COCOMPACT LATTICES IN LOCALLY PRO-p-COMPLETE RANK 2 KAC-MOODY GROUPS

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Abstract. We initiate an investigation of lattices in a new class of locally compact groups, so called locally pro-p-complete Kac-Moody groups. We discover that in rank 2 their cocompact lattices are particularly well-behaved: under mild assumptions, a cocompact lattice in this completion contains no elements of order p. This statement is still an open question for the Caprace-Rémy-Ronan completion. Using this, modulo results of Capdeboscq and Thomas, we classify edge-transitive cocompact lattices and describe a cocompact lattice of minimal covolume.

Given a locally compact group H, it is an intriguing question to find a lattice attaining minimal covolume in H, if, of course, it exists. The earliest important result along these lines is a classical theorem of Siegel that the minimal covolume lattice in $\mathrm{SL}_2(\mathbb{R})$ is the triangle $(2, 3, 7)$-group. In particular, it is cocompact.

Over a non-Archimedean local field the nature of the minimal covolume lattice changes drastically. As shown by Lubotzky [16], the lattice of minimal covolume in $\mathrm{SL}_2(\mathbb{F}_q((t)))$ is $\mathrm{SL}_2(\mathbb{F}_q[t^{-1}])$. It is no longer cocompact.

Another interesting question is to find a cocompact lattice of minimal covolume, i.e., minimal on the set of cocompact lattices. In the same paper [16] Lubotzky answers this question as well for $\mathrm{SL}_2(\mathbb{F}_q((t)))$. Notice that for locally compact groups over a non-Archimedean local field it is rare for the cocompact lattices to exist. As proven by Borel and Harder [2], the only simple Chevalley groups $G(\mathbb{F}_q((t)))$ that admit cocompact lattices are those of type A_{n-1}, i.e., the groups $\mathrm{SL}_n(\mathbb{F}_q((t)))$.

The group $\mathrm{SL}_2(\mathbb{F}_q((t)))$ is a basic example of a locally compact Kac-Moody group of rank 2 over the finite field $\mathbb{F}_q$ of $q = p^n$ elements. Let $G = G(\mathbb{F}_q)$ be a minimal Kac-Moody group of rank 2 (with a minor restriction on the Cartan matrix). Capdeboscq and Thomas [6] construct and study lattices in the topological Kac-Moody group $\overline{G}$, the Caprace-Rémy-Ronan completion of G. They find the minimal covolume lattices in $\overline{G}$ among the class of cocompact lattices without elements of order p [6]. This order p restriction is motivated by the fact that the cocompact lattices in $\mathrm{SL}_2(\mathbb{F}_q((t)))$ do not contain any elements of order p. It feels reasonable that the same phenomenon should hold in a general topological Kac-Moody rank-2 group. Capdeboscq and Thomas even conjecture that cocompact lattices in such groups contain no elements of order p.

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However, there are other topological Kac-Moody groups, for instance, the Mathieu-Rousseau group G^{ma+}, the Carbone-Garland group $G^{c\lambda}$ and the locally pro-p-complete group $\widehat{G}$, introduced by Capdeboscq and Rumynin [5]. Denoting the closure of G in G^{ma+} by $\overline{G}^+$ (note that $G^{ma+} = \overline{G}^+$ in many but not all cases), we have surjective group homomorphisms

$$\widehat{G} \twoheadrightarrow \overline{G}^+ \twoheadrightarrow G^{c\lambda} \twoheadrightarrow \overline{G}.$$

Sometimes these groups coincide, for instance, in the untwisted affine case of rank at least 3 [5, Prop. 3.2]. On other occasions, the groups are distinct, for example, $\widehat{G} \neq \overline{G}$ in rank 2. The precise relations among these groups are highly intriguing (see [4, 5, 10, 17, 20] for several discussions on this topic). It is fair to conclude that the study of all the completions is interesting: this yields new information about all of them as well as the original group G.

In the present paper we study cocompact lattices in the local pro-p-completion $\widehat{G}$ in rank 2. We prove that a cocompact lattice in $\widehat{G}$ contains no elements of order p. We establish pushing and pulling procedures connecting cocompact lattices in $\widehat{G}$ with cocompact lattices in $\overline{G}$ without elements of order p. As an upshot, we describe the minimal covolume cocompact lattices in $\widehat{G}$. This demonstrates that the lattices in the local pro-p-completion $\widehat{G}$ is an interesting subject, deserving further attention of mathematicians. It would be equally important to investigate lattices in the other complete Kac-Moody groups.

Let us describe the content of this paper section-by-section. In Section 1 we set up the scene defining the key Kac-Moody groups and discussing their structure.

In Section 2 we study the elements of order p and the cocompact lattices in $\widehat{G}$. We prove one of the main results of this paper that enable our investigation of covolumes later on:

Main Theorem 1. *(fusion of Theorems 2.3 and 2.8) Let A be a 2×2 generalised Cartan matrix with all $|a_{ij}| \geqslant 2$. Let $\mathcal{D}$ be a root datum of type A. The following statements hold for the corresponding (to $\mathcal{D}$) locally pro-p-complete Kac-Moody group $\widehat{G}$ over the field of $q = p^a$ elements:*

1. *A cocompact lattice Γ of $\widehat{G}$ contains no elements of order p.*
2. *Any element of order p in $\widehat{G}$ is contained in a conjugate of the subgroup $\mathcal{U}$ of $\widehat{G}$.*

In Section 3 we define the push-forward and the pull-back of cocompact lattices. If $\Gamma \leqslant \widehat{G}$ is a cocompact lattice, then its push-forward $\pi(\Gamma) \leqslant \overline{G}$ is a cocompact lattice. If $\Gamma \leqslant G \leqslant \overline{G}$ is a cocompact lattice, then its pull-back $\Gamma \leqslant G \leqslant \widehat{G}$ is a cocompact lattice. Notice that both push-forward and pull-back preserve the isomorphism class of a cocompact lattice.

In Section 4 we compare covolumes of cocompact lattices in $\widehat{G}$ and $\overline{G}$. After a suitable normalisation the covolumes do not change and could be computed on the set $\mathcal{X}_0$ of vertices of the Tits building of G (see Proposition 4.1):

$$\widehat{\mu}(\Gamma\backslash\widehat{G}) = \overline{\mu}(\Gamma\backslash\overline{G}) = \sum_{[\mathbf{x}]\in\Gamma\backslash\mathcal{X}_0} \frac{1}{|\Gamma_{\mathbf{x}}|} .$$

In Section 5 we utilise the results of Capdeboscq and Thomas [6] about cocompact lattices in $\overline{G}$. We prove the second main result of this paper:

Main Theorem 2. *Let A be a symmetric 2×2 generalised Cartan matrix with all $|a_{ij}| \geqslant 2$. Let $\mathcal{D}$ be a simply-connected root datum of type A. The following statements hold for the corresponding (to $\mathcal{D}$) locally pro-p-complete Kac-Moody group $\widehat{G}$ over the field of $q = p^a$ elements:*

(1) *$\widehat{G}$ admits a cocompact lattice.*
(2) *If $q \geqslant 514$, then there exist $\delta \in \{1, 2, 4\}$ such that*

$$\min\{\widehat{\mu}(\Gamma\backslash\widehat{G}) \mid \Gamma \text{ is a cocompact lattice}\} = \frac{2}{(q+1)|Z(G)|\delta} .$$

In Section 6 we discuss cocompact lattices in $\widehat{G}$ for more general Cartan matrices. We formulate several questions and conjectures facilitating further research of locally pro-p-complete groups $\widehat{G}$ and their cocompact lattices.

1. Two Completions of Kac-Moody Groups

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field, where $q = p^a$, for some prime p. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix}$$

be a 2×2 generalised Cartan matrix with $\max(a_{21}, a_{12}) \leqslant -2$.

Recall that *a root datum of type A* is a quintuple $\mathcal{D} = (A, \mathcal{Z}, \mathcal{Y}, \Pi, \Pi^\vee)$ where

- $\mathcal{Y}$ is a finitely generated free abelian group,
- $\mathcal{Z} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{Y}, \mathbb{Z})$ is its dual group,
- $\Pi = \{\alpha_1, \alpha_2\} \subset \mathcal{Z}$ is the (ordered) set of simple roots,
- $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee\} \subset \mathcal{Y}$ is the (ordered) set of simple coroots,

satisfying a single axiom $\alpha_i(\alpha_j^\vee) = a_{ji}$ for all $i, j \in \{1, 2\}$.

The Weyl group of A is the infinite dihedral group

$$W = \langle w_1, w_2 \mid w_1^2 = w_2^2 \rangle .$$

We use the standard notation $[w_i w_j]_m := w_i w_j w_i \ldots$ (m symbols w in the right hand side) for elements of W. The set of real roots is

$$\Phi = \{w\alpha_1, w\alpha_2 \mid w \in W\}.$$

Denote by Φ_+ the set of positive real roots and Φ_- the set of negative real roots. The set Φ_+ can be written as a disjoint union of the two sets Φ_+^1 and Φ_+^2:

$$\Phi_+^1 := \{\alpha_1, w_1\alpha_2, w_1w_2\alpha_1, w_1w_2w_1\alpha_2, \ldots, (w_1w_2)^n\alpha_1, (w_1w_2)^n w_1\alpha_2, \ldots\},$$

$$\Phi_+^2 := \{\alpha_2, w_2\alpha_1, w_2w_1\alpha_2, w_2w_1w_2\alpha_1, \ldots, (w_2w_1)^n\alpha_2, (w_2w_1)^n w_2\alpha_1, \ldots\}.$$

We also need the sets $-\Phi_+^1 := \{-\alpha \mid \alpha \in \Phi_+^1\}$ and $-\Phi_+^2$.

Let $G = G_{\mathcal{D}}(\mathbb{F})$ be the Kac-Moody group over the field $\mathbb{F}$ associated to a root datum $\mathcal{D}$ of type A (cf. [12, 22, 3]). To define it, we introduce an additive group

$$U_\alpha := \{x_\alpha(\mathbf{t}) \mid \mathbf{t} \in \mathbb{F}\}, \quad x_\alpha(\mathbf{t})x_\alpha(\mathbf{s}) = x_\alpha(\mathbf{t}+\mathbf{s}), \quad U_\alpha \cong (\mathbb{F}, +)$$

for each real root α and the torus

$$H = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{F}^\times .$$

Let F be the free product of H and all U_α, $\alpha \in \Phi$. We require the following elements of F defined for $\alpha \in \Phi$, $\mathbf{t} \in \mathbb{F}^\times$, $i \in \{1, 2\}$:

- $n_\alpha(\mathbf{t}) := x_\alpha(\mathbf{t})x_{-\alpha}(\mathbf{t}^{-1})x_\alpha(\mathbf{t})$, $\quad n_i(\mathbf{t}) := n_{\alpha_i}(\mathbf{t})$,
- $h_\alpha(\mathbf{t}) := n_\alpha(\mathbf{t})n_\alpha(1)^{-1}$, $\quad h_i(\mathbf{t}) := n_{\alpha_i}(\mathbf{t})$.

The Kac-Moody group $G = G_{\mathcal{D}}(\mathbb{F})$ of type A is a quotient group of F by the following relations:

(1) $h_i(\mathbf{t}) = \alpha_i^\vee \otimes \mathbf{t}$,
(2) $(y \otimes \mathbf{t}) x_{\alpha_i}(\mathbf{s})(y \otimes \mathbf{t})^{-1} = x_i(\mathbf{t}^{\alpha_i(y)}\mathbf{s})$,
(3) $n_i(1)(y \otimes \mathbf{t}) n_i^{-1}(1) = w_i(y) \otimes \mathbf{t}$,
(4) $n_i(1) x_\alpha(\mathbf{t}) n_i(1)^{-1} = x_{w_i(\alpha)}(\epsilon_{i,\alpha}\mathbf{t})$ for uniquely determined $\epsilon_{i,\alpha} \in \{-1, 1\}$,
(5) $x_\alpha(\mathbf{t}) x_\beta(\mathbf{s}) = x_\beta(\mathbf{s}) x_\alpha(\mathbf{t})$, if $\{\alpha, \beta\} \subset \Phi_+^1 \cup -\Phi_+^2$ or $\{\alpha, \beta\} \subset \Phi_+^2 \cup -\Phi_+^1$.

The choice of $\epsilon_{i,\alpha}$ in the relation (4) depends on the events in the corresponding Kac-Moody algebra $\mathfrak{g}$ over the complex numbers (cf. [12]). Let us elaborate. The Lie algebra $\mathfrak{g}$ is generated by

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1, \mathbf{f}_2 \ \text{ and } \ \mathfrak{h} = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{C}\,.$$

Using the adjoint representation $\mathrm{ad}(\mathbf{x})(\mathbf{y}) = [\mathbf{x}, \mathbf{y}]$, we define the following operators on $\mathfrak{g}$:

$$\eta_i := \exp(\mathrm{ad}(\mathbf{e}_i)) \exp(\mathrm{ad}(\mathbf{f}_i)) \exp(\mathrm{ad}(\mathbf{e}_i))\,, \quad [\eta_i \eta_j]_m := \eta_i \eta_j \eta_i \ldots \ (m \text{ symbols } \eta).$$

It is easy to observe that $\eta_i^2(\mathbf{e}_i) = \mathbf{e}_i$, $\eta_i^2(\mathbf{f}_i) = \mathbf{f}_i$ and $\eta_i^2(\mathbf{e}_j) = \pm\mathbf{e}_j$, $\eta_i^2(\mathbf{f}_j) = \pm\mathbf{f}_j$ with the same sign for $i \neq j$. Let us define the signs $\epsilon_i \in \{-1, 1\}$ by

$$\eta_1^2(\mathbf{e}_2) = \epsilon_1 \mathbf{e}_2, \quad \eta_2^2(\mathbf{e}_1) = \epsilon_2 \mathbf{e}_1\,.$$

A real root α can be written as $\alpha = w(\alpha_j)$ for unique j and $w = [w_s w_t]_m \in W$. We define the signs by

$$\epsilon_{i,\alpha} := \begin{cases} 1, \text{ if } i \neq s\,, \\ \epsilon_i^{k_t}, \text{ if } i = s, \ w_i(\alpha) = k_1\alpha_1 + k_2\alpha_2\,. \end{cases}$$

Notice that this agrees with the definition in Carter and Chen [12] because we can define the corresponding root element in $\mathfrak{g}$ by $\mathbf{e}_\alpha := [\eta_s \eta_t]_m(\mathbf{e}_j)$ so that

$$\eta_i(\mathbf{e}_\alpha) = \eta_i[\eta_s\eta_t]_m(\mathbf{e}_j) = \begin{cases} [\eta_t\eta_s]_{m+1}(\mathbf{e}_j) = \mathbf{e}_{w_i(\alpha)}, \text{ if } i \neq s\,, \\ \eta_i^2[\eta_t\eta_s]_{m-1}(\mathbf{e}_j) = \eta_i^2(\mathbf{e}_{w_i(\alpha)}) = \epsilon_i^{k_t}\mathbf{e}_{w_i(\alpha)}, \text{ if } i = s\,. \end{cases}$$

Notice that the relation (5) is simpler in our case (rank 2 and $\max(a_{21}, a_{12}) \leqslant -2$) than in the general case [12] for two reasons. First, the conditions in the relation (5) describe precisely all prenilpotent pairs of roots α, β. Second, $\alpha + \beta$ is no longer a root so that all our commutator relations are trivial.

Let $\mathbb{F} = \mathbb{F}_q$. Note that $U_\alpha \cong (\mathbb{F}_q, +)$. We also have subgroups

$$U_- = \langle U_\alpha \mid \alpha \in \Phi_- \rangle \ \text{ and } \ U := U_+ = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle.$$

The group G admits a (B, N)-pair structure, with

$$B = U \rtimes H, \quad H = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{F}^\times \text{ and } N \geqslant H \text{ with } N/H \cong W$$

where $H = B \cap N$. Then the standard maximal parabolic subgroups are

$$P_i := B \amalg B\dot{w}_i B \text{ for } i = 1, 2.$$

The (B, N)-pair structure yields an associated Tits building $\mathcal{X}$ of G. In our case (a Kac-Moody group of rank 2 over a finite field $\mathbb{F}_q$) the building $\mathcal{X}$ is a $(q+1)$-regular tree. Let $\mathcal{X}_0$ and $\mathcal{X}_1$ denote respectively the set of vertices and the set of edges of $\mathcal{X}$. The elements of $\mathcal{X}_0$ correspond to the conjugates of the parabolics P_i, $i = 1, 2$, and elements of $\mathcal{X}_1$ to the conjugates of B. In particular, we have two types of vertices - corresponding respectively to P_1 and P_2. The group G acts naturally on $\mathcal{X}$ with fundamental domain an edge. The action on $\mathcal{X}$ yields an action topology, called *the building topology* on G and the corresponding completion $\breve{G}$. The kernel of the natural map $G \to \breve{G}$ is non-trivial: it is equal to the centre of G.

Caprace and Rémy [9] define a local version of the same topology. The completion $\overline{G}$ of G with respect to this topology retains the centre. Let $\mathbf{c} \in \mathcal{X}_1$ be the edge fixed by B. For each $n \in \mathbb{N}$ define

$$U_{+,n} := \{g \in U \mid g \cdot \mathbf{c}' = \mathbf{c}' \text{ for every edge } \mathbf{c}' \text{ such that } d(\mathbf{c}, \mathbf{c}') \leqslant n\},$$

where d denotes the distance on $\mathcal{X}$. Using this Caprace and Rémy define the following left-invariant metric $d_+ : G \times G \to \mathbb{R}_+$ on G:

$$d_+(g,h) = \begin{cases} 2, \text{ if } g^{-1}h \notin U \ , \\ 2^{-n}, \text{ if } g^{-1}h \in U \ \text{ and } \ n = \max\{k \in \mathbb{N} \mid g^{-1}h \in U_{+,k}\} \end{cases}$$

for all $g, h \in G$ [9]. Write $\overline{G}$ for the completion of G with respect to this metric. It is worth noticing that $\overline{G}$ is a locally compact totally disconnected group with a (B,N)-pair $(\overline{B}, N)$, where $\overline{B}$ denotes the closure of B in $\overline{G}$ [9].

Capdeboscq and Rumynin [5] introduce the local version of the pro-p-topology on G. Let

$$\mathcal{F} := \{A \leqslant U \mid |U : A| = p^k, \text{ for some } k \in \mathbb{N}\}.$$

The set $\mathcal{F}$ is a fundamental system of neighbourhoods of 1 in U and, thus, gives a topology on B. This also defines a topology on G [5, Th. 1.2]. Capdeboscq and Rumynin show that the completion $\widehat{G}$ of G with respect to this topology is a locally compact totally disconnected group with (B,N)-pair $(\widehat{B}, N)$, where $\widehat{B}$ is the completion of B [5]. Moreover, $\widehat{B}$ is open in $\widehat{G}$ and $\widehat{B} = \widehat{U} \rtimes H$, with $\widehat{U}$ being the full pro-p completion of U [5]. We will call the group $\widehat{G}$ *the local pro-p completion* of G.

2. Behaviour of elements of order p in $\widehat{G}$

Having introduced the groups $\widehat{G}$ and $\overline{G}$, we are ready to investigate their cocompact lattices. To discuss these two groups in parallel, we talk about a group $\widetilde{G} \in \{\widehat{G}, \overline{G}\}$.

Recall that *a cocompact lattice* is a discrete subgroup $\Gamma \leqslant \widetilde{G}$ such that the quotient topological space $\Gamma \backslash \widetilde{G}$ is compact. The space $\Gamma \backslash \widetilde{G}$ admits a finite $\widetilde{G}$-invariant measure (cf. [1, Ch. 1]).

Using Φ_+^1 and Φ_+^2 from Section 1, let us define the following abelian p-groups:

$$U_i := \langle U_\alpha \mid \alpha \in \Phi_+^i \rangle \ \text{ and } \ -U_i := \langle U_\alpha \mid -\alpha \in \Phi_+^i \rangle, \text{ for } i = 1, 2.$$

We may now consider

$$\mathcal{U} = \mathcal{U}_1 := \mathrm{cl}(U_1 \times -U_2) \ \text{ and } \ \mathcal{U}_2 := \mathrm{cl}(-U_1 \times U_2),$$

where cl( ) denotes the closure in the relevant topology on the complete Kac-Moody group. We will be interested in the conjugates of $\mathcal{U}$. In particular, note that $\mathcal{U}_2 = w_1 \mathcal{U} w_1^{-1}$.

Definition 2.1. We say that a complete Kac-Moody group $\widetilde{G}$ is *p-well-behaved*, if the following conditions hold:

(P1) Cocompact lattices in $\widetilde{G}$ do not contain elements of order p.

(P2) Any element of order p in $\widetilde{G}$ is contained in a conjugate of the subgroup $\mathcal{U}$.

The aim of this section is to show that the local pro-p complete group $\widehat{G}$ is p-well-behaved. It is an open conjecture that $\overline{G}$ is p-well-behaved [6].

Lemma 2.2. *Let $U = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle$, and U_1 and U_2 are as above. Then U is a free product of U_1 and U_2.*

Proof. By [23, Prop. 4], U is an amalgamated product of U_1 and U_2 along the intersection $U_0 = U_1 \cap U_2$. However, $U_0 = 1$ [6]. The result follows. □

Now we are ready to tackle Property (P2):

Theorem 2.3. *Any element of order p in $\widehat{G}$ is contained in a conjugate of the subgroup $\mathcal{U}$ of $\widehat{G}$.*

Proof. Let $g \in \widehat{G}$ be an element of order p. Then g lies in a conjugate of the Sylow pro-p-subgroup $\widehat{U}$ of $\widehat{G}$. Thus, without loss of generality we may assume that $g \in \widehat{U}$. By Lemma 2.2, $U = U_1 * U_2$ and, thus, $\widehat{U} = \widehat{U_1 * U_2}$. Let $\amalg$ denote the free pro-p product [19, 9.1]. Notice that the pro-p completion commutes with the free product [19, 9.1.1]:

$$\widehat{U} = \widehat{U_1 * U_2} \cong \widehat{U_1} \amalg \widehat{U_2}.$$

Hertfort and Ribes [14] show that if a group decomposes as a free pro-p product of two groups, then all the torsion is contained in a conjugate of one of the factors. Hence, g is contained in a conjugate of one of $\widehat{U_i}$, $i = 1, 2$. Since $\mathcal{U} = \mathrm{cl}(U_1 \times -U_2)$, the proof is now complete. □

Our next step is to explain why Property (P2) implies Property (P1). We begin our investigation with a lemma.

Lemma 2.4. *Let G be a minimal Kac-Moody group of rank 2 over $\mathbb{F}_q$, $\widehat{G}$ its local pro-p completion and $\overline{G}$ its Caprace-Rémy completion. Let*

$$C := C(\widehat{G}, \widehat{U}) = \bigcap_{g \in \widehat{G}} g\widehat{U}g^{-1} .$$

Then

$$\widehat{G}/C \cong \overline{G}$$

as topological groups.

Proof. Recall that for two groups to be isomorphic as topological groups, we need to show existence of a continuous abstract group isomorphism with a continuous inverse. Let $\pi : \widehat{G} \twoheadrightarrow \overline{G}$ be the natural map. This is an open continuous homomorphism with kernel $\ker(\pi) = C$ [5]. Consider the group $\widehat{G}/C$ as a topological group with respect to the quotient topology coming from $\widehat{G}$. We have a commutative diagram:

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\pi} & \overline{G} \\ & \searrow^{\theta} & \uparrow{\scriptstyle \bar{\pi}} \\ & & \widehat{G}/C \end{array}$$

where θ is the quotient map and $\bar{\pi}$ is the natural map induced by π and factoring through θ. Note that since $\ker(\pi) = C$, $\bar{\pi}$ is in fact an isomorphism of abstract groups. Moreover, the map θ is a continuous, open surjection [15, 5.16, 5.17]. Thus, if $N \subseteq \overline{G}$ is open,

$$\overline{\pi}^{-1}(N) = \theta\big(\pi^{-1}(N)\big)$$

is open, giving the continuity of $\overline{\pi}$. Furthermore, $\overline{\pi}$ also has a continuous inverse. Suppose $L \subseteq \widehat{G}/C$ is open and $\psi := \overline{\pi}^{-1}$, then

$$\psi^{-1}(L) = \pi\big(\theta^{-1}(L)\big),$$

which is open by continuity of θ, openness of π and commutativity of the diagram. □

Since $C \leqslant \widehat{U} \leqslant \widehat{G}$, where $\widehat{U}$ is a compact open and closed (since $\widehat{G}$ is Hausdorff) subgroup of G, C is a closed compact pro-p subgroup of $\widehat{G}$.

Observe also that, as C is compact, the quotient map $\theta : \widehat{G} \to \widehat{G}/C$ is closed [15, 5.18]. It follows from Lemma 2.4 that the natural map $\pi : \widehat{G} \to \overline{G}$ is closed.

Lemma 2.5. *The group $\widehat{G}$ is first countable.*

Proof. It suffices to show that $\widehat{U}$ is first countable since $\widehat{U}$ is open in $\widehat{G}$.

To establish the first countability of $\widehat{U}$, it is sufficient to show that it admits a countably infinite generating set which converges to 1 [19, Rem. 2.6.7]. By Lemma 2.2,

$$\widehat{U} = \widehat{U_1 * U_2}\ .$$

The group U_i is a countably dimensional vector space over the field $\mathbb{F}_q$. Its $\mathbb{F}_p$-basis $e_k^{(i)}$ forms a countable generating sequence, converging to 1 in the pro-p-topology. It follows that $e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)} \ldots$ is a countable generating sequence of $\widehat{U}$, converging to 1. □

Since $\widehat{G}$ is first countable and, thus, metrizable, its topology is determined by sequences. In particular, in a metrizable topological space compactness is equivalent to sequential compactness.

Proposition 2.6. *(cf.* [6, Cor. 4.3.]*) Let u be an element of $\mathcal{U}$. Then there exists $g \in \widehat{G}$, such that the sequence*

$$x_n := g^n u g^{-n}\ ,\ n \in \mathbb{N}$$

has a limit point in C.

Proof. Since π is surjective, [6, Cor. 4.3] implies that there exists $g \in \widehat{G}$, such that

$$\lim_{n\to\infty} \pi(x_n) = \lim_{n\to\infty} \pi(g)^n \pi(u) \pi(g)^{-n} = 1_{\overline{G}}.$$

Take a compact open subgroup $K \leqslant \widehat{G}$. Since $\pi(K)$ is a neighbourhood of $1_{\overline{G}}$, there exists $N \in \mathbb{N}$, such that $\pi(x_n) \in \pi(K)$ for all $n > N$. Hence, $x_n \in KC$ for all $n > N$.

It remains to observe that KC is compact, and so (x_n) contains a convergent subsequence (y_n) in KC. Its limit $z = \lim y_n$ must belong to C because $\pi(z) = \lim \pi(y_n) = \lim \pi(x_n) = 1_{\overline{G}}$. □

We are nearly ready to prove property (P1) for $\widehat{G}$. The final ingredient we need is the following result for cocompact lattices:

Lemma 2.7. *(cf. [13, p. 10]) Let $\Gamma \leqslant \widehat{G}$ be a cocompact lattice. For each $u \in \Gamma$ its conjugacy class*

$$u^{\widehat{G}} = \{g^{-1}ug \mid g \in \widehat{G}\}$$

is a closed subset of $\widehat{G}$.

Proof. Let us show that Γ admits a compact fundamental domain $\widetilde{K}$ in $\widehat{G}$, i.e., a compact subset $\widetilde{K}$ such that $\widehat{G} = \Gamma\widetilde{K}$. Take a compact open subgroup $K \leqslant \widehat{G}$. Consider the quotient map $\theta : \widehat{G} \twoheadrightarrow \Gamma\backslash\widehat{G}$. For each $x \in \widehat{G}$ the set ΓxK is open and Γ-equivariant. By the definition of the quotient topology, $\theta(x)K = \theta(\Gamma xK)$ is open in $\Gamma\backslash\widehat{G}$. Thus, $\{\theta(x)K \mid x \in \widehat{G}\}$ is an open cover of $\Gamma\backslash\widehat{G}$. But Γ is cocompact, so we can choose a finite subcover $\{\theta(x_i)K \mid i = 1, \ldots, n\}$. It follows that $\widetilde{K} := \bigcup_{i=1}^{n} x_iK$ is a compact fundamental domain.

The rest of the argument follows Gelfand, Graev and Piatetsky-Shapiro [13, p. 10]. Take $x \in \mathrm{cl}(u^{\widehat{G}})$. Since $\widehat{G}$ is first countable, there exists a sequence $(g_i^{-1}ug_i)$ with $g_i \in \widehat{G}$, $i \in \mathbb{N}$, convergent to x. Let us write each g_i as u_ik_i for some $u_i \in \Gamma$, $k_i \in \widetilde{K}$. Since $\widetilde{K}$ is compact and first countable, we can choose a convergent subsequence of (k_i). Thus, without loss of generality (k_i) converges to some $k \in \widetilde{K}$. Observe that

$$u_i^{-1}uu_i = k_ig_i^{-1}ug_ik_i^{-1} = k_i(g_i^{-1}ug_i)k_i^{-1} \longrightarrow k(\lim g_i^{-1}ug_i)k^{-1} = kxk^{-1} \ .$$

Since Γ is discrete, $u_n^{-1}uu_n = kxk^{-1}$ for all sufficiently large n, so that $x \in u^{\widehat{G}}$. □

Note that since the map $\pi : \widehat{G} \to \overline{G}$ is closed, the set $\pi(u^{\widehat{G}}) = \pi(u)^{\overline{G}}$ is closed for every u from any cocompact lattice Γ.

Theorem 2.8. *A cocompact lattice in $\widehat{G}$ does not contain elements of order p.*

Proof. Let Γ be a cocompact lattice. Consider $u \in \Gamma$ such that $|u| = p$. By the proof of Theorem 2.3, u is contained in a conjugate of $\widehat{U_1}$ or $\widehat{U_2}$. Without loss of generality, $u \in \widehat{U_1}$.

By Proposition 2.6 there exists a $g \in \widehat{G}$, such that the sequence

$$x_n := g^nug^{-n}, \quad n \in \mathbb{N},$$

has a limit point $x \in C$. By construction, the sequence (x_n) lies in the closed set $u^{\widehat{G}}$. Thus $x \in u^{\widehat{G}} \cap C$. Since C acts trivially on the Tits building of G, so does x. This is a contradiction: non-identity elements of $\widehat{U_1}$ do not act trivially on the Tits building of G. □

Theorem 2.8 and Theorem 2.3 show that $\widehat{G}$ is p-well-behaved.

3. Push and pull for cocompact lattices

Given a continuous homomorphism $\theta : H \to J$ of locally compact groups with a non-discrete kernel, one should not expect a relationship between lattices in H and J. If Γ is a lattice in H, then $\theta(\Gamma)$ is not necessarily a lattice. In the opposite direction, if Γ is a lattice in J, then $\theta^{-1}(\Gamma)$ is never discrete. However, we can push and pull cocompact lattices along the quotient map.

Proposition 3.1. *Let $\theta : H \to J$ be a continuous homomorphism of locally compact groups with a compact kernel K and a closed cocompact image. If Γ is a cocompact lattice in H, then $\theta(\Gamma)$ is a cocompact lattice in J.*

Proof. Let us show that $\theta(\Gamma)\backslash J$ is compact. The multiplication defines a continuous bijective map from the fibre product:

$$\eta : F := (\theta(\Gamma)\backslash\theta(H)) \times_{\theta(H)} J \longrightarrow \theta(\Gamma)\backslash J\,, \quad (\theta(\Gamma)a, b) \mapsto \theta(\Gamma)ab\,. \tag{1}$$

It suffices to prove that the fibre product F is compact. For this we need to find a convergent subnet of an arbitrary net $(\theta(\Gamma)x_i, y_i)_{i\in\mathbb{I}}$ in F.

Since $\theta(H)\backslash J$ is compact, the net $(\theta(H)y_i)_{i\in\mathbb{I}}$ has a convergent subnet. Without loss of generality, $(\theta(H)y_i)_{i\in\mathbb{I}}$ itself is convergent. This means that each y_i can be written as $y_i = h_i z_i$ with $h_i \in \theta(H)$ and the net $(z_i)_{i\in\mathbb{I}}$ convergent in J. Since $\theta(\Gamma)\backslash\theta(H)$ is compact, the net $(\theta(\Gamma)x_i h_i)_{i\in\mathbb{I}}$ has a convergent subnet $(\theta(\Gamma)x_i h_i)_{i\in\mathbb{J}}$. Finally, since $(\theta(\Gamma)x_i, y_i) = (\theta(\Gamma)x_i h_i, z_i)$, it follows that $(\theta(\Gamma)x_i, y_i)_{i\in\mathbb{I}}$ is a convergent subnet we sought.

It remains to show that $\theta(\Gamma)$ is discrete. Suppose not. Then we can pick a net $(x_i)_{i\in\mathbb{I}}$ in Γ, convergent to some $a \in \theta(\Gamma)$, such that $\theta(x_i) \neq a$ for all i.

Choose a compact neighbourhood of identity $V \subseteq J$. The inverse image $\theta^{-1}(Va)$ is compact by the argument for (1) with the new key map

$$\eta : K \times_K \theta^{-1}(Va) \longrightarrow Va \cap \theta(H)\,, \quad (k, c) \mapsto \theta(c)\,.$$

There exists an ordinal $\mathbb{L} < \mathbb{I}$ such that $\theta(x_i) \in Va$ and, consequently, $x_i \in \theta^{-1}(Va)$ for all $i \geqslant \mathbb{L}$. Hence, we can find a convergent subnet $(y_j)_{j\in\mathbb{J}}$ (of x_i). Since Γ is discrete, there exists an ordinal $\mathbb{M} < \mathbb{J}$, such that $y_j = y_{\mathbb{M}}$ for all $j \geqslant \mathbb{M}$. Then $a = \lim \theta(x_i) = \lim \theta(y_j) = \theta(y_{\mathbb{M}})$, a contradiction. □

We can control the push-forward of cocompact lattices along the map $\pi : \widehat{G} \to \overline{G}$ more tightly than for a general map:

Lemma 3.2. *If Γ is a cocompact lattice in $\widehat{G}$, then $\Gamma \cap C = \{1_{\widehat{G}}\}$.*

Proof. First note that $\Gamma \cap C$ is finite since Γ is discrete and C is compact.Since C is the intersection of Sylow pro-p-subgroups of $\widehat{G}$, every finite order element in $\Gamma \cap C$ must have order p^k. But $\widehat{G}$ is p-well-behaved, in particular, cocompact lattices do not contain elements of order p. It follows that $\Gamma \cap C = \{1_{\widehat{G}}\}$. □

Corollary 3.3. *If $\Gamma \leqslant \widehat{G}$ is a cocompact lattice, then $\pi : \Gamma \to \pi(\Gamma)$ is an isomorphism. In particular, the cocompact lattice $\pi(\Gamma) \leqslant \overline{G}$ contains no elements of order p.*

All the cocompact lattices constructed by Capdeboscq and Thomas in $\overline{G}$ are subgroups of G [6, Th. 1.1, Th. 1.2]. In fact, their main result [6, Th. 1.3] can be interpreted as a classification of edge-transitive (see Section 5) cocompact lattices with no elements of order p in $\overline{G}$. All of them are conjugate to subgroups of G and can be lifted using the next proposition:

Proposition 3.4. *Let $\Gamma \leqslant G$ be a cocompact lattice in $\overline{G}$. Then Γ is also a cocompact lattice in $\widehat{G}$.*

Proof. Suppose $\Gamma \leqslant \widehat{G}$ is not discrete. Then there exists a sequence $x_n \in \Gamma$, convergent to $a \in \Gamma$, with $x_n \neq a$ for all n. Since $\pi(a) = \lim \pi(x_n)$ and Γ is discrete in $\overline{G}$, the sequence $\pi(x_n)$ is eventually constant: there exists N, such that $\pi(x_n) = \pi(a)$ (equivalently, $x_n \in aC$) for all $n \geqslant N$. Hence, $x_n \in aC \cap G$ for all $n \geqslant N$ since $\Gamma \subseteq G$.

Now observe that the set $aC \cap G$ has at most one element. Consider two of its elements ag and ah with $g, h \in C$. Then $(ag)^{-1}ah = g^{-1}h \in C \cap G$ that is equal to $\{1\}$ because Γ is a subgroup of $\overline{G}$ as well. Inevitably $g = h$ and $|aC \cap G| \leqslant 1$. It follows that $x_n = x_N$ for all $n \geqslant N$, a contradiction with all $x_n \neq a$. This shows that Γ is discrete in $\widehat{G}$.

The space $\Gamma\backslash\widehat{G}$ is compact because both C and $\Gamma\backslash\overline{G} \cong \Gamma\backslash\widehat{G}/C$ are compact. The proof is identical to the argument for (1) with the key map defined by

$$\eta : \Gamma\backslash\widehat{G} \times_C C \longrightarrow \Gamma\backslash\widehat{G}\,, \quad (\Gamma a, c) \mapsto \Gamma ac\,.$$

□

4. Covolumes

Following Bass and Lubotzky [1], let us discuss how the processes of pushing and pulling cocompact lattices affect their covolumes. Recall that for a locally compact group H acting on a set X with compact open stabilisers $H_{\mathbf{x}}$, $\mathbf{x} \in X$ there is a natural *covolume* of a discrete subgroup $\Gamma \leqslant H$ defined by

$$\mathrm{Vol}(\Gamma\backslash\backslash X) := \sum_{[\mathbf{x}]\in\Gamma\backslash X} \frac{1}{|\Gamma_{\mathbf{x}}|},$$

where $\Gamma_{\mathbf{x}} = H_{\mathbf{x}} \cap \Gamma$. Moreover, $\mathrm{Vol}(\Gamma\backslash\backslash X) < \infty$ if and only if Γ is a lattice (forcing H to be unimodular) and

$$\mu(H\backslash\backslash X) := \sum_{[\mathbf{x}]\in H\backslash X} \frac{1}{\mu(H_{\mathbf{x}})} < \infty,$$

where μ is a right-invariant Haar measure on H. In this case, we can choose μ on H in such a way that (see [1, 1.5])

$$\mathrm{Vol}(\Gamma\backslash\backslash X) = \mu_{\Gamma\backslash H}(\Gamma\backslash H).$$

We wish to compare covolumes of cocompact lattices in $\widehat{G}$ and $\overline{G}$. As described in Section 1, the Tits building $\mathcal{X}$ of a rank 2 Kac-Moody group is a tree, whose set of vertices $\mathcal{X}_0$ consists of conjugates of the parabolic subgroups P_1 and P_2. Let $\mathbf{x}_i$ denote the vertex corresponding to P_i and $[\mathbf{x}_i]$ its equivalence class under the action of G, $i = 1, 2$. Then

$$G\backslash\mathcal{X}_0 = [\mathbf{x}_1] \sqcup [\mathbf{x}_2].$$

Both $\overline{G}$ and $\widehat{G}$ act on $\mathcal{X}$ (cf. the discussion after [5, Cor. 1.4]). Abusing notation we also write $[\mathbf{x}_i]$ for the $\overline{G}$ and $\widehat{G}$ equivalence classes of $\mathbf{x}_i$.

Proposition 4.1. *It is possible to normalise the Haar measures $\widehat{\mu}$ on $\widehat{G}$ and $\overline{\mu}$ on $\overline{G}$ in such a way that*

$$\widehat{\mu}(\Gamma\backslash\widehat{G}) = \overline{\mu}(\Gamma\backslash\overline{G}) = \sum_{[\mathbf{x}]\in\Gamma\backslash\mathcal{X}_0} \frac{1}{|\Gamma_{\mathbf{x}}|}$$

for any cocompact lattice $\Gamma \leqslant \widehat{G}$, where, by abuse of notation, $\widehat{\mu}$ and $\overline{\mu}$ are also the induced measures on $\Gamma\backslash\widehat{G}$ and $\Gamma\backslash\overline{G}$ correspondingly.

Proof. If there exist no cocompact lattices, the statement is tautologically true.

Let $\Gamma \leqslant \widehat{G}$ be a cocompact lattice. Then $\pi(\Gamma) \cong \Gamma$ is a cocompact lattice in $\overline{G}$. The group $\overline{G}$ acts on $\mathcal{X}$ and the stabilisers $\overline{G}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathcal{X}_0$ are compact open subgroups. In particular, $\overline{\mu}(\overline{G}_{\mathbf{x}}) < \infty$. Thus,

$$\mu(\overline{G}\backslash\backslash\mathcal{X}) = \sum_{[\mathbf{x}]\in\overline{G}\backslash\mathcal{X}_0} \frac{1}{\overline{\mu}(\overline{G}_{\mathbf{x}})} = \frac{1}{\overline{\mu}(\overline{G}_{\mathbf{x}_1})} + \frac{1}{\overline{\mu}(\overline{G}_{\mathbf{x}_2})} < \infty,$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are representatives of the orbits of P_1 and P_2 under the $\overline{G}$-action respectively. It follows that $\overline{\mu}$ can be normalised so that

$$\overline{\mu}(\Gamma\backslash\overline{G}) = \sum_{[\mathbf{x}]\in\pi(\Gamma)\backslash\mathcal{X}_0} \frac{1}{|\pi(\Gamma)_{\mathbf{x}}|} = \sum_{[\mathbf{x}]\in\Gamma\backslash\mathcal{X}_0} \frac{1}{|\Gamma_{\mathbf{x}}|}.$$

Now consider the orbits of $\mathbf{x}_1$ and $\mathbf{x}_2$ under the action of $\widehat{G}$. Again, we have compact open stabilisers $\widehat{G}_{\mathbf{x}}$, for every $\mathbf{x} \in \mathcal{X}_0$. By the same argument as above

$$\mu(\widehat{G}\backslash\backslash\mathcal{X}) = \sum_{[\mathbf{x}]\in\widehat{G}\backslash\mathcal{X}_0} \frac{1}{\widehat{\mu}(\widehat{G}_{\mathbf{x}})} < \infty.$$

Consequently,

$$\widehat{\mu}(\Gamma\backslash\widehat{G}) = \sum_{[\mathbf{x}]\in\Gamma\backslash\mathcal{X}_0} \frac{1}{|\Gamma_{\mathbf{x}}|} = \overline{\mu}(\Gamma\backslash\overline{G})$$

as required.

□

5. Cocompact lattices in $\widehat{G}$, symmetric case

In this section we assume that A is symmetric ($a_{12} = a_{21}$). There is a unique (up to an isomorphism) simply connected root datum $\mathcal{D}_{sc}$ of type A: this is a root datum such that $\Pi^\vee$ forms a basis of $\mathcal{Y}$. Let G be the Kac-Moody group with the simply connected root datum. It is possible, yet requiring extra work beyond the scope of the present paper, to extend our results to an arbitrary root datum $\mathcal{D}$.

A lattice Γ is called *edge-transitive*, if it acts transitively on the set $\mathcal{X}_1$ of edges of the Tits building, described in Section 1. Capdeboscq and Thomas classify edge-transitive p-well-behaved cocompact lattices in $\overline{G}$ [6]. Now Corollary 3.3 and [6, Th. 1.3] together give us a classification of edge-transitive cocompact lattices in $\widehat{G}$.

Capdeboscq and Thomas also determine the p-well-behaved cocompact lattice of the minimal covolume in $\overline{G}$ [6, Th. 1.4]. This and the observations above yield Main Theorem 2, a similar result for all cocompact lattices in $\widehat{G}$.

Let us prove Main Theorem 2 now. Statement (1) follows from Corollary 3.3 and [6, Th. 1.1]. Statement (2) follows from Proposition 4.1 and [6, Th. 1.3]. Q.E.D.

6. Cocompact lattices in $\widehat{G}$, other cases

6.1. **Not symmetric.** Let us drop the assumption that $a_{21} = a_{12}$ while still assuming that $\max(a_{21}, a_{12}) \leqslant -2$. These are our assumptions from Section 1 through Section 4. Thus, all of our results in these sections hold.

Looking at Section 5, statement (1) of Main Theorem 2 holds but validity of statement (2) is unclear at this time. The reason is that Capdeboscq and Thomas [6] do most of their analysis only in the symmetric case. While the first two statements of [6, Th. 1.1] hold without the symmetricity assumption, yet [6, Th. 1.3] requires it. With this in light we find the following question interesting.

Question 6.1. *Determine a cocompact lattice of minimal covolume in $\widehat{G}$ and compute its covolume.*

6.2. **The case of $a_{12} = -1$.** We still assume that $\max(a_{21}, a_{12}) \leqslant -2$. The first issue with this case is that our definition of Kac-Moody group does not work in this case. If $m = a_{21}$, the reflection representation of the Weyl group in this case is given by

$$\rho(w_1) = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}, \quad \rho(w_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Hence, we have a real root that is a sum of two real roots:

$$w_1(\alpha_2) = \begin{pmatrix} m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} m-1 \\ 1 \end{pmatrix} = \alpha_1 + w_1 w_2(\alpha_1).$$

Consequently, there exist non-commuting subgroups U_α and U_β for prenilpotent pairs $\{\alpha, \beta\}$. Nevertheless, the groups $\widehat{G}$ and $\overline{G}$ still exist in this case [5] and many of our general results about them, e.g., Lemmas 2.4 and 2.7, Propositions 2.6, 3.1, 3.4 and 4.1 are still applicable. The group $\widehat{G}$ is still first countable, although Lemma 2.5 requires a new, subtler proof.

The current proof does not work because Lemma 2.2, a key result for all consequent analysis, fails. Hence, we do not know whether many major results, e.g., Theorems 2.3 and 2.8, Lemma 3.2, Corollary 3.3 remain valid. We state some of them in a series of conjectures and questions.

Conjecture 6.2. *The Kac-Moody group $\widehat{G}$ admits a cocompact lattice.*

Conjecture 6.3. *A cocompact lattice in $\widehat{G}$ does not contain an element of order p.*

Question 6.4. *Classify cocompact lattices in $\widehat{G}$.*

Question 6.5. *Determine a cocompact lattice of minimal covolume in $\widehat{G}$ and compute its covolume.*

6.3. **Higher rank.** Let us now consider a generalised Cartan matrix A of a larger size. The groups $\widehat{G}$ and $\overline{G}$ exist in this case [5]. We expect that the group $\widehat{G}$ is still first countable (cf. Lemma 2.5). Consequently, the general results about them, e.g., Lemmas 2.4 and 2.7, Propositions 2.6, 3.1, 3.4 and 4.1 are still applicable.

It would be interesting to know when these groups admit cocompact lattices. If A is affine, not of type $\widetilde{A_n}$, Borel and Harder prove that $\overline{G}$ does not admit cocompact lattices [2]. Now suppose that A has an irreducible principal minor that is affine, not of type $\widetilde{A_n}$. Caprace and Monod observe that $\overline{G}$ does not admit a cocompact lattice either [8, Rem. 4.4]. Proposition 3.1 implies that $\widehat{G}$ does not admit a cocompact lattice either. On the other hand, cocompact lattices exist in a right-angled Kac-Moody group $\overline{G}$ [7]. The right-angled Kac-Moody groups are a subclass of the groups in the following conjecture:

Conjecture 6.6. *If any irreducible principal minor of A of affine type is of type $\widetilde{A}_m$, then the Kac-Moody group $\widehat{G}$ admits a cocompact lattice.*

Question 6.7. *Determine a cocompact lattice of minimal covolume in $\widehat{G}$ and compute its covolume.*

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