Least-Squares Optimal Contrast Limited Histogram Equalisation

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Abstract

Contrast Limited Histogram Equalisation moves the input image histogram gently towards one which has a more uniform distribution. Viewed as a tone mapping operation, CLHE generates a tone curve with bounded max and min slopes. It is this boundedness which ensures that the processed images have more detail but few artefacts. Outside of limiting contrast, recent improvements to histogram equalisation include constraining the tone curve to make good whites and blacks and constraining the tone curve to be smooth.

This paper makes three contributions. First, we show that the CLHE formalism is not least-squares optimal but optimality can be achieved by reformulating the problem in a quadratic programming framework. Second, we incorporate the additional constraints of tone curve smoothness and good whites and blacks in our quadratic programming CLHE framework. Third, experiments demonstrate the utility of our method.

I - Introduction

A greyscale image consists of a single-channel where each pixel value is a scalar that represents its display intensity. The histogram can convey much information about an image and what needs to be done to improve it. If the range of dark values in the histogram are large compared to the rest of the brightness range then the image probably looks to dark and would need brightening in order to see all the details. Another important example, is when many brightnesses appear in a narrow band of histogram bins. This can happen, for example when there is a single dominant colour (e.g. a large area of sky) in the image. Unlike the 'brightening' example, we probably do not want to stretch the bins for the sky since doing so will introduce visual detail that was not apparent and this would come across as unwanted noise.

Histogram Equalisation (HE) is a contrast enhancement technique that brings out details within an image through manipulation of the associated grey level histogram. To understand how HE works we need to formally consider how histograms should be represented (and then manipulated). The number of possible values in an image is dependent on numerical encoding, although it is typical to assume 8-bit encoding and therefore 2^8=256 possible values. The histogram of an **i**nput greyscale image can be usefully thought as a vector $\mathbf{h}^i \in \mathbf{R}^{256\times 1}$, with each element in h^i_k counting the occurrences of the kth pixel intensity in the image. Rather than capturing raw counts of intensities we normalise the histogram so that it sums to 1 (it is a probability density function or PDF).

The goal of HE is to obtain a contrast-enhanced image with an associated PDF wherein the probability of each pixel value is equal, that is we map $\mathbf{h}^i$ to $\mathbf{h}$ such that $h_k = 1/256$ (remembering we use normalised histograms and we are assuming 256 quantisation levels). We say that the image histogram is made to be uniform. Intuitively, an image that uses all intensity levels equally must make image detail more conspicuous. This informal observation can be made more concrete by appealing to information theory. It is well known that encoding brightnesses in an image where the histogram is uniform requires more bits in comparison to an image with a non-uniform histogram [11]. Or, in the parlance of information theory, a uniform histogram has greater entropy (more information) than a histogram that is non-uniform. How then is histogram equalisation implemented? Well, the key observation to make is that there exists a unique tone curve - a mapping from input to output brightnesses - that results in the histogram of the output image being uniform.

We illustrate histogram equalisation in Figure 1. In Figure 1a we show an RGB colour image. From this image we calculate its brightness histogram where brightness is calculated as (R+G+B)/3. The brightness histogram of Figure 1a is shown in 1c. The tone curve which carries out histogram equalisation is shown in Figure 1e. Applying this tone curve to the input brightnesses results in the histogram shown in 1d. Note this histogram is not completely uniform as we must map all input values to the same output values and a completely uniform output histogram requires that the same input brightness must map to two different output brightnesses[12]. The tone map in Figure 1e - applied to each of the R, G, and B channels independently - when applied to 1a results in image 1b. This tone map is the cumulative histogram (CDF, or cumulative distribution function) of the input brightness histogram[13].

The image in Figure 1b is notably over-enhanced and not preferred. In HE and related methods this over-enhancement can often be attributed to large spikes in the histogram which, concomitantly, result in steep slopes in the tone curve. A consequence of these high slopes is that neighbouring intensities in the original image are mapped so far apart. This results in too much

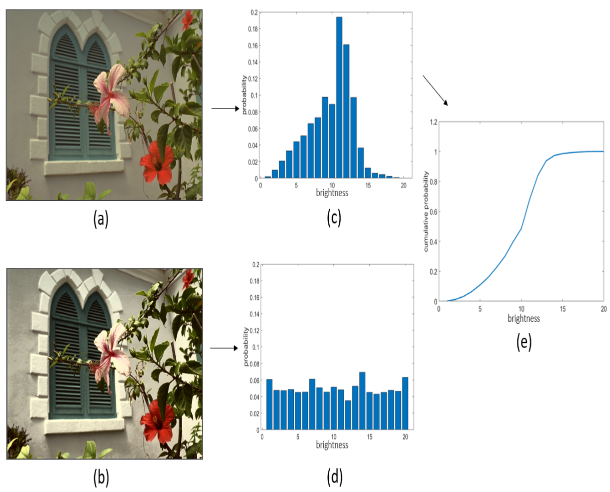

Figure 1. *Demonstration of HE intensity distribution manipulation with flower image. (a) Original image. (c) Associated PDF. (b) Enhanced image using HE. (d) Associated PDF. (e) Tone curve used for pixel mapping obtained as cumulative sum of c.*

(a) (b) (c)

Figure 2. *Comparison of boat using HE and CLHE. (a) Original image. (b) Enhanced using HE method. (c) Enhanced using CLHE method. Max and min slopes of 2 and 0.5 respectively are used.*

contrast. Equally when the slope is too low this can cause problems. For brighter values, the tone curve shown in Figure 1e is almost flat. The meaning of the tone curve in this region is that all bright values are pushed toward white. This means that some fine detail that is apparent in the input, 1a, has been lost in the histogram equalised image, 1b.

Contrast Limited Histogram Equalisation (CLHE) [4] is an extension to the HE approach that tackles the problems of high and low slopes. In CLHE, we seek to make the input brightness histogram more uniform in a way where the slope of the tone curve is bounded above and below. As we shall see, the link to the cumulative histogram and the histogram itself is important in how the CLHE method bounds the slope (see explanation in the background section). We compare HE and CLHE in Figure 2. Left, we have the input image. The histogram equalised image is shown in the middle. The image has too much contrast. CLHE - with max and min slopes of 2 and 0.5 respectively applied to the tone curve, is shown on the right.

The output of CLHE meets the slope constraints but, arguably, is still not *perfect* (we put this in italics as perfection in image enhancement is a subjective term and is not well defined). Compared with Figure 2b the white and black points are less well defined. So, without going all the way to HE, we could better define white and black in 2b and generate an even more preferred image. Additionally, CLHE does not say anything about the 2nd derivative of the tone curve. Indeed, it often introduces wiggly curves and this can manifest itself in output images as low-frequency contouring artefacts. Recent work [3] has sought to recast HE in an optimisation framework to address these issues (it strives to both deliver a good white and black, and a smooth tone curve). But the framework is disjoint from CLHE (does not bound the slope).

This paper makes several contributions. First, we consider what - over and above carrying out something like a slope limited HE - CLHE is trying to achieve. We argue - due to the structure of the algorithm (see next section) for a review - that it is trying to find a least-squares optimal solution but falls short. Our first contribution then is to place CLHE in an optimisation framework that **is** least-squares optimal.

Next we extend [3] - the method which generates smooth tone curves and better defined whites and blacks - so that it falls within our new optimisation framework. In so doing we argue that - at least for some bins - we need to relax the CLHE slope bounds. Specifically, the tone curve should have a minimum slope everywhere but it should be allowed to be slightly less than this minimum at the darkest and brightest values.

Our whole optimisation is shown to be expressible in a Quadratic Programming Framework. That is, we minimize an objective function subject to linear constraints. That we write our optimisation as a quadratic programme has two main advantages. First, quadratic objective functions solved with respect to linear constraints have a unique global optimum and this is found by the QP algorithm [14]. Further this minimum can be found rapidly[15].

Experiments demonstrate two main results. First that the difference between CLHE and least-squares optimal CLHE while in general small is systematically not 0. Second, within the optimisation framework [3] - which is not contrast limited - it is, unlike CLHE, difficult to limit the max and min slope (the only recourse is to manually experiment with the parameters that drive the method). Simply, our optimisation based extension to CLHE (by incorporating [3]) - produces better images.

Other variants of HE are presented in [16]-[18], however this paper focusses on formalising this histogram modification algorithm in CLHE [4] and the optimisation framework of [3].

In section 2 we discuss the background for our work. Our new extended and least-squares optimal CLHE is developed in section 3. In section 4 we report experimental results. The paper finishes in section 5.

II - Background

Now, let us review the CLHE algorithm. Let the histogram of all the brightnesses in the input image $\mathbf{h}^i = hist(I(x,y))$ have N bins and that we assume $I(x,y) \in [0,1]$. Further, we assume that the sum of all the bins in $\mathbf{h}_i$ is 1, $\sum_{j=1}^{N} h^i_j = 1$ (the histogram is a probability distribution). As is well known (but see [13] for a review) the cumulative histogram $\mathbf{c}^i$ defines the tone map. The cumulative histogram is defined as

$$c^i_k = \sum_{j=1}^{k} h^i_j \tag{1}$$

If we plot $(1/N, 2/N, 3/N, \cdots, (N-1/N), 1)$ against $\mathbf{c}^i$ we define the tone curve that maps the input image to the output such that the histogram of output brightnesses is (almost) uniform. That is $o(k/N) = c^i_k$ the output brightness corresponding to the input k/N (where $k \in \{1,2,\cdots,N\}$). As formulated the slope of the brightness mapping function $o()$ is relatively unconstrained. In fact at a given brightness level $o'()$ is in the range $[0,N]$ (it can't be negative as a cumulative histogram is, by definition, strictly increasing).

Let us consider the slope at a brightness level $u \in \{1/N/, 2/N, \cdots, 1\}$. The slope is simply the discrete derivative and can be written as:

$$o'(u) = \frac{c^i_{u*N} - c^i_{u*N-1}}{1/N} = \frac{h^i_{u*N} - h^i_{u*N-1}}{1/N} \tag{2}$$

Now, suppose we wish the minimum and maximum slope to be m and M respectively. Mathematically, we write:

$$m \leq \frac{h^i_{u*N} - h^i_{u*N-1}}{1/N} \leq M \tag{3}$$

Rewriting, we see that

$$L = m/N \leq h^i_{u*N} - h^i_{u*N-1} \leq M/N = U \quad (4)$$

That is, the input histogram as a whole must lie within the interval $[m/N, M/N]$ or $[L, U]$ where L and U denote the lower and upper bound for the normalised histogram count (and $L = m/N$ and $U = M/N$).

In Figure 3, we show an input histogram with 3 bins. The dashed lines are the L and U defined in Equation (4). CLHE attempts to map the input histogram so that it falls within the upper and lower bounds. With respect to Figure 3, CLHE returns bin values delimited by the blue lines.

The CLHE algorithm is discussed in [4]. But, lets recapitulate the algorithm for the upper slope limit only:

Algorithm 1: Contrast Limited Histogram Equalisation

1. $j = 0$
2. $\mathbf{h}^j = \mathbf{h}^i$
3. $j = j+1$
4. $\mathbf{h}^j = min(\mathbf{h}^{j-1}, U)$
5. $\Delta = \sum_{k=1}^N h_k^{j-1} - h_k^j$
6. $\mathbf{h}^j = \mathbf{h}^j + \Delta/N$
7. if $|\mathbf{h}^j - \mathbf{h}^{j-1}| < \varepsilon$ then stop else goto 3

The CLHE algorithm operates iteratively. First, we clip all the bin counts above the upper bound to equal this bound (see step 4). Then the 'Delta' between the original and clipped histogram is redistributed evenly across all N bins. This means the upper bound will be violated so we will need to re-clip, and then continue in this fashion until step 7, is satisfied where the resulting histogram is with a criterion amount above the upper bound. At this point $\mathbf{h}^j$ is used to generate the tone curve that is applied to enhance the original image. Because $\mathbf{h}^j$ meets (more or less) the upper bound U this means the corresponding tone curve's slope is less than the max slope M (more or less). Enforcing the lower bound is carried out analogously i.e. we clip (max) values below the lower bound and then redistribute by *subtracting* the resulting Delta from all bins equally.

We state without proof that steps 4 and 6 - the clipping and redistribution - are themselves 'least-squares optimal'. That is, the operation of CLHE step-by-step attempts to find a new histogram (that obeys the upper bound) and is also close to the original.

II.i - Histogram Modification by Optimisation

The recent literature on histogram modification has been more explicit in formulating the problem as an optimisation. Like CLHE, methods are developed to derive a new histogram from the original which has properties - such as closeness to the uniform histogram - which result in the corresponding tone curve obeying certain defined properties. For example, in "Histogram-based locality-preserving contrast enhancement" (HBLPCE) [1], it is argued that the local shape of the histogram (derived from the original, like CLHE) should be similar to the original and an optimisation is performed to enforce this condition.

A more comprehensive framework for the optimisation approach to histogram modification is presented in [3]. There, an objective function is constructed with several weighted penalty terms that can be tuned to manipulate the characteristics of the modified histogram. The optimisation is written in Equation (5).

$$\min_{\mathbf{h}} ||\mathbf{h} - \mathbf{h}^i||^2 + \lambda||\mathbf{h} - \mathbf{u}||^2 + \gamma||\nabla\mathbf{h}||^2 + \alpha||\mathbf{Sh}||^2 \quad (5)$$

The output of this optimisation is a histogram $\mathbf{h}$ that is close the original input histogram $\mathbf{h}^i$ and this *closeness* is captured by the term $||\mathbf{h} - \mathbf{h}^i||^2$. This optimisation is further conditioned by three penalty terms (which add constraints on the histogram that is derived). The penalty terms are weighted by three user defined scalars λ, γ and α. The first penalty term, $\lambda||\mathbf{h} - \mathbf{u}||^2$ teaches that the derived histogram should be close to $\mathbf{u}$, the uniform histogram. Remembering our discussion of CLHE which maps an input histogram to be within upper and lower bounds (and so also closer to the uniform histogram) this penalty term plays an analogous role. Next, in $\gamma||\nabla\mathbf{h}||^2$, ∇ denotes the first derivative operator, this constraint steers the optimisation to find a histogram (and corresponding tone curve) that is/are smooth.

The last term, $\alpha||\mathbf{Sh}||$ requires more explanation. In [3], it is argued that the tone curve should have a small gradient (close to 0) for brightnesses close to 0 and 1. This enforces a tone curve which is somewhat 's-shaped' in design, and that the input image is mapped to an output image that has good blacks and whites. For this reason they call the third penalty term "black-white stretching". One way we could capture this concept mathematically, would be to write a penalty terms as $\alpha(\sum_{k=1}^b h_k^2 + \sum_{k=w}^N h_k^2)$, where b and w respectively delimit a few brightnesses close to 0 and 1. In fact this is exactly the meaning of the third penalty term. As per our example, $\mathbf{S}$ denotes a $N \times N$ diagonal matrix. The diagonal has 1's for the first and last b and $N - w$ terms respectively and is otherwise all 0.

Notice that this minimization will not, in general, return a new histogram $\mathbf{h}$ that either sums to 1 or satisfies the upper and lower bounds as expressed in CLHE. A heuristic way to meet the upper and lower bounds would be to increase the penalty term γ (and maybe λ too). But, empirically we found this was hard to do automatically and furthermore produced very smooth histograms that had correspondingly overly smooth tone curves (where the effect of the desired enhancement was lessened).

III - Method

While the CLHE algorithm successfully produces a modified histogram subject to the upper and lower bounds, one might consider whether or not this output is the *best* we can do. In this instance the best histogram would be one that obeys the bounds whilst remaining as close as possible to the original. In Figure 3 we use a simple 3-binned histogram to demonstrate that the CLHE formulation does indeed fail to uniformly converge to the best output. With a starting histogram of [0.4, 0.6, 0] (shaded regions), and an upper and lower bound of 0.5 and 0.2 respectively, the CLHE (algorithm 1) output is [0.35, 0.45, 0.2], shown as a solid blue line. The least squares optimal solution, shown as a dashed red line, is [0.3, 0.5, 0.2]. Note that the third bin in both modified histograms has a non-0 value despite the initial value of 0 in the starting histogram, this is in compliance with the minimum slope bound.

Now, let us present our method. Broadly, we wish to adopt the optimisation framework of [3] - shown in Equation (5) and - incorporate it with CLHE. There are three technical issues we must address. First, we need to modify the optimisation to incorporate the upper and lower slope limits. Second, the optimisation should also directly return a histogram that sums to 1. Lastly, we need to modify the black-white stretch idea so that is can work in a CLHE framework.

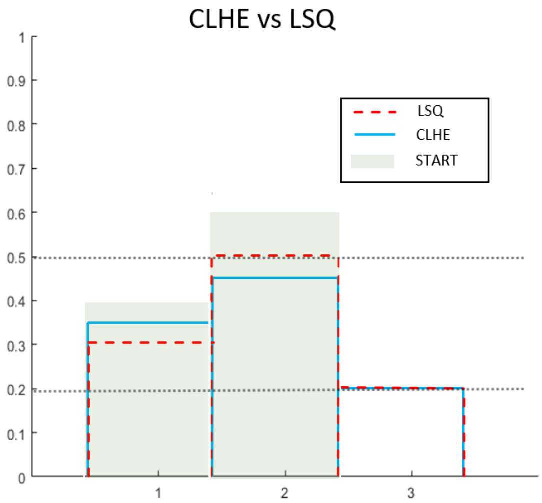

Figure 3. *Comparison of the CLHE algorithm and the least squares solution for a modified histogram. Max and min slopes of 0.5 and 0.2 respectively are used. Starting values [0.4, 0.6, 0].*

Least-squares optimal CLHE

Let us first directly rewrite CLHE as a constrained optimisation, where we use the same notation as Equation (5). As before we assume the histogram has N bins.

$$\min_{\mathbf{h}} ||\mathbf{h}-\mathbf{h}^i||^2 \qquad (6a)$$

$$s.t. \begin{cases} h_k \geq L\ ,\ k=1,2,\cdots,N \\ h_k \leq U\ ,\ k=1,2,\cdots,N \\ \sum_{k=1}^{N} h_k = 1 \end{cases} \qquad (6b)$$

The objective term in (6a) captures the idea that we wish to find a histogram $\mathbf{h}$ which is as close as possible to an input histogram $\mathbf{h}^i$ as is possible in the least-squares sense. The *constraints* in (6b) define the properties the solution has to satisfy. The first two terms refer to the upper and lower bounds. And the third term stipulates that $\mathbf{h}$ should sum to 1. Both the input and output histograms are probability density functions, they sum to 1.

Equation (6) has a quadratic objective function (6a) with linear inequality and equality constraints. That is, it defines a *Quadratic Program*. Not only do efficient algorithms exist for solving quadratic programs [15] they are also guaranteed to find the global optimum. Returning to Figure 3, the least-squares solution (shown in red dashed lines) was found by solving the quadratic program defined in Equation (6) (where $N = 3$ and $L = 0.2$ and $U = 0.5$).

Least-squares optimal CLHE with additional constraints

Now we wish to extend the least-squares optimal optimisation to incorporate the constraints set forth in Equation (5). At first glance the extension seems to be straightforward. After all, Equation (5) minimizes a quadratic objective, so we might simply use Equation (5) instead of Equation (6a). Well this is true to an extent. The optimisation in Equation (7) - using the linear constraints Equation (6a) - can be found directly using quadratic programming.

$$\min_{\mathbf{h}} ||\mathbf{h}-\mathbf{h}^i||^2 + \lambda||\mathbf{h}-\mathbf{u}||^2 \qquad (7)$$

However, the remaining two penalty terms (from Equation (5)) are not so simply transcribed. Lets consider the penalty term $\gamma||\nabla\mathbf{h}||^2$. Here ∇ denotes the derivative operator. But, this is conceptual and we need to make this concrete in order to implement the quadratic program.

Let us define the $N \times N$ *difference* matrix:

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

For our histogram the vector $\mathbf{Dh}$ denotes the discrete derivative of $\mathbf{h}$. Thus, we can add smoothness to our optimisations as:

$$\min_{\mathbf{h}} ||\mathbf{h}-\mathbf{h}^i||^2 + \lambda||\mathbf{h}-\mathbf{u}||^2 + \gamma||\mathbf{Dh}||^2 \qquad (8)$$

The final penalty term (“black white stretching” introduced in Equation (5)) was $\alpha||\mathbf{Sh}||^2$ where $\mathbf{S}$ is an $N \times N$ diagonal matrix that is all 0 except for the first and last few terms along the diagonal. If the penalty term α is large the optimisation will attempt to find a solution where the histogram is as close to 0 as possible for the brightnesses close to black and close to white. This has the effect of making a tone curve that is very flat near 0 and 1, i.e making a tone curve which makes a good white and black (is somewhat S-shaped in nature).

Of course, in our constrained quadratic programming framework the minimum slope is implicitly defined by the lower bound L. Empirically, we found that the lower slope limit is too strong a constraint for black white stretching. That is, enforcing the min slope from CLHE constraint directly can result in less well defined whites and blacks than we might wish. Thus, we suggest weakening the lower-slope limit for the first and last few bins in the histogram. We found constraining the slope in the white and black region to be 50% of the minimum slope limit was usually sufficient to provide good visual results. Thus, our final minimization, titled LSQCLHE, is written as:

$$\min_{\mathbf{h}} ||\mathbf{h}-\mathbf{h}^i||^2 + \lambda||\mathbf{h}-\mathbf{u}||^2 + \gamma||\mathbf{Dh}||^2 + \alpha||\mathbf{Sh}||^2 \qquad (9a)$$

$$s.t. \begin{cases} h_k \geq L/2\ ,\ k \in [1,b]\ or\ k \in [w,N] \\ h_k \geq L\ ,\ k \in (b,w) \\ h_k \leq U\ ,\ k \in [1,N] \\ \sum_{k=1}^{N} h_k = 1 \end{cases} \qquad (9b)$$

Note because $L = m/N$ then $L/2 \Rightarrow m/2$ i.e. we divide the limit of the minimum slope by 2 (for the brightness values near white and black).

IV - Experiments

The example shown in Figure 3 illustrates that, for the toy example shown, CLHE is not least-squares optimal. We wished to investigate whether this result holds in general and if it holds, how large is the discrepancy between the traditional CLHE and the least-squares optimal CLHE we develop here.

We calculate the % deviation of the derived histogram $\mathbf{h}$ (which meets the tone curve slope constraints) and the input histogram $\mathbf{h}^i$ as $||\mathbf{h}^i - \mathbf{h}||\ /\ ||\mathbf{h}^i||$,. We calculate this % error for CLHE and least-squares optimal CLHE for all images in the Kodak standard dataset [10]. For max and min slope parameters of 2 and 0.5 respectively we obtain the results shown in Table 1.

The results demonstrate that the difference between CLHE and our least-squares optimal variant is modest. But, for every image there is a discrepancy between the CLHE result and the

	% Error	
Kodak Image #:	**Least-squares**	**CLHE**
1: *stone building*	28.72%	30.62%
2: *red door*	84.41%	84.62%
3: *hats*	32.55%	32.89%
4: *portrait of girl in red*	28.40%	30.46%
5: *motocross bikes*	19.15%	20.28%
6: *sailboat at anchor*	40.90%	41.01%
7: *shuttered windows*	45.20%	45.23%
8: *market place*	12.35%	12.65%
9: *sailboats under spinnakers*	42.04%	43.02%
10: *off-shore sailboat race*	37.68%	38.04%
11: *sailboat at pier*	59.51%	60.15%
12: *couple on beach*	55.29%	56.02%
13: *mountain stream*	20.10%	20.78%
14: *white water rafters*	20.67%	20.99%
15: *girl with painted face*	32.06%	32.90%
16: *tropical key*	27.06%	28.84%
17: *monument*	39.24%	39.92%
18: *model in black dress*	37.40%	37.93%
19: *lighthouse in Maine*	24.98%	25.96%
20: *P51 Mustang*	90.34%	90.43%
21: *Portland Head Light*	51.94%	52.80%
22: *barn and pond*	29.37%	31.07%
23: *two macaws*	26.77%	27.47%
24: *mountain chalet*	33.76%	35.17%
Average Difference:	**0.81%**	

Table 1: Percentage Error for histograms obtained using least-squares (Equation (6)), and CLHE (algorithm 1).

best result available. Further, while indicative of general performance, it could be that there exist images where the performance difference is greater. Crucially to make CLHE least-squares optimal we reformulated the problem as an optimisation. This allows us to combine CLHE with the optimisation developed in [3] (which is more advanced but still allows arbitrary slopes and potentially arbitrary artefacts).

Next we compare images obtained through our new proposed optimisation method, Equation (9), with those delivered by [3] (which is the same optimisation but where the max and min slopes are not controlled). In Figure 4, we compare three images. Left, we have the input image. The images enhanced with the proposed method are shown in the middle. On the right we show images enhanced with the Equation (5). For our method the max and min slope is 2 and 0.5 respectively (see Figure caption for penalty parameters used). The number of histogram bins was 256.

The image in Figure 4c is clearly over-enhanced. Significant detail is lost on the doorknob and the grains in the wood are inconsistently brightened rendering the image unnatural. In 4b the image is pleasing with enhanced details and does not present with the inconsistent brightness problems of 4c. The max slope constraints of CLHE has helped the optimisation of [3] to produce a pleasing output. The histograms obtained with both methods are shown in Figure 5. The maximum slope (see the dotted grey line) is far exceeded without constraints and 54.18% of the values in the PDF lie above this point (see shaded region in Figure 5).

The images in Figure 4e and 4f, and 4h and 4i, tell a similar story. The images on the right exhibit contrast that is unnatural and not preferred. Both regions contain a large and relatively homogeneous sky region that exhibits noise artefacts when the tone curve has too great a slope (for (f) look at the sky region in the top right of the image). For our optimisation, the sky looks good in (e) and (f) and overall the reproduction is pleasing.

We make two final remarks. First, that the non sloped limited optimisation [3] can generally be tuned to give good looking outputs. But, the tuning (the setting of penalty terms) is very image dependent. By incorporating min and max slopes the precise setting of the penalty terms is much less important. Second, we have started carrying out preference experiments to test image quality more systematically. The results will be reported in the final paper. However, the initial results indicate that visually CLHE and Quadratic Programming CLHE produce similar images (preferred over the original) for tone curves with a small max slope and large min slope. In general, for the optimisation approach reported in [3] adding in slope limits almost always makes the processed image more preferred.

V - Conclusion

Histogram Equalisation (HE) is perhaps the oldest image enhancement method. The input brightness histogram is mapped to a uniform counterpart and in so doing the resulting image has *more information*. The mapping function is a simple tone curve. HE often fails because the tone curve can have arbitrarily high or low slope. Contrast Limited Histogram Equalisation (CLHE) is a method for moving the input histogram towards a uniform counterpart such that the min and max slopes are bounded. CLHE, in general, produces images that are more preferred. Other recent advances in HE - where the slope of the tone curve is **not** constrained - [3] include enforcing the output tone curve to be smooth and somewhat S-shaped in nature (thereby generating good whites and blacks).

In this paper we make 3 contributions. First, we develop a Quadratic Programming version of CLHE which - in terms of how CLHE works - is shown to be least-squares optimal. Second, within this framework, we show that it is easy to add in constraints on tone curve smoothness and to ensure the reproduced image has a well defined white and black. Thus, we arrive at a unified contrast limited optimised HE. Finally, experiments demonstrate the utility of our approach.

Acknowledgements

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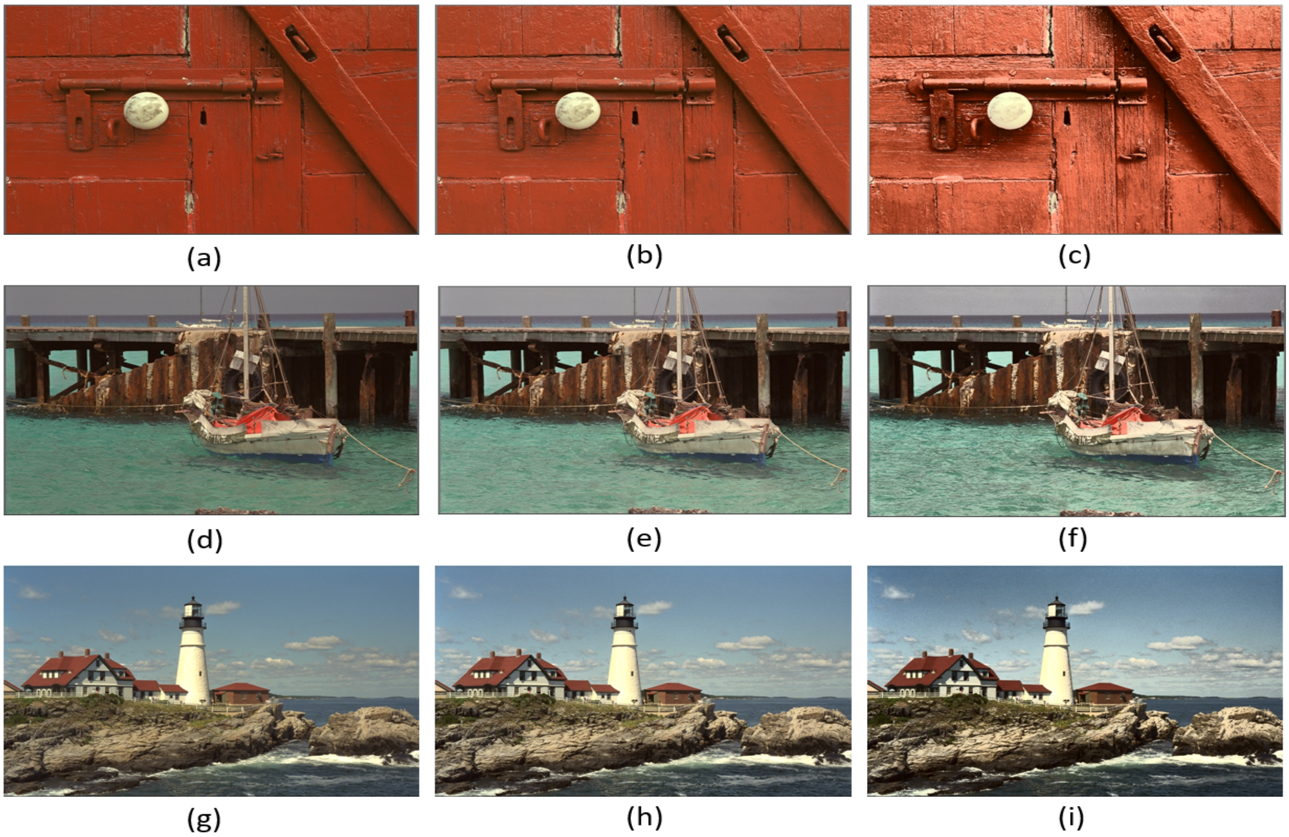

Figure 4. *Left, original images. Middle, images enhanced with the proposed method (Equation (9)). Right, images enhanced with unconstrained method presented in [3] (Equation (5)).* $\lambda = 1$, $\gamma = 5$, $\alpha = 5$, $b = 2, w = 98$, *for both equations. Min and max slope constraints of 2 and 0.5 respectively used in Equation (9).*

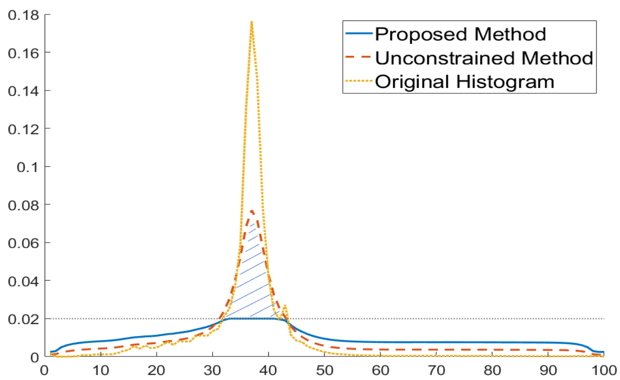

Figure 5. *Modified histograms obtained using the algorithms described in Figure 4, for images 4b (solid blue line), and 4c (dashed orange line). Input histogram shown as dotted yellow line.*

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