Markets with heterogeneous beliefs:

A necessary and sufficient condition for a trader to vanish

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Abstract

What does it take to survive in the market? Previous literature has proposed sufficient conditions for a trader to vanish, which depend on pairwise comparisons of traders' discounted beliefs. We propose a novel condition that focuses on the ratio of traders' discounted beliefs and (approximate) equilibrium prices. Unlike existing conditions, ours is both necessary and sufficient for a trader to vanish and delivers the exact rate at which vanishing traders lose their consumption shares. As an application, we analyze the performance of two intuitive behavioral strategies: the *"Follow the Leader Strategy"* that prescribes mimicking the beliefs of the most successful trader, and the *"Follow the Market Strategy"* that prescribes to use beliefs which coincide with the state price density. Further, we show that the relative performance of vanishing traders cannot be studied in isolation. Our analysis highlights an intuitive point obscured by the existing conditions: trading in financial markets is qualitatively different from bilateral trading.

JEL Classification: D51, D01, G1

1 Introduction

More than a half-century ago economists hypothesized that traders with poor forecasting abilities progressively lose wealth against traders with more accurate probabilistic views (*market selection hypothesis*, Friedman (1953)). In general equilibrium settings with complete markets and bounded aggregate endowment, previous literature has formalized this hypothesis and found sufficient conditions for a trader to vanish based on the pairwise comparison of traders' discounted beliefs. A trader vanishes if there is another trader who is more accurate (Blume and Easley (1992, 2006), Sandroni (2000), Yan (2008), Kogan et al. (2016), Massari (2016)).

This approach greatly simplifies the analysis of heterogeneous-belief economies because it does not require solving for the competitive equilibrium — traders' discounted beliefs are exogenous. However, it overlooks a fundamental aspect of competition: in financial markets, each trader interacts with all traders in the market simultaneously through prices, not in a pairwise fashion (Jouini and Napp (2007)). Therefore, in economies with more than two traders, pairwise comparisons cannot deliver a necessary and sufficient condition for a trader to vanish (Blume and Easley (2009)) or characterize the relative performance between vanishing traders (Cvitanic and Malamud (2010)).

To account for these shortcomings, we propose a novel approach that is closer to the actual trading experience in financial markets. We focus on the ratio between traders' discounted beliefs and equilibrium prices. Our approach preserves the central role of prices in market interactions and delivers a necessary and sufficient condition for a trader to vanish which also characterizes the exact rate at which vanishing traders lose their wealth. Moreover, it brings the general equilibrium analysis closer to the temporal equilibrium analysis of market selection — the latter focusing on the ratio between traders' investment strategies and equilibrium prices (Chiarella and He (2001), Evstigneev et al. (2002, 2008), Bottazzi and Dindo (2014)).

To make our condition applicable without solving for the competitive equilibrium, we provide an approximation of equilibrium prices that only depends on exogenous quantities. Our main technical contribution is demonstrating that, under standard assumptions, asymptotic equilibrium prices are well approximated by a convex combination of traders' discounted beliefs. This result proves a longstanding conjecture in economics (e.g. Blume and Easley (1993)): equilibrium prices, a risk-adjusted average of traders' beliefs, are qualitatively Bayesian, a non-risk-adjusted average of probabilities.

If all traders have the same discount factor, our condition reads: a trader vanishes if and only if his beliefs are less accurate than the probability obtained via Bayes' rule from a regular prior on the set of traders' beliefs.[1] By contrast, existing conditions read: a trader vanishes if there is another trader who is more accurate. Because traders' beliefs are exogenous and Bayesian inference is well understood, our condition is easy to verify and compute even in economies with a large number of traders — a case in which conducting pairwise comparisons between all traders in the economy might prove computationally challenging.

This paper provides two novel implications. First, our condition makes it possible to analyze the performance of traders who hold non-standard beliefs or use non-belief-based investment

[1]A prior is regular if it attaches strictly positive probability to every probability in its prior support.

strategies. Unlike existing conditions, ours precisely indicates which strategies vanish/survive when there are no perfectly rational traders in the market. As an illustration, we study the performance of two intuitive strategies: the *Follow the Leader Strategy* and the *Follow the Market Strategy*, henceforth FLS and FMS, respectively.

In a log-economy with homogeneous discount factors, the FLS coincides with mimicking the trader in the market that had the highest growth of capital from a given date. This intuitive strategy is easy to implement, and it is offered by a growing number of internet brokers.[2] The appeal of the FLS is that it guarantees to perform almost as well as the trader with the highest capital growth, the leader, provided that leaders do not change "too often". However, what happens when leaders do change "too often"? Conventional wisdom argues that the FLS would not perform well because of the transaction costs associated with changing the investment style. Here we show that the FLS leads to ruin even if there are no transaction costs. In section 6.1, we illustrate this point in a three-trader economy (traders 1,2 and FLS-trader). In the example, traders 1 and 2 alternate infinitely often as a leader because they are equally (in)accurate. The FLS-trader vanishes by the following logic. (*i*) Every time a leader changes, the FLS-trader starts copying the new leader with a small delay, because the new leader must first outperform the previous leader. (*ii*) During these delays, the FLS-trader does worse than the new leader, because he is still following the previous leader. And (*iii*), the delay-induced losses cumulate over time resulting in ruin. This argument is not compatible with the standard pairwise comparison approach because it requires comparing the FLS-trader's performance against that of more than one trader (leaders change over time). Among the existing conditions for a trader to vanish, Sandroni (2000)'s condition fails to recognize that the FLS-trader vanishes; Blume and Easley (2006)'s condition incorrectly implies that the FLS-trader dominates;[3] and Blume and Easley (2009)'s condition is inapplicable.[4]

The FMS prescribes adopting the next-period state price density as a belief. This strategy is behaviorally motivated by the *Wisdom of The Crowds* argument which states that equilibrium prices might reflect beliefs that are more accurate than the beliefs of all market participants (Galton (1907)). Our analysis of the FMS shares some similarities with the analysis of the FLS: the fate of the FMS-trader depends on the presence of a unique leader among the other traders in the economy. However, unlike the FLS-trader, there are situations in which the

[2] e.g. signal-trader, collective2 and e-toro: https://www.youtube.com/watch?v=GX041XotWd4

[3] It is known that Blume and Easley (2006)'s Theorem 8 can lead to incorrect conclusions (Massari (2013)).

[4] The elegant geometric construction behind Blume and Easley (2009)'s necessary and sufficient condition for a trader to vanish can only be used in economies in which the distribution of states' and traders' beliefs are iid.

FMS-trader dominates the market. In homogeneous discount factor, CRRA-economies with no aggregate risk in which all traders are more (less) risk averse than log, the FMS-trader dominates (vanishes) if leaders among the other traders change infinitely often.[5] The intuition goes as follow. In CRRA economies, equilibrium prices can be decomposed into a belief component (the state price density), and an endogenously determined discount factor which depends on traders' beliefs and consumption shares dispersion. When $\gamma < (>) 1$ the IES coefficient is larger (smaller) than in the log-utility case ($\gamma = 1$). As a result the demand for saving is high, interest rates are low, and the market discount factor is larger than traders' (common) discount factor: β. The FMS-trader uses the state price density to buy a constant share of the aggregate endowment, thus, his fate depends on the difference between his discount factor (β) and the market discount factor. If the consumption share/beliefs distribution quickly becomes degenerate, the endogenous component of the discount factor vanishes fast and has no effect on survival. However, if leaders alternate infinitely often, there is an infinite number of periods in which the consumption share/beliefs distribution is not degenerate. Therefore, the FMS-trader discount factor differs from the market's infinitely often and his fate depends on the sign of their difference.

Related papers that studies the effect of long run heterogeneity on equilibrium prices includes Jouini and Napp (2010), that analyze the long-run risk-return relationship in an economy with two agents have equally biased (constant) beliefs; Muraviev (2013) that derives a survival index with agents with catching up with Joneses preferences who are overconfident in interpreting a public signal; Branger et al. (2015) which study the survival issue when two agents engage in learning but commit different types of filtering errors; and He and Shi (2017) which compare the welfare between subjectively optimal portfolio strategies and an index portfolio in an economy with no perfectly rational agents.

Second, our condition is the only one that delivers the exact rate at which a vanishing trader loses his consumption share in economies with more than two traders. As much as survival is important for understanding the long-run equilibrium behavior of the market, it is equally important to understand how quickly the non-surviving traders go extinct. If the rate of extinction is low, non-surviving traders will impact equilibrium behavior for a long time (Cvitanic and Malamud (2010)). We show that the rate at which a trader vanishes depends on his discounted beliefs, his risk attitudes and equilibrium prices. Intuitively, the relative

[5] We assume constant aggregate endowment to avoid known biases due to the interaction between risk attitudes and fluctuations of the aggregate endowment.

performance between two traders that vanish cannot be analyzed in isolation because the prices they use to trade are determined by the trader that dominates.

As an application, we show that the relative performance of two traders can be reversed by the presence of a new trader in the economy. A result that question the general view according to which the market always favor traders with more accurate probabilistic views, even among vanishing traders and believed impossible within the general equilibrium setting (Cvitanic and Malamud (2010)).

Sections 2-5 introduce the model, provide our approximation of equilibrium prices and present our necessary and sufficient condition for a trader to vanish. Section 6,7 and 8 are dedicated to our analysis of the FLS-strategy, FMS-strategy, and relative extinction reversal, respectively. Proofs are in Appendices.

2 The model

2.1 The environment

Consider an infinite horizon Arrow-Debreu exchange economy with complete markets. Time is discrete and begins at date 0. At each date, the economy can be in one of S mutually exclusive states: $\mathcal{S} := \{1, ..., S\}$, with cartesian product $\mathcal{S}^t = \times^t \mathcal{S}$. The set of all infinite sequences of states, paths, is $\Sigma := \times^\infty \mathcal{S}$. $\sigma = (\sigma_1, ...)$ denotes a representative path; $\sigma^t = (\sigma_1, ..., \sigma_t)$ denotes a partial history till period t; $C(\sigma^t)$ denotes the cylinder set with base σ^t, $C(\sigma^t) = \{\sigma \in \Sigma |\, \sigma = (\sigma^t, \ldots)\}$; $\mathcal{F}_t$ denotes the σ-algebra generated by the cylinders; and $\mathcal{F}$ is the σ-algebra generated by their union. By construction $\{\mathcal{F}_t\}$ is a filtration. For any probability measure p on Σ, $p(\sigma^t) := p(\{\sigma_1 \times ... \times \sigma_t\} \times S \times S \times ...)$ denotes the marginal probability of the partial history σ^t, while $p(\sigma^t|\sigma^{t-1}) = \frac{p(\sigma^t)}{p(\sigma^{t-1})}$ denotes the conditional probability of the last observation of the partial history σ^t given its first t-1 realizations.[6] P is the true probability on $(\Sigma, \mathcal{F})$.

Next, we introduce a number of economic (random) variables with time index t. These variables are adapted to the filtration $\mathcal{F}_t$.

2.2 Traders

The economy contains a finite set of traders, $\mathcal{I}$. Each trader, i, has consumption set $\mathbb{R}_+$. A consumption plan $c : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+$ is a sequence of $\mathbb{R}_+$-valued functions $\{c_t(\sigma)\}_{t=0}^{\infty}$. Each

[6] For notation's sake, we assume that past realizations constitute all the relevant information, i.e. $\mathcal{F}_t := \sigma^t$.

trader i is characterized by a payoff function $u^i : \mathbb{R}_+ \to \mathbb{R}$ over consumption, a discount factor $\beta_i \in (0,1)$, an endowment stream $\{e^i_t(\sigma)\}_{t=0}^\infty$ and a subjective probability p^i on $(\Sigma, \mathcal{F})$, his beliefs. With an abuse of notation, $\mathcal{I}$ indicates both the set of traders and the set of traders beliefs: $\mathcal{I} = \{p^i : i \in \mathcal{I}\}$. Trader i's utility for a consumption plan c is:

$$U^i(c) = E_{p^i} \sum_{t=0}^{\infty} \beta_i^t u_i(c_t(\sigma)).$$

As is customary in the selection literature, traders can either vanish or survive.

Definition 1. *Trader* i *vanishes if* $\limsup_{t\to\infty} \frac{c^i_t(\sigma)}{\sum_{j\in\mathcal{I}} c^j_t(\sigma)} =^{P\text{-}a.s.} 0$.
He survives if $\limsup_{t\to\infty} \frac{c^i_t(\sigma)}{\sum_{j\in\mathcal{I}} c^j_t(\sigma)} >^{P\text{-}a.s.} 0$. *He dominates if* $\lim_{t\to\infty} \frac{c^i_t(\sigma)}{\sum_{j\in\mathcal{I}} c^j_t(\sigma)} =^{P\text{-}a.s.} 1$.

Finally, we rank the accuracy of beliefs according to their likelihood:

Definition 2. *Trader* i *is more accurate than trader* j *if* $\lim_{t\to\infty} \frac{p^j(\sigma^t)}{p^i(\sigma^t)} =^{P\text{-}a.s.} 0$.

This accuracy criterion is an exact version of the one adopted by Sandroni (2000) or Blume and Easley (2006), which only approximate traders' likelihood. We adopt this criterion to ensure that the inability of the pairwise comparison approach to provide a necessary and sufficient condition for a trader to vanish cannot be attributed to an approximation error. In Appendix A we discuss the relation between Definition 2 and existing criteria.

2.3 Competitive equilibrium

We derive our results using the time 0 trading setting. $q(\sigma^t)$ denotes the date 0 price of a claim that pays a unit of consumption at the end of σ^t in terms of time zero consumption. A competitive equilibrium is a sequence of prices and, for each trader, a consumption plan that is affordable, preference maximal on the budget set and mutually feasible. Sufficient assumptions for the existence of the competitive equilibrium are (Peleg and Yaari (1970)):

- **A1:** The payoff functions $u_i : \mathbb{R}_+ \to [-\infty, +\infty)$ are C^1, strictly concave, increasing and satisfy the Inada condition at 0; that is, $u_i'(c) \to \infty$ as $c \searrow 0$.[7]

- **A2:** There are numbers $0 < f \le F < +\infty$ such that for each trader i, all dates t and all paths σ, $f \le \inf_{\sigma^t} \sum_{i\in\mathcal{I}} e^i_t(\sigma) \le \sup_{\sigma^t} \sum_{i\in\mathcal{I}} e^i_t(\sigma) \le F$.

- **A3:** For all traders i, all dates t and all paths σ, $p^i(\sigma^t) > 0 \Leftrightarrow P(\sigma^t) > 0$.

[7] Defining $\log 0 = -\infty$.

A1 is a collection of standard properties for the payoff functions. There is a recent trend in the market selection literature to relax **A1** to allows for preferences which are not time-separable (Borovička (2016), Dindo (2016)). In this general setting, the market selection hypothesis can fail, and our characterization of equilibrium prices does not apply. **A2** uniformly bounds the aggregate endowment above and away from 0. This assumption is standard in part of the selection literature because it ensures that the market selection hypothesis holds. In a growing (shrinking) economy, Yan (2008) shows that the market selection hypothesis can fail because risk attitudes affect survival through their impact on optimal savings. In growing (shrinking) economies, our approximation is valid if we strengthen **A1** to require that all traders have an identical CRRA utility function. **A3** rules out pathological cases of non-existence of the competitive equilibrium due to traders' disagreement on 0 probability events.

3 Comparing the two approaches

In this section, we use an example to illustrate the main features of our approach and compare it with the existing one. In the example, all traders have log utility, and equilibrium prices can easily be obtained analytically. Our approach is shown to be more informative (we directly characterize equilibrium prices) and more precise (our condition is both necessary and sufficient for a trader to vanish) than the existing one. The main difficulty in generalizing our approach from the log case to the general case is that in most non-log economies equilibrium prices cannot be analytically obtained. In the next section, we address and solve this issue, providing an approximation of equilibrium prices that applies to every economy that satisfies **A1-A3**.

Example 1: consider an Arrow's security economy with S states with iid multinomial distribution P and constant aggregate endowment: $\forall t, \forall \sigma, \sum_{i \in \mathcal{I}} e_t^i(\sigma) = 1$. There are $\mathcal{I}$ traders with log utility, identical discount factors (β), iid beliefs p^i and positive initial consumption c_0^i. Every trader in the economy aims to solve:

$$\max_{\{c_t^i(\sigma)\}_{t=0}^{\infty}} E_{p^i} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i(\sigma)) \quad s.t. \quad \sum_{t=0} \sum_{\sigma^t \in \Sigma^t} q(\sigma^t) \left(c_t^i(\sigma) - e_t^i(\sigma) \right) \leq 0.$$

Traders' first-order conditions of the maximization problem are sufficient for the Pareto optimum and, in every path σ^t, can be expressed as $\frac{\beta^t p^i(\sigma^t)}{c_t^i(\sigma)} = \frac{q(\sigma^t)}{c_0^i}$.

Our approach:

- *First step*: to characterize equilibrium prices.

In this simple economy, equilibrium prices can be obtained explicitly from the FOC:

$$\forall t, \forall \sigma \in \Sigma, \quad q(\sigma^t) = \beta^t \sum_{i \in \mathcal{I}} p^i(\sigma^t) c_0^i.$$

Remark: As previously noted (Rubinstein (1974), Blume and Easley (2009)), in a log-economy with homogeneous discount factors, equilibrium prices coincide with the discounted probabilities ($p^B(\sigma^t)$) obtained via Bayes' rule from a prior distribution $C_0 = \{c_0^1, ..., c_0^I\}$ on $\mathcal{I} = \{p^i : i \in \mathcal{I}\}$: $q(\sigma^t) = \beta^t \sum_{i \in \mathcal{I}} p^i(\sigma^t) c_0^i := \beta^t p^B(\sigma^t)$.

- *Second step*: to use equilibrium prices to discuss survival.
  Substituting the price equation in the FOC: $c_t^i(\sigma) = \frac{\beta^t p^i(\sigma^t) c_0^i}{q(\sigma^t)} = \frac{p^i(\sigma^t) c_0^i}{p^B(\sigma^t)}$.
  Thus, $c_t^i(\sigma) \to^{P\text{-a.s.}} 0 \Leftrightarrow \frac{p^i(\sigma^t) c_0^i}{p^B(\sigma^t)} \to^{P\text{-a.s.}} 0$: trader i vanishes P-a.s. if and only if his beliefs are less accurate than the probability obtained via Bayes' rule from C_0.

The existing approach:

The existing approach skips the characterization of equilibrium prices and directly focuses on pairwise comparison of traders' discounted beliefs. Taking the pairwise ratios of the FOCs of different traders, prices simplify:

$$\frac{c_t^i(\sigma)}{c_t^j(\sigma)} = \frac{p^i(\sigma^t) c_0^i}{p^j(\sigma^t) c_0^j}. \tag{1}$$

By **A2**, the aggregate endowment is bounded, thus $\frac{p^i(\sigma^t) c_0^i}{p^j(\sigma^t) c_0^j} \to^{P\text{-a.s.}} 0 \Rightarrow c_t^i(\sigma) \to^{P\text{-a.s.}} 0$. Thus, trader i vanishes if there is another trader who is are more accurate. By taking the log of Equation 1 and approximating its RHS with the difference of traders' entropy (see Appendix A), we obtain Sandroni (2000)'s Proposition 3. By approximating its RHS with the difference of traders' sum of expected relative entropy (see Appendix A), we obtain Blume and Easley (2006)'s Theorem 8.

4 A general approximation of equilibrium prices

In this section, we provide an approximation of equilibrium prices that allows us to generalize the approach of Example 1 from the log utility/iid beliefs setting to all economies that satisfy **A1-A3**. The general setting differs from the example in two important ways. First, fluctuations of the aggregate endowment alter investment decisions. Second, risk attitudes affect the convergence rates of consumption shares: if traders are less risk averse, they trade more aggressively, and consumption shares move faster (Blume and Easley (2006)).

Theorem 1 shows that these differences are asymptotically negligible: equilibrium prices are asymptotically equivalent to a convex combination of traders' discounted beliefs. Further, our approximation is valid on every path — thus for every true probability— and for every finite[8] set of beliefs $\mathcal{I}$ that satisfies **A3**: traders' beliefs need not be iid.

Definition 3. *Given two functions $f(x)$ and $h(x)$, $f(x) \asymp h(x)$ if* $\begin{cases} \limsup\limits_{t\to\infty} \frac{f(x)}{h(x)} < \infty \\ \liminf\limits_{t\to\infty} \frac{f(x)}{h(x)} > 0 \end{cases}$.

Theorem 1. *In an economy that satisfies* ***A1-A3****,* $\forall \sigma \in \Sigma :$

$$q(\sigma^t) \asymp \sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t).$$

Proof. See Appendix B. □

In economies in which all traders have the same discount factor, Theorem 1 proves Blume and Easley (1993)'s conjecture that equilibrium prices, a risk-adjusted average of traders' beliefs, evolve in a way that is qualitatively Bayesian, a non-risk-adjusted average of probabilities.

Corollary 1. *In an homogenous discount-factors economy that satisfies* ***A1-A3****, equilibrium prices are mutually absolutely continuous with the discounted probabilities obtained via Bayes' rule from a regular prior, g, on $\mathcal{I}$.*

Proof. $\forall \sigma \in \Sigma :$, $\frac{q(\sigma^t)}{\beta^t p^B(\sigma^t)} \asymp^{By\ Th.1} \frac{\beta^t \sum\limits_{i\in\mathcal{I}} p^i(\sigma^t)}{\beta^t \sum\limits_{i\in\mathcal{I}} p^i(\sigma^t) g^i} \asymp 1.$ □

5 A necessary and sufficient condition for a trader to vanish

Our condition characterizes the dynamics of traders' consumption shares by looking at the ratio between discounted beliefs and approximate equilibrium prices. Instead of taking the ratio of traders FOCs to eliminate equilibrium prices, we approximate them and determine the fate of each trader in the economy directly from his FOCs.

[8]Massari (2016) shows that our approximation does not hold if $|\mathcal{I}| = |\mathbb{R}|$.

Theorem 2. *In an economy that satisfies **A1-A3**,*

$$(i)\ \limsup \frac{\beta_i^t p^i(\sigma^t)}{q(\sigma^t)} =^{P\text{-}a.s.} 0 \Leftrightarrow \text{ trader } i \text{ vanishes } P\text{-}a.s.;$$

$$(ii)\ \limsup \frac{\beta_i^t p^i(\sigma^t)}{\sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t)} =^{P\text{-}a.s.} 0 \Leftrightarrow \text{ trader } i \text{ vanishes } P\text{-}a.s.;$$

$$(iii)\ \limsup \frac{\beta_i^t p^i(\sigma^t)}{\max_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t)} =^{P\text{-}a.s.} 0 \Leftrightarrow \text{ trader } i \text{ vanishes } P\text{-}a.s..$$

Proof. See Appendix B. □

Consistent with previous findings, Theorem 2 highlights that risk attitudes do not affect survival. The first condition tells us that a trader vanishes P-a.s. if and only if he subjectively believes that consumption costs too much on a set of sequences that occurs P-a.s.. The second condition uses the approximation of equilibrium prices of Theorem 1 to provide a "ready to use" condition that does not depend on endogenous quantities. The last condition is similar but not equivalent to pairwise comparisons (according to which a trader i vanishes P-a.s. if $\exists j \in \mathcal{I} : \lim_{t\to\infty} \frac{\beta_i^t p^i(\sigma^t)}{\beta_j^t p^j(\sigma^t)} =^{P\text{-a.s.}} 0$). It shows that our condition is equivalent to a pairwise comparison of traders' discounted beliefs if and only if the limit between the discounted likelihood ratios between all non-vanishing traders exists.

For homogeneous discount factors economies, condition (ii) simplifies to a likelihood ratio test between trader i's beliefs and the probability obtained via Bayes' rule from a regular prior on the set of traders' beliefs. This formulation allows for direct application of known consistency results in Bayesian statistics to our selection problem. Because traders' beliefs are exogenous and Bayesian inference is well understood, this condition is easy to verify and compute.

Corollary 2. *In a homogenous discount-factors economy that satisfies **A1-A3**, trader* i *vanishes* P*-a.s. iff his beliefs are less accurate than the probabilities obtained via Bayes' rule from a regular prior,* g*, on* $\mathcal{I}$*:*

$$\text{Trader } i \text{ vanishes } P\text{-}a.s. \Leftrightarrow \limsup_{t\to\infty} \frac{p^i(\sigma^t)}{\sum_{j\in\mathcal{I}} p^j(\sigma^t) g^j} =^{P\text{-}a.s.} 0.$$

Proof. Regular prior $\Rightarrow \forall j \in \mathcal{I}, g^j \asymp 1$. The result follows from Theorem 2 (ii) simplifying the betas. □

6 An analysis of the Follow the Leader Strategy

We are now ready to apply our condition to discuss the performance of the FLS. Let's start by defining the beliefs that correspond to this strategy.

Definition 4. *The FLS-trader follows the FLS if for ever σ^{t-1} his next period beliefs are:*

$$p^{FLS}(\sigma_t|\sigma^{t-1}) = \begin{cases} p^i(\sigma_t|\sigma^{t-1}), & i : p^i(\sigma^{t-1}) = \arg\max_{j\in\mathcal{I}}\{p^j(\sigma^{t-1})\}; \\ \frac{1}{|\mathcal{K}_{t-1}|}\sum_{i\in\mathcal{K}_{t-1}} p^i(\sigma_t|\sigma^{t-1}), & \textit{if ties occur.} \end{cases}$$

Where $\mathcal{K}_{t-1}$ is the set of indexes corresponding to models with highest likelihood at σ^{t-1}:

$$\mathcal{K}_{t-1} = \{i : p^i(\sigma^{t-1}) \in \arg\max_{j\in\mathcal{I}}\{p^j(\sigma^{t-1})\}\}.$$

That is, in every period the FLS-trader performs a pairwise likelihood ratio test between all models in $\mathcal{I}$ and uses the model with highest likelihood to make next-period predictions. If ties occurs, he gives equal weight to all the accurate models.[9]

In a log-economy with a homogeneous discount factor, the investment decisions implied by the FLS-trader' beliefs coincide with the rule of thumb of mimicking the wealthiest trader in the economy. A strategy which is consistent with the tendency of some investors in financial markets to "buy winners and sell losers". The appeal of the FLS is that if there is a unique best trader/model it guarantees to perform almost as well as the best trader does. However, what happens when there is not a unique best trader in the market?

Conventional wisdom argues that in this case the FLS does not perform well because of transaction costs. Here we show that, even with zero transaction costs, the FLS leads to ruin if leaders (the best statistical models) change infinitely often and their prediction remains different enough when it matters (distinct). The precise statement of our result requires the following definitions.

Definition 5. *Given a set of beliefs $\mathcal{I}$, trader $\hat{i}_t$ is a leader at* t *if $\hat{i}_t \in arg_i \max_{i\in\mathcal{I}} p^i(\sigma^t)$.*
Leaders change infinitely often *if:*

$$arg_i \max_{i\in\mathcal{I}} p^i(\sigma^t) \neq \arg_i \max_{i\in\mathcal{I}} p^i(\sigma^{t-1}) \quad \textit{infinitely often.}$$

[9]With this tie-breaking rule, the FLS coincides with the SNML algorithm (Roos and Rissanen (2008)). Inspection of Proposition 1's proof shows that the result holds with any tie-breaking rule.

Where the inequality is as a set inequality.

Definition 6. *Beliefs* $p^{\hat{i}_t}$ *and* $p^{\hat{i}_{t-1}}$ *are* distinct *if* $\exists \epsilon > 0 : \limsup_{t\to\infty} \sum_{\sigma_t \in S} |p^{\hat{i}_t}(\sigma_t|\sigma^{t-1}) - p^{\hat{i}_{t-1}}(\sigma_t|\sigma^{t-1})| > \epsilon$ *in all periods* t *such that traders* $\hat{i}_t$ *is the leader at* t *and trader* $\hat{i}_{t-1}$ *is the leader at* $t-1$.

Proposition 1. *In a homogenous discount-factors economy that satisfies* ***A1-A3****, the FLS-trader* vanishes *if leaders change infinitely often and their beliefs are distinct.*

Proof. See Appendix B. □

This result does not rely on specific assumptions about the data-generating process. It only requires leaders to change infinitely often and to have distinct beliefs. If there is a unique leader, the-FLS trader survives because his beliefs are as accurate as the leader's. Similarly, the FLS-trader would survive if leaders change infinitely often, but their beliefs become essentially identical.

In a log-economy, the changing leader condition is equivalent to verifying that the wealthiest trader in the economy changes infinitely often. Given that successful investors in financial markets do change over time relatively often, our result tells us that a trader whose objective is to maximize the expected growth rate of his wealth (log utility) should avoid the FLS.

Example 2 illustrates Proposition 1 in a simple deterministic setting.

Example 2: consider an economy with two states $\mathcal{S} = \{a, b\}$ and three traders (1,2,FLS-trader) with identical, discount factors β and log utility. Traders 1 and 2 have fixed beliefs: $\forall t, p^1(\sigma_t = a) := p^1(a_t) = p^2(\sigma_t = b) := p^2(b_t) = \frac{1}{3}$. The sequence of states is deterministic:$\{a, b, a, b, a....\}$, which makes it is easy to verify that

$$p^{FLS}(a_t|\sigma^{t-1}) = \begin{cases} \frac{1}{2}, & if\ t-1 = even \\ p^2(a_t), & if\ t-1 = \ odd \end{cases}.$$

A quick calculation shows that the FLS-trader vanishes:

$$\lim_{t\to\infty} c_t^{FLS}(\sigma) = \lim_{t\to\infty} \frac{\beta^t_{FLS} p^{FLS}(\sigma^t) c_0^{FLS}}{q(\sigma^t)} = \lim_{t\to\infty} \frac{\frac{1}{3}\left(\frac{1}{2}\frac{1}{3}\right)^{\frac{t}{2}} \left(\frac{1}{3}\right)^{I\{t \text{ is odd}\}}}{\frac{1}{3}\left(\frac{2}{3}\frac{1}{3}\right)^{\frac{t}{2}} \left(\frac{1}{3}\right)^{I\{t \text{ is odd}\}} + \frac{1}{3}\left(\frac{1}{3}\frac{2}{3}\right)^{\frac{t}{2}} \left(\frac{2}{3}\right)^{I\{t \text{ is odd}\}} + \frac{1}{3}\left(\frac{1}{2}\frac{1}{3}\right)^{\frac{t}{2}} \left(\frac{1}{3}\right)^{I\{t \text{ is odd}\}}} = 0.$$

An intuition for the result is as follows.

Suppose you are driving in heavy traffic. There are two lines of cars (1, 2). Let's call the car in front of you *car 1* and the one next to it *car 2.* Your goal is to stay close to the car that is ahead: the leading car.[10] The FLS is, qualitatively, equivalent to the strategy that prescribes to always be in the leading car's line. This strategy can be summarized as follows.[11]

- At t=0, you are in line 1, behind car 1, which is next to car 2. If car 1 takes the lead, you remain in line 1 (behind car 1) until the two cars are next to each other again, t_1^*. If car 2 takes the lead, you change lines (loosing a position) and stay in line 2, one car behind car 2, until t_1^*.

- At t_1^*, if you are in line 1 you repeat the same strategy. Otherwise, the strategy remains essentially the same, except that you start in line 2 one car behind cars 1 and 2. If car 2 takes the lead, you remain in line 2 (one car behind car 2) until the two cars are next to each other again, t_2^*. If car 1 takes the lead, you change lines (loosing another position) and stay in line 1, two cars behind car 1, until t_2^*.

Iteratively, you lose one position against the leading car every time a change in leadership occurs. Thus, if the leading car changes infinitely often, your distance from the leader diverges.

6.1 The FLS cannot be analyzed using the standard approach

In this section, we provide an example in which the standard approach fails to correctly characterize the performance of a FLS-trader. Even when adopting our exact measure of accuracy, pairwise comparisons of individual characteristics fail to indicate that the FLS-trader vanishes. This simple example eludes all existing conditions for a trader to vanish exept ours.

Example 3: consider an economy with two states $\mathcal{S} = \{a, b\}$, iid true probabilities $P = [\frac{1}{2}, \frac{1}{2}]$ and three traders (1,2,FLS), with identical discount-factors β, and log utility. Traders 1 and 2 have iid beliefs: $\forall t, p^1(\sigma_t = a) := p^1(a_t) = p^2(\sigma_t = b) := p^2(b_t) = \frac{1}{3}$. The FLS-trader copies the beliefs of the wealthiest trader and otherwise gives the two beliefs equal weight:

$$p^{FLS}(a_t|\sigma^{t-1}) = \begin{cases} p^i(a_t); & i : p^i(\sigma^{t-1}) = \arg\max\{p^1(\sigma^{t-1}), p^2(\sigma^{t-1})\}; \\ \frac{1}{2}; & \text{if ties occur.} \end{cases}$$

Proposition 2. *In the economy of Example 3,*
(i) the FLS-trader vanishes P-a.s.;
(ii) a pairwise comparison of traders' beliefs does not imply that the FLS-trader vanishes.

[10] In our economy, the FLS-trader's consumption share is mostly affected by the leader's wealth.
[11] WLOG, we assume that when the two cars are next to each other you do not change lines.

Proof. In Appendix C we show that:
$(i) : \limsup_{t\to\infty} c^3(\sigma^t) = \limsup_{t\to\infty} \frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = \limsup_{t\to\infty} \frac{p^{FLS}(\sigma^t)}{p^1(\sigma^t)+p^2(\sigma^t)+p^{FLS}(\sigma^t)} = 0$ P-a.s.;
$(ii) : for\ i = 1, 2,\ \limsup_{t\to\infty} \frac{c_t^{FLS}(\sigma)}{c_t^i(\sigma)} = \limsup_{t\to\infty} \frac{p^{FLS}(\sigma^t)}{p^i(\sigma^t)} = \infty$ P-a.s.. □

Proposition 2 (i) is a special case of Proposition 1. It follows after noticing that trader 1 and 2 alternate infinitely often in their leadership because their beliefs are equally (in)accurate. Proposition 2 (ii) shows that the FLS-trader is infinitely often wealthier than the poorest between trader 1 and 2: his beliefs pass the pairwise comparison test against traders 1 and 2.

Together, (i) and (ii) show that a pairwise comparison of individual characteristics cannot deliver a necessary and sufficient condition for a trader to vanish. Although the FLS-trader's beliefs pass the pairwise comparison test against traders 1 and 2, he is not fit to survive against both traders simultaneously. The deviations of the empirical average from the true probability that favor him against trader 1 (2) make him lose against trader 2 (1).

Continuing with our car intuition, suppose car 1 has 50 percent chance to gain (lose) a position against car 2 in every period. (i) tells us that your distance from the leading car diverges. (ii) tells us that you are going to be ahead of the non-leading car infinitely often. Clearly, (ii) cannot be used to assess your performance against the leading car.

Remark: the existing conditions for a trader to vanish, an approximation of Proposition 2 (ii), fail to correctly characterize the performance of the FLS strategy. Specifically,

- no trader satisfies Sandroni (2000)'s condition to vanish. The average beliefs of all traders are equally accurate because the likelihood ratio diverges at a rate slower than t:

$$\ln \frac{p^{FLS}(\sigma^t)}{p^1(\sigma^t)} \asymp \sqrt{t} \Rightarrow \lim_{t\to\infty} \frac{1}{t} \ln \frac{p^{FLS}(\sigma^t)}{p^1(\sigma^t)} = \lim_{t\to\infty} [\mathcal{E}_{FLS,t} - \mathcal{E}_{1,t}] = 0;$$

- traders 1 and 2 satisfy Blume and Easley (2006)'s condition to vanish (Theorem 8). Its application leads one to incorrectly conclude that the FLS-trader dominates:

$$\lim_{t\to\infty} \left[\sum_{\tau=1}^{t} E_P \ln \frac{p^{FLS}(\sigma_\tau|\sigma^{\tau-1})}{P(\sigma_\tau)} - \sum_{\tau=1}^{t} E_P \ln \frac{p^1(\sigma_\tau|\sigma^{\tau-1})}{P(\sigma_\tau)} \right] = +\infty.^{12}$$

Example 3 shows that relying on short-period notions of accuracy can be misleading

[12] Because the FLS-trader is more accurate than trader 1 and 2 infinitely often (he uses the correct model: $p^{FLS}(\sigma_{t+1}|\sigma^{t^*}) = \frac{1}{2}$ whenever trader 1 and trader 2 are equally wealthy) and never less accurate.

(e.g. Massari (2013)). The FLS-trader's next-period beliefs are never less accurate and infinitely often more accurate than that of traders 1 and 2. Yet, the FLS-trader is not more accurate than the other traders on long sequences and vanishes P-a.s..

7 An analysis of the Follow the Market Strategy

The FMS prescribes adopting the next-period state price density as a belief. This intuitive strategy is behaviorally motivated by *Wisdom of The Crowds* argument that says equilibrium prices might reflect beliefs that are more accurate than those of all market participants (Galton (1907)). To avoid known biases due to the interaction between risk attitudes and fluctuations of the aggregate endowment, we assume constant aggregate endowment (**A2':** $\forall t, \sigma, \sum_{i\in\mathcal{I}} e_t^i(\sigma)=1$). Moreover, we assume a homogeneous discount factor and, for tractability reasons, that all traders have an identical CRRA utility function (**A1':** $\forall i \in \mathcal{I}, \beta^i=\beta^j$, and $u^i(c)=\frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma\in(0,\infty)$). Under the stated assumptions, the FMS-trader's beliefs have this analytical form.

Definition 7. *Under **A1', A2'** and **A3**, the FMS-trader's beliefs are given by:*

$$p_\gamma^{FMS}(\sigma^t)=\prod_{\tau=1}^{t} p_\gamma^{FMS}(\sigma_\tau|\sigma^{\tau-1}) \qquad ; \qquad p_\gamma^{FMS}(\sigma_t|\sigma^{t-1})=\frac{\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}{\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\in\mathcal{I}} p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}.$$

Where $c^i_{\gamma,t-1}(\sigma)=\frac{p^i(\sigma^{t-1})^{\frac{1}{\gamma}}c_0^i}{\sum_{i\in\mathcal{I}} p^i(\sigma^{t-1})^{\frac{1}{\gamma}}c_0^i}$ *is the consumption share of trader* i *at* σ^{t-1}.[13]

Next, we use **A1', A2'** to obtain an analytical equation for next-period equilibrium prices.

Lemma 1. (*i*) *Under **A1', A2'** and **A3**, next-period equilibrium prices are given by:*

$$\begin{aligned} q(\sigma_t|\sigma^{t-1}) := \frac{q(\sigma^t)}{q(\sigma^{t-1})} &= \beta\frac{\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}{\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\in\mathcal{I}} p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}\times\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\in\mathcal{I}} p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma \\ &= p_\gamma^{FMS}(\sigma_t|\sigma^{t-1})\times\beta(1+\delta_{t,\gamma}(\sigma)) \end{aligned}$$

[13] Our definition of p^{FMS} is circular: p^{FMS} appears on both sides of the equal sign. We opted for this definition to ease the comparison between p^{FMS} and equilibrium prices. It can be verified, by substitution, that our definition is equivalent to the more rigorous, $p_\gamma^{FMS}(\sigma_t|\sigma^{t-1}):=\frac{\left(\sum_{i\neq FMS} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}{\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\neq FMS} p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma)\right)^\gamma}$.

(ii) Moreover, $\delta_{t,\gamma}(\sigma) \leq (\geq) 0 \Leftrightarrow \gamma \geq (\leq) 1$ *with strict equality if and only if* $\gamma = 1$ *or the consumption share/beliefs distribution is degenerate.*

Proof. See Appendix B. □

In a log-economy (γ=1), state prices add to the common discount factor. The FMS-trader's beliefs and discount factor both coincide with those of the representative agent (Rubinstein (1974)). The FMS-trader survives in every sequence because he buys a constant share of the aggregate endowment and discount the future at the same rate of the market. If γ differs from one, state prices do not add to the common discount factor. An endogenous component of markets' discount factor, $\delta_{t,\gamma}(\sigma)$, is generated by traders' speculative incentives to trade. In this case, the FMS-trader beliefs' make him smooth consumption across states, while the difference between his discount factor and the market discount factor determine his asymptotic fate. Lemma 1 *(ii)* shows that the sign and the size of the market discount factor depends on the risk attitudes of all traders in the economy and on the consumption share/belief distribution.

In order to prove our main result, we need one last technical assumption.

A3': The next period beliefs' ratios between all traders in the economy are uniformly bounded above and below:

$$\exists \epsilon > 0 : \forall i, j \in \mathcal{I}, \forall \sigma^{t-1}, \max_{\sigma_t} \frac{p^i(\sigma_t|\sigma^{t-1})}{p^j(\sigma_t|\sigma^{t-1})} \in (\epsilon, \frac{1}{\epsilon}).$$

In iid economies, **A3'** is equivalent to **A3**. In economies in which traders' beliefs change over time, **A3'** rules out pathological cases in which some trader becomes progressively sure that one event is impossible (Sandroni (2000)). These situations are hard to analyze because traders likelihood ratios can become dominated by a single observation.

Proposition 3. *Under **A1', A2'** and **A3'**, if leaders among traders* $\mathcal{I} \setminus FMS$ *change infinitely often and have* distinct *beliefs,*

(i) $\gamma > 1 \Rightarrow$ *the FMS-trader dominates*;

(ii) $\gamma = 1 \Rightarrow$ *the FMS-trader survives without dominating*;

(iii) $\gamma < 1 \Rightarrow$ *the FMS-trader vanishes.*

Proof. Sketch: inspection of the FMS-trader's FOCs show that his fate depends on the sign and dynamic of the market discount factor: $c_t^{FMS}(\sigma) = \frac{\beta^t p_\gamma^{FMS}(\sigma^t)}{\prod_{\tau=1}^t q(\sigma_\tau|\sigma^{\tau-1})} c_0^{FMS} = \prod_{\tau=1}^t \frac{1}{(1+\delta_\tau)} c_0^{FMS}$. In particular, if leaders with distinct beliefs among the non-FMS-traders change infinitely often, the consumption share/belief distribution is not degenerate infinitely often and δ_t differs from zero infinitely often. Therefore, the FMS-trader dominates (vanishes) iff $\gamma > (<)1$ because $\gamma > (<)1 \Leftrightarrow \forall t$, $\frac{1}{(1+\delta_{t,\gamma})} \geq (\leq)1$ and $\frac{1}{(1+\delta_{t,\gamma})} > (<)1)$, infinitely often.

The formal proof — in Appendix B — maintain the same intuition but is complicated by the fact that if trader FMS dominates, then $|\delta_t| \to 0$. □

Proposition 3 can be understood in light of these observations. First, the heterogeneity of beliefs implies that traders subjectively believe assets to be mispriced and trade for speculative reasons. Second, the CRRA parameter affects the price level because it simultaneously determines traders' risk tolerance and saving rate. Third, the presence of ever-changing leaders ensures that a non-trivial amount of wealth is held by traders with different beliefs even in the long-run.[14] Intuitively the argument goes as follows.

In a log-economy ($\gamma = 1$), $\delta_{t,\gamma} = 0$ because traders optimally invest a constant share of their consumption, irrespective of equilibrium prices. The FMS-trader "buys the market" and discounts the future at the same rate as the representative agent. Therefore, he consumes a constant share of the aggregate endowment in every period and survives without dominating.

If all traders are less risk averse than log ($\gamma < 1$), they subjectively believe prices to be inaccurate and optimally decide to invest more than they would if they had log utility. In equilibrium, the price level is higher than it would be in a log economy ($\delta_{t,\gamma} > 0$). The FMS-trader vanishes because he buys the market – his beliefs coincides with the risk neutral probability— but his saving rate is lower than that of the market infinitely often.

Conversely, if all traders are more risk averse than log ($\gamma > 1$), they subjectively believe prices to be inaccurate and optimally decide to invest less than they would if they had log utility. In equilibrium, the price level is lower than it would be in a log economy ($\delta_{t,\gamma} < 0$). The FMS-trader dominates because he buys the market and his saving rate is higher than that of the market infinitely often.

If we modify Example 3 by replacing the-FLS trader with the FMS-trader and assuming that all traders have identical CRRA utility with parameter γ, we find that the standard approach can be used to discuss the performance of the FMS only in those cases in which the FMS-trader dominates ($\gamma > 1$). Pairwise comparisons of traders' discounted beliefs cannot be used to analyze the FMS-trader performance when $\gamma < 1$ by the same logic used in the analysis of the FLS. Although the FMS-trader vanishes, there is no single trader that beats him.

[14]Inspection of the proof of Proposition 3 shows that this intuition applies even when the consumption share distribution becomes degenerate because the FMS-trader dominates.

8 Relative extinction reversal

In this section, we use our condition to study the relative performance of each couple of traders.

Definition 8. *Trader i vanishes relative to trader j if* $\lim_{t\to\infty} \frac{c_t^i(\sigma)}{c_t^j(\sigma)} =^{P\text{-}a.s.} 0$.

Proposition 4. *In an economy that satisfies* ***A1-A3****, if trader i, j's utilities are CRRA with parameters γ_i, γ_j,*

$$\text{trader } i \text{ vanishes relative to trader } j \Leftrightarrow \lim_{t\to\infty} \left(\frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_i}} \Big/ \left(\frac{\beta^t p^j(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_j}} =^{P\text{-}a.s.} 0. \qquad (2)$$

Proof. It follows taking ratios of the FOC. $\square$

Proposition 4 shows that the relative performance between two traders cannot be characterized using a pairwise comparison of discounted beliefs alone. The relative performance of two vanishing traders cannot be analyzed in isolation because the prices they use to trade are not the ones they would use to trade if they were alone in the economy. The only case in which our condition coincides with the pairwise comparison of traders' discounted beliefs is when one of the two traders dominates. In this case, equilibrium prices can be approximated WLOG with the discounted beliefs of the dominating trader, and the effect of risk attitudes on traders' survival becomes mute. Otherwise, if both traders vanish, relative survival depends on their discounted beliefs, but also on their relative risk attitudes and equilibrium prices.

Proposition 4 can be understood in light of Yan (2008)'s analysis of traders' survival in growing (shrinking) economies. Although we assume constant aggregate endowment, the cumulative consumption shares of two vanishing traders shrinks over time. Therefore, the relative dynamic of two vanishing traders' consumption shares is, qualitatively, similar to the dynamic we would observe if they were trading in isolation in a shrinking economy. As in Yan (2008), we find that the relative performance of the two vanishing traders depends on their discounted beliefs and on the interaction between their relative risk tolerance and the shrinking rate of their cumulative consumption shares (aggregate endowment in Yan (2008)).

As an application, we conclude with an example showing that the relative performance of two traders can even be reversed by the presence of a new trader in the economy. A result that question the general view according to which the market always favor traders with more accurate probabilistic views, even among vanishing traders: *"...in models with intermediate* *consumption and complete markets, extinction reversal cannot occur because relative extinction is independent of characteristics of other agents"* — Cvitanic and Malamud (2010).

Example 4: consider an economy with two states $S = \{a, b\}$, two traders (i,j) with identical discount factor β, iid beliefs, $p^i(a)=\frac{6}{10}$, $p^j(a)=\frac{7}{10}$ and CRRA utility functions with $\gamma_i = 1$(log), and $\gamma_j = 5$, respectively. Rearranging the FOC we obtain condition 2: $\frac{c^i_t(\sigma)}{c^j_t(\sigma)} = \left(\frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_i}} \Big/ \left(\frac{\beta^t p^j(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_j}}$. The real probability of state a is $P(a) = \frac{1}{2}$ in every period. Because p^i is more accurate than p^j P-a.s., Theorem 2 (iii) allows to substitute $\beta^t p^i(\sigma^t)$ for $q(\sigma^t)$ and verify that *trader j vanishes relatively to trader i.*

Let's add to this economy a new trader, k, with the same discount factor and correct beliefs $p^k = P$. By Theorem 2, trader k dominates P-a.s. and we can substitute $\beta^t P(\sigma^t)$ for $q(\sigma^t)$. It is easy to verify that, at this price, it is *trader i that vanishes relative to trader j.*[15] The presence of trader k in the economy causes relative extinction reversal between i and j.

The relative extinction reversal phenomenon can be understood using the analogy we proposed between Proposition 4 and Yan (2008). Adding a dominating trader in the economy is equivalent to changing the (exogenous) shrinking rate in Yan's condition. It gives higher (relative) survival chances to the trader with a higher CRRA parameter because his higher propensity to defer consumption gets magnified by the shrinking rate of the economy.

9 Conclusions

This paper introduces an alternative approach to market selection. Instead of comparing traders' performance in a pairwise fashion, we focus on the ratio between traders' discounted beliefs and (approximate) equilibrium prices. Unlike existing conditions, ours is both sufficient and necessary for a trader to vanish and delivers the exact rate at which a vanishing trader loses his consumption share in economies with more than two traders. We present two applications to highlight an intuitive point obscured by the standard approach: trading in financial markets is qualitatively different from bilateral trading.

[15]

$$\lim_{t\to\infty} \frac{c^i_t(\sigma)}{c^j_t(\sigma)} = \lim_{t\to\infty} \frac{\left(\frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_i}}}{\left(\frac{\beta^t p^j(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_j}}} = \lim_{t\to\infty} \left(\frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}\right)^{\frac{1}{\gamma_i}} \left(\frac{q(\sigma^t)}{\beta^t p^j(\sigma^t)}\right)^{\frac{1}{\gamma_j}} =^{P\text{a.s.},\ by\ Th.1} \lim_{t\to\infty} \left(\frac{\beta^t p^i(\sigma^t)}{\beta^t P(\sigma^t)}\right)^{\frac{1}{\gamma_i}} \left(\frac{\beta^t P(\sigma^t)}{\beta^t p^j(\sigma^t)}\right)^{\frac{1}{\gamma_j}}$$

$$=^{P\text{a.s.}} \lim_{t\to\infty} \left(\frac{.6 * .4}{.5 * .5}\right)^{(1\times\frac{t}{2})} \left(\frac{.5 * .5}{.7 * .3}\right)^{(\frac{1}{5}\times\frac{t}{2})} = 0.$$

A Appendix

In this Appendix we compare our notion of accuracy with that of Sandroni (2000)'s and Blume and Easley (2006)'s. Definition 2 ranks traders' beliefs according to the asymptotic ratio of their likelihood. This criterion is more precise than Sandroni (2000)'s and Blume and Easley (2006)'s criteria, which only approximate this ratio. Our criterion relates to their as follows:

Sandroni (2000). *Entropy of trader i:*

$$\mathcal{E}_{i,t} := -\frac{1}{t}\sum_{\tau=1}^{t} E_P\left(\ln \frac{P(\sigma_\tau|\sigma^{\tau-1})}{p^i(\sigma_\tau|\sigma^{\tau-1})}\right). \tag{3}$$

If we consider $E_P\left(\ln \frac{P(\sigma_\tau|\sigma^{\tau-1})}{p^i(\sigma_\tau|\sigma^{\tau-1})}\right)$ to be a measure of a trader's expected accuracy, the comparison of traders' entropies is a comparison of the average expected accuracy of two traders. As intuition suggests, $\mathcal{E}_{i,t} - \mathcal{E}_{j,t}$ is a coarse approximation of traders' average likelihood ratio: if trader i's expected beliefs are on average more accurate than trader j's, then trader i is empirically more accurate than trader j (as an application of the Strong Law of Large Numbers for Martingale Differences, see Sandroni (2000)). However, the converse implication does not always follow because the averaging term, $\frac{1}{t}$, "kills" divergence rates that are slower than t. For example, it cannot distinguish between the different learning rates of two Bayesian traders with different prior support dimensionality (see Blume and Easley (2006)).

Blume-Easley (2006). *Sum of conditional relative entropies of trader i:*

$$D_t(P||p_i) := \sum_{\tau=1}^{t} E_P\left(\ln \frac{P(\sigma_\tau|\sigma^{\tau-1})}{p^i(\sigma_\tau|\sigma^{\tau-1})}\right). \tag{4}$$

This definition suggests that if trader i is in every period, on expectation, more accurate than trader j he should also be empirically more accurate. This intuitive argument is, however, not always correct: there are cases in which $D_t(P||p_i) - D_t(P||p_j) \to \infty$ even though $\frac{p^j(\sigma^t)}{p^i(\sigma^t)} < \infty$ (Massari (2013)). In these cases we cannot rely on this measure to discuss survival: it leads to incorrect conclusions. In section 6.1, we present a new case in which the use of this criterion leads to an incorrect conclusion (see remark 2).

The comparison between Equations 3 and 4 highlights that Blume and Easley (2006)'s criterion can lead to incorrect results only if the log likelihood divergence between two traders is o(t). Otherwise — if the log likelihood divergence rate is O(t)—, this criterion is equivalent to that of Sandroni (2000) and it can be used to delivers a correct sufficient condition for a trader to vanish.

B Appendix

Lemma 2. *In an economy that satisfies **A1-A3**:* $\exists a, b \in (0,\infty) : \forall \sigma \in \Sigma : \begin{cases} \limsup\limits_{t\to\infty} \sum\limits_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} < b \\ \liminf\limits_{t\to\infty} \sum\limits_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} > a \end{cases}.$

Proof.

(*i*) $\limsup\limits_{t\to\infty} \sum\limits_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} < \infty$: by contradiction.
$\sum\limits_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} = \infty \Leftrightarrow \exists i : u_i'(c_t(\sigma)) = 0 \Leftrightarrow c = \infty$ because each u_i is strictly concave, increasing. This condition is impossible. It violates **A2** via the market clearing condition: $\sum\limits_{i\in\mathcal{I}} c_t^i = \sum\limits_{i\in\mathcal{I}} e_t^i < F < \infty$.

(*ii*) $\liminf_{t\to\infty} \sum_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} > 0$: by contradiction.
$\sum_{i\in\mathcal{I}} \frac{1}{u_i'(c_t^i(\sigma))} = 0 \Leftrightarrow \forall i \in \mathcal{I}, u_i'(c_t^i(\sigma) = \infty$ which is true if and only if all the traders have 0 consumption and satisfy the Inada condition at 0. The first requirements is impossible as it violates the market clearing condition: $\sum_{i\in\mathcal{I}} c_t^i = \sum_{i\in\mathcal{I}} e_t^i > 0$.

□

Proof of Theorem 1

Proof. By the FOCs: $\forall i \in \mathcal{I}, \forall t, \forall \sigma \in \Sigma, \frac{q(\sigma^t)u_i'(c_0^i)}{p^i(\sigma^t)} = \beta_i^t u_i'(c_t^i)$.

Rearranging and summing over traders: $q(\sigma^t) = \frac{\sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{\sum_{j\in\mathcal{I}} \frac{1}{u_j'(c_t^j(\sigma))}}$.

By Lemma 2 $\exists a, b \in (0,\infty) : \forall t, \forall \sigma \in \Sigma,\ q(\sigma^t) \in \left[\frac{\sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{b}, \frac{\sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{a}\right]$.

Thus $q(\sigma^t) \asymp \sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t)$. □

Proof of Theorem 2:

Proof. By the FOC: $\forall \sigma^t, \frac{\beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{q(\sigma^t)} = \frac{1}{u_i'(c_t^i(\sigma))}$. Therefore, $\forall \sigma \in \Sigma$:

$\limsup_{t\to\infty} \frac{\beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{q(\sigma^t)} = 0 \Leftrightarrow \limsup \frac{1}{u_i'(c_t^i(\sigma))} = 0 \Leftrightarrow \liminf u_i'(c_t^i(\sigma)) = +\infty \Leftrightarrow^a \limsup c_t^i(\sigma) = 0$.

(*a*) By **A1** $\lim_{c\to 0} u_i'(c) = +\infty$ and the payoff functions are strictly concave, increasing.

(*i*) It follows applying the above argument to a set of sequences of measure one.

(*ii*) It follows because, by Theorem 1, we can replace $\sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t)$ for $q(\sigma^t)$ WLOG.

(*iii*) It follows noticing that $q(\sigma^t) \asymp^{By\ Th.1} \sum_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t) \asymp \max_{i\in\mathcal{I}} \beta_i^t p^i(\sigma^t)$. □

Proof of Proposition 1

Proof. By Theorem 2, the FLS-trader vanishes iff $\frac{p^{FLS}(\sigma^t)}{\sum_i p^i(\sigma^t)} \to 0$.

Let $\tilde{L}(t)$ be the number of times the leader changes before t. By the same logic of Lemma 5 $\forall \sigma^t, \ln p^{FLS}(\sigma^t) < \ln\left(\max_{i\in\mathcal{I}} p^i(\sigma^t)\right) - \tilde{L}(t)k$. By assumption $\tilde{L}(t) \to \infty$; the result follows noticing that if leaders have distinct beliefs $\exists > 0 : k > \epsilon$ infinitely often. □

Proof of Lemma 1

Proof.

(i): by the FOCs: $\forall \sigma^t, \frac{\beta_i^t p^i(\sigma^t) \frac{1}{u_i'(c_0^i)}}{q(\sigma^t)} = \frac{1}{u_i'(c_t^i(\sigma))}$. Substituting $c_t^i(\sigma)^{-\gamma}$ for $u^i(c_t^i(\sigma))'$ and $u^i(c_0^i)'$,

$$\beta^t p^j(\sigma^t) c^j(\sigma^t)^{-\gamma} = (c_0^j)^{-\gamma} q(\sigma^t) \tag{5}$$

taking ratio of traders i, j' FOCs: $\frac{\beta^t p^i(\sigma^t) c_t^i(\sigma)^{-\gamma}}{\beta^t p^j(\sigma^t) c^j(\sigma^t)^{-\gamma}} = \frac{(c_0^i)^{-\gamma} q(\sigma^t)}{(c_0^j)^{-\gamma} q(\sigma^t)}$; solving for $c^i(\sigma^t)$:

$$c_t^i(\sigma) = \left(\frac{p^i(\sigma^t)}{p^j(\sigma^t)}\right)^{\frac{1}{\gamma}} \frac{c_0^i}{c_0^j} c^j(\sigma^t). \tag{6}$$

Substituting Eq.6 in the market-clearing condition (which holds with equality because of monotonicity of u^i): $1 = \sum_{i \in \mathcal{I}} c^i_t(\sigma) = c^j(\sigma^t) \frac{\sum_{i\in\mathcal{I}} p^i(\sigma^t)^{\frac{1}{\gamma}} c^i_0}{p^j(\sigma^t)^{\frac{1}{\gamma}} c^j_0}$; solving for $c^j(\sigma^t)$ we obtain: $c^j(\sigma^t) = \frac{p^j(\sigma^t)^{\frac{1}{\gamma}} c^j_0}{\sum_{i\in\mathcal{I}} p^i(\sigma^t)^{\frac{1}{\gamma}} c^i_0}$.

Substituting $c^j(\sigma^t)$ in Eq.5 and rearranging, we obtain $q(\sigma^t) = \beta^t \left(\sum_{i\in\mathcal{I}} p^i(\sigma^t)^{\frac{1}{\gamma}} c^i_0 \right)^\gamma$. Therefore,

$$q(\sigma_t|\sigma^{t-1}) := \frac{q(\sigma^t)}{q(\sigma^{t-1})} = \frac{\beta^t \left(\sum_{i\in\mathcal{I}} p^i(\sigma^t)^{\frac{1}{\gamma}} c^i_0 \right)^\gamma}{\beta^{t-1} \left(\sum_{i\in\mathcal{I}} p^i(\sigma^{t-1})^{\frac{1}{\gamma}} c^i_0 \right)^\gamma} = \beta \left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma \tag{7}$$

Dividing and multiplying Eq. 7 by $\sum_{\hat{\sigma}_t \in \mathcal{S}} \left(\sum_{i\in\mathcal{I}} p^i(\hat{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma$ delivers the desired equation.

$$q(\sigma_t|\sigma^{t-1}) = \beta \frac{\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma}{\sum_{\hat{\sigma}_t\in\mathcal{S}} \left(\sum_{i\in\mathcal{I}} p^i(\hat{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma} \times \sum_{\hat{\sigma}_t\in\mathcal{S}} \left(\sum_{i\in\mathcal{I}} p^i(\hat{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma.$$

(ii): The last thing left to prove is that $(1+\delta_{t,\gamma}(\sigma)) = \sum_{\hat{\sigma}_t\in\mathcal{S}} \left(\sum_{i\in\mathcal{I}} p^i(\hat{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma \leq (\geq) 1 \Leftrightarrow \gamma \geq (\leq) 1$, with equality if and only if $\gamma = 1$ or the consumption share/beliefs distribution is degenerate.

Define $f(x) = x^\gamma$; $f(.)$ is continuous and strictly concave $\Leftrightarrow \gamma < 1$, linear $\Leftrightarrow \gamma = 1$ and strictly convex $\Leftrightarrow$ $\gamma > 1$. Let's rewrite $(1+\delta_{t,\gamma}(\sigma))$ as $\sum_{\sigma_t\in\mathcal{S}} \left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)^\gamma = \sum_{\sigma_t\in\mathcal{S}} f\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right)$.

Let focus on $\gamma < 1$: f(.) is strictly concave. Because $\sum_{i\in\mathcal{I}} c^i_{\gamma,t-1}(\sigma) = 1$, Jensen's inequality implies:

$$\begin{aligned}
\sum_{\sigma_t\in\mathcal{S}} f\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} c^i_{\gamma,t-1}(\sigma) \right) &\geq \sum_{\sigma_t\in\mathcal{S}} \sum_{i\in\mathcal{I}} c^i_{\gamma,t-1}(\sigma) f(p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}}) \\
&= \sum_{i\in\mathcal{I}} c^i_{\gamma,t-1}(\sigma) \sum_{\sigma_t\in\mathcal{S}} p^i(\sigma_t|\sigma^{t-1}) \\
&= 1 \qquad\qquad \text{Because } \forall i, \sum_{\sigma_t\in\mathcal{S}} p^i(\sigma_t|\sigma^{t-1}) = 1
\end{aligned}$$

With equality iff the consumption share is degenerate, or the belief of all traders with positive consumption are identical. When $\gamma < 1$, f(.) is strictly convex and the opposite inequality holds. When $\gamma = 1$, f(.) is linear and equality holds for every beliefs/consumption shares distribution.

□

Proof of Proposition 3

Proof. By FOC: $c^{FMS}_t(\sigma)^\gamma = \frac{\beta^t p^{FMS}_\gamma(\sigma^t) c^{FMS}_0(\emptyset)^\gamma}{q(\sigma^t)} = \frac{\prod_{\tau=1}^t p^{FMS}_\gamma(\sigma_\tau|\sigma^{\tau-1}) c^{FMS}_0(\emptyset)^\gamma}{\prod_{\tau=1}^t p^{FMS}_\gamma(\sigma_\tau|\sigma^{\tau-1}) * \prod_{\tau=1}^t (1+\delta_{t,\gamma}(\sigma))} = \prod_{\tau=1}^t \frac{c^{FMS}_0(\emptyset)^\gamma}{(1+\delta_{t,\gamma}(\sigma))}$.

- $\gamma = 1$: by Lemma 1, $\forall \sigma, \forall t, \delta_{t,\gamma}(\sigma) = 0$ and the result follows.

The analysis of $\gamma \neq 1$ is delicate because p^{FMS} endogenously affects the consumption share/beliefs distribution.

- $\gamma < 1$, by Lemma 1, $\forall \sigma, \forall t, (1+\delta_{t,\gamma}(\sigma)) \geq 1$, which implies that $\forall \sigma, \forall t, c^{FMS}_t(\sigma) < c^{FMS}_0$: the FMS-trader consumption share is bounded above by $c^{FMS}_0 < 1$. This observation allows to derive our result focusing exclusively on the non-FMS-traders' beliefs:

$$\forall\sigma,\forall t,(1+\delta_{t,\gamma})=\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\in\mathcal{I}}p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}}c^i_{\gamma,t-1}(\sigma)\right)^{\gamma}=\sum_{\hat\sigma_t\in\mathcal{S}}f\left(\sum_{i\in\mathcal{I}}p^i(\hat\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}}c^i_{\gamma,t-1}(\sigma)\right)$$

$$>\min_{\substack{\bar c=1-\sum_{i\neq FMS}c^i\leq c_0^{FMS}(\emptyset)\\ \bar p:\sum_{\sigma_t}\bar p(\sigma_t)=1}}\sum_{\hat\sigma_t\in\mathcal{S}}f\left(\sum_{i\neq FMS}p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}}c^i_{\gamma,t-1}(\sigma)+\bar p(\sigma_t)^{\frac{1}{\gamma}}\bar c_{t-1}(\sigma)\right)$$

With $\gamma<1$, $f(.)$ is strictly concave and continuous, therefore, by Jeffrey's inequality and continuity of f:

$$\exists\epsilon>0:\min_{\substack{c=1-\sum_{i\neq FMS}c^i\leq c_0^{FMS}(\emptyset)\\ \bar p:\sum_{\sigma_t}\bar p(\sigma_t)=1}}\sum_{\hat\sigma_t\in\mathcal{S}}\left(\sum_{i\neq FMS}p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}}c^i_{\gamma,t-1}(\sigma)+\bar p(\sigma_t)^{\frac{1}{\gamma}}\bar c_{t-1}(\sigma)\right)^{\gamma}>1+\epsilon$$

$$\Leftrightarrow\ \exists\epsilon_1>0\text{ and }i,j\in\mathcal{I}:\min\left\{c^i_{\gamma,t-1}(\sigma);c^j_{\gamma,t-1}(\sigma);\sum_{\sigma_t\in S}|p^i(\sigma_t|\sigma^{t-1})-p^j(\sigma_t|\sigma^{t-1})|\right\}>\epsilon_1.$$

The following chain of implications proves the result:

Leaders with *distinct* beliefs alternate infinitely often

$$\Rightarrow^a\ \exists\epsilon_1>0:\limsup_{t\to\infty}\left[\max_{i,j\in\mathcal{I}}\left[\min\left\{c^i_{\gamma,t-1}(\sigma);c^j_{\gamma,t-1}(\sigma);\sum_{\sigma_t\in S}|p^i(\sigma_t|\sigma^{t-1})-p^j(\sigma_t|\sigma^{t-1})|\right\}\right]\right]>\epsilon_1$$

$$\Rightarrow\forall t,(1+\delta_{t,\gamma}(\sigma))\geq1\text{ and }\exists\epsilon>0:(1+\delta_{t,\gamma}(\sigma))>1+\epsilon\ \text{ infinitely often}$$

$$\Rightarrow c_t^{FMS}(\sigma)^{\gamma}=\prod_{\tau=1}^{t}\frac{c_0^{FMS}(\emptyset)^{\gamma}}{(1+\delta_{t,\gamma}(\sigma))}\to0$$

$\Rightarrow$ the FMS-trader vanishes.

(a): Let τ denote those period in which a leader change and $\hat\imath_\tau,\hat\imath_{\tau-1}$ be the leaders at period τ and $\tau-1$, respectively. By assumption there are infinitely many periods τ. Here we prove the existence of a uniform lower bound for the three component of Equation a. To save notation, and WLOG, we assume that all traders have identical time 0 consumption: $\forall i,j\in\mathcal{I},\frac{c_0^i}{c_0^j}=1$.

- First we lower bound at τ the consumption of the trader leading at τ: $\forall\tau,\exists\epsilon_2>0:c^{\hat\imath_\tau}(\sigma)>\epsilon_2$.
  By definition of a leader and the FOCs: $\forall i\neq\hat\imath_\tau$ and FMS, $\frac{c^i_\tau(\sigma)}{c^{\hat\imath_\tau}_\tau(\sigma)}=\left(\frac{p^i(\sigma^\tau)}{p^{\hat\imath_\tau}(\sigma^\tau)}\right)^{\frac{1}{\gamma}}<1$.
  Therefore $\forall i\neq\hat\imath_\tau$ and $FMS,c^{\hat\imath_\tau}_\tau(\sigma)>c^i_\tau(\sigma)\Rightarrow c^{\hat\imath_\tau}_\tau(\sigma)>\frac{1-c_0^{FMS}}{|\mathcal{I}|}=\epsilon_2>0$.
- Next, we lower bound at τ the consumption of the trader who led at $\tau-1$: $\forall\tau,\exists\epsilon_3>0:c_\tau^{\hat\imath_{\tau-1}}(\sigma)>\epsilon_3$.

$$\frac{c^{\hat\imath_\tau}_\tau(\sigma)}{c^{\hat\imath_{\tau-1}}_\tau(\sigma)}=^{\text{FOCs}}\left(\frac{p^{\hat\imath_\tau}(\sigma^\tau)}{p^{\hat\imath_{\tau-1}}(\sigma^\tau)}\right)^{\frac{1}{\gamma}}=\left(\frac{p^{\hat\imath_\tau}(\sigma_\tau|\sigma^{\tau-1})}{p^{\hat\imath_{\tau-1}}(\sigma_\tau|\sigma^{\tau-1})}\right)^{\frac{1}{\gamma}}\left(\frac{p^{\hat\imath_\tau}(\sigma^{\tau-1})}{p^{\hat\imath_{\tau-1}}(\sigma^{\tau-1})}\right)^{\frac{1}{\gamma}}$$

$$<\left(\frac{p^{\hat\imath_\tau}(\sigma_\tau|\sigma^{\tau-1})}{p^{\hat\imath_{\tau-1}}(\sigma_\tau|\sigma^{\tau-1})}\right)^{\frac{1}{\gamma}}*1\qquad\text{Because }\hat\imath_{\tau-1}\text{ is the }\tau-1\text{ leader;}$$

$$<\left(\frac{1}{\epsilon_0}\right)^{\frac{1}{\gamma}}\qquad\text{By }\mathbf{A3'}.$$

Therefore $c_\tau^{\hat{i}_{\tau-1}}(\sigma) > \epsilon_0^{\frac{1}{\gamma}} c_\tau^{\hat{i}_\tau}(\sigma) > \epsilon_0^{\frac{1}{\gamma}} \epsilon_2 = \epsilon_3$.

  - Finally, $\forall\tau, \exists\epsilon_4 : \sum\limits_{\sigma_\tau\in S} |p^{\hat{i}_\tau}(\sigma_\tau|\sigma^{\tau-1}) - p^{\hat{i}_{\tau-1}}(\sigma_\tau|\sigma^{\tau-1})| > \epsilon_4$, because leaders' beliefs are distinct.
  - The result follows setting $\epsilon_1 = \min\{\epsilon_2, \epsilon_3, \epsilon_4\}$.

- $\gamma > 1$, by Lemma 1, $\forall\sigma, \forall t, \delta_{t,\gamma}(\sigma) \leq 1$, with equality iff the consumption share/beliefs distribution is degenerate. The main difficulty is to deal with the fact that $\delta_{t,\gamma}(\sigma) \to 0$ if the FMS-trader dominates. First, note that $\gamma > 1 \Rightarrow \forall\sigma, \forall t, c_t^{FMS}(\sigma) = \frac{c_{t-1}^{FMS}(\sigma)}{1+\delta_{t,\gamma}(\sigma)} \geq c_{t-1}^{FMS}(\sigma)$, which guarantees that $\lim\limits_{t\to\infty} c_t^{FMS}(\sigma^t)$ exists in every sequence.

  Next, proceed by contradiction assuming that the FMS-trader does not dominate: $\exists\epsilon > 0 : \lim\limits_{t\to\infty} c_t^{FMS}(\sigma^t) = 1 - \epsilon < 1$. By the same argument used above, the existence of alternating leaders implies $\forall t$, $\frac{1}{1+\delta_\tau(\sigma)} \leq 1$, with strict inequality infinitely often, so that $c_t^{FMS}(\sigma) = \prod_{\tau=1}^t \frac{1}{1+\delta_\tau(\sigma)} c_0^{FMS} \to \infty$ monotonically, which implies that exists a $t < \infty : c_t^{FMS}(\sigma) > 1 = \sum\limits_{i\in\mathcal{I}} e_t(\sigma)$, a contradiction.

$\square$

C Appendix

This Appendix is dedicated to the analysis of Example 2. Our goal is to have a P-a.s. approximation of $p^{FLS}(\sigma^t)$ and $q(\sigma^t)$ in the economy of Example 3.

Let's start by expressing the sequence of realizations as a Random Walk:

$$y_\tau = \begin{cases} 1 \; if \; \sigma_\tau = a \\ -1 \; if \; \sigma_\tau = b \end{cases}, \text{ and } S_t = \sum_{\tau=1}^t y_\tau.$$

We are interested about two RVs:

- $L(t) := \sum\limits_{\tau=1}^t I_{S_\tau=0}$: the number of times $S_t = 0$ before t (Local Time);
- $|S_t| = \left|\sum\limits_{\tau=1}^t y_\tau\right|$: the absolute distance of the random walk from 0 at t.

The relationship between $|S_t|$ and $L(t)$ is captured by Tanaka's formula, which implies Lemma 3.

Lemma 3. *Let S_t be a random walk, $|S_t|$ its absolute value at t and $L(t)$ its local time at t, then*

$$\begin{cases} \limsup\limits_{t\to\infty} |S_t| - L(t) = +\infty \quad P\text{-}a.s. \\ \liminf\limits_{t\to\infty} |S_t| - L(t) = -\infty \quad P\text{-}a.s.. \end{cases}.$$

Proof. Define the RV $z_t = y_t\,(sign(S_{t-1}))$, with $sign(x) := \begin{cases} 1 \; if \; x \geq 0 \\ -1 \; if \; x < 0 \end{cases}$.

The discrete time version of Tanaka's formula implies: $\sum\limits_{\tau=1}^t z_\tau = |S_t| - L(t) + I_{y_{T_N+1}=1}$.

Because, z_t are iid RVs with $E(z_t)$=0 and $Var(z_t)$=1, the Law of Iterated Logarithms (Williams (1991)) implies:

$$\begin{cases} \limsup\limits_{t\to\infty} \frac{\sum_{\tau=1}^t z_\tau}{\sqrt{2t\log\log t}} =^{P\text{-a.s.}} 1 \\ \liminf\limits_{t\to\infty} \frac{\sum_{\tau=1}^t z_\tau}{\sqrt{2t\log\log t}} =^{P\text{-a.s.}} -1 \end{cases}, \text{which implies the desired } \begin{cases} \limsup\limits_{t\to\infty} |S_t| - L(t) =^{P\text{-a.s.}} +\infty \\ \liminf\limits_{t\to\infty} |S_t| - L(t) =^{P\text{-a.s.}} -\infty \end{cases}.$$

$\square$

Lemma 4. *Let S_t be a random walk and $L(t)$ its local time at t, then* $\quad \lim L(t) = +\infty \quad$ *P-a.s..*

Proof. By construction, $|S| > 0$ and $L(t) > 0$; by Lemma 3:

$$\liminf_{t\to\infty} |S_t| - L(t) = -\infty \ P\text{-a.s.} \Rightarrow \lim_{t\to\infty} L(t) \to +\infty \ P\text{-a.s..}$$

$\square$

Lemma 5. *In the economy of Example 3*

$$\forall \sigma \in \Sigma, \quad \ln p^{FLS}(\sigma^t) = \ln\left(\max_{i=1,2} p^i(\sigma^t)\right) - L(t)\ln\frac{4}{3}. \tag{8}$$

Proof. Let (σ^{t-1}, a) be the sequence whose first t-1 elements coincide with σ^{t-1} and whose last element is a. That is, $(\sigma^{t-1}, a) := \{\sigma_1, ..., \sigma_{t-1}, a\}$. In this setting, p^{FLS} coincides with the SNML algorithm, (Roos and Rissanen (2008)) and can be equivalently expressed as:

$$p^{FLS}(a_t|\sigma^{t-1}) = \frac{\max_i p^i(\sigma^{t-1}, a)}{\max_i p^i(\sigma^{t-1}, a) + \max_i p^i(\sigma^{t-1}, b)} = \frac{\frac{\max_{i\in\{1,2\}} p^i(\sigma^{t-1},a)}{\max_{i\in\{1,2\}} p(\sigma^{t-1})}}{\frac{\max_{i\in\{1,2\}} p^i(\sigma^{t-1},a) + \max_{i\in\{1,2\}} p^i(\sigma^{t-1},b)}{\max_{i\in\{1,2\}} p(\sigma^{t-1})}}.$$

Where the denominator satisfies $\frac{\max_{i\in\{1,2\}} p^i(\sigma^{t-1},a) + \max_{i\in\{1,2\}} p^i(\sigma^{t-1},b)}{\max_{i\in\{1,2\}} p(\sigma^{t-1})} = \begin{cases} 1 \text{ if } S_{t-1} \neq 0 \\ \frac{4}{3} \text{ if } S_{t-1} = 0 \end{cases}$.

Thus $\ln p^{FLS}$ can be equivalently written as (telescopying):

$$\begin{aligned}
\ln p^{FLS}(\sigma^t) &= \sum_{\tau=1}^{t} \ln p^{FLS}(\sigma_\tau|\sigma^{\tau-1}) \\
&= \sum_{\tau=1}^{t} \ln \frac{\max_{i\in\{1,2\}} p^i(\sigma^t)}{\max_{i\in\{1,2\}} p^i(\sigma^{t-1})} - \sum_{\tau=1}^{t} \ln \frac{\max_{i\in\{1,2\}} p^i(\sigma^{t-1}, a) + \max_{i\in\{1,2\}} p^i(\sigma^{t-1}, b)}{\max_{i\in\{1,2\}} p^i(\sigma^{t-1})} \\
&= \ln \max_{i\in\{1,2\}} p^i(\sigma^t) - L(t)\ln\frac{4}{3}.
\end{aligned}$$

$\square$

Proof of Proposition 2:

Proof.

- (i) $\lim \frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = 0 \quad P$-a.s. :
  By construction, $q(\sigma^t) = \frac{1}{3}p^1(\sigma^t) + \frac{1}{3}p^2(\sigma^t) + \frac{1}{3}p^{FLS}(\sigma^t) \geq \frac{1}{3} \max_{i\in\{1,2\}} p^i(\sigma^t)$.
  Using the characterization of Lemma 5,

$$\frac{p^{FLS}(\sigma^t)}{q(\sigma^t)} = e^{\ln \frac{\max_{i\in\{1,2\}} p^i(\sigma^t)}{\frac{1}{3}p^1(\sigma^t)+\frac{1}{3}p^2(\sigma^t)+\frac{1}{3}p^{FLS}(\sigma^t)} - L(t)\ln\frac{4}{3}} \leq e^{\ln\frac{1}{\frac{1}{3}} - L(t)\ln\frac{4}{3}} \to^{P-\text{a.s.}} 0.$$

  Because we have shown in Lemma 4 that $L(t) \to \infty$ P-a.s..

- *(ii) For* $i=1,2$, $\limsup_{t\to\infty}\frac{p^{FLS}(\sigma^t)}{p^i(\sigma^t)}=+\infty$ P-a.s.

  WLOG, let'sfocus on $\limsup_{t\to\infty}\frac{p^{FLS}(\sigma^t)}{p^1(\sigma^t)}$. Trader's 1 likelihood is given by:

$$
\begin{aligned}
p^1(\sigma^t) &= \left(\frac{1}{3}\right)^{\sum_{\tau=1}^{t} I_{a_\tau}}\left(\frac{2}{3}\right)^{\sum_{\tau=1}^{t} I_{b_\tau}} \\
&= \left(\frac{1}{3}\frac{2}{3}\right)^{\min_{j=a,b}\left\{\sum_{\tau=1}^{t} I_{\sigma_\tau=j}\right\}}\left(\frac{1}{3}\right)^{\sum_{\tau=1}^{t} I_{a_\tau}-\min_{j=a,b}\left\{\sum_{\tau=1}^{t} I_{\sigma_\tau=j}\right\}}\left(\frac{2}{3}\right)^{\sum_{\tau=1}^{t} I_{b_\tau}-\min_{j=a,b}\left\{\sum_{\tau=1}^{t} I_{\sigma_\tau=j}\right\}} \\
&= \begin{cases} \left(\frac{2}{3}\frac{1}{3}\right)^{\frac{t-|S_t|}{2}}\left(\frac{1}{3}\right)^{|S_t|} & if\ S_t>0 \\ \left(\frac{2}{3}\frac{1}{3}\right)^{\frac{t-|S|}{2}}\left(\frac{2}{3}\right)^{|S_t|} & if\ S_t<0 \end{cases}.
\end{aligned}
$$

Because we are focusing on the lim sup of the likelihood ratio we can assume WLOG that $S_t>0$, so that

$$
\begin{cases} \ln p^1(\sigma^t)=\ln\left(\frac{2}{3}\frac{1}{3}\right)^{\frac{t-|S|}{2}}\left(\frac{1}{3}\right)^{|S_t|} \\ \ln p^{FLS}(\sigma^t) =^{By\ Lem5} \ln\max_{i=\{1,2\}} p^i(\sigma^t)-L(t)\ln\frac{4}{3}=\ln p^2(\sigma^t)-L(t)\ln\frac{4}{3}=\left(\frac{1}{3}\frac{2}{3}\right)^{t-|S_t|}\left(\frac{2}{3}\right)^{|S_t|}-L(t)\ln\frac{4}{3} \end{cases}.
$$

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{p^{FLS}(\sigma^t)}{p^1(\sigma^t)} &= \limsup_{t\to\infty} e^{\ln\frac{\max_{i\in 1,2} p^i(\sigma^t)}{p^1(\sigma^t)}-L(t)\ln\frac{4}{3}} \\
&= \limsup_{t\to\infty} e^{\ln\frac{\left(\frac{1}{3}\frac{2}{3}\right)^{t-|S_t|}\left(\frac{2}{3}\right)^{|S_t|}}{\left(\frac{1}{3}\frac{2}{3}\right)^{t-|S_t|}\left(\frac{1}{3}\right)^{|S_t|}}-L(t)\ln\frac{4}{3}} \\
&= \limsup_{t\to\infty} e^{|S_t|\ln 2-L(t)\ln\frac{4}{3}}=+\infty \qquad P\text{-a.s., By Lemma 3.}
\end{aligned}
$$

□

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