Debt constraints and monetary policy*

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Abstract

In the present paper we show how simple monetary policies can mitigate real effects of credit frictions. We consider stationary overlapping generations economies in which consumers are not equally efficient in producing capital and cannot commit to repay loans. The presence of money in itself does not mitigate the real effects of credit frictions. Equilibrium allocations are generally not Pareto optimal unless the returns on money and capital production are identical for more productive consumers. However, printing money and distributing it to young consumers increases their incomes allowing young more productive consumers to produce more capital. Consequently money printing increases output.

Keywords Financial Frictions · Monetary Policy · Overlapping Generations Economies

JEL Classification D5 · E4 · E5

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1 Introduction

Overview of paper Simple monetary policies in form of printing money can mitigate the effects of frictions in credit markets in situations like the financial crisis where agents had limited access to credit. How money transfers are distributed is important for the effectiveness of these policies. In the present paper we study economies in which consumers cannot commit to repay loans and examine the effects on output and welfare of monetary policies in form of printing money and distributing it to agents.

We consider overlapping generations (OG) economies with production. There are four commodities: a consumption good, capital, labor and money. Consumers live for two dates, consume both when young and old, and work when young. Moreover they produce capital by use of the consumption good. Some consumers are more productive than others in production of capital. Firms produce the consumption good by use of capital and labor. Productivities of consumers are common knowledge. Therefore from an efficiency point of view less productive consumers should leave capital production to more productive consumers.

We assume that there is a friction in the credit market: consumers cannot commit to repay loans. Since consumers live for two dates, usual penalties such as exclusion from taking loans or exclusion from both taking and making loans are ineffective. Hence debt is constrained to zero. Consequently there is no transfer of resources between consumers in the same generation so both kinds of consumers can very well be investing in capital production. The presence of money does not induce less productive consumers to move their savings from capital production by themselves to capital production by more productive consumers. However printing money and distributing it to more productive consumers enables them to buy consumption goods from less productive consumers. Thereby more productive consumers produce more capital and less productive consumers hold more money. Hence printing money mitigates the effects of the debt constraints.

Equilibria are prices, consumption plans and production plans such that consumers maximize utilities, firm maximize profits and markets clear. Money need not be valuable in equilibrium. Steady states are equilibria where consumption and production are constant across dates. In Theorem 1 we show that there are steady states where money is valuable. To study the impact of changing monetary policy from the passive policy with no transfers to an active policy with non-trivial transfers we consider a scenario with the passive policy before some date and an active policy after that date. There are two steps: first in Theorem 2 we show monetary steady states with active monetary policies exist and active monetary policies increase output compared to the passive monetary policy; and, second we show that output increases in the transition from a steady state with the passive policy to a steady state with the active policy. Active policies have several effects on consumers: additional income for consumers who receive transfers as well as higher wages and lower returns on savings for all consumers.

Our results in Theorems 1 and 2 are obtained under reasonably general assumptions. In Section 5 we study an example to further illustrate our findings as well as the welfare implications of monetary policies. Consumers consume only when old. Thereby the negative effect of monetary policies on the returns on savings is maximized. Consumption goods are produced by use of Cobb-Douglas production functions. Scenarios with transfers to all young consumers and transfers solely to more productive young consumers are considered. If the elasticity of capital is larger than the fraction of more productive consumers, then printing money increases welfare of all consumers in both scenarios.

Returns on money and producing capital are determined in equilibrium. Unless these returns are equal for the more productive consumers, the two kinds of consumers face different returns on savings. Therefore it is possible to improve welfare by changing consumption at any two consecutive dates without changing production at any date or consumption at any other date than the two consecutive dates. In general, monetary policies in form of printing money and distributing it to young consumers cannot make returns equal. However these policies are simple and mitigate the real effects of debt constraints. Monetary policies involving money taxes eliminate the need for printing money, but enforcement of tax payment is needed. Hence institutions enforcing payment of taxes have to be available.

Related literature Our model is a degenerate example of a model with endogenous debt limits in which debt limits are zero because consumers live for two dates. However if consumers lived longer and the penalty for not repaying loans was exclusion from both taking and making loans, then debt constraints would become endogenous and depend on future prices. Kehoe and Levine (1993) study pure exchange optimal growth economies where consumers cannot commit to repay loans and defaulting consumers are excluded from future trading in financial markets.

Kocherlakota (2009), Farhi and Tirole (2012), Liu and Wang (2014) and Miao and Wang (2012, 2017) consider economies with endogenous debt constraints where some assets may be used as collateral. The assets used as collateral vary from worthless land to output of and shares in firms taking loans. The main finding is that bubbles in the assets used as collateral can mitigate the effects of debt constraints.

Kunieda (2008) and Martin and Ventura (2012, 2016) consider OG economies with production where consumers differ with respect to their productivity and face debt constraints. In Kunieda (2008) there is a continuum of different productivities and dynamic efficiency of steady-state equilibria is studied. With constant nominal money supply, too few consumers engage in capital production and too many consumers hold money. Monetary policies in form of distributing money to less productive old consumers work in that they decrease the return on holding money, so some of the less productive consumers move their savings to capital production from money. It is shown that there is a monetary policy that achieves constrained dynamic efficiency. However, the policy is not a Pareto improvement compared to constant money supply. Most of the analysis is carried out with Cobb-Douglas economies and focuses on steady states. We study a different channel through which monetary policy can work. Indeed, we show monetary policies in form of distributing money to young consumers work in that they enable more productive young consumers to produce more capital. The channel is that the monetary transfers increase incomes of more productive young consumers so they increase their savings. Since more productive young consumers have a comparative advantage in capital production, they use their increases in savings to increase their engagement in capital production. Most of the analysis is carried out for general economies and focuses on transitional dynamics as well as steady states.

In Martin and Ventura (2012, 2016), there are two productivities and bubbles on assets are studied. Bubbles are stochastic in that old bubbles may die and new bubbles may arise. These bubbles can be interpreted as bubbles in financial assets like shares, but they cannot be interpreted as money because they affect some assets and not all assets. Such bubbles affect the value of assets used as collateral in credit transactions. Their welfare analysis focuses primarily on steady states. In terms of policy, they look at a possible macro-prudential role of a Lender of Last Resort, which guarantees credit when collateral is scarce and taxes credit when it is excessive. While we consider a similar mechanism, we study equilibria where money is a deterministic bubble that is controlled by monetary policy. Also, our setup is more general and we analyze the existence of equilibria and the real effects of monetary policy in the steady state and in transition. Our aim is to provide a rationale for an active monetary policy.

One strand of the literature on the role of money emphasizes differences in liquidity of financial assets. Kiyotaki and Moore (2012) consider optimal growth economies with firms operated by entrepreneurs under endogenous debt constraints. Opportunities to produce capital arrive randomly. Entrepreneurs can finance production of capital by selling their portfolio of money and equity in other firms or by issuing equity. However they can sell only a fraction of their equity in other firms and commit to pay their shareholders only a fraction of future returns on capital. The fraction of future returns on capital that can be used as collateral varies randomly. Printing money and distributing it like helicopter drops has no real effects.

Another strand is formed by the New Monetarist literature. There, micro-founded search-theoretical models of money are often used to evaluate monetary policies. For these models Gu et al. (2016) find that money and credit are perfect substitutes and that this finding is robust to variations in policy, debt limit and use of collateral. Chiu and Molico (2010) find that mildly expansionary redistributive monetary policy can potentially be welfare improving by relaxing the liquidity constraint of some agents. Williamson (2008) finds that market segregation, where some agents have access to money markets and other do not, implies that printing money and distributing it through money markets have distributional effects. In our economies, the presence of money is not a perfect remedy for debt constraints, and the presence of perfectly functioning credit markets would not necessarily eliminate dynamic inefficiency of equilibrium allocations. Consequently money and credit are not perfect substitutes.

Sorger (2005) considers stability and determinacy of monetary steady states in OG economies with homogeneous consumers and without financial frictions. Monetary policy is described by inflation targeting or inflation forecast targeting. Unlike conventional wisdom, it is shown that active inflation forecast targeting can lead to indeterminacy. Monetary policy is implemented by printing money and distributing it to consumers when old. We consider heterogeneous consumers who cannot commit to repay debt. Monetary policy that distributes money to young consumers is shown to be welfare-improving. We show that real outcomes can be different for the same inflation rate, and that the real effect of printing money depends on how it is distributed.

Plan of the paper In Section 2 our model is introduced. In Section 3 equilibria, steady states and equilibrium dynamics are defined and studied for economies without money printing. In Section 4 the effects of printing money and distributing it to young more productive consumers are studied. In Section 5 an example is considered and the output and welfare effects of monetary policies consisting of printing money and distributing it to young consumers, both more productive and less productive, are worked out. Section 6 contains some final remarks.

2 The model

We consider stationary overlapping generations economies with perfectly competitive markets. Our economies have two distinctive features: consumers are not equally efficient in producing capital and consumers cannot commit to repay loans. In the present section we introduce our model and examine the decision problems of the agents.

Setup

Time is discrete and extends from $-\infty$ to $+\infty$. At every date there are a consumption good, capital, labor and possibly money. The consumption good is used as numeraire. Let $(q_t, r_t, w_t) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$ be the prices of money, capital and labor at date t. Markets are assumed to be perfectly competitive.

At every date a continuum of identical consumers is born. They live for two subsequent dates, consume when young and old and work when young. The mass of consumers in every generation is normalized to one. A consumer is described by her consumption set $X = \mathbb{R}^2_{++}$, homothetic utility function $u : X \to \mathbb{R}$ and her endowment of one unit of labor when young. There is no utility or disutility associated with working so the labor supply is equal to one.

Consumers are supposed to satisfy the following assumptions:

(C.1) $u \in C^2(X, \mathbb{R})$ with $Du(c) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ for all c and $v^T D^2 u(c) v < 0$ for all c and $v \neq 0$ with $v \cdot Du(c) = 0$.

(C.2) $u^{-1}(a) = \{ c \in X \mid u(c) = a \}$ is closed in $\mathbb{R}^2$ for all a.

The first assumption states that the utility function is twice differentiable, strongly monotone and strictly quasi-concave. The second assumption implies indifference curves do not tend to the axes.

Savings can be in the consumption good and in money. Consumers in generation t can transform the consumption good x_t at date t into capital k_t at date $t+1$ by use of a linear technology $k_t = \beta x_t$ with $\beta > 0$ and sell the capital to firms at price r_{t+1}. The real return on producing capital is $r_{t+1}\beta$. Moreover consumers can exchange the consumption good for money m_t. Money is a durable good that yields no utility. For $q_t > 0$ the real return on money is q_{t+1}/q_t.

There is a continuum of identical firms that transform capital and labor at date t into the consumption good at date t. A firm is described by its production function $F : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ with $Y = F(K, L)$. Firms are supposed to satisfy the following assumptions:

(F.1) $F(aK, aL) = aF(K, L)$ for all $a > 0$ and (K, L).

(F.2) $F \in C^2(\mathbb{R}^2_{++}, \mathbb{R}_{++})$ with $DF(K, L) \in \mathbb{R}^2_{++}$ for all (K, L) and $v^T D^2 F(K, L) v < 0$ for all (K, L) and $v \neq 0$ with $v \cdot DF(K, L) = 0$.

(F.3) $\lim_{K \to 0} D_K F(K, 1) = \lim_{L \to 0} D_L F(1, L) = \infty$ and $\lim_{K \to \infty} D_K F(K, 1) = \lim_{L \to \infty} D_L F(1, L) = 0$.

The first assumption states that the production function has constant returns to scale. The second assumption states that the production function is twice differentiable, strongly monotone and strictly quasi-concave. The third assumption is the standard Inada conditions. Capital depreciates completely during production.

Our economies have two distinctive features. The first feature is that there are two technologies for producing capital, β_L and β_H, with $\beta_L < \beta_H$. Some consumers, denoted L-consumers, have access to the low-productive technology, and others, denoted H-consumers, have access to the high-productive technology. The proportions of consumers with access to the two technologies are $\delta_L, \delta_H > 0$.

The second feature is that consumers cannot commit to repay loans. Therefore L-consumers are not willing to lend their savings to H-consumers although the latter have access to a superior technology compared with the former. Consequently, consumers save by producing capital on their own or by holding money, but not by lending. From an efficiency point of view L-consumers should lend to H-consumers rather than produce capital themselves.

An economy is a description of consumers and firms $\mathscr{E} = (u, \beta_L, \beta_H, \delta_L, \delta_H, F)$. In Sections 3 and 3 there is a fixed stock of money $M > 0$ and in Section 4 there is money printing.

The consumer problem

The problem of an i-consumer with $i \in \{L, H\}$ in generation t is

$$\max_{(c^y_{it}, c^o_{it}, m_{it}, x_{it}, k_{it})} \quad u(c^y_{it}, c^o_{it})$$
$$\text{s.t.} \quad \left\{ \begin{array}{rcl} c^y_{it} + q_t m_{it} + x_{it} & \leq & w_t \\ c^o_{it} & \leq & q_{t+1} m_{it} + r_{t+1} k_{it} \\ k_{it} & \leq & \beta_i x_{it} \\ m_{it}, x_{it} & \geq & 0. \end{array} \right. \tag{1}$$

A consumption plan for an i-consumer in generation t is a consumption bundle, input and output in capital production and nominal savings $(c_{it}, x_{it}, k_{it}, m_{it})$ satisfying the inequalities in Problem (1).

Since the utility function is homothetic implying the demand function is homogeneous of degree one in income, it is helpful to consider the following normalized consumer problem

$$\max_{(c^y, c^o)} \quad u(c^y, c^o)$$
$$\text{s.t.} \quad pc^y + c^o \leq p.$$

Assumptions (C.1) and (C.2) imply there is a differentiable function $f : \mathbb{R}_{++} \to X$ such that $f(p) = (f^y(p), f^o(p))$ is a solution to the problem (see Tvede, 2010). The function $s : \mathbb{R}_{++} \to]0,1[$ defined by $s(p) = 1 - f^y(p)$ is related to the function f by $f^y(p) = 1 - s(p)$ and $f^o(p) = ps(p)$. The real interest rate is $p - 1$ and if income is one when young and zero when old, then real savings are $s(p)$. Real savings are assumed to be bounded away from zero for high real interest rates:

(C.3) $\liminf_{p\to\infty} s(p) > 0.$

The assumption is satisfied for many utility functions, including Cobb–Douglas and CES utility functions with positive elasticity of substitution.

For convenience, consumption is assumed to be positive both when young and when old. Savings $w_t - c_t^y$ are positive for all real interest rates because consumers have no income when old. However, since we use real savings rather than demand in our analysis, it is not important whether consumers consume both when young and when old $X = \mathbb{R}^2_{++}$ or solely when old $X = \{0\} \times \mathbb{R}_{++}$. In the latter case real savings are constant and equal to one, making real savings independent of real interest rates.

Let $p_{t+1} = q_{t+1}/q_t$ be the return on savings for those holding money and $r_{t+1}\beta_i$ the return on savings for those making productive investments. If $r_{t+1}\beta_i < p_{t+1}$, then the solution to the consumer problem (1) is

$$\begin{cases} c_{it} &= f(p_{t+1})w_t \\ x_{it} &= 0 \\ k_{it} &= 0 \\ m_{it} &= s(p_{t+1})\dfrac{w_t}{q_t}. \end{cases}$$

If $r_{t+1}\beta_i > p_{t+1}$ the solution is

$$\begin{cases} c_{it} &= f(r_{t+1}\beta_i)w_t \\ x_{it} &= s(r_{t+1}\beta_i)w_t \\ k_{it} &= \beta_i s(r_{t+1}\beta_i)w_t \\ m_{it} &= 0. \end{cases}$$

If $r_{t+1}\beta_i = p_{t+1}$, then the return on capital production is equal to the return on money and all convex combinations of the two solutions are solutions.

For $r_{t+1}\beta_L > p_{t+1}$ all consumers save by producing capital. For $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$ L-consumers save by holding money and H-consumers save by producing capital. Finally, for $r_{t+1}\beta_H < p_{t+1}$ all consumers save by holding money.

The firm problem

The problem of a firm at date t is

$$\begin{aligned}\max_{(K_t,L_t,Y_t)} \quad & Y_t - r_t K_t - w_t L_t \\ \text{s.t.} \quad & Y_t \;\le\; F(K_t, L_t).\end{aligned}$$

A production plan for a firm at date t is inputs and output (K_t, L_t, Y_t) satisfying the inequality in the problem of the firm. The marginal products of capital and labor $\eta_K, \eta_L : \mathbb{R}_{++} \to \mathbb{R}_{++}$ are defined by

$$\begin{aligned}\eta_K(K) &= D_K F(K,1) \\ \eta_L(K) &= D_L F(K,1).\end{aligned}$$

(F.1)–(F.3) imply $\eta_L'(K) > 0 > \eta_K'(K)$ for all K, $\lim_{K\to 0} \eta_K(K) = \lim_{K\to\infty} \eta_L(K) = \infty$ and $\lim_{K\to\infty} \eta_K(K) = \lim_{K\to 0} \eta_L(K) = 0$.

The capital share of production is assumed to be bounded away from one for low levels of capital:

(F.4) $\limsup_{K\to 0} \dfrac{D_K F(K,1)K}{F(K,1)} < 1.$

The assumption is satisfied for many production functions, including both Cobb–Douglas and CES production functions.

Profit maximization implies that the solution to the problem of the firm satisfies the standard first-order conditions

$$\begin{aligned}r_t &= \eta_K(K_t) \\ w_t &= \eta_L(K_t) \;=\; \eta_L \circ \eta_K^{-1}(r_t).\end{aligned}$$

3 Equilibria

In the present section we study existence of equilibria and steady states as well as equilibrium dynamics.

Definition of equilibrium

Equilibria are sequences of prices, consumption plans and production plans such that consumers maximize their utilities, firms maximize their profits and markets clear.

Definition 1 *An **equilibrium** is a sequence of prices, consumption plans and production plans*

$$((\bar{q}_t, \bar{r}_t, \bar{w}_t)_{t\in\mathbb{Z}}, (\bar{c}_{it}, \bar{x}_{it}, \bar{k}_{it}, \bar{m}_{it})_{i\in\{L,H\},t\in\mathbb{Z}}, (\bar{K}_t, \bar{L}_t, \bar{Y}_t)_{t\in\mathbb{Z}}),$$

such that

- *Consumers maximize utility:* $(\bar{c}_{it}, \bar{x}_{it}, \bar{k}_{it}, \bar{m}_{it})$ *is a solution to the problem of an i-consumer in generation t for both i and every t.*

- *Firms maximize profits:* $(\bar{K}_t, \bar{L}_t, \bar{Y}_t)$ *is a solution to the problem for a firm at date t for every t.*

- *Markets clear:*

$$\begin{cases} \sum_i \delta_i(\bar{c}^y_{it} + \bar{c}^o_{it-1} + \bar{x}_{it}) & = & \bar{Y}_t \\ \bar{K}_{t+1} & = & \sum_i \delta_i \bar{k}_{it} \\ \bar{L}_t & = & 1 \\ \sum_i \delta_i \bar{m}_{it} & = & M \end{cases}$$

  for every t.

Equilibria can be *monetary* or *real* depending on whether money has value at every date or not. We focus exclusively on monetary equilibria.

Let λ_{it}, respectively $(1-\lambda_{it})$, be the share of savings of an i-consumer that goes into production of capital, respectively into holding money. Then equilibria are associated with solutions $(q_t, r_t, \lambda_{Lt}, \lambda_{Ht})_{t\in\mathbb{Z}}$ to the following difference equation:

$$\begin{array}{rcl} \eta_K^{-1}(r_{t+1}) & = & (\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L) + \lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H))\, \eta_L \circ \eta_K^{-1}(r_t) \\ q_t M & = & ((1-\lambda_{Lt})\delta_L + (1-\lambda_{Ht})\delta_H)\, s\left(\frac{q_{t+1}}{q_t}\right) \eta_L \circ \eta_K^{-1}(r_t) \\ \lambda_{it} & \in & \begin{cases} \{0\} & \text{for } r_{t+1} < \frac{q_{t+1}}{q_t}\frac{1}{\beta_i} \\ [0,1] & \text{for } r_{t+1} = \frac{q_{t+1}}{q_t}\frac{1}{\beta_i} \\ \{1\} & \text{for } r_{t+1} > \frac{q_{t+1}}{q_t}\frac{1}{\beta_i} \end{cases} \quad \text{for both } i\text{'s.} \end{array} \tag{2}$$

The first equation in Equation (2) is market clearing for the capital market with the capital demanded by firms on the left and the capital supplied by consumers on the right. The first part on the right is share of income allocated to investment in capital production and the second part is the real wage. The second equation is market clearing for the money market with real money supplied by old consumers and real money demanded by young consumers. The first part on the right is the share of income allocated to holding money and the second part is the real wage. The third equation describes how the share of income invested in capital production is determined for both i's. For fixed r_{t+1} and q_{t+1} the three equations determine r_t, q_t and λ_{it} for both i's. Money allows transfers of resources across generations (Samuelson, 1958; Diamond, 1965; Gale, 1973; Balasko and Shell 1980; Tirole, 1985).

Steady states

Steady states are equilibria for which consumption possibilities are identical across generations.

Definition 2 *A* ***steady state*** *is an equilibrium*

$$((\bar{q}_t,\bar{r}_t,\bar{w}_t)_{t\in\mathbb{Z}},(\bar{c}_{it},x_{it},\bar{k}_{it},\bar{m}_{it})_{i\in\{L,H\},t\in\mathbb{Z}},(\bar{K}_t,\bar{L}_t,\bar{Y}_t)_{t\in\mathbb{Z}})$$

for which there are $((\bar{p},\bar{r},\bar{w}),(\bar{c}_i,\bar{x}_i,\bar{k}_i)_{i\in\{L,H\}},(\bar{K},\bar{L},\bar{Y}))$ *such that* $(\bar{p}_t,\bar{r}_t,\bar{w}_t)=(\bar{p},\bar{r},\bar{w})$ *and* $(\bar{c}_{it},\bar{x}_{it},\bar{k}_{it})=(\bar{c}_i,\bar{x}_i,\bar{k}_i)$ *for both i and* $(\bar{K}_t,\bar{L}_t,\bar{Y}_t)=(\bar{K},\bar{L},\bar{Y})$ *for every t.*

Our first result is that all economies, for which savings are strictly larger than demand for capital at the real return on capital equal to $1/\beta_L$, have monetary steady states.

Theorem 1 *Consider an economy $\mathscr{E}$. There are monetary steady states provided that for $r=1/\beta_L$,*

$$\eta_K^{-1}(r) \;<\; (\beta_L\delta_L s(r\beta_L)+\beta_H\delta_H s(r\beta_H))\,\eta_L\circ\eta_K^{-1}(r).$$

Proof: At monetary steady states $\bar{q}_{t+1}/\bar{q}_1=1$ for every t because otherwise $\bar{q}_t M$ is not constant across dates. Therefore $1/\beta_H\leq\bar{r}\leq 1/\beta_L$ at monetary steady states, otherwise either nobody would hold money or nobody would produce capital. The first and the third difference equations in (2) can be used to find (r,λ_L,λ_H) with $\lambda_H+\lambda_L\in(0,2)$. The second difference equation in (2) can be used to find $q>0$.

Suppose that for $r=1/\beta_L$,

$$\frac{1}{\beta_H\delta_H s(r\beta_H)}\,\eta_K^{-1}(r) \;\geq\; \eta_L\circ\eta_K^{-1}(r).$$

Then the condition in Theorem 1 implies that there is $\lambda_L\in[0,1[$ such that

$$\frac{1}{\lambda_L\beta_L\delta_L s(r\beta_L)+\beta_H\delta_H s(r\beta_H)}\,\eta_K^{-1}(r) \;=\; \eta_L\circ\eta_K^{-1}(r).$$

Hence there is a monetary steady state with $\bar{r}=1/\beta_L$ where part of the L-consumers saves by holding money and the rest saves by producing capital.

Suppose that for $r = 1/\beta_L$ and $r' = 1/\beta_H$,

$$\begin{aligned}
\frac{1}{\beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) &< \eta_L \circ \eta_K^{-1}(r) \\
\frac{1}{\beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') &> \eta_L \circ \eta_K^{-1}(r').
\end{aligned}$$

Then there is $r'' \in]1/\beta_L, 1/\beta_H[$ such that

$$\frac{1}{\beta_H \delta_H s(r''\beta_H)} \eta_K^{-1}(r'') = \eta_L \circ \eta_K^{-1}(r'').$$

Hence, by continuity, there is a monetary steady state with $\bar{r} = r''$ where all L-consumers save by holding money and all H-consumers save by producing capital.

Suppose that for $r = 1/\beta_L$ and $r' = 1/\beta_H$,

$$\begin{aligned}
\frac{1}{\beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) &< \eta_L \circ \eta_K^{-1}(r) \\
\frac{1}{\beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') &\leq \eta_L \circ \eta_K^{-1}(r').
\end{aligned}$$

Then there is $\lambda_H \in]0, 1]$ such that

$$\frac{1}{\lambda_H \beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') = \eta_L \circ \eta_K^{-1}(r').$$

Hence the condition in Theorem 1 implies there is a monetary steady state with $\bar{r} = r'$ where part of the H-consumers saves by producing capital and the rest saves by holding money. $\square$

There are three kinds of monetary steady states: L-consumers are indifferent between saving by holding money and saving by producing capital $\bar{p} = \beta_L \bar{r}$; L-consumers prefer to save by holding money and H-consumers prefer to save by producing capital $\bar{p} \in]\beta_L \bar{r}, \beta_H \bar{r}[$; and, finally, H-consumers are indifferent between saving by holding money and saving by producing capital $\bar{p} = \beta_H \bar{r}$.

Weakly Pareto optimal allocations, introduced in Balasko and Shell (1980), are allocation for which it is not possible to improve welfare by changing consumption at finitely many dates. Without frictions, allocations are weakly Pareto optimal if and only if they can be supported by prices. Consumption in the two first types of monetary steady states is not weakly Pareto optimal because L-consumers and H-consumers in the same generation have different marginal rates of substitution. Transfers of resources from L-consumers to H-consumers when young and from H-consumers to L-consumers when old could improve welfare for everybody. Consumption in the last type of monetary steady states is weakly Pareto optimal because L-consumers and H-consumers in the same generation have identical marginal rates of substitution.

Dynamics

Equilibria can be everything from steady states to chaotic. However, complicated dynamics are eliminated by the assumption that savings are non-decreasing in the real interest rate. From the first-order condition of the normalized problem

$$-D_y u(1-s, ps) + pD_o u(1-s, ps) \;=\; 0.$$

it follows that savings are non-decreasing in the real interest rate if and only if

$$D^2_{yo} u(1-s, ps) - D^2_{oo} u(1-s, ps) - D_o u(1-s, ps) \;\leq\; 0.$$

The condition is mathematically, but not economically, clear.

Monetary dynamics depend on whether consumers prefer to produce capital, hold money or both. In the case where L-consumers are indifferent between holding money and producing capital and H-consumers prefer to produce capital, so $r_{t+1}\beta_L = p_{t+1}$. Equation (2) without the last equation is equivalent to

$$\begin{aligned} \frac{1}{\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L)+\beta_H\delta_H s(r_{t+1}\beta_H)}\,\eta_K^{-1}(r_{t+1}) \;&=\; \eta_L \circ \eta_K^{-1}(r_t) \\ (1-\lambda_{Lt})\delta_L s(r_{t+1}\beta_L) \;&=\; \frac{1}{\eta_L \circ \eta_K^{-1}(r_t)}\, q_t M. \end{aligned}$$

For fixed r_t and q_t, the two equations determine r_{t+1} and λ_{Lt} while q_{t+1} is determined by $q_{t+1} = q_t r_{t+1}\beta_L$. In a steady state $\bar{p} = 1$ so $\bar{r} = 1/\beta_L$. Therefore there is at most one monetary steady state where L-consumers are indifferent between holding money and producing capital and H-consumers prefer to produce capital independent of whether s is non-decreasing or not. Concerning the forward dynamics there is at most one r_{t+1}, λ_{Lt} and q_{t+1} for fixed r_t and q_t provided s is non-decreasing.

In the case where L-consumers prefer to hold money and H-consumers prefer to produce capital, $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$. Equation (2) without the last equation is equivalent to

$$\begin{aligned} \frac{1}{\beta_H\delta_H s(r_{t+1}\beta_H)}\,\eta_K^{-1}(r_{t+1}) \;&=\; \eta_L \circ \eta_K^{-1}(r_t) \\ \delta_L s\left(\frac{q_{t+1}}{q_t}\right) \;&=\; \frac{1}{\eta_L \circ \eta_K^{-1}(r_t)}\, q_t M. \end{aligned}$$

The first equation defines the evolution of r_t. For a fixed evolution of r_t the second equation defines the evolution of q_t. For the first equation, the first part on the left need not be monotonic in r_{t+1}, the second part on the left is decreasing in r_{t+1} and the part on the right is decreasing in r_t. Since the range of $\eta_L \circ \eta_K^{-1}$ is $]0, \infty[$ and the part on the right is decreasing in r_t, for all r_{t+1} there is a unique r_t such that (r_t, r_{t+1}) is a solution to the first equation. Consequently the backward equilibrium dynamics are well defined.

If savings are not non-decreasing in the real interest rate, then the first part on the left of the first equation can be non-monotonic in r_{t+1}. Consequently the forward equilibrium dynamics need not be well defined and there can be multiple steady states and complex dynamics. If savings are non-decreasing in the real rate of interest, then the first part on the left is non-increasing in r_{t+1} implying the part on the left is decreasing in r_{t+1}. Since the range of η_K^{-1} is $]0,\infty[$ and part on the left is decreasing in r_{t+1}, for all r_t there is a unique r_{t+1} such that (r_t, r_{t+1}) is a solution to the first equation. Consequently the forward dynamics are well defined.

In the case where L-consumers prefer to hold money and H-consumers are indifferent between holding money and producing in capital, $r_{t+1}\beta_H = p_{t+1}$. Equation (2) without the last equation is equivalent to

$$\begin{aligned}\frac{1}{\lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)}\eta_K^{-1}(r_{t+1}) &= \eta_L\circ\eta_K^{-1}(r_t)\\ \delta_L s(r_{t+1}\beta_L)+(1-\lambda_{Ht})\delta_H s(r_{t+1}\beta_H) &= \frac{1}{\eta_L\circ\eta_K^{-1}(r_t)}q_t M.\end{aligned}$$

For fixed r_t and q_t the two equations determine r_{t+1} and λ_{Ht} while q_{t+1} is determined by $q_{t+1} = q_t r_{t+1}\beta_H$. In a steady state $\bar{p} = 1$ so $\bar{r} = 1/\beta_H$. Therefore there is at most one monetary steady state where L-consumers prefer to hold money and and H-consumers are indifferent between holding money and investing in capital. Concerning the forward dynamics there is at most one r_{t+1}, λ_{Lt} and q_{t+1} for fixed r_t and q_t provided s is non-decreasing.

4 Printing money

Since our setup and the setup used by Martin and Ventura (2012, 2016) are similar, printing money and distributing it to young productive consumers could be expected to work more or less as new bubbles created by young productive consumers in their model. However, money is a bubble directly influenced by monetary policy and is not subject to random creation and random bursts. Also, bubbles associated with financial assets cannot be negative while monetary transfers can be.

To assess the implications of monetary policy for the real economy in steady states as well as during the transition, we consider a scenario with a passive monetary policy before some date and an active monetary policy after that date. First in Theorem 2 we show monetary steady states with active policies exist and in monetary steady states active policies increase output compared to the passive policy. Second we show output increases at every date in the transition from a steady state for the passive policy to a steady state for the active policy.

Steady states with money printing

Let N^y_{it} (N^o_{it}) be the monetary transfers to young (old) i-consumers at date t. For fixed prices monetary transfers increase incomes of consumers receiving them and these income increases lead to higher consumption both when young and when old provided consumption when young and consumption when old are normal goods. Therefore for fixed prices monetary transfers to young consumers, but not to old consumers, lead to higher savings so monetary transfers to young H-consumers lead to more production of capital.

Three scenarios of monetary policies are considered, namely newly printed money is distributed: 1. Exclusively to young H-consumers; 2. To both kinds of young consumers in equal amounts; and, 3. Exclusively to young L-consumers. In the first and third scenarios monetary transfers depend on the kind of as well as the age of the consumers and in the second scenario monetary transfers depend solely on the age of the consumers. In the first scenario $N^y_{Ht} \geq 0$ and $N^y_{Lt} = N^o_{Ht} = N^o_{Lt} = 0$ for every t; in the second $N^y_{Ht} = N^y_{Lt} \geq 0$ and $N^o_{Ht} = N^o_{Lt} = 0$ for every t; and, in the third $N^y_{Lt} \geq 0$ and $N^y_{Ht} = N^o_{Ht} = N^o_{Lt} = 0$ for every t.

Monetary equilibria are associated with solutions $(q_t, r_t, \lambda_{Lt}, \lambda_{Ht})_{t\in\mathbb{Z}}$ to the following difference equation:

$$
\begin{aligned}
\eta_K^{-1}(r_{t+1}) - \lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L)\, q_t N^y_{Lt} - \lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)\, q_t N^y_{Ht} & \\
= \; (\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L) + \lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H))\, \eta_L \circ \eta_K^{-1}(r_t) & \\
q_t M_t - (1-\lambda_{Lt})\delta_L\, s\left(\frac{q_{t+1}}{q_t}\right) q_t N^y_{Lt} - (1-\lambda_{Ht})\delta_H\, s\left(\frac{q_{t+1}}{q_t}\right) q_t N^y_{Ht} & \\
= \; ((1-\lambda_{Lt})\delta_L + (1-\lambda_{Ht})\delta_H)\, s\left(\frac{q_{t+1}}{q_t}\right) \eta_L \circ \eta_K^{-1}(r_t) & \\
\lambda_{it} \; \in \; \begin{cases} \{0\} & \text{for } r_{t+1} < \dfrac{q_{t+1}}{q_t}\dfrac{1}{\beta_i} \\ [0,1] & \text{for } r_{t+1} = \dfrac{q_{t+1}}{q_t}\dfrac{1}{\beta_i} \quad \text{for both } i\text{'s} \\ \{1\} & \text{for } r_{t+1} > \dfrac{q_{t+1}}{q_t}\dfrac{1}{\beta_i} \end{cases} & \\
M_{t+1} \; = \; M_t + \delta_L N^y_{Lt+1} + \delta_H N^y_{Ht+1}. &
\end{aligned}
\tag{3}
$$

The first equation in Equation (3) is market clearing for the capital market with demand for capital of firms minus supply of capital financed by money transfers on the left and supply of capital financed by wage income on the right. The second equation is market clearing for the money market with supply of money minus demand for money financed by money transfers on the left and demand for money financed by wage income on the right. The third equation describes how the share of income invested in capital production is determined for both i's. The fourth equation describes how the stock of money evolves.

For our result on existence of and output at monetary steady states with monetary transfers to young consumers we consider monetary policies of the form $N_{it}^y = \varepsilon_i^y N_t$ for both i with $\varepsilon_i \geq 0$ for both i and $\delta_L \varepsilon_L + \delta_H \varepsilon_H = 1$.

Theorem 2 *Consider an economy $\mathscr{E}$. Let $N_t/M_t = n$ for every t and assume for $r = 1/\beta_L$,*

$$\eta_K^{-1}(r) \ < \ (\beta_L \delta_L s(r\beta_L) + \beta_H \delta_H s(r\beta_H))\, \eta_L \circ \eta_K^{-1}(r).$$

Then there is a neighborhood of the passive monetary policy $n = 0$ such that

- *There are monetary steady states with money printing for all policies in the neighborhood.*
- *Assume $\varepsilon_H > 0$. Minimum (maximum) output for all active policies with $n > 0$ in the neighborhood is larger than minimum (maximum) output for the passive policy.*
- *Assume $\varepsilon_H = 0$. Then a active policy $n > 0$ removes monetary steady states with $r > (1-n)/\beta_L$ for the passive policy and does not change steady states with $r < (1-n)/\beta_L$ for the passive policy.*

Proof: Let $\mu \in]0,1[$ be defined by $\lambda_L \geq \mu$ implies for $r = 1/\beta_L$,

$$\eta_K^{-1}(r) \ < \ (\lambda_L \beta_L s(r\beta_L) + \beta_H s(r\beta_H)) \eta_L \circ \eta_K^{-1}(r).$$

Let $\nu \in]0,1[$ be defined by $n \in [0,\nu]$ implies $1 - (\delta_L \varepsilon_L + \delta_H \varepsilon_H) s(1-n) n > 0$ and for all $r \in [(1-n)/\beta_L, 1/\beta_L]$ and $\lambda_L \geq \mu$,

$$\eta_K^{-1}(r) \ < \ (\lambda_L \beta_L s(r\beta_L) + \beta_H s(r\beta_H)) \eta_L \circ \eta_K^{-1}(r).$$

Let $Q : \mathbb{R}_{++} \times [0,\nu] \times [0,\mu] \times [0,1] \to \mathbb{R}_{++}$ be defined by

$$Q(r,n,\lambda_L,\lambda_H) \ = \ \frac{((1-\lambda_L)\delta_L + (1-\lambda_H)\delta_H) s(1-n)}{1 - ((1-\lambda_L)\delta_L \varepsilon_L + (1-\lambda_H)\delta_H \varepsilon_H) s(1-n) n} \frac{\eta_L \circ \eta_K^{-1}(r)}{M}.$$

Then $Q(r,n,\lambda_L,\lambda_H)$ is the steady state price for money for real return on capital r, policy n and fractions of consumers saving by producing capital (λ_L, λ_H). Let the correspondence

$\Lambda : \mathbb{R}_{++}\times[0,\nu] \to [0,\mu]\times[0,1]$ be defined by

$$\Lambda(r,n) \;=\; \begin{cases} \{(0,0)\} & \text{for } r < \dfrac{1-n}{\beta_H} \\[2ex] \{0\}\times[0,1] & \text{for } r = \dfrac{1-n}{\beta_H} \\[2ex] \{(0,1)\} & \text{for } \dfrac{1-n}{\beta_H} < r < \dfrac{1-n}{\beta_L} \\[2ex] [0,\mu]\times\{1\} & \text{for } r = \dfrac{1-n}{\beta_L} \\[2ex] \{(\mu,1)\} & \text{for } r > \dfrac{1-n}{\beta_H}. \end{cases}$$

Then $\Lambda(r,n)$ is the steady state fractions of consumers saving by producing capital for real return on capital r and policy n – except λ_L is restricted to $[0,\mu]$. Moreover, Λ is upper hemi-continuous with convex values and Λ is single valued at (r,n) for $r \notin \{(1-n)/\beta_H, (1-n)/\beta_L\}$. Let function $E : \mathbb{R}^2_{++}\times[0,\nu]\times[0,\mu]\times[0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned} E(r,q,n,\lambda_L,\lambda_H) \;=\; & \eta_K^{-1}(r) - (\lambda_L\beta_L s(r\beta_L)\varepsilon_L + \lambda_H\beta_H s(r\beta_H)\varepsilon_H)qMn \\ & -(\lambda_L\beta_L s(r\beta_L)+\lambda_L\beta_L s(r\beta_L))\eta_L\circ\eta_K^{-1}(r). \end{aligned}$$

Then $E(r,q,n,\lambda_L,\lambda_H)$ is the excess demand for capital for real return on capital r, price of money q, policy n and fractions of consumers saving by producing capital (λ_L,λ_H). Moreover for $r = (1-n)/\beta_H$ and $n \in [0,\mu]$, $E(r,Q(r,n,0,0),n,0,0) > 0$ with $(0,0) \in \Lambda(r,n)$ and for $r = (1-n)/\beta_L$ and all $n \in [0,\mu]$, $E(r,Q(r,n,\mu,1),n,\mu,1) < 0$ with $(\mu,1) \in \Lambda(r,n)$.

Let the correspondence $\Theta : \mathbb{R}_{++}\times[0,\nu] \to \mathbb{R}$ be defined by

$$\Theta(r,n) \;=\; \bigcup_{(\lambda_L,\lambda_H)\in\Lambda(r,n)} E(r,Q(r,n,\lambda_L,\lambda_H),n,\lambda_L,\lambda_H).$$

Then there is a monetary steady state with return on capital r and policy n if and only if $0 \in \Theta(r,n)$. Moreover Θ is upper hemi-continuous and convex valued where $v_0 \in \Theta(r,0)$ and $v_n \in \Theta(r,n)$ for $n > 0$ implies $v_n < v_0$ for all r. Since $\Theta(r,n) > 0$ for $r = (1-n)/\beta_H$ and all $n \in [0,\mu]$ and $\Theta(r,n) < 0$ for $r = (1-n)/\beta_L$ and all $n \in [0,\mu]$, there is a monetary steady state with money printing for all $n \in [0,\mu]$. For $\varepsilon_H > 0$ the minimum (maximum) real return on capital for all active policies is smaller than the minimum (maximum) real return on capital for the passive policy, because $v_0 \in \Theta(r,0)$ and $v_n \in \Theta(r,n)$ for $n > 0$ implies $v_n < v_0$ for all r as well as $\Theta(r,n) > 0$ for $r = (1-n)/\beta_H$ and all $n \in [0,\mu]$ and $\Theta(r,n) < 0$ for $r = (1-n)/\beta_L$ and all $n \in [0,\mu]$. For $\varepsilon_H = 0$ an active policy n removes monetary steady states for the passive policy with $r > (1-n)/\beta_L$ and does not change steady states for the passive policy with $r < (1-n)/\beta_L$. $\square$

Transitional dynamics

In the discussion below we assume that: savings are non-decreasing in returns on savings; and, the return on capital production for L-consumers is lower than the return on money and the return on money is lower than the return on capital production for H consumers. The second assumption implies that L-consumers save by holding money and H-consumers save by producing capital. Output as well as net-output, where net-output is output minus input in capital production, are increased by active policies compared to the passive policy provided young H-consumers receive transfers, because the return in output of input in capital production is larger than one.

In order to study the dynamic effects of changing monetary policy consider a locally stable steady state. Then there is a neighborhood of the passive policy $n = 0$ such that the steady states with money printing are locally stable for all policies in that neighborhood. Hence if money printing is started at date t, then there is a transition from the steady state without money printing to the steady state with money printing. Indeed young consumers adjust their demands and savings from date t and forward and prices adjust from date $t+1$ and forward. The transition is monotonic with output increasing from date $t+1$ and forward as illustrated in Figure 1.

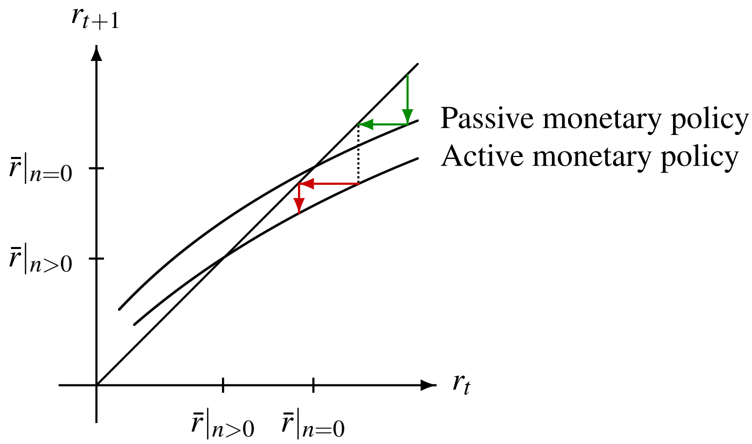

Figure 1: Monetary policy and monetary dynamics

At monetary steady states with money printing, transfers to consumers have a direct effect on the return of money. Since the real value of money has to be constant across dates, the price of the consumption good has to increase faster with monetary transfers than without monetary transfers. Therefore printing money decreases the return on existing money. All in all, distributing newly printed money solely to young H-consumers or to both kinds of young consumers increases output by reducing the inefficiencies in capital production, and, distributing money solely to young L-consumers does not change output. Since printing money increases output, the stock of capital increases. Hence the real wage is increased.

Therefore for consumers there are two main effects of money printing, namely lower returns on savings and higher real income, where the higher income comes from higher real wage as well as transfers.

It is straightforward and quite tedious to calculate the changes in indirect utilities caused by money printing. However the expressions are not easy to evaluate. As explained above, there are two effects pointing in opposite directions. In the next section, we consider an example where the negative effect from lower returns on savings is maximized by assuming that consumers solely consume when old and derive a condition on the output elasticity of capital for Cobb-Douglas production functions that ensure welfare is increased for both kinds of consumers.

5 An example

To discuss boundaries for monetary policy as well as optimal policies and inflation rates, we reconsider the model economy as put forward in Martin and Ventura (2012) in this section. We also show by means of numerical examples that the business cycle fluctuations which Martin and Ventura (2012) attribute to stochastic assets bubbles can also arise in perfect foresight equilibria with deterministic monetary policy rules.

Setup

Consumers care only about consumption when old, i.e. $X = \{0\} \times \mathbb{R}_{++}$. Therefore all income received when young will be saved, i.e. $s(p) = 1$ for all p. This means that the welfare results of our example are conservative in that the negative effects of money printing are maximized. Second, production of goods takes place according to a Cobb-Douglas function

$$F(K_t, L_t) = K_t^{\sigma} L_t^{1-\sigma}$$

with $\sigma \in (0,1)$ being the output elasticity of capital.

For convenience, transfers to young consumers are written as multiples $\gamma \in \mathbb{R}$ of total wages when young, i.e. $N_{it}^{y} = \gamma_{it}^{y} w_t / q_t$. If $\gamma > 0$, a consumer receives money and if $\gamma < 0$ she pays money. As before, transfers to old consumers are not considered as they would have redistributive effects only.

The consumer problem at date t for a young consumer is

$$\max_{(c^o_{it}, m_{it}, x_{it}, k_{it})} \quad c^o_{it}$$
$$\text{s.t.} \begin{cases} x_{it} + q_t m_{it} & \leq & w_t + q_t N^y_{it} \\ c^o_{it} & \leq & q_{t+1} m_{it} + r_{t+1} k_{it} \\ k_{it} & \leq & \beta_i x_{it} \\ m_{it}, x_{it} & \geq & 0. \end{cases} \tag{4}$$

Since $s(p) = 1$ so $f^y(p) = 0$ and $f^o(p) = p$, the solution to the consumer problem for $r_{t+1}\beta_i < p_{t+1}$ is

$$\begin{cases} c^o_{it} & = & p_{t+1}(1+\gamma^y_{it})w_t \\ x_{it} & = & 0 \\ k_{it} & = & 0 \\ m_{it} & = & (1+\gamma^y_{it})w_t/q_t. \end{cases}$$

For $r_{t+1}\beta_i > p_{t+1}$ the solution is

$$\begin{cases} c^o_{it} & = & r_{t+1}\beta_i(1+\gamma^y_{it})w_t \\ x_{it} & = & (1+\gamma^y_{it})w_t \\ k_{it} & = & \beta_i(1+\gamma^y_{it})w_t \\ m_{it} & = & 0. \end{cases}$$

For $r_{t+1}\beta_i = p_{t+1}$, the return on capital production is equal to the return on money and all convex combinations of the two solutions are solutions.

The solution to the firm problem at date t satisfies

$$\begin{aligned} r_t &= \eta_K(K_t) &= \sigma K_t^{\sigma-1}, \\ w_t &= \eta_L(K_t) &= (1-\sigma)K_t^{\sigma}. \end{aligned}$$

Finally, the path for the aggregate money stock, $\{M_t\}_{t\in\mathbb{Z}}$ is described by

$$M_{t+1} = M_t + \sum_i \delta_i N^y_{it+1}.$$

Monetary equilibria

In this example, we restrict our attention to equilibria in which L-consumers prefer to save by holding money and H-consumers by producing capital, i.e., $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$ for every t. As will be shown, this is the only equilibria where active monetary policy could be expansionary in our setting.

The equilibrium conditions can be summarized as

$$\begin{aligned} \bar{K}_{t+1}/\bar{K}_t &= (1+\gamma^y_{Ht})\delta_H\beta_H(1-\sigma)\bar{K}_t^{\sigma-1}, \\ p_{t+1} &= \left(1-\frac{\delta_H}{\delta_L}\gamma^y_{Ht+1}\right)\left(1+\gamma^y_{Lt}\right)^{-1}\left(\frac{\bar{K}_{t+1}}{\bar{K}_t}\right)^{\sigma}, \end{aligned} \tag{5}$$

implying that the saddle path for the return on money is given by

$$p_{t+1} = \left(1-\frac{\delta_H}{\delta_L}\gamma^y_{Ht+1}\right)\left(1+\gamma^y_{Lt}\right)^{-1}\left((1+\gamma^y_{Ht})\delta_H\beta_H(1-\sigma)\bar{K}_t^{\sigma-1}\right)^{\sigma}.$$

Since $\sigma(\sigma-1)\in]-1,0[$, monetary steady states, provided they exist, are thus unique and characterized by

$$\begin{aligned} \bar{p}_M &= \left(1-\frac{\delta_H}{\delta_L}\gamma^y_H\right)(1+\gamma^y_L)^{-1} \\ \bar{K}_M &= \left((1+\gamma^y_H)\delta_H\beta_H(1-\sigma)\right)^{1/(1-\sigma)} \\ \bar{r}_M &= \frac{\sigma}{1-\sigma}\left(\delta_H\beta_H(1+\gamma^y_H)\right)^{-1} \end{aligned}$$

with $\gamma^y_{Ht}=\gamma^y_H$ and $\gamma^y_{Lt}=\gamma^y_L$ for all t. Accordingly, a monetary expansion increases capital and output at steady-state provided if new money is transferred to young H-consumers. However the return on money and consequently inflation depend on all drivers of the stock of money.

The equilibrium allocations associated with monetary steady states are dynamically efficient provided the return on capital is greater than one, $\beta_H\bar{r}_M>1$ or equivalently

$$\frac{\sigma}{1-\sigma}\frac{1}{\delta_H(1+\gamma^y_H)} > 1. \tag{6}$$

It can be shown that equilibrium allocations associated with real steady states are dynamically inefficient and equilibrium allocations associated with monetary steady states are dynamically efficient provided

$$\delta_H(1+\gamma^y_H) \leq \frac{\sigma}{1-\sigma} < \frac{\bar{\beta}}{\beta_L}.$$

Money restores efficiency because L-consumers switch their savings from investing in capital production to holding money.

Implications for monetary policy

We now look at the conditions for which monetary steady states exist. We restrict attention to monetary policies for which the associated allocation is dynamically efficient. Otherwise, the incremental output due to monetary policy would necessarily be welfare-reducing.

Let the function $F : \mathbb{R}_{++} \to \mathbb{R}$ and the real number $\xi \in \mathbb{R} \setminus \{0\}$ be defined by

$$\begin{aligned} F(\gamma) &= \left(\frac{\delta_L}{\delta_H} - \gamma\right)(1+\gamma), \\ \xi &= \left(\delta_H \beta_H \frac{1-\sigma}{\sigma}\right)^{-1} \frac{\delta_L}{\delta_H}(1+\gamma_L^y). \end{aligned}$$

There are monetary steady states provided $\beta_L r_M < p_M < \beta_H r_M$ or, equivalently, if

$$0 < F(\gamma_H^y)\xi^{-1} - \beta_L < \beta_H - \beta_L.$$

Let $G : \mathbb{R} \to \mathbb{R}$ be defined by $G(\gamma) = F(\gamma)\xi^{-1} - \beta$, and $(\gamma_1(\beta), \gamma_2(\beta))$ the two solutions to $G(\gamma) = 0$. There are two cases, namely $\xi > 0$ and $\xi < 0$. Suppose $\xi > 0$. If $(2\delta_H)^{-2}\xi^{-1} < \beta_L$, then there is no γ_H^y for which a monetary steady state exists. Let $\nu \equiv (\delta_L \beta_L)/(\delta_H \beta_H)$. If $\gamma_2(\beta_L) > \nu$, then there are monetary steady states and output is higher than at the real steady state with output being maximal for $\gamma_H^y = \gamma_2(\beta_L)$. In all other cases where $\xi > 0$ there are monetary steady states, but output is lower than at real steady states. Suppose $\xi < 0$. Then $\gamma_2(\beta_H) > \nu$ if and only if there are monetary steady states with $\gamma_H^y = \gamma_2(\beta_H)$ at which output is higher than at real steady states. Otherwise there are monetary steady states, but output is lower than at real steady state.

Whether monetary policies which increase output exist depends on γ_L^y because it determines ξ and consequently the solution to $G(\gamma) = 0$. Output converges to its supremum for ξ converging to zero. Policies that result in ξ tending to zero tax away income of young L-consumers (γ_L^y close to minus one). For γ_H^y converging to δ_L/δ_H, the steady-state interest rate and the return on money converge to

$$\begin{aligned} \bar{r}_M &= \frac{\sigma}{1-\sigma} \frac{1}{\beta_H} \\ \bar{p}_M &= \frac{\sigma}{1-\sigma} \frac{\beta_L}{\beta_H}. \end{aligned}$$

Therefore the supremum output is dynamically efficient provided $\sigma/(1-\sigma) \geq 1$ so the rate of return on capital production $\beta_H \bar{r}_M$ is at least 1.

Finally, assume that transfers to the two types of consumers have to be identical, perhaps because productivity in capital production is private information or because of political reluctance to discriminate between consumers. Hence $\gamma_i^y = \gamma^y$ for some γ^y. Output is maximized for

$$\gamma^y \;=\; \frac{1}{\delta_H}\left(\delta_L - \frac{\sigma}{1-\sigma}\,\nu\right).$$

The steady-state interest rate is

$$\bar{r}_M \;=\; \frac{\sigma}{1-\sigma}\,\frac{\delta_H}{\delta_H\beta_H + (\sigma/(1-\sigma))\delta_L\beta_L}$$

and the supremum output is thus dynamically efficient provided

$$\frac{\sigma}{1-\sigma}(1-\nu) \;\geq\; 1.$$

Welfare comparison of steady states

The following welfare comparison of steady states shows that there is a wide range of parameters for which monetary policies with relatively small money transfers improve welfare for both consumer types in the long-run. The reference scenario is a passive monetary policy without money transfers, i.e. where $\gamma_L^y = \gamma_H^y = \gamma^I = 0$. We consider two active monetary policies: one policy with money transfers solely to young H-consumers, i.e. $\gamma_H^y = \gamma^A > 0, \gamma_L^y = 0$; and another policy with equal money transfers to both types of young consumers, i.e. where $\gamma_H^y = \gamma_L^y = \gamma^A > 0$.

Welfare of H-consumers is higher for active monetary policies compared to the reference scenario, since the ratio of the respective steady-state consumption levels for both scenarios is

$$\frac{\bar{c}_H^o(\gamma^A)}{\bar{c}_H^o(\gamma^I)} \;=\; (1+\gamma^A)^{\sigma/(1-\sigma)}$$

which is larger one. As for L-consumers, the welfare effects are more delicate. Active monetary policies make them better off provided

$$\frac{\bar{c}_L^o(\gamma^A)}{\bar{c}_L^o(\gamma^I)} \;=\; \left(1 - \frac{\delta_H}{\delta_L}\gamma^A\right)(1+\gamma^A)^{\sigma/(1-\sigma)} > 1,$$

which holds if and only if

$$\frac{\delta_H}{\delta_L} \;<\; \frac{(1+\gamma_H^y)^{\sigma/(1-\sigma)} - 1}{(1+\gamma_H^y)^{\sigma/(1-\sigma)}\gamma_H^y}.$$

The expression on the right hand side is decreasing in γ_H^y and, according to L'Hôpital's rule, converges to $\sigma/(1-\sigma)$ as γ_H^y converges to zero. Therefore, provided $\sigma > \delta_H$, there is an active monetary policy already in the neighborhood of the passive monetary policy such

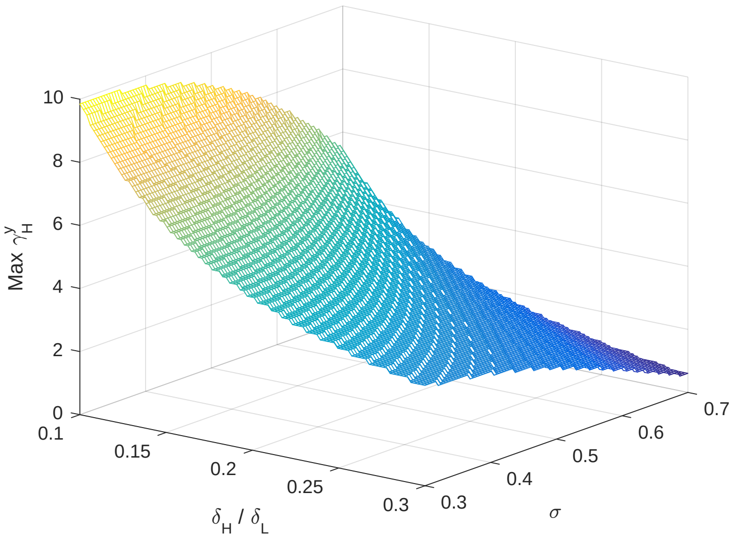

Figure 2: Monetary policies for which $\bar{c}_L(\gamma^A)/\bar{c}_L(\gamma^I) = 1$

that both types of consumers are better off at steady states. Moreover, for $\sigma > \delta_H$ the ratio $\bar{c}^o_H(\gamma^A)/\bar{c}^o_H(\gamma^I)$ is also increasing in γ^A and $\bar{c}^o_L(\gamma^A)/\bar{c}^o_L(\gamma^I)$ is maximized for

$$\gamma_A = \frac{\sigma - \delta_H}{\delta_H}.$$

Hence, monetary policies with $\gamma^A = (\sigma - \delta_H)/\delta_H$ are optimal in that no other monetary policy is preferred by both consumer types.

Figure 2 shows the supremum value of γ^A as a function of $(\delta_H/\delta_L, \sigma)$ such that welfare of both consumer types is higher with an active monetary policy for all positive γ^A smaller than the supremum of γ^A.

Welfare during transition

To assess the welfare implications of a change in monetary policy during the transition, consider an economy that is initially at a monetary steady state with a passive monetary policy, $\gamma^y_L = \gamma^y_H = \gamma^I = 0$. At some date $t = T$, the policy maker unexpectedly implements an active policy described in the previous section for all $t \geq T$, which can be either a transfer only to H-consumers or an equal transfer to both types of consumers. Equation (5) implies that the equilibrium trajectory of the economy along the saddle path, with an initial capital stock of $\bar{K}_T = (\delta_H \beta_H (1-\sigma))^{1/(1-\sigma)}$, reads

$$p_t = \begin{cases} 1 - (\delta_H/\delta_L)\gamma_H^y & \text{for } t = T \\ \dfrac{1-(\delta_H/\delta_L)\gamma_H^y}{1+\gamma_L^y}(1+\gamma_H^y)^{\sigma^{t-T}} & \text{for } t \geq T+1 \end{cases}$$

$$\bar{K}_t = (1+\gamma_H^y)^{\frac{1-\sigma^{t-T}}{1-\sigma}} (\delta_H \beta_H (1-\sigma))^{1/(1-\sigma)} \qquad \text{for } t \geq T+1.$$

All H-consumers born at dates $t \geq T$ enjoy a higher welfare for both types of active policies since

$$\frac{\bar{c}_{Ht}(\gamma^A)}{\bar{c}_H(\gamma^I)} = (1+\gamma_H^y)^{\frac{\sigma(1-\sigma^{t+1})}{1-\sigma}} > 1.$$

As for L-consumers, we have

$$\frac{\bar{c}_{Lt}(\gamma^A)}{\bar{c}_L(\gamma^I)} = \left(1 - \frac{\delta_H}{\delta_L}\gamma_H^y\right)(1+\gamma_H^y)^{\frac{\sigma(1-\sigma^{t+1})}{1-\sigma}}.$$

Hence, a necessary and sufficient condition that L-consumers are better off throughout the entire transition period as well as at the new steady state is

$$\frac{\delta_H}{\delta_L} < \frac{(1+\gamma_H^y)^\sigma - 1}{(1+\gamma_H^y)^\sigma \gamma_H^y}.$$

According to L'Hôpital's rule, there is thus an active monetary policy in the neighborhood of the passive monetary policy that is always welfare improving provided $\sigma > \delta_H/\delta_L$. This condition does not depend on whether young L-consumers obtain transfers or not.

Consumers born at dates $t < T-1$ are not affected by the policy change because they do not live to see the policy change. H-consumers born at date $T-1$ are also unaffected because the capital stock and thus the return on capital do not change before date $T+1$. However, L-consumers born at date $T-1$ are strictly worse off due to the change in monetary policy because printing money and giving it to H-consumers at date T induces an immediate fall in the return on holding money. This loss cannot be compensated by monetary transfers. To see this suppose monetary transfers γ_{Lt}^o to old L-consumers were possible at date T. The return on holding money according to Equation (5) at date T would read

$$p_T = 1 - \frac{\delta_H}{\delta_L}\gamma_{HT}^y - \gamma_{LT}^o.$$

Hence, monetary transfers $\gamma_{LT}^o > 0$ lead to a one-for-one drop in the return on money, leaving the purchasing power in the hands of old L-consumers at date T, and thus their consumption $c_{LT-1}^o = (p_T + \gamma_{LT}^o)w_T$, unchanged. Only real transfers, for example a one-off lump sum transferred from the initial young consumers, can achieve Pareto-improvements.

Numerical simulation exercises

To better understand the impact of the money printing policy, we perform a series of numerical simulation exercises using the perfect foresight version of economy in our example. The parameters are calibrated to match certain characteristics of the US economy. The parameter σ is set 1/3 to match the observed long-run capital share of output. We set $\delta_H = 0.13$ to mirror the fraction of entrepreneurs in the US population. Alternatively, it is possible for us to exploit the well-known positive sorting between ability and education attainment. Incidentally, we obtain a very similar calibration of 12.8% if we use the percentage of the population with masters, doctorates and/or professional degrees.

One 'generation' is set to 15 years. We assume that the economy is initially at its steady state with $\gamma^y_{H0} = \gamma^y_{L0} = \gamma_0$. The value of γ_0 is calibrated to reproduce the US average annual inflation of 1.9% over the recent 15 year period, ending in 2018. We discipline β_H by the average annualized real interest rate of 1.4% during the same period. For simplicity, $\gamma^o_H = \gamma^o_L = 0$ for all our exercises. β_L is set to ensure $\beta_L r_t < p_t < \beta_H r_t$ for all t so that the economy stays in monetary equilibrium. L is chosen so that the GDP is one in initial steady state. It happens that $\beta_H r > 1$ under our choice of parameters, so that the economy is dynamically efficient. Table 1 summarizes the baseline calibration.

Parameter	Value	Remark
σ	1/3	Long-run capital share of output
δ_H	0.13	The fraction of entrepreneurs in the US (Global Entrepreneur Monitor, 2016), or the fraction of US population holding masters, doctorates, or professional degrees (Educational Attainment in the United States: 2017, US Census Bureau)
γ_0	0.272	US average annual inflation of 1.9% between 2004-2018 based on GDP deflator (FRED, St Louis Fed)
γ_1	0.352	To generate steady state inflation of 2.4% (i.e. 50 basis point higher than 1.9%) under Policy A
β_H	2.454	US average annual real interest rate of 1.4% between 2004-2018 (Treasury inflation indexed long-term average yield. FRED, St Louis Fed)
β_L	0.491	$\beta_L = \beta_H/5$, to ensure $\beta_L r_t < p_t < \beta_H r_t$

Table 1: Baseline calibration

First we examine the effect of unanticipated expansionary money printing policies. Policy A is an egalitarian policy such that $\gamma^y_H = \gamma^y_L$ for all t. The policy maker increases the money supply multiplier γ permanently to γ_1 from γ_0, which results in a 50 basis point increase in the steady state inflation from the baseline. Policy B maintains the same monetary

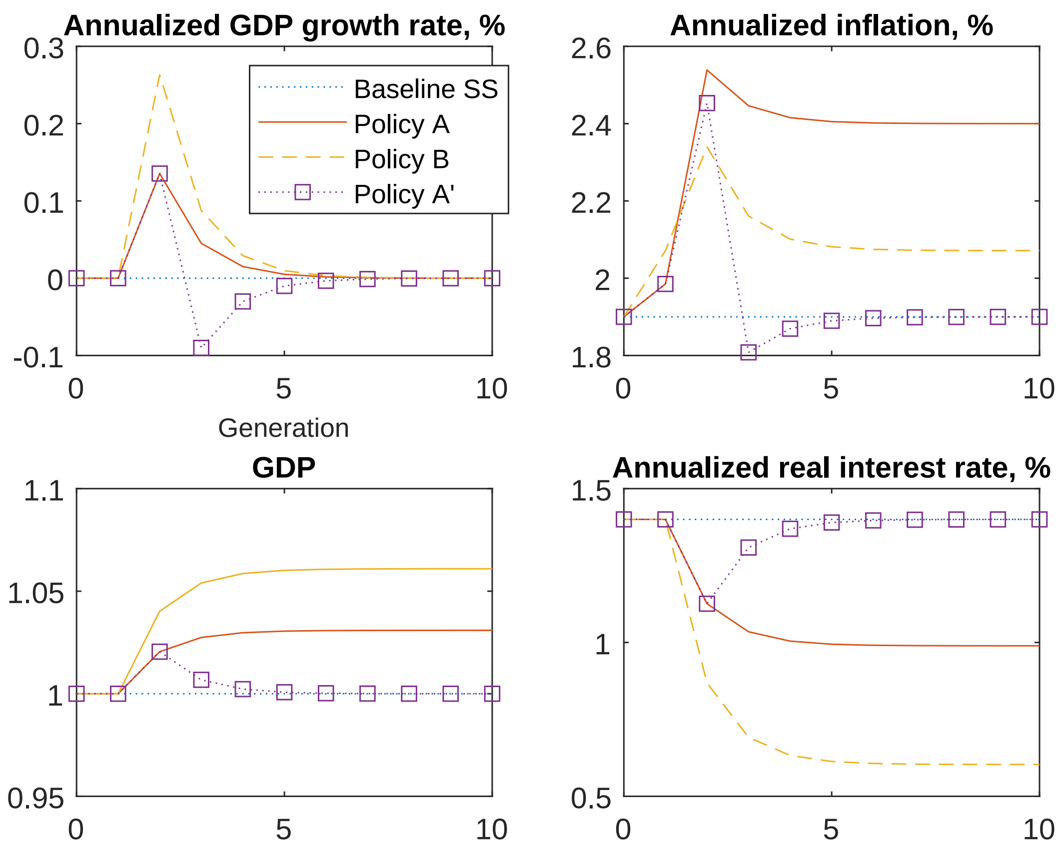

Figure 3: Effect of active money printing policy

supply as Policy A in that $\gamma_H^y + \gamma_L^y = 2\gamma_1$. But extra monetary injection is given exclusively to H consumers. Figure 3 plots the dynamics of key variables under such policies. The dotted blue lines represents the economy in steady state under the baseline calibration. The economies under Policies A and B are shown in the solid red line and the dashed orange line. The figure shows that there exists is a Phillips curve type trade-off between inflation, $1/p-1$, and GDP. Note that, unlike the conventional models which requires price frictions for the Phillips curve, our model generates this relationship through financial frictions with perfectly flexible prices. The egalitarian Policy A increases the long-run GDP by about 3% while it increases annual inflation by 0.5%. The selective Policy B is able to achieve about the twice of GDP increments with about a half of inflation rises. The policy effects are heavily concentrated in the first several periods. The real interest rate is nothing but marginal product of capital net of depreciation. Even though there is no active loan markets in our economy due to financial frictions, this captures the (shadow) rate of return on savings.

Next, we consider the effect of one-off expansionary policy. In particular, the policy maker sets $\gamma_H^y = \gamma_L^y = \gamma_1$ for $t = 1$, then reverts back to $\gamma_H^y = \gamma_L^y = \gamma_0$ for $t > 1$, which we denote Policy A'. In Figure 3, this one-off policy is depicted in violet dotted lines with square markers. Again, the dynamics of real variables such as GDP and real interest rates closely mirror the monetary dynamics summarized in inflation. Notably this one-off money policy has persistent effects even though capital fully depreciates. This is because money is injected in constant multiples of wages and they are the function of capital formed in the previous period.

An alternative monetary policy strategy is to target some rate of inflation and to control the nominal interest rate as a policy instrument. In this strategy, the quantity of money or its expansion path is unimportant, both as a target and as an instrument. In economies plagued with financial frictions, money plays an explicit role above and beyond controlling interest rates. While monetary policy can achieve its inflation target, it can do more. For any inflation rate, monetary policy can further manipulate consumption plans, and thus output and even welfare, by varying the quantities of money that different types of consumers obtain over their lifetime in a way as described in the analysis above. To illustrate this, we also compare a constant money printing policy vis-à-vis two inflation targeting policies. In the inflation targeting regime, the policy maker sets money injection using the feedback rule as follows:

$$\gamma_{Ht}^{y} = \gamma_{Lt}^{y} = \gamma_i \left(\frac{1/p_{t-1}}{1/p^*} \right)^{\phi} \text{ for } t \geq 1 \tag{7}$$

where $1/p^*$ is the target inflation rate. This is a generalization of the inflation targeting feedback rule commonly used in the central banking literature. The typical inflation targeting rule requires the central bank to relax (tighten) the economy when the inflation is below (above) the target, which translates into $\phi < 0$. This is our Inflation Targeting 1. However, one can argue that the conventional inflation targeting rule, in fact, requires $\phi > 0$. The logic is that the traditional inflation targeting rules are implemented by setting the higher (lower) nominal interest rates when the economy is above (below) the target. Since the real interest rate is predetermined in our model, the policy maker can increase the (shadow) nominal interest rate by causing inflation, through the Fisher equation. Under this reasoning, the coefficient ϕ must be positive because inflation and money growth are positively related. Hence we also examine the model economy under $\phi > 0$. This is Inflation Targeting 2. As before, the economy is at the initial steady state inflation of 1.9% implied by γ_0. And the policy maker wishes to maneuver the economy to a new inflation target of 2.4%. This means $\gamma_i = \gamma_1$ for Equation (7). We set $\phi = -2$ for Inflation Targeting 1 and $\phi = 2$ for Inflation Targeting 2. This is broadly within the range of the Taylor rule policy coefficient on inflation (Carlstrom and Jacobson, 2015).

Figure 4 illustrates the dynamics of economy under these alternative policy rules in comparison with the constant money printing rule (Policy A as in Figure 3). Unlike the conventional wisdom, both types of inflation targeting rules do not perform well in our case. The rule 1 introduces unnecessary fluctuations by repeating over/undershooting the

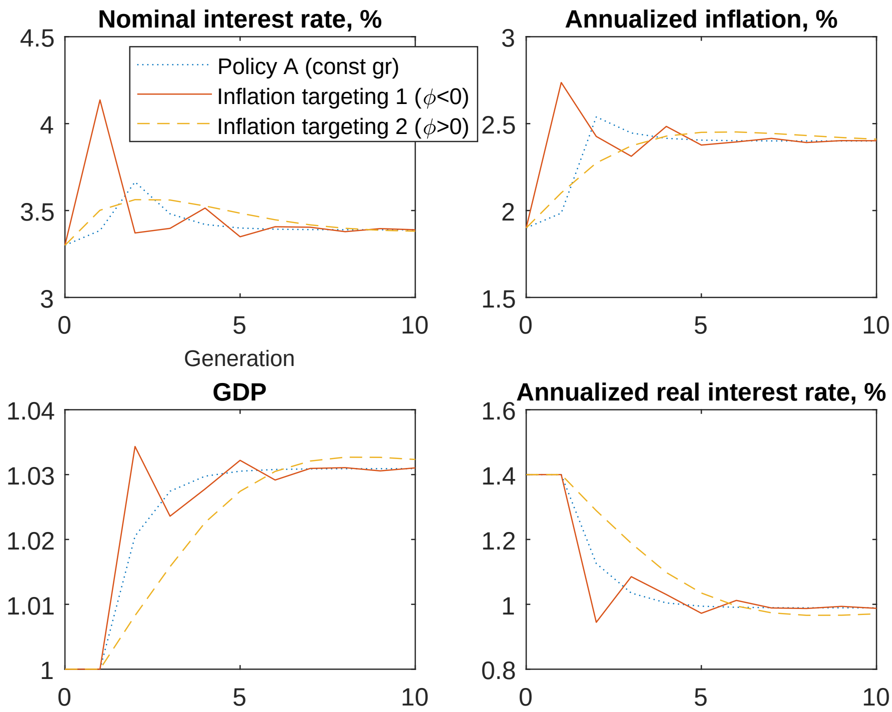

Figure 4: Constant money growth policy vs. inflation targeting

inflation target. A smaller (absolute) value of ϕ does reduce the policy induced oscillation but does not remove it completely. On the other hand, the rule 2 helps smooth the transition dynamics. But this comes at the expense of much slower adjustments in the real economy. The inflation targeting has further complications in our economy. As discussed earlier, there can be many different real outcomes for each 'inflation targeting'. For instance, given the same values of γ_i and ϕ, the policy maker can assign different weights to γ_H^y and γ_L^y. In this sense, it appears that constant money growth policy analogous to the Friedman rule is much more straightforward and thus favored in this setting.

Discussion

A brief, concluding discussion of the similarities with and differences to Martin and Ventura (2012) is due. First, new money injections to young consumers N_{it}^y in our model correspond to new bubbles in Martin and Ventura (2012) while the aggregate money stock M_{t+1} matches their total size of bubbles. Following the long tradition in economics since Samuelson, we thus treat money as bubble and exploit the mechanism in Martin and Ventura (2012) by which new bubbles first available to young consumers can be expansionary in the presence of financial frictions. There are, however, three key differences. First, unlike in Martin and Ventura (2012), our money is a non-random policy tool. Thus, in principle, new money injection can be either positive (transfer) or negative (tax). This cannot be the case in financial bubbles where consumers have only limited liabilities. Second, consumer preferences in our example are rather extreme, just like in Martin and Ventura (2012). Theorem 2 has shown that active monetary policies are expansionary under more general preferences, but such extreme preferences penalize us most when it comes to the welfare evaluation of monetary policies. Finally, in Martin and Ventura (2012) it is the stochastic nature of the bubble which drives investment and business cycles, whereas we have provided simple examples in which deterministic monetary policy rules can generate such cycles in perfect foresight equilibria.

6 Final remarks

Money matters in economies where efficient allocation of savings across investment opportunities is prevented by frictions in credit markets. The presence of money in itself does not help mitigate the effects of frictions. However, we have shown that simple monetary policies in form of printing money and distributing it appropriately do help. Monetary policies in form of money taxes and money transfers can eliminate the effects of frictions, but these policies are institutionally much more demanding.

We have demonstrated that different real outcomes can be obtained under the same inflation target. Indeed the real effect of printing money depends on how it is distributed. Therefore the discussion of inflation targeting versus monetary targeting needs to be qualified: central bankers need to go beyond solely monitoring inflation.

We have considered a rather rudimentary asset market with two assets, namely investment in capital production and money. An additional financial asset would be needed to study quantitative easing, an unconventional policy recently adopted by some central banks consisting of exchanging money for financial assets. A natural candidate would be private credit, but the introduction of private credit would demand easing the frictions in the credit markets so part of investments could be used as collateral for loans.

References

Balasko, Y., Shell, K.: The Overlapping-Generations Model, I: The Case of Pure Exchange without Money, Journal of Economic Theory **23**, 281–306 (1980).

Carlstrom, C., Jacobson M.: Do Forecasters Agree on a Taylor Rule?, Economic Commentary 2015-10 (2015).

Chiu, J., Molico, M.: Liquidity, redistribution, and the welfare cost of inflation, Journal of Monetary Economics **57**, 428–438 (2010).

Diamond, P.: National Debt in a Neoclassical Growth Model, American Economic Review **55**, 1126–1150 (1965).

Farhi, E., Tirole, J.: Bubbly Liquidity, Review of Economic Studies **79**, 678–706 (2016).

Gale, D.: Pure Exchange Equilibrium of Dynamic Economic Models, Journal of Economic Theory **6**, 12–36 (1973).

Gu, C., Mattesini, F., Wright, R.: Money and Credit Redux, Econometrica **84**, 1–32 (2016).

Kehoe, T., Levine D.: Debt-Constrained Asset Markets, Review of Economic Studies **60**, 865–888 (1993).

Kiyotaki, N., Moore, J.; Liquidity, Business Cycles, and Monetary Policy, Journal of Political Economy (in press).

Kocherlakota, N.: Bursting Bubbles: Consequences and Cures, unpublished manuscript (2009).

Kunieda, T.: Asset Bubbles and Borrowing Constraints, Journal of Mathematical Economics **44**, 112–131 (2008).

Liu, Z., Wang, P.; Credit Constraints and Self-Fulfilling Business Cycles, American Economic Journal: Macroeconomics **6**, 32–69 (2014).

Martin, A., and Ventura, J.: Economic Growth with Bubbles, American Economic Review **102**, 3033–3058 (2012).

Martin, A., Ventura, J.: Managing Credit Bubbles, Journal of the European Economic Association **14**, 753–789 (2016).

Miao, J., Wang, P.: Bubbles and Total Factor Productivity, American Economic Review **102**, 82–87 (2012).

Miao, J., Wang, P.: Asset Bubbles and Credit Constraints, unpublished manuscript (2017).

Samuelson, P.: An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money, Journal of Political Economy **66**, 467–482 (1958).

Sorger, G.: Active and Passive Monetary Policy in an Overlapping Generations Model, Review of Economic Dynamics **8**, 731–748 (2005).

Tirole, J.: Asset Bubbles and Overlapping Generations, Econometrica **53**, 1499–1528 (1985).

Tvede, M.: Overlapping Generations Economies, Palgrave Macmillan, London (2010).

Williamson, S.: Monetary policy and distribution, Journal of Monetary Economics **55**, 1038–1053 (2008).