2-CATEGORIES OF SYMMETRIC BIMODULES AND THEIR 2-REPRESENTATIONS

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Abstract. In this article we analyze the structure of 2-categories of symmetric projective bimodules over a finite dimensional algebra with respect to the action of a finite abelian group. We determine under which condition the resulting 2-category is fiat (in the sense of [MM1]) and classify simple transitive 2-representations of this 2-category (under some mild technical assumption). We also study several classes of examples in detail.

1. Introduction and description of the results

In the last 20 years, many exciting breakthroughs in representation theory, see e.g. [Kh, CR, BS, EW, Wi], have originated from the idea of categorification. This has inspired the subject of 2-representation theory which studies 2-categories with suitable finiteness conditions. An appropriate 2-analogue of a finite dimensional algebra was defined in [MM1] and called finitary 2-category. Various aspects of the structure and 2-representation theory of finitary and, more specifically, fiat 2-categories have been studied in [MM1, MM2, MM3, MM4, MM5, MM6, MMMT, MMMZ, ChMa, ChMi], see also references therein. In particular, [MM5] introduces the notion of simple transitive 2-representation which is an appropriate categorification of the concept of an irreducible representation. A natural and interesting problem is the classification of simple transitive 2-representations for various classes of 2-categories, see [Ma].

One interesting example of a fiat 2-category is the 2-category of Soergel bimodules associated to the coinvariant algebra of a finite Coxeter system, see [So1, So2, EW]. For these 2-categories, simple transitive 2-representations have been classified in several special cases including type A, the dihedral types and some small classical types, see [MM5, KMMZ, MT, MMMZ]. The article [MMMZ] develops a reduction technique that reduces the classification problem to smaller 2-categories which, in practice, are often given by "symmetric bimodules" as defined in a special case in [KMMZ].

Inspired by this, in the present article we formalize the concept of symmetric bimodules under the action of an abelian group. While defining symmetric bimodules with respect to a nonabelian group action is possible, decompositions will no longer simply rely on the Pontryagin dual group and the techniques of this paper would need to be changed significantly. We study the resulting 2-categories of projective symmetric bimodules and their 2-representations. We show that these 2-categories are weakly fiat provided that the underlying algebra is self-injective and fiat if the underlying algebra is weakly symmetric, mirroring the situation for the 2-category of all projective bimodules from [MM1, Subsection 7.3] and [MM6, Subsection 2.8]. Using [MMMZ], we reduce the classification of simple transitive 2-representations in the weakly symmetric case to the classification of module categories over the 2-category $\mathrm{Rep}(G)$ from [Os].

One of the main results of [MM3] classifies a class of "simple" 2-categories with a particularly nice combinatorial structure. Here we study one of the smallest families of

2-categories which do not fit into the setup of [MM3]. We show that these can always be realized inside a 2-category of symmetric bimodules.

The paper is organized as follows. In Section 2 we collect the necessary preliminaries. In Section 3 we introduce symmetric bimodules and study their structure and simple transitive 2-representations. The latter are classified in Theorem 17. In Section 4 we study 2-categories with one object which, apart from the identity 1-morphism, have precisely two indecomposable 1-morphisms up to isomorphism and these two form a biadjoint pair and their own left/right/two-sided cell. In Theorem 25 we realize such 2-categories as 2-subcategories of certain symmetric bimodules.

Acknowledgements. The first author is supported by the Swedish Research Council, Göran Gustafsson Foundation and Vergstiftelsen. The second author is supported by EPSRC grant EP/S017216/1. The third author is supported by Göran Gustafsson Foundation.

2. Preliminaries

2.1. **Setup.** We work over an algebraically closed field $\Bbbk$.

2.2. **Finitary 2-categories.** A $\Bbbk$-linear category is called *finitary* if it is small and equivalent to the category of finitely generated projective modules over some finite dimensional associative $\Bbbk$-algebra.

We call a 2-category $\mathscr{C}$ *finitary* (over $\Bbbk$) if it has finitely many objects, each morphism category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is a finitary $\Bbbk$-linear category, all compositions are (bi)additive and $\Bbbk$-linear and all identity 1-morphisms are indecomposable (cf. [MM1, Subsection 2.2]).

We say that $\mathscr{C}$ is *weakly fiat* if it is finitary and has a weak antiautomorphism $(\ )^\star$ of finite order and adjunction morphisms, see [MM6, Subsection 2.5]. If $(\ )^\star$ is involutive, we say that $\mathscr{C}$ is *fiat*, see [MM1, Subsection 2.4].

2.3. 2-**representations.** Let $\mathscr{C}$ be a finitary 2-category. A *finitary 2-representation* of $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to the 2-category $\mathfrak{A}^f_\Bbbk$ whose

- objects are finitary $\Bbbk$-linear categories;
- 1-morphisms are additive $\Bbbk$-linear functors;
- 2-morphisms are natural transformations of functors.

Such 2-representations form a 2-category denoted $\mathscr{C}$-afmod, see [MM3, Subsection 2.3].

Similarly we define an *abelian 2-representation* of $\mathscr{C}$ as a 2-functor from $\mathscr{C}$ to the 2-category whose

- objects are categories equivalent to categories of finitely generated modules over finite dimensional $\Bbbk$-algebras;
- 1-morphisms are right exact $\Bbbk$-linear functors;
- 2-morphisms are natural transformations of functors.

Such 2-representations form a 2-category denoted $\mathscr{C}$-mod, see [MM3, Subsection 2.3]. Following [MM2, Subsection 4.2], we denote by $\overline{\cdot}$ the *abelianization* 2-functor from $\mathscr{C}$-afmod to $\mathscr{C}$-mod.

A finitary 2-representation $\mathbf{M}$ is called *simple transitive* provided that $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ has no non-trivial $\mathscr{C}$-invariant ideals, see [MM5, Subsection 3.5].

2.4. **Cells and cell 2-representations.** Given two indecomposable 1-morphisms F and G in $\mathscr{C}$, we define $\mathrm{F} \geq_L \mathrm{G}$ if F is isomorphic to a direct summand of $\mathrm{H} \circ \mathrm{G}$, for some 1-morphism H. This produces the *left* preorder $\geq_L$, of which the equivalence classes are called *left cells*. Similarly one obtains the *right* preorder $\geq_R$ and the corresponding *right cells*, and the *two-sided* preorder $\geq_J$ and the corresponding *two-sided cells*.

For each simple transitive 2-representation $\mathbf{M}$, there is a unique two-sided cell which is maximal, with respect to $\geq_J$, among those two-sided cells whose 1-morphisms are not annihilated by $\mathbf{M}$. This two-sided cell is called the *apex* of $\mathbf{M}$, see [ChMa, Subsection 3.2].

If $\mathcal{J}$ is a two-sided cell of $\mathscr{C}$, we say that $\mathscr{C}$ is *$\mathcal{J}$-simple* if any non-zero 2-ideal of $\mathscr{C}$ contains the identity 2-morphisms of all 1-morphisms in $\mathcal{J}$.

Each left cell of a fiat 2-category contains a so-called *Duflo involution*, see [MM1, Subsection 4.5].

For a left cell $\mathcal{L}$ in $\mathscr{C}$, there exists $\mathtt{i} \in \mathscr{C}$ such that all 1-morphisms in $\mathcal{L}$ starts at $\mathtt{i}$. Denote by $\mathbf{N}_{\mathcal{L}}$ the 2-subrepresentation of $\mathbf{P}_{\mathtt{i}}$ which is defined as for each $\mathtt{j} \in \mathscr{C}$ the category $\mathbf{N}_{\mathcal{L}}(\mathtt{j})$ is the additive closure of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ consisting of all 1-morphisms F with $\mathrm{F} \geq_{\mathcal{L}} \mathcal{L}$. From [MM5, Lemma 3], we know that the 2-representation $\mathbf{N}_{\mathcal{L}}$ contains a unique maximal ideal which does not contain any id_{F} for $\mathrm{F} \in \mathcal{L}$, denoted $\mathcal{I}_{\mathcal{L}}$. The quotient $\mathbf{C}_{\mathcal{L}} := \mathbf{N}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}$ is called the *cell 2-representation* associated to $\mathcal{L}$.

3. Symmetric bimodules and their simple transitive 2-representations

3.1. **Symmetric bimodules.** Let A be a finite dimensional, unital, associative $\Bbbk$-algebra. We assume that A is basic and that $\{e_1, e_2, \dots, e_k\}$ is a complete set of pairwise orthogonal primitive idempotents in A.

Let G be a finite abelian subgroup of the group of automorphisms of A. Assume that $\mathrm{char}(\Bbbk)$ does not divide $|G|$. The action of G on A induces an action of G on the category of A-A-bimodules via $M \mapsto {}^{\varphi}M^{\varphi}$, where the action of A on ${}^{\varphi}M^{\varphi}$ is given by

$$a \cdot m \cdot b := \varphi(a) m \varphi(b), \qquad \text{for all } a, b \in A \text{ and } m \in M.$$

We will write ${}^{\varphi}f^{\varphi}$ for the translate of a morphism f under the action of $\varphi \in G$.

Let $\mathcal{X}$ denote the category whose objects are A-A-bimodules and morphisms between A-A-bimodules M and N are defined by

$$\mathrm{Hom}_{\mathcal{X}}(M, N) := \bigoplus_{\varphi \in G} \mathrm{Hom}_{A\text{-}A}(M, {}^{\varphi}N^{\varphi}).$$

An element $f \in \mathrm{Hom}_{\mathcal{X}}(M, N)$ is thus represented by a tuple $(f_\varphi)_{\varphi \in G}$, where the component f_φ is in $\mathrm{Hom}_{A\text{-}A}(M, {}^{\varphi}N^{\varphi})$. For any $f \in \mathrm{Hom}_{\mathcal{X}}(M, N)$ and $g \in \mathrm{Hom}_{\mathcal{X}}(N, K)$, considering

$$\begin{array}{ccc} \mathrm{Hom}_{A\text{-}A}(N, {}^{\psi}K^{\psi}) \otimes \mathrm{Hom}_{A\text{-}A}(M, {}^{\varphi}N^{\varphi}) & \to & \mathrm{Hom}_{A\text{-}A}(M, {}^{\varphi\psi}K^{\varphi\psi}) \\ g_\psi \otimes f_\varphi & \mapsto & {}^{\varphi}(g_\psi)^{\varphi} \circ f_\varphi, \end{array}$$

where we use $\varphi\psi = \psi\varphi$ on the right hand side, the composition $g \circ f$ is given by

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{X}}(N,K) \otimes \mathrm{Hom}_{\mathcal{X}}(M,N) & \to & \mathrm{Hom}_{\mathcal{X}}(M,K) \\ (g_\psi)_{\psi\in G} \otimes (f_\varphi)_{\varphi\in G} & \mapsto & \big(\sum_{\varphi\in G} {}^{\varphi}(g_{\sigma\varphi^{-1}})^{\varphi} \circ f_\varphi\big)_{\sigma\in G}. \end{array} \tag{1}$$

This composition can be depicted by the diagram

$$M \xrightarrow{(f_\varphi)_{\varphi\in G}} \bigoplus_{\varphi\in G} {}^{\varphi}N^{\varphi} \xrightarrow{({}^{\varphi}(g_{\sigma\varphi^{-1}})^{\varphi})_{\varphi,\sigma\in G}} \bigoplus_{\sigma\in G} {}^{\sigma}K^{\sigma} .$$

We refer to [CiMa] for details. In $\mathcal{X}$, we have an isomorphism

$$M \cong {}^{\varphi}M^{\varphi},$$

for all $\varphi \in G$, since id_M can appear in the φ^{-1}-component of $\mathrm{Hom}_{\mathcal{X}}(M, {}^{\varphi}M^{\varphi})$. Furthermore, there is a faithful embedding of $\mathcal{X}$ into the category of all A-A-bimodules by sending an object $M \in \mathcal{X}$ to the A-A-bimodule $\bigoplus_{\varphi\in G} {}^{\varphi}M^{\varphi}$ and each morphism $f = (f_\sigma)_{\sigma\in G} \in \mathrm{Hom}_{\mathcal{X}}(M,N)$ to the A-A-bimodule homomorphism $\big({}^{\varphi}(f_{\varphi^{-1}\psi})^{\varphi}\big)_{\varphi,\psi\in G}$ in $\mathrm{Hom}_{A\text{-}A}(\bigoplus_{\varphi\in G} {}^{\varphi}M^{\varphi}, \bigoplus_{\psi\in G} {}^{\psi}N^{\psi})$.

We denote by $\tilde{\mathcal{X}}$ the idempotent completion of $\mathcal{X}$, i.e. an object of $\tilde{\mathcal{X}}$ is given by a pair (M,e) where M is an A-A-bimodule and e is an idempotent in $\mathrm{End}_{\mathcal{X}}(M)$. For an A-A-bimodule M, set

$$G_M := \{\varphi \in G \,|\, M \cong {}^{\varphi}M^{\varphi}\}$$

which is a subgroup of G.

Remark 1. As we will often encounter and use in this article, computation of homomorphism in $\tilde{\mathcal{X}}$ using homomorphisms in $\mathcal{X}$ requires care. Given M and M' in $\mathcal{X}$ and idempotents e and e' in $\mathrm{End}_{\mathcal{X}}(M)$ and $\mathrm{End}_{\mathcal{X}}(M')$, respectively, in the computation of $\mathrm{Hom}_{\tilde{\mathcal{X}}}((M,e),(M',e'))$ using $\mathrm{Hom}_{\mathcal{X}}(M,M')$ it is very important to make sure that the elements from $\mathrm{Hom}_{\mathcal{X}}(M,M')$ one works with do belong to $e' \circ \mathrm{Hom}_{\mathcal{X}}(M,M') \circ e$. This is usually achieved by pre- and postcomposing the elements one works with e and e', respectively. Moreover, for any element $\boldsymbol{f}$ in $\mathrm{Hom}_{\tilde{\mathcal{X}}}((M,e),(M',e'))$, we have

$$\boldsymbol{f} \circ \mathrm{id}_{(M,e)} = \boldsymbol{f} = \mathrm{id}_{(M',e')} \circ \boldsymbol{f}, \tag{2}$$

where, in fact, $\mathrm{id}_{(M,e)} = e$ and $\mathrm{id}_{(M',e')} = e'$.

As usual, we denote by $\hat{G}$ the *Pontryagin dual* of G whose elements are all group homomorphisms from G to $\Bbbk^*$ with respect to point-wise multiplication. As G is finite and abelian, the group $\hat{G}$ is (non-canonically) isomorphic to G and $\hat{G}$ is canonically isomorphic to the group of isomorphism classes of simple G-modules with respect to taking tensor products.

The group algebra $\Bbbk[G]$ is commutative and semi-simple and admits a unique decomposition into a product of $|G|$ copies of $\Bbbk$. Let $\{\pi_\chi, \chi \in \hat{G}\}$ be the corresponding primitive idempotents. Each π_χ has the form $\frac{1}{|G|}\sum_{\alpha\in G} \chi(\alpha)\alpha$ and hence defines an idempotent $\tilde{\pi}_\chi$ in $\mathrm{End}_{\mathcal{X}}(A)$ given by the tuple $\left(\frac{\chi(\alpha)}{|G|}\alpha\right)_{\alpha\in G}$.

For an arbitrary subgroup H of G, we have a natural surjection $\hat{G} \to \hat{H}$ given by restriction. For $\zeta \in \hat{G}$ and $\chi \in \hat{H}$, we define $\chi\zeta \in \hat{H}$ via $\chi\zeta(\alpha) := \chi(\alpha)\zeta(\alpha)$, for $\alpha \in H$.

Lemma 2.

(*i*) *Let M be an indecomposable A-A-bimodule. Then there is an isomorphism of algebras*

$$\mathrm{End}_{\mathcal{X}}(M)/\mathrm{Rad}(\mathrm{End}_{\mathcal{X}}(M)) \cong \Bbbk[G_M]/\mathrm{Rad}(\Bbbk[G_M]) \cong \Bbbk[G_M].$$

(*ii*) *Indecomposable objects of $\tilde{\mathcal{X}}$ are of the form $M_{\varepsilon_\chi} := (M, \varepsilon_\chi)$, where M is an indecomposable A-A-bimodule and $\chi \in \hat{G}_M$. Here, for $\alpha \in G$, the α-component of ε_χ is $\frac{\chi(\alpha)}{|G_M|}\alpha$, if $\alpha \in G_M$, and zero otherwise.*

Proof. Note that, for $\alpha \in G$, if ${}^{\alpha}M^{\alpha}$ is not isomorphic to M, then the α-component of any endomorphism of M belongs to the radical of $\mathrm{End}_{\mathcal{X}}(M)$. Therefore Claim (i) follows from (1). Claim (ii) follows from (1) and the definitions. □

The category of all A-A-bimodules has a natural monoidal structure given by the tensor product over A. We define a tensor product on $\mathcal{X}$ by

- $M \otimes_{\mathcal{X}} N := \bigoplus_{\varphi \in G} \big(M \otimes_A {}^{\varphi}N^{\varphi}\big)$, for any A-A-bimodules M and N,
- $f \otimes_{\mathcal{X}} g := \big(f_\alpha \otimes {}^{\gamma}(g_{\beta\gamma^{-1}})^{\gamma}\big)_{\alpha,\beta,\gamma \in G}$, where
$$f_\alpha \otimes {}^{\gamma}(g_{\beta\gamma^{-1}})^{\gamma} : M \otimes_A {}^{\gamma}N^{\gamma} \to {}^{\alpha}(M')^{\alpha} \otimes_A {}^{\beta}(N')^{\beta},$$
for any A-A-bimodules M, M', N and N' and morphisms
$$f = (f_\alpha)_{\alpha \in G} \in \mathrm{Hom}_{\mathcal{X}}(M, M'), \quad g = (g_\beta)_{\beta \in G} \in \mathrm{Hom}_{\mathcal{X}}(N, N').$$

Note that there is no identity object with respect to the tensor product $\otimes_{\mathcal{X}}$ unless G is trivial. In general, $\otimes_{\mathcal{X}}$ does not define a monoidal structure on $\mathcal{X}$. However, we will see in Propositions 6 and 7 that $\tilde{\mathcal{X}}$ has an identity given by tensoring with $(A, \tilde{\pi}_{1_{\hat{G}}})$. The asymmetry of the above definition is only notational as the following lemma shows.

Lemma 3. *In the category $\mathcal{X}$, there is an isomorphism*

$$\bigoplus_{\varphi \in G} \big(M \otimes_A {}^{\varphi}N^{\varphi}\big) \cong \bigoplus_{\varphi \in G} \big({}^{\varphi^{-1}}M^{\varphi^{-1}} \otimes_A N\big).$$

Proof. We first note that the map $m \otimes n \mapsto m \otimes n$ gives rise to an isomorphism of A-A-bimodules from $M \otimes_A {}^{\varphi}N$ to $M^{\varphi^{-1}} \otimes_A N$. Thus we have an isomorphism

$$(3) \qquad {}^{\varphi^{-1}}M^{\varphi^{-1}} \otimes_A N \cong {}^{\varphi^{-1}}\,(M \otimes_A {}^{\varphi}N^{\varphi})^{\varphi^{-1}}$$

of A-A-bimodules. We hence have an isomorphism

$${}^{\varphi^{-1}}M^{\varphi^{-1}} \otimes_A N \xrightarrow{(f_\psi)_{\psi \in G}} M \otimes_A {}^{\varphi}N^{\varphi}$$

where $f_{\varphi^{-1}}$ is given by (3) and the remaining components are zero. □

Remark 4. Under the isomorphism provided by Lemma 3, the morphism $f \otimes_{\mathcal{X}} g$ in $\mathrm{Hom}_{\mathcal{X}}(M \otimes_{\mathcal{X}} N, M' \otimes_{\mathcal{X}} N')$ has components of the form

$${}^{\gamma}(f_{\alpha\gamma^{-1}})^{\gamma} \otimes g_\beta : {}^{\gamma}M^{\gamma} \otimes_A N \to {}^{\alpha}(M')^{\alpha} \otimes_A {}^{\beta}(N')^{\beta}.$$

Lemma 5.

(*i*) *The operation $\otimes_{\mathcal{X}}$ is bifunctorial.*

(*ii*) *If e and f are idempotents in $\mathcal{X}$, then so is $e \otimes_{\mathcal{X}} f$. Hence $\otimes_{\mathcal{X}}$ extends to a bifunctor $\otimes_{\tilde{\mathcal{X}}} : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ given by $(M, e) \otimes_{\tilde{\mathcal{X}}} (N, f) = (M \otimes_{\mathcal{X}} N, e \otimes_{\mathcal{X}} f)$.*

Proof. Let M, N, K, L, X and Y be objects in $\mathcal{X}$. Let $f : M \to K$, $g : N \to L$, $h : K \to X$ and $l : L \to Y$ be morphisms in $\mathcal{X}$ with their only non-zero components being f_α, $g_{\gamma\varphi^{-1}}$, $h_{\beta\alpha^{-1}}$ and $l_{\delta\gamma^{-1}}$, respectively. Consider the diagram

$$\begin{array}{ccc}
M & \otimes_A & {}^\varphi N^\varphi \\
\downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle {}^\varphi(g_{\gamma\varphi^{-1}})^\varphi} \\
{}^\alpha K^\alpha & \otimes_A & {}^\gamma L^\gamma \\
\downarrow{\scriptstyle {}^\alpha(h_{\beta\alpha^{-1}})^\alpha} & & \downarrow{\scriptstyle {}^\gamma(l_{\delta\gamma^{-1}})^\gamma} \\
{}^\beta X^\beta & \otimes_A & {}^\delta Y^\delta .
\end{array}$$

Bifunctoriality of $\otimes_A$ yields

(4)
$$\left({}^\alpha(h_{\beta\alpha^{-1}})^\alpha \circ f_\alpha\right) \otimes \left({}^\gamma(l_{\delta\gamma^{-1}})^\gamma \circ {}^\varphi(g_{\gamma\varphi^{-1}})^\varphi\right) = \left({}^\alpha(h_{\beta\alpha^{-1}})^\alpha \otimes {}^\gamma(l_{\delta\gamma^{-1}})^\gamma\right) \circ \left(f_\alpha \otimes {}^\varphi(g_{\gamma\varphi^{-1}})^\varphi\right).$$

The left hand side and the right hand side of (4) coincide with

$$(h \circ f)_\beta \otimes {}^\varphi(l \circ g)^\varphi_{\delta\varphi^{-1}} \quad \text{and} \quad {}^\alpha(h \otimes l)^\alpha_{\beta\alpha^{-1}} \circ (f \otimes g)_\alpha,$$

respectively. As these are the only non-zero components in $(h \circ f) \otimes_{\mathcal{X}} (l \circ g)$ and $(h \otimes_{\mathcal{X}} l) \circ (f \otimes_{\mathcal{X}} g)$, respectively, and have the same source and target, we obtain

$$(h \circ f) \otimes_{\mathcal{X}} (l \circ g) = (h \otimes_{\mathcal{X}} l) \circ (f \otimes_{\mathcal{X}} g) \tag{5}$$

in this case. The general case of equality (5) follows by linearity, implying Claim (i).

Claim (ii) follows from equality (5). □

Proposition 6. *For any $\chi, \zeta \in \hat{G}$, we have*

$$(A, \tilde{\pi}_\chi) \otimes_{\tilde{\mathcal{X}}} (A, \tilde{\pi}_\zeta) \cong (A, \tilde{\pi}_{\chi\zeta}). \tag{6}$$

Proof. We start by constructing a morphism $\boldsymbol{f}$ from the right hand side of (6) to the left hand side, which is defined as follows:

$$\boldsymbol{f} := (f_{\sigma,\tau})_{\sigma,\tau\in G} \colon A \to \bigoplus_{\sigma,\tau\in G} {}^\sigma A^\sigma \otimes_A {}^\tau A^\tau$$

where $f_{\sigma,\tau}$ is given by

$$1 \mapsto \frac{1}{|G|^2}\chi(\sigma)\zeta(\tau)(1 \otimes 1) \in {}^\sigma A^\sigma \otimes_A {}^\tau A^\tau. \tag{7}$$

Consider the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\tilde{\pi}_{\chi\zeta}} & \bigoplus_{\beta\in G} {}^\beta A^\beta \\
\downarrow{\scriptstyle (f_{\sigma,\tau})_{\sigma,\tau\in G}} & & \downarrow{\scriptstyle \left({}^\beta(f_{\alpha\beta^{-1},\delta\beta^{-1}})^\beta_{\alpha\beta^{-1},\delta\beta^{-1}\in G}\right)_{\beta\in G}} \\
\bigoplus_{\sigma,\tau\in G} {}^\sigma A^\sigma \otimes_A {}^\tau A^\tau & \xrightarrow{({}^\sigma(\tilde{\pi}_\chi \otimes_{\mathcal{X}} \tilde{\pi}_\zeta)^\sigma)_{\sigma\in G}} & \bigoplus_{\alpha,\delta\in G} {}^\alpha A^\alpha \otimes_A {}^\delta A^\delta .
\end{array}$$

By definition, the $\tau\sigma^{-1}, \alpha\sigma^{-1}, \delta\sigma^{-1}$-component of $\tilde{\pi}_\chi \otimes_{\mathcal{X}} \tilde{\pi}_\zeta$ is multiplication by the scalar

$$\frac{1}{|G|^2}\chi(\alpha\sigma^{-1})\zeta(\delta\tau^{-1}) = \frac{1}{|G|^2}\chi(\alpha)\chi(\sigma^{-1})\zeta(\delta)\zeta(\tau^{-1}). \tag{8}$$

Now we compute the components of the two compositions $(\tilde{\pi}_\chi \otimes_{\mathcal{X}} \tilde{\pi}_\zeta) \circ \boldsymbol{f}$ (corresponding to the path going down and then right) and $\boldsymbol{f} \circ \tilde{\pi}_{\chi\zeta}$ (corresponding to the path going right and then down) in our diagram which end up in a specific ${}^{\alpha}A^{\alpha} \otimes_A {}^{\delta}A^{\delta}$. One way around, using (7) and (8), we obtain that 1 is sent to

$$\begin{aligned} \tfrac{1}{|G|^4} \sum_{\sigma,\tau\in G} \chi(\sigma)\chi(\alpha)\chi(\sigma^{-1})\zeta(\tau)\zeta(\delta)\zeta(\tau^{-1})(1\otimes 1) &= \tfrac{|G|^2}{|G|^4}\chi(\alpha)\zeta(\delta)(1\otimes 1) \\ &= \tfrac{1}{|G|^2}\chi(\alpha)\zeta(\delta)(1\otimes 1). \end{aligned}$$

The other way around, using (7) we obtain that 1 is sent to

$$\tfrac{1}{|G|^3} \sum_{\beta\in G} \chi\zeta(\beta)\chi(\alpha\beta^{-1})\zeta(\delta\beta^{-1})(1\otimes 1) = \tfrac{1}{|G|^2}\chi(\alpha)\zeta(\delta)(1\otimes 1).$$

Hence the diagram commutes and, moreover, (2) is satisfied. Thus $\boldsymbol{f}$ represents a morphism from the right hand side of (6) to the left hand side.

We proceed by constructing a morphism $\boldsymbol{g}$ from the left hand side of (6) to the right hand side. Consider the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha\in G}(A\otimes_A {}^{\alpha}A^{\alpha}) & \xrightarrow{\tilde\pi_\chi\otimes_{\mathcal{X}}\tilde\pi_\zeta} & \bigoplus_{\gamma,\delta\in G} {}^{\gamma}A^{\gamma}\otimes_A {}^{\delta}A^{\delta} \\ \Big\downarrow \tfrac{1}{|G|}\left(\chi(\beta)\zeta(\beta\alpha^{-1})\right)_{\alpha,\beta\in G} & & \Big\downarrow \tfrac{1}{|G|}\left({}^{\gamma}(\chi(\sigma\gamma^{-1})\zeta(\sigma\delta^{-1}))^{\gamma}_{\delta\gamma^{-1},\sigma\gamma^{-1}\in G}\right)_{\gamma\in G} \\ \bigoplus_{\beta\in G} {}^{\beta}A^{\beta} & \xrightarrow{({}^{\beta}\tilde\pi^{\beta}_{\chi\zeta})_{\beta\in G}} & \bigoplus_{\sigma\in G} {}^{\sigma}A^{\sigma} \end{array}$$

whose vertical part defines $\boldsymbol{g}$, with its α,β-component sending $1\otimes 1$ to $\frac{1}{|G|}\chi(\beta)\zeta(\beta\alpha^{-1})1$. For fixed $\alpha,\sigma\in G$, going one way around, using (8) we obtain

$$\sum_{\gamma,\delta\in G}\frac{1}{|G|^3}\chi(\gamma)\zeta(\delta\alpha^{-1})\chi(\sigma\gamma^{-1})\zeta(\sigma\delta^{-1}) = \frac{1}{|G|}\chi(\sigma)\zeta(\sigma\alpha^{-1}).$$

The other way around yields

$$\frac{1}{|G|^2}\sum_{\beta\in G}\chi(\beta)\zeta(\beta\alpha^{-1})\chi\zeta(\sigma\beta^{-1}) = \frac{1}{|G|}\chi(\sigma)\zeta(\sigma\alpha^{-1})$$

which implies that the diagram commutes and, moreover, $\boldsymbol{g}\circ(\tilde\pi_\chi\otimes_{\mathcal{X}}\tilde\pi_\zeta) = \tilde\pi_{\chi\zeta}\circ\boldsymbol{g} = \boldsymbol{g}$.

Now we claim that both compositions $\boldsymbol{f}\circ\boldsymbol{g}$ and $\boldsymbol{g}\circ\boldsymbol{f}$ are the identities, i.e. of the respective idempotents. The φ-component of the composition $\boldsymbol{g}\circ\boldsymbol{f}$ sends 1 to

$$\frac{1}{|G|^3}\sum_{\sigma,\tau\in G}\chi(\sigma)\zeta(\tau)\chi(\varphi\sigma^{-1})\zeta(\varphi\tau^{-1}) = \frac{1}{|G|}\chi\zeta(\varphi).$$

The α,σ,τ-component of the composition $\boldsymbol{f}\circ\boldsymbol{g}$ sends $1\otimes 1$ to

$$\frac{1}{|G|^3}\sum_{\beta\in G}\chi(\beta)\zeta(\beta\alpha^{-1})\chi(\sigma\beta^{-1})\zeta(\tau\beta^{-1})(1\otimes 1) = \frac{1}{|G|^2}\chi(\sigma)\zeta(\tau\alpha^{-1})(1\otimes 1).$$

The claim follows. □

Proposition 7. *Let $i,j\in\{1,2,\dots,k\}$ and $M = Ae_i\otimes_{\Bbbk} e_jA$. Let further $\chi\in\hat G_M$ and $\zeta\in\hat G$. Then*

$$(M,\varepsilon_\chi)\otimes_{\tilde{\mathcal{X}}}(A,\tilde\pi_\zeta)\cong(M,\varepsilon_{\chi\zeta}). \tag{9}$$

Proof. We follow the proof of Proposition 6. We start by constructing a morphism $\boldsymbol{f}$ from the right hand side of (9) to the left hand side. Consider the morphism

$$\boldsymbol{f} := (f_{\sigma,\tau})_{\sigma,\tau\in G}\colon M\to\bigoplus_{\sigma,\tau\in G}{}^{\sigma}M^{\sigma}\otimes_A {}^{\tau}A^{\tau}$$

where $f_{\sigma,\tau}$ is given by

$$e_i \otimes e_j \mapsto \frac{1}{|G_M||G|}\chi(\sigma)\zeta(\tau)(\sigma(e_i) \otimes \sigma(e_j) \otimes 1) \in {}^{\sigma}M^{\sigma} \otimes_A {}^{\tau}A^{\tau}.$$

if $\sigma \in G_M$ and zero otherwise. Consider the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\varepsilon_{\chi\zeta}} & \bigoplus_{\beta\in G} {}^{\beta}M^{\beta} \\
\Big\downarrow{\scriptstyle (f_{\sigma,\tau})_{\sigma,\tau\in G}} & & \Big\downarrow{\scriptstyle ({}^{\beta}(f_{\alpha\beta^{-1},\delta\beta^{-1}})^{\beta}_{\alpha\beta^{-1},\delta\beta^{-1}\in G})_{\beta\in G}} \\
\bigoplus_{\sigma,\tau\in G} {}^{\sigma}M^{\sigma} \otimes_A {}^{\tau}A^{\tau} & \xrightarrow{({}^{\sigma}(\varepsilon_{\chi}\otimes_{\mathcal{X}}\tilde{\pi}_{\zeta})^{\sigma})_{\sigma\in G}} & \bigoplus_{\alpha,\delta\in G} {}^{\alpha}M^{\alpha} \otimes_A {}^{\delta}A^{\delta}.
\end{array}$$

By definition, the $\tau\sigma^{-1}, \alpha\sigma^{-1}, \delta\sigma^{-1}$-component of $\varepsilon_{\chi} \otimes_{\mathcal{X}} \tilde{\pi}_{\zeta}$ sends $e_i \otimes e_j \otimes 1$ to

$$\frac{1}{|G_M||G|}\chi(\alpha)\chi(\sigma^{-1})\zeta(\delta)\zeta(\tau^{-1})(\alpha\sigma^{-1}(e_i) \otimes \alpha\sigma^{-1}(e_j) \otimes 1)$$

if $\alpha\sigma^{-1} \in G_M$ and zero otherwise. Now, going to the right and then down, the α, δ-component of the composition $\boldsymbol{f} \circ \varepsilon_{\chi\zeta}$ sends $e_i \otimes e_j$ to

$$\frac{1}{|G_M||G|}\chi(\alpha)\zeta(\delta)(\alpha(e_i) \otimes \alpha(e_j) \otimes 1),$$

if α is in G_M, and zero otherwise. Going down and then to the right, the α, δ-component of $(\varepsilon_{\chi} \otimes_{\mathcal{X}} \tilde{\pi}_{\zeta}) \circ \boldsymbol{f}$ gives the same result, which also equals $f_{\alpha,\delta}$.

To construct a morphism $\boldsymbol{g}$ from the left hand side of (9) to the right hand side, consider the diagram

$$\begin{array}{ccc}
\bigoplus_{\alpha\in G}(M \otimes_A {}^{\alpha}A^{\alpha}) & \xrightarrow{\varepsilon_{\chi}\otimes_{\mathcal{X}}\tilde{\pi}_{\zeta}} & \bigoplus_{\gamma,\delta\in G} {}^{\gamma}M^{\gamma} \otimes_A {}^{\delta}A^{\delta} \\
\Big\downarrow{\scriptstyle (g_{\beta,\alpha})_{\alpha,\beta\in G}} & & \Big\downarrow{\scriptstyle \left({}^{\gamma}(g_{\sigma\gamma^{-1},\delta\gamma^{-1}})^{\gamma}_{\delta\gamma^{-1},\sigma\gamma^{-1}\in G}\right)_{\delta\in G}} \\
\bigoplus_{\beta\in G} {}^{\beta}M^{\beta} & \xrightarrow{({}^{\beta}\varepsilon^{\beta}_{\chi\zeta})_{\beta\in G}} & \bigoplus_{\sigma\in G} {}^{\sigma}M^{\sigma}.
\end{array}$$

where $g_{\beta,\alpha}$ sends $e_i \otimes e_j \otimes 1 \in M \otimes_A {}^{\alpha}A^{\alpha}$ to $\frac{1}{|G_M|}\chi(\beta)\zeta(\beta\alpha^{-1})(\beta(e_i) \otimes \beta(e_j)) \in {}^{\beta}M^{\beta}$, if $\beta \in G_M$, and to zero otherwise.

For fixed $\alpha, \sigma \in G$, going one way around we obtain the map which sends $e_i \otimes e_j \otimes 1$ to

$$\frac{1}{|G_M|}\chi(\sigma)\zeta(\sigma\alpha^{-1})(\sigma(e_i) \otimes \sigma(e_j)),$$

if $\sigma \in G_M$, and to zero otherwise, which coincides with $g_{\sigma,\alpha}$. The other way around gives the same result.

Checking that both compositions $\boldsymbol{f} \circ \boldsymbol{g}$ and $\boldsymbol{g} \circ \boldsymbol{f}$ are the respective idempotents is similar to the proof of Proposition 6. $\square$

3.2. **Tensoring symmetric bimodules with A-modules.**

Proposition 8.

(*i*) *There is a bifunctor* $\otimes^{(r)} : \text{mod-}A \times \mathcal{X} \to \text{mod-}A$ *defined by*

$$\begin{array}{rcl}
(V, M) & \mapsto & V \otimes^{(r)} M := \bigoplus_{\varphi\in G} V \otimes_A {}^{\varphi}M^{\varphi} \\
(f, g) & \mapsto & f \otimes^{(r)} g := \big(f \otimes {}^{\varphi}(g_{\alpha\varphi^{-1}})^{\varphi}\big)_{\varphi,\alpha\in G}.
\end{array}$$

(*ii*) *There is a bifunctor* $\otimes^{(l)} : \mathcal{X} \times A\text{-mod} \to A\text{-mod}$ *defined by*

$$\begin{array}{rcl}(M,V) & \mapsto & M \otimes^{(l)} V := \bigoplus_{\varphi\in G} {}^{\varphi}M^{\varphi} \otimes_A V \\ (g,f) & \mapsto & g \otimes^{(l)} f := \left({}^{\varphi}(g_{\alpha\varphi^{-1}})^{\varphi} \otimes f\right)_{\alpha,\varphi\in G}.\end{array}$$

Proof. We note that we use Lemma 3 for the formulation of Claim (ii). The proof of both claims is similar to the proof of Lemma 5(i). □

Proposition 9.

(*i*) *The bifunctor* $\otimes^{(r)}$ *induces a bifunctor* $\text{mod-}A \times \tilde{\mathcal{X}} \to \text{mod-}A$ *(which we will denote by the same symbol abusing notation).*

(*ii*) *The bifunctor* $\otimes^{(l)}$ *induces a bifunctor* $\tilde{\mathcal{X}} \times A\text{-mod} \to A\text{-mod}$ *(which we will denote by the same symbol abusing notation).*

Proof. Let $(M, e) \in \tilde{\mathcal{X}}$. Then, for any $V \in \text{mod-}A$, the endomorphism $\text{id}_V \otimes^{(r)} e$ is an idempotent endomorphism of $V \otimes^{(r)} M$, so we can define $V \otimes^{(r)} (M, e)$ as the image of this idempotent. It is easy to check that this does the job for Claim (i). Claim (ii) is similar. □

3.3. **The 2-category $\mathscr{G}_A$ of projective symmetric bimodules.** Assume that we are in the setup of Subsection 3.1. Let

$$A = A_1 \times A_2 \times \cdots \times A_n$$

be the (unique up to permutation of factors) decomposition of A into a direct product of indecomposable algebras. Assume that the action of each $\varphi \in G$ preserves each A_i. Also assume that none of the A_i is simple. We also consider the algebra $B := A \times \Bbbk$ which will play a crucial role in the proof of Theorem 17.

For each $i \in \{1, 2, \ldots, n\}$, fix a small category $\mathcal{C}_i$ equivalent to $A_i\text{-proj}$. Define the 2-category $\mathscr{G}_A$ to have

- objects $\mathtt{1}, \mathtt{2}, \ldots, \mathtt{n}$, where we identify $\mathtt{i}$ with $\mathcal{C}_i$;
- 1-morphisms are endofunctors of $\mathcal{C} := \coprod_{\mathtt{i}} \mathcal{C}_i$ isomorphic to functors $X \otimes^{(l)} {}_-$, where X is in the additive closure of $\left(A \oplus (A \otimes_\Bbbk A), \text{id}_{A\oplus(A\otimes_\Bbbk A)}\right)$ inside $\tilde{\mathcal{X}}$;
- 2-morphisms are given by morphisms between X and X' in $\tilde{\mathcal{X}}$;
- horizontal composition is just composition of functors;
- vertical composition is inherited from $\tilde{\mathcal{X}}$;
- the identity 1-morphism in $\mathscr{G}_A(\mathtt{i}, \mathtt{i})$ is isomorphic to $(A_i, \tilde{\pi}_{1_{\hat{G}}}) \otimes^{(l)} {}_-$.

Note that the restriction on $\text{char}(\Bbbk)$ as not dividing the order of G is necessary to have identity 1-morphisms. Observe further that $A \oplus (A \otimes_\Bbbk A)$ is invariant, up to isomorphism, under the functor $M \mapsto {}^{\varphi}M^{\varphi}$, for any $\varphi \in G$. The fact that this defines a 2-category is justified by Proposition 7, showing that $(A_i, \tilde{\pi}_{1_{\hat{G}}}) \otimes^{(l)} {}_-$ is indeed an identity, and the following lemma.

Lemma 10. *Let* X *and* Y *be in the additive closure of* $\left(A \oplus (A \otimes_\Bbbk A), \text{id}_{A\oplus(A\otimes_\Bbbk A)}\right)$ *inside* $\tilde{\mathcal{X}}$*. Then there is an isomorphism*

$$(X \otimes^{(l)} {}_-) \circ (Y \otimes^{(l)} {}_-) \cong (X \otimes_{\tilde{\mathcal{X}}} Y) \otimes^{(l)} {}_-$$

of endofunctors of $\mathcal{C}$.

Proof. First we assume that X and Y are in $\mathcal{X}$. Then, for any $P \in \mathcal{C}$, we have

$$(X \otimes_{\mathcal{X}} Y) \otimes^{(l)} P = \left(\bigoplus_{\varphi \in G} X \otimes_A {}^{\varphi}Y^{\varphi} \right) \otimes^{(l)} P = \bigoplus_{\varphi,\psi \in G} {}^{\psi}(X \otimes_A {}^{\varphi}Y^{\varphi})^{\psi} \otimes_A P$$

and

$$X \otimes^{(l)} (Y \otimes^{(l)} P) = X \otimes^{(l)} \left(\bigoplus_{\varphi \in G} {}^{\varphi}Y^{\varphi} \otimes_A P \right) = \bigoplus_{\varphi,\psi \in G} {}^{\psi}X^{\psi} \otimes_A {}^{\varphi}Y^{\varphi} \otimes_A P.$$

Choosing an isomorphism

$$\bigoplus_{\varphi,\psi \in G} {}^{\psi}(X \otimes_A {}^{\varphi}Y^{\varphi})^{\psi} \cong \bigoplus_{\varphi,\psi \in G} {}^{\psi}X^{\psi} \otimes_A {}^{\varphi}Y^{\varphi}$$

of A-A-bimodules yields the desired isomorphism of functors.

Now, let e and f be idempotents in $\mathrm{End}_{\mathcal{X}}(X)$ and $\mathrm{End}_{\mathcal{X}}(Y)$, respectively. Consider

$$\begin{aligned} (e \otimes_{\mathcal{X}} f) \otimes^{(l)} \mathrm{id}_P &= \left({}^{\gamma}(e_{\alpha\gamma^{-1}})^{\gamma} \otimes f_{\beta}\right)_{\alpha,\beta,\gamma \in G} \otimes^{(l)} \mathrm{id}_P \\ &= \left({}^{\delta}\left({}^{\gamma}(e_{\alpha\gamma^{-1}})^{\gamma} \otimes f_{\beta}\right)^{\delta}_{\beta\delta^{-1}} \otimes \mathrm{id}_P\right)_{\alpha,\beta,\gamma,\delta \in G} \\ &= \left({}^{\gamma\delta}(e_{\alpha\gamma^{-1}})^{\gamma\delta} \otimes {}^{\delta}(f_{\beta\delta^{-1}})^{\delta} \otimes \mathrm{id}_P\right)_{\alpha,\beta,\gamma,\delta \in G} \end{aligned}$$

and

$$\begin{aligned} e \otimes^{(l)} (f \otimes^{(l)} \mathrm{id}_P) &= e \otimes^{(l)} \left({}^{\psi}(f_{\varphi\psi^{-1}})^{\psi} \otimes \mathrm{id}_P\right)_{\varphi,\psi \in G} \\ &= \left({}^{\tau}(e_{\sigma\tau^{-1}})^{\tau} \otimes {}^{\psi}(f_{\varphi\psi^{-1}})^{\psi} \otimes \mathrm{id}_P\right)_{\sigma,\tau,\varphi,\psi \in G}, \end{aligned}$$

we see that

$$(e \otimes_{\mathcal{X}} f) \otimes^{(l)} \mathrm{id}_P = e \otimes^{(l)} (f \otimes^{(l)} \mathrm{id}_P)$$

and hence the isomorphism in the previous paragraph descends to the summands (X, e) and (Y, f). □

The 2-category $\mathscr{G}_B$ is defined similarly. To distinguish the underlying categories of bimodules, we use the notation $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ for the corresponding categories of symmetric B-B-bimodules.

3.4. Two-sided cells in $\mathscr{G}_A$.

We recall the notation introduced just before Lemma 2.

Proposition 11. *The 2-category $\mathscr{G}_A$ has $n+1$ two-sided cells, namely*

(*a*) *for $i = 1, 2, \dots, n$, the two-sided cell $\mathcal{J}_{\mathtt{i}}$ consisting of $|G|$ elements $(\mathbb{1}_{\mathtt{i}}, \tilde{\pi}_{\varphi})$, where $\varphi \in G$,*

(*b*) *the two-sided cell $\mathcal{J}_0$ consisting of all isomorphism classes of indecomposable 1-morphisms in the additive closure of $(A \otimes_{\Bbbk} A, \mathrm{id}_{A \otimes_{\Bbbk} A})$ inside $\tilde{\mathcal{X}}$.*

Proof. Since tensor products in which one of the factors is a projective bimodule never contain a copy of the regular bimodule as a direct summand, the existence of two-sided cells as claimed in Part (a) follows from Proposition 6. To complete the proof of the proposition, it remains to show that all isomorphism classes of indecomposable 1-morphisms in the additive closure of $(A \otimes_{\Bbbk} A, \mathrm{id}_{A \otimes_{\Bbbk} A})$ inside $\tilde{\mathcal{X}}$ belong to the same two-sided cell. Ignoring idempotents in $\mathcal{X}$, the claim follows directly from [MM5, Subsection 5.1]. In full generality, the statement is then proved using Proposition 7. □

3.5. **Adjunctions.** In this subsection, we study adjunctions in the 2-category $\mathscr{G}_A$ under the assumption that A is self-injective. We assume that A is basic and that there is a fixed complete G-invariant set $\mathtt{E}$ of primitive idempotents. We denote by ν the bijection on $\mathtt{E}$ which is induced by the Nakayama automorphism of A given by

$$\mathrm{Hom}_{\Bbbk}(eA, \Bbbk) \cong A\nu(e), \quad \text{for } e \in \mathtt{E}.$$

For a primitive idempotent $e \in A$, we denote by ε_e the idempotent in $\mathrm{End}_{\mathcal{Y}}(Ae)$ or $\mathrm{End}_{\mathcal{Y}}(eA)$ corresponding to the trivial character of $G_{A\nu(e)} = G_{Ae} = G_{eA} =: G_e$. We denote by $\mathbf{m}$ the multiplication map in B.

Proposition 12. *We have adjunctions*

(*a*) $((Ae, \varepsilon_e), (eA, \varepsilon_e))$ *in* $\tilde{\mathcal{Y}}$;

(*b*) $((eA, \varepsilon_e), (A\nu(e), \varepsilon_{\nu(e)}))$ *in* $\tilde{\mathcal{Y}}$.

Proof. We first define the counit

$$\epsilon : (Ae \otimes_{\mathcal{Y}} eA, \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_e) \to (A, \tilde{\pi}_{1_{\hat{G}}}).$$

This is defined by the vertical part of the diagram

$$\begin{array}{ccc}
\bigoplus_{\varphi} Ae \otimes_{\Bbbk} {}^{\varphi}eA^{\varphi} & \xrightarrow{\frac{1}{|G_e|^2}(\alpha\otimes\beta\varphi^{-1})_{\alpha,\beta\varphi^{-1}\in G_e}} & \bigoplus_{\alpha,\beta} {}^{\alpha}Ae^{\alpha} \otimes_{\Bbbk} {}^{\beta}eA^{\beta} \\
\Big\downarrow{\scriptstyle \frac{1}{|G_e|}(\gamma\circ\mathbf{m})_{\gamma,\varphi\in G}} & & \Big\downarrow{\scriptstyle \frac{1}{|G_e|}(\delta\circ\mathbf{m}\circ(\alpha\otimes\alpha)^{-1})_{\delta,\alpha,\beta\in G}} \\
\bigoplus_{\gamma} {}^{\gamma}A^{\gamma} & \xrightarrow{\frac{1}{|G|}(\delta\gamma^{-1})_{\delta,\gamma\in G}} & \bigoplus_{\delta} {}^{\delta}A^{\delta}
\end{array}$$

where here and in the rest of the proof all elements indexing direct sums run through G, and the notation $(\alpha \otimes \beta\varphi^{-1})_{\alpha,\beta\varphi^{-1}\in G_e}$ should be read as the (α, β, φ)-component of the map being defined as zero if the conditions $\alpha, \beta\varphi^{-1} \in G_e$ are not satisfied.

To check that the vertical arrows define a morphism $(Ae \otimes_{\mathcal{Y}} eA, \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_e) \to (A, \tilde{\pi}_{1_{\hat{G}}})$, we need to verify that the diagram commutes and the result coincides with the original map, namely, satisfying (2). We consider the (δ, φ)-component of both compositions. First going to the right and then down, $e \otimes \varphi(e)$ is mapped to

$$\frac{1}{|G_e|^3} \sum_{\alpha\in G_e, \beta\in\varphi G_e} \delta(e\alpha^{-1}\beta(e)) = \begin{cases} \frac{1}{|G_e|}\delta(e), & \text{if } \varphi \in G_e, \\ 0, & \text{otherwise.} \end{cases}$$

The other way around, $e \otimes \varphi(e)$ is sent to

$$\frac{1}{|G_e||G|} \sum_{\gamma\in G} \delta(e\varphi(e)) = \begin{cases} \frac{1}{|G_e|}\delta(e), & \text{if } \varphi \in G_e, \\ 0, & \text{otherwise,} \end{cases}$$

which coincides with the image of $\frac{1}{|G_e|}(\delta \circ \mathbf{m})_{\delta,\varphi\in G}$ on $e \otimes \varphi(e)$. So our counit ϵ is, indeed, well-defined.

We now define the unit

$$\eta : (\Bbbk, \tilde{\pi}_{1_{\hat{G}}}) \to (eA \otimes_{\mathcal{Y}} Ae, \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_e)$$

by the vertical part of the diagram

$$\begin{array}{ccc}
\Bbbk & \xrightarrow{\frac{1}{|G|}(\varphi)_{\varphi\in G}} & \bigoplus\limits_{\varphi} {}^{\varphi}\Bbbk^{\varphi} \\
\Big\downarrow{\scriptstyle (\eta_1^{\delta,\alpha})_{\delta\alpha^{-1}\in G_e}} & & \Big\downarrow{\scriptstyle (\eta_\varphi^{\beta,\gamma})_{\beta\gamma^{-1}\in G_e,\varphi\in G}} \\
\bigoplus\limits_{\delta,\alpha} {}^{\delta}eA^{\delta}\otimes_A {}^{\alpha}Ae^{\alpha} & \xrightarrow{\frac{1}{|G_e|^2}(\beta\delta^{-1}\otimes\gamma\alpha^{-1})_{\beta\delta^{-1},\gamma\alpha^{-1}\in G_e}} & \bigoplus\limits_{\beta,\gamma} {}^{\beta}eA^{\beta}\otimes_A {}^{\gamma}Ae^{\gamma}
\end{array}$$

where

$$\eta_\varphi^{\beta,\gamma}(\varphi(1)) = \begin{cases} \frac{1}{|G_e|}\beta(e)\otimes\gamma(e), & \text{if } \beta\gamma^{-1}\in G_e, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

We again check that this defines a morphism $(\Bbbk,\tilde{\pi}_{1_{\hat{G}}}) \to (eA\otimes_{\mathcal{Y}} Ae, \varepsilon_e\otimes_{\mathcal{Y}}\varepsilon_e)$ by verifying (2). Computing the (β,γ)-component of the path first going to the right and then down, we see that

$$1 \mapsto \begin{cases} \frac{1}{|G_e|}\beta(e)\otimes\gamma(e), & \text{if } \beta\gamma^{-1}\in G_e, \\ 0, & \text{otherwise,} \end{cases}$$

that is, $(\eta_1^{\beta,\gamma})_{\beta\gamma^{-1}\in G_e}$. The other way around,

$$1 \mapsto \sum_{\delta\in\beta G_e,\alpha\in\gamma G_e} \frac{1}{|G_e|^3}\beta(e)\otimes\gamma(e) = \begin{cases} \frac{1}{|G_e|}\beta(e)\otimes\gamma(e), & \text{if } \beta\gamma^{-1}\in G_e, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the condition $\delta\alpha^{-1}\in G_e$ is automatically satisfied for $\beta\gamma^{-1}\in G_e$ and $\delta\in\beta G_e, \alpha\in\gamma G_e$. Thus our unit η is well-defined as well.

Now we need to check the adjunction axioms. Denoting (Ae,ε_e) by F and (eA,ε_e) by G, we first verify

$$\mathrm{F}\to\mathrm{F}\mathbb{1}_{\mathtt{j}}\to\mathrm{FGF}\to\mathbb{1}_{\mathtt{i}}\mathrm{F}\to\mathrm{F}$$

is the identity, for appropriate $\mathtt{i}$ and $\mathtt{j}$. To this end, consider the commutative diagram

$$\begin{array}{ccc}
Ae & \xrightarrow{\frac{1}{|G_e|}(\iota)_{\iota\in G_e}} & \bigoplus\limits_{\iota} {}^{\iota}Ae^{\iota} \\
\Big\downarrow{\scriptstyle \frac{1}{|G_e||G|}(\kappa\otimes\varphi(1))_{\kappa\in G_e,\varphi\in G}} & & \Big\downarrow{\scriptstyle \frac{1}{|G_e||G|}(\alpha\iota^{-1}\otimes\psi(1))_{\alpha\iota^{-1}\in G_e,\psi\in G}} \\
\bigoplus\limits_{\kappa,\varphi} {}^{\kappa}Ae^{\kappa}\otimes_{\Bbbk} {}^{\varphi}\Bbbk^{\varphi} & \xrightarrow{\frac{1}{|G_e||G|}(\alpha\kappa^{-1}\otimes\psi\varphi^{-1})_{\alpha\kappa^{-1}\in G_e,\psi,\varphi\in G}} & \bigoplus\limits_{\alpha,\psi} {}^{\alpha}Ae^{\alpha}\otimes_{\Bbbk} {}^{\psi}\Bbbk^{\psi} \\
\Big\downarrow{\scriptstyle \frac{1}{|G_e|}(\beta\kappa^{-1}\otimes\eta_\varphi^{\gamma,\delta})_{\beta\kappa^{-1},\gamma\delta^{-1}\in G_e,\varphi\in G}} & & \Big\downarrow{\scriptstyle \frac{1}{|G_e|}(\lambda\alpha^{-1}\otimes\eta_\psi^{\mu,\upsilon})_{\lambda\alpha^{-1},\mu\upsilon^{-1}\in G_e,\psi\in G}} \\
\bigoplus\limits_{\beta,\gamma,\delta} {}^{\beta}Ae^{\beta}\otimes_{\Bbbk} {}^{\gamma}eA^{\gamma}\otimes_A {}^{\delta}Ae^{\delta} & \xrightarrow{\frac{1}{|G_e|^3}(\lambda\beta^{-1}\otimes\mu\gamma^{-1}\otimes\upsilon\delta^{-1})} & \bigoplus\limits_{\lambda,\mu,\upsilon} {}^{\lambda}Ae^{\lambda}\otimes_{\Bbbk} {}^{\mu}eA^{\mu}\otimes_A {}^{\upsilon}Ae^{\upsilon} \\
\Big\downarrow{\scriptstyle \frac{1}{|G_e|^2}\left((\xi\beta^{-1}\circ\mathbf{m})\otimes\theta\delta^{-1}\right)_{\theta\delta^{-1}\in G_e,\xi,\beta,\gamma\in G}} & & \Big\downarrow{\scriptstyle \frac{1}{|G_e|^2}\left((\sigma\lambda^{-1}\circ\mathbf{m})\otimes\tau\upsilon^{-1}\right)_{\tau\upsilon^{-1}\in G_e,\sigma,\lambda,\mu\in G}} \\
\bigoplus\limits_{\xi,\theta} {}^{\xi}A^{\xi}\otimes_A {}^{\theta}Ae^{\theta} & \xrightarrow{\frac{1}{|G_e||G|}(\sigma\xi^{-1}\otimes\tau\theta^{-1})_{\tau\theta^{-1}\in G_e,\sigma,\xi\in G}} & \bigoplus\limits_{\sigma,\tau} {}^{\sigma}A^{\sigma}\otimes_A {}^{\tau}Ae^{\tau} \\
\Big\downarrow{\scriptstyle \frac{1}{|G_e||G|}\left(\mathbf{m}\circ(\rho\xi^{-1}\otimes\rho\theta^{-1})\right)_{\rho\theta^{-1}\in G_e,\xi\in G}} & & \Big\downarrow{\scriptstyle \frac{1}{|G_e||G|}\left(\mathbf{m}\circ(\omega\sigma^{-1}\otimes\omega\tau^{-1})\right)_{\omega\tau^{-1}\in G_e,\sigma\in G}} \\
\bigoplus\limits_{\rho} {}^{\rho}Ae^{\rho} & \xrightarrow{\frac{1}{|G_e|}(\omega\rho^{-1})_{\omega\rho^{-1}\in G_e}} & \bigoplus\limits_{\omega} {}^{\omega}Ae^{\omega}
\end{array}$$

where in the third horizontal arrow the conditions are $\lambda\beta^{-1},\mu\gamma^{-1},\upsilon\delta^{-1}\in G_e$.

We want the ρ-component of the composition on the left hand side of the diagram to be given by $\frac{1}{|G_e|}\rho$, if $\rho \in G_e$, and by zero otherwise. To see this, first notice that multiplication $\mathbf{m}$ in the third map will give something non-zero only if $\beta\gamma^{-1} \in G_e$. Taking into account all conditions specified in the diagram, this forces $\kappa, \beta, \gamma, \delta, \theta, \rho \in G_e$ in order for the ρ-component to be non-zero. Each choice of $\kappa, \beta, \gamma, \delta, \theta \in G_e$ and $\varphi, \xi \in G$ yields a summand $\frac{1}{|G|^2|G_e|^6}\rho$ in the composition (recall the factor $\frac{1}{|G_e|}$ in (10)). Summing over all these possibilities hence produces the desired result.

The fact that the composition

$$\mathrm{G} \to \mathbb{1}_{\mathtt{j}}\mathrm{G} \to \mathrm{GFG} \to \mathrm{G}\mathbb{1}_{\mathtt{i}} \to \mathrm{G}$$

is the identity follows as above by flipping all tensor factors and replacing Ae by eA in appropriate places. This proves part (a).

Assume that $\mathtt{E} = \{e_1, e_2, \dots, e_k\}$. For any $1 \leq i \leq k$, we choose a Jordan-Hölder series of each Ae_i by

$$Ae_i = X_{i,0} \supsetneq X_{i,1} \supsetneq X_{i,2} \supsetneq \cdots \supsetneq X_{i,m_i} \supsetneq X_{i,m_i+1} = 0.$$

As our algebra A is basic, each $\mathbb{k}$-space $X_{i,j}/X_{i,j+1}$, where $0 \leq j \leq m_i$, is of dimension one. For each i, we fix some basis $\mathtt{E}_i := \{e_i, x_{i,j} : 1 \leq j \leq m_i\}$ of Ae_i such that we have $x_{i,j} \in X_{i,j} \setminus X_{i,j+1}$, for every j. Then $\mathtt{A} := \bigcup_{i=1}^{k} \mathtt{E}_i \supset \mathtt{E}$ is a basis of A. Let $\mathbf{t} : A \to \mathbb{k}$ be the unique linear map such that, for all $a \in \mathtt{A}$, we have

$$\mathbf{t}(a) = \begin{cases} 1, & a = x_{i,m_i}, \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

For $a \in \mathtt{A}$, we denote by a^* the unique element in A which satisfies

$$\mathbf{t}(ba^*) = \begin{cases} 1, & b = a; \\ 0, & b \in \mathtt{A} \setminus \{a\}. \end{cases}$$

We now define the unit

$$\tilde{\eta} : (A, \tilde{\pi}_{1_{\hat{G}}}) \to (A\nu(e) \otimes_{\mathcal{Y}} eA, \varepsilon_{\nu(e)} \otimes_{\mathcal{Y}} \varepsilon_e)$$

by the vertical part of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\frac{1}{|G|}(\varphi)_{\varphi \in G}} & \bigoplus_{\varphi} {}^{\varphi}A^{\varphi} \\ \Big\downarrow {\scriptstyle (\tilde{\eta}_1^{\alpha,\beta})_{\alpha\beta^{-1} \in G_e}} & & \Big\downarrow {\scriptstyle (\tilde{\eta}_{\varphi}^{\gamma,\delta})_{\gamma\delta^{-1} \in G_e, \varphi \in G}} \\ \bigoplus_{\alpha,\beta} {}^{\alpha}A\nu(e)^{\alpha} \otimes_{\mathbb{k}} {}^{\beta}eA^{\beta} & \xrightarrow{\frac{1}{|G_e|^2}(\gamma\alpha^{-1} \otimes \delta\beta^{-1})_{\gamma\alpha^{-1}, \delta\beta^{-1} \in G_e}} & \bigoplus_{\gamma,\delta} {}^{\gamma}A\nu(e)^{\gamma} \otimes_{\mathbb{k}} {}^{\delta}eA^{\delta} \end{array}$$

where

$$\tilde{\eta}_{\varphi}^{\gamma,\delta}(\varphi(1)) = \begin{cases} \dfrac{1}{|G_e|} \sum_{a \in \mathtt{A}} \gamma(a^*\nu(e)) \otimes \delta(ea), & \text{if } \gamma\delta^{-1} \in G_e; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Going right and then down, summing over $\varphi \in G$ cancels the scalar $\frac{1}{|G|}$ and hence the image of 1 in the (γ, δ)-component is given by the right hand side of (11) and equals the (γ, δ)-component of $(\tilde{\eta}_1^{\gamma,\delta}(1))_{\gamma\delta^{-1} \in G_e}$, which implies the first equality of (2). Going down and then right, to obtain a non-zero contribution, we need $\alpha\beta^{-1}, \gamma\alpha^{-1}, \delta\beta^{-1} \in G_e$, which yields $\delta\gamma^{-1} \in G_e$. Summing over all such choices of α and β, the image of 1 in the (γ, δ)-component is again given by the right hand side of (11), implying the second equality of (2). Hence $\tilde{\eta}$ is well-defined.

Now we define the counit

$$\tilde{\epsilon} : (eA \otimes_{\mathcal{Y}} A\nu(e), \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_{\nu(e)}) \to (\mathbb{k}, \tilde{\pi}_{1_{\hat{G}}})$$

by the vertical part of the diagram

$$\begin{array}{ccc}
\bigoplus_{\varphi} eA \otimes_A {}^{\varphi}A\nu(e)^{\varphi} & \xrightarrow{\frac{1}{|G_e|^2}(\rho\otimes\psi\varphi^{-1})_{\rho,\psi\varphi^{-1}\in G_e}} & \bigoplus_{\rho,\psi} {}^{\rho}eA^{\rho} \otimes_{\mathbb{k}} {}^{\psi}A\nu(e)^{\psi} \\
\Big\downarrow \frac{1}{|G_e|}(\alpha\circ\mathbf{t}\circ\mathbf{m})_{\alpha,\varphi\in G} & & \frac{1}{|G_e|}(\beta\circ\mathbf{t}\circ\mathbf{m}\circ(\rho\otimes\rho)^{-1})_{\beta,\psi,\rho\in G} \Big\downarrow \\
\bigoplus_{\alpha} {}^{\alpha}\mathbb{k}^{\alpha} & \xrightarrow{\frac{1}{|G|}(\beta\alpha^{-1})_{\alpha,\beta\in G}} & \bigoplus_{\beta} {}^{\beta}\mathbb{k}^{\beta}.
\end{array}$$

Consider the (β, φ)-component of both compositions. First going down and then to the right, the first map is zero, for $\varphi \notin G_e$, as $\mathbf{t}(eAf) = 0$ unless $f = \nu(e)$. If $\varphi \in G_e$, then each α contributes $\frac{1}{|G||G_e|}\beta \circ \mathbf{t} \circ \mathbf{m}$. Hence the resulting map is $\frac{1}{|G_e|}\beta \circ \mathbf{t} \circ \mathbf{m}$ and the second equality of (2) is satisfied. The right vertical map is zero unless $\psi\rho^{-1} \in G_e$ which, together with the conditions on the upper horizontal map, forces $\varphi, \rho, \psi \in G_e$. Summing over the choices for ρ and ψ, we obtain the same resulting map and thus the diagram commutes, in which case the first equality of (2) holds. Hence $\tilde{\epsilon}$ is well-defined.

We now verify the adjunction axioms. Denoting (eA, ε_e) by $\tilde{\mathrm{F}}$ and $(A\nu(e), \varepsilon_{\nu(e)})$ by $\tilde{\mathrm{G}}$, we need to show that

$$\tilde{\mathrm{F}} \to \tilde{\mathrm{F}}\mathbb{1}_{\mathtt{i}} \to \tilde{\mathrm{F}}\tilde{\mathrm{G}}\tilde{\mathrm{F}} \to \mathbb{1}_{\mathtt{j}}\tilde{\mathrm{F}} \to \tilde{\mathrm{F}}$$

is the identity. To this end, we assemble our maps in a large commutative diagram as in part (a) and compute the left hand side. This is given by the composition

$$\begin{array}{ccc}
eA & \xrightarrow{\frac{1}{|G||G_e|}(\delta\otimes\varphi(1))_{\delta\in G_e,\varphi\in G}} & \bigoplus_{\delta,\varphi} {}^{\delta}eA^{\delta} \otimes_A {}^{\varphi}A^{\varphi} \\
 & & \Big\downarrow \frac{1}{|G_e|}(\alpha\delta^{-1}\otimes\tilde{\eta}_{\varphi}^{\beta,\gamma})_{\alpha\delta^{-1},\beta\gamma^{-1}\in G_e,\varphi\in G} \\
\bigoplus_{\xi,\theta} {}^{\xi}\mathbb{k}^{\xi} \otimes_{\mathbb{k}} {}^{\theta}eA^{\theta} & \xleftarrow{\frac{1}{|G_e|^2}\left((\xi\circ\mathbf{t}\circ\mathbf{m}\circ(\alpha\otimes\alpha)^{-1})\otimes\theta\gamma^{-1}\right)} & \bigoplus_{\alpha,\beta,\gamma} {}^{\alpha}eA^{\alpha} \otimes_A {}^{\beta}A\nu(e)^{\beta} \otimes_{\mathbb{k}} {}^{\gamma}eA^{\gamma} \\
\Big\downarrow \frac{1}{|G_e||G|}\left(\mathbf{m}\circ(\omega\xi^{-1}\otimes\omega\theta^{-1})\right)_{\omega\theta^{-1}\in G_e,\xi\in G} & & \\
\bigoplus_{\omega} {}^{\omega}eA^{\omega} & &
\end{array}$$

where in the third map the conditions are $\theta\gamma^{-1} \in G_e$ and $\alpha, \beta, \xi \in G$. However, the third map is zero unless $\alpha\beta^{-1} \in G_e$ (for the same reason involving $\mathbf{t}$ as used above), which, together with the other conditions in the diagram, shows that we have a non-zero contribution to the ω-component only if $\delta, \alpha, \beta, \gamma, \theta, \omega \in G_e$. For $\omega \in G_e$, the contribution of a fixed choice of $\delta, \varphi, \alpha, \beta, \gamma, \xi, \theta$ to the image of e is

$$\frac{1}{|G|^2|G_e|^6}\omega\left(\sum_{a\in\mathtt{A}} \mathbf{t}(e\beta\alpha^{-1}(a^*\nu(e)))ea\right).$$

Observing that, by our choice of $\mathtt{A}$, we have

$$\mathbf{t}(e\beta\alpha^{-1}(a^*\nu(e))) = \begin{cases} 1, & \text{if } a = e; \\ 0, & \text{otherwise,} \end{cases}$$

and summing over all choices of $\delta, \varphi, \alpha, \beta, \gamma, \xi, \theta$, we obtain that the image of e is $\frac{1}{|G_e|}\omega(e)$, as desired.

Now we verify the other axiom. The composition

$$\tilde{\mathrm{G}} \to \mathbb{1}_{\mathtt{i}}\tilde{\mathrm{G}} \to \tilde{\mathrm{G}}\tilde{\mathrm{F}}\tilde{\mathrm{G}} \to \tilde{\mathrm{G}}\mathbb{1}_{\mathtt{j}} \to \tilde{\mathrm{G}}$$

is given by the diagram, which consists of the left hand side of a large commutative diagram as in part (a),

$$\begin{array}{ccc}
A\nu(e) & \xrightarrow{\frac{1}{|G||G_e|}(\alpha(1)\otimes\varphi)_{\alpha\in G,\varphi\in G_e}} & \bigoplus_{\alpha,\varphi} {}^{\alpha}A^{\alpha} \otimes_A {}^{\varphi}A\nu(e)^{\varphi} \\
 & & \Big\downarrow \frac{1}{|G_e|}(\tilde{\eta}_{\alpha}^{\delta,\beta}\otimes\gamma\varphi^{-1})_{\delta\beta^{-1},\gamma\varphi^{-1}\in G_e,\alpha\in G} \\
\bigoplus_{\xi,\theta} {}^{\theta}A\nu(e)^{\theta} \otimes_{\mathbb{k}} {}^{\xi}\mathbb{k}^{\xi} & \xleftarrow{\frac{1}{|G_e|^2}\left(\theta\delta^{-1}\otimes(\xi\circ\mathbf{t}\circ\mathbf{m}\circ(\beta\otimes\beta)^{-1})\right)} & \bigoplus_{\delta,\beta,\gamma} {}^{\delta}A\nu(e)^{\delta} \otimes_{\mathbb{k}} {}^{\beta}eA^{\beta} \otimes_A {}^{\gamma}A\nu(e)^{\gamma} \\
\Big\downarrow \frac{1}{|G_e||G|}\left(\mathbf{m}\circ(\omega\theta^{-1}\otimes\omega\xi^{-1})\right)_{\omega\theta^{-1}\in G_e,\xi\in G} & & \\
\bigoplus_{\omega} {}^{\omega}A\nu(e)^{\omega} & &
\end{array}$$

where in the third map the conditions are $\theta\delta^{-1} \in G_e$ and $\beta, \gamma, \xi \in G$. Note that the third map is zero unless $\beta\gamma^{-1} \in G_e$. Taking into account all conditions in the diagram, this shows that we have a non-zero contribution to the ω-component only if $\varphi, \gamma, \beta, \delta, \theta, \omega \in G_e$. For $\omega \in G_e$, the contribution of a fixed choice of $\alpha, \varphi, \gamma, \beta, \delta, \theta, \xi$ to the image of $\nu(e)$ is

$$\frac{1}{|G|^2|G_e|^6}\omega\left(\sum_{a\in\mathtt{A}} a^*\nu(e)\mathbf{t}(ea\beta^{-1}\gamma(\nu(e)))\right).$$

By our choice of $\mathtt{A}$, we have

$$\mathbf{t}(ea\beta^{-1}\gamma(\nu(e))) = \begin{cases} 1, & \text{if } a^* = \nu(e); \\ 0, & \text{otherwise.} \end{cases}$$

Summing over all choices of $\alpha, \varphi, \gamma, \beta, \delta, \theta, \xi$, we obtain that the image of $\nu(e)$ is $\frac{1}{|G_e|}\omega(\nu(e))$, as desired. This completes the proof. □

We now consider tensor products of indecomposable projective symmetric B-B-bimodules with simple quotients of projective A-$\mathbb{k}$-bimodules. To this end, we extend our notation to $G_{fe} := G_{Af\otimes_{\mathbb{k}}eA} (= G_e \cap G_f)$, for $e, f \in \mathtt{E}$, and denote the simple quotient of (Ae, ε_e) by (L_e, ε_e). As each $\varphi \in G$ is an automorphism of A, we have the induced action of φ on $\{(L_e, \varepsilon_e) : e \in \mathtt{E}\}$ which maps each vector space L_e to the vector space $L_{\varphi(e)}$.

Lemma 13. *In $\tilde{\mathcal{Y}}$, there is an isomorphism*

$$(Af \otimes_{\mathbb{k}} eA, \varepsilon_{fe}) \otimes_{\tilde{\mathcal{Y}}} (L_e, \varepsilon_e) \cong (Af, \frac{1}{|G_{fe}|}(\gamma)_{\gamma\in G_{fe}}) \cong \bigoplus_{\xi}(Af, \varepsilon_{\xi})$$

where ξ runs over all characters appearing in the induction of the trivial G_{fe}-module to G_f.

Proof. We first construct an isomorphism between $(Af \otimes_{\Bbbk} eA, \varepsilon_{fe}) \otimes_{\tilde{\mathcal{Y}}} (L_e, \varepsilon_e)$ and $(Af, \frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}})$. In one direction, the morphism $\boldsymbol{g}$ is given by the diagram

$$\begin{array}{ccc}
Af & \xrightarrow{\frac{1}{|G_{fe}|}(\gamma)_{\gamma\in G_{fe}}} & \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma} \\
\Big\downarrow {\scriptstyle \frac{1}{|G_{fe}||G_e|}(\psi\otimes\psi(e)\otimes\varphi(l))_{\psi\in G_{fe},\varphi\in G_e}} & & \Big\downarrow {\scriptstyle \frac{1}{|G_{fe}||G_e|}(\alpha\gamma^{-1}\otimes\alpha(e)\otimes\beta(l))_{\alpha\gamma^{-1}\in G_{fe},\beta\in G_e}} \\
\bigoplus_{\varphi,\psi} {}^{\psi}Af \otimes_{\Bbbk} eA^{\psi} \otimes_A {}^{\varphi}(L_e)^{\varphi} & \xrightarrow{\frac{1}{|G_{fe}||G_e|}(\alpha\psi^{-1}\otimes\alpha\psi^{-1}\otimes\beta\varphi^{-1})} & \bigoplus_{\alpha,\beta} {}^{\alpha}Af \otimes_{\Bbbk} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta}
\end{array}$$

where the lower horizontal map is indexed by $\alpha\psi^{-1} \in G_{fe}, \beta\varphi^{-1} \in G_e$, and l denotes the canonical generator of the one-dimensional module L_e (the image of e in L_e). To see that the diagram commutes, first notice that the (α, β)-component of the object in the lower right-hand corner is nonzero if and only if $\alpha\beta^{-1} \in G_e$. If this is the case, then the (α, β)-component of the map going first to the right and then down is given by

$$f \mapsto \begin{cases} \frac{1}{|G_{fe}||G_e|}\alpha(f) \otimes \alpha(e) \otimes \beta(l), & \text{if } \alpha \in G_{fe}, \\ 0, & \text{otherwise.} \end{cases}$$

First going down and then to the right, we notice that $\varphi \in G_e, \psi \in G_{fe}$ forces $\beta \in G_e, \alpha \in G_{fe}$, so we obtain the same result, which verifies (2).

A morphism $\boldsymbol{h}$ in the other direction is given by

$$\begin{array}{ccc}
\bigoplus_{\varphi} Af \otimes_{\Bbbk} eA \otimes_A {}^{\varphi}(L_e)^{\varphi} & \xrightarrow{\frac{1}{|G_{fe}||G_e|}(\alpha\otimes\alpha\otimes\beta\varphi^{-1})_{\alpha\in G_{fe},\beta\varphi^{-1}\in G_e}} & \bigoplus_{\alpha,\beta} {}^{\alpha}Af \otimes_{\Bbbk} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta} \\
\Big\downarrow {\scriptstyle \frac{1}{|G_{fe}|}(\delta\otimes\mathbf{m})_{\delta\in G_{fe},\varphi\in G}} & & \Big\downarrow {\scriptstyle \frac{1}{|G_{fe}|}(\gamma\alpha^{-1}\otimes\mathbf{m}\circ(\alpha\otimes\alpha)^{-1})_{\gamma\alpha^{-1}\in G_{fe},\beta\in G}} \\
\bigoplus {}^{\delta}Af^{\delta} & \xrightarrow{\frac{1}{|G_{fe}|}(\gamma\delta^{-1})_{\gamma\delta^{-1}\in G_{fe}}} & \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma}
\end{array}$$

A nonzero contribution to the (γ, φ)-component, when going first down and then to the right can only happen for $\varphi \in G_e$ and $\gamma \in G_{fe}$, in which case the generator $f \otimes e \otimes \varphi(l)$ (as an A-$\Bbbk$-bimodule) gets sent to $\frac{1}{|G_{fe}|}\gamma(f)$. Similarly, going first to the right and then down, a nonzero contribution only occurs for $\alpha\beta^{-1} \in G_e$, which, together with the conditions in the diagram, again forces $\beta, \varphi \in G_e, \alpha, \gamma \in G_{fe}$. Hence, summing over such α and β, we obtain the same result which verifies (2).

To check that both compositions of $\boldsymbol{g}$ and $\boldsymbol{h}$ are the respective identities (that is, the correct idempotents), it suffices to consider the compositions of the left hand side of the diagrams.

Starting with $\boldsymbol{g} \circ \boldsymbol{h}$ and considering

$$\bigoplus_{\varphi} Af \otimes_{\Bbbk} eA \otimes_A {}^{\varphi}(L_e)^{\varphi} \to \bigoplus_{\delta} {}^{\delta}Af^{\delta} \to \bigoplus_{\alpha,\beta} {}^{\alpha}Af \otimes_{\Bbbk} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta},$$

the (α, β, φ)-component of the composition is zero unless $\varphi, \beta \in G_e, \alpha \in G_{fe}$, in which case the generator $f \otimes e \otimes \varphi(l)$ is mapped to $\frac{1}{|G_{fe}||G_e|}\alpha(f) \otimes \alpha(e) \otimes \beta(l)$, as desired.

For $\boldsymbol{h} \circ \boldsymbol{g}$, we consider

$$Af \to \bigoplus_{\varphi,\psi} {}^{\psi}Af \otimes_{\Bbbk} eA^{\psi} \otimes_A {}^{\varphi}(L_e)^{\varphi} \to \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma}$$

and verify that, in the γ-component, f is indeed sent to $\frac{1}{|G_{fe}|}\gamma(f)$, if $\gamma \in G_{fe}$, and zero otherwise, as claimed.

Hence we have an isomorphism $(Af \otimes_{\Bbbk} eA, \varepsilon_{fe}) \otimes_{\tilde{\mathcal{Y}}} (L_e, \varepsilon_e) \cong (Af, \frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}})$, as stated.

Now notice that $\dfrac{1}{|G_{fe}|}\displaystyle\sum_{\gamma \in G_{fe}} \gamma$ is a trivial idempotent on G_{fe}. When viewed as an idempotent of the larger group G_f, it decomposes into precisely the (multiplicity-free) sum of those idempotents affording characters ξ of G_f which appear in the induction of the trivial character from G_{fe} to G_f. This proves the proposition. □

Proposition 14. *We have adjunctions* $((Af \otimes_{\Bbbk} eA, \varepsilon_{fe}), (A\nu(e) \otimes_{\Bbbk} fA, \varepsilon_{\nu(e)f}))$, *for idempotents* $e, f \in A$.

Proof. From the defining action of $\mathscr{G}_B$ on $B\text{-mod}$ we have that $(Af \otimes_{\Bbbk} eA, \varepsilon_{fe})$ is left adjoint to $(A\nu(e) \otimes_{\Bbbk} fA, \varepsilon_{\chi})$, for some $\chi \in \hat{G}_{\nu(e)f}$. We thus have an isomorphism of nonzero spaces of homomorphisms,

$$\begin{aligned}\mathrm{Hom}_{\tilde{\mathcal{Y}}}((Af, \frac{1}{|G_{fe}|}(\beta)_{\beta \in G_{fe}}), (L_f, \varepsilon_f)) &\cong \mathrm{Hom}_{\tilde{\mathcal{Y}}}((Af \otimes_{\Bbbk} eA, \varepsilon_{fe}) \otimes_{\tilde{\mathcal{Y}}} (L_e, \varepsilon_e), (L_f, \varepsilon_f)) \\ &\cong \mathrm{Hom}_{\tilde{\mathcal{Y}}}((L_e, \varepsilon_e), (A\nu(e) \otimes_{\Bbbk} fA, \varepsilon_{\chi}) \otimes_{\tilde{\mathcal{Y}}} (L_f, \varepsilon_f)),\end{aligned}$$

where the first isomorphism follows from Lemma 13.

By (the opposite of) Proposition 7, noting that $G_{fe} = G_{\nu(e)f}$ and $G_{\nu(e)} = G_e$, there are isomorphisms

$$\begin{aligned}(A\nu(e) \otimes_{\Bbbk} fA, \varepsilon_{\chi}) \otimes_{\tilde{\mathcal{Y}}} (L_f, \varepsilon_f)) &\cong (A, \tilde{\pi}_{\chi}) \otimes_{\tilde{\mathcal{Y}}} (A\nu(e) \otimes_{\Bbbk} fA, \varepsilon_{\nu(e)f}) \otimes_{\tilde{\mathcal{Y}}} (L_f, \varepsilon_f) \\ &\cong (A, \tilde{\pi}_{\chi}) \otimes_{\tilde{\mathcal{Y}}} (A\nu(e), \frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}}) \\ &\cong (A\nu(e), \frac{1}{|G_{fe}|}(\chi(\gamma)\gamma)_{\gamma \in G_{fe}}),\end{aligned}$$

yielding the isomorphism

$$\mathrm{Hom}_{\tilde{\mathcal{Y}}}((Af, \frac{1}{|G_{fe}|}(\beta)_{\beta \in G_{fe}}), (L_f, \varepsilon_f)) \cong \mathrm{Hom}_{\tilde{\mathcal{Y}}}((L_e, \varepsilon_e), (A\nu(e), \frac{1}{|G_{fe}|}(\chi(\gamma)\gamma)_{\gamma \in G_{fe}})).$$

Denote the left hand side by U and the right hand side by V.

Now we claim that $\dim_{\Bbbk} U = 1$. By (2), for any morphism $\boldsymbol{g} \in U$, we have

$$\mathrm{id}_{(L_f, \varepsilon_f)} \circ \boldsymbol{g} = \boldsymbol{g} = \boldsymbol{g} \circ \mathrm{id}_{(Af, \frac{1}{|G_{fe}|}(\beta)_{\beta \in G_{fe}})}. \tag{12}$$

Assume that $\boldsymbol{g} = (k_{\alpha}\alpha)_{\alpha \in G_f}$, where $k_{\alpha} \in \Bbbk$, and consider the diagram

$$\begin{array}{ccc} Af & \xrightarrow{\frac{1}{|G_{fe}|}(\beta)_{\beta \in G_{fe}}} & \bigoplus_{\beta} {}^{\beta}Af^{\beta} \\ \Big\downarrow {\scriptstyle (k_{\alpha}\alpha)_{\alpha \in G_f}} & & \Big\downarrow {\scriptstyle (k_{\gamma\beta^{-1}}\gamma\beta^{-1})_{\gamma\beta^{-1} \in G_f}} \\ \bigoplus_{\alpha} {}^{\alpha}(L_f)^{\alpha} & \xrightarrow{\frac{1}{|G_f|}(\gamma\alpha^{-1})_{\gamma\alpha^{-1} \in G_f}} & \bigoplus_{\gamma} {}^{\gamma}(L_f)^{\gamma}. \end{array}$$

The morphism $\mathrm{id}_{(L_f, \varepsilon_f)} \circ \boldsymbol{g}$ is exactly the path going down and then right, taking into account the fact that $\alpha \in G_f$ forces $\gamma \in G_f$, and the γ-component of this morphism is given by

$$f \mapsto \begin{cases} \frac{1}{|G_f|}\Big(\sum_{\alpha\in G_f} k_\alpha\Big)\gamma(f), & \text{if } \gamma \in G_f; \\ 0, & \text{otherwise.} \end{cases}$$

The first equality of (12) shows that $\frac{1}{|G_f|}\sum_{\alpha\in G_f} k_\alpha = k_\gamma$, for all $\gamma \in G_f$, and hence $k_\gamma = k_{\gamma'}$ for all $\gamma, \gamma' \in G_f$. Then we obtain $\boldsymbol{g} = (k\alpha)_{\alpha\in G_f}$, where $k \in \Bbbk$, and the first equality is automatically satisfied. Going right and then down and using the fact that $\beta \in G_{fe}, \gamma\beta^{-1} \in G_f$ implies $\gamma \in G_f$, the second equality of (12) is easily verified. The claim follows.

As $U \cong V$, we have $\dim_\Bbbk V = 1$. Using (2), for any morphism $\boldsymbol{h} \in V$, we have

$$\mathrm{id}_{(A\nu(e), \frac{1}{|G_{fe}|}(\chi(\gamma)\gamma)_{\gamma\in G_{fe}})} \circ \boldsymbol{h} = \boldsymbol{h} = \boldsymbol{h} \circ \mathrm{id}_{(L_e,\varepsilon_e)}. \tag{13}$$

Assume that $\boldsymbol{h} = (l_\alpha\alpha)_{\alpha\in G_e}$, where $l_\alpha \in \Bbbk$, and consider the diagram

$$\begin{array}{ccc} L_e & \xrightarrow{\frac{1}{|G_e|}(\beta)_{\beta\in G_e}} & \bigoplus_\beta {}^\beta(L_e)^\beta \\ \Big\downarrow{\scriptstyle (l_\alpha\alpha)_{\alpha\in G_e}} & & \Big\downarrow{\scriptstyle (l_{\delta\beta^{-1}}\delta\beta^{-1})_{\delta\beta^{-1}\in G_e}} \\ \bigoplus_\alpha {}^\alpha A\nu(e)^\alpha & \xrightarrow{\frac{1}{|G_{fe}|}(\chi(\delta\alpha^{-1})\delta\alpha^{-1})_{\delta\alpha^{-1}\in G_{fe}}} & \bigoplus_\delta {}^\delta A\nu(e)^\delta. \end{array}$$

The morphism $\boldsymbol{h} \circ \mathrm{id}_{(L_e,\varepsilon_e)}$ coincides with the path going to the right and then down. Note that $\beta \in G_e, \delta\beta^{-1} \in G_e$ forces $\delta \in G_e$. Then the δ-component of this composition is given by

$$e \mapsto \begin{cases} \frac{1}{|G_e|}\Big(\sum_{\beta\in G_e} l_{\delta\beta^{-1}}\Big)\delta(e), & \text{if } \delta \in G_e; \\ 0, & \text{otherwise.} \end{cases}$$

By re-indexing, the second equality of (13) shows that, for all $\delta \in G_e$, we have $\frac{1}{|G_e|}\sum_{\sigma\in G_e} l_\sigma = l_\delta$, and thus $l_\delta = l_{\delta'}$, for all $\delta, \delta' \in G_e$. Therefore we have $\boldsymbol{h} = (l\alpha)_{\alpha\in G_e}$, where $l \in \Bbbk$, and the second equality holds. Due to $\dim_\Bbbk V = 1$, the first equality of (13) should also hold for any $l \in \Bbbk^*$. Going down and then to the right, the δ-component of $\mathrm{id}_{(A\nu(e), \frac{1}{|G_{fe}|}(\chi(\gamma)\gamma)_{\gamma\in G_{fe}})} \circ \boldsymbol{h}$ is zero unless $\delta \in G_e$, in which case e is sent to

$$\frac{1}{|G_{fe}|}\Big(\sum_{\alpha\in\delta G_{fe}} \chi(\delta\alpha^{-1})\Big) l\delta(e) = \frac{1}{|G_{fe}|}\Big(\sum_{\varphi\in G_{fe}} \chi(\varphi)\Big) l\delta(e).$$

The first equality implies that $\frac{1}{|G_{fe}|}\Big(\sum_{\varphi\in G_{fe}} \chi(\varphi)\Big) = 1$. By multiplying any $\chi(\gamma)$, where $\gamma \in G_{fe}$, to both side of the latter, we obtain

$$\chi(\gamma) = \frac{1}{|G_{fe}|}\Big(\sum_{\varphi\in G_{fe}} \chi(\gamma\varphi)\Big) = \frac{1}{|G_{fe}|}\Big(\sum_{\psi\in G_{fe}} \chi(\psi)\Big) = 1.$$

Therefore $\chi = \varepsilon_{\nu(e)f}$ and the proof is complete. □

Proposition 15. *In $\mathscr{G}_B$, we have adjunctions $((A_i, \tilde{\pi}_\chi), (A_i, \tilde{\pi}_{\chi^{-1}}))$, for each $\chi \in \hat{G}$ and $i = 1, \ldots, n$. Similarly, we have adjunction $((\Bbbk, \tilde{\pi}_\chi), (\Bbbk, \tilde{\pi}_{\chi^{-1}}))$, for each $\chi \in \hat{G}$.*

Proof. By Proposition 6, we have

$$(A_i, \tilde{\pi}_\chi) \otimes_{\tilde{\mathcal{Y}}} (A_i, \tilde{\pi}_{\chi^{-1}}) \cong (A_i, \tilde{\pi}_{1_{\hat{G}}}) \cong (A_i, \tilde{\pi}_{\chi^{-1}}) \otimes_{\tilde{\mathcal{Y}}} (A_i, \tilde{\pi}_\chi)$$

and similarly for $\Bbbk$. Both unit and counit are then just identities and the claim is immediate. □

Proposition 16. *If A is self-injective, then the 2-categories $\mathscr{G}_A$ and $\mathscr{G}_B$ are weakly fiat. If A is weakly symmetric, then both $\mathscr{G}_A$ and $\mathscr{G}_B$ are fiat.*

Proof. Assume A is self-injective. Proposition 7 shows that any indecomposable 1-morphism can be written as a product of those treated in Propositions 12, 14 and 15. This implies that $\mathscr{G}_A$ and $\mathscr{G}_B$ are weakly fiat. If A is weakly symmetric, then ν is the identity, and all adjunctions given in Propositions 12, 14 and 15 become (weakly) involutive, proving fiatness. □

3.6. **Simple transitive 2-representations of $\mathscr{G}_A$.** Now we can formulate our first main result. We assume that A is weakly symmetric, basic and there is a fixed complete G-invariant set $\mathtt{E}$ of primitive idempotents, so that $\mathscr{G}_A$ and $\mathscr{G}_B$ are fiat.

Theorem 17. *Under the above assumptions, for every two-sided cell $\mathcal{J}$ in $\mathscr{G}_A$, there is a natural bijection between equivalence classes of simple transitive 2-representations of $\mathscr{G}_A$ with apex $\mathcal{J}$ and pairs (K, ω), where K is a subgroup of G and $\omega \in H^2(K, \Bbbk^*)$.*

Proof. For $\mathcal{J} = \mathcal{J}_{\mathtt{i}}$, where $i = 1, 2, \dots, n$, Proposition 6 shows that the $\mathcal{J}$-simple quotient of $\mathscr{G}_A$ is biequivalent to the 2-category $\mathrm{Rep}(G)$ from [Os]. Therefore the statement follows from [Os, Theorem 2].

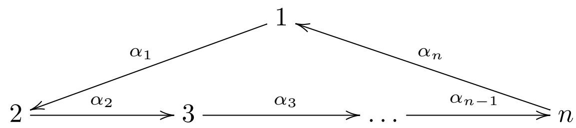
For $\mathcal{J} = \mathcal{J}_0$, consider $B = A \times \Bbbk$. Then we can realize $\mathscr{G}_A$ as a both 1- and 2-full subcategory of $\mathscr{G}_B$ in the obvious way. Let $\mathtt{j}$ denote the object corresponding to the additional factor $\Bbbk$. Let $\mathcal{H}_1$ be the $\mathcal{H}$-class in $\mathscr{G}_B$ containing the identity 1-morphism on $\Bbbk$. By Lemma 2, $\mathcal{H}_1$ contains $|G|$ indecomposable 1-morphisms, moreover, $\mathcal{H}_1$ is contained in $\mathcal{J}_0^{(B)}$, the two-sided cell of projective bimodules in $\mathscr{G}_B$. Note that the 1- and 2-full 2-subcategory $\mathscr{A}_{\mathcal{H}_1}$ of $\mathscr{G}_B$ with object $\mathtt{j}$ is biequivalent to the 2-category $\mathrm{Rep}(G)$ as above. Hence, by [Os, Theorem 2], there is a natural bijection between equivalence classes of simple transitive 2-representations of $\mathscr{A}_{\mathcal{H}_1}$ with apex $\mathcal{H}_1$ and pairs (K, ω) as in the theorem. By [MMMZ, Theorem 15], there is a bijection between equivalence classes of simple transitive 2-representations of $\mathscr{A}_{\mathcal{H}_1}$ and equivalence classes of simple transitive 2-representations of $\mathscr{G}_B$ with apex $\mathcal{J}_0^{(B)}$.

Let $\mathcal{H}_2$ be any self-dual $\mathcal{H}$-class in $\mathscr{G}_A$ contained in $\mathcal{J}_0$. Notice that this is also a self-dual $\mathcal{H}$-class in $\mathscr{G}_B$ contained in $\mathcal{J}_0^{(B)}$. Let $\mathscr{A}_{\mathcal{H}_2}$ be the corresponding 1- and 2-full 2-subcategory of $\mathscr{G}_A$ (and of $\mathscr{G}_B$), cf. [MMMZ, Subsection 4.2]. By [MMMZ, Theorem 15], there is a bijection between equivalence classes of simple transitive 2-representations of $\mathscr{A}_{\mathcal{H}_2}$ and equivalence classes of simple transitive 2-representations of $\mathscr{G}_B$ with apex $\mathcal{J}_0^{(B)}$, and also of $\mathscr{G}_A$ with apex $\mathcal{J}_0$. The claim follows. □

To prove Theorem 17, one could alternatively use [MMMT, Corollary 12].

Remark 18. An analogue of Theorem 17 is also true in the weakly fiat case, that is when A is just self-injective but not necessarily weakly symmetric. However, the proof requires an adjustment of the results of [MMMZ, Theorem 15] to the case when instead of one diagonal $\mathcal{H}$-cell one considers a diagonal block which is stable under $\star$. One could carefully go through the proof [MMMZ, Theorem 15] and check that everything works.

3.7. **A class of examples.** Fix a positive integer $n > 1$ and let A be the quotient of the path algebra of the cyclic quiver



modulo the ideal generated by all paths of length n. Now we let G be the cyclic group of order n whose generator φ acts on A by sending e_i to e_{i+1} and α_i to α_{i+1} (where we compute indices modulo n).

For $i = 1, 2, \dots, n$, we denote by F_i the indecomposable 1-morphism in $\mathscr{G}_A$ corresponding to tensoring with $Ae_1 \otimes_{\Bbbk} e_iA$ (we omit the idempotents since the action of G is free). Then $(\mathrm{F}_i, \mathrm{F}_{n+1-i})$ is an adjoint pair (and, indeed, biadjoint), for each i. Hence $\mathscr{G}_A$ is fiat.

Note that every subgroup of G is cyclic and $H^2(\mathbb{Z}/k\mathbb{Z}, \Bbbk^*) \cong \Bbbk^*/(\Bbbk^*)^k \cong \{e\}$ since $\Bbbk$ is algebraically closed. Therefore, simple transitive 2-representations of $\mathscr{G}_A$ are in bijection with divisors of n. For $d|n$, the algebra underlying the simple transitive 2-representations of $\mathscr{G}_A$ corresponding to d is the algebra $A^{\langle \varphi^{\frac{n}{d}} \rangle}$ with the obvious action of $\mathscr{G}_A$. Here $A^{\langle \varphi^{\frac{n}{d}} \rangle}$ denotes the invariant subalgebra of A under the action of the subgroup $\langle \varphi^{\frac{n}{d}} \rangle$ of G, which is generated by $\varphi^{\frac{n}{d}}$.

4. Two-element $\mathcal{H}$-cells with no self-adjoint elements

4.1. **Basic combinatorics.**

Proposition 19. *Let $\mathscr{C}$ be a fiat 2-category such that*

- *$\mathscr{C}$ has one object $\mathtt{i}$;*
- *$\mathscr{C}$ has two two-sided cells, each of which is also a right cell and a left cell, one being $\{\mathbb{1}_{\mathtt{i}}\}$ and the other one given by $\{\mathrm{F}, \mathrm{G}\}$ with $\mathrm{F} \not\cong \mathrm{G}$;*
- *$\mathrm{F}^\star \cong \mathrm{G}$.*

Then there exists $n \in \mathbb{Z}_{>0}$ such that

$$\mathrm{FF} \cong \mathrm{FG} \cong \mathrm{GF} \cong \mathrm{GG} \cong (\mathrm{F} \oplus \mathrm{G})^{\oplus n}. \tag{14}$$

Proof. We have

$$\mathrm{FF} \cong \mathrm{F}^{\oplus a_1} \oplus \mathrm{G}^{\oplus a_2}, \qquad \mathrm{FG} \cong \mathrm{F}^{\oplus b_1} \oplus \mathrm{G}^{\oplus b_2},$$
$$\mathrm{GF} \cong \mathrm{F}^{\oplus c_1} \oplus \mathrm{G}^{\oplus c_2}, \qquad \mathrm{GG} \cong \mathrm{F}^{\oplus d_1} \oplus \mathrm{G}^{\oplus d_2},$$

for some $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Z}_{\geq 0}$.

From $\mathrm{F}^\star \cong \mathrm{G}$, we see that $(\mathrm{FG})^\star \cong \mathrm{FG}$ and $(\mathrm{GF})^\star \cong \mathrm{GF}$. This implies $b_1 = b_2 =: b$ and $c_1 = c_2 =: c$. Furthermore, $(\mathrm{FF})^\star \cong \mathrm{GG}$, which implies $a_1 = d_2 =: x$ and $a_2 = d_1 =: y$.

As G is in the same left cell as F, we obtain $y + c > 0$ and $y + b > 0$.

Case 1: $y = 0$**.** In this case we have $c, b > 0$ by the above, and

$$\mathrm{FF} \cong \mathrm{F}^{\oplus x}, \quad \mathrm{FG} \cong \mathrm{F}^{\oplus b} \oplus \mathrm{G}^{\oplus b}, \quad \mathrm{GF} \cong \mathrm{F}^{\oplus c} \oplus \mathrm{G}^{\oplus c}, \quad \mathrm{GG} \cong \mathrm{G}^{\oplus x}.$$

We use this to compute both sides of the isomorphism $(\mathrm{FG})\mathrm{G} \cong \mathrm{F}(\mathrm{GG})$. This yields $b^2 = xb$ (by comparing the coefficients as F) and $b^2 + xb = xb$ (by comparing the coefficients as G). Hence $b = 0$, a contradiction. Therefore this case cannot occur.

Case 2: $y > 0$**.** In this case we have

$$\mathrm{FF} \cong \mathrm{F}^{\oplus x} \oplus \mathrm{G}^{\oplus y}, \quad \mathrm{FG} \cong \mathrm{F}^{\oplus b} \oplus \mathrm{G}^{\oplus b}, \quad \mathrm{GF} \cong \mathrm{F}^{\oplus c} \oplus \mathrm{G}^{\oplus c}, \quad \mathrm{GG} \cong \mathrm{F}^{\oplus y} \oplus \mathrm{G}^{\oplus x}.$$

We use this to compute both sides of the isomorphism $(\mathrm{FG})\mathrm{F} \cong \mathrm{F}(\mathrm{GF})$, and obtain $xc = xb$ (by comparing the coefficients as F) and $yc = yb$ (by comparing the coefficients as G). As $y > 0$, we have $c = b$.

Finally, we compute both sides of the isomorphism $(\mathrm{FG})\mathrm{G} \cong \mathrm{F}(\mathrm{GG})$. This implies $b^2 + by = xy + bx$ (by comparing the coefficients as F) and $b^2 + bx = y^2 + bx$ (by comparing the coefficients as G). As $y > 0$ and $b \geq 0$, from the second equation we deduce $b = y$. Using $y > 0$ and $b = y$, the first equation yields $b = x$. The claim follows. □

4.2. **The algebra of the cell 2-representation.** Let $\mathscr{C}$ be a fiat 2-category as in Proposition 19. Consider the cell 2-representation $\mathbf{C}_{\mathcal{H}}$ of $\mathscr{C}$, where $\mathcal{H} = \{\mathrm{F}, \mathrm{G}\}$. Denote by A its underlying basic algebra with a fixed decomposition $1_A = e_{\mathrm{F}} + e_{\mathrm{G}}$ of the identity into primitive orthogonal idempotents. Let P_{F} and P_{G} denote the corresponding indecomposable projective A-modules and L_{F} and L_{G} their respective simple tops. Note that fiatness of $\mathscr{C}$ implies self-injectivity of A (cf. [KMMZ, Theorem 2]).

From (14) we obtain that the matrix describing the action of both F and G in the cell 2-representation (in the basis of indecomposable projective modules) is

$$\begin{pmatrix} n & n \\ n & n \end{pmatrix}. \tag{15}$$

By [MM5, Lemma 10], the same matrix describes the action of both F and G in the abelianization of the cell 2-representation in the basis of simple modules. Without loss of generality, assume that G is the Duflo involution of the left cell $\mathcal{H}$. In the cell 2-representation, we then have $\mathrm{G}L_{\mathrm{G}} \cong P_{\mathrm{G}}$ and $\mathrm{F}L_{\mathrm{G}} \cong P_{\mathrm{F}}$. This, together with the description of the matrix of the action in (15), shows that

$$[P_{\mathrm{G}} : L_{\mathrm{F}}] = [P_{\mathrm{F}} : L_{\mathrm{F}}] = [P_{\mathrm{G}} : L_{\mathrm{G}}] = [P_{\mathrm{F}} : L_{\mathrm{G}}] = n. \tag{16}$$

Therefore, the Cartan matrix of A is given by (15).

As the bimodules X and Y, representing F and G, respectively, are projective, see [KMMZ, Theorem 2] and [MM5, Lemma 13] for details, we deduce that $Ae_{\mathrm{F}} \otimes_{\Bbbk} e_{\mathrm{G}}A$ appears as a direct summand of X and $Ae_{\mathrm{G}} \otimes_{\Bbbk} e_{\mathrm{G}}A$ appears as a direct summand of Y. Due to $\mathrm{G}^\star \cong \mathrm{F}$, and

$$0 \neq \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(\mathrm{G}L_{\mathrm{G}}, L_{\mathrm{G}}) \cong \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(L_{\mathrm{G}}, \mathrm{F}L_{\mathrm{G}}) \cong \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(L_{\mathrm{G}}, P_{\mathrm{F}}),$$

the algebra A is not weakly symmetric. Furthermore, we have

$$0 = \mathrm{Hom}_A(P_{\mathrm{G}}, L_{\mathrm{F}}) \cong \mathrm{Hom}_A(\mathrm{G}L_{\mathrm{G}}, L_{\mathrm{F}}) \cong \mathrm{Hom}_A(L_{\mathrm{G}}, \mathrm{F}L_{\mathrm{F}}),$$

so $\mathrm{F}L_{\mathrm{F}}$ is a direct sum of copies of P_G. Comparing the Cartan matrix of A with the matrix of the action of F in the basis of simples (both given by (15)), we see that $\mathrm{F}L_{\mathrm{F}} \cong P_{\mathrm{G}}$. Similarly we deduce $\mathrm{G}L_{\mathrm{F}} \cong P_{\mathrm{F}}$. Hence, we have

$$X \cong Ae_{\mathrm{F}} \otimes_{\Bbbk} e_{\mathrm{G}}A \oplus Ae_{\mathrm{G}} \otimes_{\Bbbk} e_{\mathrm{F}}A \quad \text{and} \quad Y \cong Ae_{\mathrm{F}} \otimes_{\Bbbk} e_{\mathrm{F}}A \oplus Ae_{\mathrm{G}} \otimes_{\Bbbk} e_{\mathrm{G}}A. \tag{17}$$

4.3. **Functors isomorphic to the identity endomorphism of** 2**-representations.** In this subsection, we will formulate a general result for an arbitrary finitary 2-category $\mathscr{C}$. This result will be needed for Subsection 4.5. For simplicity, we assume that $\mathscr{C}$ has only one object $\mathtt{i}$. Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$. Let $(\mathrm{Id}_{\mathbf{M}}, \eta) : \mathbf{M} \to \mathbf{M}$ be the identity endomorphism of $\mathbf{M}$. Here η is given by the family $\{\eta_{\mathrm{F}},\, \mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{i})\}$ of natural transformations where each η_{F} is the identity natural transformation of $\mathbf{M}(\mathrm{F})$.

Lemma 20. *Let* $\Phi : \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{i})$ *be a functor isomorphic to the identity functor* $\mathrm{Id}_{\mathbf{M}(\mathtt{i})}$. *Then there exists a family of natural isomorphisms* $\{\zeta_{\mathrm{F}}, \mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{i})\} =: \zeta$ *such that* (Φ, ζ) *is an endomorphism of the* 2*-representation* $\mathbf{M}$.

Proof. Note that $\Phi \cong \mathrm{Id}_{\mathbf{M}(\mathtt{i})}$ as a functor. Let $\theta : \mathrm{Id}_{\mathbf{M}(\mathtt{i})} \to \Phi$ be a fixed natural isomorphism and set $\nu := \theta^{-1}$. For any 1-morphisms F, G and 2-morphism $\alpha : \mathrm{F} \to \mathrm{G}$, consider the diagram

$$\begin{array}{ccccc} \Phi \circ \mathbf{M}(\mathrm{F}) & \xrightarrow{\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{F})}} & \mathbf{M}(\mathrm{F}) & \xrightarrow{\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \theta} & \mathbf{M}(\mathrm{F}) \circ \Phi \\ {\scriptstyle \mathrm{id}_{\Phi} \circ_{\mathrm{h}} \mathbf{M}(\alpha)} \downarrow & & {\scriptstyle \mathbf{M}(\alpha)} \downarrow & & \downarrow {\scriptstyle \mathbf{M}(\alpha) \circ_{\mathrm{h}} \mathrm{id}_{\Phi}} \\ \Phi \circ \mathbf{M}(\mathrm{G}) & \xrightarrow{\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}} & \mathbf{M}(\mathrm{G}) & \xrightarrow{\mathrm{id}_{\mathbf{M}(\mathrm{G})} \circ_{\mathrm{h}} \theta} & \mathbf{M}(\mathrm{G}) \circ \Phi. \end{array} \tag{18}$$

Here in the middle column we use the fact that, for any 1-morphism H, we have

$$\mathrm{Id}_{\mathbf{M}(\mathtt{i})} \circ_{\mathrm{h}} \mathbf{M}(\mathrm{H}) = \mathbf{M}(\mathrm{H}) = \mathbf{M}(\mathrm{H}) \circ_{\mathrm{h}} \mathrm{Id}_{\mathbf{M}(\mathtt{i})}.$$

Diagram (18) commutes thanks to the interchange law, indeed, both paths in the left square are equal to $\nu \circ_{\mathrm{h}} \mathbf{M}(\alpha)$ and both paths in the right square are equal to $\mathbf{M}(\alpha) \circ_{\mathrm{h}} \theta$.

For each 1-morphism F, define

$$\zeta_{\mathrm{F}} := (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{F})}) : \Phi \circ \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{F}) \circ \Phi. \tag{19}$$

Now we claim that (Φ, ζ) is an endomorphism of the 2-representation $\mathbf{M}$. Commutativity of (18) gives $(\mathbf{M}(\alpha) \circ_{\mathrm{h}} \mathrm{id}_{\Phi}) \circ_{\mathrm{v}} \zeta_{\mathrm{F}} = \zeta_{\mathrm{G}} \circ_{\mathrm{v}} (\mathrm{id}_{\Phi} \circ_{\mathrm{h}} \mathbf{M}(\alpha))$. We are hence left to check the equality

$$\zeta_{\mathrm{F} \circ \mathrm{G}} = (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \zeta_{\mathrm{G}}) \circ_{\mathrm{v}} (\zeta_{\mathrm{F}} \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}). \tag{20}$$

Here, by definition, we have

$$\begin{aligned} \mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \zeta_{\mathrm{G}} &= (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \\ &= (\mathrm{id}_{\mathbf{M}(\mathrm{F} \circ \mathrm{G})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \end{aligned}$$

and

$$\begin{aligned} \zeta_{\mathrm{F}} \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})} &= (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \theta \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \circ_{\mathrm{v}} (\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \\ &= (\mathrm{id}_{\mathbf{M}(\mathrm{F})} \circ_{\mathrm{h}} \theta \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \circ_{\mathrm{v}} (\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{F} \circ \mathrm{G})}). \end{aligned}$$

Now (20) follows from the fact that $\nu\theta = \mathrm{Id}_{\mathbf{M}(\mathtt{i})}$. The proof is complete. □

Remark 21.

(i) The natural isomorphism $\theta : \mathrm{Id}_{\mathbf{M}(\mathtt{i})} \to \Phi$ defines a modification from $(\mathrm{Id}_{\mathbf{M}}, \eta)$ to (Φ, ζ) whose inverse is given by $\nu : \Phi \to \mathrm{Id}_{\mathbf{M}(\mathtt{i})}$. Indeed, for any 1-morphisms F, G and any 2-morphism $\alpha : \mathrm{F} \to \mathrm{G}$, we have

$$\begin{aligned} (\mathbf{M}(\alpha) \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} \eta_{\mathrm{F}} &= \mathbf{M}(\alpha) \circ_{\mathrm{h}} \theta \\ &= (\mathrm{id}_{\mathbf{M}(\mathrm{G})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\mathbf{M}(\alpha) \circ_{\mathrm{h}} \mathrm{Id}_{\mathbf{M}(\mathtt{i})}) \\ &= (\mathrm{id}_{\mathbf{M}(\mathrm{G})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\mathrm{Id}_{\mathbf{M}(\mathtt{i})} \circ_{\mathrm{h}} \mathbf{M}(\alpha)) \\ &= (\mathrm{id}_{\mathbf{M}(\mathrm{G})} \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} (\nu \circ_{\mathrm{h}} \mathrm{id}_{\mathbf{M}(\mathrm{G})}) \circ_{\mathrm{v}} (\theta \circ_{\mathrm{h}} \mathbf{M}(\alpha)) \\ &= \zeta_{\mathrm{G}} \circ_{\mathrm{v}} (\theta \circ_{\mathrm{h}} \mathbf{M}(\alpha)), \end{aligned}$$

(ii) Any invertible modification θ from $(\mathrm{Id}_{\mathbf{M}}, \eta)$ to some $(\Phi, \zeta) \in \mathrm{End}_{\mathscr{C}\text{-afmod}}(\mathbf{M})$ defines a natural isomorphism from $\mathrm{Id}_{\mathbf{M}(\mathtt{i})}$ to Φ. Moreover, from the fact that $(\mathbf{M}(\mathrm{id}_{\mathrm{F}}) \circ_{\mathrm{h}} \theta) \circ_{\mathrm{v}} \eta_{\mathrm{F}} = \zeta_{\mathrm{F}} \circ_{\mathrm{v}} (\theta \circ_{\mathrm{h}} \mathbf{M}(\mathrm{id}_{\mathrm{F}}))$, it follows that each ζ_{F} is uniquely defined by (19) with $\nu := \theta^{-1}$.

4.4. **Inductive limit construction for 2-representations.** Assume that we are in the same setup as in Subsection 4.3. For any finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$, we denote by $\overline{\mathbf{M}}^{\mathsf{pr}}$ the 2-subrepresentation of $\overline{\mathbf{M}}$ with the action of $\mathscr{C}$ restricted to the category $\overline{\mathbf{M}}^{\mathsf{pr}}(\mathtt{i})$ consisting of projective objects in $\overline{\mathbf{M}}(\mathtt{i})$. There exists a strict 2-natural transformation $\Upsilon\colon \mathbf{M} \to \overline{\mathbf{M}}^{\mathsf{pr}}$ given by sending an object X to the diagram $0 \to X$ with the obvious assignment on morphisms. Similarly to [MaMa, Subsection 5.8], we have a direct system

$$\mathbf{M} \to \overline{\mathbf{M}}^{\mathsf{pr}} \to \overline{(\overline{\mathbf{M}}^{\mathsf{pr}})}^{\mathsf{pr}} \to \cdots, \tag{21}$$

where each arrow is given by Υ with $\mathbf{M}$ replaced by the starting point corresponding to this arrow. We denote by $\underrightarrow{\mathbf{M}}$ the inductive limit of (21). This is a 2-representation of $\mathscr{C}$ and the natural embedding of $\mathbf{M}$ into $\underrightarrow{\mathbf{M}}$ is an equivalence. Let $\mathcal{L}$ be a left cell of $\mathscr{C}$ and $\mathbf{C}_{\mathcal{L}} := \mathbf{N}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}$ the corresponding cell 2-representation. By Yoneda Lemma, see [MM2, Lemma 9], for any object X in $\mathbf{M}(\mathtt{i})$ there exists a strict 2-natural transformation $\Lambda_X\colon \mathbf{P}_{\mathtt{i}} \to \mathbf{M}$ which sends $\mathbb{1}_{\mathtt{i}}$ to X and, moreover, any morphism $f\colon X \to Y$ in $\mathbf{M}(\mathtt{i})$ extends to a modification $\theta_f\colon \Lambda_X \to \Lambda_Y$. If $\mathcal{I}_{\mathcal{L}}$ annihilates X, then Λ_X induces a strict 2-natural transformation Λ'_X from $\mathbf{C}_{\mathcal{L}}$ to $\mathbf{M}$. Indeed, we have the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{N}_{\mathcal{L}} & \xhookrightarrow{\ \Xi\ } & \mathbf{P}_{\mathtt{i}} & \xrightarrow{\ \Lambda_X\ } & \mathbf{M}. \\
{\scriptstyle\Pi}\downarrow & & & \nearrow{\scriptstyle \Lambda'_X} & \\
\mathbf{C}_{\mathcal{L}} & & & &
\end{array}$$

If $\mathcal{I}_{\mathcal{L}}$ also annihilates Y, then Λ_Y gives rise to a strict 2-natural transformation Λ'_Y from $\mathbf{C}_{\mathcal{L}}$ to $\mathbf{M}$ such that $\Lambda_Y\Xi = \Lambda'_Y\Pi$. Due to surjectivity of Π, the modification $\theta_f \circ_{\mathrm{h}} \mathrm{id}_\Xi$ from $\Lambda_X\Xi = \Lambda'_X\Pi$ to $\Lambda_Y\Xi = \Lambda'_Y\Pi$ induces a modification θ'_f from Λ'_X to Λ'_Y in $\mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\mathbf{C}_{\mathcal{L}}, \mathbf{M})$. By functoriality of the abelianization, via the limiting construction (21) we thus obtain two 2-natural transformations $\underrightarrow{\Lambda'_X}, \underrightarrow{\Lambda'_Y} \in \mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\underrightarrow{\mathbf{C}_{\mathcal{L}}}, \underrightarrow{\mathbf{M}})$ and the modification $\underrightarrow{\theta'_f}\colon \underrightarrow{\Lambda'_X} \to \underrightarrow{\Lambda'_Y}$.

4.5. **Symmetries of the cell 2-representation.**

Lemma 22. *The annihilators in $\mathscr{C}$ of L_{F} and L_{G} coincide.*

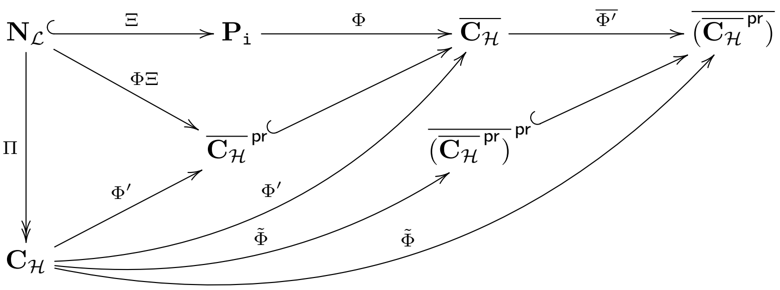
Proof. Since G is the Duflo involution in $\mathcal{H}$, it follows from [MM2, Subsection 6.5] that the annihilator of L_{F} is contained in the annihilator of L_{G} (as the latter is a certain unique maximal left ideal by [MM2, Proposition 21]). Furthermore, the evaluation at L_{G}, inside the abelianized cell 2-representation, of $\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}_1, \mathrm{H}_2)$ is full for all $\mathrm{H}_1, \mathrm{H}_2 \in \{\mathrm{F}, \mathrm{G}\}$ by [MM2, Subsection 6.5].

Hence, if the annihilator of L_{F} were strictly contained in the annihilator of L_{G}, the dimension of the endomorphism space (in the cell 2-representation) of $(\mathrm{F} \oplus \mathrm{G})L_{\mathrm{F}}$ would be strictly bigger than the dimension of the endomorphism space of $(\mathrm{F} \oplus \mathrm{G})L_{\mathrm{G}}$. However, from Subsection 4.2 we know that $(\mathrm{F} \oplus \mathrm{G})L_{\mathrm{F}} \cong (\mathrm{F} \oplus \mathrm{G})L_{\mathrm{G}}$. The claim follows. □

On the one hand, by [MM2, Lemma 9], sending $\mathbb{1}_{\mathtt{i}}$ to L_{F} extends to a strict 2-natural transformation $\Phi\colon \mathbf{P}_{\mathtt{i}} \to \overline{\mathbf{C}_{\mathcal{H}}}$. By Lemma 22, we know that $\Phi\Xi$ factors through $\mathbf{C}_{\mathcal{H}}$ and obtain a strict 2-natural transformation Φ' from $\mathbf{C}_{\mathcal{H}}$ to $\overline{\mathbf{C}_{\mathcal{H}}}$. Note that Φ sends both F and G to projective objects in $\overline{\mathbf{C}_{\mathcal{H}}}(\mathtt{i})$. Therefore Φ' is also a strict

2-natural transformation from $\mathbf{C}_{\mathcal{H}}$ to $\overline{\mathbf{C}_{\mathcal{H}}}^{\mathsf{pr}}$ and we have the following commutative diagram:

(22)



Applying the procedure in Subsection 4.4 to $\Phi' \colon \mathbf{C}_{\mathcal{H}} \to \overline{\mathbf{C}_{\mathcal{H}}}^{\mathsf{pr}}$, we obtain a strict 2-natural transformation $\underrightarrow{\Phi'}$ in $\mathrm{End}_{\mathscr{C}\text{-afmod}}(\underrightarrow{\mathbf{C}_{\mathcal{H}}})$ which swaps the isomorphism classes of indecomposable projectives.

On the other hand, sending $\mathbb{1}_{\mathtt{i}}$ to L_{G} extends to a strict 2-natural transformation $\Psi \colon \mathbf{P}_{\mathtt{i}} \to \overline{\mathbf{C}_{\mathcal{H}}}$, and the latter induces a strict 2-natural transformation Ψ' from $\mathbf{C}_{\mathcal{H}}$ to $\overline{\mathbf{C}_{\mathcal{H}}}$ (which factors through $\overline{\mathbf{C}_{\mathcal{H}}}^{\mathsf{pr}}$). For Ψ, we have a diagram similar to the one in (22). Note that $\overline{\Psi'}\Psi(\mathbb{1}_{\mathtt{i}}) = L_{\mathrm{G}}$ and $\overline{\Phi'}(L_{\mathrm{F}}) \cong L_{\mathrm{G}}$. For a fixed isomorphism $\alpha \colon L_{\mathrm{G}} \to \overline{\Phi'}(L_{\mathrm{F}})$, by Subsection 4.4 there exists an invertible modification ϑ from $\tilde{\Psi} = \overline{\Psi'}\Psi'$ to $\tilde{\Phi} = \overline{\Phi'}\Phi'$ (here both equalities are in $\mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\mathbf{C}_{\mathcal{H}}, \overline{(\overline{\mathbf{C}_{\mathcal{H}}}^{\mathsf{pr}})}^{\mathsf{pr}})$). Note that the limiting construction (21) applied to $\tilde{\Psi}$ gives a functor isomorphic to $\mathrm{Id}_{\underrightarrow{\mathbf{C}_{\mathcal{H}}}}$. Using Subsections 4.3 and 4.4, we thus get an invertible modification $\underrightarrow{\vartheta}$ from $\mathrm{Id}_{\underrightarrow{\mathbf{C}_{\mathcal{H}}}}$ to $(\underrightarrow{\Phi'})^2$. Following [MaMa, Lemma 18] and the proof of [MaMa, Proposition 19], we obtain that

(a) for any $\upsilon \in \mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\mathrm{Id}_{\underrightarrow{\mathbf{C}_{\mathcal{H}}}}, (\underrightarrow{\Phi'})^2)$, we have $\mathrm{id}_{(\underrightarrow{\Phi'})^2} \circ_{\mathrm{h}} \upsilon = \upsilon \circ_{\mathrm{h}} \mathrm{id}_{(\underrightarrow{\Phi'})^2}$;

(b) there exists an invertible modification $\upsilon \in \mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\mathrm{Id}_{\underrightarrow{\mathbf{C}_{\mathcal{H}}}}, (\underrightarrow{\Phi'})^2)$ such that we have either $\mathrm{id}_{\underrightarrow{\Phi'}} \circ_{\mathrm{h}} \upsilon = \upsilon \circ_{\mathrm{h}} \mathrm{id}_{\underrightarrow{\Phi'}}$ or $\mathrm{id}_{\underrightarrow{\Phi'}} \circ_{\mathrm{h}} \upsilon = -\upsilon \circ_{\mathrm{h}} \mathrm{id}_{\underrightarrow{\Phi'}}$.

Note that $(\underrightarrow{\Phi'})^2$ preserves the isomorphism classes of projectives and hence defines an auto-equivalence of $\underrightarrow{\mathbf{C}_{\mathcal{H}}}$ which is isomorphic to the identity. Therefore $\underrightarrow{\Phi'}$ induces an automorphism φ of A and such that φ^2, corresponding to $(\underrightarrow{\Phi'})^2$, is an inner automorphism of A, cf. [Zi, Lemma 1.10.9]. Assume that the inner automorphism φ^2 is of the form $x \mapsto axa^{-1}$, where $x \in A$, for some fixed invertible element $a \in A$. Similarly to the paragraph before [KMMZ, Proposition 39], there exists an element $b \in A$ which is a polynomial in a^{-1} and such that $b^2 = a^{-1}$. Let σ be the inner automorphism of A given by $x \mapsto bxb^{-1}$, for $x \in A$.

Lemma 23. *We have* $(\sigma\varphi)^4 = \mathrm{id}_A$.

Proof. The obvious fact that φ and φ^2 commute is equivalent to the requirement that $t := \varphi(a^{-1})a$ belongs to the center of A. We have

$$\varphi(t) = \varphi^2(a^{-1})\varphi(a) = aa^{-1}a^{-1}\varphi(a) = a^{-1}\varphi(a) = t^{-1}.$$

Therefore $\varphi^2(t) = t = a\varphi(a^{-1})aa^{-1} = a\varphi(a^{-1})$, which implies that $\varphi(a^{-1})$ and a commute. Consequently, $\varphi(a^{-1})$ and a^{-1} commute. This implies that any polynomial in $\varphi(a^{-1})$ commutes with any polynomial in a^{-1}. Therefore $\varphi(b)$ and b commute and thus $\varphi(b^{-1})$ and b commute as well. Hence the elements a, a^{-1}, b, b^{-1}, $\varphi(a)$, $\varphi(a^{-1})$, $\varphi(b)$, $\varphi(b^{-1})$ all commute.

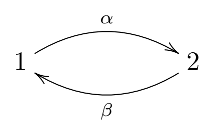
A direct computation shows that the action of $(\sigma\varphi)^4$ on A is given by conjugation with

$$b\varphi(b)aba^{-1}\varphi(a)\varphi(b)\varphi(a^{-1})a^2.$$

Using commutativity of the factors, this reduces to $a\varphi(a^{-1})$ which is central. The claim follows. □

The functor of twisting A-modules by σ is isomorphic to the identity functor as σ is inner. By Lemma 20, the functor of twisting by σ gives rise to an endomorphism Σ of $\underrightarrow{\mathbf{C}_{\mathcal{H}}}$ which preserves the isomorphism classes of projectives. Then the 2-natural transformation $\Omega := \Sigma\underrightarrow{\Phi'} \in \mathrm{End}_{\mathscr{C}\text{-afmod}}(\underrightarrow{\mathbf{C}_{\mathcal{H}}})$ induces an automorphism on A given by $\sigma\varphi$. We denote this automorphism by ι.

Example 24. Let A be the quotient of the path algebra of the quiver



modulo the relations $\alpha\beta = \beta\alpha = 0$. Let φ be the automorphism of A defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(\alpha) = -\beta$ and $\varphi(\beta) = \alpha$. Then $\varphi^4 = \mathrm{id}_A$ but $\varphi^2 \neq \mathrm{id}_A$. In fact, φ^2 is conjugation by $a = e_1 - e_2$. Note that the element $\varphi(a^{-1})a = -e_1 - e_2$ is central. This example shows that $\underrightarrow{\Phi'}$ does not necessarily correspond to an automorphism of order 2.

4.6. **Connection to $\mathscr{G}_A$.** Set G to be the cyclic group generated by ι (note that $|G| = 2$ or $|G| = 4$) and consider the fiat 2-category $\mathscr{G}_A$, where A is the underlying algebra of $\mathbf{C}_{\mathcal{H}}$. Let $\mathscr{H}_A$ denote the full and faithful 2-subcategory of $\mathscr{G}_A$ generated by $(A, \tilde{\pi}_{1_{\hat{G}}})$ and 1-morphisms in the two-sided cell $\mathcal{J}_0$, referring to Subsection 3.3 and Subsection 3.4 for notation.

Theorem 25. *If $\mathscr{C}$ is $\mathcal{H}$-simple, then $\mathscr{C}$ is biequivalent to a 2-subcategory of $\mathscr{H}_A$.*

Proof. As mentioned above, $\underrightarrow{\mathbf{C}_{\mathcal{H}}}$ is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{H}}$. As $\mathscr{C}$ is $\mathcal{H}$-simple, the 2-representation $\underrightarrow{\mathbf{C}_{\mathcal{H}}}$ gives a faithful 2-functor from $\mathscr{C}$ to $\mathscr{C}_A$. Note that the 1-morphisms F and G are represented, respectively, by X, Y in (17) under the 2-functor $\underrightarrow{\mathbf{C}_{\mathcal{H}}}$. Assume that the family of natural isomorphisms

$$\eta := \{\eta_{\mathrm{H}} \colon \Omega \circ \underrightarrow{\mathbf{C}_{\mathcal{H}}}(\mathrm{H}) \to \underrightarrow{\mathbf{C}_{\mathcal{H}}}(\mathrm{H}) \circ \Omega, \ \mathrm{H} \in \mathscr{C}(\mathtt{i}, \mathtt{i})\}$$

is the data associated to the 2-natural transformation $\Omega \in \mathrm{End}_{\mathscr{C}\text{-afmod}}(\underrightarrow{\mathbf{C}_{\mathcal{H}}})$ constructed above. Thus, for any 2-morphism $\alpha : \mathrm{H} \to \mathrm{K}$ in $\mathscr{C}$, we have

$$(\underrightarrow{\mathbf{C}_{\mathcal{H}}}(\alpha) \circ_{\mathrm{h}} \Omega) \circ_{\mathrm{v}} \eta_{\mathrm{H}} = \eta_{\mathrm{K}} \circ_{\mathrm{v}} (\Omega \circ_{\mathrm{h}} \underrightarrow{\mathbf{C}_{\mathcal{H}}}(\alpha)). \tag{23}$$

Due to the fact that Ω swaps the isomorphism classes of projectives in $\underrightarrow{\mathbf{C}_{\mathcal{H}}}(\mathtt{i})$, all A-A-bimodule homomorphisms corresponding to non-zero 2-morphisms in $\mathscr{C}$ are symmetric in the sense that they are uniquely determined by their images on the representatives of distinct G-orbits of indecomposable direct summands of the source and the images on the remaining indecomposable summands of the source can be obtained by (23). □

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