the iterative method in detail R4. Choose an example of quadratic equation that cannot be factorized R5. Solve examples on how to rearrange equations to the form x=… R6. Use equation roots after rearranging the equation R7. Use the equation roots to show the iterative method works R8. Ask students to pick rearrangements and starting point R9. After rearranging the equation, use graphs to show where the equations cross R10. Show students how to make and complete a table for x_n and x_{n+1} R11. Explain about the decimal places in the answer R12. Choose equations spontaneously during the lesson R13. Use Autograph with different re-arrangements and starting R14. Try to zoom in on each of the two roots of the quadratic equation R15. Use Autograph to show the graphs, their staircases and tables R16. Explain how to complete the table using calculators R17. Choose exercises from the textbook, and give a handout with graphs to students *General:* R18. Check if students did their homework and set up one for next week R19. Start introducing the "Numerical Methods" and its four subsections R20. Solve some examples for the students, give them tips, and show past-exam questions R21. Use students' contributions, answer questions and let them practice independently R22. Emphasise that students can and should use calculators	*Specific:* O1. The numerical methods are useful to solve equations that are difficult to solve algebraically O2. With a quadratic equation, roots can be found using the quadratic formula O3. Students should know how to rearrange equations O4. Equation roots can be used to show how the iterative method zooms in on the roots O5. The graphs help students see where two equations cross O6. Solving different examples helps show the students that some re-arrangements work and some do not O7. The use of Autograph with different re-arrangements helps show that different starting points may lead to different results O8. Autograph shows what happens in the background O9. It is important to explain the use of calculators for the iterative method, as this is what they use in exams O10. The use of exam-style questions is in response to students' needs, and to give them practice for the exam *General:* O11. Students should not be given too much homework in one week

Table 13: George's scheme during lesson 8 on iteration, no inferences were observed

his students to create different re-arrangements of the equation. This is a way to *making connections between procedures*. George was always *responding to students' ideas*, commenting on when two rearrangements were the same and just written differently and on when a student completed the square instead of rearranging the equation. After finding the different arrangements, George again tried to create a *connection* when he used the equation's solutions the students found on their calculators to show that the iterative methods would "zoom in on the answers". In this case, we also notice George's *awareness of purpose*. George was demonstrating about iteration to students while at the same time inviting them to contribute. These *students' contributions* here as *resources* that George used during the lesson, along with instructional materials. Afterwards, he asked his students to solve some textbook questions, this showed how he how he used the textbook (*use of instructional materials*) when it came to choosing questions for practice. Autograph was another *instructional material* that George used in order to further explain about the iterative method and show how the different rearrangements would work (*use of instructional materials*). He also used different representations (*choice of representation*) like tables and graphs; and tried *making connections between procedures and* making connections between different resources. However, the rearrangement $x = \sqrt{-4x - 1}$ that he claimed (based on Autograph) would not work, would have worked if he tried to do it using the formula and table in a similar way to what he did in Figure 35. The issue here is that Autograph (in Figure 33) only showed the positive part of the graph of $y = \sqrt{-4x - 1}$, hence the rearrangement seemed not to work. This issue can be attributed to both mathematical knowledge and knowledge of the software. After showing the iterative method procedure on the board George decided to show what happens if the iterative method was done on Autograph. This use of Autograph showed teacher's *knowledge about the software* affordances and its use. We also notice during the use of Autograph *concentration on procedures*. One *contingency* incident happened when George entered the equations from one arrangement on Autograph and noticed that the graphs of these equations did not cross, in this case he commented that some arrangements did not work and asked the students to pick a different arrangement (use of instructional materials/ use of resources). Another example of contingent moment (*responding to students' ideas*) was when a student asked what would happen if $x_n + 4 = 0$ in $x_{n+1} = \frac{-1}{x_n+4}$. At that point, George showed the graph on Autograph to show that it would not work. In response to this student, George used Autograph (*use of instructional materials*) and was *making connections between concepts* (dividing by zero, tan 90, parallel lines, asymptotes of the graph of the tangent line).

Here, a combination of spontaneous *choice of examples* and a choice of Autograph *(use of instructional materials)* is noticed. Then, a student's response was used as a *resource* and the teacher connected that to what Autograph could do for the same questions, by displaying both on the screen and using Autograph to identify any errors (*identifying errors*). During this process, Autograph gave "overflow" as an outcome, the teacher did not comment on that during the lesson but he zoomed out and tried again. George did not *use* this *opportunity* to explain what that meant during the lesson. But when asked about it in the post-observation interview, he said it indicates that it diverges at that point "instead of converging it will diverge". However, this was not the case during the lesson as Autograph then gave overflow because the point was outside the range on Autograph screen.

During the interview, George's comment on his spontaneous choice of equation reflected George's confidence in not following a specific agenda which can create more contingent moments during the lesson. His *awareness of purpose* was not very clear here, as with a different spontaneous example maybe a chosen arrangement would have worked to zoom to roots. Would he have in this case looked for an example that would not work? Also, George did not explain to the students the meaning of this getting "overflow" on Autograph. His focus was on getting a number and comparing that to one of the equation roots, which could be due to him focusing on procedure (*concentration on procedure*).

4.1.6 Episode 7- from lesson 2: In relation to volume of revolution

During the first lesson on volume of revolution, Student A asked about Figure 18:

> "The graph on the board, if you rotated that around $y = 0$ to get like a donut shape… you could do the whole thing and then minus the bottom. So, you could do rotation between 0 and 1 and then between 0 and 2 and take them away from each other, but will it work? I don't know…".

The teacher approved to start with by saying "you could… cylinder". Student A objected the cylinder idea because "the edge was slanted". At that point the following conversation took place:

George: You confused two types of questions, I think so. This is rotates vertically, whereas you're asking me about rotating around the x axis. Let me see if there is one like that

Student A: I don't think there will be because there is a gap in the middle

George: It is quite easy to do because you could do your volume of the whole thing take away the cylinder. And the cylinder would be quite easy because it's just what's the volume of the cylinder? $\pi r^2 h$. So, you're r here is constant throughout you literally just gonna work that out. Your h is whatever the length of the cylinder is. So, you do the big volume, the whole thing, take away the volume of the cylinder

Student B: He's saying you're wrong, Sir!

Student A: You're assuming it's a cylinder when it's not… it's slanted it's not exact…

George then went on the school website looking for more past examination question trying to find a question that has an idea similar to what Student A asked about, but he could not. He only found what he described as classic questions. So, he decided to try to use Autograph with the same graph he initially used to create the "packman" shape, but instead of rotating the graph around the x axis, he rotated around a line parallel to the x-axis and got the shape in Figure 37, which was not similar to the one Student A asked about.

Student A: This is bigger in the middle, I think it's just confusing

George: Yes, it is confusing

Then, George went back to the original rotation with the packman.

Student B: Delete it all, and start doing it again

George: I'm not sure if there is a way of making a hole in the middle. So, it doesn't work, don't even ask… Student A for your one, for that one

Student A: It's fine I don't need to know, it's just this… no one will get a question like that

George: You won't get a question like that, because you need to know that, end of it… What we're gonna do now, find your y^2 or your x^2 in this case, plug it in and work it out… What I need from you guys can you promise not to forget this stuff between today and tomorrow, ok?

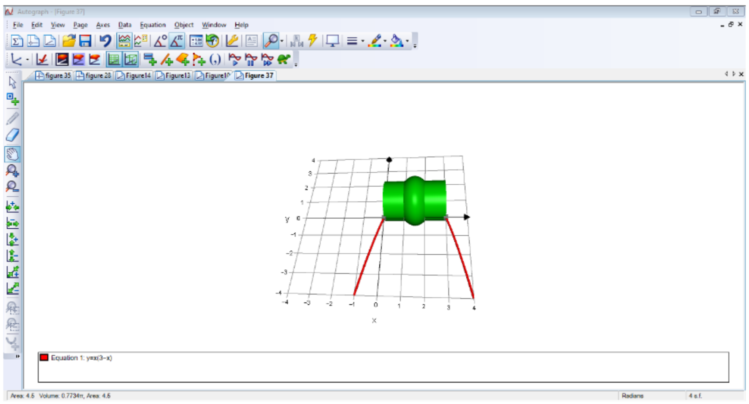


Figure 37: The shape George created on Autograph in an attempt to answer student A question (reproduction of George's work)

The teacher then asked if the students needed copies of the formula sheet and they said yes. The lesson finished at that point, the teacher did not comment of the figure the student was asking about. Student A was told that this does not come in the examination, so he did not insist on getting an answer.

4.1.6.1 *Analysis*

A view through the Documentational Approach lens is summarized in Table 9. It shows that this comes under the teacher's general rule of action: to answer students' questions. George tried to answer the question on the board, then on Autograph but finally he said that this question was not required for the examination. In terms of the Knowledge Quartet, this represents a contingent moment. George tried to respond to student A question and demonstrated his answer (*responding to students' ideas*); but student A spotted a mistake in it. George tried then to use Autograph (*use of instructional materials)* to clarify his demonstration; and allowed students' agency and contributions to help with that but the commands he used on Autograph did not give a shape similar to the one student A asked about. This led George to say that the question student A asked was not required for the examination. George used examination requirement as a way out of answering the question. This indicated how examination requirements influenced this teacher's knowledge and way of dealing with such a contingent moment. So, he used examination requirement to manage the moment and control the anxiety that this question can lead him or his students to feel.

4.1.7 Discussion and Summary

The resources that are common in George's teaching of the different episodes include paper based and electronic resources: interactive white board, board, curriculum of year 13, textbook, past examination questions and mark schemes, past teaching-experience, students' contributions (including students' previous knowledge), calculators, notebooks, and Autograph. School policy and textbook were also considered by George, and were part of his resources. In terms of the five points I addressed in the discussion of George' characteristic lesson, episodes 1 to 7 helped reflect further on them.

First, the use of the Documentational Approach proposed looking at students' contributions (including their previous knowledge) as resources on which the teacher acts and builds the lesson. The seven episodes showed how throughout his teaching, George gave an important role to students' contribution, and used these as resources (as we saw in the trigonometry lesson and episode 6 on iteration), and this was a characteristic of his teaching. George was inviting his students to contribute and based some of his *demonstration* on those contributions.

The contributions here were within a range of expected answers, for example in episode 6 George asked for all the possible rearrangements of a specific equation. To analyse this characteristic, the code "*use of instructional materials*" should be combined with "use of resources" to include students' contributions.

Second, the Knowledge Quartet analysis suggested paying attention to how the teacher created connections between resources (including students' contributions). The example that promoted this idea in the characteristic lesson was George's connection of students' estimate of the equation on a graph and the actual equation of graph on Autograph. His connection and comparison between the two helped him propose that the two were equivalent. More examples were addressed in the seven episodes. One is in episodes 2 to 4 on volume of revolution, when George "re-schemed" his lesson on the volume of revolution in episode 3 based on the experience he had teaching the same topic in episode 2. We saw then a connection created between his past teaching experience and how the students' reacted to it, Autograph and the textbook. Another example was addressed in episode 5, when George tried to create connection between the textbook question, Autograph and the students' contribution. Also, in episode 6, George used students' contributions (choice of rearrangement) and tried to connect that to Autograph and the iteration method. The connection between resources created in the characteristic lesson may seem more significant, because it involved reinforcing students' previous knowledge on trigonometry and connected that to a new concept and procedure. While, the example taken from episode 5 on differential equation was not well-considered and lead to a moment of *Contingency*. These examples strengthen my idea that we may need to add more codes to the Knowledge Quartet in order to reflect and address such key details in the context of George's teaching. This code was suggested in Kayali and Biza (in press) to be "connection between resources".

Third, and regarding the Knowledge Quartet, the analysis proposed paying attention to how the teacher dealt with examination requirements and school policy (including school policy about homework), or the institutional factors the teacher has to deal with. Regarding examination requirements, these were one of George's resources through his use of past-examination papers. Despite his *overt display of subject knowledge*, George in two episodes could not answer students' question and he justified that by saying that the two questions were not required in the examination. Thus, a code like "adhering to examination requirements" under the *Contingency* dimension would help analyse situations in which examination requirements are set as boundaries of what students need to know. This code is not to judge teachers' knowledge, but only to reflect how answering unexpected questions outside the examination requirements is dealt with in cases similar to the two we saw with George. Having said that, it still seems that teachers' knowledge about the curriculum guidelines and examination requirements comes to the front while they are teaching and shapes their subject knowledge. Hence, it might be worth adding a code in relation to the knowledge about examinations requirements and policies.

Fourth, also regarding the Knowledge Quartet, the analysis proposed considering the teacher's knowledge about mathematics and the software. This is an aspect that becomes clear when mathematics-education software is used in the class. It is about knowing the software affordances in relation to specific mathematical content. For example, in episode 6 about iteration zooming out was important to avoid getting "overflow" on the screen; if the starting point for iteration was not already within the display. Otherwise, the "overflow" in this case could be confused with the same outcome the "overflow" that implied non-convergence.

Fifth, another aspect in relation to the Knowledge Quartet is about contingent moments and how they can be identified. Identifying them is especially challenging in the cases when the teacher responds confidently to unexpected questions; like the case pointed to in the last paragraph of the analysis (about the student's questioning what to do when asked about tangents). Also, in terms of *Contingency*, George had prepared some resources to help avoid contingency like having trigonometry formulae typed up for reference; having pre-prepared graphs on Autograph; and having past-examination questions and textbook as references. In addition to that, George was confident to use spontaneous and live work in the class like in the episode about iteration and the live work on volume of revolution in episode 1, which led sometimes to contingent moments. It was hard to decide whether some moments in George's teaching were contingent or not. For example, in the differential equations lesson there was no evidence whether having one differential equation representation on Autograph appearing as vertical line segments on the screen was something George anticipated or not. He responded quickly to this appearance by zooming in to make the representation clearer. Thus, it is challenging to identify whether or not this was a contingent moment (i.e. *responding to students' ideas*) to which the teacher responded efficiently and with confidence, because the unexpected element in it was not clear or identifiable. Finally, contingent moments resulted in "rescheming" in the episodes about the volume of revolution, where the rescheming reflected the dynamics of teaching in George's teaching. Rescheming can be helpful to address changes in series of lessons on one topic.

In summary, the five themes addressed in the characteristic lesson were evident in the rest of episodes with George and we summarize them as:

1. Students' contributions are resources.
2. The code "connection between resources" will be added to the *Connection* dimension in the Knowledge Quartet.
3. The code "adhering to examination requirement" will be added under the *Contingency* dimension. The code "knowledge of examination requirement" will be added under the *Foundation* dimension.
4. The code "Over Software Knowledge" will be added under the *Foundation* dimension.
5. Two aspects in relation to contingent moments: they can lead to "rescheming"; and they can be unidentifiable to the researcher when the teacher responds with confidence to the moment's surprizing element being unidentifiable by the researcher.

These five points will be addressed later with analysis of lessons from other teachers in the next sections. For the three other teachers (Martin, Adam and Charlie); teaching episodes will be described using the Knowledge Quartet language and codes with the Documentational Approach analysis summarized afterwards.

4.2 Second teacher data - Martin

Martin was a mathematics teacher with one-year teaching experience, during which he taught students aged 16-18 years. He held a PGCE in secondary mathematics teaching. The school he worked for provided interactive whiteboards, iPads for students' use, and Autograph software (www.autograph-math.com).

Martin's lessons include six lessons, and were about integration, sets, volume of revolution, and mechanics. In two of these lessons the teacher used the mathematics-education software GeoGebra (https://www.geogebra.org/): the lesson about volume of revolution (lesson 4), and the one about mechanics (lesson 6). One of these lessons (lesson 4 on the volume of revolution) is included in the data for discussion below. The lesson was chosen for two reasons: first it covered a topic that was taught by George who worked with Martin at the same school; second the use of technology in the lesson on mechanics was based on the use of an applet found online and the teacher input in that was minimal. So, next is an overview of lesson 4 described using the Knowledge Quartet language and then analysed using the Documentational Approach, followed by a discussion.

4.2.1 Lesson 4: Volume of revolution

4.2.1.1 *Lesson overview*

In this lesson Martin was introducing volume of revolution to his year 13 students. There were sixteen students in this group. He started by showing a table that states the learning objectives of the lesson. He summarized his objectives in three items: "1. revolution around the x-axis, 2. revolution around the y-axis, 3. Integration: end of chapter questions. Find questions you can't do"; which reflected his *decision about sequencing*. He had also prepared a question (*choice of examples*) about volume of revolution and displayed it on the board, below the learning objectives table (Table 14).

In his work towards achieving his first objective, Martin chose to work on the equation of a circle $x^2 + y^2 = r^2$. Martin started explaining that in a circle $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$ and $y = \pm\sqrt{r^2 - x^2}$. He added that in this lesson students "can disregard negative solutions" and write $y = \sqrt{r^2 - x^2}$ (Figure 38), then substitute it in the formula for volume of revolution using the limits $-r$ and $+r$. After that demonstration, Martin gave a worksheet (*use of instructional materials*) to his students and asked them to work on finding V for the rotation of circle first around x-axis, and then around y-axis (Table 14). All students had individual mini whiteboards to work on (use of resources). Martin then advised his students to ask for student X help as he was familiar with the topic (use of resources). Student X did not belong to this year 13 group, and was there for this lesson in order to do more work on mathematics questions and ask for Martin's help if needed. Martin left the classroom to photocopy something as he said. The students were working on the worksheet questions on until the teacher came back.

Martin then started explaining to a group of students about volume of revolution: "Do you remember when we said whenever we were looking at integration, initially it was an area and it was little slices, remember that?". He added that in this lesson they were interested in the volume of the shape that results from rotating a function and they were going to use integration. One student from this group started writing on a mini whiteboard the formula for volume of revolution (Figure 39). So far, Martin's *demonstration* was mainly focused on showing the students how to rearrange the equation of a circle to the form $y = \sqrt{r^2 - x^2}$, considering only the positive solution of the square root, and using that in the formula for volume of revolution $V = \pi \int_a^b y^2 \, dx$ where $a = -r$ and $b = +r$. This reflected *concentration on procedures* at the beginning of the lesson. It showed how Martin got his students working on finding volume of revolution by using the *demonstration* he gave about rearranging the formula of a circle, the introduction to the formula of volume of revolution, the worksheet he prepared, and students' contributions (including students' previous knowledge about integration, and student X knowledge). This is a connection between resources.

You already know the equation of a circle is $x^2 + y^2 = r^2$	
Rotation about x	Rotation about y
$V = \pi \int_a^b y^2\,dx$ Hint: You'll need to rearrange the equation for a circle to make y the subject. You can disregard the negative solution. Make a sketch of your function.	$V = \pi \int_a^b x^2\,dy$ Hint: You'll need to rearrange the equation for a circle to make x the subject. You can disregard the negative solution. Make a sketch of your function.

Table 14: Martin learning objectives for the lesson on volume of revolution

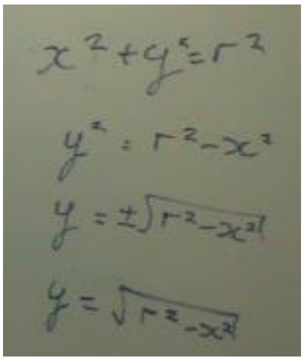


Figure 38: Martin's work on solving $x^2 + y^2 = r^2$ considering the positive solution only

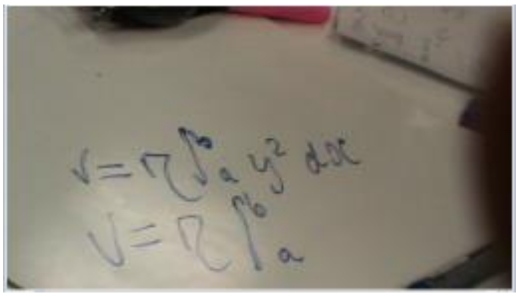


Figure 39: Martin's starting to demonstrate the application of the volume of revolution formula

Martin recommended sketching the shape. He then commented on another student's work from the same group: "Is this a circle though? Look at what [student Y] has drawn, so if I rotate that around the x-axis what do I get? It's going to make a sphere isn't it?". A student asked: "When you say round the x-axis are you assuming that the graph isn't flat?". Martin answered: "Yeah, yeah … so it's 3 dimensional, so what shape are you going to make?". The students now answered that it was going to make a sphere and Martin said that in this case the result of the integration is the "volume of a sphere". Martin tried to create a connection between area, integration and volume of revolution. This showed him *making connections between concepts and making connections between procedures*. He also tried to connect concepts (volume of revolution and volume of a sphere). His choice of *representation* included using different representations (algebraic and graphs), to show this connection.

Then, the teacher made sure that every student had the worksheet to work on. Students were working independently, and Martin was going around answering their questions and checking their work. Figure 40 shows some examples of students' work on their mini whiteboards.

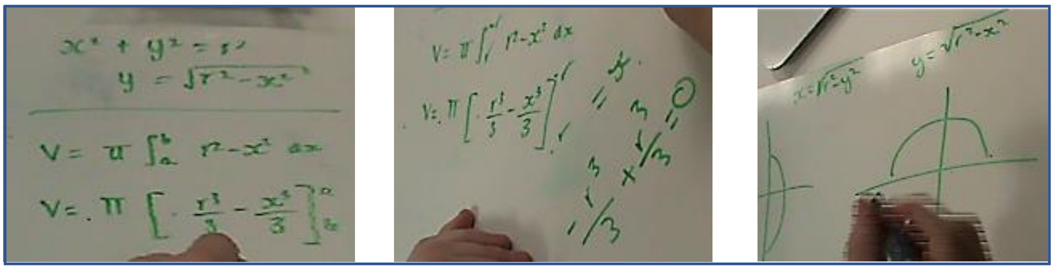


Figure 40: examples of students' work

After about four minutes, Martin asked:

> "Guys, do you want to see actually what you are doing? Would it be helpful if I gave you a big nudge as to exactly what you are doing? So, look at the board, you're going to have to turn around".

Martin was using GeoGebra (Figure 41-42):

> "So, I have drawn it in two dimensions for you on the left-hand side [Figure 41] and whatever way you actually did the working we found that it was $y = \sqrt{r^2 - x^2}$, yeah. What that is if we just take the positive solution is just a semi-circle [Figure 41], so we are just taking the positive solution. Obviously, the negative solution would make the bottom half of the circle, so we are just interested in the positive half. Now, what is going to happen if I rotate that about the x-axis? So, the red line here, is the x-axis and what you are going to see, or should see is this [Figure 42],

is what actually you are doing. You are taking that semi-circle and you are rotating it through three dimensions and it's going to make a sphere. And, it's actually the volume that's contained within that rotation that you are calculating. And that's what you and I were talking about a minute ago because what would you expect the volume of that to come out as? You would expect it to come out at $V = \frac{4}{3}\pi r^3$ which is a volume of a sphere. Are there any questions about that about what has actually happened there?"

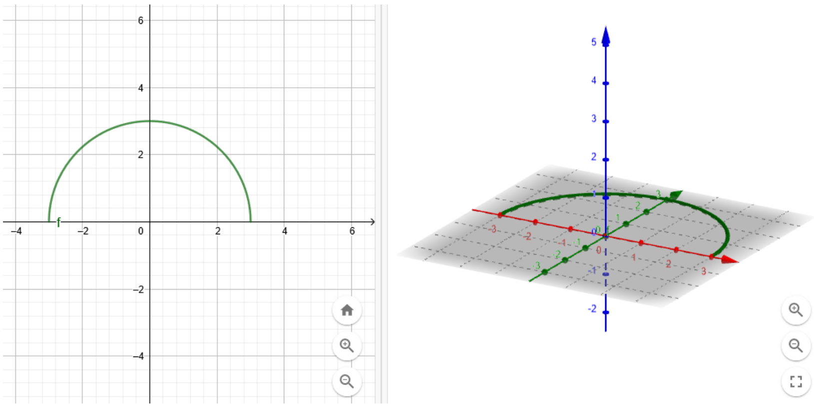


Figure 41: Martin's display of a semi-circle on GeoGebra (reproduced for clarity)

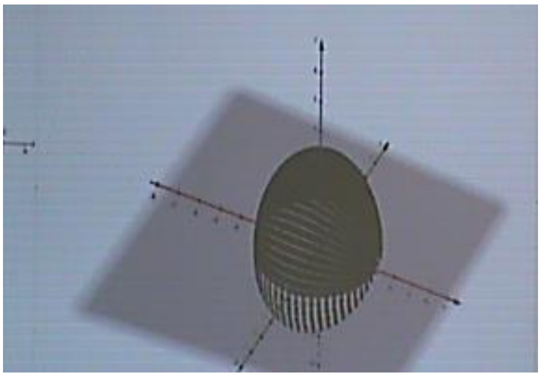


Figure 42: The rotation of a semi-circle on GeoGebra (reproduction of Martin's work)

Student N questioned: "Why have you just used the positive solution … because you're going to get the same volume aren't you?"

Martin responded:

"Well, you need to think about that circle is actually two functions. You have positive and negative because of the square root, don't you? So, it's actually two separate functions and if you do the negative one obviously you just do the same thing. So, we can disregard one of them because it's going to overlap the same. You can certainly do it with the negative one if you wanted to try, but you would get the same result, is that alright? Student Z are you happy with that? is it a bit clearer now? What you are doing? Student M does that make a bit more sense? Not really, okay. I'll come and sit with you, shall I? yeah."

Martin's response was that doing the negative solution would give the same answer, however he did not comment enough on rotating a full circle. From Martin's response to the student's idea, it seemed that he did not consider rotating the full circle (this choice is explored further in the post-observations interview). This could be because he was not ready for it (subject knowledge) or he thought the student was asking about the negative solution only. The activity on GeoGebra created some confusion, Student M asked: "I get the rotation bit, but I don't really know what that has to do with this", referring to the formula of volume of revolution.

Martin replied:

"You don't know what that has to do with that? So, you've used the formula first semi-circle, haven't you? And you've plugged it into that integral between a and b. And we said that the r goes between $-r$ and $+r$, because if you think about it if you look at that circle if that's $+r$ then this is $-r$ isn't it? So, y varies between $-r$ and $+r$. That's the integral and then by integrating and multiplying by π you actually create a sphere"

Student M: Okay

Martin: So, actually you are creating an algebra

Student M: That's fine.

Martin's response about the connection between the graph and shape on GeoGebra and the formula of volume of revolution, was based on explaining the steps of using the formula. Although Martin tried *responding to students' ideas* here, his response did not explain the connection the student questioned. Instead, it was focused on the formula. Martin here did not use the opportunity to create the connection. He only commented that by doing the steps in the formula they were "creating an algebra", this showed his *concentration on procedures*.

Students now went back to working independently, and Martin went around to answer questions about the formula. He also told the students that what they were doing is harder than what is expected in examinations. At some point, Martin came to talk to me about the lesson, he said:

> "This is quite a bit harder than the one they see in the exam…. They will be given for instance, I'm making them work with algebraic limits which stretches them straight away. They will be given the numerical limits, they'll be given a simpler function, they wouldn't be expected to take that and make it themselves".

Martin's *choice of examples* harder than examination questions reflected his consideration of examination policy and requirements. And, when I asked him about his reason for choosing this specific example he said: "Because it's harder if they can do this they understand, and it's easier for me to do on GeoGebra. I have a two-dimensional trapezium". This reflected his knowledge of GeoGebra, as he chose a function he could easily do on the software he is familiar with.

After some time, Martin asked his students again how they were doing. Student A replied: "I don't think too good". Martin walked towards student A and started explaining to him and his group:

> "You don't think too good, so why are you sat there doing nothing then? Okay, so which one are you working on? What one are we working on x-axis? Okay, so I did tell you that your limits of integration are -r and +r the thing you're bumping into is that you're integrating r, what should you integrate with respect to?

Student A: should it be just r^2x

Martin: should it? because if you were integrating just say 3 it would just be $3x$, wouldn't it? r is a constant so r^2x …

Student A: Oh, because you times it by x

Martin: That's what you've bumped into. So, it's not actually this, it's your integration skills letting the side down

Martin went to talk to another group now. When looking at some students work from another group, Martin noticed that some had the same difficulty with integration that student A had. So, he said to them:

> "If we were integrating say $3dx$ what would you get? $3x$, yeah, exactly it's a constant not a variable that is why this is hard… Think about it we have a circle we have to find our circle $x^2+y^2=r^2$ that's fixed isn't it? r is a constant it's not a variable. r doesn't change if I change x and y is that okay? are you sure?"

Student B: So, if we integrated this twice

Martin: No, no you haven't integrated it twice all he's done there is ... no, no you don't integrate with respect to r just integrate with respect to x … no, no this is very hard and not what you're going to be expected to do in the exam. That's why I am doing it, this is properly hard

When students had difficulty integrating $r\,dx$, Martin's response showed how he was *responding to students' ideas* and *identifying errors* by: trying to compare; *making connections between concepts and making connections between procedures*. However, Martin's comment about this question not being expected in the "exam" reflected how he tried to relieve himself from the pressure of this moment by reducing the significance of this question, using examination requirement as a reference.

After about 20 minutes of students work in groups, Martin said:

> "Not just today we are going to do it over a couple of lessons you just need to be really familiar with this kind of problems. Guys when you are ready to start some actual problems page 110 in the core 3 core 4 book".

Students were asked to move to the textbook questions now. In the meantime, Martin was showing the trapezium rotation on GeoGebra to one student who was ready for it, but on the main board. It seemed that students' questions and progresses were treated by Martin as resources that he used to help each student or group of students' progress and move to the next step.

Some students were then saying that no one from their group understood the idea. Martin was now working with them and explaining the rotation of semicircle. While GeoGebra was playing the rotation of trapezium, Martin continued working with the groups until the end of the lesson. In a comment on his work on GeoGebra, Martin told me after the lesson: "So, that's the way I use GeoGebra. I don't really do anything live with them on it, it's really just to run in the background and support the idea". By doing this, Martin did not fully exploit the dynamicity of

GeoGebra. He connected between resources, but in a procedural way. By doing so he is connecting resources (prepared task & student questions) and this influences the flow of his lesson.

In this lesson, the teacher made *decisions about sequencing* (worksheet then textbook questions) and *anticipation of complexity* (reassuring students that they would work on this for 2-3 lessons). Also, the rotation of trapezium on the board (*use of instructional materials*/ use of resources), was not showed to the whole class but only to one student who seemed ready for it (students' contributions as resources). Students so far seemed to have questioned the connection between the display on GeoGebra, the choice of semicircle, and the integration of rdx.

4.2.2 Post observation interview

In the post-observations interview, I asked Martin about the reasons behind his choice of semi-circle and trapezium for this lesson, he said:

> "So, the, I think part of it was I didn't want to choose anything too weird and wonderful. I could have chosen any shape really, but I had to choose something that was familiar to the students, something they could work out the area of and then do the rotation themselves. And, for instance choosing the trapezium was because they didn't necessarily need to be able to integrate to get into that task, they didn't know they could work the area of the trapezium. And, I think sometimes I can't assume that they can, that their skills are yes I can integrate that straight away. It's got to be that they have other access points into the lesson because otherwise you're going to end up with somebody sitting doing nothing which is a real danger".

His *choice of examples* was based on this example being familiar to students, more difficult than examination question so if the students solve it they would be able to solve examination questions.

When asked about choosing a semi-circle rather than a circle, Martin said it was "because then you would just get the full circle out again, you couldn't actually see the animation". When asked about the way he used pre-prepared graphs on GeoGebra, Martin described his method as follows:

> “I give them the task, and then without giving them any instruction will start the animation running. So, if they are looking for support or they’re looking for, it’s there for them. But I don’t use it to teach it; because it reinforces the concept, but it isn’t the whole concept and I don’t want them restricting their thinking to just something moving on a board. It doesn’t work like that in reality”

He also added that he would not do the functions live in the class “because I guess what if it doesn’t work; I don’t have time in a lesson to sit and fiddle with GeoGebra and make it work […]. If I was more proficient probably I could do it live”.

In a question on whether the students connect the image they see on GeoGebra with the formula, Martin answered:

> “It’s more important that they are familiar with the formula and are able to express it using the formula and think about what is happening with the mathematics than be able to visualise something spinning in their head, because they are not going to have that when they are examined. Do you know what I mean? They need to be able to use that formula and that’s the crutch it’s a lovely way of demonstrating it, but it is a crutch it’s not what they need to know”.

Martin commented in the interview on his 40 second rule: if he could not explain something to the students in 40 seconds it means he did not understand it enough. This meant he relied on minimal teaching. He said:

> “So, whenever I was training as a teacher, something I struggled with was basically letting the students get on with it trying to cover every possibility and every permutation. And, then I was thinking about that and thinking I am working far harder than they are. So, I started trying to cut down the explanations. And, at A level trying to explain something in less than a minute or in around 40 seconds is really, really challenging it’s really, really, really hard to do. Yeah, it’s really hard to do but something like integration is really dull. If somebody is trying to go here and you do this one people switch off after about 40 seconds anyway. My concept of teaching at the moment is that all they need is the first step to get into something the rest they can just do”.

To minimise his talking time, Martin said:

> "What I try to do is, I try to design tasks that sort of link on into each other, so that they will when they are working I can just keep them working. If the whole class is struggling, then I will obviously stop them and do some teaching. But, that's the last resort. Yeah, it depends on how they are going but I try and design lessons so that they feed on. So, you've seen my power point slides even are one big continual everything links together".

It seemed that students' questions were treated by Martin as resources that he used to help each student or group of students' progress and move to the next step. This was justified by him saying that he avoided talking for long and preferred to give tasks for the students to do. This meant that Martin would give his students a brief introduction and a task to engage with and complete independently, and then explain ideas to them in groups or individually based on what questions they came up with. So, Martin particularly seemed to use students' questions (part of students' contributions) as resources.

On his use of the textbook, Martin said:

> "I only use the questions, I don't use the explanations from the textbook. I don't rely on the textbooks for learning very much […]. Because there is so much that is incorrect, like they mixed up tangent and normal and things like that. Textbooks are very unreliable".

According to Martin, his only online resource was the scheme which he described as "a list of topics and I just go on to teach that"

Martin said: "I teach quite quickly the pace of lessons that I teach moves quite quickly compared to some of the other teachers because I don't do the talking rubbish for half an hour". He also said that the homework was usually past-examination questions.

On his use of GeoGebra, I asked him to elaborate on why he would not use it "live" and he said:

> "Because I do something very similar for it. So, I give them the task and then without giving them any instruction will start the animation running so if they are looking for support or they're looking for it's there for them but I don't use it to teach it because it reinforces the concept but it isn't the whole concept and I don't want them restricting their thinking to just something moving on a board, it doesn't work like that in reality… It's more important that they are familiar with the formula and are able to express it using the formula and think about what is happening

with the mathematics than be able to visualise something spinning in their head because they are not going to have that when they are examined do you know what I mean they need to be able to use that formula and that's the crutch. It's a lovely way of demonstrating it but it is a crutch it's not what they need to know, does that make sense?"

This showed how Martin used GeoGebra as instructional material (*use of instructional materials*), but had at the same time examination questions in his thoughts. That lead him to *concentration on procedures*. His knowledge about the software was not enough for him to use it live in class, so he relied on graphs prepared and displayed in class to avoid contingent moments that might result from the software not working as he wishes.

On his reasons for using GeoGebra rather than other software like Autograph which was available at his school, he commented:

"The reason why I use GeoGebra is because it works well with Mac and the graphics and everything work really fluidly on the Mac. I've tried other things and they don't work quite as well. And, also GeoGebra is free. […]. I have Autograph, I just think it's very limited. I don't really like Autograph, GeoGebra is far more powerful and I can play around with the JavaScript and make it do whatever- exactly what I want it to do. Autograph doesn't have that facility".

He added that he should use GeoGebra more in the future:

"More, more way more […]. I can make them or get ones online and tweak them to what I want them to be or whatever. But as I can do that I'll get more and more and more and more. They are so time consuming to make".

In summary, Martin used textbooks to find questions for students to work on. He used the scheme of work provided by the school and past-examination questions for homework (provided on a website created by the head of mathematics/school policy and examination requirements). He relied on minimal talking by him, and structured tasks for students. While students were working on the tasks, he went around *identifying errors* and explaining more. His explanation was prompted by his students' questions, and it reflected his knowledge of the topic he was teaching and the software he used. His choice of an example familiar to the students (semi-circle) was based on a plan of how students can proceed with the volume of revolution and how they can see the animation on GeoGebra. His *anticipation of complexity* lead him to devote more than one lesson to it. He made *decisions about sequencing* the tasks and moving from worksheet to textbook questions. He made *making connections between procedures* (area, volume, integration). And he connected between resources (worksheet & GeoGebra). Martin frequently responded to students' ideas and questions. His response to the student's question about why he rotated just the "positive" half a circle, considered rotating just the "negative" half but not the full circle. In the interview, Martin said he did not consider the full circle because he wanted to avoid overlapping and because rotating a semicircle is clearer and easier in GeoGebra.

The *choice of examples* (semicircle rotation as a difficult one) created confusion between students regarding the integration and whether r is a variable. Going for a "difficult" question to start teaching the concept reflected inexperience in students' needs. It is an unexperienced approach that created complications during the lesson. The choice of this approach can be due to unfamiliarity with students' needs and lack of experience, or it could be due to the teacher's attempt to save time by going for a difficult question straight away. However, this choice reflected that there was no *recognition of conceptual appropriateness*. Moreover, students could not see the purpose of the formula and how it was deduced to find the volume. The teacher's explanation focused on the procedural level (*concentration on procedures*).

The way the teacher used the formula of the volume of a sphere to show that the formula for the volume of revolution works, showed how he used these *making connections between procedures* without giving attention to the concepts behind these. Even when students asked about the connection between the formula and the rotation on GeoGebra, the teacher's response showed *concentration on the procedures* and formula and not on the concepts. The teacher did not consider explaining how the formula can be deduced either because he did not know or because he did not think it was important part of the lesson. His explicit *concentration on procedures* was supported by use of GeoGebra for *demonstration* only without showing how the displayed graphs and figures could be constructed.

The teacher recommended sketching the graph at some point, but that did not seem something he integrated within the lesson. He used it in a way similar to the way he used GeoGebra, i.e. to show that the rotation of a semicircle is a sphere and that using the formula for volume of revolution will produce the formula for the volume of a sphere.

The connection he tried to create by mentioning that integration is used to find the volume, as it is used to find the area did not go into the details of the concepts and was limited to procedures (*making connections between procedures)*. Thus, we see prioritization of procedures over substance and concepts.

Martin was trying to avoid *Contingency* when it came to the use of technology, by using ready-made shapes. This use of ready resources contradicted his claim of using his own resources. It seems he was searching and choosing resources, but he did not design or create ones. This could be due to time management, as he commented in the interview.

4.2.2.1 *Analysis*

Using the Documentational Approach

Resources included: interactive white board, board, mini whiteboards, curriculum of year 13, textbooks, past-examination questions and mark schemes, past teaching-experience, students' contributions, calculators, notebooks, GeoGebra, worksheet, and the school website.

Martin's scheme of work for this lesson is summarized in Table 15. His specific aim was to introduce the volume of revolution, and his general aims were to prepare students for the examination and to minimise the teaching and get students to practice. His first general rules of actions included giving enough information to students in order to let them work independently. This included specific rules of actions: briefly introducing the lesson objectives and showing the formulae for volume of revolution; starting with a challenging question; reminding the students of the equation of a circle, showing the students how to rearrange the equation of a circle to the form y=… considering the positive solution only, asking students to use that to find the volume of revolution around the x-axis using the formula given in the table, connecting volume of revolution with students' previous knowledge about integration and area; connecting the result of the integration to students' previous knowledge about the volume of a sphere.

His second general rule of action was to use GeoGebra for demonstration, including: using GeoGebra to show the students a semicircle prepared and drawn in 3D; rotating a semicircle around the x-axis; showing another example pre-prepared using the rotation of a trapezium. His third general rule of action was to let students work independently, including: students can ask student X for help; students work independently on the worksheet, students solve the questions on the worksheet; students solve the handout and textbook questions. His fourth general rule of action was to answer students' questions. His fifth general rule of action was not to use the interactive function of the whiteboard.

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A4. To introduce the volume of revolution *General:* A5. To prepare students for exam A6. To minimise the teaching and get students to practice.	*General:* R1. Give enough information to students and then let them work R2. Use GeoGebra for demonstration R3. Enable students to work independently on the worksheet R4. Answer students' questions R5. Avoid using the interactive function of the whiteboard *Specific:* R6. Introduce the lesson objectives showing the formulae for volume of revolution R7. Remind the students of the equation of a circle R8. Show the students how to rearrange the equation of a circle to the form y=… considering the positive solution only R9. Ask students to use that to find the volume of revolution around the x-axis using the formula given in the table R10. Students can ask student X for help R11. Connect volume of revolution with students' previous knowledge about integration and area R12. Use GeoGebra to show the students a semicircle in 3D R13. Rotate a semicircle around the x axis R14. Connect result of integration to students' previous knowledge about sphere R15. Start with a challenging question R16. Show another example prepared (rotation of a trapezium) R17. Students solve the questions on the worksheet R18. Students solve the handout and textbook questions R19. Answer students' questions	*General:* O12. Longer talk might lead to students being distracted O13. Using a familiar graph helps reinforcing previous knowledge and makes new ideas more accessible to students O14. Using graphs drawn before helps save time O15. Using GeoGebra in the background is a crutch that supports the idea but does not teach the concept O16. GeoGebra works well with graphics and MAC, it is powerful while Autograph is limited O17. GeoGebra not used live because of time limitation and risk of software not working O18. If students do a challenging question they can then do exam questions which are easier O19. Textbook is good sources of exercises, but not for explanations as textbooks are unreliable O20. Interactive whiteboards never work and they do not add to the lesson *Specific:* O21. Using a semicircle offers a harder question than exam questions because it has algebraic limits, and it is easier to do and see on GeoGebra	

Table 15: Martin's scheme of work for his lesson on volume of revolution

In terms of his operational invariants, Martin thought longer talk might lead to students being distracted. His choice of semicircle was based on offering a harder question than examination questions because of its algebraic limits, but easier to do and see on GeoGebra. Also, he thought using a familiar graph helps reinforcing previous knowledge and makes new ideas more accessible to students. By using pre-prepared graphs Martin was aiming to save time and avoid the risk of software not working. He thought that using GeoGebra in the background is a crutch that supports the idea but does not teach the concept. His choice of GeoGebra was because it worked well with graphics and MAC, it was powerful while Autograph was limited.

Martin's view on GeoGebra seemed to be based on professional development training he received. Martin thought textbook is good sources of exercises, but not for explanations as textbooks are unreliable. Martin thought interactive whiteboards never worked and they did not add to the lesson. No inferences were noticed in this lesson. However, some of the operational invariants seem to be based on inferences from previous experience like spending less time talking and demonstrating; what works with MAC; and views about potential problems with GeoGebra.

4.2.3 Discussion and Summary

The resources that are frequently noticed in Martin's teaching include paper-based and electronic resources: interactive white board, board, mini whiteboards, curriculum, textbooks, past-examination questions and mark schemes, past teaching-experience, students' contributions, calculators, notebooks, GeoGebra, and the school website. The five points addressed in the discussion about George's teaching also came up in the analysis from Martin as explained below.

First, throughout his teaching, Martin gave an important role to students' contribution, and used these as resources. His scheme was influenced by these resources. So, he relied, as he explained, on minimal *demonstration* and talking by the teacher followed by students working individually or on groups to solve exercises. While students were working, he would go around the class to answer questions and he relied on students' question to find out what he needed to explain further. These were expected questions (unlike those that come under *Contingency*), for example asking about how to integrate a function given by the teacher. These questions were Martin's resource to decide when to answer a question to the individual student who asked it (e.g. when the rest of the class seem to know the answer); and when to demonstrate it on the main board to the whole class (e.g. when the same question is asked by more than one student). This is a special characteristic of Martin's teaching that involved scheming on the spot, it showed the way he used students' contributions as resources in a way different from

George’s way. This in turn supported replacing the code “*use of instructional materials*” by “use of resources” to include students’ contributions (including students’ questions and students’ previous knowledge).

Second, is how Martin created connections between resources. We show these connections in Martin’s teaching less, and when they happened they were sometimes procedural or unsuccessful (like his attempt to connect the rotation on GeoGebra with the formula of volume of revolution). When comparing this with George’s connections between resources, we notice that George’s were deeper, meaningful and reflected the teacher’s confidence. Thus, we can conjecture that the deeper the connections between resources are, the more confidence the teacher have in balancing his resources and using them for *demonstration* that goes beyond emphasize on the procedures.

Third, in terms of textbook use, Martin adopted this persona, in other words, he wanted to create his own resources because according to him the textbooks are not always accurate and included errors. But, in practice he did not reflect that entirely. The textbook in addition to the scheme of work, past-examination questions and school website are resources related to the school policy and examination requirements that Martin used in his lesson. Regarding examination requirements, these were one of Martin’s resources through his use of past-examination papers and his reference to examination requirements during the lessons. The code “adhering to examination requirements” under the *Contingency* dimension is also relevant in Martin’s case, to help analyse situations in which examination requirements are set as boundaries of what students need to know (like the unclear connection between the animation on GeoGebra and the formula of volume of revolution). Like with George’s case, this code would help reflect how answering unexpected questions outside the examination requirements is dealt with (in Martin’s case by adhering to the procedure). This code is more obvious in Martin’s data due to his *concentration on procedures* that are part of examination requirements.

Fourth, in terms of GeoGebra, Martin’s view of the software as a powerful tool seemed to be influenced by professional development training that he received. His use of the software was pre-prepared and not dependent on students’ contribution, unlike what we saw with George. Martin justified this by time restrictions, which is a factor related to school and examination policies. He also explained it by his knowledge of the software (i.e. needing to know more about GeoGebra) and attempts to avoid *Contingency*.

Fifth, in terms of *Contingency*, Martin used ready-made resources (accessed online or from the textbook) to help avoid *Contingency* and had past-examination questions and textbook as references. In comparison with George, Martin was less confident to use spontaneous and live work in the class and felt that he needed more software knowledge to be able to do that. However, his use of ready-made resources led sometimes to contingency moments (like when he was questioned about the connection between the animation on GeoGebra and the formula for volume of revolution).

Finally, beside the differences addressed above between Martin's and George's practices, there were also differences addressed between the perspective emerging from the interview and what happened during the lessons. The latter differences include:

- Claiming that textbook is "rubbish" verses observing the textbook being always used in class as the source of exercises
- Claiming that GeoGebra is very good and powerful but using it in a procedural way due to time restrictions
- Claims related to creating own resources verses using ready-made resources, either accessed online or from the textbook

These differences show the potency of the analysis that looks at both aspects. The differences could be explained by arguing that although the teacher had good knowledge about mathematics and its teaching, he just needed more time to reify his resources and balance them. Or, they could be explained by the teacher *concentration on procedures* and examinations requirement and lacking the knowledge about the software. Such differences are not visible in George's case, which can probably be attributed to more experience that lead to better balance of resources.

4.3 Third teacher data - Adam

Adam was a mathematics teacher with four years' experience, during which he taught students aged 12-18 years. He held a PGCE in secondary mathematics teaching and was about to finish MA in education including a dissertation which focuses on mathematics education. The school he worked for had interactive whiteboards, a computer lab available for booking, and GeoGebra (www.geogebra.org) and Autograph software installed on all computers (www.autograph-math.com).

From a set of six lesson observations with Adam, three episodes will be discussed and analysed: two episodes from lessons 4 and 5 on integration; and one episode from the pre-observation lesson which was a revision lesson on simultaneous modulus equations. The rest of lessons will not be included because mathematics-education software was not used. The next section describes the three episodes using the Knowledge Quartet lens and then analyses them using the Documentational Approach; starting with the two episodes on integration.

4.3.1 Two episodes on Integration

The episodes are taken from two lessons to one group of sixteen Year 12 students. While the first episode addresses a full lesson in which integration was introduced to the students; the second is a part of a lesson in which Adam used Autograph to continue working on integration with the same group and do further demonstration on the constant of integration c.

4.3.1.1 *Episode 1- Lesson 4*

Starter activity

Adam started by asking his students to solve a starter question on dividing fractions (Figure 43) commenting:

> "You will definitely need to be able to do this, that's why it is on the board, so remember make sure you know how to divide fractions… in about three minutes we're going to go on, but you need to be able to divide fractions before we do anything today".

This reflected the connections he was aiming to make with students' previous knowledge. The students solved the starter individually, in pairs or in groups. Adam was going around checking students' work, then he showed the answers on the board to allow students to check if they got correct answers. So, he used the resources available (activity and board), to allow students to check their answers independently.

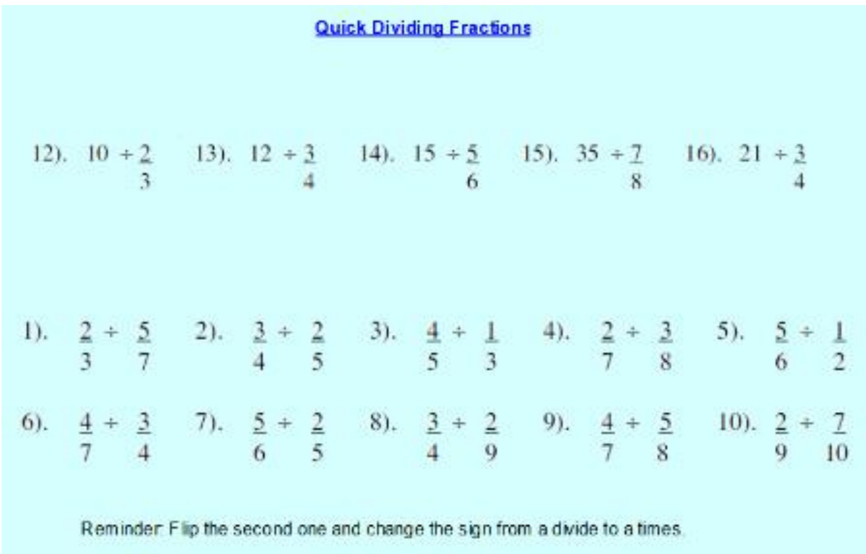


Figure 43: Starter question on dividing fractions

Introduction to integration

To start introducing integration, Adam used his students' contributions as resources. So, he asked for two students to volunteer to go outside the classroom. When two students left the room, Adam drew a graph on Autograph and asked his students to tell him the equation of that graph. A student replied the equation of the graph was $y = (x - 5)^2$. Adam commented the answer was right "because that translates it to the left 5 yeah along the…", in this answer he was *making connections between concepts* (equation of a graph, translation). Then, he asked if anyone wrote it differently, a student replied that it could be written $y = x^2 - 10x + 25$. So, Adam asked "then what is the gradient going to be?", and said it would be "$2x - 10$". By asking this question about the gradient Adam was *making connections between concepts* (gradient, differentiation) and *making connections between representations* (equation and graph). At that point, Adam hid the equation of the graph and the graph and invited the two students outside the class back into the room; and asked them about $2x - 10$: "We've differentiated something, can you tell me what it was we differentiated?". Adam asked the two students "talk about it out loud" and try answer this question, so he could use their contributions as resources. The two students were guessing ($x^2 - 5$, $x^2 - 5x$,…), and then decided it should be $x^2 - 10x + 25$, the teacher asked the students why it would be "how did you get the x^2 bit?".

Student: 'Cause, umm I don't know. I just know if you differentiate that [meaning x^2] and then you $-\ 1$ so it just becomes x to the power of 1, just x and then there is an invisible one here (in reference to the exponent) and times that by 2 that's $2x$

Adam: And how did you get the $10x$?

Student: Like you just – the 1 here, so it takes the power of 0, and then you times 1 by 10 so it's 10 and then that is if you just put it into brackets like that it will be

Adam: How did you get 25?

Student: I just did that in my head… that will be that

Adam commented that he understood why the students got x^2 and 10x, but he was not sure why they went for 25 as a constant:

> "I get what you've done here, but it doesn't have to be factorised, this does it. So, so far you've got this but and these two are linked, aren't they? But, you don't know what this is here [referring to the constant of integration], Do you?"

In this comment we see Adam *responding to students' ideas* in what could be a contingent moment; demonstrating to students that the constant of integration is unknown at this point. The students seemed to agree with Adam's point. In response, one student suggested: "So, is that just $+n$". Adam responded:

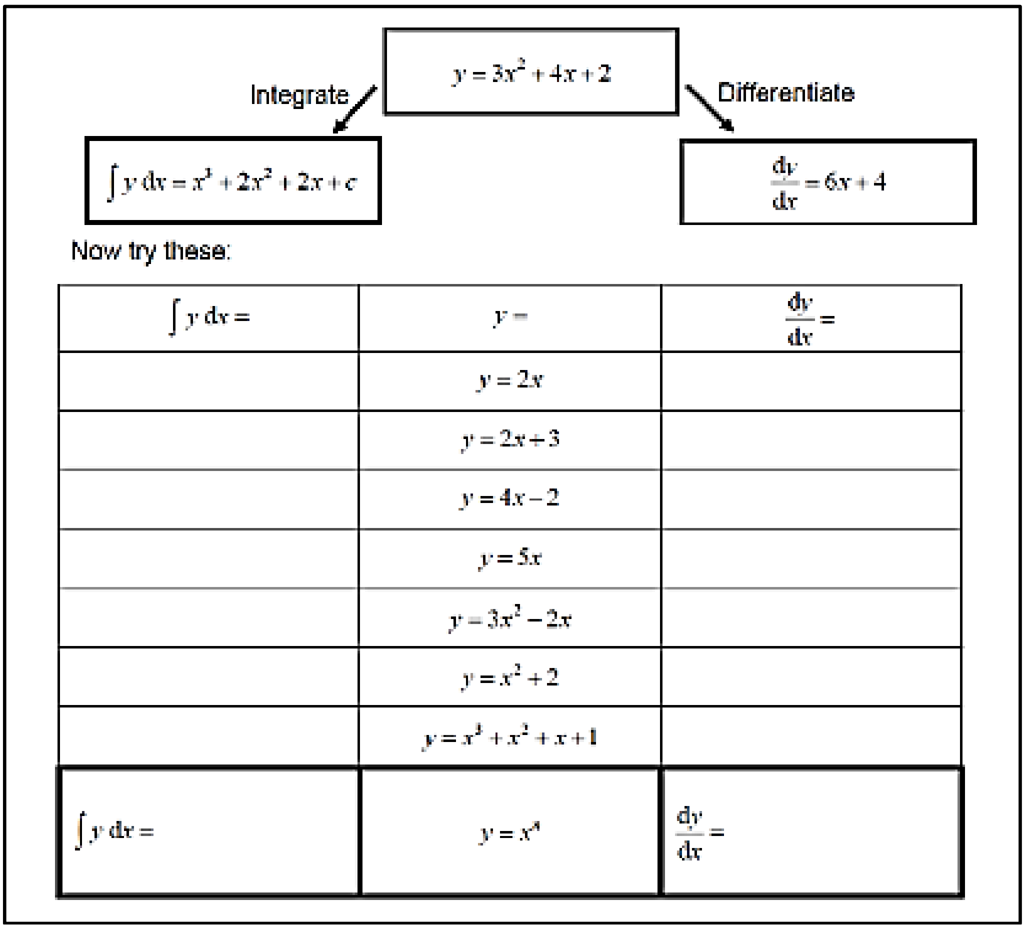
> "Yeah, and we actually call it $+c$. It's a constant the integration because you don't know what it is do you. But, yeah well done, that is brilliant. That's what hopefully you'll be able to do by the end of the lesson. So, if we had this let's pretend that you were all outside now and we got $dy/dx =$, I don't know, $\frac{dy}{dx} = 3x^2 + 2x + 5$ see if you can tell me what the y is, I will give you two minutes on your tables"

After two minutes, Adam mentioned that this process is called integration. When asked whether they need to add c, Adam said "Yes, you will lose one mark in the examination if you don't". Adam's last comment reflected his attention to examination policy. The students integrated the example (which Adam created spontaneously during the lesson) $\frac{dy}{dx} = 3x^2 + 2x + 5$ correctly. The conversations between Adam and his students so far showed Adam connecting resources to introduce integration (students' contributions at different points, the function and its differential, the graph on Autograph, examination requirements). He was *responding to students' ideas*, for example when one of the two students who were trying to guess the equation suggested 25 to be the constant of integration, Adam felt that the student was trying to give an equation that could be factorised (in this case it could be written as a square $(x-5)^2$). So, he asked why and then commented that: "I get what you've done here but it doesn't have to be factorised ", demonstrating to the students that the constant of integration should be kept as c at this point. But, was the student trying to get an equation that could be "factorised" or was he trying to complete the square? Also, the *use of mathematical terminology* in Adam's last statement in the dialogue above could be confusing because he used the term "constant of integration" before he introduced the term "integration".

Next, Adam asked his students to tell him what the "rule" was, a student said: "Add 1 to the power of, then divide by the power". Adam asked for a clarification about "the power", the student said it was "the new power". Adam approved the rule and used it (using students' contributions as resources) to demonstrate the example the students solved, then he asked them to solve two more examples.

Time for practice

Next, Adam asked the students to solve three questions that he spontaneously created and wrote on the board. In the third one of these questions (integrate $\frac{dy}{dx} = 6x^{1/2} + 12x$) the students needed to divide fractions and Adam commented that the three questions were getting gradually more difficult. He then gave them a worksheet (Figure 44) (*use of instructional materials*) to solve, commenting: “You can write on the sheet. You got two things; you want to differentiate one of them and integrate the other. Differentiate on the right integrate on the left, it’s a bit like reversing a car”. So, he was *making connections between procedures* (integration, differentiation). He gave the students some time to practice working independently on the worksheet (Figure 44) and textbook questions (Figure 45) and he was going around answering students’ questions and *identifying errors*. Adam frequently commented on forgetting to add the constant c. For example, he said “There is about 20.000 people in the country that always do that and miss out on their A in AF maths, because they forget the $+c$. Do not forget your $+c$”; which reflected his attention to examinations and examination requirements. His attention to examinations was also reflected in his request that students would be doing assessments for homework.


$y = 3x^2 + 4x + 2$

Integrate

Differentiate

$\int y \, dx = x^3 + 2x^2 + 2x + c$

$\frac{dy}{dx} = 6x + 4$

Now try these:

$\int y \, dx =$	$y =$	$\frac{dy}{dx} =$
	$y = 2x$	
	$y = 2x + 3$	
	$y = 4x - 2$	
	$y = 5x$	
	$y = 3x^2 - 2x$	
	$y = x^2 + 2$	
	$y = x^3 + x^2 + x + 1$	
$\int y \, dx =$	$y = x^n$	$\frac{dy}{dx} =$

Figure 44: the worksheet Adam gave his students

Exercise 8.1

1 These are all expressions for $\frac{dy}{dx}$.

Integrate to find an equation for y in each case.

a x^4 **b** x^7 **c** $3x^2$

d x^{10} **e** $5x^4$ **f** $\frac{1}{2}x^5$

g $-2x^5$ **h** $-\frac{1}{4}x^7$ **i** $4x$

2 These are all expressions for $\frac{dy}{dx}$.

Integrate to find an equation for y in each case.

a x^{-3} **b** $3x^{-4}$ **c** $\frac{1}{x^2}$

d $\frac{2}{x^3}$ **e** $-x^{-5}$ **f** $-\frac{5}{x^3}$

g $\frac{2}{3}x^{-3}$ **h** $-\frac{4}{5x^3}$ **i** 3

3 Simplify the right-hand side of each equation and then integrate to find an equation for y.

a $\frac{dy}{dx}=(x^2)^2$ **b** $\frac{dy}{dx}=(2x)^3$ **c** $\frac{dy}{dx}=(x^{-2})^3$ **d** $\frac{dy}{dx}=\left(\frac{3}{x}\right)^2$

e $\frac{dy}{dx}=\left(\frac{2}{x}\right)^3$ **f** $\frac{dy}{dx}=\left(\frac{2x}{3}\right)^3$ **g** $\frac{dy}{dx}=\left(-\frac{1}{x}\right)^4$ **h** $\frac{dy}{dx}=(3\sqrt{x})^4$

4 Integrate each expression with respect to x.

a $\sqrt{x}$ **b** $(\sqrt{x})^{\frac{1}{3}}$ **c** $\frac{1}{2\sqrt{x}}$

d $x\sqrt{x}$ **e** $-\frac{1}{x^2\sqrt{x}}$ **f** $\left(\frac{2}{\sqrt{x}}\right)^3$

Figure 45: Textbook questions

After that Adam asked students to play a race game (ActiVote pads: *use of instructional materials*). The race game involved competition on answering integration questions. In terms of the choice of these questions, Adam mentioned in the interview that "there is a software with a bank of questions which you can use", but not for A level questions where he had to create the questions himself:

> "You have to answer it quickly, and I try to make questions quite simple. All of them you can answer quite quickly, and I do it almost like a check. But, actually you can't make them gradually harder because if you put ten questions in then it automatically just choses whichever in any random order"

Adam also commented that this software added fun to the lesson and that he used it when students seemed bored to give them "a bit of a break, something a bit different rather than a lot more questions".

4.3.1.2 *Episode 2: The constant of integration c*

In this episode, Adam continued working on integration with the aim of highlighting the importance of c the constant of integration, to the group he taught in the previous episode. Adam used Autograph and students' contributions for that purpose. He said: "So, can I have you looking at the screen then. Let's have a look at this, I was playing around with Autograph and I went $dy/dx = 1$, what do you think? What graph is going to come up here?". A student suggested it would be a straight line, another suggested $y = x$, another said it was going to be horizontal and another said it would be "$y = -$ something".

Adam: I'm going to show you what happens, and I expect you to give me a good explanation as to why. Ready? So, I'm giving you the answer then I want you to tell me why $dy/dx = 1$ is this [referring to Figure 46].

Student A: Oh my God

Student B: I don't have an explanation for that

Student C: I don't know, but it looks pretty…

Adam: I'll do $y = x$. I'll put $y = x$ on there [referring to Autograph [see Figure 47], because I actually think that actually made a bit of sense. $y = x$, who said $y = x$ again, there's $y = x$.

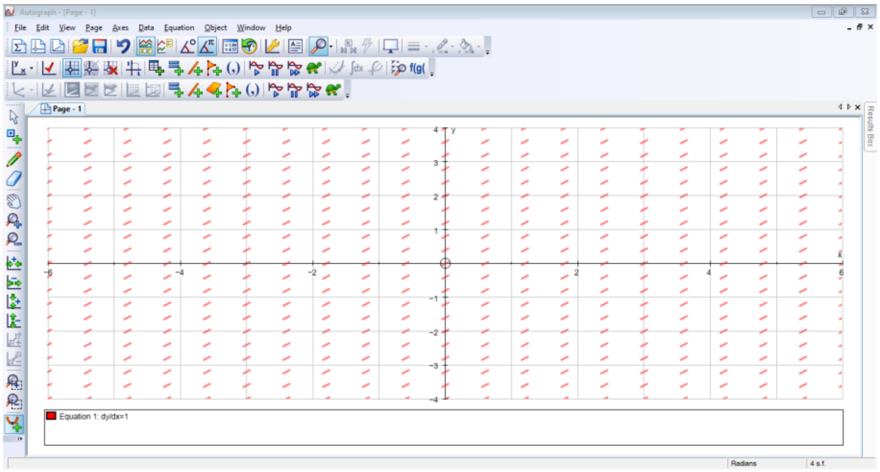


Figure 46: $dy/dx = 1$ on Autograph (reproduction of Adam's work)

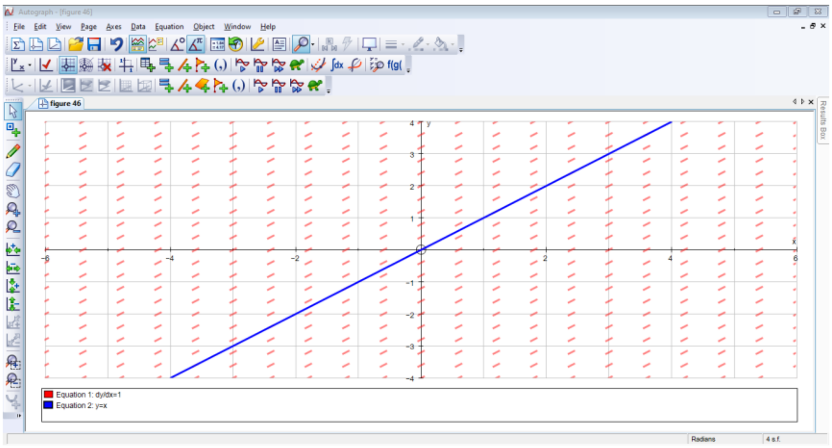


Figure 47: $dy/dx = 1$ and $y = x$ on Autograph (reproduction of Adam's work)

Student D: Gradient, doesn't have a point

Adam: Oh, what was that? Say that again, that made sense.

Student D: It's just a gradient, it doesn't have a point to go through.

Adam: Yeah, so it's a gradient function, isn't it? But, you actually don't know where. Because if we integrated this, so dy/dx is what? Sorry, what did I put?

Pupil: 1

Adam: 1, therefore then we integrated it $y = x + c$ didn't we? Does that make sense? We don't know what the C is. So, actually it could be $y = x$, that was a little bit right but what else could it be? x?

Student E: $+1$

Adam: $+ 1$, that could've been the equation [referring to Figure 48], couldn't it? What else could it have been?

Student E: $x + 2$

Adam entered both $y = x + 1$ and $y = x + 2$ on Autograph (see Figures 48 and 49).

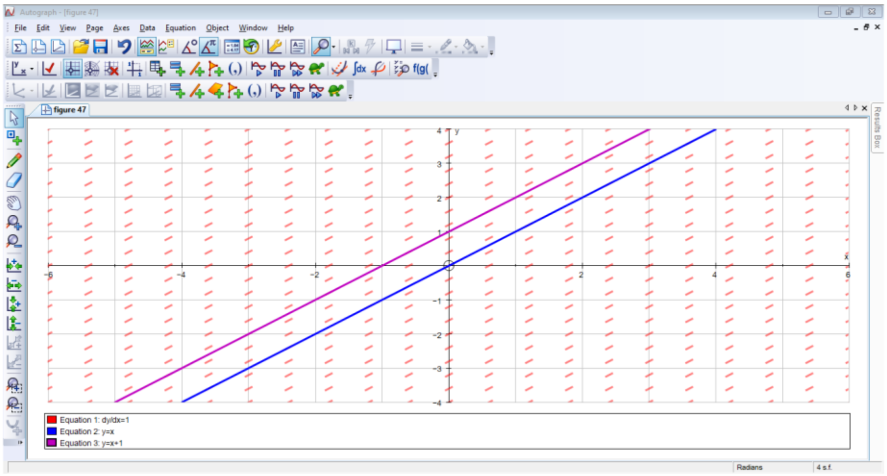


Figure 48: $dy/dx = 1$, $y = x$ and $y = x + 1$ on Autograph (reproduction of Adam's work)

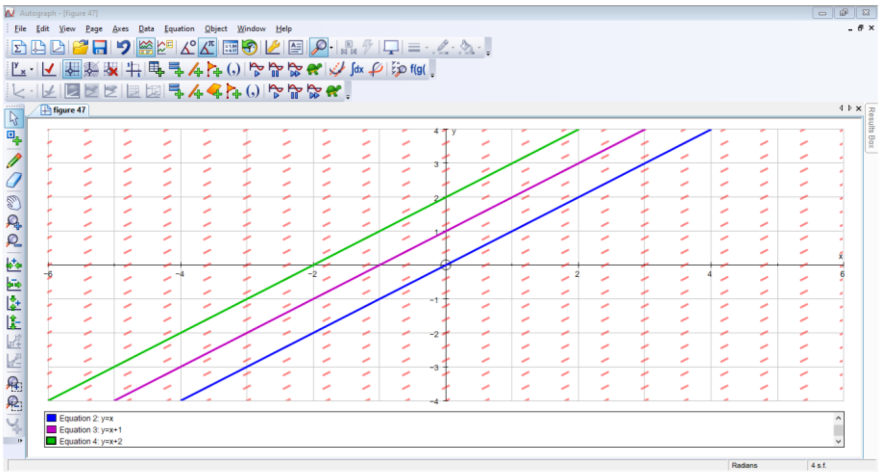


Figure 49: $dy/dx = 1$, $y = x$ and $y = x + 2$ on Autograph (reproduction of Adam's work)

Adam: $x + 2$. Do you see [referring to Figure 49]? When you integrate it gives you a family of curves, doesn't it? It could be loads of them there all related. How many of this happy family are $y = x + c$? There is an infinite amount in this family. You can see them all, we've only picked out three. How would you? What do you think the exam will do to make? What information would they have to give you so then you would have to work out exactly what graph it is? An x and y co-ordinates, would they give you one or two or how many x and y co-ordinates do you think they would give you?

Pupil: 1

Adam: Yeah, good. So, if they say $f'(x)$ is this, find the equation of the curve $f(x)$ you don't need the $+c$ anymore, do you? You might have to use it, because of that you can find a family of curves then hopefully you can find what the C is.

In this *demonstration* about integration, Adam frequently asked for students' contributions and used these as resources (e.g. to explain what they saw in Figure 46). He used Autograph as a resource and connected between Autograph and students' contributions to show students why they need to include ($+c$) when they do integration questions. He also connected between these and examination requirements to explain what information the students would need in the examination to figure out a specific "graph". So, he demonstrated that when given the coordinates of a point from the function the value of c can be calculated. Adam also connected between the algebraic and graphic representations (*making connections between representations*).

In the post-observations interview, Adam commented on his use of Autograph in this episode. In terms of his choice of this activity, he said:

> "I think I got this from the MEI website, and I think the point with this because it shows the gradient at different points, doesn't it? It shows a family of gradients and I think it makes it clear. [...] I think it makes a point that actually, we can't, there must be a family of curves that is the solution and not just one and that is the main point of this".

In terms of how Adam found the activity, he said:

> "I think my thinking was if you show them this and you show them the family of curves then they will be more inclined to understand it and then don't forget the $+c$. I think if you just kind of know how to integrate and differentiate and you know just the bog standard add one to the

power divide it by the new power you would be inclined to forget the c. Whereas if you show them something like this, show them the family of curves and they actually understand - oh actually look when that goes therefore when I integrate I don't know what that is - I think then they would understand it more. My thinking you know at a more comprehensive level and therefore not forget the $+c$ but whether that is true or not I don't know that was my thinking"

Using the Documentational Approach

In these episodes, Adam's resources were the textbook, his computer, Autograph, interactive whiteboard, board, his past experiences with students, the worksheet, the examination questions and grades, Curriculum of year 12, past teaching-experience, students' contributions including their previous knowledge, notebooks, the school's scheme of work, voting software ExamView (www.turningtechnologies.com/examview) and hardware (ActiVote pads), and MEI website (Mathematics in Education and Industry website https://mei.org.uk/).

Adam's scheme of work in both of episodes 1 and 2 is summarized in Table 16. In the first episode, Adam's specific aim was to introduce integration, and his general aim was to prepare students for the examination. His general rules of action included reminding students of fractions division; introducing integration; giving students some questions for practice; setting up homework/assessment; giving a concluding activity (voting race game). To remind students of fractions division, Adam followed the specific rules of action: to show the question; to give students time to solve it then to show the correct answers on the board. To introduce integration, Adam followed the specific rules of action: to ask two students to volunteer to leave the class; to draw a graph on Autograph and ask students to guess its equation; to ask students about the "gradient" of the graph, to hide the equation of the graph; to invite the volunteer students to come back into class and guess the graph's equation from its gradient's equation; to give the students an expression to integrate; to introduce the "rule" of integration; and to emphasise the importance of the constant of integration c. Adam's specific operational invariant was that many students forget c. His general operational invariants were that ActiVote adds fun to the lesson; and to connect new concepts with students' previous knowledge. One inference (students forget to add c the constant of integration) that resulted from previous experience of teaching integration lead Adam to do the activity on Autograph in the second episode, this could potentially be an inference from the first episode as well (because Adam noticed students forgetting to add c). In the second episode, Adam's aims were: specific - to continue working on integration; and general- to prepare students for the

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A1. To introduce integration *General:* A2. To prepare students for exam	*Specific:* R1. Ask two students to volunteer to leave the class R2. Draw a graph on autograph and ask students to guess its equation R3. Ask students about the "gradient" of the graph R4. Hide the equation of the graph and invite the volunteer students to come back into class and guess the graph's equation from its gradient's equation R5. Give the students an expression to integrate R6. Introduce the "rule" of integration R7. Emphasise the importance of c *General:* R8. To give students some questions for practice R9. To set up homework/assessment R10. Play voting race game	*General:* O1. Autograph ease of use as a tool for visual representation O2. ActiVote adds fun to the lesson *Specific:* O3. Many students forget c O4. It is useful to connect new topics with students' previous knowledge	Many students forget to add c the constant of integration
	General R11. To continue demonstration about integration *Specific* R12. Ask two students to solve questions about integration R13. Use Autograph to graph dy/dx=1 R14. Ask students for an explanation R15. Use Autograph to explain why students need +c *R16.* Demonstrate how to find the value of c	*Specific* *O5.* Autograph use makes the idea behind c clearer and works as a visual reminder	

Table 16: Adam's scheme of work in episodes 1 (in row 2) & episode 2 (in row 3)

examination. His general rule of action was to continue his demonstration about integration focusing on the constant of integration. Under that he had the specific rules of actions: to use Autograph to graph dy/dx=1; to ask students for an explanation; to use Autograph to explain why students need +c; to demonstrate how to find the value of c. Beside the operational invariant noticed in the first episode, there was one more operational invariant: Autograph use makes the idea behind c clearer and works as a visual reminder.

4.3.1.3 *Episode 3 – Modulus equations*

This episode is taken from a 75-minute revision lesson about solving simultaneous linear and modulus equations. Preliminary analysis of this lesson was published in Kayali & Biza (2017). The lesson was audio-recorded and was with a group of nine Year 12 students (17-18 years old). In this episode, Adam asked his students to solve questions on simultaneous equation, some of these questions were from the textbook (*use of instructional materials*). After giving the students some time to work on the questions, Adam started solving the questions on the board. First, he would use Autograph to show the graphs of the equations (*use of instructional materials*), then he used the graphical solution as a starting point and a way to check the algebraic solution (*making connections between procedures* and making *connections between representations*). For example, when the teacher was solving $y = |x + 2|$ and $y = 3$ simultaneously, he asked students to draw it and see the answers in the graph before solving algebraically: "You will get two points, you can see this graphically". Then, he started to write one of his student's algebraic answers on the board: "$3 = x + 2$ or $-3 = x + 2$". He then commented on the student's answer: "So, math says x = 1 or -1... What textbook would say is $y = (x + 2)$ or $y = -(x + 2)$. Textbook would just say that, so I'll probably do it this way".

In his *demonstration* about the algebraic solution, he frequently asked for students' contributions and used these as resources. While solving questions, Adam commented on which questions were "exam questions" and what grade they would help the students achieve (knowledge of examination requirements). When a student asked why they needed to learn about modulus equations, Adam chose to use Google to find out the answer. Based on his Google search Adam said that modulus equations were used for "distance" and "currency exchange". When students asked why one would use graphs and modulus equations to measure distance, Adam responded because distance had to be a positive number. The students expressed that they were not convinced because they "already know" that distance would be positive, and they did not feel they would need modulus equations for that. This showed how Adam tried *responding to students' ideas* by using Google as a resource, his attempt in this case was not convincing. This was a contingent moment in which Adam attempted to briefly respond to the student's question. It reflected his subject knowledge and lack of *awareness of purpose*, which were bounded by examination requirements, this matches his aim of preparing students for the examination. Although Adam did not say clearly that he was "adhering to examination requirements", the way he dealt with this contingent moment, reflected such adherence.

Later in the lesson, when Adam felt that some students finished answering all the questions he initially asked them to do, he decided to come up with extension questions to keep them engaged. In other words, this was a contingent moment in which he was *responding to the availability of tools and resources* (i.e. students' behaviour showing they finished their work). Adam responded by asking his students to give him two different modulus functions that do not intersect, and two that intersect once. One student (student A) gave a correct example that Adam approved after showing the graphs on Autograph (Figure 50):

Student A: $y = |x|$ and $y = 2|x|$, shift across

Adam: Oh, yeah it is.

Student A: Yeah, you've translated it.

Another student (student B) gave an example that showed one intersection points on the Autograph display but had another that was not visible on the screen (Figure 51):

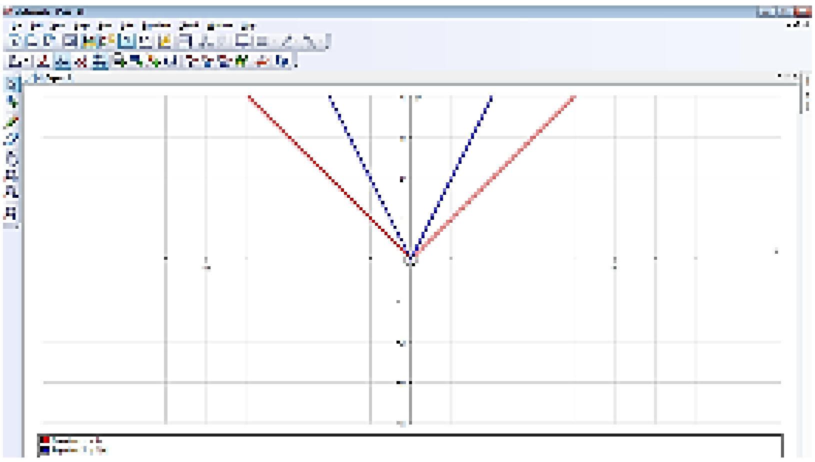


Figure 50: Student A answer on Autograph (reproduction of Adam's work)

Student B: $y = |x - 4|$ and $y = 2|x|$

Adam looked at the graphs on Autograph and nodded in what seemed like a hesitant agreement.

Student C: Change the slope.

Adam amended the equations as student C suggested and wrote y = 2 | x – 4 | and y = 2 | x | without commenting on student's B answer. This was a contingent moment that shows Adam *responding to students' ideas*, but he did not use this opportunity to demonstrate to the rest of the class why student's B answer was amended. He also did not utilize the zooming in/out feature to show the students how many intersection points there are for student's B answer. This could be due to lack of knowledge and fluency in the use of the software.

After that Adam asked students to play a Bingo game (Figure 52) he had prepared on simultaneous equations. He used Excel to list the functions and was revealing these gradually when needed, asking the students for quick answers to the simultaneous equations in the Bingo game (*use of instructional materials*).

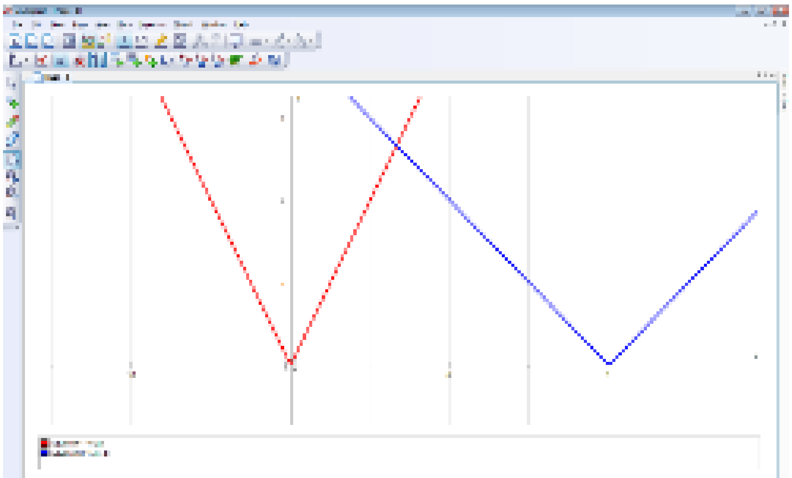


Figure 51: Student B answer on Autograph (reproduction of Adam's work)

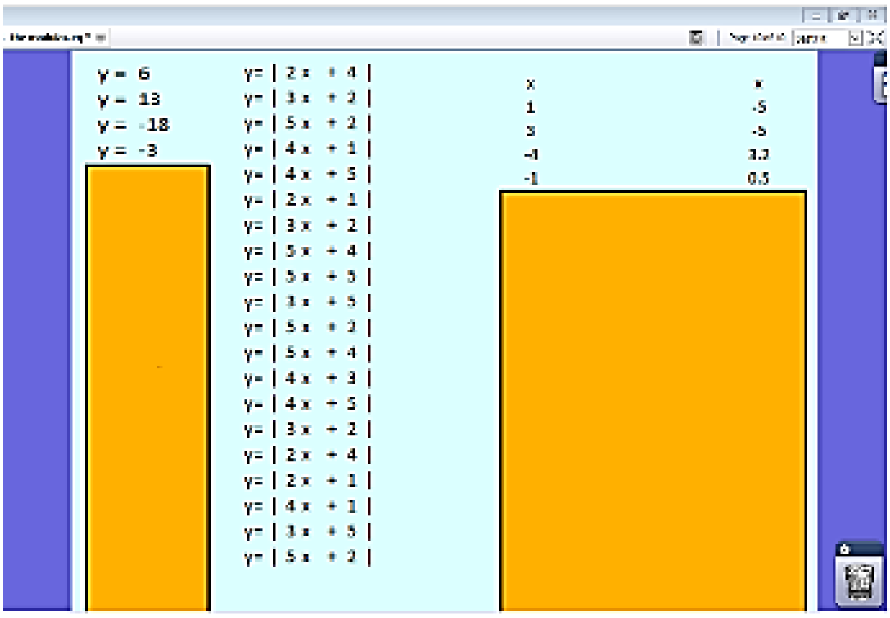


Figure 52: The bingo game

Using the Documentational Approach

Adam's resources were the textbook, his computer, Autograph, Excel, interactive whiteboard, his past experiences with students, the Bingo game, the extension question which he seemed to have made up at the point he felt it was needed, and the examination questions and grades.

Adam's scheme of work in both of episodes 1 and 2 is summarized in Table 17. His specific aim was to do some revision on simultaneous and modulus equations, and his general aim was to prepare students for examination. His specific rules of action were to ask students to solve questions on simultaneous equations; to show the graphs and algebraic solutions of the simultaneous equations given; to have the students playing a Bingo game on simultaneous equations. His general rules of action were to make connections with examination questions and grades; to answer students' questions; and to use Autograph to show the graphs of functions and connect these with the algebraic solutions. Adam had a general operational invariant regarding Autograph's ease of use as a tool for visual representation, and a specific operational invariant regarding using graphical solutions as a starting point for algebraic ones and for the construction of meaning about c the constant of integration (*choice of representation*). No specific inferences were noticed from this lesson.

4.3.2 Discussion

From the three episodes above we see how Adam's resources included the interactive whiteboard, board, Autograph, ActiVote, ExamView, textbook, curriculum, examination requirements, past teaching experience, notebooks, scheme of work, MEI website and students' contributions. Adam used the textbook as an instructional material (*use of instructional materials*), and this use was combined with the use of questions and activities that Adam designed, which reflected his confidence in creating teaching materials. For example, Adam commented on the starter activities he used at the beginning of each lesson, like the one he used at the beginning on episode one (Figure 43). He said he usually designed these activities based on his previous experience and inferences after trying to teach a concept in one lesson and finding out that the students needed reminding of a concept from their previous knowledge first. To give an example, Adam said in the interview that he would design a starter activity on adding fractions if he knew students would need that already known concept in order to learn a new concept. By doing this he aimed to:

"remind how to add fractions because they still make mistakes with it unfortunately. That would be an example yeah, maybe it would've been a car crash three years ago and I've gone none of them can add fractions, let's have a quick reminder".

<table>
<tr><th colspan="4">Scheme</th></tr>
<tr><th>Aims</th><th>Rules of action</th><th>Operational invariants</th><th>Inferences</th></tr>
<tr>
<td>Specific:
A1. Revision on simultaneous modulus equations
General:
A2. To prepare students for exam</td>
<td>Specific:
R1. To remind students of the graphs of some functions
R2. To show the graphs and algebraic solutions of the simultaneous equations given
R3. To use autograph to show the graphs of functions including modulus ones
General:
R4. To ask students to solve questions on simultaneous equations
R5. To make comments on exam questions and grades
R6. To answer students' questions
R7. To have the students playing a Bingo game on simultaneous equations
R8. To use Autograph to show the graphs of functions and connect these with the algebraic solutions</td>
<td>Specific:
O1. A graphical solution can be the starting point for an algebraic one.
General:
O2. Autograph is easy to use as a tool for visual representation</td>
<td></td>
</tr>
</table>

Table 17: Adam's scheme of work during the first lesson observed for him

The “car crash” would be recorded by Adam to remind himself of why the lesson did not go well, and what he needed to do better next time. The “car crash” in other words would lead Adam to inferences for the future, these inferences would later become operational invariant in his practice.

Having examination requirements and the MEI website as resources, reflected Adam’s attention to examination and school policy. His attention to examination requirement was reflected in his comments about how questions are asked in the examination and what mistakes students widely make (e.g. forgetting to add c the constant of integration, what information are given in the examination to find c the constant of integration). Also, the code “adhere to examination requirement” comes up when Adam was asked about the why one should learn about modulus equations. As such a question was not required in the examination, Adam did not know the answer and he chose to use Google to search for one. A similar contingent moment happened in one of George’s lesson on the volume of revolution, when a student asked him why they needed to learn that concept. George’s response was it could be used by builders buying building materials to know for example how much cement they would need. George further commented, dismissing his previous statement, that builders would just usually estimate how much they would need and buy more later if needed. From these two examples from George and Adam, it seems that questions outside the curriculum, specifically about the use of mathematical concepts in practical life could be challenging for teachers.

Adam used students’ contributions as resources in two ways: to answer questions he prepared or from the textbook; and to demonstrate ideas (like what he did in both episodes on integration). The latter reflects confidence in having higher chances of *Contingency*. This shows similarity to the way George used students’ contributions as resources, however George seemed more open to work live with his students in different and several scenarios.

Adam connected between resources in a way that helped him demonstrate ideas. The episodes showed him *making connections between representations*, students’ contributions and examination requirements. Such connections were consistent with Adam’s teaching aims. Adam’s knowledge of the software he used (Autograph) was essential for him to create such connections and to demonstrate ideas in a way coherent with the topic he was teaching. However, there are two points to address in relation to the use of Autograph:

- In the episode about simultaneous equations, a contingent moment was observed, and the teacher did not use the software features to reflect on it in class (i.e. he could have zoomed out to show his students one intersection point that they could not otherwise see). This was a missed opportunity that he did not use to reflect on having graphical or algebraic solutions and to create connections between representations.

- In the second episode, Adam used the same feature in Autograph as the one George used in his work on differential equations. Adam mentioned the gradient to demonstrate to the students what they saw on Autograph, unlike George who did not mention the gradient in his *demonstration,* even when students questioned what they saw on Autograph. The example Adam used on Autograph was for a linear function, unlike the examples George used which were cubic or square functions. In George's case, the representation on Autograph created confusion and was questioned by students. This did not happen in Adam's lesson, and this could be due to: the *choice of examples*; the dynamics of the taught group; or the *demonstration* about the connection between the representation on Autograph and the gradient function.

In general, Autograph was used by Adam also as an instructional material (*use of instructional materials*) that helped student visualize graphs and supported the *demonstration* (like in the episodes on integration). During the interview, Adam was asked about his use of Autograph, he commented that he used it for specific topics and that he was still learning how to use it:

> "I use it for certain things that I know how to use. So, once I have built it in to my lesson, I use it. So, I might make a note to myself oh I'll use Autograph for this, and I will use Autograph for it. So, for example the other day I was doing mod graphs and I must've done it before, where it's almost like mod football where you have to put a graph that goes through a certain point and then into a goal or go through two points then the goal, and because I had done that before I go 'ahh yeah that's really good' and I do it again. But, it's almost once I've planned my lessons I don't then, I would have to sit down and really think about it. So, for example iteration in C4 I know that you can use iteration and do spider diagrams and step diagrams for iteration, but I haven't done that yet. So, you know they are still in the back of my mind, it's something that I need to improve but it's just coming to it. You know, I might go oh I'm teaching tomorrow, oh I've got that lesson that's quite good rather than actually because I have to teach myself how to use Autograph for that specific skill. You know, it's not something that you can just pick up still and go. So, I'm getting there I feel like I am slowly getting better. But, also, as well, especially with A level, I find that I do it, I will do that lesson once a year say rotating something round the

x-axis. I know how to do it but then I have to remind myself oh how do you do that again, and it takes a good 10-15 minutes even though I've done it 5 times before. I've only ever done it 5 times before, so I'm still forgetting how to use Autograph to rotate something round so it is still a big hindrance. But, then sometimes it is worth the effort you know. Autograph is for that skill rotating something, it does really visually show this is what's happening".

I asked Adam if in his last example about rotations he was referring to the volume of revolution, he said: "Yeah, that's what I mean. So, sometimes it is really worth doing and sometimes it's just almost like a gimmick. It depends on what you're teaching I find"

The last two quotes from Adam reflect how his knowledge of the software influenced how often he used it. He used it with resources or activities that he was already aware of and used before. But he did not use it when such use was more challenging. In other words, when he did not remember exactly how to use it or when he had to learn about or develop an activity himself. These two situations are also relevant to teachers' struggle with time and how they prioritize their work. The priority and time are not easily given to tasks that do not follow the examinations style and requirements, like in many cases with technology. Another aspect in these two quotes is about Adam's appreciation of the software. Adam expressed his appreciation of mathematics-education software as visual mediators. But he also expressed that software use was "sometimes" worth doing, so he did not always find it as useful, adding that it could be "like a gimmick". These comments go along with the way Adam described his use of technology once to me as mainly to graph functions and offer visual representations to the students. Furthermore, his use of digital resources other than the software, such as in the case of the race game, seemed "like a gimmick" that kept the students engaged using questions that followed examinations styles and requirements with Adam being the game master.

The three episodes above reflected Adam's subject knowledge and knowledge of the software (Autograph here), the latter was associated with specific mathematical topics. They also showed him using students' contributions and creating connections between resources in a way similar to George's. They reflected the influence of examination requirements on what Adam knew and demonstrated in the lessons; and shed light on two contingent moments he faced. For example, one contingent moment was reported in the lesson about simultaneous equations; when a student asked about the use of modulus equations and Adam relied on Google search to give an answer. One example of a possibly contingent moment was in the first episode when the students integrated $2x - 10$, and decided that 25 was the constant of integration. There is no evidence to suggest whether Adam expected this answer or not. There is a chance that more contingent moments happened; but went unnoticed in cases where the teacher responded confidently without showing or addressing what he did not expect.

4.4 Fourth teacher data - Charlie

Charlie was a mathematics teacher with twenty years of teaching experience, during which he taught students aged 12-18 years. He had a BSc in mathematics and a PGCE in secondary mathematics teaching. He was the head of mathematics at his school during the data collection. The school he worked for provided interactive whiteboards, iPads for students' use, and Explain Everything (www.explaineverything.com) and Autograph software (www.autograph-math.com).

Out of nine lessons observed for Charlie, episodes from four lessons have been selected to be included in this chapter: two lessons on polynomials (lessons 5 and 8), one lesson on transformations (lesson 9), and one lesson on tangent and normal lines (lesson 3). In the rest of the lessons (about quadrilaterals and inequalities) either mathematics education software was not used, or the teacher's input was minimal and mainly students were practising and solving questions.

4.4.1 Two Lessons on Polynomials

4.4.1.1 *Lessons 5 overview*

This episode includes an overview of one full lesson (fifth lesson) that was observed for Charlie. The lesson was 50-minutes long and it was to teach a group of twelve year 12 students, namely (GC1) about polynomials.

4.4.1.1.1 Episode 1- Homework

Charlie started this lesson by checking if students had any questions about their homework. One question was related to the answer given at the back of the book (by Rayner and Williams (2004)) of question 14. The teacher commented on that "some of us disagree with the answer at the back of the book…. I don't know what the issue is; but I don't agree with them. It might be that I am making error". The teacher then collected the pieces of homework from the students for him to mark and then moved to introduce a new chapter. Charlie relied on the textbook as the source of exercises for homework (*use of instructional materials*); and the choice of exercises for homework was set for the mathematics department. However, he was *identifying errors* that were in the textbook and telling the students about these.

In relation to homework, Charlie commented:

> "So, the way that we set up the homework here with the A level is we have a number of small homeworks which I don't take in. I'll say you can look at this if you like, you can try some questions but if you don't say anything I won't be chasing it up, so that would be let's say exercise 7A I won't take in 7B, I won't take in 7C, the only exercise I will take in is called an assignment and it's a summary of the chapter. Now, those are formal pieces of work and they come in, so in the meantime I am saying to them 'look if you are completely fine with this and you think you've got it then you've got other things to do, fine you've still got to do the assignment. If you're not so happy and you want to do some questions and you found a weird one here, then I want to know about it'. So, that's what that's for and they know that's the set up, and it's really good for us so it allows them to raise any problems that they find. In fact, I hope I've said to them you know when you're doing those questions, you really should be doing a couple just to check you know what you are doing but then go find the weird ones because that's where you do the learning".

This suggests how Charlie used homework to identify "weird" questions, and help the students understand them. Students' contributions are in this case are resources in the form of the "weird questions" identified by students for Charlie to demonstrate. Charlie elaborated:

> "that's for you as a student to find what you can't do so I can help you with it. But, without me slogging through question 1 which they know is right, question 2 they know it's right, and they only want help with question 17. So, there is no point me marking all that stuff they should be marking it ticking it and it's only the stuff they can't do that I'm interested in."

This meant that students work on homework provided a resource that Charlie used to identify what questions students found difficult. This is *connection between resources* (textbook questions & students' work on homework). In another comment about homework, Charlie said: "well it's actually almost always, it's past A level questions. So, it's a group of past A level questions on that chapter on that topic". We see here past-examination questions being used as a resource.

4.4.1.1.2 Episode 2- Introduction to polynomials

Charlie said to the class that he was introducing a “rubbish chapter […] Chapter 9 is Polynomials, it’s how many? It’s like three pages lone, it’s not really a thing”. He then explained that poly meant “many” and nomials meant “numbers” and asked “What are many numbers? That’s not really very helpful, is it? If I give you a question like this how many solutions?”. Charlie’s *anticipation of complexity* (i.e. simplicity here) of this chapter he was introducing was clear in his comments “rubbish chapter… not really a thing”. In his *use of mathematical terminology*, he wanted to demonstrate to the students what “polynomials” meant, so he tried to address the meaning of poly as many and the meaning of nomials as numbers, while nomials meant terms (numbers and variables) (Oxford University, 2000). To clarify the meaning of “many numbers”, Charlie chose a spontaneous example to graph on Autograph (*choice of representation+ use of instructional materials*) $y = 2x + 1$ and $y = 10$, and asked the students about the number of common solutions the two equations had “where they intersect is the solution”. He then sketched the graphs on Autograph (Figure 53) and noted that the two intersected at 4.5, which meant there was one solution. Charlie asked a student to tell him how many solutions he would get for $ax + b = c$, student A answered: “one solution”. The students recognized that there was one common solution.

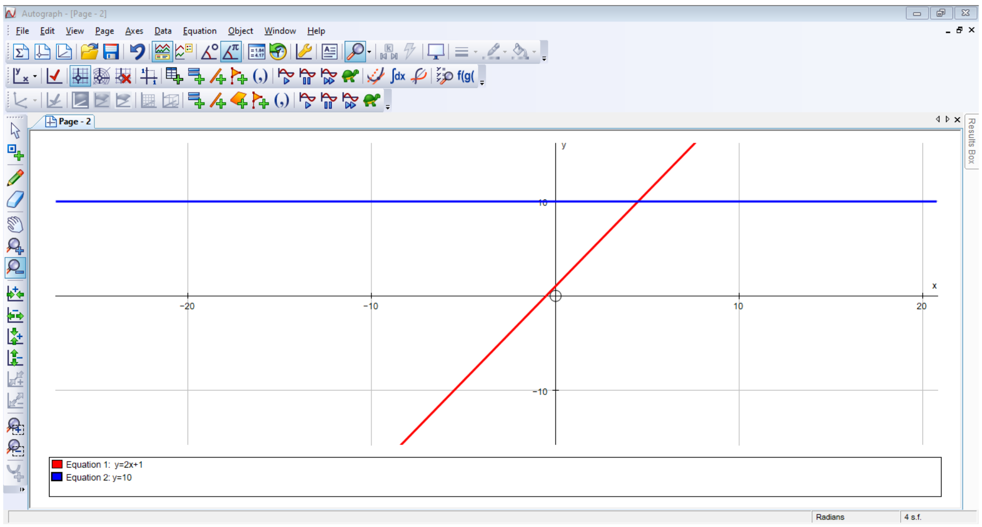


Figure 53: $y = 2x + 1$ and $y = 10$ on Autograph (reproduction of Charlie’s work)

Asking the same question about common solutions, Charlie this time spontaneously chose $x^2 = 8$, wrote it as $y = x^2$ and $y = 8$, and graphed the latter two on Autograph (Figure 54). Having the graphs on Autograph intersecting at two points lead him to conclude that there are “two solutions”. At that point he asked: “However, are there always two solutions?”. He then moved on to ask about the number of solutions for $y = x^3 - k$. Student answered “three”.

Charlie wrote the two equations $y = x^3$ and $y = k$, sketched these on Autograph (Figure 55) and showed the students that a cubic equation does not always have three solutions. He showed different cases by showing different cubic functions and by changing the constant k in the equations $y = k$. He said: "I might have to write here up to 3 solutions" (*subject knowledge*+ knowledge about software affordances). He then added that it "seems to be a pattern". By doing this, Charlie was attempting making connections between concepts (the number of solutions and the degree of the polynomial). However, he did not comment on this connection at this point, but he decided to define polynomials first (*decisions about sequencing*). He wrote "Polynomials are any algebraic expressions… you have to have integer powers, fair enough? There was a question years ago, is it true if I have any polynomial of degree n, the highest power here, has at most n solutions?". Thus, Charlie introduced the degree of the polynomial (*use of mathematical terminology*), then connected that to the number of solutions found (*making connections between concepts*). But, polynomials are not equations and there was no comment during the lesson about the difference between the two. The connection between polynomial degree and number of solutions could create confusion between the two concepts; it showed that the conceptual appropriateness was not well recognized by the teacher here (*recognition of conceptual appropriateness*). The question about the number of solutions made Charlie mention stories about mathematicians, and how we could say "any polynomial of degree n has n solutions just not all of them are real" (*overt display of subject knowledge*). He added "all powers must be whole numbers, positive numbers" (*use of mathematical terminology*). Charlie was aware of his purpose for this part of the lesson (define polynomials and degree of polynomial), however *decisions about sequencing* (meaning of the term polynomial, graphing & common solutions, definition of polynomial, then degree of polynomial & number of solution) and *choice of examples* (graphing equations) both seem tentative and reflect a vague *awareness of purpose*.

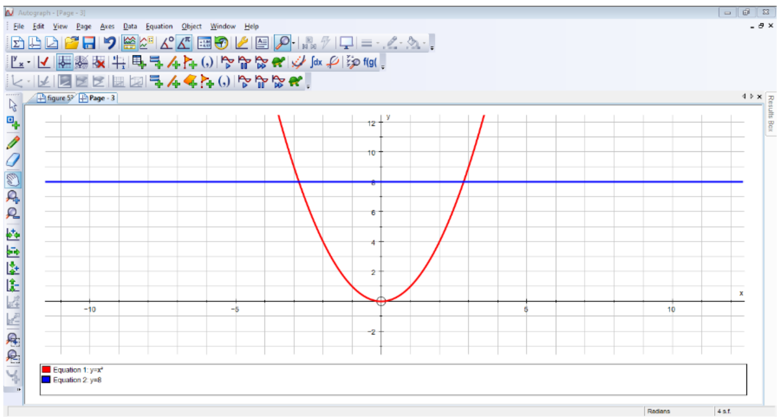


Figure 54: $y = x^2$ and $y = 8$ on Autograph (reproduction of Charlie's work)

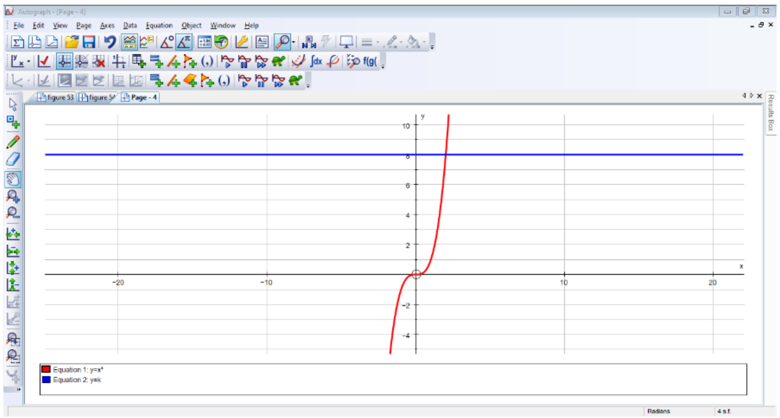


Figure 55: $y = x^3$ and $y = k$ on Autograph (reproduction of Charlie's work)

When I asked Charlie to reflect on this Autograph activity during the interview, he said he did not remember its details. After a brief reminder of the steps he followed, he mentioned that his use of Autograph was in order to do some "live" work during the class. He said:

> "I really think that it is important for them to see not only the algebraic solution which they could do I would assume, they could do that more easily. But, I would assume they are less familiar with the visual approach and the graphical modelling of the same thing. And, then also I am mindful of the A level content which is they really need to look at these sometimes as equations but often they need to look at it as formally as pictures, sorry as graphs. So, for instance, I know that for these guys who are in year 12 when they get onto year 13 they need to look at graphs of functions and graphs of inverse functions; and it's crucial then that they understand the graphical representation [...] So, it's looking ahead that's the key [...] We talked about algebra and the graphs and then just putting them together and they need to see both so that is what that is for so yeah is there possibly a simpler solutions but looking at the bigger learning points that's what that's for ".

This comment highlights Charlie's *awareness of purpose* of the activity, which again seemed vague and tentative, both during the lesson and the interview. It also clarified Charlie's *choice of representation*; making *connections between representations* and making connections between resources (i.e. year 12 & year 13 curriculum) as well as his attempt at *making connections between concepts* (i.e. equations, functions and polynomials). The latter was not well demonstrated to the students and could have been supported by further *demonstration* of the differences between these concepts.

4.4.1.1.3 Episode 3- Introducing the adding, subtracting and multiplying polynomials

The next part of the lesson was about introducing how to add, subtract and multiply polynomials and was done through the textbook. Charlie asked students to do exercise 9 in Rayner and Williams (2004) “and see the questions you like, and go yeah I know that […] we can go through the ones you don’t like”. We notice how Charlie here used resources (students input regarding what they know how to do and the textbook) and connected these in order to make decision about what questions to solve in class. Whenever Charlie solved questions on the board, he was picking students to answer those questions and he was explaining ideas when needed. He did some questions on the board and asked students to try to solve questions independently then discuss them between each other. He again asked them to skip the questions they find easy and do the ones that are “worth doing”. He then started asking students to pick a question to do on the board. One student picked question F. The teacher solved the question on the board and had students contributing to the answer. Thus, he used students’ contributions as resources to solve questions on the board. These contributions were used to help Charlie *demonstrate* how to add, subtract and multiply polynomials.

4.4.1.1.4 Episode 4- Time for practice

Students continued working independently and Charlie encouraged them to ask questions (contributions as resources) when they needed. A student was confused when she saw questions with function notation (Figure 56). Charlie commented:

> “Have you heard of a function of x? … 4x+2 if x=1 y=6 if x=2 y=10 and so on. […] That is a function of x. […] I called one function 4x+1 another function… I called that one f(x) […]. If I start with p(x) it might stand for polynomial; […] lazy way to say polynomial […] Then say well…that is the graph of the function, that’s the function”.

Here, Charlie attempted *making connections between concepts* (here the two concepts polynomials and functions) following the textbook example. However, this connection was only briefly demonstrated and was in response to a student question (students’ contributions as resources).

Miscellaneous exercise 9

1 Let $p(x) = 3x^2 + 2x - 1$ and $q(x) = x^2 - 2x + 3$. Find $p(x) + q(x)$, $p(x) - q(x)$ and $p(x)q(x)$.

2 The polynomials $f(x)$ and $g(x)$ are $2x^2 + ax - 3$ and $3x^2 - bx - 2$ respectively, where a and b are constants. In the product $f(x)g(x)$, the coefficient of x^3 is 6 and the coefficient of x is 1. Find the coefficient of x^2.

Figure 56: Textbook question with function notation in Rayner and Williams (2004)

The second textbook question in Figure 56 included the use of simultaneous equations, so Charlie commented that there are two variables and two pieces of information:

> "so, this is sneaky cheeky simultaneous equations, you have two variables with two pieces of information… I've got a function of x… if we multiply them together, we get ()(), expand… Shall we do it the long way or the super quick way".

The student asked for the quick way. Charlie said: "We only need information about the coefficient of x^3 and x". Charlie was writing the terms that included x^3 and x on the board and asking for students' contributions to the answer. He added: "We can group together all the x^3 and all the x terms". So, Charlie wrote that $3ax^3 - 2bx^3 = 6$ and $-2ax + 3bx = 1$. Charlie's continued working on finding a and b and then finding the coefficient of x^2. During that work, he pointed at one student saying: "You've just made the face of understanding". In this work, we see Charlie *making connections between procedures* (multiplication of polynomials + simultaneous equations) and showing *concentration on procedures*. We also notice how "the face of understanding" was as a resource here for him to tell that the student understood how to answer the question. Charlie continued *responding to students' ideas* until the lesson finished. In the post-observation interview, Charlie commented as follows:

> "that's because so that's partly because I really want to celebrate this idea that in maths when you've overcome a difficult task, and you've been in that uncertain area, it is a real joy and it's fleeting it doesn't last for very long, but it's a real joy to have overcome that. And I want to celebrate that because again I don't want them to just think that people who are possibly good at maths just automatically get this and they don't feel the same way, and it's also partly because yeah you've been in that zone of uncertainty whatever the phrase might be but you should go through that period. So, it is worth celebrating when you come out. And, I may have said even in the lessons, I'm not sure, and it's because some said it to me we're not sure in how many

other subjects do you get that … now it may happen in every other subject, but in maths I always make a point in saying you don't get that expression in many other subjects".

Charlie's experience in teaching (resource) made him feel that it was good to "celebrate" "understanding" concepts in mathematics.

4.4.1.2 *Episode 5- Lesson 8*

This episode was also taken from a year 12 lesson on polynomials. The group that was taught this lesson, let us call it here GC2 (with five students), was different from the group in the episode before GC1. The lesson was the eighth lesson I observed for Charlie, and it was taught without the use of Autograph.

Charlie started by solving question from the homework and he solved what he or the students felt was challenging. Like in the previous episode, Charlie relied on the textbook for choice of exercises for class practice and homework (*use of instructional materials*). However, he was *identifying errors* that were in the textbook and telling the students about these.

He then moved to the new chapter "polynomials"; and said that the chapter was on three pages explaining about polynomials. His teaching aims for the "polynomials" chapter was clear: "Say what they are, then do addition and subtraction multiplication". In his *use of mathematical terminology*, he wanted to demonstrate what polynomials meant, so he asked: "what is polynomial? Student S you've got up to two minutes to answer that. Anyway, what's polynomial? what does poly mean?". Student S answered "many". Charlie asked "Nomial, what does that mean?". A student replied: "Poly means many, nomials means terms, many terms". Charlie wrote some spontaneous example and defined polynomials as a "collection of terms but the terms have to have whole numbers powers, no negative powers and no fractions" (*decisions about sequencing*). Afterward, he asked the students to do some textbook questions (use of resources). He was asked if 7 is a polynomial and he said it was "because it could be regarded as $7x^0$" (*subject knowledge*). Then he introduced the meaning of "coefficient" (*use of mathematical terminology*) as "the number and the sign", pointing to some coefficients from the examples he had on the board.

After that, students continued working on the textbook questions, and Charlie was answering questions on the board whenever a student asked for help. Thus, he used students' contributions as resources that indicated what needed further explanation. A question the students needed help with was the same question Charlie solved in the previous episode (question 2 in Figure 56); about finding the coefficient of x^3. A student suggested to "times it all out" and "put $6x^3$" for the term with x^3. Charlie agreed but added:

> "How many pieces of information are you given? How many unknowns have you got? In your experience in maths, if you have two pieces of information and two unknowns, when you have two bits of information and two unknowns, we tend to solve these in what very very common way to solve the question? Two equations and two unknowns?".

A student said: "simultaneous equations" (*making connections between procedures and making connections between concepts*). So, Charlie commented:

> "this is a classic A-level set up: two equations two unknowns, expect simultaneous equations… this is a classic A-level question, that is a challenging A-level question because you've got a's and b's, but look at the individual elements they are all GCSE, put it together and it is an A level question, and it is definitely a challenging question"

This reflects his consideration of examination requirements (i.e. A-level set up) and the connections he created between resources (i.e. past-examination questions & textbook questions, GCSE & A-level curricula). Charlie then demonstrated that students can either find all terms (9 here) and then work out the coefficient of x^3, or they could only look at the terms with x^3 and find its coefficient. He explained that although he "liked" the latter, there was nothing wrong with the first way. A student commented that the first way was "more work". Charlie emphasized that he "should like" the first method as well. He also recommended what he called "doctor A's advice" which implied solving any question with unknown elements the way you would solve it if those elements were known. The use of doctor A here seemed to refer to advice on how to deal with A level examinations. Charlie and the class continued by solving the question on the board, at the end Charlie commented: "None of the elements are any more difficult than GCSE; but they are put together in such a way that it is difficult; it is a big challenging A level question. None of the maths techniques is difficult". Charlie then mentioned what he called "doctors A's advice" which implied that solving A level questions was like walking through a dark tunnel that you "cannot see" the end of it, unlike GCSE questions where the "tunnel is straight" and you "see exactly where you can get to". For that reason, it is recommended that the students should do "the most obvious thing" (in this case to "expand and factorise") to solve A-level questions, to "eventually hit light at the end of the tunnel". This showed Charlie trying to create connections between GCSE and A level examination questions (*making connections between resources/ procedures*). Students then continued solving question from the textbook until the end of the lesson.

Using the Documentational Approach

Charlie's resources in episodes 1 to 5 were: interactive white board, iPad, board, curriculum of year 12, textbooks, past teaching-experience, students' contributions, notebooks, Autograph, and the school's scheme of work.

Charlie's scheme of work for lessons 5 and 8 are summarized in Tables 18 and 19 respectively. His specific aim in both lessons was to introduce polynomials including their definition and degree and how to add, subtract and multiply polynomials. His general aim was to prepare students for examination.

His general rules of action included: to check homework; to introduce polynomials; to solve some questions on the board with the help of students; students to work independently on textbook questions; and to answer students' questions on the board. In the first lesson on polynomials, under the general aim "to introduce polynomials" there were specific rules of action: to introduce the lesson topic; to explain the meaning of poly & nomial; to help students discover how many solutions a polynomial can have using Autograph; to define polynomials; to introduce the degree of polynomial and connect it with the number of solutions; to connect polynomials and functions; and to introduce how to add, subtract and multiply polynomials. In the second lesson, the specific rules of action did not include "to help students discover how many solutions a polynomial can have using Autograph" or "to connect polynomials and functions" but included all the rest. Charlie justified not using Autograph in the second episode by the lack of time with the second group G2, as they were "behind". And he did not talk about functions because none of the students in this group asked him about functions, which confirms his use of students' questions as resources to build his *demonstration* on.

In the first lesson, Charlie's operational invariants were general: school and parents expect students will be given homework; after doing homework students come back with questions they cannot do and need help with; Autograph is dynamic; visual and graphical representations should be appreciated by students as well as algebraic ones; polynomials and function are connected and it is good to talk about this connection in year 12.

Scheme			
Aims	*Rules of action*	*Operational invariants*	*Inferences*
Specific: A1. To introduce polynomials: • definitions • polynomial degree • add, subtract and multiply polynomials *General:* A3. To prepare students for exam	*Specific:* R1. Introduce the lesson topic: learning about polynomials R2. Explain the meaning of poly & nomial R3. Help students discover how many solutions a polynomial can have, using Autograph R4. Define polynomials R5. Introduce the degree of polynomial and connect it with the number of solutions R6. Connect polynomials and functions *General:* R7. Check homework R8. Introduce polynomials R9. Introduce how to add, subtract and multiply polynomials R10. Solve some questions on the board with the help of students R11. Enable student to work independently on textbook questions R12. Answer students' questions on the board	*General:* O1. School and parents expect students will be given homework O2. After doing homework, students come back with questions they cannot do and need help with O3. Autograph is dynamic O4. Visual & graphical representations should be appreciated by students, as well as algebraic ones O5. Polynomials & function are connected, students will learn about functions in year 13. So, it is good to talk about this connection in year 12	

Table 18: Charlie's scheme of use in lesson 5

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: *A1.* To introduce polynomials: • definitions • polynomial degree • add, subtract and multiply polynomials *General:* A4. To prepare students for exam	*Specific* R1. Explain the meaning of poly & nomial R2. Give some examples of polynomials R3. Define polynomials R4. Not use Autograph as this class is behind *General* R5. Check homework R6. Introduce polynomials R7. Solve some questions on the board with the help of students R8. Enable student to work independently on textbook questions R9. Answer students' questions on the board	*General* O1. School and parents expect students will be given homework O2. After doing homework, students come back with questions they cannot do and need help with	Many students forget to add c the constant of integration

Table 19: Charlie's scheme of use in lesson 8

In the second lesson, only the first two operational invariants (in relation to the homework) were in action. As Autograph was not used, the operational invariants in relation to it were not in action. The same applied to the operational invariant in relation to functions and their connection with polynomials. The re-scheming between episodes was done due to the context of the second lesson, and looking at schemes-in-action showed the difference between the schemes activated in both lessons and the justification of the differences.

4.4.2 Lesson 9 on Transformations

4.4.2.1 *Episode 6-Introduction to Translation*

This lesson was taught to the five students in GC2. After homework check, Charlie started talking about computer games. He used that to introduce how transformations are used in animation. He asked his students if they wanted to move a function that could be a "Pacman[3]" five units to the left what would they do, a student answered $x-5$. Then, he asked if he wanted the centre of the Pacman to move from $(-2{,}3)$ to $(0{,}0)$, a student responded that x needs to be replaced by $x+2$ and y needs to be replaced by $y-3$. Charlie added that that should be also done to the equation of the circle representing Pacman $x^2+\ y^2=r^2$. Charlie then used Autograph (Figure 57) to show the students how the circle (Pacman) moved around the screen when translated by $(a\ ,b)$. Students' contributions were resources that supported Charlie's teaching, including students' previous knowledge of concepts like the circle. And, Autograph was used as an instructional material (*use of instructional materials*).

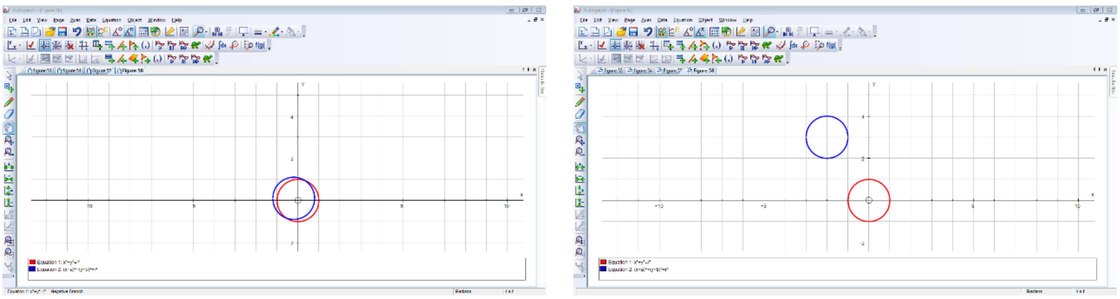


Figure 57: Charlie's work on circle translation on Autograph (reproduction of Charlie's work)

4.4.2.2 *Episode 7- Introduction to Stretch*

Charlie then decided to change the character from Pacman to Mario[6] brothers and asked the students what would happen if Mario ate mushrooms. The students responded that he would

[6] In reference to Mario, the hero character in a series of video games released since the 1980s. See www.mario.nintendo.com

"get bigger". Charlie responded in an attempt to connect video games with the use of transformations and specifically "stretch":

> "He gets bigger. So, let's stretch Mario. Okay, Pacman never got stretched but let's stretch Mario. So, as you well know I am no computer programmer so I can't, I haven't got a Mario for you although I suppose we could do this but I'm not going to. You've seen the Batman equation, right? Have you seen the Batman equation? So, there's a funky equation, bit of a long winded one but we can stretch the Batman symbol. Okay, stretch it out that way stretch it out that way Batman symbol. Have you seen the love heart equation? That's quite nice. Have you seen the love heart equation, no? I'll have to find that for you sometime. There is an equation for a love heart, so there is probably an equation for Mario. But, there's Mario that's the best I can do. So, we are talking about transformation of a curve it doesn't matter what the curve is. It's the principles that are important. So, it might be y= this, it might be what's that wiggly one we did the other day? you know do you remember we looked at a wiggly curve? Could be anything".

Charlie's talk about the batman equation and heart-shape equation was not supported by a *demonstration* of what these were, which could be due to his lack of knowledge of the equations.

Charlie then hand- sketched a "wiggly" curve on the board which had Autograph screen on display (Figure 58). Charlie added:

> "Now, we know how to move this about on the screen so let's stretch it now. Now, has anyone got an elastic band? Have you got one or a hair clip? Yeah, hair band yeah. Sorry, I'm not familiar with these things. Now if you stretch something like an elastic band or a hair band I don't know if you've ever been aware of this but if I keep one end still if I keep that end still and it originally looks like really exciting if it was this side and stretch it in the x direction scale factor two again the point on the y-axis stays the same and it goes that way okay? Anything on that y-axis stays the same the final thing is what if it straddles the y-axis stretches in the x direction scale factor two. The points here don't move but everything else gets doubled, okay. Now, I'll give you some more information about the stretch in the x direction, it's going to involve the following, Okay,

so we're going to look at a stretch in the x direction here is some random curve doesn't matter what it is okay. Any old random curve I don't know what the equation is for that; but I don't really care [...] Now we will stretch it in that direction; but what does that mean? We'll take some points on whatever the curve is if we stretch it by scale factor say two in the x direction. That simply means this start at the line x=0 which is all the y-axis here, measure the x distance from there to the point that you are thinking of stretch scale factor to the x direction, means we just double that distance and that distance was there […] so a stretch is quite specific thing, probably something you're familiar with I can do this for all the points".

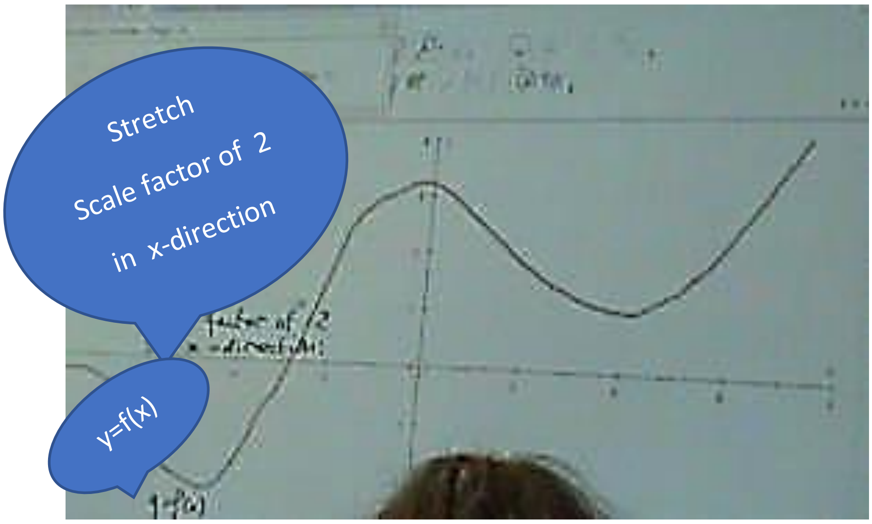


Figure 58: A hand-sketch by Charlie on the board with Autograph in the background, with bubbles added to clarify what he wrote next to the graph

Charlie was demonstrating how to do the stretch by hand for some of the curve's points (one point at a time) and in his *demonstration* asking for students' contribution to answer questions about length, distance and scale factor. He was also *making connections between representations and procedures* when he said:

"Now, graphically how do we get the stretch? How do we get Mario to double in height when he eats the mushroom? So, algebraically how do we do this? And do you know, are you familiar with this algebraically? Translation; replace x with $x - k$ and y with $y - k$. How do we do the stretch and replace what with what?".

To answer the question and demonstrate to the students, he decided to replace x by $2x$ and do it "free hand" and see what would happen. Charlie commented that the result was not what the students expected: "It seems to have done a stretch, but it hasn't changed the height at all and it looks like to me that this point has come in here by what fraction? Probably a half I don't know". Charlie checked for some points and commented that it "looks like all those distances there look like they've been halved". To give further examples about stretch factor, Charlie decided to "play around" and try to replace x by ½x this time. He commented that the distances seemed to have "doubled". He then he decided to consider k and $1/k$ and asked the student about the "rule", which he confirmed was:

> "Excellent, so what is our rule for a stretch in the x direction if we want to. If we want to stretch anywhere in the x direction I've put a little star there because we need to clarify what that actually means by a scale factor of two, three, four, five or K then student I what do we do?"

The student responded that they would multiply by "one over the scale factor". Charlie approved:

> "One over scale factor that's a nice way to describe it. Then, we replace, this is how I think of it but I'll put yours in there as well we replace x with x/k, that is actually how I think of it but you actually said possibly something more helpful you said then we multiply the x term by $1/k$, good stuff."

Charlie continued *demonstration* and wrote the "rule" for a stretch in the y direction: "Replace y with y/k, or if you prefer multiply the y term that's quite a nice design I quite like that…", he meant here multiply y by $1/k$. Thus, Charlie created a connection (i.e. $x/k = 1/k\ x$). After that, Charlie solved two examples on the board in which he applied the "rules" about transformations.

In a comment about teaching transformations Charlie reflected on his previous experience of teaching this topic:

> "I really don't like teaching that because I know having taught it for years students really struggle. And when you've got it, it's the simplest maths in the world. There is no maths to do, but they struggle with it at GCSE. They see it in year 12, see it in year 13, they hate it every single year"

Claiming that "there is no maths to do" in this chapter reflected how Charlie felt that mathematics was linked with difficulty and that the easiness of this chapter implied that there was "no maths" in it. His previous experience in teaching this topic was a resource that Charlie used to make *decisions about sequencing* for the lessons on transformations. So, he would introduce transformations to the students, then would dedicate a lesson for students to practise solving questions on transformations independently:

> "So, the next lesson I don't intend to say anything really […] It's all old school stuff. I remember when the first video games came out and it was literally a pixel going across the screen. And, that subject is so abstract and really dull. I always pull it back to those computer games and I don't think anyone is any more interested in it".

When asked about his choice to sketch the graph by hand, although Autograph was available and on, he justified that by saying:

> "I want to focus on the idea rather than that equation. And, also having said all of this when I get presented with a new function and very often in the exam they are just given y=f(x) and they will have to manipulate y=f(x) so they do a stretch in the x direction they need to know that line doesn't change but this one if it's a scale factor of half what's the distance to the y axis. Its new position will be there, so again that is modelling kind of what they will need to do in the exam as well".

Charlie's focus on the idea here seemed a *concentration on procedure*s. It reflected his attention to examination requirement which was also mentioned in the interview:

> "There are two possibly competing aspirations for a maths lesson, one is for me just to share the experience the joy and the interest and the deductive reasoning and having time to do fun activities and things I love to do all that and if I knew how to find the time to do all that I would much rather do that than any exam. Okay, and in fact the whole school says we're more than exams. I also know which side my bread is buttered and for me I know that I have to deliver outstanding results and we have done that. So, I think summer just gone before we had 85% A* a level and we had other […] maths departments coming to see me to say well how are your results so good. So, that definitely has a positive effect and I want to keep that. But, there is a

> tension between what I would like to do which is maybe start an A level lesson with an investigation bit of you know, it could be IT led, it could be classic sort of puzzles that you rearrange on the table. But, I don't know how to put all of those in and still cover all the material that I want to cover without having lots more lessons and I don't get more lessons. I put more bids in for more lessons; but I don't get them and I understand that. So, at the moment I run the model just the best way I can which is there is a discrepancy between the two. What is going to give, and for me and I am fully prepared to stand up and say I'm doing this wrong. But, I think the exam results and the performance really are why the students have chosen the subject they want to get to University. The parents want outstanding bragging rights for how their students have got on. And, yeah if the head said to me look how about we prepare to reduce your maybe attainment at the expense of greater enrichment; if it comes from the head I'll do that. Whilst it's not coming from the head or the director of studies, then I will stick with this model because I know it produces something but at A level the content is so heavy I personally can't see how to get through it otherwise"

This reflected the tension between Charlie's aim to deliver outstanding examination results and his aspiration to share the joy of mathematics with his students. Examination results took priority in this case, and, based on that, Charlie balanced his use of resources.

Using the Documentational Approach

In episodes 6 and 7, Charlie's resources included a rubber band and computer games, as well as the resources he used in previous episodes. Charlie's scheme of work for this lesson is summarized in Table 20. His specific aim was to introduce transformations to his students (translation and stretch for this lesson). His general aim was to prepare students for examination. His specific rules of action were: to connect transformations with computer games (Pacman and Mario); to demonstrate and introduce the rule for translation; to demonstrate and introduce the rule for stretch. These were under the general rule of action: to introduce transformations quickly. Other general rules of action include: solving some examples on the board with the help of students, and for students to work independently on textbook questions in the next lesson. The last general rule of action was justified by the specific operational invariant: students do not like transformations, introduce it quickly and next lesson is for students' practice. This operational invariant was an inference from previous teaching

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A1. To introduce transformations: ○ translation ○ stretch ○ reflection *General:* A2. To prepare students for exam	*Specific:* R1. Connect transformations with computer games (Pacman and Mario) R2. Demonstrate and introduce the rule for translation R3. Demonstrate and introduce the rule for enlargement R4. Demonstrate and introduce the rule for stretch *General:* R5. Demonstrate transformation rules on the board R6. Solve some examples on the board with the help of students R7. Enable students to work independently on textbook questions in the next lesson	*General:* O1. Students do not like transformations, it is better to introduce it quickly and leave the next lesson for students' practice	

Table 20: Charlie's scheme of work during his eighth lesson on transformations

experience that made Charlie feel that students do not like this topic; and it was best to give a quick introduction to transformations and then leave students to practice independently. More general operational invariant were identified in the interview including: Charlie's personal appreciation of the value of enjoying mathematics and enriching its teaching; his considerations of time pressure and curriculum guidelines; his commitment to working in line with the expectations of parents and head of school, which meant here a commitment to delivering outstanding examinations results; his belief that having more mathematics lessons per week would help him resolve the tension between teaching students mathematics that they would enjoy and teaching to deliver excellent examinations results; and his understanding that although his school expressed that they were not teaching just for examinations they still needed to deliver excellent results.

4.4.3 Episode 8- From lesson 3

In this lesson, Charlie had only one year-12 student. He should have had four students, but three of them were absent. The teacher decided to go on and teach because he was using a software called Explain Everything, which worked as an online whiteboard and that could be shared with his students (use of resources). This, in his view, would allow him to send his students a summary of what was taught in that lesson.

The teacher was covering another teacher and was not sure what he was going to teach, neither was the student aware of what she was taught the lesson before. However, with some discussion the student could remember the chapter she was doing with the other teacher in the previous lesson. So, the topic was decided to be on the equations of tangent and normal. The student was asked to pick an exercise from the textbook that she found challenging (students' contributions as a resource), and so she did. The teacher started telling her about what he called a "MANTRA" which was " $y - y_1 = m(x - x_1)$ [he usually said it with a special voice tone]". He added: "You need to know two things which are a point and the slope which you can usually get from differentiating the equation of the line". The student then had to practice solving another question on her own. Afterwards, she had a question about the equation of the normal. At that point, Charlie tried to use Desmos to explain the idea to her, but he could not because he was not familiar with the software (use of resources, knowledge of the software). He then decided to use Autograph (use of resources) and showed the relation between the tangent and normal using a quadratic function $y = x^2$ (Figures 59) and then trigonometric function $y = sinx$ (Figure 60); and commenting that it was "like a sailing boat" (*making connections between concepts*). Charlie kept hinting and asking about how the slopes of the tangent and normal lines would be related. He attempted hinting to the steepness of the two lines in Figure 60-while he was dragging these on Autograph; and asking which was steeper (the tangent or normal line) at different positions on the graphs and whether that would mean a greater or smaller slope. But, the student did not spot the relation he wanted her to reach between the slopes of the tangent and normal lines of a graph. So, Charlie's attempt to demonstrate the connection between tangent and normal of a line by making connections between resources (Autograph and student contributions as a resource), was not successful. This created a contingent moment that resulted from the unavailability of resources (student contributions here). Charlie appeared *responding to the unavailability of resources* in this moment by going back to telling the student explicitly about the connection between tangent and normal. He said (referring to Figure 60):

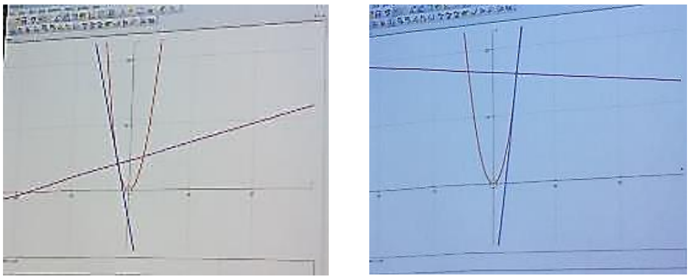


Figure 59: Charlie's display on the connection between tangent and normal of $y = x^2$

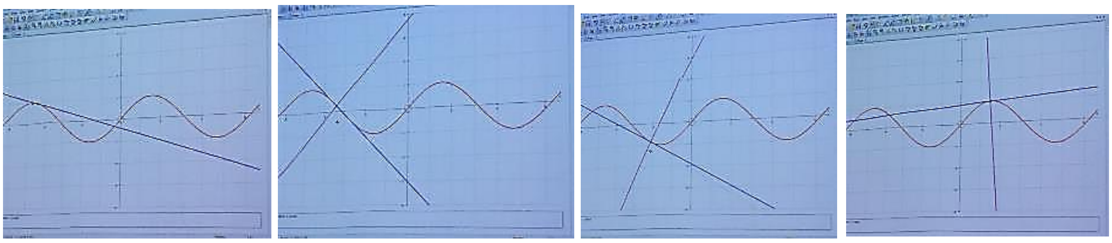


Figure 60: Charlie's display on the connection between tangent and normal of $y = sinx$

> "As I drag this point, you can see the blue points [on the tangent line]… the purply [the normal line] is always 90 degrees to it… Have you seen that? […] Imagine the blue line is just horizontal so the purple line is?"

Charlie then asked the student if she knew how to work out the gradient of a function and explained that when finding the gradient, one cannot divide by zero. He then stopped using Autograph and said:

> "Right, that was a bit getting off the track, what is the relationship connection between the gradient and tangent and the gradient of the normal?... always at 90 degrees […] They are always

perpendicular to each other. And what does that mean if I give you a gradient of one of them? […] What if the gradient of the tangent was a fraction h/k?"

Student: -1, oh $-k/h$

Charlie: Fantastic, and I don't know why k and h, I remember a million years ago my teacher used k/h so I got stuck with those … this is the most useful thing about gradient. Two perpendicular gradients multiply to get, take these two they multiply to?

Student: -1

Charlie: The other two?

Student: -1

Charlie: Good, flip it and change the sign. Shall we try a question now? Find the equation of the normal; and at that point your brain should be going $y - y1 =$…

Charlie's moved from using graphs to show the connection between tangents and normal (*making connections between concepts*) to using algebraic formulae (*making connections between procedures*), including the use of a prescribed method i.e. "flip it and change the sign", with a connection between the two representation through his reference to the slope and steepness on a line (*making connections between representations*). This was a safe zone for Charlie to move to, after the contingent moment that resulted from the use of technology.

Using the Documentational Approach

Charlie did not have a pre-prepared scheme of work. So, he devised a scheme of work (summarized in Table 21) during the lesson based on his previous experience. His resources included: Explain Everything, students' contributions, Autograph, Desmos and the textbook. His aim was to teach about tangent and normal lines. His rules of action included: inviting the student to choose a textbook exercise that she found challenging to be solved on the board; using an activity on a software to explain the connection between normal and tangent lines of a graph; explaining the connection between tangents and normal algebraically; then, asking the student to work independently. His operational invariants included: students can follow what they missed by looking at Explain Everything summary; students would remember better if the "MANTRA" is said in a special tone during the lesson. One inference could be in relation to the use of software; it seemed that Charlie felt more comfortable using Autograph than Desmos for this lesson.

Scheme			
Aims	***Rules of action***	***Operational invariants***	***Inferences***
Specific: A1. To explain about normal and tangent lines *General:* A2. To prepare students for exam	*Specific:* R1. Invite students to choose a textbook exercise that she found challenging to be solved on the board R2. Use an activity on a software to explain the connection between normal and tangent lines of a graph *General:* R3. Solve some examples on the board with the help of the student R4. Allow student to work independently on textbook questions in the next lesson R5. Explain the connection between tangents and normal algebraically R6. Ask the student to work independently.	*General:* O1. Students can follow what they missed by looking at Explain Everything summary O2. Students would remember better if the “MANTRA” is said in a special tone during the lesson.	I1. To use Autograph rather than Desmos

Table 21: Charlie’s scheme of work during his third lesson on tangent and normal lines

4.4.4 Discussion and Summary

In the episodes above, we see how Charlie's resources included the interactive whiteboard, board, Autograph, Desmos, Explain Everything, iPad, textbook, homework policy, curriculum, examination requirements, past teaching experience, notebooks, and the school's scheme of work for mathematics teaching. Having examinations requirements and school's and parents' expectations about homework as resources, reflected Charlie's attention to examination and school policy. His attention to examination requirements was reflected in his comments about how A-level questions are formed and their connection with GCSE questions, through doctor A's advice. His commitments to institutional policies and factors were evident during the interview when he expressed operational invariant including: appreciation of the value of enjoying mathematics and enriching its teaching; considerations of time pressure and curriculum guidelines; commitment to working in line with the expectations of parents and head of school; his belief that having more mathematics lessons per week would help him resolve the tension between teaching students mathematics that they would enjoy and teaching to deliver excellent examinations results; and his understanding that although his school expressed that they were not teaching just for examinations they still needed to deliver excellent results. The code "adhere to examination requirement" does not come up in Charlie's data as under *Contingency*. But his commitment to examination requirements was apparent throughout his teaching, for example, in his use of hand sketching in the third episode.

Charlie used students' contributions as resources in two ways: first to answer questions he prepared or from the textbook; and, second to demonstrate ideas (like with the polynomials). The first way, implied that Charlie's *demonstration* was to some extent dependent on students' contributions. For example, the student question about the function notation in the first episode initiated a difference in the scheme between the first and the second episodes. The second way in relation to demonstrating ideas was within Charlie's comfort zone and under his control. Charlie's use of students' contributions as resources did not reflect confidence in having higher chances of *Contingency*. His way seemed closer to Adam's; while George seemed more open to work live with his students in different and several scenarios. However, Charlie had to work live in his last episode where the scheme of work was very dynamic because it was not pre-prepared and relied heavily on students' contributions.

Charlie also connected between resources in a way that helped him demonstrate ideas. For example, he tried to connect stretching a hair band with the activity on stretch he did on the board and with the algebraic "rule" in relation to stretch. The episodes also showed how he was *making connections between representations* (algebraic and graphic in the episodes on transformation). In the eighth episode, Charlie tried to create connection between student contributions and his work on Autograph but that did not work and created a moment of *Contingency* where student contribution was not available.

Charlie's knowledge of the software he used (Autograph) was essential for him to create such connections and to demonstrate ideas, like what he did in the first episode on polynomials. However, the connection Charlie tried to create during the first episode on polynomials between the intersection points of two equations on Autograph and the degree of polynomial was superficial and confusing. His knowledge of Autograph use in relation to the topic of polynomial did not seem well-established. This could explain why he did not use the same activity in episode 2 on polynomials, and did not remember it in the interview. The third episode showed Charlie using the pen to sketch a graph and stretch it on the board (Figure 58), in order to demonstrate about stretch to the students. In the eighth episode, Charlie did not know how to use Desmos and his use of Autograph was not well-integrated within the lesson and led to a contingent moment. From the interview, it seemed that Charlie appreciated hand sketching because it was the way the students would use in the examination. In general, Charlie used Autograph (*use of instructional materials*) to help students visualize graphs and supported the *demonstration* (like in the episode on transformation). It was not used to create discussions or initiate conversations with the students.

The three episodes above reflected Charlie's subject knowledge and knowledge of the software (Autograph here). They also showed him using students' contributions and creating connections between resources and reflected the influence of examination requirements on Charlie's teaching. One contingent moment was reported in these episodes (in episode 8). That moment resulted from the unavailability of resources (student's contribution). Charlie's commitment to a specific teaching style that is highly dependent on the textbook and on what he described as a "repertoire of resources" that he established over the years of his teaching reduced the likelihood of contingent moments in his lessons.

5 Findings and Conclusion

In this chapter, I first focus on the findings and on answering the research questions of this study. Then I overview this study's contribution to theory, its limitations, future research, and implications for policy and practice. I also summarize my personal reflections on conducting this study.

5.1 Overview of findings

This section draws a view based on the five themes addressed throughout the analysis of the data from the four participants: students' contributions as resources; examination requirements; connections between resources; knowledge of the software; contingent moments. I then aim to answer the research questions of this study. First, I include a general discussion of the five points based on the differences and similarities among the four teachers, in relations to the five points. Some of the similarities and differences have been mentioned briefly in the individual teacher's data discussion; here all of them are summarized within the next five subsections.

5.1.1 Students' contributions as a resource

Students' contributions seemed to represent an important part of a teacher's resources. The Documentational Approach lens facilitated seeing these contributions, which are part of the social interactions, as a resource (Gueudet & Trouche, 2009) that had a great impact on the schemes of work the teachers followed. Data from the four participants showed evidence of this use. Each of the four participants followed an approach in which he invited students to participate in a way that supported his plan for the lesson, but the teachers' approaches to using students' contributions as resources had some differences. George invited them to participate by giving him answers that he expected and used to demonstrate concepts and create discussion in the class (in episode 6 on iteration, for example, he asked for the different rearrangement of one equation). He also invited such contributions to answer practice questions on the board. Martin's use of students' contributions was different; he did not use these to create discussions. Instead, he only invited students to contribute after he finished his usually brief *demonstration* and gave the students the time to practice working independently on some exercises. Then, he would listen to the students' questions, and he would use these questions to decide what ideas to demonstrate on the board; what to demonstrate to individual students or groups, and what did not need further demonstration. Adam, like George, used students' contributions as resources to support his *demonstration* or to answer exercises on the board. But, in the data I observed, Adam's use of students' contribution to support his *demonstration* was less frequent than his use of those contributions to support answering exercises on the board (two out of the six lessons included using students' contributions to support the *demonstration* of mathematical ideas, compared to six out of the six lessons that included the use of students' contributions for solving exercises on the board). Charlie sometimes used students' contributions to support his demonstration; he also used them to help solve exercises on the board. Like Martin, he used students' contributions, specifically questions here, to find out which ideas needed further explanation from the teacher. I noted that the use of students' contributions as resources was directed to working on the applications and/or the meaning of mathematical ideas to different degrees among the teachers. George's use of these resources tried to balance between considering both the introduction of new mathematical ideas and their meaning, and their application to solve exercises. Martin's was focused on the application. Charlie's and Adam's focused on both giving higher priority to applications in exercises compared to George, within the lessons included in this study.

Based on the differences addressed in the previous paragraph, these contributions were in three forms: contributions invited by the teacher to help solve questions on the board; contributions invited by the teacher to support *demonstration* of new ideas; students' questions on a new concept that would lead the teacher to demonstrate ideas to one student or to the class. In what follows I summarize what each of these forms include.

- *Contributions invited by the teacher to help solve questions on the board*

The four teachers used these contributions to solve questions on the board for practice, and one can see instances of that in almost all episodes. Whenever any of the four participants was solving a question on the board, he asked for students' contributions to help with the work. These resources give the teacher indications about the students' progress. Inviting students to contribute to the solving process is a way to keep the students' engaged; assess their progress; and, give them feedback.

- *Contributions invited by the teacher to support demonstration of new ideas*

This includes the use of students' contributions to demonstrate ideas and their meaning to the class, instead of relying on *demonstration* done solely by the teacher. It seemed that using students' contributions to demonstrate new ideas was less common in the data than the other two forms. We see instances of it in some of George's work, in his lesson on trigonometry and in his episode on iteration (episode 6). We also see that in Adam's first and second episodes. Also, Charlie tried to use his student contribution in his eighth episode to demonstrate the connection between the normal and tangent equations, based on his work on Autograph. Charlie's attempt was not successful as the student did not reach the conclusion he wanted her to reach (i.e. the normal and tangent lines are perpendicular).

- *Students' questions on new concept that would lead the teacher to demonstrate ideas to one student or to the class*

In some of the episodes, the teacher would try to minimize his *demonstration* time, by giving a brief introduction to the concept (or just the procedure) and then asking the students to practice solving exercises independently from the teacher. The teacher would then explain the ideas the students' ask about during their independent work. This was the case in the second participant's (Martin's) lesson. Also, Charlie followed a similar strategy in episode 5 when he explained what polynomials are, and then asked his students to work on the textbook questions and tell him which questions they wanted to see on the board. Charlie completed the questions the students felt they needed help with on the board. This form of contribution gives the lesson a dynamic flow that is dependent on what students ask about. But, this dynamic flow is usually within the teachers' range of expected scenarios.

5.1.2 Examination requirements and other institutional factors

Also, in relation to resources, institutional factors, and specifically examination policy and school policy had great contribution to teaching approaches. One can see how examination requirements and past-examination papers were resources that the four teachers used. The teachers even expressed the importance of these during the interviews. All four teachers considered requirements as important. They all used past-examination papers as resources. Some of them commented even on the examination's mark scheme. For example, when Adam (in his second episode) justified the need to add c, the constant of integration, by mentioning that those who do would lose one mark in the examination. The use of past-examination papers was because the students encouraged that, according to George and Charlie, to get some practice for the examination. Examination requirements had impact on the teachers' decisions on what to explain (or not explain) in their lessons based on examination requirements. Due to their commitment to these requirements, teachers' knowledge seemed to some extent bounded by these. This led to contingent moments in which the teachers backed away from answering questions because they were not required in the examination and they did not know how to answer them. One example of this is in George's seventh episode when a student asked him a question he could not answer, and he said it was not required in the examination. A second example is in Martin's lesson on the volume of revolution, when he did not create a connection between the activity he showed on GeoGebra and the formula for volume of revolution. His justification of that was based on that the work on GeoGebra was not required in the examination. There is a difference between the influence of examination requirements in the two examples. In the first, George was trying to address connections between ideas that were not required in the examination (e.g. rotation on Autograph and the formula of volume of revolution). However, he did not know the answer to the student's question, so he used examination requirements as the reason not to answer that question. While in Martin's case, Martin was pre-decided that such a connection (between the rotation on GeoGebra and the formula of volume of revolution) was not important because it was not required in the examination. So, he did not even try to address it when he was questioned by one student about this connection. Martin's use of technology was focused on displaying a shape (circle, then trapezium) and their rotations and playing these in the background while the students were working on applying the formula to solve exercises from the textbook.

5.1.3 Teachers' knowledge of the software

This is important specifically in relation to the taught content, and it increases the teacher's confidence in using software in practice. Such knowledge is what made George do the activity on trigonometry in his characteristic lesson. Due to the lack of such knowledge of Desmos, Charlie could not use Desmos to do his activity in episode 8 on the normal and tangent lines. While he could do the same activity on Autograph which he was familiar with. I note that teachers' knowledge of the software varied among the four participants. George seemed the most confident when using Autograph within the range topics he introduced during this study. Martin was not familiar with Autograph, he said he was familiar with GeoGebra. However, his use of GeoGebra seemed procedural and did not support the construction of mathematical meaning. For example, when he used GeoGebra in the volume of revolution lesson, his work on GeoGebra seemed to the students disconnected from the formula they were learning. Adam was familiar with Autograph and he used it to support his *demonstration* about integration. But, he said that when he used Autograph in other lesson, his use was to graph functions only. Charlie tried to use Desmos once, but he did not manage to. Then, he said he was not familiar with it. So, he moved to use Autograph. This use was for graphing functions mainly. Charlie's knowledge of Autograph did not seem to support the integration of the software affordances in the demonstration. For example, in episode 7, with Autograph on the board he hand-sketched on a function on the board and transformed some of its points one by one by hand. Also, in his first lesson on polynomials, his use of Autograph seemed tentative and was not well integrated within the lesson.

5.1.4 Connections between resources

These are the connections a teacher creates or tries to create between some of the resources available to him during a lesson in order to achieve a specific purpose. The purpose is usually to support the introduction of new concepts or procedures. From the data, I note that all four participants tried to create such connections. George's connections between resources seemed well-thought and balanced in some of his lesson (the lesson on trigonometry for example), but not so well-thought in others (in episode 2 on volume of revolution for example). For the other three participants, these connections were less frequent and mostly to explain the application of a procedure or formula in exercises. For example, Martin did not create connections between the volume of revolution activity on GeoGebra and the students' work on the volume of revolution formula on the worksheet he prepared.

The data showed that connections between resources could be crucial for the flow of the lesson, and if they are not created, or vaguely created, the flow of the lesson can be affected. For example, having not created connection between resources (prepared task and GeoGebra activity), Martin tried to focus on the formula of volume of revolution and its application in exercises. This confused the students who questioned this connection but did not elicit a response. In contrast, the connections created in George's lesson on Trigonometry, when he connected students' contributions with Autograph helped the students deduce the trigonometric formula under scrutiny. George also connected between resources (students' contributions, Autograph) in his sixth episode on iteration in section 4.1.5. Also, he connected between resources (Autograph, textbook, previous experience of teaching the same topic, formulae) in the episodes on volume of revolution. Adam's connection in the differentiation lesson (between students' contribution and Autograph) is another example that supported students' learning (about integration in this case).

5.1.5 Contingent moments

All four teachers had moments of *Contingency*. Some moments were tricky to identify as contingent or not, due to the expectancy element not being confirmed by the teacher. An example of this is in George's trigonometry lesson, when he was asked about having a tangent in an exercise, instead of sine or cosine. Among the four teachers, Charlie was the one who would have fewer contingent moments because he was following a specific and well-controlled teaching approach during his lessons. The only contingent moment noticed in his lessons was due to his attempt to use Autograph to demonstrate about the equations of tangents and normals. His reaction was to go back to his telling and prescription of methods approach with which he felt more secure. On the other hand, George's openness to use students' contributions as resources at different stages of the lesson lead to more contingent moments. The latter case reflected confidence in working live with students, even with spontaneous examples. Martin tried to avoid contingent moments by giving a limited and brief demonstration, mostly based on the applications, and then, leaving the students to do the work on exercises, during this work he would answer students' question individually first. This meant that he would have the chance to decide whether the question would be explained on the board or not. Adam showed more confidence with demonstrating for longer and discussing questions on the board. However, he did not reflect as much confidence as George to work live with students. I noticed that some contingent moments were the result of students asking questions that were not included in the plan, and some resulted from not knowing how to use the software for the teaching of a specific idea. Based on that, it can be argued that contingent moments may result from students asking questions that are not included in the examinations and/or curriculum, or not knowing how to use a mathematics-education software for the teaching of a specific content.

5.2 Answering the research question

The five points above has enabled me to answer the research questions of this study as detailed below. The answer to the main research question is done through answering two sub-questions:

RQ: How do secondary mathematics teachers work with resources including technology in their classroom?

i. What are the teachers' resource systems and schemes of work?
ii. How do they enact their schemes of work in class?

The two sub-questions were identified having in mind the theoretical framework of this study: the Documentational Approach and Knowledge Quartet. These two lenses afforded a broader view on teaching approaches by focusing on the more general aspects (i.e. resources, scheme of work) as well as the detailed actions (i.e. *Foundation*, *Transformation*, *Connection* and *Contingency*). They facilitated answering my main research question, by offering me the tools to form and answer the two sub-questions.

In response to the first sub-question, this study offered views of teachers' schemes of work during lessons and their reflections on these in post-observation interviews. It afforded looking at all aspects of schemes: resources, rules of actions, operational invariants and inferences. The study highlighted the resources, used by the participants. These resources included both paper-based and digital resources including for example a textbook and mathematics-education software. Common resources between the four participants include students' contributions to the lessons, examination requirements and past-examination papers. Three of the four participants used the mathematics-education software Autograph, while one used GeoGebra although he had access to Autograph as well. The rules of actions the teachers had in their schemes always included giving the students' time to practice solving exercises independently. Two of the teachers (Martin and Charlie) used this time for independent practice to get questions from the students and demonstrate ideas further based on these questions. In the four cases, students' contributions added a dynamic factor to the lesson's rules of action. In terms of the operational invariants, one common operational invariant between the four participants was related to their appreciation of the software affordances. However, this appreciation did not always lead to well-integrated use of the software which can be attributed to the teachers' knowledge of the software in relation to the mathematical content they taught in a specific lesson. Another common operational invariant is about connecting new mathematical concepts with previous knowledge. Inferences varied depending on the topic and did not always occur.

In response to the second sub-question, I will summarize here some aspects in relation to how the teachers enacted their schemes of work at upper secondary level, and specifically when digital resources were used along with other resources. The data showed the teachers using students' contributions as a resource along with other resources to demonstrate ideas; to support the application of procedures and formulae; or to find out what needed further explanation by the teacher. This implies that in the context of upper secondary mathematics teaching, the code "*use of instructional materials*" under the *Transformation* dimension, could be extended to "use of resources" in order to shed light on students' contributions as a resource. The study showed that teachers' knowledge included knowledge of examination requirements and school policy as well as knowledge of any software used for teaching mathematics. So, this encouraged the addition of two new codes to the *Foundation* dimension "knowledge of examination requirement and school policy" and "knowledge of the software". The study also offered a view on how teachers connected between different resources including students' contributions, to make their lessons more comprehensible to the students. This led me to suggesting the code "connection between resources" to be added under the *Connection* dimension. Also, the contingent moments identified in this study included some that resulted from the teacher being exposed to question that are not required for the examination, or from the teacher not knowing how to use the mathematics-education software for a specific purpose dependent on the lesson context. This in turn inspired the possibility of adding one code to the *Contingency* dimension “adhering to examination requirements”. Generally, in order to enact their schemes of work, the teachers were trying to balance their resources and to prioritize all the time. The balance or unbalance was evident when mathematics-education was used along with other resources. In such cases, the teachers needed to make decisions about how much time they devoted to working on activities on the software when the activities were not directly included in the examination. Similar issues related to time pressure were reported in Bretscher (2014). The balance was also dependent on students’ contributions, which meant it was dynamic and changing. This dynamic nature inspired the definition of a “scheme-in-action” as the scheme that is activated and applied during a lesson, and is usually dependent on the context and the available resources. Teachers also needed to change schemes from one lesson to another on one topic, due to inferences in one lesson or to other institutional factors. Changing or amending a scheme from one lesson to another is called here “re-scheming” whether it is done due to inferences, institutional constraints or even new resources that become available to the teacher.

In summary, the overview of the five points revealed some aspects of teachers practices and knowledge-in-action, which in turn inspired some theoretical findings. These theoretical findings are discussed in the following section.

5.3 Contribution to theory

The use of the two lenses (the Documentational Approach and the Knowledge Quartet) in this work highlighted the following aspects in relation to the theories used: suggestions of new codes for the Knowledge Quartet; the definition of scheme-in-action; and the definition of re-scheming. In this section, I outline these three aspects and connect them to theoretical frameworks and constructs, I then discuss how the use of the two lenses facilitated these findings and enriched the analysis.

5.3.1 Suggestion of new codes to the Knowledge Quartet

The discussion of the main five themes in Table 7 inspired some suggestions in relation to the codes of the Knowledge Quartet in the context of using technology for secondary mathematics teaching. New codes or extensions of existing codes are suggested below, this resulted from the context of the data and from looking at the data using the Documentational Approach as well as the Knowledge Quartet. First, this study included data in which technology was used, this facilitated viewing additional issues related to the use of technology for teaching secondary school mathematics. In relation to the teachers' knowledge, this encouraged the consideration of TPCK by (Mishra & Koehler, 2006; Shulman, 1986), instead of Shulman's (Mishra & Koehler, 2006; Shulman, 1986) categorization of knowledge that was originally considered by Rowland et al. (2009). Second, the suggestions resulted from the use of the Documentational Approach which afforded a broader view on resources that included social and institutional resources (Gueudet & Trouche, 2009). Although adding new codes to the Knowledge Quartet might overload it (by increasing the number of codes from 22 to 26), I find these codes crucial in the context of this study. The suggestions in relation to the codes are:

- *Use of resources*

The use of students' contributions as resources in most of the episodes, supports the "use of resources" as a possible code under the Knowledge Quartet instead of "*use of instructional materials*" under *Transformation* dimension. This is not a completely new code, but it extends an existing code. "Use of resources" affords looking at teachers' interactions with students which is a significant aspect of teaching. I envisage that it would, in future research, facilitate the inclusion of other social interactions as resources; such as conversations with other teachers or teacher assistants.

- *Connections between resources*

Rowland et al. (2009) suggest that the connection dimension reflects essential connections that are required to achieve progression in learning. In the context of this study, 'connection between resources' seems to be one of these essential connections. So, I tentatively suggest it as a code under the *Connection* dimension. 'Connections between resources' concerns the connections a teacher creates or tries to create between different resources available to her/him in order to achieve her/his teaching aims during a lesson. Those aims can be about supporting the introduction of mathematical meanings, or about their applications. Connections between resources can affect the delivery of the teaching aims and the flow of the lesson, as the flow of the lesson can be disturbed if such connections are not well-created (as noticed in Martin's lesson).

- *Adhering to examination requirements*

This is a code suggested under the Knowledge Quartet *Contingency* dimension, to help analyze moments in which the teachers cannot answer students' questions and justify that by the fact that the questions asked are not required in the examinations. An example of this was in George's data, in the episode 7 which was on volume of revolution. Another example was in Martin's lesson, when Martin did not connect his use of GeoGebra with the formula of volume of revolution, because GeoGebra use was not required in the examination. Jaworski and Potari (2009) suggest that when alerted to classroom tensions, teachers tend to draw “attention to the so-called unchangeable factors which include school and educational systems” (p.234), including examinations systems. Such attention or adherence can be more visible with classes closer to taking external examinations (Ruthven et al., 2008), such as the Year 13 classes presented in the data. “In those cases involving older classes closer to external examinations, the emphasis was on avoiding student work with the software or making it manageable, and on structuring dynamic geometry use more directly towards standard mathematical tasks” (Ruthven et al., 2008, p. 315).

- *Knowledge of examination requirements and school policy (or knowledge of curricular and institutional policies)*

Examination requirements is one aspect of teachers’ knowledge, it helps teachers prepare students for the examination by addressing how examination questions are phrased and formed. These include “grading and testing practices” (Engeström, 1998, p. 76). From the episodes in this study, sharing this aspect of knowledge with the students was done by all the four participants who also reflected on it as a fundamental practice that is even required by students. School policy is also one institutional aspect that affects teachers’ choices (Bretscher, 2014; Jaworski & Potari, 2009). For example, in relation to the homework in Charlie’s case and in relation to the school website in Martin’s case. Both examination requirements and school policy are in the seven categories of Shulman’s knowledge base: examination requirements as part of curricular knowledge; and school policy as part of knowledge of educational contexts (Shulman, 1987). Jaworski and Potari (2009) point to teachers’ tendencies to consider and draw on the different elements of the educational and institutional systems. Engeström (1998) referred to:

> "grading and testing practices, patterning and punctuation of time, uses (not contents) of textbooks, bounding and use of the physical space, grouping of students, patterns of discipline and control, connections to the world outside the school, and interaction among teachers as well as between teachers and parents." (p.76)

He considered these important factors that impact on classroom practices, and on making sense of these practices. Supported by these theoretical arguments and the evidence from the data, I propose “Knowledge of examination requirements and school policy” as a code under the *Foundation* dimension.

- *Knowledge of software*

This is an essential aspect that facilitates the use of mathematics-education software for mathematics teaching. This aspect is not included in Shulman's categorization of knowledge, which was the basis of the knowledge Quartet codes. But, it is an important aspect of Mishra and Koehler (2006) TPCK, which informs the addition of the "knowledge of software" code. Mishra and Koehler (2006) argue for the need of such knowledge and add that "thoughtful pedagogical uses of technology require the development of a complex, multifaceted, and situated nature of this knowledge" (p.1017).This knowledge is in-action in one lesson in relation to the taught content or the topic of the that specific lesson, where "content, pedagogy and technology" interplay (Mishra & Koehler, 2006, p. 1017). The episodes in this study showed cases of when teachers knew how to use a software for specific content, for example George and Adam had some knowledge about how to use Autograph to talk about integration (specifically the constant of integration c). Charlie did not know how to use Desmos in episode 8 (about normal and tangent lines), but he knew how to use Autograph for that purpose. However, even his use of Autograph was not fully integrated within the lesson and the student did not seem to reach the conclusion he aimed for her to reach regarding the connection between tangent and normal equations. These examples show how digital resources: add to the complexity of teaching (Fuglestad et al., 2010); can lead to hiccups (Clark-Wilson & Noss, 2015) or contingent moments (Rowland et al., 2015); and interplay with teachers' professional knowledge (Gueudet & Trouche, 2009). Hence, I suggest that the analysis of this aspect of teachers' knowledge can be supported by adding the code "knowledge of software" under the *Foundation* dimension.

5.3.2 Scheme-in-action

The use of the Knowledge Quartet in this study afforded looking at the details of the teachers work in class. This included a view of their "knowledge-in-action" (Rowland et al., 2005, p. 261), which led to a detailed consideration of what the teachers did in class and why they did it, and how they interacted with their students. The combination of such a view with the view through the Documentational Approach lens, afforded looking at "schemes-in-action". A clear example of a scheme-in-action is the scheme followed by Charlie in his two lessons on polynomials, specifically in his last rule of action in the scheme where he answered students question on the board. It was noticed that the different questions asked by the students in the two lessons led Charlie to focus on different ideas in his demonstration. So, in the first lesson on polynomials he connected these with function, because a student asked about the relation between the two. While in the second lesson he did not mention that connection, because the students did not ask about it. This was not done in the second lesson, although Charlie addressed the importance of this connection during the interview. In other words, Charlie's scheme-in-action in his first lesson on polynomials included answering students' questions on the board and those questions made him explain the connection between polynomials and functions. However, the same rule of action "answering students' questions on the board" did not lead him to the same *demonstration* in the second lesson, where the students did not inquire about such connection. So, although Charlie was following the same general rules of actions, some specific rules of action (in this case creating a connection between functions and polynomials) was not activated or applied.

Therefore, inspired by Rowland's knowledge-in-action (Rowland et al., 2005), a scheme-in-action is the scheme activated and applied during a lesson. It is documented based on the steps the teacher does in class. A scheme-in-action of a lesson on a specific topic can slightly differ from one lesson to another on the same topic, without any intentions from the teacher to amend or modify the scheme. Instead, the slight differences or changes are 'live' and in response to the available resources within the lesson (for example students' contributions). Based on that, scheme-in-action sheds light on the dynamic nature of schemes of work that teachers follow.

5.3.3 Re-scheming

The use of the Documentational Approach lens in this study afforded looking at the changes in the teachers' schemes of work from one lesson to another, in other words "re-scheming" (Kayali & Biza, 2018a). Rescheming involves amending the scheme from one lesson to another on the same topic. It can be a result of an inference the teacher reaches in one lesson and decides to act on by changing his scheme for the next lessons on the same topic. For example, in George's three lesson on volume of revolution, George re-schemed his second lesson based on his experience in the first lesson. In his second episode, he tried to use Autograph to explain the formula of volume of revolution. But, he did not feel that his students could see the link between the activity on Autograph and the formula of volume of revolution. Due to the activity not working as he expected, George decided not to use it for that purpose in his third episode, and to use an image from the textbook instead. The Documentational Approach facilitated seeing the inference George reached based on his experience teaching the second episode, and this inference led him to avoid using Autograph to explain the formula for volume of revolution in his third episode on volume of revolution. It seems that the inference in this case became an operational invariant later, with the teacher believing that relying on the textbook image to explain the formula was better than "anything" he could use on Autograph. Another example is in Charlie's rescheming between his two lesson on polynomials. His first lesson's scheme included using Autograph to introduce polynomials; while his second lesson did not. This was justified by him by the lack of time in the second lesson. So, he had to re-scheme because the second group was "behind" in terms of the curriculum.

Therefore, inspired by the definition of "re-source" in Adler (2000, p. 207), re-scheming is defined as scheming "again or differently" from one lesson to another on the same topic. It can be justified by inferences from previous lessons (like in the example from George in the previous paragraph). Also, justified by institutional factors (like time constraints in the example from Charlie in the previous paragraph). I conjecture that rescheming can also happen due to the teacher having access to a new resource that s/he decides to integrate in lessons.

5.3.4 The use of Documentational approach in tandem with the Knowledge Quartet

Overall, looking at the data from the four participants through two lenses, showed how the findings from each of these lenses supported and complemented the finding from the other. On one hand, the Documentational Approach lens afforded looking at the students' contributions as resources; looking at institutional factors including examination requirements and past-papers; and looking at teachers' schemes of work in the lesson. It also afforded looking at the changes in the schemes of work from one lesson to another, in other words "re-scheming" (Kayali & Biza, 2018a) and was justified by the inferences or time constraints. For example, in George's three lesson on volume of revolution, George re-schemed his second lesson based on his experience in the first lesson (i.e. his use of Autograph did not help explain the formula of volume of revolution). The Documentational Approach allowed me to see the inference the teacher reached based on his experience teaching the first lesson, and this inference led him to avoid using Autograph to explain the formula for volume of revolution in his second lesson. In summary, the First and Second points of the findings (i.e. students' contributions as a resource, and examination requirements and school policy) were provoked by the use of the Documentational Approach.

The use of the Knowledge Quartet afforded looking at the details of the teachers work in class. This included a view of their "knowledge-in-action" (Rowland et al., 2005, p. 261), which led to a detailed consideration of what the teachers did in class and why they did it, and how they interacted with their students. The combination of this with the view through the Documentational Approach lens, afforded looking at "schemes-in-action". A clear example of a scheme-in-action is the scheme followed by Charlie in his two lessons on polynomials, specifically under his general rule of action "to answer students' questions on the board" in the scheme (summarized Tables 18 and 19) where he answered these on the board. I noticed that the different questions asked by the students in the two lessons led Charlie to focus on different ideas in his demonstration. So, in the first lesson on polynomials he connected these with function, because a student asked about the relation between the two. In the second lesson, he did not mention that connection, maybe because the students did not ask about it. This was not done in the second lesson, although Charlie addressed the importance of this connection during the interview. The difference between the two lesson's schemes was not due to re-scheming, but due to specific ideas (here connection with functions) being in-action due to the resources (students' contribution) available in that lesson (*responding to the (un)availability of tools and resources*). Re-scheming is a pre-decided scheming done by the teacher (like when Charlie decided not to use Autograph in his second lesson on polynomials due to time constraints), while scheming-in-action is giving a scheme a dynamic dimension by allowing it to vary from one lesson to another based on students' contributions for example.

In summary, the use of the Documentational Approach and Knowledge Quartet afforded a broader view on teaching approaches by focusing on the more general aspects (i.e. resources, scheme of work) as well as the detailed actions (i.e. *Foundation*, *Transformation*, *Connection* and *Contingency*). The findings were based on five points: students' contributions as a resource; examination requirements and school policy; knowledge of the software; connection between resources; and contingent moments. This led to three main findings. First, the suggestions of new codes for the Knowledge Quartet including: use of resources (under *Transformation*), knowledge of examination requirements (under *Foundation*), knowledge of software (under *Foundation*), connections between resources (under *Connections*). Second, the definition of scheme-in-action as the scheme that is activated and applied within a lesson. Third, the definition of re-scheming as to scheme again or differently from one lesson to another on one topic. These theoretical contributions are accompanied by the contributions to practice that I summarize next.

5.4 Implications for policy and practice

This work sheds light on aspects of practice including teachers' resources, schemes of work, knowledge-in-action (through the Knowledge Quartet's *Foundation*, *Transformation*, *Connection*, and *Contingency*), scheme-in-action, re-scheming. In other words, it offered a view on what resources teachers used and how they used them. It took a holistic view on resources using the definition from Adler (2000) adopted by Gueudet & Trouche (2009) that lead to including students' contributions as a resource. And, facilitated a view on how teachers integrated resources in their lessons and how they (un)balanced their resources while they were enacting their schemes of work. This (un)balance can be a learning example for other teachers. In other words, the critical incidents in the teaching episodes in this study can be developed into teaching scenarios to be used for professional development and teacher training sessions (Biza et al., 2007; Goodell, 2006; Tripp, 2012). For example, the trigonometry episode from George's data was used as a base to create three tasks (trigonometry 1, trigonometry 2 and trigonometry 3) and Adam's third episode on modulus equations was used to create one task (on simultaneous modulus equations) under the MathTASK project (Biza et al., in press; Biza et al., 2007), these tasks are available on the MathTASK website under the technology and resources strand ("MathTASK- Technology and resources," 2017). Personally, as a mathematics teacher, each of these examples is a rich learning example that will inform my future teaching practices. As a new researcher, these examples and findings will enlighten my future research as I explain in section 5.6.

In relation to the use of mathematics-education software, this study supports previous research findings in Bretscher (2014) and Trouche and Drijvers (2014) that such use is still limited. It seems that the "integration of digital technology into classroom practice is also subject to political, social and cultural factors related to mathematics, digital technologies and to teaching and learning" (Hoyles & Lagrange, 2009, p. 351). This was evident in the data through the teacher-centered uses of software, for example in Martin's lesson. In the data, there were frequent references to institutional dimensions influencing teachers' decisions about such use including time pressure, curriculum and examinations requirements. Consequently, an institutional acknowledgement of well-established mathematics-education software uses is the way promote such use. (Vale, Julie, Buteau, & Ridgway, 2009) suggest that the "implementation of mathematics afforded by digital technologies is more likely to occur when and where there is a shared vision among political leaders, education authorities, mathematicians and mathematics teachers" (p.357). A similar conclusion, specifically about the use of dynamic geometry software, was suggested by Ruthven et al. (2008):

> "… explicit curricular recognition is required for the use of dynamic geometry as a disciplinary tool, and for the need for students to develop an associated instrumental knowledge, if dynamic geometry is to move from being a marginal amplifier of established practice to become a more integral organiser of a renewed practice of classroom mathematics" (p. 315)

However, these implications do come with the limitations I summarize in the next section.

5.5 Limitations

One limitation in this study is related to the fifth point of the findings, contingent moments. It was not always possible to decide whether a specific moment is contingent or not. This was because the teacher did not show that it was an unexpected moment, although it resulted from a student contribution that could have been unexpected. One example is in the George's characteristic lesson on trigonometry, when he was asked about the tangent case after he taught the sine and cosine cases. George seemed confident in his answer, so it was not possible to decide whether he expected this question or not. Talking to the teacher straight after the lesson could have helped identifying if a moment like this was a contingent or not. But, due to how busy teachers' timetables were, this was not practically possible. And to ask about it in the post-observation interview would not have been helpful as the teacher would not have remembered the exact moment. This is in fact a second limitation of this study, as pre- and post- lesson interviews would have helped in having further views of what the teachers planned to do during the lessons and what they actually did. But, in relation to both of these limitations, the question is: would asking the teacher about whether s/he expected the question or not have helped identify more contingent moments?

A third limitation is related to the methodological approach of this study. Specifically, there are chances that my own views and biases and the theoretical lenses that informed this study have influenced my analysis of the data. These subjectivities cannot be eliminated, but "it is important to identify them and monitor them in relation to the theoretical framework and in light of the researcher's own interests, to make clear how they may be shaping the collection and interpretation of data" (Merriam & Tisdell, 2016, p. 16).

A fourth limitation is related to the use of semi-structured interviews for data collection, which meant a "greater degree of latitude […] and a need to understand the context and content of the interview" (May, 2010, p. 135). This means that there are chances of the latitude not being fully exploited, for example, by asking further questions on the content or context. Finally, due to the use of case study, further work needs to be done to validate any generalisations from this data set.

The fifth limitation is related to the analysis of the data set of this study. The data included the teachers using special expressions, gestures and voice tones to express ideas and give their students hints and reminders about the topics they were teaching. The two theoretical frameworks used in this study facilitated limited analysis of these. A discursive analysis would afford commenting further on the fine details of the use of words and gestures in the data.

5.6 Future research

In future research and with the use of more data, further examination of the theoretical findings of this study will help validate and/or refine the findings. The suggestion of new codes to the Knowledge Quartet is one aspect that needs further validation and refining. This may include amending the suggested new codes to include more cases. Another aspect is related to looking at further cases of scheming-in-action and/or rescheming to refine the definition of these terms and look at diverse scenarios that include them. Future research can also look further at the teaching episodes in this study and create more tasks to be used for professional development and teacher training (for example, under the MathTASK project (Biza et al., in press; "MathTASK," 2016), or even to create teaching scenarios for students to learn from and reflect on. Finally, research using a theory that affords a more micro analysis of the data set in this study would be helpful to uncover the aspect of the teachers' expressions, phrases, gestures and voice tone and how these contributed to the teaching/learning process.

5.7 Personal reflections

The journey of this work has been a treasured one for me. The diverse learning experiences I have had since I joined the UEA as a PhD student are experiences to cherish. Academically, doing this research gave me the opportunity to further learn about research in education, and specifically mathematics education research. In my first year, reading about different theories in mathematics education and education enriched my knowledge in this field. Specifically, when I thought about how these theories would have worked in situations when I was the teacher. Some theories and theoretical constructs have become 'alive' and impacted on how I saw daily events that are not related to teaching. For example: I see resources in different contexts with a broader view; and the instrumental approach's instrumentation/ instrumentalization are applicable to every tool in daily life. Progressing through the years of my PhD studies, I participated in UK-based and European conferences on mathematics education including: the 10th Congress of the European Society for Research in Mathematics Education (CERME 10); the 11th Congress of the European Society for Research in Mathematics Education (CERME 11); the 9th British Congress on Mathematics Education (BCME 9); and the 42nd Conference of the International Group for the Psychology of Mathematics Education (PME 42). For each of these conferences I co-authored a paper, presented it, and reviewed papers written by other researchers in mathematics education. The papers I presented in these conferences reported on different stages of this study and allowed me to hear feedback from researchers in mathematics education and develop my work accordingly. I also attended meetings with the research in mathematics education group at UEA regularly. These meetings have offered me invaluable chances for sharing experiences: to hear about the work of other group members, and to get advice on my own work. My study period at UEA was heightened through my participation as a research associate in two research projects, MathTASK (www.uea.ac.uk/education/mathtask) and CAPTeaM (www.uea.ac.uk/capteam). The two projects gave me opportunities to meet practicing teachers; listen to their reflections on teaching practices; join workshops for teachers and/or researchers; take part in publications relevant to the two projects (e.g. (Biza et al., in press)); work with a team of teachers and researchers on task development for MathTASK; and develop my views on teaching mathematics for students with diverse needs.

In relation to my teaching practice, the experience I had during this research study made me reflect on teaching situations that my participants went through and also on situations I have been through. Each moment I spent in class observing a participant was a learning experience for me, and I believe this will have long term impact on my future teaching practices. Additionally, during my study period I had the opportunity to work as an associate tutor for some sessions both at masters and undergraduate levels. This gave me the chance to work on planning and delivering lectures and seminars, and on the marking of assessments of masters and undergraduate students. In relation to research practices, conducting the study was an eye-opener for me on the challenges faced by researchers. This include the challenges related to: choosing a theoretical framework, ethics and getting ethical approval for the study, recruiting participants, dealing with the participants throughout the study, collecting data, organizing data, and the lengthy process of analysing data. These challenges helped in refining my skills as a new researcher. Overall, this work has impacted hugely on different aspects of my life and inspired me to look forward to further work in research as well as teaching.

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Appendices

List of Appendices_

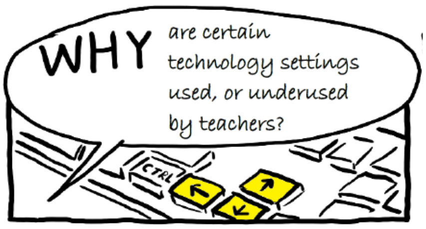
Appendix A: Study poster

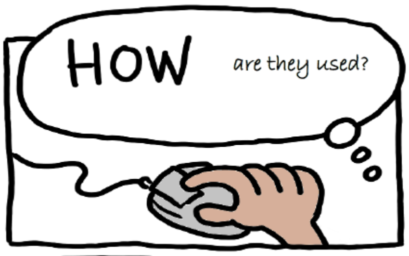
INVITATION TO PARTICIPATE IN A RESEARCH PROJECT

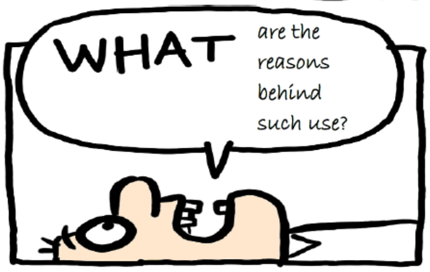
SECONDARY MATHEMATICS TEACHERS AND CLASSROOM TECHNOLOGY

Dear Mathematics teachers,

You are invited to take part in a research study about secondary mathematics teachers' use of digital technologies for mathematics teaching. The study aims to answer the following questions:







So, if you are currently teaching mathematics at secondary level, your participation will be highly appreciated. Your contribution involves having one 40-minute interview with me. The interview will take place at the UEA or at your school, at a time that is convenient to you, and it will be audio recorded. You will be asked questions relating to the types of material you use for teaching, especially the digital ones. You will be also asked to reflect on 2 classroom scenarios. You can review the transcript of your interview, if you wish, to ensure they are an accurate reflection of it.

Being in this study is completely voluntary and you do not have to take part.

Information sheets and consent forms are available at your school office. You can also request these by contacting me on L.Kayali@uea.ac.uk. Should you wish to participate, please fill in the consent form and email it to me or give to me when I visit your school.

Thank you!

The study is being carried out by Lina Kayali, PhD student, School of Education and Lifelong Learning, University of East Anglia; under the supervision of Dr. Irene Biza. Information sheets and consent forms can be requested by emailing L.Kayali@uea.ac.uk

Appendix B: Interview opt in form

Lina Kayali

PhD student

15.04.16

Faculty of Social Sciences

School of Education and Lifelong Learning

University of East Anglia

Norwich Research Park

Norwich NR4 7TJ

Using Technology in Secondary Mathematics Lessons

PARTICIPANT INFORMATION STATEMENT – Teacher Interview

(1) What is this study about?

You are invited to take part in a research study about secondary mathematics teachers' use of digital technologies for mathematics teaching. I am interested in understanding why are certain technology settings used, or underused by teachers? How are they used? And what are the reasons behind such use? In order to answer these questions, I aim to look at teachers' use of computational learning environment in mathematics lessons. This will be done through interviewing the teachers; analysing their resources, lesson plans, and actions in classroom; and listening to their interpretations and justification of how their lessons went. So, the aim is to reflect teachers' views as expressed by the teachers themselves.

You have been invited to participate in the interview part of this study because you are currently teaching mathematics at secondary level. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the study. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about. Participation in this research study is voluntary. By giving consent to take part in this study you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined below.
- ✓ Agree to the use of your personal information as described.

(2) Who is running the study?

The study is being carried out by Lina Kayali, PhD student, School of Education and Lifelong Learning, University of East Anglia; under the supervision of Dr. Irene Biza. School of Education and Lifelong Learning, University of East Anglia.

(3) What will the study involve for me?

Your participation involves having one interview with me. The interview will take place at the UEA or at your school, at a time that is convenient to you. The interviews will be audio recorded.

You will be asked questions relating to the types of material you use for teaching, especially the digital ones. You will be also asked to reflect on 2-3 classroom scenarios. You will be able to review the transcript of your interviews, if you wish, to ensure they are an accurate reflection of the discussion.

(4) How much of my time will the study take?

It is expected that the interview will take about 40 mins.

(5) Do I have to be in the study? Can I withdraw from the study once I've started?

Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with the researchers or anyone else at the University of East Anglia. If you decide to take part in the study and then change your mind later, you are free to withdraw at any time. You can do this by letting me know by email (L.Kayali@uea.ac.uk) or by phone (01603 592924). You are free to stop the interview at any time. Unless you say that you want me to keep them, any recordings will be erased and the information you have provided will not be included in the study results. You may also refuse to answer any questions that you do not wish to answer during the interview. If you decide at a later time to withdraw from the study your information will be removed from our records and will not be included in any results, up to the point I have analysed and published the results.

(6) Are there any risks or costs associated with being in the study?

Aside from giving up your time, I do not expect that there will be any risks or costs associated with taking part in this study.

(7) Are there any benefits associated with being in the study?

I would hope that by talking about your experiences that it will allow you to reflect on those areas that have helped as well as those areas that might need additional support. The study will also contribute to the effectiveness of using technology for mathematics teaching in the future.

(8) What will happen to information about me that is collected during the study?

By providing your consent, you are agreeing to me collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 1998 Data Protection Act and the University of East Anglia Research Data Management Policy (2013). Your information will be stored securely and your identity/information will only be disclosed with your permission, except as required by law. Study findings may be published, but you will not be identified in these publications unless you agree to this using the tick box on the consent form. Although your data will be anonymised, there is a chance that through the study context you will be identifiable by people who know you and know about the study. In this instance, data will be stored for a period of 10 years and then destroyed.

(9) What if I would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me on L.Kayali@uea.ac.uk or 01603 592924.

(10) Will I be told the results of the study?

You have a right to receive feedback about the overall results of this study. You can tell me that you wish to receive feedback by providing a contact detail on the consent section of this information sheet. This feedback will be in the form of a one page lay summary of the findings. You will receive this feedback after the study is finished.

(11) What if I have a complaint or any concerns about the study?

The ethical aspects of this study have been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

If there is a problem please let me know. You can contact me via the University at the following address:

Lina Kayali

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

L.Kayali@uea.ac.uk

If you would like to speak to someone else you can contact my supervisor:

Dr. Irene Biza

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0) 1603 591741

Email: I.Biza@uea.ac.uk

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact please contact the Head of the School of Education and Lifelong Learning, Dr Nalini Boodhoo, at n.boodhoo@uea.ac.uk.

(12) OK, I want to take part – what do I do next?
You need to fill in one copy of the consent form and give to me when I visit your school. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARTICIPANT CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.

✓ The researchers have answered any questions that I had about the study and I am happy with the answers.

✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia now or in the future.

✓ I understand that I can withdraw from the study at any time.

✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published, but these publications will not contain my name or any identifiable information about me unless I consent to being identified using the "Yes" checkbox below.

☐ Yes, I am happy to be identified.
☐ No, I don't want to be identified. Please keep my identity anonymous.

I consent to:

- **Audio-recording** YES ☐ NO ☐
- **Reviewing transcripts** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

....

Signature **PRINT name** **Date**

PARTICIPANT CONSENT FORM (2nd Copy to Participant)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.

✓ The researchers have answered any questions that I had about the study and I am happy with the answers.

✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia now or in the future.

✓ I understand that I can withdraw from the study at any time.

✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published, but these publications will not contain my name or any identifiable information about me unless I consent to being identified using the "Yes" checkbox below.

☐ Yes, I am happy to be identified.
☐ No, I don't want to be identified. Please keep my identity anonymous.

I consent to:

- **Audio-recording** YES ☐ NO ☐

- **Reviewing transcripts** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

....

Signature **PRINT name** **Date**

Appendix C: Lesson-observations opt in form

Lina Kayali

PhD student

15.04.16

Faculty of Social Sciences

School of Education and Lifelong Learning

University of East Anglia

Norwich Research Park

Norwich NR4 7TJ

Using Technology in Secondary Mathematics Lessons

PARTICIPANT INFORMATION STATEMENT – Teacher Observations

(1) What is this study about?

You are invited to take part in the second phase of a research study about secondary mathematics teachers' use of digital technologies for mathematics teaching. I am interested in understanding why are certain technology settings used, or underused by teachers? How are they used? And what are the reasons behind such use? In order to answer these questions, I aim to look at teachers' use of computational learning environment in mathematics lessons. This will be done through interviewing the teachers; analyzing their resources, lesson plans, and actions in classroom; and listening to their interpretations and justification of how their lessons went. So, the aim is to reflect teachers' views as expressed by the teachers themselves.

You have been invited to participate in this study because you are currently teaching mathematics at secondary level, and you participated in phase one of this study. This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the study. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about. Participation in this research study is voluntary. By giving consent to take part in this study you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined below.
- ✓ Agree to the use of your personal information as described.

(2) Who is running the study?

The study is being carried out by Lina Kayali, PhD student, School of Education and Lifelong Learning, University of East Anglia; under the supervision of Dr. Irene Biza. School of Education and Lifelong Learning, University of East Anglia.

(3) What will the study involve for me?

Your participation will involve you completing your day to day mathematics classes as normal. I will be present in up to 11 of your classes across the term (starting in September) and I will be taking notes about how the class is structured, examples used, lesson plans, worksheets and how the digital technology is used. With your permission I would also like to video record some of these sessions (I will let you know which ones) so that I can see how all of the class interact with each other. You can get to see any notes I take that are specifically about you. You will also be asked, before and after each lesson, about what you think and feel about your own mathematics lessons and planning, what went/did not go well and why.

(4) How much of my time will the study take?

I will be in your mathematics classes for 1 lesson of pilot observations before you start teaching [to be inserted: a specific mathematical topic e.g.: geometry], then I may observe 10 more lessons of your [to be inserted: a specific mathematical topic] lessons (starting in September) but this will be part of your normal everyday teaching. Additional time required is for pre- and post-lesson conversations, that will together take about 15 minutes per lesson, so that will mean no more than 2.5 hours of your time during the observation period.

(5) Do I have to be in the study? Can I withdraw from the study once I've started?

Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with the researchers or anyone else at the University of East Anglia. If you decide to take part in the study and then change your mind later, you are free to withdraw at any time. You can do this by letting me know by email (L.Kayali@uea.ac.uk) or by phone (01603 591039). You are free to stop the interview and/or the observations at any time. Unless you say that you want us to keep them, any recordings will be erased and the information you have provided will not be included in the study results. You may also refuse to answer any questions that you do not wish to answer during the interview. If you decide at a later time to withdraw from the study (interview and observation) your information will be removed from our records and will not be included in any results, up to the point we have analysed and published the results.

(6) Are there any risks or costs associated with being in the study?

Aside from giving up your time, we do not expect that there will be any risks or costs associated with taking part in this study.

(7) Are there any benefits associated with being in the study?

I would hope that by talking about your experiences that it will allow you to reflect on those areas that have helped as well as those areas that might need additional support. The study will also contribute to the effectiveness of using technology for mathematics teaching in the future.

(8) What will happen to information about me that is collected during the study?

By providing your consent, you are agreeing to me collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise. Data management will follow the 1998 Data Protection Act and the University of East Anglia Research Data Management Policy (2013). Your information will be stored securely and your identity/information will only be disclosed with your permission, except as required by law. Study findings may be published, but you will not be identified in these publications unless you agree to this using the tick box on the consent form. Although your data will be anonymised, there is a chance that through the study context you will be identifiable by people who know you and know about the study. In this instance, data will be stored for a period of 10 years and then destroyed.

(9) What if I would like further information about the study?

When you have read this information, I will be available to discuss it with you further and answer any questions you may have. You can contact me on L.Kayali@uea.ac.uk or 01603 591039.

(10) Will I be told the results of the study?

You have a right to receive feedback about the overall results of this study. You can tell me that you wish to receive feedback by providing a contact detail on the consent section of this information sheet. This feedback will be in the form of a one page lay summary of the findings. You will receive this feedback after the study is finished.

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If you would like to speak to someone else you can contact my supervisor:

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NORWICH NR4 7TJ

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(12) OK, I want to take part – what do I do next?
You need to fill in one copy of the consent form and give to me when I visit your school. Please keep the letter, information sheet and the 2nd copy of the consent form for your information.

This information sheet is for you to keep

PARTICIPANT CONSENT FORM (1st Copy to Researcher)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.

✓ The researchers have answered any questions that I had about the study and I am happy with the answers.

✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia now or in the future.

✓ I understand that I can withdraw from the study at any time.

✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don’t wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published, but these publications will not contain my name or any identifiable information about me unless I consent to being identified using the “Yes” checkbox below.

☐ Yes, I am happy to be identified.
☐ No, I don't want to be identified. Please keep my identity anonymous.

I consent to:

- **Audio-recording** YES ☐
  NO ☐
- **Video-recording** YES ☐
  NO ☐
- **Reviewing transcripts** YES ☐
  NO ☐
- **Would you like to receive feedback about the overall results of this study?**
  YES ☐
  NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

☐ Email: ____

……………………………………………………… ………………………………………………………………
………………………..

Signature **PRINT name** **Date**

PARTICIPANT CONSENT FORM (2nd Copy to Participant)

I, ... [PRINT NAME], agree to take part in this research study.

In giving my consent I state that:

✓ I understand the purpose of the study, what I will be asked to do, and any risks/benefits involved.

✓ I have read the Participant Information Statement and have been able to discuss my involvement in the study with the researchers if I wished to do so.

✓ The researchers have answered any questions that I had about the study and I am happy with the answers.

✓ I understand that being in this study is completely voluntary and I do not have to take part. My decision whether to be in the study will not affect my relationship with the researchers or anyone else at the University of East Anglia now or in the future.

✓ I understand that I can withdraw from the study at any time.

✓ I understand that I may stop the interview at any time if I do not wish to continue, and that unless I indicate otherwise any recordings will then be erased and the information provided will not be included in the study. I also understand that I may refuse to answer any questions I don't wish to answer.

✓ I understand that personal information about me that is collected over the course of this project will be stored securely and will only be used for purposes that I have agreed to. I understand that information about me will only be told to others with my permission, except as required by law.

✓ I understand that the results of this study may be published, but these publications will not contain my name or any identifiable information about me unless I consent to being identified using the "Yes" checkbox below.

☐ Yes, I am happy to be identified.
☐ No, I don't want to be identified. Please keep my identity anonymous.

I consent to:

- **Audio-recording** YES ☐ NO ☐

- **Video-recording** YES ☐ NO ☐
- **Reviewing transcripts** YES ☐ NO ☐
- **Would you like to receive feedback about the overall results of this study?** YES ☐ NO ☐

If you answered **YES**, please indicate your preferred form of feedback and address:

☐ Postal: ____

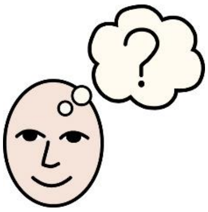
☐ Email: ____

....

Signature **PRINT name** **Date**

Appendix D: Student consent form for lesson observations

Study Information Sheet: My maths class



Hello. My name is Lina Kayali. I am doing a research study to find out more about teachers' use of digital technology in maths classes. You are invited to take part in a research study about secondary mathematics teachers' use of digital technologies for mathematics teaching. The study investigates why are certain technology settings used, or underused by teachers? How are they used? And what are the teachers' reasons behind such use?

I am asking you to be in our study because you are a student in [to be inserted: teacher's name]'s maths class.

You can decide if you want to take part in the study or not. You don't have to - it's up to you.

This sheet tells you what we will ask you to do if you decide to take part in the study. Please read it carefully so that you can make up your mind about whether you want to take part.

If you decide you want to be in the study and then you change your mind later, that's ok. All you need to do is tell us that you don't want to be in the study anymore.

If you have any questions, you can ask us or your family or someone else who looks after you. If you want to, you can contact me any time on L.Kayali@uea.ac.uk or 01603 591039.

What will happen if I say that I want to be in the study?

If you decide that you want to be in my study, I will ask you to do these things:

- Let me come and sit in your classroom and make some notes about what happens. This will be mainly about the teacher, but it could include things that you say or do. I may also want to take some video of your lessons so I can look at them later on.
- If you wish to, you can come along to one interview in which you will talk about what you think about your maths lessons.

When I ask you questions, you can choose which ones you want to answer. If you don't want to talk about something, that's ok. You can stop talking to me at any time if you don't want to talk to me anymore.

If you say it's ok, I will record what you say with a tape recorder.
If you say it's ok, I will make a video of you in the class with a video recorder.

You are free to stop participating at any stage or to refuse to answer any of the questions. If you decide at a later time to withdraw from the study your information will be removed from our records and will not be included in any results, up to the point we have analysed and published the results. (To be deleted: When I video record the lesson, I won't be able to take out the things you say after you have said them. This is because you will be talking in a group and our notes will have all the things that everyone else said as well).

Will anyone else know what I say in the study?
I won't tell anyone else what you say to me, except if you talk about someone hurting you or about you hurting yourself or someone else. Then I might need to tell someone to keep you and other people safe.

All of the information that I have about you from the study will be stored in a safe place and I will look after it very carefully. I will write a report about the study and show it to other people but I won't say your name in the report and no one will know that you were in the study, unless you tell me that it's ok for me to say your name.

How long will the study take?

I will be in 10-11 of your maths classes. If you talk with me about what you think the maths class is like, this will take 10-15 minutes.

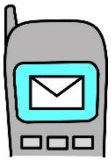
Are there any good things about being in the study?

I think you'll like talking about your maths lessons and you will also be helping me do my research.

Are there any bad things about being in the study?
This study will take up some of your time, but I don't think it will be bad for you or cost you anything.

Will you tell me what you learnt in the study at the end?
Yes, I will if you want me to. There is a question on the next page that asks you if you want me to tell you what I learnt in the study. If you circle Yes, when I finish the study I will tell your teacher what I learnt and s/he will share this with you.

What if I am not happy with the study or the people doing the study?



If you are not happy with how I am doing the study or how I treat you, then you or the person who looks after you can:

- **Call** the university on 01603 591039
- Write an **email** to n.boodhoo@uea.ac.uk

OK, I want to take part – what do I do next?

If you're happy to fill in the 2 forms below and give number 1 to me in the next class. You can keep this letter and the form 2 to remind you about the study.

This sheet is for you to keep.

Study Information Sheet: My maths class

Consent Form 1

If you are happy to be in the study, please

- **write** your **name** in the space below
- **sign** your **name** at the bottom of the next page
- put the **date** at the bottom of the next page.

You should only say 'yes' to being in the study if you know what it is about and you want to be in it. If you don't want to be in the study, don't sign the form.

I, ...[PRINT NAME], am happy to be in this research study.

In saying yes to being in the study, I am saying that:

✓ I know what the study is about.

✓ I know what I will be asked to do.

✓ Someone has talked to me about the study.

✓ My questions have been answered.

✓ I know that I don't have to be in the study if I don't want to.

✓ I know that I can pull out of the study at any time if I don't want to do it anymore.

✓ I know that I don't have to answer any questions that I don't want to answer.

✓ I know that the researchers won't tell anyone what I say when we talk to each other, unless I talk about being hurt by someone or hurting myself or someone else.

Now we are going to ask you if you are happy to do a few other things in the study. Please circle 'Yes' or 'No' to tell us what you would like.

Are you happy for me to make **videos** of you during lessons? **Yes** **No**

Are you happy for me to **tape record** your voice during lessons and interview? **Yes**
No

Do you want me to tell you what I **learnt** in the study? **Yes**
No

……………………………………………………… ………………………………………………….

Signature **Date**

Study Information Sheet: My maths class

Consent Form 2

If you are happy to be in the study, please

- **write** your **name** in the space below
- **sign** your **name** at the bottom of the next page
- put the **date** at the bottom of the next page.

You should only say 'yes' to being in the study if you know what it is about and you want to be in it. If you don't want to be in the study, don't sign the form.

I, ...[PRINT NAME], am happy to be in this research study.

In saying yes to being in the study, I am saying that:

✓ I know what the study is about.

✓ I know what I will be asked to do.

✓ Someone has talked to me about the study.

✓ My questions have been answered.

✓ I know that I don't have to be in the study if I don't want to.

✓ I know that I can pull out of the study at any time if I don't want to do it anymore.

✓ I know that I don't have to answer any questions that I don't want to answer.

✓ I know that the researchers won't tell anyone what I say when we talk to each other, unless I talk about being hurt by someone or hurting myself or someone else.

Now we are going to ask you if you are happy to do a few other things in the study. Please circle 'Yes' or 'No' to tell us what you would like.

Are you happy for me to make **videos** of you during lessons? **Yes** **No**

Are you happy for me to **tape record** your voice during lessons and interview? **Yes** **No**

Do you want me to tell you what I **learnt** in the study? **Yes** **No**

……………………………………………………… ……………………………………………………….

Signature **Date**

Appendix E: Sample of Pre-observation interview questions

Questions asked to Adam to clarify his answers to scenario-based interviews during the second phase of the study

In relation to the 3D scenario

Q1. So, when you say here this task needs scaffolding, what kind of scaffolding can you offer the student in this case? What sort of scaffolding will make it easier for them?
Q2. Why do you think it is more valuable when you do it with the paper?
Q3. Why would you not go technology with that instead of paper?
Q4. Is that a modification you are suggesting because the problem here suggest that the volume is one litre?
Q5. With this question, what do you think the student is thinking of when they give you this answer?
Q6. What do you expect the answer is according to the classes you are teaching now?
Q7. You are saying here that you can use GeoGebra to detail the changing in length and see the effect on volume, what do you mean?
Q8. Would you use it as a group activity?
Q9. "Students won't be able to learn the software quick enough to be effective", what do you mean?
Q10. Would you expect ICT facilities not to be available?
Q11. "Unreliable", what kind of issues do you usually have?
Q12. And "students not able to access due to literacy", could you explain?
Q13. You say here "if the student doesn't know how to use the software", how would you tackle this?

In relation to the scenario on square

Q14. You say here the scenario is not very common is that in terms of demands, the way the square was done or is it in terms of behaviour?
Q15. So, how would you react in this case?
Q16. Do you think it is good idea to use GeoGebra in this case? Or is it the same as in the first problem you said like it is better to use paper?
Q17. Can you see the use of the computer in a different question? Would you rephrase the question maybe to make it more useful to use the computer?

Appendix F: Sample of the post-observation interview

Questions asked to Charlie based on the critical incidents identified in his lessons

Q1. The use of Autograph to discover how many solutions an equation (polynomial) has? Why was it done in that way? Was it useful? How?

Q2. iPad not used, why?

Q3. There are clear expectations related to homework, how does that help?

Q4. A lot of "crazy" things in mathematics like "proof" and "the crazy rule of inequalities", why?

Q5. The teacher uses many of his own special terms like: MANTRA for the equation of a line, crazy rule of inequalities, sheep, snail (signs, numbers, individual letters), Jellyfish, billy Bedmas, why?

Q6. The teacher tends to write common mistakes that students usually make, show why they are wrong and then write the correct answer, why? Ex. Reversing inequality when multiplying by a negative number, and the solution of a quadratic equation.

Q7. The teacher said they, as a maths department, aim to create a repertoire of activities and resources. How is that helpful? What are the criteria for it? Who can amend it?

Q8. Students are always taking notes, despite that the teacher emails the presentations to them. Why?

Q9. With a quadratic inequality the teacher said always sketch. Why not check the values within specific ranges?

Q10. When the teacher tried to use Autograph to show the relation between tangent and normal, the student was able to spot they were perpendicular but not able to discover the relation between their slopes, so the teacher switched to using two pens and helped her to reach a conclusion, why did he change from Autograph to pens?

Q11. The teacher frequently commented on students making "the face of understanding in maths" or "the noise of understanding in maths", and asked students to feel free to express them…

Q12. The teacher talks a lot about examinations, past papers, what examiners like/ don't like, what is ok to reset (C1 & C2) and what is not ok to reset (S1 and S2). "What we want to do is to be awesome at doing maths... But the way you're tested on that is looking at papers and the annoying thing is those three questions you almost never get on papers". How important are examination papers?

Q13. The difference between "Have I made a mistake?" said during the lesson, and "I do not agree with the answer at the back of the book, but it could be that I made an error". How are the mistakes in the book dealt with? Is there any other way to confirm what the right answer is?

Q14. I noticed a pattern of lessons: the first 5-10 minutes checking if there is any work to be submitted and any questions to be discussed before staring the new lesson, then teacher introducing the new ideas but frequently asking questions so that the students are involved in discovering the new concepts and a student is picked randomly to answer at any moment, then students work independently or in pairs/groups on solving some textbook exercises and they ask questions on the go if needed, finally homework is set. The textbook is the main source of exercises… Is that right?

Q15. The scheme of work is amendable by any mathematics teacher, not only by the head of maths, how does that help? Does it create issues?

Q16. The teacher frequently commented on resources he used in previous schools and found useful to use in this one. Could you elaborate?

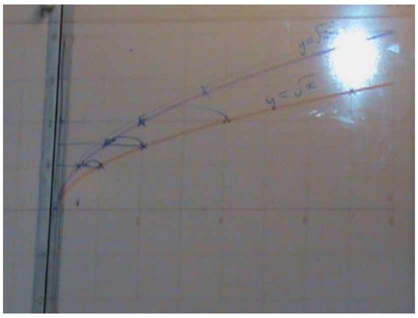
Q17. Photomath (https://www.photomath.net/en/) solved $x^2=4$ as $x=2$. “Really good, I must use it … I never use it but good point”. How is it useful? Has it been used since? Why, why not?

Q18. Classes grouped so that there are 4 students in one group and 12 in another, when asked the teacher said this is because the 12-student group is a higher achieving one. How does that help?

Q19. Lack of calculators only in A level is an issue because students only did exams with calculators and they tend to change fractions into decimals. Would you recommend calculators or changing teaching strategy to rely more on decimals?

Q20. The teacher mentioned you like to be involved in research because it’s a good thing… so was involved in a CASIO calculators’ trial. How does it add to the school?

Q21. When the teacher spoke about transformations, he was very quick introducing all, he used Autograph for the original graph but was sketching roughly the ones resulting from transformations. He said this was a difficult topic and over the years he felt it’s best to introduce it quickly and then do exercises. Why did he not use Autograph to show the resulting graphs?



Appendix G: Sample of the analysis using the Knowledge Quartet

Teacher	George
Topic	Chapter: Numerical Methods- Four methods: • Change of sign method • Iterative method • Mid-ordinate rule • Simpson rule
Year	13

Narrative of part of the lesson

The chapter he was introducing "Numerical Methods", he explained that in this chapter there are four subsections or four methods: "one is called change of sign method, another one is called the iterative method, another one is the mid-ordinate rule and the last one is the Simpson rule". George said that "the whole chapter is about things that you can't solve by doing the algebra to them, or just by integrating them, ok. And, it turns out that there are way more functions and things that you can't solve algebraically than there are ones that you can, ok. There are loads of horrible things to integrate that are too difficult to integrate, that it's just easier to do a numerical thing and kind of approximately, a bit like a trapezium rule". George reminded the students that they saw this rule during the lesson about the volume of revolution. He used Simpson rule on Autograph when he was trying to show the volume of revolution on the software.

George further explained that Simpson rule and the mid-ordinate rule were about finding the area and extension of the trapezium rule, while the change of sign method and the iterative method were about solving equations. He then started talking about the iterative methods, and mentioned that for that one he needs "not a nice equation, one that is difficult to solve. I'm trying to think of one now, for example x^3, let's go 4 x^3-, I don't know, $10x^2$ + 3=0, maybe there is a way of factorising it maybe you can solve it but maybe you can't ok and what we do is we say ok there could be a root between $x = 1$ and $x = 2$ for example we might know that information already, I don't know if this particular one is. I'm just making the numbers up but it could be the case there, and then we have to show that there is a route, okay?". So, George started working on the example that he seems to have spontaneously came up with $4x^3 - 10x^2 + 3 = 0$. He emphasised that for this lesson he was going to focus on the iterative method, not the change of sign method. Then he came up with an example of an equation that does not factorize and it was $x^2 + 4x + 1 = 0$.

Lina Kayali (EDU - Student)
Awareness of purpose

Lina Kayali (EDU - Student)
Connection-concepts

Lina Kayali (EDU - Student)
Connection-concepts

Lina Kayali (EDU - Student)
Awareness of purpose

Lina Kayali (EDU - Student)
Awareness of purpose

Lina Kayali (EDU - Student)
Choice of examples

Lina Kayali (EDU - Student)
Awareness of purpose/ decision about sequencing

Lina Kayali (EDU - Student)
Choice of examples

"I'll give you an example so here it says $x^2 + 4x + 1 = 0$, it does not factorise, okay. now I know this example you could just push it through the quadratic formula, but we are going to use it as an example of one that pretend you can't, okay".

George then asked about the different ways to rearrange the equation: "so my first question to you now is then can you take that can you rearrange it so that it is x equals?". He started by showing a way to rearrange this example. So, he wrote:

$$x^2 = -4x - 1$$
$$x = \sqrt{-4x - 1}$$

A student suggested another rearrangement $x = \frac{-1}{x+4}$, so George asked how he reached that and the student explained his method. Students were asked after that to find as many rearrangements into $x =$... as possible. Students found 3, 4 or 5 rearrangements. The teacher emphasized he wanted different ones, and then asked students to give the rearrangements they got.

One student said $x = \frac{-x^2 - 1}{4}$, and explained how the method used to get it. Another student completed the square and gave the roots. At this point, the teacher said: "You've done it right you're all fine you've got the answer you've just done it differently to what I was expecting, I was hoping for lots of rearrangements like this you've actually found the answer gone one step further you're not

Lina Kayali (EDU - Student)
Choice of examples

Lina Kayali (EDU - Student)
Awareness of purpose/ demonstration/ conceptual appropriateness

Lina Kayali (EDU - Student)
Use of terminology

Lina Kayali (EDU - Student)
RSI

Lina Kayali (EDU - Student)
Awareness of purpose/conceptual appropriateness

Lina Kayali (EDU - Student)
RSI

wrong though, ok. I'm not going to write it on there. Alright, any other rearrangements?". The teacher then suggested $x = \frac{-1}{x} - 4$. Then, a student suggested $x = \frac{-4x - 1}{x}$. The teacher said he was happy with that and he wrote it down, he also said that it could be written as $x = -4 - \frac{1}{x}$, which is the same as the previous rearrangement. George now has five rearrangements on the board:

$$x = \sqrt{-4x - 1}$$

$$x = \frac{-1}{x + 4}$$

$$x = \frac{-x^2 - 1}{4}$$

$$x = \frac{-1}{x} - 4$$

$$x = \frac{-4x - 1}{x}$$

George added that he was going to use one of these arrangements to "zoom in on the answers". He asked someone with a calculator to find the roots, the answers came from the students " - 0.267" or "-3.732" using completing the square method. He asked his students to pick one rearrangement for him to start with. The students chose:

$$x = \sqrt{-4x - 1}$$

George was then interrupted by a colleague who wanted to talk to him. So, he gave the students a question from the textbook to do while they were waiting for him (5 questions from page 127). He explained that they only needed to do the rearrangements questions until he finished talking to the other teacher. Once George came back, the students asked him what were they supposed to do. So, he picked two questions from the book and explained that the students were asked to show how each of them was rearranged. So, the first one was $x^2 - 6x + 1 = 0$ and the students were asked to show how it can be rearranged to $x = \frac{1}{6-x}$. While the next question was how to re-arrange $x^2 + 10x - 3 = 0$ to $x = \frac{3-x^2}{10}$.

Lina Kayali (EDU - Student)
demonstration

Lina Kayali (EDU - Student)
RSI

Lina Kayali (EDU - Student)
Awareness of purpose/demonstration

Lina Kayali (EDU - Student)
Connections- procedures/concepts

Lina Kayali (EDU - Student)
textbook

Lina Kayali (EDU - Student)
demostration

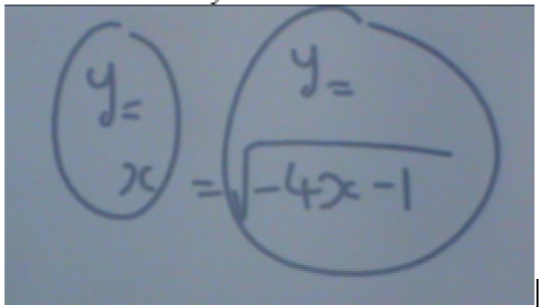
The two questions were done on the board with the students contributing. For the first one, these were the steps followed:

$$x^2 - 6x + 1 = 0$$
$$x^2 - 6x = -1$$
$$x(x-6) = -1$$
$$x = \frac{-1}{x-6}$$

For the second question, the following steps were done:

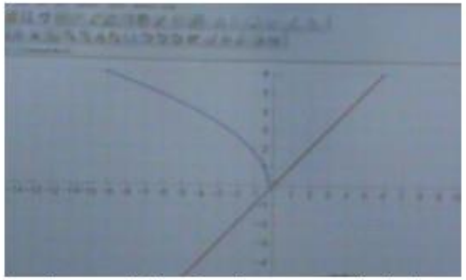
$$x^2 + 10x - 3 = 0$$
$$x^2 + 10x = 3$$
$$10x = 3 - x^2$$
$$x = \frac{3 - x^2}{10}$$

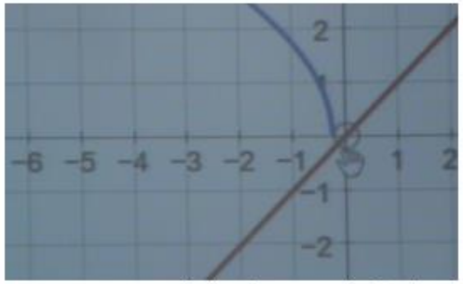
George the commented "if you can do that you're half way there with this method… Alright then, now it gets to the interesting bit where we can start thinking about graphs. Now, what we are going to do is we are going to say imagine plotting the graph of $y = x$. And, then we've also got the graph of y = that length and we are saying where do those two graphs cross, okay? So, let's just have a look at what that looks like so $y = x$ is our red line and the other one is $y = \sqrt{-4x-1}$".



George inserted both functions on Autograph:

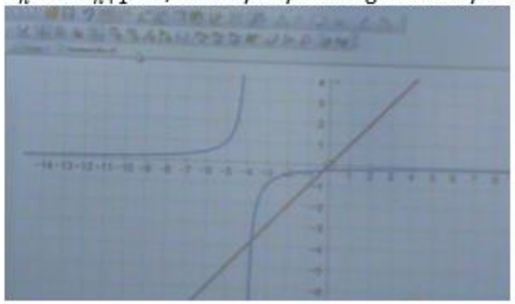
Lina Kayali (EDU - Student)
choice of examples





He then said: “We've chosen a really bad rearrangement. One of the things we do find out with this is... some of this work and some of these don't. Now, I can see that the blue one does not quite, I think it almost touches, but not quite touches the red one, okay. So, they in fact don't cross so for that particular rearrangement sadly it's not going to work out. So, let's not do that one, let's pick a different one. Which one do you want?”.

A student picked $y = \frac{-1}{x+4}$. George graphed $y = x$ and $y = \frac{-1}{x+4}$ on Autograph, and commented that the two graphs crossed this time at two points. Having noted that the two pints are close to $x = -2$ and $x = -3$, George explained the method: “Now how this process goes, and this is where those weird x's come in, let's just leave that one there. We are going to choose an x value we are going to choose a starting value, and we are going to put it into this ok that's going to be x_n and when we put that on the grid. It's going to spit another number out, and that will be what we will call our next x value. And, then what we'll do is we'll do it again we'll go round and round and round until our numbers to converge and we get nice numbers, okay. So, actually let's just try something out x and x_n and x_{n+1}. So, let's try any starting number you want”.



Lina Kayali (EDU - Student)
demonstration

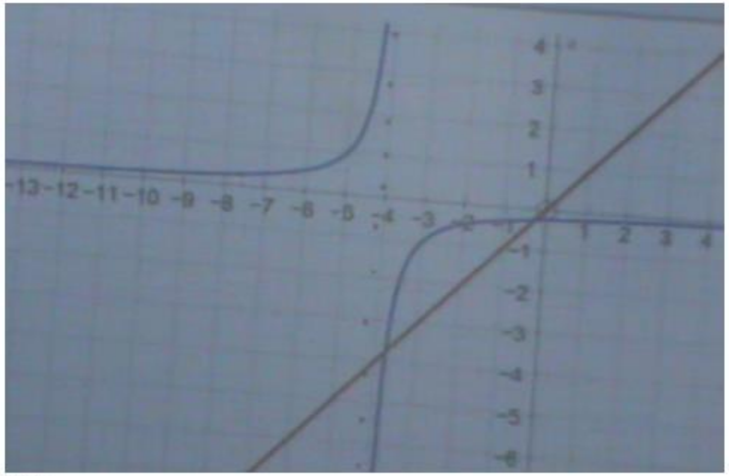
George wrote $x_{n+1} = \frac{-1}{x_n+4}$ and started completing the following table:

x_n	x_{n+1}
1	$-\frac{1}{5}$

At that point, a student asked what would happen if $x_{n+1} + 4 = 0$. The teacher replied "It just wouldn't work out. And, actually, what value doesn't work out -4?". Upon asking this question, the George showed again the graphs on Autograph, and the following conversation took place:

George: Does that make sense there, what happens if x is -4?
Student A: It never reaches
George: Yeah! Do you know what the and I could come up here and draw something, do you know what that would be called that line?

Lina Kayali (EDU - Student)
RSI & use of opportunity



Pupil: ...

Teacher: You got it, yaaay! you know like when you know your tan graph at 90 degrees, that 90 degrees on a tan graph is effectively saying we've got an infinitely tall triangle which wouldn't be a triangle at all, because it would be at 90 degrees. You're talking two parallel lines that's why tan of 90 doesn't work... So, what we've got there is your value of -4, as you see there, because we can't divide by 0 it wouldn't work out. But, this one does. Put the 1 into it, our next value is going to be minus a fifth, okay. So, minus a fifth. Anyone with a calculator can you stick that into a calculator for us? Now were working out $-\frac{1}{5}$".

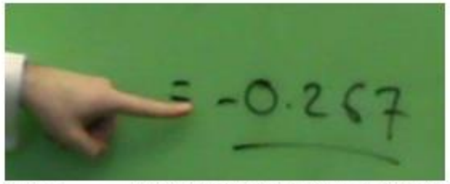
Lina Kayali (EDU - Student)
demonstration

The work continued on the table until the following values were found and written as decimals:

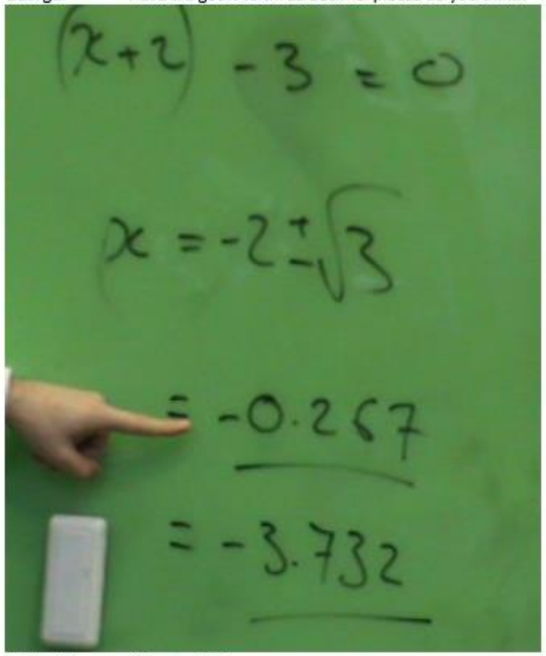
x_n	x_{n+1}
1	$-\frac{1}{5} = -0.2$
$-\frac{1}{5}$	$-\frac{5}{19} = -0.26315$
$-\frac{5}{19}$	$-\frac{19}{71} = -0.2676$
$-\frac{19}{71}$	-0.26769

George then pointed to one of the roots found using completing the square, showing the students that it is approximately equal to the last two numbers in the table

Lina Kayali (EDU - Student)
connection-procedures



George: Have we got it to three decimal places do you think?



Student A: Got what sir?

Student B: That is not the actual answer 0.26794

Student C: Oh yeah it's close enough though.

George then explained that staring at $x = 1$, we go values close to the first root -0.267, while starting at a different value might lead to the other root. He also decided to do the iterative method on Autograph, to show what happens in the background.

Lina Kayali (EDU - Student)
RSI

Lina Kayali (EDU - Student)
Demonstration/use of instructional materials/ connection-procedures

Analysis of lines 27-139 using the Knowledge Quartet

In his explanation about the iterative method, George was spontaneous in his choice of example (the equation $x^2 + 4x + 1 = 0$ in this case). Within the lesson, he created an equation that could not be factorized but can be solved using the quadratic formula (34-39). This choice was based on a specific purpose which is to show the students that the iterative methods works and gives them an approximation of the equation's solutions. Then, he asked his students to create different arrangements of the equation. This is a way to connect procedures (40-48). George was always responding to students' ideas (45-56), commenting on when two rearrangements were the same and just written differently and on when a student solved the equation instead of rearranging it. After finding the different arrangements, George again tried to create a connection when he used the equation's solutions the students found on their calculators to show that the iterative methods would "zoom in on the answers" (62-65). George was demonstrating about iteration to students (70-90, 99-109, 127-132) while at the same time inviting them to contribute (use of instructional materials/ use of resources) (70-90). After that, he asked his students to solve some textbook questions (68-71), this showed how he adhered to textbooks (use of instructional materials) (67-70) when it came to choosing questions for practice. In order to further explain about the iterative method and show how the different rearrangements George and his students had found would work, George used Autograph as an instructional material (92-93, 99-100, 110-125), and tried to create connections between procedures (129-130) and between different resources (138-139). So, after showing the iterative method procedure on the board he decided to show what happens if the iterative method was done on Autograph (140-148), we also notice in these lines concentration on procedure. One contingency incident (use of opportunity) happened when George entered the equations from one arrangement on Autograph and noticed that the graphs of these equations did not cross (92-98), in this case he commented that some arrangements did not work and asked the students to pick a different arrangement (use of instructional materials/ use of resources). Another example of contingency moment (responding to students' ideas) was when a student asked what would happen if $x_n + 4 = 0$ in $x_{n+1} = \frac{-1}{x_n+4}$ (110-125). At that point George showed the graph on Autograph to show that it would not work. In response to this student, George used Autograph as instructional material and also tried to connect concepts (dividing by zero, tan 90, parallel lines). Another example of responding to students' ideas was when a student asked why they were not getting the actual equation roots, and George commented that the answers they were getting using the iterative method were "close" to the roots (134-138).

In summary, George showed good foundation knowledge. In his transformation of ideas, he was spontaneous in his choice of examples and representations. He also asked for students' contributions and used these as a resource that he connected with other resources and instructional materials including Autograph. He frequently connected concepts and procedures. He was responding to students' ideas and sometimes using these as opportunities to create connections between concepts.

Appendix H: Application for ethical approval of a research project

UNIVERSITY OF EAST ANGLIA

SCHOOL OF EDUCATION AND LIFELONG LEARNING RESEARCH ETHICS COMMITTEE

APPLICATION FOR ETHICAL APPROVAL OF A RESEARCH PROJECT

This form is for all staff and students across the UEA who are planning educational research. Applicants are advised to consult the school and university guidelines before preparing their application by visiting https://www.uea.ac.uk/research/our-research-integrity and reading the EDU Research Ethics Handbook. The Research Ethics page of the EDU website provides links to the University Research Ethics Committee, the UEA ethics policy guidelines, ethics guidelines from BERA and the ESRC, and resources from the academic literature, as well as relevant policy updates: www.uea.ac.uk/edu/research/researchethics. If you are involved in counselling research you should consult the BACP Guidelines for Research Ethics: www.bacp.co.uk/research/ethical_guidelines.php.

Applications must be approved by the Research Ethics Committee before beginning data generation or approaching potential research participants.

- Staff and Postgraduate (PGR) student applications (including the required attachments) must be submitted electronically to Dawn Corby d.corby@uea.ac.uk, two weeks before a scheduled committee meeting.
- Undergraduate students and other students must follow the procedures determined by their course of study.

APPLICANT DETAILS	
Name:	Lina Kayali
School:	EDU
Current Status:	**PGR Student**
UEA Email address:	**L.Kayali@uea.ac.uk**
If PGR Student, name of primary supervisor and programme of study: **Primary supervisor: Dr. Irene Biza** **Program of study: PhD in mathematics Education**	
If UG student or other student, name of Course and Module:	

The following paperwork must be submitted to EDU REC **BEFORE** the application can be approved. Applications with missing/incomplete sections will be returned to the applicant for submission at the next EDU REC meeting.	
Required paperwork	**✓Applicant Tick to confirm**
Application Form (fully completed)	✓
Participant Information sheet (EDU template)	✓
Participant Consent form (EDU templates appropriate for nature of participants i.e. adult/parent/carer etc.)	✓
Other supporting documents (for e.g. questionnaires, interview/focus group questions, stimulus materials, observation checklists, letters of invitation, recruitment posters etc)	✓

2. PROPOSED RESEARCH PROJECT DETAILS:	
Title:	Forming Identities -The Case of Secondary Mathematics Teachers and Classroom Technology
Start/End Dates:	May 2016 - Sep 2019

3. FUNDER DETAILS (IF APPLICABLE):	
Funder:	UEA studentship
	Has funding been applied for? YES Application Date: Feb 2015
	Has funding been awarded? YES
Will ethical approval also be sought for this project from another source? NO	
	If "yes" what is this source?

4. APPLICATION FORM FOR RESEARCH INVOLVING HUMAN PARTICIPANTS:

4.1 Briefly outline, using lay language, your research focus and questions or aims (no more than 300 words).

I am researching teachers' use of computational learning settings in secondary mathematics lessons.

My research questions are:

- Why are certain settings used, or underused by the teachers?
- How are these settings used?
- What are the reasons behind such use?

In order to answer these questions, I aim to investigate teachers' beliefs and practices when using computational learning environment in mathematics teaching. This will be done through interviewing the teachers; analyzing their resources, lesson plans, and actions in classroom; and listening to their interpretations and justification of how their lessons went. The aim is to reflect teachers' views as expressed by the teachers themselves.

Data collection is divided into the following two phases:

The First Phase

This is split into three parts.

The First part: I will develop classroom scenarios (tasks) in order to use them for the semi-structured interviews with teachers in the next part. These scenarios will be based on two components: tasks taken from previous research on using technology in mathematics education; and issues and difficulties faced by teachers and addressed in previous research. This will create classroom-like situations with issues to discuss.

The Second part: I will interview 7-10 secondary school mathematics teachers. The interview will be semi-structured and based on reflections on the tasks developed in the first part.

The Third part: I will do one-lesson pilot observations with up to 7 of the teachers who joined the second part. This will give me the chance to listen to the teachers I interviewed again, and accordingly identify 2-3 teachers to participate in the next part.

The Second Phase

I will conduct pre and post lesson interviews, with lesson observations in between. During lesson observations, I will act as an observer. The pre- and post-lesson interviews will be designed to allow the teachers to reflect on their lessons.

4.2 Briefly outline your proposed research methods, including who will be your research participants and where you will be working (no more than 300 words).

- **Please provide details of any relevant demographic detail of participants (age, gender, race, ethnicity etc)**

The participants are secondary school mathematics teachers (KS 3, 4 & 5). They will be interviewed, then 2-3 of them will be observed over 10-11 lessons. The work will take place in 1-2 schools. I will work on providing qualitative findings established on an interpretative research methodology. Data will be collected using interviews and lesson observations as follows: • Audio-recorded semi-structured interviews based on tasks that will be used as a trigger of discussion. This will be done with 7-10 teachers. Written answers to the tasks will be required, followed by a discussion of the teacher's answers. The data collected at this stage, will help me identify the teachers who are going to be invited to participate in the next stage. Interview duration is 30-40 minutes. • Audio-recorded pilot observations (1 lesson observation) done with up to 7 teachers that are likely to participate in the next stage. These will help me identify the participants for the next stage based on the teachers' actions in classroom and their ways of using technology for mathematics teaching. • Audio-recorded or video-recorded lesson observations (over 10 lessons) with pre-lesson and post-lesson interviews. The interviews in this stage are informal conversations (over up to 15 minutes per lesson) with teachers about their planning, expectations, outcomes and evaluations of the lessons. They also include reflection on events observed during lessons. Data will be collected from the teachers' comments, actions, worksheets, materials, resources, as well as lesson plans. The choice between video or audio recordings will be made depending on the consents I get from the teachers, students, parents and schools. I might have some of the observed teacher's students interviewed. This is done to seek students' comments on their mathematics lessons and get more data about the teachers' work. It will happen only if some students volunteer to say something about their teacher or their lessons; and will be analysed in comparison with the teachers' aims and comments.

4.3 Briefly explain how you plan to gain access to prospective research participants. (no more than 300 words).

- **If children/young people (or other vulnerable people, such as people with mental illness) are to be involved, give details of how gatekeeper permission will be obtained. Please provide any relevant documentation (letters of invite, emails etc) that might be relevant**

- **Is there any sense in which participants might be 'obliged' to participate – as in the case of students, prisoners or patients – or are volunteers being recruited? How will you ensure fully informed and freely given consent in the recruitment process? Entitlement to withdraw consent must be indicated and when that entitlement lapses.**

I am planning to recruit participants for part two of the first phase (interviews) in one of two ways:

1- I will look for secondary school contact details online, then contact the head of school to check if the school is willing to take part in the study. Hence, a poster will be handed to the head teacher to post it in the notice board. Also, letters of invite, information sheet and consent forms will be posted or handed in to the schools and to the mathematics teachers in these schools.

2- Some mathematics teacher might be contacted directly, without contacting their schools first. This will happen if their email addresses are available online, or by passing the information sheets to them through my social network.

By trying to approach teachers directly, or using a poster in a school notice board, the chances of those teachers feeling obliged to participate because their head teachers want them to will be reduced.

For the second phase of the study, I will approach the teachers who participated in the first phase and check if they (upon the approval of their head teachers) are willing to take part in the second phase of this study. Accordingly, information sheets and consent forms will be given to head teachers and teachers.

Before I start lesson observations, letters of invites will be prepared and given to students (KS3, 4 & 5) for parental/adult consent. Although, this study is about teachers and there is no direct data collection about children, volunteers from students might be sought if the progress of the data collection implies a need for students' feedback on their mathematics lessons. There should not be any chances of students feeling obliged to participate as this will be on a completely voluntary basis, if needed. And the invitation will be in the form of a sentence displayed on the board saying "If you wish, you can talk to Lina about your mathematics lesson after this class finishes".

4.4 Please state who will have access to the data and what measures will be adopted to maintain the confidentiality of the research subject and to comply with data protection requirements e.g. will the data be anonymised? (No more than 300 words.)

I will be the only one who have access the raw date. However, my supervisors will have access to some of the data (transcripts & recordings). Videotapes, audiotapes and transcripts will be securely stored, and then destroyed 10 years after the end of the work. Any names to be mentioned in the thesis or publications will be substitutes for the originals.

Data will be stored in a secure place. Hard copies will be kept in a secure locker. While electronic data will be kept is a password secured folder in my personal computer and on a hard drive.

4.5 Will you require access to data on participants held by a third party? In cases where participants will be identified from information held by another party (for example, a doctor or school) describe the arrangements you intend to make to gain access to this information (no more than 300 words).

No

4.6 Please give details of how consent is to be obtained (no more than 300 words).

Copies of proposed information sheets and consent forms, written in simple, non-technical language, MUST accompany this proposal form. You may need more than one information sheet and consent form for different types of participants. (Do not include the text of these documents in this space).

For the first phase of this study: Information sheets and consent forms will be given the teachers who are going to be interviewed and/or to their schools' heads if they are contacted through schools. For the second phase: information sheets and consent forms will be handed in to the teachers, their students and also their school managers.

4.7 If any payment or incentive will be made to any participant, please explain what it is and provide the justification (no more than 300 words).

As participating in this study adds more work to the teachers' schedule, and as a thank you gesture, I wish to give those who participate in both of the first and second phases £10 gift vouchers. This will be given at the end of the work and will not be mentioned before or during the data collection, in order to avoid the risks of the teachers feeling that these vouchers are given to them in order to say or do specific things.

4.8 What is the anticipated use of the data, forms of publication and dissemination of findings etc.? (No more than 300 words.)

PhD thesis, conferences posters and papers, and journal articles.

4.9 Findings of this research/project would usually be made available to participants. Please provide details of the form and timescale for feedback. What commitments will be made to participants regarding feedback? How will these obligations be verified? If findings are not to be provided to participants, explain why. (No more than 300 words.)

Findings will be made available to participants. A list of recommendations and summary of outcomes will be handed to participants by summer 2017.

4.10 Please add here any other ethical considerations the ethics committee may need to be made aware of (no more than 300 words).

- **If you are conducting research in a space where individuals may also choose not to participate, how will you ensure they will not be included in any data collection or adversely affected by non-participation? An example of this might be in a classroom where observation and video recording of a new teaching strategy is being assessed. If consent for all students to be videoed is not received, how will you ensure that a) those children will not be videoed and/or b) that if they are removed from that space, that they are not negatively affected by that?**

A discrete microphone will be provided to the teachers. In case of video recording, students who do not wish to participate will be seated in a way that the camera does not capture them and their data will not be used at all in this study.

4.11 What risks or costs to the participants are entailed in involvement in the research/project? Are there any potential physical, psychological or disclosure dangers that can be anticipated? What is the possible harm to the participant or society from their participation or from the project as a whole? What procedures have been established for the care and protection of participants (e.g. insurance, medical cover, counselling or other support) and the control of any information gained from them or about them?

The risks include stress due to the presence of a researcher within the classroom, this applies to teachers and students. The information will be only available to the researcher. However, upon publication, and because there are not many participants taking part in this study, there is a chance that the participants will be identified by people in their small community who know about the study and read it.

4.12 What is the possible benefit to the participant or society from their participation or from the project as a whole?

Possible benefits include: recommendations suggested by the researcher, and personal reflections on practice and beliefs that help the teachers rethink their work and progress it.

4.13 Comment on any cultural, social or gender-based characteristics of the participants which have affected the design of the project or which may affect its conduct. This may be particularly relevant if conducting research overseas or with a particular cultural group

- **You should also comment on any cultural, social or gender-based characteristics of you as the researcher that may also affect the design of the project or which may affect its conduct**

The study is conducted in the UK, at British schools. However, I am not looking at specific cultural or social class, or gender.

4.14 Identify any significant environmental impacts arising from your research/project and the measures you will take to minimise risk of impact.

No environmental impact anticipated.

4.15 Please state any precautions being taken to protect your health and safety. Have you taken out travel and health insurance for the full period of the research? If not, why not. Have you read and acted upon FCO travel advice (https://www.gov.uk/foreign-travel-advice)? If acted upon, how?

- **Provide details including the date that you have accessed information from FCO or other relevant organization**

- **If you have undertaken the EDU Risk Assessment form for Field Study activities, please indicate if this was approved and date of approval**

As my study is taking part in the UK, there is no need for health or travel insurance. But, I will make sure that my family is aware of my travelling plans and schedule and I have a working mobile phone to contact people and be contacted when needed.

4.16 Please state any precautions being taken to protect the health and safety of other researchers and others associated with the project (as distinct from the participants or the applicant).

No other researchers in this project.

4.17 The UEA's staff and students will seek to comply with travel and research guidance provided by the British Government and the Governments (and Embassies) of host countries. This pertains to research permission, in-country ethical clearance, visas, health and safety information, and other travel advisory notices where applicable. If this research project is being undertaken outside the UK, has formal permission/a research permit been sought to conduct this research? Please describe the action you have taken and if a formal permit has not been sought please explain why this is not necessary/appropriate (for very short studies it is not always appropriate to apply for formal clearance, for example).

Not applicable.

4.18 Are there any procedures in place for external monitoring of the research, for instance by a funding agency?

No

5. DECLARATION:

Please complete the following boxes with YES, NO, or NOT APPLICABLE:

I have read (and discussed with my supervisor if student) the University's Research Ethics Policy, Principle and Procedures, and consulted the British Educational Research Association's Revised Ethical Guidelines for Educational Research and other available documentation on the EDU Research Ethics webpage and, when appropriate, the BACP Guidelines for Research Ethics.	Y
I am aware of the relevant sections of the Data Protection Act (1998): http://www.hmso.gov.uk/acts/acts1998/19980029.htm and Freedom of Information Act (2005).	Y
Data gathering activities involving schools and other organizations will be carried out only with the agreement of the head of school/organization, or an authorised representative, and after adequate notice has been given.	Y
The purpose and procedures of the research, and the potential benefits and costs of participating (e.g. the amount of their time involved), will be fully explained to prospective research participants at the outset.	Y
My full identity will be revealed to potential participants.	Y
Prospective participants will be informed that data collected will be treated in the strictest confidence and will only be reported in anonymised form unless identified explicitly and agreed upon	Y
All potential participants will be asked to give their explicit, written consent to participating in the research, and, where consent is given, separate copies of this will be retained by both researcher and participant.	Y
In addition to the consent of the individuals concerned, the signed consent of a parent/carer will be required to sanction the participation of minors (i.e. persons under 16 years of age).	Y
Undue pressure will not be placed on individuals or institutions to participate in research activities.	Y
The treatment of potential research participants will in no way be prejudiced if they choose not to participate in the project.	Y
I will provide participants with my UEA contact details (*not* my personal contact details) and those of my supervisor, in order that they are able to make contact in relation to any aspect of the research, should they wish to do so. I will notify participants that complaints can be made to the Head of School.	Y

Participants will be made aware that they may freely withdraw from the project at any time without risk or prejudice.	Y
Research will be carried out with regard for mutually convenient times and negotiated in a way that seeks to minimise disruption to schedules and burdens on participants	Y
At all times during the conduct of the research I will behave in an appropriate, professional manner and take steps to ensure that neither myself nor research participants are placed at risk.	Y
The dignity and interests of research participants will be respected at all times, and steps will be taken to ensure that no harm will result from participating in the research	Y
The views of all participants in the research will be respected.	Y
Special efforts will be made to be sensitive to differences relating to age, culture, disability, race, sex, religion and sexual orientation, amongst research participants, when planning, conducting and reporting on the research.	Y
Data generated by the research (e.g. transcripts of research interviews) will be kept in a safe and secure location and will be used purely for the purposes of the research project (including dissemination of findings). No-one other than research colleagues, professional transcribers and supervisors will have access to any identifiable raw data collected, unless written permission has been explicitly given by the identified research participant.	Y
Research participants will have the right of access to any data pertaining to them.	Y
All necessary steps will be taken to protect the privacy and ensure the anonymity and non-traceability of participants – e.g. by the use of pseudonyms, for both individual and institutional participants, in any written reports of the research and other forms of dissemination.	Y

I am satisfied that all ethical issues have been identified and that satisfactory procedures are in place to deal with those issues in this research project. I will abide by the procedures described in this form.

Name of Applicant:	**Lina Kayali**
Date:	**27/04/2016**

PGR Supervisor declaration (for PGR student research only)

I have discussed the ethics of the proposed research with the student and am satisfied that all ethical issues have been identified and that satisfactory procedures are in place to deal with those issues in this research project.

Name of PGR Supervisor:	**Dr. Irene Biza**
Date:	**27/04/2016**

EDU ETHICS COMMITTEE 2014/15

Appendix I: Ethics application feedback

EDU ETHICS APPLICATION FEEDBACK 2015-2016

APPLICANT DETAILS	
Name:	Lina Kayali
School:	EDU
Current Status:	**PGR Student**
UEA Email address:	**L.Kayali@uea.ac.uk**

EDU Recommendation	✓
Approved, data collection can begin	✓
Minor revisions/further details required (see feedback below)	
Not Approved, resubmission required (see feedback below)	

EDU REC feedback to applicant: Chair review date 2.6.16……………………….

Comments: the Chair thanks you for your responses and can now give full approval; you may begin data collection. Please ensure you use the revised PIS forms and address the additional point under 4.2. within any that you use.

Application form

4.2 - the first part of the descriptor is confusing and doesn't match to what was said in 4.1 in regards to overall how many – it appears that the first section is describing the 'proper' project rather than the pilot? Please clarify

The second sentence in the descriptor has been amended to "They will be interviewed, then I will do a pilot observation with up to 7 of them, and finally 2-3 of them will be observed over 10 lessons" – addressed and approved

4.2 – please clarify the indication of the 'I might have some of the observed teacher's students interviewed'. Will you or won't you?

I will if I get parents' and students' consent, if not I will reflect on the class's engagement and overall participation. So, the last paragraph in 4.2 was changed to "Audio recorded interviews with students. This is done to seek students' comments on the mathematics lessons I observe, and get more data about the teachers' work. It will happen only if some students volunteer to say something about their teacher or their lessons; and will be analyzed in comparison with the teachers' aims and comments. If no students volunteer, I will not conduct any interviews with students and instead I will reflect on students' engagement in classroom" - – addressed and approved and you need to ensure that the 'possibility' of this is identified in the relevant PIS

4.11 – even though the risks may be minimal, can you reflect on what those issues may be in more detail and what might you do to alleviate them?

More details were added and highlighted:

- The risks include stress due to the presence of a researcher within the classroom, this applies to teachers and students. Regarding the teachers, I will assure them that this study is not done to evaluate their performance. Also, having the teachers interviewed will help minimize this risk as they will be familiar with the researcher by the time of the observation. Also having informal interviews before and after the lesson will help making the teachers more comfortable. Regarding students, I will do my best to make my presence in the class unnoticed by acting as an observer, in a way that does not disturb their usual lesson flow.

- There is a chance that teachers might be pushed by their school management to take part in the study. In an attempt to avoid this, I will try to approach teachers individually or through posters in schools.
- The chances of students feeling obliged to participate in interviews should be minimal as this will be on a completely voluntary basis. And the invitation will be in the form of a sentence displayed on the board saying "If you wish, you can talk to Lina about your mathematics lesson after this class finishes". I will also assure the students that I will not share what they say with their teachers and their names will be anonymized. I will phrase the interview questions in a way that makes the students feel comfortable.
- The information will be only available to the researcher. However, upon publication, and because there are not many participants taking part in this study, there is a chance that the participants will be identified by people in their small community who know about the study and read it. – addressed and approved

PIS parents

(13) Who is running the study? Should have your supervisor's details too

(14) Supervisor details inseted "under the supervision of Dr. Irene Biza, School of Education and Lifelong Learning, University of East Anglia". – addressed and approved

(15) Section '**Does my child have to be in the study? Can they withdraw from the study once they've**

(16) **started?** If you're able to identify students so they won't be included in any note taking etc, then

(17) you should be able to identify any student that you videoed in the classroom observations. This

(18) needs to be adjusted. You need to clarify what you mean by an interview conversation and then the

(19) 'lesson recordings'. If the latter is the video observations you should state this and clarify the

(20) withdrawal process.

Lesson recordings are lesson observations that are audio or video recorded, depending on what the participants agree on. Interviews are individual interviews (to talk about the child's maths lesson). The sentence "However, it will not be possible to withdraw their individual comments from our lesson recordings once they have started, as it is a group discussion." was deleted and replaced by "In the latter case, data from individual child will not be used, but the data about the general group activity will be used to comment on teachers' use of technology". – addressed and approved

(21)

(22) Complaint process – this should be you, then your supervisor and then Nalini

Supervisor details inserted:

"If you would like to speak to someone else you can contact my supervisor:

Dr. Irene Biza

School of Education and Lifelong Learning

University of East Anglia

NORWICH NR4 7TJ

Tel: +44 (0) 1603 591741

Email: I.Biza@uea.ac.uk" – addressed and approved

(23) Consent section – in the first you have 'interview' and in the second you have 'focus group'. Clarify

(24) which one and adjust forms accordingly

(25) Adjusted as "interview" – addressed and approved

(26)

PIS child

Videoing the session – why can't you remove their data afterwards? See point identified in PIS parents

Deleted the sentence: "When I video record the lesson, I won't be able to take out the things you say after you have said them. This is because you will be talking in a group and our notes will have all the things that everyone else said as well". You are free to stop participating at any stage or to refuse to answer any of the questions. If you decide at a later time to withdraw from the study your information will be removed from our records and will not be included in any results, up to the point we have analysed and published the results". – *addressed and approved*

How will you plan to tell the children about the outcome of the study? You could indicate that you will give the feedback to the teacher who will share this with them

Amended the last sentence "when I finish the study I will tell your teacher what I learnt and s/he will share this with you." – addressed and approved

PIS teacher interviews phase 1

Section 2 Should have your supervisor's details too

Supervisor details inserted – addressed and approved

Section 11 - this should be you, then your supervisor and then Nalini

Supervisor details inserted – addressed and approved

PIS teachers phase 2

Section 2 Should have your supervisor's details too

Supervisor details inserted – addressed and approved

Section 5 – what about withdrawal from the observations?

Withdrawal from study includes withdrawal from both observations and interviews. – addressed and approved

Section 11 - this should be you, then your supervisor and then Nalini

Supervisor details inserted – addressed and approved

Action required by applicant:

Address requests and points of clarification

Adjust PIS documents

Ethical approval has now been given:

Signed: Kate Russell .... EDU Chair, Research Ethics Committee