Conceptualising the Discourse at the Mathematical Horizon: Looking at one teacher's actions "beyond the mathematics of the moment"

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This report introduces part of a larger study on secondary teachers' mathematical and pedagogical discourses that are significant to the coherence of mathematical ideas and practices across educational levels. The study draws on the literature related to what is mostly called *Horizon Content Knowledge* and specifically on the theoretical construct of the *Discourse at the Mathematical Horizon.* The aim of this report is to propose and exemplify an analytical approach that conceptualises and identifies the characteristics of this discourse in a lesson observation and an interview with one newly qualified mathematics teacher. The proposed analytical approach illustrates the teacher's actions of interpreting and giving meaning to students' unexpected ideas and how these actions can lead in the identification of discursive patterns of how the teacher goes beyond the content of a specific teaching situation.

Keywords: teachers' discourses; mathematical horizon; teaching; secondary mathematics

Introduction

The coherent teaching and learning of mathematics across educational levels dominates the current narrative of mathematics education in the UK. The focus is on making connections across the curriculum and creating resources for teachers and students that support such connections (e.g. Cambridge Mathematics, 2015). In a coherent approach to teaching, discussions in the classroom might hint at unexpected links between mathematical ideas or practices. What could the teacher do then?

The study we report here is part of the PhD research of the first author. The purpose of the larger project is to explore in-service teachers' mathematical and pedagogical discourses that are significant to teaching practices beyond the content of a specific teaching and learning situation. Specifically, the main focus is on reconceptualising 'Discourse at the Mathematical Horizon' (Cooper, 2016), with reference to the UK context, and exploring its significance to teaching of mathematics that supports coherence across educational levels. This report focuses on the development of an analytical approach for the first part of the project that involves interviews and classroom observations. We first discuss key elements from the literature and briefly recount the rationale behind the choice of the theoretical framework. We then describe the methodology and the method of analysis exemplified in an episode from Liz, a newly qualified teacher, and her Year 7 class.

Horizon Content Knowledge

The Mathematical Knowledge for Teaching (MKT) model (Ball, Thames, & Phelps, 2008) is one of the first and still popular models to describe mathematics teachers' knowledge. This model proposes a domain of teacher's knowledge, Horizon Content Knowledge (HCK), which addresses situations where the mathematics goes beyond "the mathematics of the moment" (Ball & Bass, 2009, p.6), namely beyond the content of a specific teaching and learning situation.

HCK was originally described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403). A year later the notion was defined as:

> … an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (Ball & Bass, 2009, p.6)

This awareness was hypothesised as aiding teachers in taking the following actions (original formatting):

- Making judgments about mathematical importance
- Hearing mathematical significance in what students are saying
- Highlighting and underscoring key points
- Anticipating and making connections
- Noticing and evaluating mathematical opportunities
- Catching mathematical distortions or possible precursors to later mathematical confusion or misrepresentation (Ball & Bass, 2009, p.6).

Despite researchers' attempts to develop and describe HCK, the idea remains a 'grey area' compared to the other domains of the MKT model. Papadaki (Papadaki, 2019) proposed examples that showcase the variations of narratives, sometimes conflicting, around the notion of HCK. In brief, HCK has been described as an "awareness" (Ball & Bass, 2009), "familiarity" and "orientation" (Jakobsen, Thames, Ribeiro, & Delaney, 2012) or as "advanced mathematical knowledge" (Zazkis & Mamolo, 2011). These descriptors convey different meanings about the nature of the knowledge attributed to HCK. In another conceptualisation, HCK "shapes the MKT from a continuous mathematical education point of view" (Fernández, Figueiras, Deulofeu, & Martínez, 2011, p.2645). From this perspective, HCK is a knowledge connecting past, present and following mathematical levels. Finally, Papadaki (Papadaki, 2019) suggests connections between the standpoint the researchers adopt and the metaphors they use to describe HCK.

Given the aforementioned developments related to HCK, why should anyone keep looking? Even with the lack of clarity in its definition, HCK is considered part of the MKT model to this day. As the work around the other domains is growing, HCK acts as a 'placeholder', as a reminder that the rest of the domains do not span every aspect of the professional knowledge needed for teaching mathematics. We want to get more insight into this professional knowledge, and we propose that looking into the ideas of HCK through a discursive lens might aid in resolving some of the conflicts mentioned earlier.

Discourse at the Mathematical Horizon

According to the theory of commognition, discourses are "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by the participants" (Sfard, 2008, p.93). Cooper (2014) proposed an adaptation of the MKT model using a commognitive approach and called it Mathematical Discourse for Teaching. Within this model, a working definition of Discourse at the Mathematical Horizon is "patterns of mathematical communication that are appropriate in a higher grade level" (Cooper & Karsenty, 2018, p.242).

There is little empirical evidence of how Discourse at the Mathematical Horizon is operationalised to account for teachers' discursive actions. One of the aims of the larger project is to refine the definition of Discourse at the Mathematical Horizon based on evidence from secondary teachers and teacher educators in the UK. In other words, what are the routines – sets of rules defining a discursive pattern (Sfard, 2008) – that govern the Discourse at the Mathematical Horizon?

Methodology

The purpose of this paper is to exemplify an attempt on exploring the characteristics of Discourse at the Mathematical Horizon, based on evidence from the classroom and reflections of teachers. The PhD study is conducted in England and participants are secondary school mathematics teachers and teacher educators. Here, we focus on a classroom episode between one participant, Liz, and one of her Year 7 students, Steven (both pseudonyms). The data consist of a lesson observation and a post-lesson interview, both audio-recorded. Liz is a Newly Qualified Teacher in her first year of teaching. She has a degree in Mathematics, and she worked as a data analyst for several years before she decided to get a teaching qualification.

The preliminary look at the data aimed to identify classroom episodes where the discussion went beyond the objectives of the lesson. Then, each episode was analysed by using themes based on the actions Ball & Bass (2009) initially linked with HCK. For the purpose of this report, we focus on one episode. This episode is situated around the task shown in Figure 1. Finally, we looked for discursive patterns in each theme and throughout the episode.

Find four numbers that add up to 360º.

How many different ways can you find to do this?

130º	131º	105º	158º
141º	179º	50º	29º
101º	91º	39º	1º
99º	82º	79º	89º
109º	127º	81º	98º
117º	122º	71º	53º
152º	55º	63º	58º
75º	22º	28º	125º

Figure 1: Starter task

The task

Before we go into details about the episode, we present the task and our interpretation of its potentials. The task is explorative, and it was given as a starter to a lesson about the angles in quadrilaterals. The starter links to the fact that the sum of the angles in a quadrilateral is equal to 360º. The task has two sub-questions, first the students are asked to find four numbers from the table that add up to 360. The second part prompts the students to find different sets of such numbers.

There are multiple ways to tackle the problem. One can start checking numbers in random to find quadruplets that add up to 360, e.g. 91 + 22 + 122 + 125 = 360. This process is time consuming but checking all the possible combinations would provide every possible answer to the first part of the question. However, finding all the combinations is not an objective of this task. In this case, a less time-consuming approach that still gives many combinations is looking for pairs of numbers that sum to a multiple of ten and then combine the pairs that add up to 360, e.g. 82 + 58 = 140 and 141 + 79 = 220.

There is a hidden pattern in the choice of numbers of the table in Figure 1. After the rearrangement of the numbers based on their units in Figure 2, we observe that almost all the numbers can make pairs that add up to 180. Therefore, identifying the pairs of numbers that add up to 180 and then creating different combinations would give many quadruplets very quickly. This pattern was not explicitly stated in the task and Liz did not mentioned it in the introduction of the task to the students. The episode we describe in the next section starts with Steven, a student, noticing such pairs.

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
130	141	152	63		105		117	28	39
50	101	122	53		125		127	58	79
	91	82			75			98	89
	81	22			55			158	99
	71								109
	1								179
	131								29

Figure 2: Arrangement by units

The episode

While the students are working on the starter, Liz is going around asking the students to explain their working. Most of them are noticing pairs of numbers that sum to multiples of 10, for example 109+81=190, and then looking for another pair that adds up to 210. At Steven's desk the following conversation takes place:

Liz:	How many ways have you found Steven?
Steven:	I just found most two [numbers] do one eighty [180]. You can do different things.
Liz:	Yeah.
Steven:	It is gonna be over th-... thirty [ways].
Liz:	Can you find a way where they don't equal one eighty?

Later she asks the student to share his approach with the classroom.

Liz:	Um, Right, I am gonna go to Steven because he spotted this really quickly […] Steven, how did you do it?
Steven:	I just found a hundred and eighty.
Liz:	He just found two numbers that added to a hundred and eighty. So, then what else did you need to do?
Steven:	[inaudible]
Liz:	And he needs to find another two that add to a hundred and eighty. So, he did it really quickly. Eem, what was your first one?

In her post-lesson interview, Liz reflects on this situation:

Liz:	[...] I liked the starter... and Steven... immediately... just went I'm gonna find pairs of one hundred eighty. And he was one, I think he was the only... pupil in the room that was just, could see that you could just, and I thought that was quite nice.

Analysis of the episode

The context of the activity is angles in quadrilaterals. However, the discussion on the task goes beyond the specific topic in many ways. For example, neither the student nor the teacher talks about angles or mention the word 'degrees'. The focus is mostly on the problem-solving technique used and there is no evidence of connection of the task to quadrilaterals. We decided to look into the dialogue more closely to explore emerging patterns in Liz's actions.

Liz's reaction to Steven's approach was coded as 'hearing mathematical significance in what students are saying'. Liz seems impressed by Steven's method. Her appreciation to his work is visible in her reflection – "I thought that was quite nice". Liz realises the effectiveness, or at least some of the benefits, of this method and shares her interpretation of Steven's actions with the class.

Focusing on the utterances, there is a shift from what the student does to what the teacher shares in the classroom and later in her reflection. Specifically, Steven's exploration of the numbers leads him to notice a repeating pattern in the table – "I just found most two [numbers] do one eighty [180]. He then uses this observation to find quadruplets. However, Liz is interpreting Steven's actions as if his goal was to look for pairs that add up to 180 – "Steven... immediately... just went I'm gonna find pairs of one hundred eighty". Aware of the hidden pattern of the numbers in the table (Figure 2), Lis seems to have a certain expectation from the students, to "find pairs of one hundred eighty". Liz attributes to Steven her own way of thinking as she transforms the student's actions to a rule to be shared with the class.

In this episode, Liz takes two mathematically informed decisions. The first is to ask Steven to look for pairs that do not add up to 180, probably because of what the other students were saying to her about their methods. Her second decision was to share Steven's method with the whole class. Both actions were coded as 'making judgement about mathematical importance' and 'noticing and evaluating mathematical opportunities'. An emerging pattern shared in both cases is that Liz asks her students to consider a different method than the one previously used. In both cases, alternative methods are proposed, and students are let free to choose what works for them. A tentative interpretation is that Liz's seems to be aware that both methods are valuable for different reasons. However, more evidence is needed for such claim. Another emerging pattern is the shift of attention from individual students to the whole class, for example, what Liz hears from other students informs what she suggests to Steven and what Steven says is transformed to a rule that she shares with the whole class.

Discussion

The episode illustrates how noticing and acting upon a student's remark could give an opportunity for discussion beyond the mathematics of the moment. Driven by the student's observation, the teacher spent some time on the methods behind finding quadruplets in this activity before moving on to the focus of the lesson.

The analysis of the episode is a preliminary attempt to utilise the literature on HCK to identify potential patterns in teacher actions that might be applicable across episodes. Our aim in the next steps of the analysis is to identify how such patterns form routines of the Discourse at the Mathematical Horizon. Data from Liz's lesson observations and interviews show that she is keen on discussing aspects of problem solving with her students. Problem solving might be the backbone that connects her teaching sessions, that drives her beyond "the mathematics of the moment". This is not necessarily the case for other participants. Therefore, the analytical approach we trailed in this episode should be tested in a larger set of data to look for characteristics within participants or shared across participants.

Finally, focusing on classroom episodes and post observation interviews explores teachers mathematical and pedagogical discourses. Part of the larger project is to go beyond the exploration of established discourses by challenging these discourses in situations where the teachers have the time to reflect and discuss with each other.

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