Zariski Geometries on Strongly Minimal Unars

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Abeer Albalahi

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Abstract

In this thesis, we will axiomatize the theory of a strongly minimal unar, that is, a structure $\mathcal{A}$ in the language $L = \langle f \rangle$ where f is a unary function. We will first classify the strongly minimal unars where f is injective and give complete axiomatizations for them. Then we will show that these theories have quantifier elimination after adding some relation symbols. Then we will topologize $\mathcal{A}$ and prove that this topology satisfies the Zariski axioms. Finally we will classify all the strongly minimal unars, not necessarily the injective ones, and give an axiomatization of some of the cases of the classification.

Contents

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1

Introduction

Strongly minimal sets play an important role in modern model theory where the local properties of the geometry of strongly minimal sets have an influence on the global properties of structures.[12, p.289]

Motivation 1.1. Morley's categoricity theorem states that if T is a countable theory, then T is κ-categorical for some uncountable κ if and only if it is κ categorical for all uncountable κ.
By [12, Corollary 6.1.16 and Theorem 6.1.18] we can deduce that strongly minimal sets exist in every uncountable categorical theory. By [12, Corollary 6.1.12] we can see that any model of a strongly minimal theory is determined up to isomorphism by its dimension.
A geometry on the algebraic closure of strongly minimal sets can be a disintegrated geometry, a locally modular (not disintegrated) geometry, or a non-locally modular geometry. Examples of such geometries are a pure set, vector spaces, and algebraically closed fields.
The trichotomy conjecture by *Zilber* is that every strongly minimal set which is not locally modular interprets an algebraically closed field. However, this conjecture is false as *Hrushovski* in 1993 [4] found counter examples where he constructed nonlocally modular strongly minimal sets which do not interpret even a group. In 1996 [6], *Hrushovski* and *Zilber* defined Zariski geometries and proved that if M is a strongly minimal Zariski geometry and not locally modular then M interprets an algebraically closed field.

Some locally modular strongly minimal sets satisfy the axioms of Zariski geometries such as a pure infinite set, vector spaces, and their affine spaces.

However, we do not know if there are examples of locally modular strongly minimal sets which are not Zariski geometries. The table below summarises what is known. This project is aiming to see if all locally modular strongly minimal sets are Zariski geometries, or if we have new interesting examples of strongly minimal sets which are not Zariski geometries.

More precisely, the question is "*Is every locally modular strongly minimal set a Zariski Geometry?*". This thesis starts to answer this question.

	Strongly minimal set which is a Zariski geometry	Strongly minimal set which is not a Zariski geometry
Non-locally modular	Algebraically closed fields	New strongly minimal sets (interesting behaviour)
Locally modular (not disintegrated)	Vector spaces Affine spaces	?
Disintegrated	Pure sets Unars studied in this thesis	?

Table 1.1: Strongly Minimal Sets

We aim in this thesis to classify strongly minimal sets in the language of a unary function symbol f, then topologize these structures so they satisfy the Zariski geometry axioms.

The thesis consists of seven chapters.

Chapter 2 is divided into two subsections. The first subsection is about the geometry of strongly minimal sets with basic definitions and lemmas. We also give a brief discussion of the pregeometry on the basic examples of strongly minimal sets. Then we prove that a strongly minimal theory is uncountably categorical. The second subsection is about Morley rank and degree with basic definitions and lemmas particularly in strongly minimal theories.

In Chapter 3 we review the papers which were our main sources in this project. The main paper is "Categorical Theories of a Function" by Yu. E. Shishmarev where he classifies the categorical unars in general whereas we classify the strongly minimal unars.

In Chapter 4, we take a pure set and topologize it then we prove that this topology satisfies the Zariski axioms.

In Chapter 5, we work on the classification of strongly minimal injective unars. We first axiomatize the theory of an injective unar especially the strongly minimal ones and then we prove that this theory is complete. Then we show this theory with some conditions has quantifier elimination.

Chapter 6 is about Zariski geometry on strongly minimal injective unars. We will topologize a strongly minimal injective unar and we will add a condition named (*), which will be defined and justified in this chapter, to this topology. Then we will prove that this topology satisfies the Zariski axioms.

In Chapter 7, we will give the classification of all strongly minimal unars, not only the injective ones. The attempt in this chapter was to give a complete axiomatization for all cases of our classification but as we lack time we will give an axiomatization in some cases.

2

Background Material

2.1 Geometry of Strongly Minimal Sets

The notations and conventions used in this thesis are from [12] but they are standard. A structure is a set equipped with relations, functions, and constants corresponding to the symbols of a first order language L. For example, $\langle \mathbb{N}, s, 0 \rangle$ is the structure of the natural numbers with the successor function s and constant symbol 0. We use a curly letter to denote the structure and a Roman letter to denote the domain of the structure such as $\mathcal{N} = \langle \mathbb{N}, s, 0 \rangle$. We write $A \subseteq B$ to mean that A is a subset of B. The notation $\mathbb{N}^+$ stands for the set of positive integers. We write $\overline{a}$ to denote a finite sequence $(a_1, ..., a_n)$, and $\overline{a} \in A$ to denote $(a_1, ..., a_n) \in A^n$. For a structure $\mathcal{A}$, $|A|$ is the cardinality of the domain A. We will use $\overline{x}$, $\overline{y}$ as tuples of variables and $\varphi(\overline{x}, \overline{y})$ as a formula in the variables $\overline{x}$ and $\overline{y}$. We write $\mathcal{A} \vDash \varphi(\overline{a})$ to mean that $\varphi(\overline{a})$ is true in $\mathcal{A}$. Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Then L_A is the language obtained by adding constant symbols to L for each element in A and $Th_A(\mathcal{M})$ is the set of all L_A-sentences true in $\mathcal{M}$. We say that an L-theory T is satisfiable if there is an L-structure $\mathcal{M}$ such that $\mathcal{M} \vDash T$.

Definition 2.1. [12, p.115] A set p of L_A-formulas in free variables $x_1, ..., x_n$ is called an n-type if $p \cup Th_A(\mathcal{M})$ is satisfiable. The set p is called a complete n-type if for all L_A-formulas φ with free variables from $x_1, ..., x_n$ either $\varphi \in p$

or $\neg\varphi \in p$. The set of all complete n-types is denoted by $S_n^{\mathcal{M}}(A)$.

Suppose p is an n-type over A. Then p is realised in $\mathcal{M}$ if there exists $\overline{a} \in M^n$ such that $\mathcal{M} \vDash \varphi(\overline{a})$ for all $\varphi \in p$.

Let T be a complete theory with an infinite model in a countable language L.

Definition 2.2. [12, p.138] Let κ be an infinite cardinal. A model $\mathcal{M}$ of T is called κ-saturated if for all $A \subseteq M$ with $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realised in $\mathcal{M}$.

Definition 2.3. [9, p.117] Let $\mathcal{M}$ be an L-structure and $A \subseteq M$. Let $X \subseteq M^n$. We say X is definable with parameters from A if and only if there is an L-formula $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ and elements $a_1, ..., a_m \in A$ such that $X = \{(x_1, ..., x_n) \in M^n : \mathcal{M} \vDash \varphi(x_1, ..., x_n, a_1, ..., a_m)\}$.

Definition 2.4. [16, p.1] An infinite definable set $X \subseteq M^n$, where X is definable with parameters, is called minimal if every definable (with parameters) subset of X is either finite or cofinite. If $\varphi(\overline{x}, \overline{a})$ is the formula that defines X, then $\varphi(\overline{x}, \overline{a})$ is minimal. We say that X and $\varphi(\overline{x}, \overline{a})$ are strongly minimal if $\varphi(\overline{x}, \overline{a})$ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$.

Typical examples of strongly minimal sets are as follows:

1. An infinite set in the language of equality.
2. The structure $\langle \mathbb{N}; s, 0\rangle$ consisting of the set of natural nembers equipped with the successor function.
3. An infinite vector space over a field K.
4. An algebraically closed field.

Definition 2.5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and A be a subset of M. We say that $b \in M$ is *algebraic* over A if there is an $\mathcal{L}$-formula $\varphi(v, \bar{w})$ and $\bar{a} \in A$ such that $\mathcal{M} \models \varphi(b, \bar{a})$ and $\{y \in M : \mathcal{M} \models \varphi(y, \bar{a})\}$ is finite.

So, the algebraic closure, denoted by $acl(A)$, is the union of all finite A-definable subsets. In algebraically closed fields, this is equal to the field theoretic algebraic closure.
In strongly minimal structures, acl gives rise to a special feature, called a 'pregeometry'.

Definition 2.6. [11, p.192] Let X be a set and let $cl : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be an operator on the power set of X. We say that (X, cl) is a pregeometry if the following conditions hold:

i) If $A \subseteq X$, then $A \subseteq cl(A)$ and if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$. $(Monotonicity)$

ii) If $A \subseteq X$, then $cl(cl(A)) = cl(A)$.

iii) If $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\})$, then $a \in cl(A)$ or $b \in cl(A \cup \{a\})$. $(Exchange)$

iv) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$. $(Finite\ nature\ of\ closure)$

Remark 2.7. In Definition 2.6, the properties (i), (ii), and (iv) are true of algebraic closure in any structure M. Also, exchange holds in any strongly minimal set.

Lemma 2.8. [11, p.192] If D is a strongly minimal set, then (D, acl) is a pregeometry.

Using Remark 2.7, (i), (ii), and (iv) hold in (D, acl). So we only need to show that (iii) holds and thus (D, acl) is a pregeometry.

Proof. See[11, p.192] □

Definition 2.9. [12, p.290] If (X, cl) is a pregeometry, then $A \subseteq X$ is independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$. We say that B is a basis for $Y \subseteq X$ if $B \subseteq Y$ is independent and $Y \subseteq acl(B)$.

The dimension of a pregeometry Y, denoted by $\dim(Y)$, is the cardinality of its basis and this dimension is well-defined. In other words, any two bases for Y have the same cardinality.

Lemma 2.10. [12, p.210] Let $A, B \subseteq D$ be independent with $A \subseteq acl(B)$.

i) Suppose that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cup B_0$ is a basis for $acl(B)$ and $a \in A \setminus A_0$. Then, there is $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ is a basis for $acl(B)$.

ii) $|A| \leq |B|$.

iii) If A and B are bases for $Y \subseteq D$, then $|A| = |B|$.

Proof. See [12, p.210] □

Definition 2.11. Let (X, cl) be a pregeometry and $A \subseteq X$. We define the localization $cl_A(B) = cl(A \cup B)$.

The following are important properties of pregeometry:

Definition 2.12. [12, p.290] Let (X, cl) be a pregeometry:

i) We say that (X, cl) is trivial if $cl(A) = \bigcup_{a \in A} cl(\{a\})$ for any $A \subseteq X$.

ii) We say that (X, cl) is modular if for any finite dimensional closed $A, B \subseteq X$,

$$\dim(A \cup B) = \dim A + \dim B - \dim (A \cap B).$$

iii) We say that (X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Now we will examine some examples of strongly minimal sets to see what type of pregeometry they give rise to.

Example 2.13. [12, p.291]

i) The *acl* on $\langle X, = \rangle$ and $\langle \mathbb{Z}, s \rangle$ is a trivial pregeometry. For $\langle X, = \rangle$, we have $acl(x) = \{x\}$ for all $x \in X$ and $acl(\varnothing) = \varnothing$ and $acl(Y) = Y$ for all $Y \subseteq X$. For $\langle \mathbb{Z}, s \rangle$, we have $acl(z) = \{s^n(z) : n \in \mathbb{Z}\}$, $acl(\varnothing) = \varnothing$, and $acl(Y) = \{s^n(y) : y \in Y, n \in \mathbb{Z}\} = \bigcup_{y \in Y} \{s^n(y) : n \in \mathbb{Z}\} = \bigcup_{y \in Y} acl(\{y\})$ for all $Y \subseteq \mathbb{Z}$. Note that the algebraic closure in $\langle X, = \rangle$ for the empty set is empty and for each element in X is itself. So *acl* is a trivial geometry.

ii) Let $\mathcal{V} = \langle V, +, 0, \lambda_k : k \in K \rangle$ where V is an infinite vector space over a division ring K. Then V is a strongly minimal set. For a subset $A \subseteq V$, $span(A)$ is the set of all K-linear combinations of elements of A. So $acl(A) = span(A)$, $cl(\varnothing) = \{0\}$, and for each $a \in V \setminus \{0\}$ and for each $\lambda_k \in K$ we have $cl(a) = \{\lambda_k a : k \in K\}$ which is a line through a and 0. So $(\mathcal{V}, cl)$ is a pregeometry. Let A and B be subspaces of a finite dimensional vector space V. By the dimension theorem for intersections of linear subspaces, $dim(A + B) = dimA + dimB - dim(A \cap B)$. So $(\mathcal{V}, cl)$ is a modular pregeometry.

iii) Now let $a, b, c \in V$ be non-collinear. Consider the affine geometry on V where $cl(A)$ is the smallest affine space containing A, $cl(\varnothing) = \varnothing$ and $cl(a) = \{a\}$. This geometry is not modular as $dim(a, b, c, c+b-a) \neq dim(a, b) + dim(c, c+b-a) - dim((a, b) \cap (c, c+b-a))$. However, if we localize this geometry at 0 we will get the vector space pregeometry which is modular. So affine geometry on V is locally modular.

iv) Let K be an algebraically closed field of infinite transcendence degree. We will localize the pregeometry at k where k is a subfield of K of finite transcendence degree. Let a, b, x be algebraically independent over k and $y = ax + b$. So $dim(k(x, y, a, b)/k) = 3$ and $dim(k(x, y)/k) = dim(k(a, b)/k) = 2$ but $acl(k(x, y)) \cap acl(k(a, b)) = k$. So algebraically closed fields are not locally modular.

The following theorem states a significant property of a strongly minimal theory T which is that for each uncountable cardinal we have a unique model of T up to isomorphism. Hence, T is an uncountably categorical theory.

Theorem 2.14. [12, p.211] Suppose T is a strongly minimal theory in a countable language. If $\kappa \geq \aleph_1$ and $\mathcal{M}, \mathcal{N} \models T$ with $|\mathcal{M}| = |\mathcal{N}| = \kappa$, then $\mathcal{M} \cong \mathcal{N}$.

First we need to show that the cardinality of a basis of $\mathcal{M}$ is equal to the cardinality of $\mathcal{M}$. Then we prove that if $\mathcal{M}$ and $\mathcal{N}$ have the same dimension, then they are isomorphic. The proof of this theorem in [12, p.211] use Zorn's Lemma but here we will use the back and forth method to prove this theorem which is an important method and we will use it later in this thesis. The ideas in the following proof are not new.

Proof. Let B be a basis for $\mathcal{M}$. Then, $\mathcal{M} = acl(B)$. So for each $a \in M$ there is a formula $\varphi(x, \bar{y})$ and a tuple $\bar{b}$ from B such that $\varphi(M, \bar{b})$ is finite and $a \in \varphi(M, \bar{b})$. There are $|T| = \aleph_0$ such formulas $\varphi(x, \bar{y})$ and $|B|$ finite tuples from B. Thus, $|M| \leq \aleph_0 \cdot |B| = |B|$. So $|M| \leq |B|$. But $B \subseteq M$ which means $|B| \leq |M|$. Hence $|B| = |M|$.
Let $\mathcal{M}, \mathcal{N} \models T$ with $|M| = |N| = \kappa \geq \aleph_1$. Let B be a basis for $\mathcal{M}$ and C be a basis for $\mathcal{N}$. By the first part of the proof, we have $|B| = |C| = \kappa$. Thus we can choose a bijection $f : B \longrightarrow C$. By [12, Corollary 6.1.7, p.210], f is partial elementary. List M as $(a_\alpha)_{\alpha<\kappa}$ where $\kappa = |M|$. For

each $\alpha < \kappa$ we define a partial elementary map

$$f_\alpha : B \cup \{a_\beta : \beta < \alpha\} \longrightarrow N.$$

Let $f_0 = f$ where f is partial elementary.

If α is a limit ordinal, take $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then f_α is partial elementary. If α is not a limit ordinal, let $\alpha = \beta + 1$. We have f_β is partial elementary and want to define f_α on a_β.

Since $a_\beta \in acl(B)$, $a_\beta \in (B \cup \{a_\gamma : \gamma < \beta\})$. So $tp(a_\beta/B \cup \{a_\gamma : \gamma < \beta\})$ is isolated by a formula $\varphi(x, \bar{d})$ where $\bar{d}$ is a tuple from $B \cup \{a_\gamma : \gamma < \beta\}$. Thus $M \models \varphi(a_\beta, \bar{d})$. So $M \models \exists x\ \varphi(x, \bar{d})$ which means $N \models \exists x\ \varphi(x, f_\beta(\bar{d}))$. Choose $e \in N$ such that$N \models \varphi(e, f_\beta(\bar{d}))$. Since $\varphi(x, \bar{d})$ isolates $tp(a_\beta/B \cup \{a_\gamma : \gamma < \beta\})$ and f_β is partial elementary, $\varphi(x, f_\beta(\bar{d}))$ isolates $tp(e/C \cup \{f_\beta(a_\gamma) : \gamma < \beta\})$. So define $f_\alpha(a_\beta) = e$ and f_α is partial elementary as required. Now let $g = \bigcup_{\alpha<\kappa} f_\alpha$. Then $g : M \longrightarrow N$ is an elementary map. For surjectivity, if $e \in N$, there is a finite tuple $\bar{c}$ from C such that $e \in acl(\bar{c})$ as $N = acl(C)$. Let $\varphi(x, \bar{c})$ be an algebraic formula for e with m realisations. So $N \models \exists^{=m} x[\varphi(x, \bar{c})] \wedge \varphi(e, \bar{c})$. So $M \models \exists^{=m} x[\varphi(x, f^{-1}(\bar{c}))]$.

Suppose $a_1, ..., a_m$ are the realisations in M. Then one of $g(a_1), ..., g(a_m)$ must be e. So g is surjective and hence $M \cong N$. □

2.2 Morley Rank And Morley Degree In Strongly Minimal Theories

Morley Rank

The main references for this section are [12, section 6.2], and [3, section 2, p.23]. Morley rank is an important tool in model theory to analyse ω-stable

theories. It was introduced by M. Morley in his study of complete countable theory T such that T is κ-categorical for some uncountable κ.
Morley rank is an ordinal value to measure the complexity of a definable subset. It generalizes the notion of dimension in algebraic geometry. The idea is that if a definable set S has infinitely many pairwise disjoint definable subsets of rank n, then the rank of S is at least $n+1$.
Morley rank is defined by induction as follow:

Definition 2.15. Let $\mathcal{A}$ be an L-structure and $\varphi(\overline{x})$ be an L_A-formula. The Morley rank for the formula φ in $\mathcal{A}$, denoted by $RM^{\mathcal{A}}(\varphi(\overline{x}))$ is either an ordinal, -1, or ∞. First, we define $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \alpha$ for an ordinal α by induction:

i) $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq 0$ if and only if $\varphi(\mathcal{A})$ is non empty;

ii) if α is a limit ordinal, then $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \alpha$ if and only if $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \beta$ for all $\beta < \alpha$;

iii) for any ordinal α, $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \alpha + 1$ if and only if there are $L_{\mathcal{A}}$-formulas $\psi_1(\overline{x}), \psi_2(\overline{x}), ...$ such that $\psi_1(\mathcal{A}), \psi_2(\mathcal{A}), ...$ is an infinite family of pairwise disjoint subsets of $\varphi(\mathcal{A})$ and $RM^{\mathcal{A}}(\psi_i(\overline{x})) \geq \alpha$ for all i.

Remark 2.16. • If $\varphi(\mathcal{A}) = \varnothing$, then $RM^{\mathcal{A}}(\varphi(\overline{x})) = -1$.
• If $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \alpha$ but $RM^{\mathcal{A}}(\varphi(\overline{x})) \not\geq \alpha + 1$, then $RM^{\mathcal{A}}(\varphi(\overline{x})) = \alpha$.
• If $\varphi(\mathcal{A})$ is finite and not empty, then $RM^{\mathcal{A}}(\varphi(\overline{x})) = 0$.
• If $\varphi(\mathcal{A})$ is infinite but does not contains an infinite family of disjoint infinite definable subsets, then $RM^{\mathcal{A}}(\varphi(\overline{x})) = 1$.
• If $RM^{\mathcal{A}}(\varphi(\overline{x})) \geq \alpha$ for all ordinals α, then $RM^{\mathcal{A}}(\varphi(\overline{x})) = \infty$.

Now we will define Morley rank of φ rather than defining it depending on the model that contains parameters realised in φ.

Definition 2.17. If $\mathcal{A}$ is an L-structure and $\varphi(\overline{x})$ is any L_A-formula, we define $RM(\varphi(\overline{x}))$, the Morley rank of φ, to be $RM^{\mathcal{B}}(\varphi(\overline{x}))$ where $\mathcal{B}$ is any ω-saturated elementary extension of $\mathcal{A}$.

By [12, Corollary 6.2.4], $RM(\varphi(\overline{x}))$ does not depend on the choice of ω-saturated elementary extension $\mathcal{B}$.

Definition 2.18. Let $\mathcal{A} \models T$ and $S \subseteq A^n$ be defined by the L_A-formula $\varphi(\overline{x})$. Then we define the Morley rank of S, denoted by $RM(S)$, to be $RM(\varphi(\overline{x}))$.

In other words, for an ω-saturated model $\mathcal{A}$ and a definable subset $S \subseteq A^n$, $RM(S) \geq \alpha + 1$ if and only if there are $S_1, S_2, \ldots$ pairwise disjoint definable subsets of S such that $RM(S_i) \geq \alpha$ for $i \in \mathbb{N}$.

Morley rank has some properties which we will introduce as follow:

Lemma 2.19. Let $\mathcal{A}$ be an L-structure and let S_1 and S_2 be definable subsets of A^n. Then:

i) If $S_1 \subseteq S_2$, then $RM(S_1) \leq RM(S_2)$.

ii) $RM(S_1 \cup S_2)$ is the maximum of $RM(S_1)$ and $RM(S_2)$.

iii) If S_1 is non empty, then $RM(S_1) = 0$ if and only if S_1 is finite.

Proof. see [12, p.218] □

Morley Degree

Definition 2.20. Let S be a definable set such that $MR(S) = \alpha$. The Morley degree of S, denoted by $MD(S)$ is the maximal number d such that S cannot be partitioned into more than d definable sets of Morley rank α.

The following Proposition shows that the definition of Morley degree in 2.20 is well-defined.

Proposition 2.21. Let φ be an $L_{\mathcal{A}}$-formula with $RM(\varphi) = \alpha$ for some ordinal α. There is a natural number d such that if $\psi_1, ..., \psi_n$ are $L_{\mathcal{A}}$-formulas such that $\psi_1(\mathcal{A}), ..., \psi_n(\mathcal{A})$ are disjoint subsets of $\varphi(\mathcal{A})$ such that $RM(\psi_i) = \alpha$ for all i, then $n \leq d$.

Proof. See [12, p.220]. □

Morley Rank and Degree in Strongly Minimal Theories

In strongly minimal theories Morley rank is the same thing as dimension. Moreover, strongly minimal sets can be defined using Morley rank and Morley degree.

Lemma 2.22. A formula $\varphi(\overline{x})$ is strongly minimal if and only if $RM(\varphi) = 1$ and $MD(\varphi) = 1$.

Proof. Suppose $\varphi(\overline{x})$ is a strongly minimal formula. Then $RM(\mathcal{A}) \geq 1$ as $\varphi(\mathcal{A})$ is infinite. But $\varphi(\mathcal{A})$ cannot be partitioned into more than one infinite definable set. So $RM(\varphi) = MD(\varphi) = 1$.
Now suppose that $RM(\varphi) = MD(\varphi) = 1$. Then by Remark 2.16, $\varphi(\mathcal{A})$ is infinite and cannot be partitioned into more than one infinite definable set. Thus φ is strongly minimal. □

Let $\mathcal{M}$ be a model of a theory T and $A \subset M$.

Definition 2.23. For $p \in S_n(A)$, define $RM(p) = inf\{RM(\varphi) : \varphi \in p\}$. If $RM(p) < \infty$, then $deg_M(p) = inf\{deg_M(\varphi) : \varphi \in p \text{ and } RM(\varphi) = RM(p)\}$.

Definition 2.24. If $A \subset M$ and $\overline{a} \in \mathcal{M}$, then we define $RM(\overline{a})$ to be $RM(tp(\overline{a}))$ and $RM(\overline{a}/A)$ to be $RM(tp(\overline{a}/A))$.

Now let T be a strongly minimal theory, $\mathcal{M} \models T$, $A \subseteq M$ and $\overline{a} \in M^n$. Define $dim(\overline{a}/A)$ to be the cardinality of its basis, with respect to the pregeometry.

In strongly minimal theories, Morley rank is the same as dimension.

Theorem 2.25. Suppose that T is a strongly minimal theory. If $A \subset M$ and $\overline{a} \in M$, then $RM(\overline{a}/A) = dim(\overline{a}/A)$.

Proof. See [12, p.224]. □

3

Papers Review

Let L_f be the language with a single unary function symbol f. A unar is an L_f-structure. The class of unars is a rich source of examples and a class that can be used in solving many problems in model theory [14].

In this section we will analyse three papers which have results about theories of unars. Two papers are "Complete Theories of Unars" and "Totally Transcendental Theories of Unars" by A. A. Ivanov, and "Categorical Theories of a Function" by Yu. E. Shishmarev.

The aim of [14] is to classify the unars whose theory is $\aleph_0$-categorical and/or uncountably categorical. The method for doing this was to describe a complete L_f-theory T as either limited or not limited.

Definition 3.1. [14] A theory T is limited if there is $N \in \mathbb{N}$ such that $T \vdash \forall x[\bigvee_{n,m=1}^{N} (f^n(x) = f^{n+m}(x))]$ and T is not limited otherwise.

The 'root' of an element was the key point in proving categoricity.

Definition 3.2. [14] The root of depth n of an element x is the set

$$K_n(x) = \{y \in A | \exists i \leq n \text{ such that } f^i(y) = x\}.$$

The root of x is

$$K(x) = \bigcup_{n \in \mathbb{N}} K_n(x).$$

Definition 3.3. [14] A connected subset of the root $K_n(x)$ that contains x

is called a subroot of depth n of the element x.

The main results of [14] are the following theorems.

Theorem 3.4. A theory T of unars is countably categorical if and only if T is limited and for any $A \models T$, the set $K(A)$ of isomorphism classes of roots $K(a)$ for $a \in A$ is finite.

The idea of proving that T is countably categorical is using the two conditions which are T is limited and $K(A)$ is finite. As $K(A)$ is finite, there is $N_0 \in \mathbb{N}$ such that for all $x \in A$ either $\{y \in A : f(y) = x\}$ is an infinite set or $\leq N_0$. As T is limited, then every point is a pre-periodic or on a cycle. Thus the set of types in A is finite.Then the proof is conducted by using partition on these types. Then the proof is concluded by constructing connected components using roots such that these connected components satisfies four conditions which guarantee that T is countably categorical.

Theorem 3.5. A limited theory T is uncountably categorical if and only if it satisfies the following conditions:

i) $|f^{-1}(a)|$ is infinite for at most one a, and is otherwise bounded.

ii) If $|f^{-1}(a)|$ is finite for all a, then all except one type of connected component of A are finite.

iii) If there is $a \in A$ such that $|f^{-1}(a)|$ is infinite, then all types of connected components of A are finite and all $K(y)$ for $f(y) = a$ are isomorphic except for finitely many a.

A cycle of a connected component X is a set consisting of all $x \in X$ such that $f^n(x) = x$ where $n \in \mathbb{N}^+$.

A set of N-neighborhood of $X \subseteq A$ is the set $\{y \in A : \exists x \in X \text{ such that } \bigvee_{n,m}^{N} f^n(x) = f^m(y)\}$.

Theorem 3.6. An unlimited theory T is uncountably categorical if it satisfies the following conditions:

i) $|f^{-1}(a)|$ is bounded.

ii) For each $n \in \mathbb{N}$ there are only finitely many connected components whose cycle consist of n elements.

iii) There exists a finite set $X_0 \subseteq A$, a set $Y \subseteq A$, an $m \in \mathbb{N}$, and a set $\{P_a : a \in Y\}$ such that $A = X_0 \cup \bigcup_{a \in Y} P_a$, P_a is a subroot of depth m for $a \in Y$, and for $a, b \in Y$, the subroots P_a and P_b are isomorphic and this isomorphism can be continued to an isomorphism of their $2m$-neighborhoods.

Shishmarev proves the necessity of these conditions using the fact that any uncountably categorical structure $\mathcal{A}$, is homogeneous and any infinite definable subset of $\mathcal{A}$ must have the same cardinality as $\mathcal{A}$. (See [12], Corollary 4.3.39). In this thesis, we use similar ideas, but strong minimality is more powerful than just uncountable categoricity. Shishmarev leaves the sufficiency of these conditions as an exercise. We prove a similar theorem as Proposition 5.5 of this thesis to classify the injective unars.

Another paper on theory of unars is "Totally Transcendental Theories of Unars" by A. A. Ivanov. Its purpose is to study totally transcendental theories of unars, in particular almost categorical theories.
We will introduce some terminology used in [8].

Definition 3.7. [12] A theory T is totally transcendental if, for all $\mathcal{A} \models T$, every $L_{\mathcal{A}}$-formula has ordinal Morley rank (not ∞).

Definition 3.8. [2] A theory T' is a principal extension of T if T' is an extension of T by constants $c_0, c_1, ..., c_{n-1}$ which realize a principal n-type of T.

Definition 3.9. [2] Let Φ be a set of formulas in one variable. We say that Φ is two-cardinal in the theory T if there is a model $\mathcal{A}$ of T and a proper elementary extension $\mathcal{B}$ such that $\varphi(\mathcal{A}) = \varphi(\mathcal{B})$ for any $\varphi \in \Phi$.

Definition 3.10. [8] A complete theory T is almost categorical if there exists a principal extension T' of T and formulas $\varphi_1(x), ..., \varphi_m(x)$ that are strongly minimal in T' such that the formula $\Phi_1(x) \vee ... \vee \Phi_m(x)$ is not two-cardinal in T'.

The class of almost categorical theories of unars corresponds to the class of totally transcendental theories of unars of rank 2. This is a consequence of Theorem 1 in [2] which states that "Any totally transcendental theory of rank 2 is almost categorical". However, there is a counter example to the converse which is an infinite periodic abelian group with period p such that p is not divisible by the square of any prime number. This group is almost categorical but not uncountably categorical and it has an arbitrarily large finite rank [8].

The main theorem in [8] is that the class of almost categorical theories of unars corresponds with the class of $\omega-$stable theories of unars whose Morley rank is 2. This theorem is proved by investigating the set of complete non-algebraic 1-types over a model $\mathcal{A}$ of T and show that this set which contains formulas of the form $f^n(x) = a$ for $a \in A$ is finite as well as the set which does not contains the formulas of the form $f^n(x) = a$.

Theorem 1.1 in [8] states that two unars $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if and only if $\mathcal{M}$ and $\mathcal{N}$ either have the same finite number of $(k, l)-$roots of the same type or there are infinitely many $(k, l)-$ roots of the same type. (One can refer to [8].page 1 and 2 for definition of (k, l)-roots and (k, l)-ranks). So in the case of injective unars, we have that if $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent this means that $k_r(\mathcal{M}) = k_r(\mathcal{N})$ where $r \in \mathbb{N}$ and k_r is finite or both $k_r(\mathcal{M})$ and $k_r(\mathcal{N})$ are infinite.

It is also proved in [8] that the theory of unars admits quantifier elimination after expanding the language $L = \langle f \rangle$ with all predicates defined by certain formulas called basis formulas and the proof is analogous to [12, Corollary 3.6.3] which we also use to prove quantifier elimination in the theory of injective unars. The paper conclude with Theorem 4.1 which gives a condition on a complete theory T of unars for it to be totally transcendental of rank 2.
The second paper by A. A. Ivanov, "Complete Theories of Unars", concentrates on a technical proof regarding (k, l)-roots and (k, l)-rank which were the main source for the work in [8]. Also, the main result in [7] is proving the criterion for two unars to be elementary equivalent.

4

Zariski Geometry on a Pure set

Zariski geometry was introduced by Ehud Hrushovski and Boris Zilber in[5] [17]. It gives a characterisation of the Zariski topology on an algebraic curve and all its powers. As the idea of Zariski geometry is linked to algebraic geometry, we will give a brief introduction about algebraic geometry.

4.1 Introduction

The main references for this section are [1] and [13]. Algebra and geometry are important subjects in mathematics and the connection between them has a significant role in studying mathematical objects. Algebraic geometry is a branch of mathematics which is classically the study of the sets of zeros of polynomial rings. Now modern algebraic geometry uses abstract techniques from commutative algebra for solving geometrical problems about the sets of zeros. Algebraic geometry was motivated by Fermat and Descartes where they investigated the properties of algebraic curves such as conics and cubics. Now algebraic geometry is involved in almost all other branches of mathematics either directly or indirectly[10].

The real beginning of algebraic geometry was in the 19th century where David Hilbert established his fundamental theorems, the " Hilbert Basis

Theorem" and the "Hilbert Nullstellensatz". His results connect algebraic geometry to commutative algebra.

A ring R is Noetherian if it satisfies the ascending chain condition on ideals, that is, given a chain of ideals $I_1, I_2, ...,$ there exists n such that $I_n = I_{n+1} = ...$.

Definition 4.1. [9] Let P be a set of polynomials from $K[x_1, ..., x_n]$. The set $V(P) = \{\overline{a} \in K^n : f(\overline{a}) = 0 \text{ for all } f(\overline{x}) \in P\}$ is the zero set of P. The subset $V(P) \subseteq K^n$ is called an affine algebraic veriety.

Definition 4.2. [9] Let I be an ideal of a ring R. Then I is radical ideal if, for an $m \in \mathbb{N}^+$, for all $f \in R$, if $f^m \in I$, then $f \in I$.

Theorem 4.3. (Hilbert's Basis Theorem:) If R is a Noetherian ring then the polynomial ring $R[x]$ is Noetherian.

It follows from the theorem that every affine algebraic variety is the common zero set of finitely many polynomials.

The set of polynomials vanishing on an affine algebraic variety V forms an ideal in the polynomial ring and this ideal is radical.

Now we state the fundamental theorem "Hilbert's Nullstellensatz".

Theorem 4.4. (Hilbert's Nullstellensatz:) For any ideal $J \subset \mathbb{C}[x_1, ..., x_n]$, $I(V(J)) = \sqrt{J}$ and if J is radical then $I(V(J)) = J$.

Hilbert's Nullstellensatz implies two important results. First, there is a one-to-one correspondence between affine algebraic varieties in $\mathbb{A}^n$ and radical ideals in $\mathbb{C}[x_1, ..., x_n]$. Second, every maximal ideal has the form $(x_1 - a_1, ..., x_n - a_n)$ for $(a_1, ..., a_n) \in \mathbb{A}^n$.

Hilbert's Nullstellensatz can be applied over any field beside the complex field. Given any field k, let K be an algebraically closed field extension and consider the polynomial ring $k[x_1, ..., x_n]$. Let I be an ideal in this ring. Then $V(I) = \{(a_1, ..., a_n) \in K^n | f(a_1, ..., a_n) = 0 \text{ for all } f \in I\}$. If p is a polynomial in $k[x_1, ..., x_n]$ such that $p(a_1, ..., a_n) = 0$ then $p \in I$ and I is finitely generated.

Theorem 4.5. (Fundamental Theorem of Algebra:) Every non-constant polynomial in one variable with complex coefficients has at least one complex root.

This is equivalent to saying that the complex field is an algebraically closed field. We can see there is a link between Hilbert's Nullstellensatz and the Fundamental Theorem of Algebra. For instant, if we take the real field, which is not algebraically closed, we can see that (x^2+1) is a maximal ideal in $\mathbb{R}[x]$ but is not of the form $(x-a)$ which means Hilbert's Nullstellensatz has failed in this case. If we take the algebraic set $V(I(S))$ where $S \subset k^n$ not necessarily an algebraic set then $V(I(S))$ is the smallest algebraic set which contains S and is called the Zariski closure of S. The algebraic subsets of k^n define the closed sets of Zariski topology on k^n. This has established a correspondence between geometric objects namely algebraic sets and algebraic objects namely ideals in an algebraically closed field.

4.2 Zariski Geometry on a Pure set

Since we are working on Zariski geometry on strongly minimal injective unars, it will be helpful if we work first on Zariski geometry on pure sets where it is simpler.
The rest of this chapter is given as an exercise in [12] but as far as we know the details have not been written out before.
Let X be an infinite set. We can topologize X^n by taking the closed sets to be the sets defined by positive quantifier-free formulas in the language of equality. We will show that this topology will determine a Zariski geometry.
A topological space is Noetherian if there is no infinite descending chain of closed sets. A closed set S is irreducible if there are no proper closed subsets S_0 and S_1 such that $S = S_0 \cup S_1$. The closure of a set S in a topological space, denoted by $\overline{S}$, is the smallest closed set such that $S \subseteq \overline{S}$.

Definition 4.6. [12, p.306] A Zariski geometry is an infinite set D and a sequence of Noetherian topologies on $D, D^2, D^3, ...$ such that the following axioms hold.

(Z0) i) If $\pi : D^n \longrightarrow D^m$ is defined by $\pi(x) = (\pi_1(x), ..., \pi_m(x))$ where each $\pi_i : D^n \longrightarrow D$ is either constant or coordinate projection, then π is continuous.

ii) Each diagonal $\Delta^n_{i,j} = \{x \in D^n : x_i = x_j\}$ is closed.

(Z1) (Weak QE): If $C \subseteq D^n$ is closed and irreducible, and $\pi : D^n \longrightarrow D^m$ is a projection, then there is a closed $F \subset \overline{\pi(C)}$ such that $\pi(C) \supseteq \overline{\pi(C)} \setminus F$.

(Z2) (Uniform one-dimensionality):

i) D is irreducible.

ii) Let $C \subseteq D^n \times D$ be closed and irreducible. For $a \in D^n$, let $C(a) = \{x \in D : (a, x) \in C\}$. There is a number N such that, for all $a \in D^n$, either $|C(a)| \leq N$ or $C(a) = D$. In particular, any proper closed subset of D is finite.

(Z3) (Dimension theorem): Let $C \subseteq D^n$ be closed and irreducible. Let W be a nonempty irreducible component of $C \cap \Delta^n_{i,j}$. Then $dimW \geq dimC - 1$ where dim is the same as Morley rank.

Definition 4.7. A Zariski geometry on an L-structure $\mathcal{A}$ is a Zariski geometry on the domain of $\mathcal{A}$ such that every closed set is a definable (with parameters) set in $\mathcal{A}$ and every definable subset $S \subseteq A^n$ is a finite Boolean combination of closed sets.

Definition 4.8. Let $\mathcal{C}_n$ be the collection of positive quantifier-free definable subsets of X^n, for $n \in \mathbb{N}^+$, and $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ in the language of equality with parameters.

This chapter is devoted to proving the following Proposition.

Proposition 4.9. i) For each $n \in \mathbb{N}^+$, $\mathcal{C}_n$ is the set of closed sets of a topology on X^n.

ii) The topology is Noetherian.

iii) The topology satisfies the axioms (Z0), (Z1), (Z2), and (Z3), so make X into a Zariski geometry.

4.2.1 Topology on $\langle X, = \rangle$

In this section, we will prove parts i) and ii) of Proposition 4.9.

A positive quantifier-free formula is built from atomic formulas using $\wedge$ and $\vee$ only. The atomic formulas are precisely $x_i = x_j$ for $i, j = 1, ..., n$, or $x_i = a_i$ for $i = 1, ..., n$ where a_i is a parameter from X, in the structure $\langle X, = \rangle$.

Definition 4.10. (Disjunctive Normal Form; DNF):[15, p.25] A formula of the form $\bigvee_{i=1}^{r} \bigwedge_{j=1}^{s_i} q_{ij}$, where each q_{ij} is either an atomic formula or the negation of atomic formula, is said to be in disjunctive normal form.

Lemma 4.11. (DNF Lemma):[15, p.25] Every quantifier free formula is equivalent to one in DNF, and every positive quantifier free formula is equivalent to a disjunction of conjunctions of atomic formulas.

Let $S \subseteq X^n$ be in $\mathcal{C}_n$. Then, by the Disjunctive Normal Form Lemma, S can be written in the form $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ where $\varphi_{k,l}(\overline{x})$ are atomic formulas and $r, s \in \mathbb{N}$. The set S corresponds to the finite union of S_k where S_k is defined by $\bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$. Hence, it is sufficient to describe sets of the form S_k, which for simplicity we rewrite as $\bigwedge_{l=1}^{s} \varphi_l(\overline{x})$.

Now we will introduce the notion of formulas in special form in $L = \langle = \rangle$.

Definition 4.12. Given a subset $Fix \subseteq \{1, ..., n\}$ and an equivalence relation $\sim$ on $\{1, ..., n\} \setminus Fix$, and $a_i \in X$ for $i \in Fix$, let $\varphi_{Fix,\overline{a},\sim}$ be the formula given by $\bigwedge\limits_{i \in Fix} x_i = a_i \wedge \varphi_\sim$, where $\varphi_\sim$ is a conjunction of atomic formulas $x_i = x_j$ such that

$$\varphi_\sim \vdash x_i = x_j \text{ if and only if } i \sim j.$$

Any formula φ is in special form if there are $Fix, \overline{a}$ and $\sim$ such that $\varphi = \varphi_{Fix,\overline{a},\sim}$.

Let m be the total number of equivalence classes of $\sim$. So if $\varphi_{Fix,\overline{a},\sim}$ defines the subset $S \subseteq X^n$, then $dimS = m$ where $dimS$ denotes the dimension of S.

The dimension of S is the same thing as Morley rank, and S is in definable bijection to X^m. We can see this from Definition 4.12. If x_i is fixed for some $i \in \{1, ..., n\}$ then we take it out from n. If $x_i = x_j$ is one of the formulas of S, then i and j are in the same equivalence class.

Proposition 4.13. Suppose $S \subseteq X^n$ and $S \neq \varnothing$ is defined by a conjunction of atomic formulas. Then there is a formula $\varphi(x_1, ..., x_n)$ in special form defining S.

Proof. If $S = X^n$, then we can take φ to be $x_1 = x_1$ which is in special form. Suppose S is defined by a conjunction of atomic formulas $\bigwedge\limits_{l=1}^{r} \varphi_l$ and $S \neq \varnothing$ and $S \neq X^n$. Define $Fix \subseteq \{1, ..., n\}$ by

$$Fix = \{i : \text{ for some } a \in X, \varphi \vdash x_i = a\}.$$

For $i \in Fix$, let $a_i \in X$ be such that $\varphi \vdash x_i = a_i$. Define $\sim$ on $\{1, ..., n\} \setminus Fix$ by $i \sim j$ if and only if $\varphi \vdash x_i = x_j$.

Let $\varphi_{Fix,\bar{a},\sim}$ be

$$\bigwedge_{i\in Fix} x_i = a_i \wedge \bigwedge_{\{(i,j)|i\sim j\}} x_i = x_j.$$

Then $\varphi_{Fix,\bar{a},\sim}$ is in special form and defines the same set as φ. $\square$

Definition 4.14. Given $\varphi = \varphi_{Fix,\bar{a},\sim}$ in special form define the rank of φ to be $rk\varphi = \omega^m$ where $\sim$ has m equivalence classes.

Definition 4.15. Given $S \in \mathcal{C}_n$ and φ defining S, with $\varphi = \bigvee_{k=1}^{r} \varphi_k$ with each φ_k in special form, define $rk\varphi = \sum_{k=1}^{r} rk\varphi_k$ such that $rk\varphi_1 \geq rk\varphi_2 \geq ... \geq rk\varphi_k$. Define $rkS = min\{rk\varphi|\varphi$ is of the above form and φ defines $S\}$.

Remark 4.16. In Definition 4.15, it is important that $rk\varphi_1 \geq rk\varphi_2 \geq ... \geq rk\varphi_k$ as the ordinal sum is not commutative. For instant, $\omega + \omega^2 = \omega^2$ but $\omega^2 + \omega \neq \omega^2$.

Example 4.17. We will take a closed set S in X^3 as an example. Suppose $\varphi(x_1, x_2, x_3)$ is the formula

$$(x_1 = x_2) \vee (x_2 = x_3),$$

and $\varphi'(x_1, x_2, x_3)$ is the formula

$$\varphi(x_1, x_2, x_3) \vee (x_1 = x_2 \wedge x_3 = a_3) \vee (x_1 = a_1 \wedge x_2 = a_2 \wedge x_3 = a_3).$$

Both $\varphi(x_1, x_2, x_3)$ and $\varphi'(x_1, x_2, x_3)$ are disjunction of formulas in special form. So

$$rk(\varphi) = \omega^2 + \omega^2 = 2\omega^2$$

and

$$rk(\varphi') = 2\omega^2 + \omega + 1.$$

Note that $\varphi(x_1, x_2, x_3)$ and $\varphi'(x_1, x_2, x_3)$ define the same subset S of X^3. Thus $rk(S) = 2\omega^2$. The set S consists of two planes in X^3. So the dimension of these planes are 2. So $rk(S)$ can't be smaller than $2\omega^2$.

Lemma 4.18. If $S_1, S_2 \in \mathcal{C}_n$ then $rk(S_1 \cup S_2) \leq rkS_1 + rkS_2$.

Proof. Let $S_1, S_2 \in \mathcal{C}_n$. Choose φ_1, φ_2, disjunctions of formulas in special form, $\varphi_1 = \bigvee_{k=1}^{r_1} \psi_{k1}$ and $\varphi_2 = \bigvee_{k=1}^{r_2} \psi_{k2}$ with each ψ_{kj} in special form such that $\varphi_1(X) = S_1$ and $\varphi_2(X) = S_2$ and $rkS_1 = rk\varphi_1$ and $rkS_2 = rk\varphi_2$. $S_1 \cup S_2$ is defined by

$$\begin{aligned} \varphi &= \varphi_1 \vee \varphi_2 \\ &= \bigvee_{k=1}^{r_1} \psi_{k1} \vee \bigvee_{k=1}^{r_2} \psi_{k2} \end{aligned}$$

So

$$\begin{aligned} rk\varphi &= \sum rk\psi_{k1} + \sum rk\psi_{k2} \\ &= rk\varphi_1 + rk\varphi_2 \\ &= rkS_1 + rkS_2 \end{aligned}$$

Thus

$$rk(S_1 \cup S_2) \leq rk\varphi$$

□

Lemma 4.19. Suppose $C, S \in \mathcal{C}_n$, and S is defined by a conjunction of atomic formulas,and $C \subsetneq S$. Then $rkC < rkS$.

Proof. Let $C \subsetneq S$. By Proposition 4.13 S is defined by a formula φ in special form. We need to show $rkC < rkS$. Suppose C is defined by $\psi = \bigvee_{k=1}^{r} \psi_k$ each ψ_k in special form. Let C_k be defined by $\psi_k(X)$ for $k = 1, ..., r$. So $C = C_1 \cup ... \cup C_r$. Since C_k is a proper closed subset of S, then

$$X \vDash \forall x_1, ..., x_n[\psi_k(\overline{x}) \longrightarrow \varphi(\overline{x})].$$

So we can assume all conjuncts in φ (atomic formulas) are also in ψ_k. As C_k is a proper subset of S, there must be at least one more atomic formula, say θ in ψ_k. So we have

$$X \models \forall \overline{x} \left[\psi_k(\overline{x}) \longrightarrow (\theta(\overline{x}) \wedge \varphi(\overline{x})) \right].$$

So we can assume $\psi_k = \theta \wedge \varphi$. So θ is either $x_i = a_i$ for some $i \notin Fix(\varphi)$ or θ is $x_i = x_j$ such that $i \nsim_\varphi j$. If θ is $x_i = a_i$ for some $i \notin Fix(\varphi)$, then all j in the equivalence class of i in $\sim_\varphi$ are in $Fix(\psi_k)$ and the number of equivalence classes of $\sim_{\psi_k}$ is the number of equivalence classes of $\sim_\varphi -1$. If θ is $x_i = x_j$ such that $i \nsim_\varphi j$, then $[i]_{\sim_\varphi} \cup [j]_{\sim_\varphi} = [i]_{\sim_{\psi_k}}$. That is, the equivalence classes of i and j under $\sim_\varphi$ are contained in one equivalence class under $\sim_{\psi_k}$. So the number of equivalence classes for $\sim_{\psi_k}$ is the number of equivalence classes for $\sim_\varphi -1$. Let $m = dimS$. Then $rkS = rk\varphi = \omega^m$. For each k, $rkC_k = rk\psi_k = \omega^{dimC_k}$, but $dimC_k < m$, so $rkC_k \leq \omega^{m-1}$. So by Lemma 4.18,

$$rkC \leq \sum_{k=1}^{r} rkC_k \leq r.\omega^{m-1} < \omega^m = rkS.$$

So $rkC < rkS$.

$\square$

Corollary 4.20. If $S \in \mathcal{C}_n$ is defined by a conjunction of atomic formulas then S is irreducible: If $C_1, C_2 \in \mathcal{C}_n$ and $S = C_1 \cup C_2$ then either $C_1 = S$ or $C_2 = S$.

Proof. Suppose $S = C_1 \cup C_2$ with $C_1, C_2 \subsetneq S$, and S is defined by a conjunction of atomic formulas. By Lemma 4.19, $rkC_1 < rkS$. By Definition 4.14, $rkS = \omega^m$ for some $m \in \mathbb{N}^+$. Now $rkC_1, rkC_2 \in \mathbb{N}[\omega]$, say

$$rkC_1 = \sum_{i=0}^{m-1} \gamma_i \omega^i < \omega^m$$

and

$$rkC_2 = \sum_{i=0}^{m-1} \delta_i \omega^i < \omega^m.$$

So

$$rk(C_1 \cup C_2) \leq \sum_{i=0}^{m-1} (\gamma_i + \delta_i)\omega^i < \omega^m = rkS.$$

So $C_1 \cup C_2 \neq S$, a contradiction. Thus, S is irreducible. □

It follows from the definition of $\mathcal{C}_n$ that it contains X^n and is closed under finite unions and finite intersections. Also $\mathcal{C}_n$ contains $\varnothing$ as $\varnothing$ is defined by $x_1 = a_1 \wedge x_1 = a_2$ for $a_1 \neq a_2 \in X$ which is a positive quantifier free formula. To show it is a topology we must show it is closed under infinite intersections.

Proposition 4.21. If $S_1 \subsetneq S_2 \subseteq X^n$ with $S_1, S_2 \in \mathcal{C}_n$, then $rk(S_1) < rk(S_2)$.

Proof. Suppose $S_1 \subsetneq S_2 \subseteq X^n$ are in $\mathcal{C}_n$. First we consider the case where S_2 is defined by formula in special form. So by Corollary 4.20, S_2 is irreducible. So by Lemma 4.19, $rkS_1 < rkS_2$. Now we consider the case where S_2 is defined by a disjunction of formulas in special form, then, say S_2 has dimension D and decomposition $S_2 = \underbrace{S'_1 \cup ... \cup S'_d}_{dim=D} \cup \underbrace{S'_{d+1} \cup ... \cup S'_k}_{dim<D}$. Then either $dim(S_1) < D$, so $rk(S_1) < \omega^D \leq rk(S_2)$, or $dim(S_1) = D$. Then the irreducible components of S_1 of dimension D are some of $S'_1, ..., S'_d$. If not all $S'_1, ..., S'_d$ are subsets of S_1 then $rk(S_1) < \omega^D.d \leq rk(S_2)$. Otherwise, $S'_1 \cup ... \cup S'_d \subseteq S_1$. So let r be the largest number such that there is an S'_i in the irreducible decomposition of S_2 which is not in S_1 of dimension r. Then $rk(S_2) = \omega^D.d + ... + \omega^{r+1}.k + \omega^r.l + ...$ and $rk(S_1) = \omega^D.d + ... + \omega^{r+1}.k + \omega^r.l' + ...$ where $l' < l$. So $rk(S_1) < rk(S_2)$. □

Proposition 4.22. An infinite intersection of members of $\mathcal{C}_n$ is in $\mathcal{C}_n$.

Proof. Let $S_i \in \mathcal{C}_n$ for $i \in I$. So $S_i \subseteq X^n$, for all $i \in I$. Each S_i is defined by a positive quantifier free formula. We want to show there is a finite $I_0 \subseteq I$ such that $\bigcap_{i \in I} S_i = \bigcap_{i \in I_0} S_i$. Assume that I is an ordinal. Let $C_\alpha = \bigcap_{\beta < \alpha} S_\beta$.

Thus $C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$. Using Lemma 4.19, we have $rkC_1 \geq rkC_2 \geq ....$ So, since rk is ordinal valued, there are only finitely many $i \in I$ such that $rkC_{i-1} > rkC_i$, say $i_1, ..., i_k$. Let $I_0 = \{i_l | l = 1, ..., k\}$. So $rkC_{i_k} = rkC_j$ for all $j > i_k$ and $C_{i_k} = C_j$ for all $j \geq i_k$. So $\bigcap\limits_{i_l \in I_0} S_{i_l} = \bigcap\limits_{i \in I} S_i$. So $\mathcal{C}_n$ is closed under infinite intersections, so is a topology and it is Noetherian. □

We are interested in closed set but also we are more interested in irreducible closed sets. In general, from the classifications of closed sets above, the closed set S can be written as $\bigvee\limits_{k=1}^{r} \bigwedge\limits_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ for the least possible value of r such that if $r = 1$, then S is irreducible, and if $r > 1$, then the conjunctions $\bigwedge\limits_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ give the r irreducible components of S.

Lemma 4.23. If S_1 and S_2 are closed sets and S_1 is irreducible, and $f : S_1 \longrightarrow S_2$ is continuous and surjective then S_2 is irreducible.

Proof. Suppose $S_2 = C_1 \cup C_2$ where C_1, C_2 are closed sets. Then

$$S_1 = f^{-1}(C_1) \cup f^{-1}(C_2).$$

As f is continuous, $f^{-1}(C_1)$ and $f^{-1}(C_1)$ are closed in S_1. Thus either $f^{-1}(C_1) = S_1$ or $f^{-1}(C_2) = S_1$. So $f(S_1) = C_1$ or $f(S_1) = C_2$. That is $S_2 = C_1$ or $S_2 = C_2$. So S_2 is irreducible. □

Proposition 4.24. The irreducible closed sets are exactly those defined by conjunctions of atomic formulas.

Proof. Let $S \subseteq X^n$ and $S \neq \varnothing$. If S is defined by a conjunction of atomic formulas, then by Corollary 4.20 S is irreducible.
Suppose S is closed and irreducible. As S is closed, by DNF theorem, it is defined by a formula of the form $\bigvee\limits_{k=1}^{r} \bigwedge\limits_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ where each $\varphi_{k,l}(\overline{x})$ is atomic. Let S_k be defined by the conjunction $\bigwedge\limits_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$. Then $S = \bigcup\limits_{k=1}^{r} S_k$. By Proposition 4.13, each S_k is defined by a formula in special form. As S is

irreducible, it is equal to one S_k. So S is defined by a formula in special form.

□

4.2.2 Zariski Axioms on $\langle X, = \rangle$

We will show that $\langle X, = \rangle$ satisfies Zariski axioms.

We need the following definition to prove (Z0).

Definition 4.25. Given a function $\sigma : \{1, ..., m\} \longrightarrow \{1, ..., n\}$, we can get a projection map $\pi_\sigma : X^n \longrightarrow X^m$ given by $\pi_\sigma(x_1, ..., x_n) = (x_{\sigma(1)}, ..., x_{\sigma(n)})$. As we need to consider each coordinate projection or constant, and show it is continuous, we define

$$\pi_{\alpha,i}(x_1, ..., x_n) = \begin{cases} x_{\alpha(i)} & \text{if } \alpha(i) \in \{1, ..., n\} \\ a & \text{if } \alpha(i) = a, a \in X \end{cases}$$

where $\alpha : \{1, ..., m\} \longrightarrow \{1, ..., n\} \cup X$. So $\pi_\alpha(x_1, ..., x_n) = (\pi_{\alpha,1}(\overline{x}), ..., \pi_{\alpha,m}(\overline{x}))$ where $\overline{x} = (x_1, ..., x_n)$.

Lemma 4.26. (Z0) holds for $\langle X, = \rangle$.

Proof. Let $\psi_{k,l}(\overline{x})$ be the negation of atomic formulas

$$\neg x_{i_{k,l}} = x_{j_{k,l}},$$

or

$$\neg x_{i_{k,l}} = a_{k,l}$$

for $i_{k,l}, j_{k,l} \in \{1, ..., n\}, a_{k,l} \in X$. Any open set $U \subseteq X^m$ has the form $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \psi_{k,l}(\overline{x})$, which is a finite positive boolean combination of basic open sets. So

$$\pi_\alpha^{-1}(U) = \{(\overline{x} \in X^n | \bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \psi_{k,l}(\pi_\alpha(\overline{x}))\}$$

which is also a finite positive boolean combination of basic open sets in X^n, as we shall see, from which it follows that π_σ is continuous. We need to see that $\psi_{k,l}(\pi_\alpha(\overline{x}))$ is an open set. But $\psi_{k,l}(\pi_\alpha(\overline{x}))$ is the formula $\neg(x_i = x_j)$ with substitution of terms $\pi_{\alpha,i}(\overline{x})$ for x_i and $\pi_{\alpha,j}(\overline{x})$ for x_j. So it is $\neg(\pi_{\alpha,i}(\overline{x}) = \pi_{\alpha,j}(\overline{x}))$ or $\neg(\pi_{\alpha,i}(\overline{x}) = a_{k,l})$. These are negations of atomic formulas, so they are basic open sets. Thus, π_α is continuous. Therefore, axiom (Z0) holds for $\langle X, = \rangle$. □

Lemma 4.27. (Z1) holds in $\langle X, = \rangle$.

Proof. Let $C \subseteq X^n$ be irreducible closed set. Let $\pi : X^n \longrightarrow X^{n-1}$ and $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-1})$. By Proposition 4.24, there is a formula φ in special form which defines C. Then φ has one of the following forms:
$First$: The formula φ can be the form of $\theta(x_1, ..., x_{n-1})$ where $\theta(x_1, ..., x_{n-1})$ is the conjunction of atomic formulas. Thus $\pi(C)$ is defined by $\theta(x_1, ..., x_{n-1})$ which is a closed set.
Second : The formula φ can have the form $\theta(x_1, ..., x_{n-1}) \wedge x_n = t$ where t is either a constant or x_i where $i = 1, ..., n-1$. So $\pi(C)$ will be defined by eliminating x_n and so $\pi(C)$ is defined by $\theta(x_1, ..., x_{n-1})$ which is closed set. Thus in both cases, $\pi(C)$ is closed and hence $\overline{\pi(C)} = \pi(C)$. Therefore, take $F = \varnothing$ which is a closed set and $\pi(C) \supseteq \overline{\pi(C)} \setminus \varnothing$ as needed. So we have proved that Z1 holds in $\langle X, = \rangle$. □

Lemma 4.28. (Z2) holds in $\langle X, = \rangle$.

Proof.

1. By Corollary 4.20, X is irreducible.

2. Back to $C \subseteq X^{n-1} \times X$ and the projection $\pi : X^n \longrightarrow X^{n-1}$. For $\overline{a} \in X^{n-1}$, let $C(a) = \{x_n \in X | (\overline{a}, x_n) \in C\}$. We consider the same two cases for the form of φ as in Lemma 4.27. In the first form, as the same φ defines $\pi(C)$ we get $dim(\pi(C)) = dim(C) - 1$. Thus $C = \pi(C) \times X$ and $C(a) = X$.
   In the second form, $\pi \restriction_C : C \longrightarrow \pi(C)$ is a bijection and $dim(\pi(C)) =$

$dim(C)$. So $C(a) = \{b\}$ if t is a constant symbol and $t^X = b$ or $C(a) = \{a_i\}$ if t is x_i. So $C(a) = \{t^X(\overline{a})\}$. Therefore, $|C(a)| = 1$.

Hence Z2 holds in $\langle X, = \rangle$. □

Lemma 4.29. (Z3) holds in $\langle X, = \rangle$.

Proof. Let W be a non empty irreducible component of $C \cap \Delta^n_{i,j}$ where $\Delta^n_{i,j} = \{x \in X^n : x_i = x_j\}$ and C is irreducible. Let θ be the formula defining $C \cap \Delta^n_{i,j}$. We will examine the intersection of the diagonal with the irreducible closed sets. If $W = X^n \cap \Delta^n_{i,j}$, then $dim(W) = dim(X^n) - 1 = n - 1$ and so $dim(X^n) \leq dim(W) + 1$. If $C = \varnothing$ then $W = \varnothing \cap \Delta^n_{i,j} = \varnothing$ but we assumed that W is non-empty. If $W = C \cap \Delta^n_{i,j}$ where $C \subsetneq X^n$ and $C \neq \varnothing$, then looking at the forms of the formula φ in special form defining C we will have four cases:

Case 1: If $i, j \in Fix_C$, then we will have the formula $x_i = a_i \wedge x_j = a_j$ in φ. Since we assumed that $W \neq \varnothing$, we must have $a_i = a_j$. So $W = C$. So $dim(W) = dim(C)$.

Case 2: If $i \in Fix_C$ and $j \notin Fix_C$ then we will have the formula $x_i = a_i$ in φ. So θ will be equivalent to

$$\bigwedge_{k \in Fix_C} x_k = a_k \wedge \bigwedge_{k \sim_C j} x_k = a_i \wedge \bigwedge_{k \sim_C l, k \nsim_C j, k<l} x_k = x_l$$

which means that $i, j \in Fix_W$. So

$$Fix_W = Fix_C \cup \{ \text{ the } \sim_C - \text{ equivalence classes of } j\}$$

and $C \cap \Delta^n_{i,j}$ is irreducible as θ is in special form. So $dim(W) = dim(C) - 1$.

Case 3: If $i, j \notin Fix_C$ and $i \sim_C j$ then φ will imply the formula $x_i = x_j$. So θ will imply the formula $x_i = x_j$. So $i \sim_W j$. So $W = C$. So

$dim(W) = dim(C)$.

Case 4: If $i, j \notin Fix_C$ and $i \nsim_C j$, then θ is equivalent to

$$\bigwedge_{i \in Fix_C} x_i = a_i \wedge \bigwedge_{k \sim_C l, k<l} x_k = x_l \wedge \bigwedge_{\{(k,l): k \sim i, l \sim j, k<l\}} x_k = x_l.$$

So $i \sim_W j$ which means that i and j are in one equivalence class in W. As the above formula is in special form, $C \cap \Delta^n_{i,j}$ is irreducible. So the number of equivalence classes of $\sim_W$ is equal to the number of equivalence classes of $\sim_C -1$. So $dim(W) = dim(C) - 1$.

Thus Z3 holds in $\langle X^n, = \rangle$. □

In conclusion, we can deduce that $\langle X, = \rangle$ is a Noetherian topological structure and by Lemma 4.26, 4.27, 4.28, and 4.29 it satisfies the Zariski axioms.

5

Classification of injective unars

In this chapter we will give first a condition for two injective unars to be isomorphic. Then we will give an axiomatization of the theory of a strongly minimal injective unar $\mathcal{A}$ and prove this theory is complete. We will then prove that this theory has quantifier elimination after adding unary relational symbols R_n to the language $L = \langle f \rangle$.
Let L_f be the language with a single unary function symbol.

Definition 5.1. A unar is an L_f-structure. Let $\langle \mathcal{A}, f \rangle$ be a unar. If f is injective then $\langle \mathcal{A}, f \rangle$ will be called an injective unar.

This terminology comes from [7] and [8].

Definition 5.2. Let $X \subseteq A$ and $x, y \in A$. We say that x, y are connected if there is $n, m \in \mathbb{N}$ such that $f^n(x) = f^m(y)$. The set $X \subseteq A$ is connected if any two elements of X are connected. A maximal connected set is called a connected component of $\mathcal{A}$.

When f is injective, the connected components in $\mathcal{A}$ can be classified to be either a copy of $\mathbb{N}$, $\mathbb{Z}$, or a cycle of period r where $r \in \mathbb{N}^+$.

Lemma 5.3. Let $\langle \mathcal{A}, f \rangle$ be an injective unar. Then every connected component of $\mathcal{A}$ is either

1. a copy of $\langle \mathbb{N}, succ \rangle$,

2. a copy of $\langle \mathbb{Z}, succ \rangle$,

3. a cycle of period r for some $r \in \mathbb{N}^+$.

Proof. Suppose X is a connected component of $\mathcal{A}$ and X is finite. Let $x \in X$. We have $x = f^0(x), f(x), f^2(x), f^3(x), ... \in X$. So $f^n(x) \in X$ for all $n \in \mathbb{N}$. Let r be the smallest number in $\mathbb{N}$ such that $f^r(x) = f^s(x)$ for some $s \in \{0, ..., r-1\}$. We claim $s = 0$. If not, $f(f^{r-1}(x)) = f(f^{s-1}(x))$. So, by injectivity of f, $f^{r-1}(x) = f^{s-1}(x)$, and $s - 1 \in \{0, ..., r-2\}$, contradicting that r is the smallest such number. Then X contains a cycle of period r. Since f is injective, the cycle is all of X. Now suppose X is infinite. Let $x \in X$. Consider $f^0(x) = x, f(x), f^2(x), ..., f^n(x), ..$ for $n \in \mathbb{N}$. If $f^r(x) = f^s(x)$ for some $r, s \in \mathbb{N}$, and $r \neq s$, then by the previous argument, X is finite. So all $f^n(x)$ are distinct for $n \in \mathbb{N}$. There are two cases. If there is $x_0 \in X$ which is not in the image of f, then X is $\{x_0, f(x_0), ..., f^n(x_0), ...\}$, i.e, $X = \{f^n(x_0) : n \in \mathbb{N}\}$ which is a copy of $\mathbb{N}$. Otherwise, choosing any $x \in X$, $f^{-1}(x)$ exists and is unique. So we get $f^{-n}(x) \in X$ for each $n \in \mathbb{N}$ and for any $n, m \in \mathbb{Z}$, $f^n(x) \neq f^m(x)$ unless $n = m$. So, X is a copy of $\mathbb{Z}$. □

The number of copies of each connected component in $\mathcal{A}$ plays an important rôle in classifying injective unars. So we will define a sequence of $\mathcal{A}$ depending on the number of copies of the connected components in $\mathcal{A}$.

Definition 5.4. Let $\langle \mathcal{A}, f \rangle$ be an injective unar. Define $\sigma(\mathcal{A}) = (k_r)_{r \in \mathbb{N} \cup \{\infty\}}$ a sequence of cardinals, where k_0 and k_∞ represents the numbers of copies of $\mathbb{N}$ and $\mathbb{Z}$ respectively and k_r represents the number of copies of cycles with period r for $r \in \mathbb{N}^+$.

We need to know the condition for two injective unars to be isomorphic. This will be helpful to prove the completeness of the theory T_σ of injective unar.

Proposition 5.5. Suppose $\mathcal{A}$ and $\mathcal{B}$ are injective unars such that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. Then $\mathcal{A} \cong \mathcal{B}$.

Proof. Suppose $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. We need to show that $\mathcal{A} \cong \mathcal{B}$. For $r = 0$, enumerate the $0s$ elements of the copies of $\mathbb{N}$ in $\mathcal{A}$ as $a_1^0, ..., a_{k_0}^0$ and the $0s$ elements of the copies of $\mathbb{N}$ in $\mathcal{B}$ as $b_1^0, ..., b_{k_0}^0$. Define

$$\pi_0 : \bigcup_{i=1}^{k_0} \mathbb{N}_i^{\mathcal{A}} \longrightarrow \bigcup_{i=1}^{k_0} \mathbb{N}_i^{\mathcal{B}},$$

where $\mathbb{N}_i^{\mathcal{A}}$ and $\mathbb{N}_i^{\mathcal{B}}$ are the *ith* copy of natural numbers in $\mathcal{A}$ and $\mathcal{B}$ respectively, by $\pi_0(a_i^0) = b_i^0$ and $\pi_0(f^n(a_i^0)) = f^n(b_i^0)$ where $i = 1, ..., k_0$ and $n \in \mathbb{N}$. For $r \in \mathbb{N}^+$, choose an element from each cycle of length r in $\mathcal{A}$ and enumerate these elements as $a_1^r, ..., a_{k_r}^r$. Choose an element from each cycle of length r in $\mathcal{B}$ and enumerate them as $b_1^r, ..., b_{k_r}^r$. Define

$$\pi_r : \bigcup_{i=1}^{k_r} C_{r_i}^{\mathcal{A}} \longrightarrow \bigcup_{i=1}^{k_r} C_{r_i}^{\mathcal{B}},$$

where $C_{r_i}^{\mathcal{A}}$ and $C_{r_i}^{\mathcal{B}}$ are the *ith* copy of the cycle of length r in $\mathcal{A}$ and $\mathcal{B}$ respectively, by $\pi_r(a_i^r) = b_i^r$ and $\pi_r(f^n(a_i^r)) = f^n(b_i^r)$ where $i = 1, ..., k_r$ and $n \in \mathbb{N}$. For $r = \infty$, choose an element from each copy of $\mathbb{Z}$ in $\mathcal{A}$ and enumerate these elements as $a_1^\infty, ..., a_{k_\infty}^\infty$. Choose an element from each copy of $\mathbb{Z}$ in $\mathcal{B}$ and enumerate them as $b_1^\infty, ..., b_{k_\infty}^\infty$. Define

$$\pi_\infty : \bigcup_{i=1}^{k_\infty} \mathbb{Z}_i^{\mathcal{A}} \longrightarrow \bigcup_{i=1}^{k_\infty} \mathbb{Z}_i^{\mathcal{B}},$$

where $\mathbb{Z}_i^{\mathcal{A}}$ and $\mathbb{Z}_i^{\mathcal{B}}$ are the *ith* copy of integers in $\mathcal{A}$ and $\mathcal{B}$ respectively, by $\pi_\infty(a_i^\infty) = b_i^\infty$ and $\pi_\infty(f^n(a_i^\infty)) = f^n(b_i^\infty)$ where $i = 1, ..., k_\infty$ and $n \in \mathbb{Z}$. From the definition of π_r where $r \in \mathbb{N} \cup \{\infty\}$, π_r is well defined and is a bijection. Let $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map such that

$$\pi = \bigcup_{r \in \mathbb{N} \cup \{\infty\}} \pi_r$$

where $\pi(a_i^r) = \pi_r(a_i^r)$. Now π is a bijection as each π_r is a bijection. Also, $\pi(f^n(a_i^r)) = \pi_r(f^n(a_i^r)) = f^n(b_i^r) = f^n(\pi_r(a_i^r)) = f^n(\pi(a_i^r))$. So π commutes with f and hence is an isomorphism. □

5.1 The theory of an injective unar

Given a unar $\mathcal{A}$, especially a strongly minimal unar, we want to give an axiomatization of its complete first-order theory, $Th(\mathcal{A})$.

Definition 5.6. For $r \in \mathbb{N}^+$, let $\psi_r(x)$ be the formula

$$f^r(x) = x \wedge \bigwedge_{s|r, s \neq r} f^s(x) \neq x.$$

The formula $\psi_r(x)$ defines the set of points which lie on a cycle of period r.

Lemma 5.7. Let $\mathcal{A}$ be an injective unar.

1. If $k_0 < \infty$, then $\mathcal{A} \models \exists^{=k_0} x[\neg\exists y[f(y) = x]]$.
2. If k_0 is infinite, then $\mathcal{A} \models \exists^{\geq n} x[\neg\exists y[f(y) = x]]$ for each $n \in \mathbb{N}^+$.
3. If $k_r < \infty$, then $\mathcal{A} \models \exists^{=n} x[\psi_r(x)]$ where $n = k_r.r$.
4. If k_r is infinite, then $\mathcal{A} \models \exists^{\geq n} x[\psi_r(x)]$ for each $n \in \mathbb{N}^+$.

Proof. Immediate from the definition of k_r. □

Lemma 5.7 says that the values of $k_0(\mathcal{A})$ and $k_r(\mathcal{A})$ for $r \in \mathbb{N}^+$ can be determined by the first-order theory of $\mathcal{A}$ if these values are finite. However, if these values are infinite, the first-order theory of $\mathcal{A}$ can only says these values are infinite but it cannot determine which infinite value $k_0(\mathcal{A})$ or $k_r(\mathcal{A})$ is. The sequence $\sigma(\mathcal{A})$ is a sequence of cardinals and it gives $k_\infty(\mathcal{A})$ but $k_\infty(\mathcal{A})$ cannot be determined in the first-order theory of $\mathcal{A}$.

Definition 5.8. Given a sequence $\sigma = (k_r)_{r\in\mathbb{N}}$ where each $k_r \in \mathbb{N} \cup \{\infty\}$, let T_σ be the theory axiomatized by

1. $\forall xy[f(x) = f(y) \longrightarrow x = y]$
2. If k_0 is finite, then $\exists^{=k_0}x[\neg\exists y[f(y) = x]]$.
3. If k_0 is infinite, then $\exists^{\geq n}x[\neg\exists y[f(y) = x]]$ for each $n \in \mathbb{N}^+$.
4. For each $r \in \mathbb{N}^+$, if k_r is finite, then $\exists^{=n}x[\psi_r(x)]$ where $n = k_r.r$.
5. For each $r \in \mathbb{N}^+$, if k_r is infinite, then $\exists^{\geq n}x[\psi_r(x)]$ for each $n \in \mathbb{N}^+$.
6. If $k_0 = 0$ and there is $N \in \mathbb{N}$ such that for all $r > N$, $k_r = 0$ and $\sum_{r=1}^{N} k_r$ is infinite, then $\forall x \bigvee_{i=1}^{N} \psi_i(x)$.
7. For each $n \in \mathbb{N}^+$, $\exists^{\geq n}x[x = x]$.

Lemma 5.9. Suppose $\mathcal{A}$ is a strongly minimal injective unar. Then $\sigma(\mathcal{A})$ satisfies the following:

1. k_0 is finite.
2. If some k_r is infinite for $r \in \mathbb{N}^+$, then $k_0 = 0$, $k_\infty = 0$ and $\sum_{q\neq r} k_q$ is finite.

Furthermore, $T_{\sigma(\mathcal{A})} \subseteq Th(\mathcal{A})$.

Proof. 1. Let $\varphi(x)$ be the formula $\exists y[f(y) = x]$. Since f is injective, $\varphi(x)$ defines an infinite subset of $\mathcal{A}$. As T_σ is strongly minimal, $\neg\varphi(x)$ defines a finite set. Each element which satisfies $\neg\varphi(x)$ is the zero of a connected component which is a copy of $\mathbb{N}$. So k_0 is the size of the set $\neg\varphi(\mathcal{A})$ which is finite.

2. Suppose $k_r = \infty$ for some $r \in \mathbb{N}^+$. Then $\mathcal{A} \models \psi_r(a)$ if and only if a is in a cycle of length r. Now $\psi_r(\mathcal{A})$ is the union of the infinitely many cycles of length r so is an infinite definable subset of $\mathcal{A}$. As $\mathcal{A}$ is strongly minimal, $\neg\psi_r(x)$ is finite. So $k_0 = 0, k_\infty = 0$ and $\sum_{q \neq r} k_q$ is finite.

To show that $T_{\sigma(\mathcal{A})} \subseteq Th(\mathcal{A})$, axioms $1, 2, 4, 5$ and axiom 7 are immediate from Definition 5.8. We just need to check axiom 6. Suppose that $k_0 = 0$ and there is $N \in \mathbb{N}$ such that for all $r > N$, $k_r = 0$ and $\sum_{r=1}^{N} k_r$ is infinite. Let $\Psi(x)$ be the formula $\bigvee_{r=1}^{N} \psi_r(x)$. So $\Psi(x)$ defines the set of points on any finite cycle. Since $\sum_{r=1}^{N} k_r$ is infinite, $|\Psi(\mathcal{A})|$ is infinite so it is cofinite. So $|\neg\Psi(\mathcal{A})|$ is finite. Suppose $\neg\Psi(\mathcal{A})$ is non-empty. So $\exists x[\bigwedge_{i=1}^{N} \neg\psi_i(x)]$. So there is at least one copy of $\mathbb{Z}$, contrary to strong minimality. So $\forall x \bigvee_{i=1}^{N} \psi_i(x)$. □

The Upward Löwenheim-Skolem Theorem indicates that for every infinite L-structure and cardinal $\kappa \geq |L|$ there is an elementary extension of cardinality at least κ. In $L = \langle f \rangle$, we will prove that T_σ with certain conditions is complete. This is important as it will help to capture the models of T_σ.

Proposition 5.10. Suppose σ satisfies properties 1 and 2 from Lemma 5.9. Then T_σ is categorical in all uncountable cardinals and is complete.

Proof. There are four cases.

Case 1: $k_0(\sigma) > 0$. So for all $r \in \mathbb{N}$, $k_r(\sigma)$ is finite. If $\mathcal{A} \models T_\sigma$ then $k_r(\mathcal{A}) = k_r(\sigma)$ for all $r \in \mathbb{N}$. Let $\lambda = k_\infty(\mathcal{A})$ and write $\mathcal{A}_\lambda$ for this $\mathcal{A}$. Then the models of T_σ are exactly $\mathcal{A}_\lambda$ for λ any cardinal. Then, $|\mathcal{A}_\lambda| = \aleph_0 \cdot k_0(\sigma) + \sum_{r \in \mathbb{N}^+} r \cdot k_r(\sigma) + \lambda \cdot \aleph_0 = \aleph_0 + \lambda$. So T_σ is uncountably categorical and not countably categorical.

Case 2: $k_0(\sigma) = 0$ and some $k_r(\sigma)$ is infinite for $r \in \mathbb{N}^+$. So $\sum_{q \neq r} k_q(\sigma)$ is finite. So there is $N \in \mathbb{N}$ such that if $r > N$ then $k_r(\sigma) = 0$. So $T_\sigma \vdash \forall x \bigvee_{i=1}^{N} \psi_i(x)$. So $k_\infty(\mathcal{A}) = 0$. Let $\lambda = k_r(\mathcal{A})$ and write $\mathcal{A}_\lambda$ for this $\mathcal{A}$.

So $|\mathcal{A}_\lambda| = \aleph_0 \cdot k_0(\sigma) + \sum_{q \neq r} q \cdot k_q(\sigma) + \lambda \cdot r + 0 \cdot \aleph_0 = \lambda$ because λ is infinite. So T_σ is totally categorical.

Case 3: If $k_0(\sigma) = 0$ and $\sum_{r \in \mathbb{N}^+} k_r(\sigma)$ is finite, then for $\mathcal{A} \models T_\sigma$ we have $k_0(\mathcal{A}) = 0$ and $k_r(\mathcal{A}) = k_r(\sigma)$. There is $N \in \mathbb{N}$ such that if $r > N$ then $k_r = 0$. Let $\Psi(x)$ be the formula $\bigvee_{r=1}^{N} \psi_r(x)$. Given $\mathcal{A} \models T_\sigma$, $\Psi(\mathcal{A})$ is finite. Since $\mathcal{A}$ is infinite, $\neg\Psi(\mathcal{A})$ is non-empty. So there is at least one copy of $\mathbb{Z}$. Let $\lambda = k_\infty(\mathcal{A})$ and write $\mathcal{A}_\lambda$ for this $\mathcal{A}$. Then, $|\mathcal{A}_\lambda| = \aleph_0 \cdot 0 + \sum_{r \in \mathbb{N}^+} r \cdot k_r(\sigma) + \lambda \cdot \aleph_0 = \aleph_0 + \lambda$. So T_σ is uncountably categorical but not countably categorical.

Case 4: $k_0(\sigma) = 0$ and no $k_r(\sigma)$ is infinite for $r \in \mathbb{N}^+$ and $\sum_{r \in \mathbb{N}^+} k_r(\sigma)$ is infinite. Then, $|\mathcal{A}_\lambda| = \aleph_0 \cdot 0 + \sum_{r \in \mathbb{N}^+} r \cdot k_r(\sigma) + \lambda \cdot \aleph_0 = \aleph_0 + \lambda$. So T_σ is uncountably categorical but not countably categorical.

By axiom 7, T_σ has no finite model. In all cases, T_σ is uncountably categorical, so by the Łos -Vaught test, T_σ is complete. □

5.2 Quantifier Elimination

For the rest of this section, we assume σ satisfies the conditions in Lemma 5.9.

We need to prove that T_σ admits quantifier elimination. However, the elements in $\langle \mathbb{N}, succ \rangle$ cannot be defined without using quantifiers. Thus we need to expand the language $L_f = \langle f \rangle$ to $L_{f,R} = \langle f, (R_n)_{n \in \mathbb{N}} \rangle$ in order to eliminate the quantifiers.

Definition 5.11. The language $L_{f,R} = \langle f, (R_n)_{n\in\mathbb{N}}\rangle$ consists of a unary function symbol f and unary relation symbols R_n for each $n \in \mathbb{N}$.

Given a model of T_σ, we make an expansion-by-definitions to an $L_{f,R}$-structure which is a model of $T_{\sigma,R}$ as follows where R_n names the set of all numbers n's in copies of $\mathbb{N}$.

Definition 5.12. Let $T_{\sigma,R}$ be the theory axiomatized by the axioms of T_σ in addition to the axioms

1. $\forall x[R_0(x) \leftrightarrow \neg\exists y[f(y) = x]]$.
2. $\forall x[R_n(x) \leftrightarrow \exists y[x = f^n(y) \wedge R_0(y)]]$ for each $n \in \mathbb{N}^+$.

We need to examine the 1-types in $T_{\sigma,R}$. We will show that the principal formulas in $T_{\sigma,R}$ are $R_n(x)$ for $n \in \mathbb{N}$, and $\psi_r(x)$ for $r \in \mathbb{N}^+$. Each of these principal formulas gives a complete 1-type in $T_{\sigma,R}$. We also need to examine non-principal 1-types in $T_{\sigma,R}$. We will show there is only one such 1-type. To prove these statements we will use automorphisms.

Automorphisms of $\mathcal{A}$:

For a given sequence of cardinals $\sigma = (k_r)_{r\in\mathbb{N}\cup\{\infty\}}$, take the model $\mathcal{A}_\sigma$ to be

$$(\{0\} \times k_0 \times \mathbb{N}) \cup \bigcup_{r\in\mathbb{N}^+} (\{r\} \times k_r \times C_r) \cup (\{\infty\} \times k_\infty \times \mathbb{Z})$$

where k_r means $\{i \in Ord | i < k_r\}$. An element of $\mathcal{A}_\sigma$ is then a triple (r, i, n) where $r \in \mathbb{N} \cup \{\infty\}$, $i = 0, 1, ..., k_r - 1$ (if k_r is finite) and $n \in \mathbb{N}$ or $n \in \mathbb{Z}$ or $n \in C_r = \{0, 1, ..., r-1\}$ considered as an r-cycle. To specify $\pi \in Aut(\mathcal{A}_\sigma)$, we need the following: For each $r \in \mathbb{N} \cup \{\infty\}$, we choose a permutation ρ_r of k_r. For each $r \in \mathbb{N}^+$, and each $i < k_r$, we choose $m_{r,i} \in \{0, 1, ..., r-1\}$. For each i, k_∞, we choose $m_{\infty,i} \in \mathbb{Z}$.

Proposition 5.13.

$$\pi(r,i,n)=\begin{cases}(0,\rho_0(i),n) & \text{if } r=0\\ (r,\rho_r(i),n+m_i(\mathrm{mod} r)) & \text{if } r\in\mathbb{N}^+\\ (\infty,\rho_\infty(i),n+m_i) & \text{if } r=\infty\end{cases}$$

defines an automorphism of $\mathcal{A}_\sigma$. Furthermore, every automorphism of $\mathcal{A}_\sigma$ is of this form.

Proof. Each π is bijective and preserves f. So it is an automorphism. The "furthermore" part seems clear, but we do not give a proof because we do not need to use it. □

1-Types in $T_{\sigma,R}$:

Proposition 5.14.

1. For each $n \in \mathbb{N}$, if $k_0 \neq 0$ then $R_n(x)$ is a principal formula.
2. For each $r \in \mathbb{N}^+$, if $k_r \neq 0$ then $\psi_r(x)$ is a principal formula.
3. $p_{\mathbb{Z}} = \{\neg R_n(x) : n \in \mathbb{N}\} \cup \{\neg\psi_r(x) : r \in \mathbb{N}^+\}$ is the type of an element in a copy of $\mathbb{Z}$.

There are no other complete 1-types of $T_{\sigma,R}$.

Proof.

1. Let $n \in \mathbb{N}$. Suppose $a, b \in \mathcal{A}_\sigma$ such that $\mathcal{A}_\sigma \models R_n(a)$ and $\mathcal{A}_\sigma \models R_n(b)$. So there are $i, j < k_0$ and $n \in \mathbb{N}$ such that $a = (0, i, n)$ and $b = (0, j, n)$. By Proposition 5.13 there is $\pi \in Aut(\mathcal{A}_\sigma)$ such that $\pi(a) = b$. So $tp(a) = tp(b)$. So $R_n(x)$ is a complete type.
2. Let $r \in \mathbb{N}^+$. Suppose $a, b \in \mathcal{A}_\sigma$ such that $\mathcal{A}_\sigma \models \psi_r(a)$ and $\mathcal{A}_\sigma \models \psi_r(b)$. So there are $i, j < k_r$ and $n, m \in \{0, ..., r-1\}$ such that $a = (r, i, n)$

and $b = (r, j, m)$. By Proposition 5.13 there is $\pi \in Aut(\mathcal{A}_\sigma)$ such that $\pi(a) = b$. So $tp(a) = tp(b)$. So $\psi_r(x)$ is a complete type.

3. Suppose $a, b \in \mathcal{A}_\sigma$ such that $\mathcal{A}_\sigma \models p_{\mathbb{Z}}(a)$ and $\mathcal{A}_\sigma \models p_{\mathbb{Z}}(b)$. So there are $i, j < k_\infty$ and $n, m \in \mathbb{Z}$ such that $a = (\infty, i, n)$ and $b = (\infty, j, m)$. By Proposition 5.13 there is $\pi \in Aut(\mathcal{A}_\sigma)$ such that $\pi(a) = b$. So $tp(a) = tp(b)$. So $p_{\mathbb{Z}}(x)$ is a complete type.

Every element of any model $\mathcal{A} \models T_{\sigma,R}$ has one of these types, so there are no other types. □

Quantifier elimination plays an important rôle in studying definable sets as definable sets which are defined by quantified formulas can be complicated. If an L-theory has quantifier elimination this means that every L-formula is equivalent to a quantifier-free L-formula.

Definition 5.15. Suppose $\mathcal{M} \models T_{\sigma,R}$, and $\mathcal{A}$ is an $L_{f,R}$-substructure of $\mathcal{M}$. The connected hull of $\mathcal{A}$ in $\mathcal{M}$, denoted by $ConHull_{\mathcal{M}}(\mathcal{A})$ is the $L_{f,R}$-substructure of $\mathcal{M}$ consisting of all the connected components of $\mathcal{M}$ which meet $\mathcal{A}$. Equivalently, $ConHull_{\mathcal{M}}(\mathcal{A}) = \{(f^{\mathcal{M}})^n(a) : n \in \mathbb{Z}, a \in A\}$

Lemma 5.16. Suppose $\mathcal{M}, \mathcal{N} \models T_{\sigma,R}$, and $\mathcal{A}$ is a common $L_{f,R}$-substructure of $\mathcal{M}$ and $\mathcal{N}$. Then $ConHull_{\mathcal{M}}(\mathcal{A}) \cong ConHull_{\mathcal{N}}(\mathcal{A})$.

Proof. Let $a \in A$ and let E_a be the connected component of A containing a. We need to define an isomorphism

$$\pi : ConHull_{\mathcal{M}}(\mathcal{A}) \longrightarrow ConHull_{\mathcal{N}}(\mathcal{A})$$

such that

$$\pi = \bigcup_{a \in A} \pi_a$$

and

$$\pi_a : ConHull_{\mathcal{M}}(a) \longrightarrow ConHull_{\mathcal{N}}(a).$$

If E_a is a cycle C_r for $r \in \mathbb{N}^+$, then E_a is a cycle in $\mathcal{M}$ and $\mathcal{N}$. So take π_a to be the identity on E_a. If E_a is in a copy of $\mathbb{N}$, then there is $n \in \mathbb{N}$ such that $\mathcal{M} \models R_n(a)$. As $\mathcal{A}$ is an $L_{f,R}$-substructure of $\mathcal{M}$, $\mathcal{A} \models R_n(a)$. As $\mathcal{A}$ is an $L_{f,R}$-substructure of $\mathcal{N}$, $\mathcal{N} \models R_n(a)$. So since $\mathcal{M}, \mathcal{N} \models T_{\sigma,R}$, the connected components of a in $\mathcal{M}$ and $\mathcal{N}$ are copies of $\mathbb{N}$, and a is the $(n+1)^{th}$ element. So define π_a by $\pi_a(f^{\mathcal{M}})^m(a) = (f^{\mathcal{N}})^m(a)$ for $m \geq -n$. So π_a is an isomorphism from $ConHull_{\mathcal{M}}(a)$ to $ConHull_{\mathcal{N}}(a)$. If E_a is in a copy of $\mathbb{Z}$, then $\mathcal{M} \models \neg(R_n(a) \vee \psi_r(a))$ for all $n \in \mathbb{N}, r \in \mathbb{N}^+$. As $\mathcal{A}$ is an $L_{f,R}$-substructure of $\mathcal{M}$, $\mathcal{A} \models \neg(R_n(a) \vee \psi_r(a))$. As $\mathcal{A}$ is an $L_{f,R}$-substructure of $\mathcal{N}$, $\mathcal{N} \models \neg(R_n(a) \vee \psi_r(a))$. Since $\mathcal{M}, \mathcal{N} \models T_{\sigma,R}$, the connected components of a in $\mathcal{M}$ and $\mathcal{N}$ are copies of $\mathbb{Z}$ and a can be any element of these copies. So define π_a by $\pi_a(f^{\mathcal{M}})^m(a) = (f^{\mathcal{N}})^m(a)$ for $n \in \mathbb{Z}$. So π_a is an isomorphism from $ConHull_{\mathcal{M}}(a)$ to $ConHull_{\mathcal{N}}(a)$. Thus $ConHull_{\mathcal{M}}(\mathcal{A}) \cong ConHull_{\mathcal{N}}(\mathcal{A})$. □

Lemma 5.17. Suppose $\mathcal{M}, \mathcal{N} \models T_{\sigma,R}$, and let $\varphi(\bar{x}, y)$ be a quantifier-free $L_{f,R}$-formula and $\bar{a} \in \mathcal{A}$ where $\mathcal{A}$ is a common $L_{f,R}$-substructure of $\mathcal{M}$ and $\mathcal{N}$. Then either $|\varphi(\bar{a}, \mathcal{M})|$ and $|\varphi(\bar{a}, \mathcal{N})|$ are finite or $|\neg\varphi(\bar{a}, \mathcal{M})|$ and $|\neg\varphi(\bar{a}, \mathcal{N})|$ are finite. In other words, the size of the set defined by $\varphi(\bar{a}, y)$ depends on T_σ not on the models of T_σ.

Proof. First we consider the case where φ is an atomic formula θ. Atomic formulas in $L_{f,R}$ are $R_n(y)$ for $n \in \mathbb{N}$, $y = f^m(a)$ for $m \in \mathbb{Z}$ and $a \in A$, and $f^r(y) = y$ for $r \in \mathbb{N}^+$.

Case 1: $\theta(\bar{a}, y)$ is the formula $R_n(y)$ for $n \in \mathbb{N}$. In T_σ, k_0 is finite. So $R_n(y)$ defines a finite set of size k_0. So $|\theta(\bar{a}, \mathcal{M})| = |\theta(\bar{a}, \mathcal{N})| = k_0$.

Case 2: $\theta(\bar{a}, y)$ is the formula $f^r(y) = y$ for $r \in \mathbb{N}^+$. We have two possibilities. First, if k_s is finite for all $s \in \mathbb{N}^+$, then $f^r(y) = y$ defines a finite set of size $\sum_{s|r} s \cdot k_s$. So $|\theta(\bar{a}, \mathcal{M})| = |\theta(\bar{a}, \mathcal{N})| = \sum_{s|r} s \cdot k_s$. Second, if k_t is infinite for some $t \in \mathbb{N}^+$, then $k_0 = 0$ and $\sum_{q \neq t} k_q$ is finite. So $f^r(y) = y$

defines an infinite set for r such that $t|r$ and a finite set of size $\sum_{s|r} s \cdot k_s$ if $t \nmid r$. If $t|r$ then the set defined by $\neg[f^r(y) = y]$ has size $\sum_{q \nmid r} q \cdot k_q$ which is a finite sum. So $|\neg\theta(\bar{a}, \mathcal{M})| = |\neg\theta(\bar{a}, \mathcal{N})| = \sum_{q \nmid r} q \cdot k_q$.

Case 3: $\theta(\bar{a}, y)$ is the formula $y = f^m(a)$ for some $m \in \mathbb{Z}$. So $\theta(\bar{a}, y)$ defines a singleton. So $|\theta(\bar{a}, \mathcal{M})| = |\theta(\bar{a}, \mathcal{N})| = 1$.

So either $|\theta(\bar{a}, \mathcal{M})| = |\theta(\bar{a}, \mathcal{N})|$ is finite or $|\neg\theta(\bar{a}, \mathcal{M})| = |\neg\theta(\bar{a}, \mathcal{N})|$ is finite. So the size of the sets defined by $\theta(\bar{a}, y)$ depends only on T_σ.

Now we consider an arbitrary quantifier-free formula φ. In disjunctive normal form, $\varphi(\bar{a}, y)$ can be written as $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, y)$ where $\bigwedge_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, y)$ is a finite conjunction of atomic or negated atomic formulas. So, if at least one of the $\varphi_{k,l}(\bar{a}, y)$ for $l = 1, ..., s_k$ defines a finite set then both of

$$\Big|\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{M})\Big| \text{ and } \Big|\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{N})\Big| \text{ have size } \leq m_k$$

where

$$m_k = min\{n_l | n_l \text{ is the size of finite sets defined by } \varphi_{k,l}(\bar{a}, y)\}.$$

If all of $\varphi_{k,l}(\bar{a}, y)$ define cofinite sets then

$$\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{M}) \text{ and } \bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{N}) \text{ are cofinite }.$$

So, if all of $\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, y)$ define finite sets then both of

$$\Big|\bigcup_{k=1}^{r}\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{M})\Big| \text{ and } \Big|\bigcup_{k=1}^{r}\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{N})\Big| \text{ have size } \leq \sum_{k=1}^{r} m_k.$$

If at least one of $\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, y)$ define a cofinite set then

$$\bigcup_{k=1}^{r}\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{M}) \text{ and } \bigcup_{k=1}^{r}\bigcap_{l=1}^{s_k} \varphi_{k,l}(\bar{a}, \mathcal{N}) \text{ are cofinite}$$

So the size of the sets defined by $\varphi(\bar{a}, y)$ depends on T_σ.

□

Theorem 5.18. $T_{\sigma,R}$ has quantifier elimination.

Proof. Suppose $\mathcal{M}, \mathcal{N} \models T_{\sigma,R}$, and $\mathcal{A}$ is a common $L_{f,R}$-substructure of $\mathcal{M}$ and $\mathcal{N}$. By [12, Corollary 3.1.6], we need to show that if $\varphi(\bar{x}, y)$ is a quantifier free formula and $\bar{a} \in A$ and there is $b \in M$ such that $\mathcal{M} \models \varphi(\bar{a}, b)$, then there is $c \in N$ such that $\mathcal{N} \models \varphi(\bar{a}, c)$. In some cases we define an isomorphism

$$\pi : ConHull_{\mathcal{M}}(\mathcal{A} \cup \{b\}) \longrightarrow ConHull_{\mathcal{N}}(\mathcal{A} \cup \{c\})$$

such that $\pi \restriction_{ConHull(\mathcal{A})}$ is the identity and $\pi(b) = c$. Then since $\varphi(\bar{x}, y)$ is a quantifier-free $L_{f,R}$-formula and $ConHull_{\mathcal{M}}(\mathcal{A} \cup \{b\})$ is an $L_{f,R}$-substructure of $\mathcal{M}$, $ConHull_{\mathcal{M}}(\mathcal{A} \cup \{b\}) \models \varphi(\bar{a}, b)$. Then by the isomorphism π, $ConHull_{\mathcal{N}}(\mathcal{A} \cup \{c\}) \models \varphi(\bar{a}, c)$ and as $ConHull_{\mathcal{N}}(\mathcal{A} \cup \{c\}) \subseteq \mathcal{N}$, we get $\mathcal{N} \models \varphi(\bar{a}, c)$. We have three cases for b.

Case 1: If $b \in ConHull_{\mathcal{M}}(\mathcal{A})$, by Lemma 5.16 we can take $c = b$.

Case 2: If $b \notin ConHull_{\mathcal{M}}(\mathcal{A})$ but $\mathcal{M} \models R_m(b)$ for some $m \in \mathbb{N}$. Then it is in a copy of $\mathbb{N}$ which is in $\mathcal{M} \setminus ConHull_{\mathcal{M}}(\mathcal{A})$. Choose a copy of $\mathbb{N}$ from $\mathcal{N} \setminus ConHull_{\mathcal{N}}(\mathcal{A})$. This exists as $k_0(\mathcal{N}) = k_0(\mathcal{M})$. Let c be the element in this copy of $\mathbb{N}$ such that $\mathcal{N} \models R_m(c)$. Define π by $\pi \restriction_{ConHull_{\mathcal{M}}(\mathcal{A})}$ is π_0 and $\pi(f^r(b)) = f^r(c)$ where $r \geq -m$. Then π is an isomorphism.

Case 3: If $b \notin ConHull_{\mathcal{M}}(\mathcal{A})$ but $\mathcal{M} \models \psi_r(b)$ for some $r \in \mathbb{N}^+$. Then it is in a cycle of length r which is in $\mathcal{M} \backslash ConHull_{\mathcal{M}}(\mathcal{A})$. If $k_r(\sigma)$ is finite, then $k_r(\mathcal{N}) = k_r(\mathcal{M})$. So choose a cycle of length r from $\mathcal{N} \backslash ConHull_{\mathcal{N}}(\mathcal{A})$ and let c be in this cycle. If $k_r(\sigma)$ is infinite, then it is not necessarily the case that $k_r(\mathcal{N}) = k_r(\mathcal{M})$. However, there are only finitely many cycles of length r which contain some $a \in A$. So choose a cycle of length r from $\mathcal{N} \backslash ConHull_{\mathcal{N}}(\mathcal{A})$ and let c be an element in this cycle. Define π by $\pi \restriction_{ConHull_{\mathcal{M}}(\mathcal{A})}$ is π_0 and $\pi(f^r(b)) = f^r(c)$ where $r \in \mathbb{N}$. Then π is an isomorphism.

Otherwise, b is in a copy of $\mathbb{Z}$ not in $ConHull_{\mathcal{M}}(\mathcal{A})$. If $\mathcal{N}$ has a copy of $\mathbb{Z}$ which is not in $ConHull_{\mathcal{N}}(\mathcal{A})$, then choose c to be an element in this copy of $\mathbb{Z}$. If $\mathcal{N}$ does not have any such copy of $\mathbb{Z}$, we need to use the formula $\varphi(\bar{a}, y)$. We will construct an automorphism π of $\mathcal{M}$. Define $\pi : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$\pi(x) = \begin{cases} x & \text{if } x \notin ConHull_{\mathcal{M}}(b) \\ f(x) & \text{if } x \in ConHull_{\mathcal{M}}(b) \end{cases}.$$

Then π is an automorphism of $\mathcal{M}$ over $\mathcal{A}$. So $\pi(b) \in \varphi(\bar{a}, \mathcal{M})$ and $\pi^r(b) \in \varphi(\bar{a}, \mathcal{M})$ for all $r \in \mathbb{Z}$. So $\varphi(\bar{a}, \mathcal{M})$ is an infinite subset of $\mathcal{M}$. So $\varphi(\bar{a}, \mathcal{M})$ is cofinite. So, by Lemma 5.17, $\varphi(\bar{a}, \mathcal{N})$ is cofinite. As $\mathcal{N}$ is infinite, $\varphi(\bar{a}, \mathcal{N})$is infinite. So choose c in this set. Then $\mathcal{N} \models \varphi(\bar{a}, c)$.

□

Theorem 5.19. Suppose σ satisfies the following:

1. k_0 is finite.

2. If some k_r is infinite for $r \in \mathbb{N}^+$, then $k_0 = 0$ and $\sum_{q \neq r} k_q$ is finite.

Then T_σ is strongly minimal, and has quantifier elimination in the language $L_{f,R}$.

Proof. By Theorem 5.18, $T_{\sigma,R}$ has quantifier elimination. So every L_f-formula is equivalent to a quantifier-free formula in $L_{f,R}$. So every definable subset of $\mathcal{M}$ is defined by a quantifier-free $L_{f,R}$-formula. By Lemma 5.17, the subset of $\mathcal{M}$ defined by an $L_{f,R}$-quantifier-free formula $\varphi(\bar{a}, y)$ for $\bar{a} \in A$ is finite or cofinite. So T_σ is strongly minimal. $\square$

6

Zariski Geometry on Strongly Minimal Injective Unars

In the previous chapter, we worked on the theory of injective unars T_σ and proved that T_σ with certain conditions is strongly minimal. In this chapter, we define a topology on a strongly minimal injective unar $\mathcal{A}$ and show that $\mathcal{A}$ is a Noetherian topological structure which satisfies the axioms for a Zariski geometry.

The first part of this chapter is devoted to topologizing $\mathcal{A}$. In order to topologize $\mathcal{A}$ and prove this topology is Noetherian, we first introduce the notion of a characteristic of $\mathcal{A}$ and we introduce the closed sets $\mathcal{C}_n$. Then we define the notion of a formula being in special form which is important in defining the irreducible closed sets. We also define the dimension and rank of the irreducible closed sets which are defined by a formula in special form.

In the second part of the chapter we prove that the topology on $\mathcal{A}$ with assumption (*) satisfies the Zariski geometry axioms.

This chapter is similar to chapter 4, except that the presence of f in the language means that the closed sets are more complicated than when the language is empty. However, the dimension and rank of closed sets, and Noetherianity, are to some extent similar to those in chapter 4.

6.1 Topology on Injective Unars

In this section, we will introduce the topology on $\mathcal{A}$.

6.1.1 Characteristic of $\mathcal{A}$ and Assumption (*)

Definition 6.1. Let $\mathcal{A}$ be a strongly minimal injective unar. Then $\mathcal{A}$ is said to have positive characteristic r if $k_r(\mathcal{A})$ is infinite for some $r \in \mathbb{N}^+$. Otherwise $\mathcal{A}$ is said to have characteristic ∞. We write $char(\mathcal{A})$ for the characteristic of $\mathcal{A}$.

Definition 6.2. We define assumption (*) by: either $char(\mathcal{A}) = \infty$ or $char(\mathcal{A}) = r$ where $r \in \mathbb{N}^+$ and for all $q \in \mathbb{N}$, $k_q = 0$ unless $q|r$.

Example 6.3. If $k_6(\mathcal{A})$ is infinite, then the formula $x_i = f^6(x_i)$ defines an infinite, hence cofinite set which we need to be open. But it is a positive formula, so we also want the set to be closed. Also the formula $x_i = f^s(x_i)$ where $6 \mid s$ defines an infinite set as well.

Remark 6.4. If (*) holds and $r = char(\mathcal{A})$, then the formula $f^r(x_i) = x_i$ is equivalent to $x_i = x_i$.

Example 6.5. Again back to Example 6.3, the formula $x_i = f^6(x_i)$ is equivalent to $x_i = x_i$ as we have all $k_q = 0$ for $q \in \mathbb{N}^+$ except k_1, k_2 and k_3, and for any x where x is in either a cycle of length $1, 2$ or 3, x satisfies the formula $x = f^6(x)$.

Lemma 6.6. If $\mathcal{A}$ is a strongly minimal injective unar then after removing finitely many points it satisfies (*).

Proof. Suppose $\mathcal{A}$ is a strongly minimal injective unar then by Lemma 5.9 there are two cases:

Case 1: k_0 and k_r are finite for all $r \in \mathbb{N}^+$. So $char(\mathcal{A})$ is ∞, so (*) holds.

Case 2: k_r is infinite for some $r \in \mathbb{N}^+$. So $char(\mathcal{A}) = r$. Let X be the subset of A defined by the formula $x_i \neq f^r(x_i)$. The size of X is $\sum_{q \nmid r} q \cdot k_q$ which is a finite sum as we have $\sum_{q \neq r} k_q$ is finite. So $A \setminus X$ satisfies (*). □

6.1.2 Definition of The Closed Sets $\mathcal{C}_n$

In $L = \langle f \rangle$, the closed sets are the sets defined by a positive quantifier-free formula.

Definition 6.7. For $n \in \mathbb{N}^+$, let $\mathcal{C}_n$ be the collection of subsets of A^n which are defined by a positive Boolean combination of atomic L_f-formulas (with parameters), and let $\mathcal{C} = \bigcup_{n \in \mathbb{N}^+} \mathcal{C}_n$.

This chapter is devoted to the proof of the following theorem.

Theorem 6.8. Assume $\mathcal{A}$ is a strongly minimal injective unar which satisfies (*), then the following hold:

(i) For each $n \in \mathbb{N}^+$, $\mathcal{C}_n$ is the set of closed sets of a topology on A^n.

(ii) The topology is Noetherian.

(iii) The topologies satisfy the axioms (Z0), (Z1), (Z2), and (Z3), which make $\mathcal{A}$ into a Zariski geometry.

The proof of Theorem 6.8 will take the whole of this chapter.

A positive quantifier -free formula is built from atomic formulas using $\wedge$ and $\vee$ only. The atomic formulas are $x_i = f^m(x_j)$ for $m \in \mathbb{N}$ and $i, j = 1, ..., n$, or $x_i = a_i$ for $i = 1, ..., n$ and a_i is a parameter from A, in the structure $\langle \mathcal{A}, f \rangle$. For $m \in \mathbb{N}^+$ we write $x_i = f^{-m}(x_j)$ to mean $f^m(x_i) = x_j$. This makes sense because f is injective. If f is not injective, then we also need to consider the atomic formula $f^n(x_i) = f^m(x_j)$ for $n, m \in \mathbb{N}$ and $i, j = 1, ..., n$ as well.

Lemma 6.9. Let $\mathcal{A}$ be an injective unar, and $\varphi(\overline{x})$ an atomic $L_f(A)$-formula. Then $\varphi(\overline{x})$ is equivalent to a formula of the form $x_i = f^m(x_j)$ for some $m \in \mathbb{N}$, or $x_i = a$ for some $a \in A$.

Proof. By injectivity of $f^{\mathcal{A}}$, $f^{m_i}(x_i) = f^{m_j}(x_j)$ is equivalent to $x_i = f^{m_{ij}}(x_j)$ where $m_{ij} = m_j - m_i$ and $m_j \geq m_i$. □

We refer to Definition 4.10 and Lemma 4.11.

Let $S \subseteq A^n$ be in $\mathcal{C}_n$. Then, by the Disjunctive Normal Form Lemma **??**, S can be written in the form $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ where $\varphi_{k,l}(\overline{x})$ are atomic formulas and $r, s_k \in \mathbb{N}$. The set S corresponds to the finite union of S_k where S_k is defined by $\bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$. Hence, it is sufficient to describe sets of the form S_k. In this case S will be defined by $\bigwedge_{l=1}^{r} \varphi_l(\overline{x})$.

Proposition 6.10. Let $\mathcal{A}$ be a strongly minimal injective unar. Then:

(i) Every $S \in \mathcal{C}_n$ is definable with parameters.

(ii) Every definable (with parameters) subset $S \subseteq A^n$ is a finite Boolean combination of sets in $\mathcal{C}_n$.

Proof. (i) Immediate by definition of $\mathcal{C}_n$.

(ii) By Theorem 5.19, T_σ has quantifier elimination in $L_{f,R}$. So S is defined by a finite Boolean combination of $L_{f,R}$-atomic formulas. So it is enough to show that $L_{f,R}$-atomic formulas define subsets in $\mathcal{C}_n$, or finite Boolean combination of them.

Case 1: If the $L_{f,R}$-atomic formula is $R_n(x_i)$, we list the set $R_n(\mathcal{A})$ as $a_1, ..., a_{k_0(\mathcal{A})}$. Then $R_n(x_i)$ defines the same subset of $\mathcal{A}^n$ as $\bigvee_{j=1}^{k_0(\mathcal{A})} x_i = a_j$, which is in $\mathcal{C}_n$.

Case 2: Otherwise, the atomic formula is an L_f-atomic formula which defines a set in $\mathcal{C}_n$ by definition.

□

6.1.3 Formulas In Special Form

In theory of algebraically closed fields, definable sets are converted to ideals where ideals allow the use of the Hilbert Basis Theorem to show the topology is Noetherian. In the theory of unars, we need to find an analogue for the Hilbert Basis Theorem. The special form formulas φ which define irreducible closed sets play the same role as prime ideals.

Definition 6.11. Let Fix be a subset of $\{1, ..., n\}$. For each $i \in Fix$, let $a_i \in A$. Let $\sim$ be an equivalence relation on $\{1, ..., n\} \setminus Fix$. For each i, j where $i \sim j$ let $m_{ij} \in \mathbb{Z}$ be such that the following holds:

(i) If $char(\mathcal{A}) = \infty$ and $i \sim j$ and $j \sim k$ then $m_{ij} + m_{jk} = m_{ik}$.

(ii) If $char(\mathcal{A}) = r$ where $r \in \mathbb{N}^+$ then all $m_{ij} \in \{0, ..., r-1\}$ and if $i \sim j$ and $j \sim k$ then $m_{ij} + m_{jk} = m_{ik} (\mathrm{mod} r)$.

Given the above data, we define the formula $\varphi_{Fix,\overline{a},\sim,\overline{m}}(\overline{x})$ to be

$$\bigwedge_{i \in Fix} x_i = a_i \wedge \bigwedge_{\{(i,j) | i \sim j, i<j\}} x_i = f^{m_{ij}}(x_j).$$

Any formula $\varphi(x_1, ..., x_n)$ is in special form if there are $Fix, \overline{a}, \sim$ and $\overline{m}$ such that $\varphi(x_1, ..., x_n) = \varphi_{Fix,\overline{a},\sim,\overline{m}}$.

Proposition 6.12. Suppose that $S \subseteq A^n$ is non-empty and defined by a conjunction of atomic formulas. Then there are finitely many subsets $S_1, \ldots, S_t$ of S such that $S = \bigcup_{k=1}^{t} S_k$ and each S_k is defined by a formula in special form.

Proof. Suppose $S \subseteq A^n$ is non-empty and defined by a conjunction $\varphi = \bigwedge_{l=1}^{N} \varphi_l$ of atomic formulas. We need to find formulas $\theta_1, ..., \theta_k$ with each θ_k of the form $\varphi_{Fix,\bar{a},\sim,\overline{m}}$.

Define $Fix \subseteq \{1, ..., n\}$ by

$$Fix_1 = \{i| \text{ for some } a \in A, \varphi \vdash x_i = a\},$$

$$Fix_2 = \{i| \text{ for some } q \in \mathbb{N}^+ \text{ and } r \nmid q, \varphi \vdash x_i = f^q(x_i)\} \text{ if } char(\mathcal{A}) = r,$$

or

$$Fix_2 = \{i| \text{ for some } q \in \mathbb{N}^+, \varphi \vdash x_i = f^q(x_i)\} \text{ if } char(\mathcal{A}) = \infty.$$

Then define $Fix = Fix_1 \cup Fix_2$.

Define $\sim$ on $\{1, ..., n\} \setminus Fix$ by $i \sim j$ if and only if there is $m_{ij} \in \mathbb{Z}$ such that $\varphi \vdash (x_i = f^{m_{ij}}(x_j))$. This also defines the m_{ij}.

Let

$$\varphi_{\sim,\overline{m}} = \bigwedge_{\{(i,j)|i<j, i\sim j\}} x_i = f^{m_{ij}}(x_j).$$

So $\varphi \vdash \varphi_{\sim,\overline{m}}$.

Now we define the number t and the tuple $\bar{a}_k$.

For each $i \in Fix_2$, set

$$t_i = \sum_{s|q_i} s \cdot k_s,$$

where $q_i \in \mathbb{N}^+$ is the least such that $\varphi \vdash x_i = f^{q_i}(x_i)$.

Since $S \neq \varnothing$, $t_i \neq 0$. Let

$$t = \prod_{i \in Fix_2} t_i.$$

Then we choose the $a_{i,k}$ for $i \in Fix_2$ and $k = 1, ..., t$ to list all the t tuples from A with

$$\bigwedge_{i \in Fix_2} f^{q_i}(a_{i,k}) = a_{i,k}.$$

For $i \in Fix_1$, set $a_{i,k}$ to be the $a \in A$ such that $\varphi \vdash x_i = a$, for each

$k = 1, ..., t$. Then for $k = 1, ..., t$, let θ_k be

$$\varphi_{\sim,\overline{m}} \wedge \bigwedge_{i \in Fix} x_i = a_{i,k}.$$

Let $S_k = \theta_k(A)$. We need to show that

$$S = \bigcup_{k=1}^{t} S_k.$$

First we show that

$$S \subseteq \bigcup_{k=1}^{t} S_k.$$

Suppose $\bar{a} = (a_1, ..., a_n) \in S$. So $A \models \varphi(\bar{a})$. Since $\varphi \vdash \varphi_{\sim,\overline{m}}$, $A \models \varphi_{\sim,\overline{m}}(\bar{a})$. If $i \in Fix_1$, then $\varphi \vdash x_i = a_{i,k}$ for all k. For each $i \in Fix_2$, $\varphi \vdash f^{q_i}(x_i) = x_i$, so $A \models f^{q_i}(a_i) = a_i$. So by the choice of the $a_{i,k}$, there is $k \in \{1, ..., t\}$ such that $A \models \bigwedge_{i \in Fix_2} a_i = a_{i,k}$. So then $\bar{a} \in S_k$. So

$$S \subseteq \bigcup_{k=1}^{t} S_k.$$

Now we show that

$$\bigcup_{k=1}^{t} S_k \subseteq S.$$

Let $\bar{a} \in S_k$ for some k. We must show that $A \models \bigwedge_{l=1}^{N} \varphi_l(\bar{a})$. By Lemma 6.9, there are three possibilities for φ_l up to equivalence under T_σ. If φ_l is $x_i = a$ for some i and some $a \in A$ then, by definition of Fix_1, $i \in Fix_1$ and $a = a_{i,k}$. So $A \models \varphi_l(\bar{a})$. If φ_l is $x_i = f^{m_{ij}}(x_j)$ for some $i \neq j$ and some $m_{ij} \in \mathbb{N}$, then $\varphi_{\sim,\overline{m}} \vdash \varphi_l$. So $\theta_k \vdash \varphi_l$. So $A \models \varphi_l(\bar{a})$. If φ_l is $x_i = f^{q_i}(x_i)$ for some $q_i \in \mathbb{N}^+$, then $i \in Fix_2$. So $f^{q_i}(a_{i,k}) = a_{i,k}$. So $A \models \varphi_l(\bar{a})$. So

$$\bigcup_{k=1}^{t} S_k \subseteq S$$

as required. Hence

$$S = \bigcup_{k=1}^{t} S_k.$$

□

Example 6.13. (i) If $char(\mathcal{A}) = \infty$, then k_r is finite for each $r \in \mathbb{N}^+$.

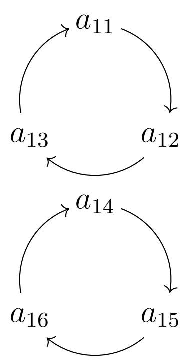
Let $S \subseteq A^4$ and S be defined by the formula

$$x_1 = f^2(x_3) \wedge x_3 = f(x_1) \wedge x_2 = f^5(x_4).$$

So S is also defined by

$$x_1 = f^3(x_1) \wedge x_3 = f(x_1) \wedge x_2 = f^5(x_4).$$

Suppose $k_3 = 2$ and we list the elements on 3-cycles as $a_{1,1}, ..., a_{1,6}$.



Then the special form is equivalent to

$$x_1 = a_{1,k} \wedge x_3 = a_{3,k} \wedge x_2 = f^5(x_4)$$

for $k = 1, ..., 6$ and $a_{3,k} = f(a_{1,k})$. So S will be defined by the formula

$$\bigvee_{k=1}^{6} \left[x_1 = a_{1,k} \wedge x_3 = a_{3,k} \wedge x_2 = f^5(x_4)\right].$$

(ii) Let $char(\mathcal{A}) = 6$. So $k_6 = \infty$, and k_1, k_2, k_3 are finite and each $k_r = 0$ for $r \in \mathbb{N}^+ \setminus \{1, 2, 3, 6\}$. So if S is defined by the formula $x_1 = f^6(x_1)$

then the special form will be $x_1 = x_1$ as all the elements in S satisfy the formula $x_1 = f^6(x_1)$.

6.2 Dimension and Rank of Closed Sets in Injective Unars

The dimension of sets defined by formulas in special form is needed so we can determine the rank of these sets. We can use the number of equivalence classes in the formula in special form to define their dimension.

Lemma 6.14. Suppose $S \subseteq A^n$ is defined by $\varphi_{Fix,\overline{a},\sim,\overline{m}}$ in special form. Let d be the number of equivalence classes for $\sim$. Then

(i) S is in definable bijection with A^d.

(ii) The Morley rank, $MR(S) = d$

(iii) The Morley degree, $MDeg(S) = 1$.

Proof. (i) For each $\sim -$ class, choose j such that $m_{ij} \geq 0$ for all i such that $i \sim j$. Let $j_1, ..., j_d$ be the set of these representatives of the $\sim -$ classes. Then $\pi : S \longrightarrow A^d$ given by $\pi(x_1, ..., x_n) = (x_{j_1}, ..., x_{j_d})$ is a bijection. To show this, define $\theta : A^d \longrightarrow S$ as follows: Let $(b_1, ..., b_d) \in A^d$. Define $\theta(b_1, ..., b_d) = (e_1, ..., e_n)$ where

$$e_i = \begin{cases} a_i & \text{if } i \in Fix \\ f^{m_{ij_r}}(b_r) & \text{if } i \sim j_r \text{ for some } r = 1, ..., d. \end{cases}$$

Then $\theta(\pi(x_1, ..., x_n)) = \theta(x_{j_1}, ..., x_{j_d}) = (x_1, ..., x_n)$. So $\theta \circ \pi = Id_S$. Also $\pi(\theta(b_1, ..., b_d)) = \pi(e_1, ..., e_n) = (e_{j_1}, ..., e_{j_d}) = (b_1, ..., b_d)$. So $\pi \circ \theta = Id_{A^d}$. Thus π is a bijection. Parts (ii) and (iii) follow from (i) because A is strongly minimal.

□

Classifying closed sets is important as we can then determine the irreducible closed ones which is important in proving the Zariski geometry axioms as well as Noetherianity. This can be done by defining their rank. We write $dim(S)$ for $MR(S)$.

Definition 6.15. Given $\varphi = \varphi_{Fix,\overline{a},\sim,\overline{m}}$ in special form, defining S, define the rank of φ and of S to be $rk\varphi = rkS = \omega^{dim(S)}$.

Definition 6.16. Given $S \in \mathcal{C}_n$ and φ defining S, with $\varphi = \bigvee_{k=1}^{r} \varphi_k$ where each φ_k is in special form, define $rk\varphi = \sum_{k=1}^{r} rk\varphi_k$ such that $rk\varphi_1 \geq rk\varphi_2 \geq ... \geq rk\varphi_k$. Define $rkS = min\{rk\varphi | \varphi$ is of the above form and φ defines $S\}$.

Remark 6.17. In Definition 6.16, it is important that $rk\varphi_1 \geq rk\varphi_2 \geq ... \geq rk\varphi_k$ as the ordinal sum is not commutative.

Example 6.18. We will take a closed set S in A^3 as an example. Suppose $\varphi(x_1, x_2, x_3)$ is the formula

$$x_1 = f(x_2) \vee (x_2 = a_2 \wedge x_3 = a_3),$$

and $\varphi'(x_1, x_2, x_3)$ is the formula

$$\varphi(x_1, x_2, x_3) \vee (x_1 = a_1 \wedge x_2 = a_2 \wedge x_3 = a_3) \vee (x_1 = f(x_2) \wedge x_2 = f^3(x_3)).$$

Both $\varphi(x_1, x_2, x_3)$ and $\varphi'(x_1, x_2, x_3)$ are in special form. So

$$rk(\varphi) = \omega^2 + \omega$$

and

$$rk(\varphi') = \omega^2 + 2\omega + 1.$$

Note that $\varphi(x_1, x_2, x_3)$ and $\varphi'(x_1, x_2, x_3)$ define the same subset S of A^3. Therefore $rk(S) = \omega^2 + \omega$.

Lemma 6.19. If $S_1, S_2 \in \mathcal{C}_n$ then $rk(S_1 \cup S_2) \leq rkS_1 + rkS_2$.

Proof. Let $S_1, S_2 \in \mathcal{C}_n$. Choose φ_1, φ_2, disjunctions of formulas in special form, $\varphi_1 = \bigvee_{k=1}^{r_1} \psi_{k1}$ and $\varphi_2 = \bigvee_{k=1}^{r_2} \psi_{k2}$ with each ψ_{kj} in special form such that $\varphi_1(\mathcal{A}) = S_1$ and $\varphi_2(\mathcal{A}) = S_2$ and $rkS_1 = rk\varphi_1$ and $rkS_2 = rk\varphi_2$. $S_1 \cup S_2$ is defined by

$$\begin{aligned} \varphi &= \varphi_1 \vee \varphi_2 \\ &= \bigvee_{k=1}^{r_1} \psi_{k1} \vee \bigvee_{k=1}^{r_2} \psi_{k2} \end{aligned}$$

So

$$\begin{aligned} rk\varphi &= \sum rk\psi_{k1} + \sum rk\psi_{k2} \\ &= rk\varphi_1 + rk\varphi_2 \\ &= rkS_1 + rkS_2 \end{aligned}$$

Therefore

$$rk(S_1 \cup S_2) \leq rk\varphi$$

□

In the theory of algebraically closed fields, the descending chain condition for closed sets is proved first by converting the definable sets to ideals then by using the Hilbert basis theorem. In the theory of injective unars, the special form of formulas plays the role of the prime ideals and Proposition 6.22 plays the role of the Hilbert basis theorem.

Lemma 6.20. Suppose $C, S \in \mathcal{C}_n$, and S is defined by a formula in special form,and $C \subsetneq S$. Then $rkC < rkS$.

Proof. Let $C \subsetneq S$. Suppose S is defined by the formula φ in special form. We need to show $rkC < rkS$. Suppose C is defined by $\psi = \bigvee_{k=1}^{r} \psi_k$ where each ψ_k is in special form. Let $C_k = \psi_k(A)$, $k = 1, ..., r$. So $C = C_1 \cup ... \cup C_r$.

Since C_k is a proper closed subset of S, $\mathcal{A} \vDash \forall x_1, ..., x_n[\psi_k(\overline{x}) \longrightarrow \varphi(\overline{x})]$. So we can assume all conjuncts in φ (atomic formulas) are also in ψ_k. As C_k is a proper subset of S, there must be at least one more atomic formula, say θ in ψ_k. So we have $\mathcal{A} \vDash \forall\overline{x}[\psi_k(\overline{x}) \longrightarrow (\theta(\overline{x}) \wedge \varphi(\overline{x}))]$. So we can assume $\psi_k = \theta \wedge \varphi$. So θ is either $x_i = a_i$ for some $i \notin Fix(\varphi)$, then all j in the equivalence class of i in $\sim_\varphi$ are in $Fix(\psi_k)$ and the number of equivalence classes of $\sim_{\psi_k}$ is the number of equivalence classes of $\sim_\varphi -1$; or θ is $x_i = f^m(x_j)$ such that $i \nsim_\varphi j$. Then $[i]_{\sim_\varphi} \cup [j]_{\sim_\varphi} = [i]_{\sim_{\psi_k}}$. That is, the equivalence classes of i and j under $\sim_\varphi$ are contained into one equivalence class under $\sim_{\psi_k}$. So the number of equivalence classes for $\sim_{\psi_k}$ is the number of equivalence classes for $\sim_\varphi -1$. Let $m = dimS$. Then $rkS = rk\varphi = \omega^m$. For each k, $rkC_k = rk\psi_k = \omega^{dimC_k}$, but $dimC_k < m$, so $rkC_k \leq \omega^{m-1}$. So by Lemma 6.19,

$$rkC \leq \sum_{k=1}^{r} rkC_k \leq r.\omega^{m-1} < \omega^m = rkS.$$

So $rkC < rkS$. □

Lemma 6.21. If $S \in \mathcal{C}_n$ is defined by a formula in special form then S is irreducible: If $C_1, C_2 \in \mathcal{C}_n$ and $S = C_1 \cup C_2$ then either $C_1 = S$ or $C_2 = S$.

Proof. Suppose $S = C_1 \cup C_2$ with $C_1, C_2 \subsetneq S$, and S is defined by a formula in special form. By Lemma 6.20, $rkC_1 < rkS$. By Definition 6.15, $rkS = \omega^{dim(S)}$ where $dim(S) \in \mathbb{N}^+$. Now $rkC_1, rkC_2 \in \mathbb{N}[\omega]$, say

$$rkC_1 = \sum_{i=0}^{dim(S)-1} \gamma_i\omega^i < \omega^{dim(S)}$$

and

$$rkC_2 = \sum_{i=0}^{dim(S)-1} \delta_i\omega^i < \omega^{dim(S)}.$$

So, by Lemma 6.19,

$$rk(C_1 \cup C_2) \leq \sum_{i=0}^{dim(S)-1} (\gamma_i + \delta_i)\omega^i < \omega^{dim(S)} = rkS.$$

So $C_1 \cup C_2 \neq S$, a contradiction. Thus, S is irreducible. □

It follows from Proposition 6.12 that $\mathcal{C}_n$ contains $\varnothing$ and A^n and is closed under finite unions and finite intersections. To show it is a topology we must show it is closed under infinite intersections.

Proposition 6.22. If $S_1 \subsetneq S_2 \subseteq A^n$ with $S_1, S_2 \in \mathcal{C}_n$. Then $rk(S_1) < rk(S_2)$.

Proof. Suppose $S_1 \subsetneq S_2 \subseteq A^n$ are in $\mathcal{C}_n$. First we consider the case where S_2 is defined by formula in special form. So by Corollary 6.21, S_2 is irreducible. So by Lemma 6.20, $rkS_1 < rkS_2$. Now we consider the case where S_2 is defined by a disjunction of formulas in special form, then, say S_2 has dimension D and decomposition $S_2 = \underbrace{S_1' \cup ... \cup S_d'}_{dim=D} \cup \underbrace{S_{d+1}' \cup ... \cup S_k'}_{dim<D}$. Then either $dim(S_1) < D$, so $rk(S_1) < \omega^D \leq rk(S_2)$, or $dim(S_1) = D$. Then the irreducible components of S_1 of dimension D are some of $S_1', ..., S_d'$. If not all $S_1', ..., S_d'$ are subsets of S_1 then $rk(S_1) < \omega^D.d \leq rk(S_2)$. Otherwise, $S_1' \cup ... \cup S_d' \subseteq S_1$. So let r be the largest number such that there is an S_i' in the irreducible decomposition of S_2 which is not in S_1 of dimension r. Then $rk(S_2) = \omega^D.d + ... + \omega^{r+1}.k + \omega^r.l + ...$ and $rk(S_1) = \omega^D.d + ... + \omega^{r+1}.k + \omega^r.l' + ...$ where $l' < l$. So $rk(S_1) < rk(S_2)$. □

Proposition 6.23. An infinite intersection of members of $\mathcal{C}_n$ is in $\mathcal{C}_n$. Furthermore, $\mathcal{C}_n$ is the set of closed subsets for a Noetherian topology on A^n.

Proof. Let $S_i \in \mathcal{C}_n$ for $i \in I$. So $S_i \subseteq A^n$, for all $i \in I$. Each S_i is defined by a positive quantifier free formula. We want to show there is a finite $I_0 \subseteq I$

such that $\bigcap_{i\in I} S_i = \bigcap_{i\in I_0} S_i$. Assume that I is an ordinal. Let $C_\alpha = \bigcap_{\beta<\alpha} S_\beta$. Also $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$. Using Lemma 6.20, we have $rkC_1 \geq rkC_2 \geq \ldots$. So, since rk is ordinal valued, there are only finitely many $i \in I$ such that $rkC_{i-1} > rkC_i$, say $i_1, ..., i_k$. Let $I_0 = \{i_l | l = 1, ..., k\}$. So $rkC_{i_k} = rkC_j$ for all $j > i_k$ and $C_{i_k} = C_j$ for all $j \geq i_k$. So $\bigcap_{i_l\in I_0} S_{i_l} = \bigcap_{i\in I} S_i$. So $\mathcal{C}_n$ is closed under infinite intersections, so is a topology and it is Noetherian. □

We will now prove that the irreducible closed sets are exactly those defined by a formula in special form.

Proposition 6.24. Let $S \subseteq A^n$, and $S \neq \varnothing$. Then S is closed and irreducible if and only if S is defined by a formula in special form.

Proof. If S is defined by a formula in special form then by Lemma 6.21 S is irreducible.
Suppose S is closed and irreducible. Since S is closed, by the DNF theorem, it is defined by a formula of the form $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$ where each $\varphi_{k,l}(\overline{x})$ is atomic. Let S_k be defined by $\bigwedge_{l=1}^{s_k} \varphi_{k,l}(\overline{x})$. Then $S = \bigcup_{k=1}^{r} S_k$. By Proposition 6.12, each S_k is of the form $S_k = \bigcup_{j=1}^{t_k} S_{k,j}$ such that each $S_{k,j}$ is defined by a formula in special form. So $S = \bigcup_{k=1}^{r} \bigcup_{j=1}^{t_k} S_{k,j}$. But S is irreducible, so is equal to one of the $S_{k,j}$. So S is defined by a formula in special form. □

6.3 Zariski Geometry Axioms on Injective Unars

This section is devoted to proving that the structure of injective unars with the condition (*) satisfies the Zariski geometry axioms.
The definition of Zariski geometry was given in Definition 4.6. We repeat it here for convenience.

Definition 6.25. [12, p.306] A Zariski geometry is an infinite set A and a sequence of Noetherian topologies on $A, A^2, A^3, ...$ such that the following axioms hold.

(Z0) i) If $\pi : A^n \longrightarrow A^m$ is defined by $\pi(x) = (\pi_1(x), ..., \pi_m(x))$ where each $\pi_i : A^n \longrightarrow A$ is either constant or coordinate projection, then π is continuous.

ii) Each diagonal $\Delta^n_{i,j} = \{x \in A^n : x_i = x_j\}$ is closed.

(Z1) (Weak QE): If $C \subseteq A^n$ is closed and irreducible, and $\pi : A^n \longrightarrow A^m$ is a projection, then there is a closed $F \subset \overline{\pi(C)}$ such that $\pi(C) \supseteq \overline{\pi(C)} \setminus F$.

(Z2) (Uniform one-dimensionality):

i) A is irreducible.

ii) Let $C \subseteq A^n \times A$ be closed and irreducible. For $a \in A^n$, let $C(a) = \{x \in A : (a, x) \in C\}$. There is a number N such that, for all $a \in A^n$, either $|C(a)| \leq N$ or $C(a) = A$. In particular, any proper closed subset of A is finite.

(Z3) (Dimension theorem): Let $C \subseteq A^n$ be closed and irreducible. Let W be a nonempty irreducible component of $C \cap \Delta^n_{i,j}$. Then $dimW \geq dimC - 1$ where dim is the same as Morley rank.

We recall from Definition 4.25 some notation for the projection maps in axiom (Z0).

Definition 6.26. Given a function $\sigma : \{1, ..., m\} \longrightarrow \{1, ..., n\}$, we can get a projection map $\pi_\sigma : A^n \longrightarrow A^m$ given by $\pi_\sigma(x_1, ..., x_n) = (x_{\sigma(1)}, ..., x_{\sigma(m)})$. As we need to consider each map which is a co-ordinate projection or a constant, and show it is continuous, we define

$$\pi_{\alpha,i}(x_1, ..., x_n) = \begin{cases} x_{\alpha(i)} & \text{if } \alpha(i) \in \{1, ..., n\} \\ a & \text{if } \alpha(i) = a, a \in A \end{cases}$$

where $\alpha : \{1,...,m\} \longrightarrow \{1,...,n\} \cup A$. So $\pi_\alpha(x_1,...,x_n) = (\pi_{\alpha,1}(\overline{x}),...,\pi_{\alpha,m}(\overline{x}))$ where $\overline{x} = (x_1,...,x_n)$.

Lemma 6.27. (Z0) holds for $\langle A, f\rangle$.

Proof. i) Let $\psi(\overline{x})$ be the negation of atomic formulas

$$\neg[x_i = f^m(x_j)] \text{ for } m \in \mathbb{Z}$$

or

$$\neg[x_i = a]$$

where $i, j \in \{1,...,n\}, a \in A$. Any open set $U \subseteq A^m$ can be defined by a positive boolean combination $\bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \psi_{k,l}(\overline{x})$ of basic open sets. Now

$$\pi_\alpha^{-1}(U) = \{(\overline{x} \in A^n| \bigvee_{k=1}^{r} \bigwedge_{l=1}^{s_k} \psi_{k,l}(\pi_\alpha(\overline{x}))\}.$$

So we need to see that $\psi_{k,l}(\pi_\alpha(\overline{x}))$ is an open set. But $\psi_{k,l}(\pi_\alpha(\overline{x}))$ is the formula $\neg[x_i = f^m(x_j)]$ or $\neg[x_i = a_i]$ with substitution of terms $\pi_{\alpha,i}$ for x_i and $\pi_{\alpha,j}$ for x_j. So it is $\neg[\pi_{\alpha,i}(\overline{x}) = f^m(\pi_{\alpha,j}(\overline{x}))]$ or $\neg[\pi_{\alpha,i}(\overline{x}) = a_{k,l}]$. These are negations of atomic formulas, so they are basic open sets. Thus, π_α is continuous.

ii) The diagonal $\Delta^n_{i,j}$ is defined by the atomic formula $x_i = x_j$ which is in special form. So $\Delta^n_{i,j}$ is closed.

Therefore, axiom (Z0) holds for $\langle A, f\rangle$. □

Lemma 6.28. (Z1) holds in $\langle A, f\rangle$.

First we give proof for the easier case $m = n - 1$.

Proof for $m = n-1$: Let $C \subseteq A^n$ be irreducible closed set. If $C = \varnothing$ then $\pi(C) = \varnothing$ which is closed. Otherwise let $\varphi(\overline{x})$ be a formula in special form defining C. Let $\pi : A^n \longrightarrow A^{n-1}$ and $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-1})$. Then we will have the following cases:

Case 1: If $n \in Fix$, then $\pi(C)$ is defined by

$$\bigwedge_{i \in Fix \setminus \{n\}} x_i = a_i \wedge \bigwedge_{\{(i,j)|i \sim j, i<j\}} x_i = f^{m_{ij}}(x_j)$$

which is in special form. So $\pi(C)$ is closed and we can take $F = \varnothing$.

Case 2: If $n \notin Fix$, then write $\varphi'(x_1, ..., x_{n-1})$ for the formula

$$\bigwedge_{i \in Fix} x_i = a_i \wedge \bigwedge_{\{(i,j)|i \sim j, i<j<n\}} x_i = f^{m_{ij}}(x_j).$$

Then $\varphi(\overline{x})$ is the formula

$$\varphi'(x_1, ..., x_{n-1}) \wedge \bigwedge_{\{i|i \sim n, i<n\}} x_i = f^{m_{in}}(x_n).$$

So $\pi(C)$ is defined by $\exists x_n \varphi(\overline{x})$, which is equivalent to

$$\varphi'(x_1, ..., x_{n-1}) \wedge \exists x_n \left[\bigwedge_{i \sim n, i<n} x_i = f^{m_{in}}(x_n) \right].$$

The formula

$$\exists x_n \left[\bigwedge_{i \sim n, i<n} x_i = f^{m_{in}}(x_n) \right]$$

is equivalent to

$$\exists x_n \left[\bigwedge_{i \sim n, i<n} x_n = f^{m_{ni}}(x_i) \right].$$

If some $m_{ni} \geq 0$, then the projection will be defined by eliminating x_n and $\pi(C)$ will be defined by $\varphi'(x_1, ..., x_{n-1})$ which is in special form. So $\pi(C)$ is closed and we can take $F = \varnothing$. Also if $char(\mathcal{A}) \neq \infty$, then f is surjective and again we can eliminate x_n in the projection and $\pi(C)$ will be defined by $\varphi'(x_1, ..., x_{n-1})$. So $\pi(C)$ is closed and hence $F = \varnothing$. Now suppose all $m_{ni} < 0$. Choose l such that m_{nl} is smallest.

So

$$\varphi'(x_1, ..., x_{n-1}) \wedge \exists x_n \left[\bigwedge_{i \sim n, i<n} x_i = f^{m_{in}}(x_n) \right]$$

is equivalent to

$$\varphi'(x_1, ..., x_{n-1}) \wedge \exists x_n \left[x_l = f^{m_{ln}}(x_n) \right]$$

which is equivalent to

$$\varphi'(x_1, ..., x_{n-1}) \wedge \neg \bigvee_{j=0}^{m_{ln}-1} R_j(x_l).$$

So take F to be the set defined by

$$\varphi'(x_1, ..., x_{n-1}) \wedge \bigvee_{j=0}^{m_{ln}-1} R_j(x_l)$$

which is closed. Hence, $\pi(C) = \overline{\pi(C)} \setminus F$. We can also see that $\overline{\pi(C)}$ is defined by $\varphi'(x_1, ..., x_{n-1})$.

□

Example 6.29. Suppose C defined by the formula

$$x_1 = f^5(x_3) \wedge x_3 = f^2(x_4).$$

So the special form is

$$x_1 = f^5(x_3) \wedge x_1 = f^7(x_4) \wedge x_3 = f^2(x_4).$$

So the projection of C from A^4 to A^3 will be defined by

$$x_1 = f^5(x_3) \wedge \exists x_4 \left[x_4 = f^{-7}(x_1) \wedge x_4 = f^{-2}(x_3) \right]$$

We choose the least m_{4i} which is $m_{41} = -7$. So we get for the projection

$$x_1 = f^5(x_3) \wedge \neg \bigvee_{j=0}^{6} R_j(x_1).$$

Now we will examine the projection $\pi : A^n \longrightarrow A^m$ where $\pi(x_1, ..., x_n) = (x_1, ..., x_m)$.

Proof for arbitrary m: Let $C \subseteq A^n$ be an irreducible closed set. If $C = \varnothing$ then $\pi(C) = \varnothing$ which is closed. Otherwise let $\varphi(\overline{x})$ be a formula in special form defining C. Write $\varphi'(x_1, ..., x_m)$ for the formula

$$\bigwedge_{i \in Fix \cap \{1,...,m\}} x_i = a_i \wedge \bigwedge_{\{(i,j) | i \sim j, i < j \leq m\}} x_i = f^{m_{ij}}(x_j).$$

Then $\varphi(\overline{x})$ is the formula

$$\varphi'(x_1, ..., x_m) \wedge \bigwedge_{i \in Fix \cap \{m+1,...,n\}} x_i = a_i \wedge \bigwedge_{\{(i,k) | i \sim k, i < k, k > m\}} x_i = f^{m_{ik}}(x_k).$$

Let $K = \{k \in \{m+1, ..., n\} | \exists i \in \{1, ..., m\}$ such that $i \sim k\}$.
Then $\varphi(\overline{x})$ is the formula

$$\varphi'(x_1, ..., x_m) \wedge \bigwedge_{i \in Fix \cap \{m+1,...,n\}} x_i = a_i \wedge$$
$$\bigwedge_{\{(j,k) | j \leq m, k \in K, j \sim k\}} x_j = f^{m_{jk}}(x_k) \wedge \bigwedge_{\{(i,k) \in \{m+1,...,n\}^2 | i \sim k, i < k\}} x_i = f^{m_{ik}}(x_k).$$

Then under projection, the conjunctions

$$\bigwedge_{i \in Fix \cap \{m+1,...,n\}} x_i = a_i$$

and

$$\bigwedge_{\{(i,k) \in \{m+1,...,n\}^2 | i \sim k, i < k\}} x_i = f^{m_{ik}}(x_k)$$

are eliminated.

List K as $k_1, ..., k_r$. So $\pi(C)$ is defined by

$$\varphi'(x_1, ..., x_m) \wedge \exists x_{k_1}, ..., x_{k_r} \left[\bigwedge_{\{(j,k)|j \leq m, k \in K, j \sim k\}} x_j = f^{m_{jk}}(x_k) \right].$$

The formula

$$\exists x_{k_1}, ..., x_{k_r} \left[\bigwedge_{\{(j,k)|j \leq m, k \in K, j \sim k\}} x_j = f^{m_{jk}}(x_k) \right]$$

is equivalent to

$$\exists x_{k_1}, ..., x_{k_r} \left[\bigwedge_{\{(k,j)|j \leq m, k \in K, j \sim k\}} x_k = f^{m_{kj}}(x_j) \right].$$

If $char(\mathcal{A}) \neq \infty$ then f is surjective. So $\pi(C)$ is defined by $\varphi'(x_1, ..., x_m)$ which is in special form so $\pi(C)$ is closed and we can take $F = \varnothing$. Otherwise for each $m_{kj} \geq 0$ we eliminate such m_{kj} in $\pi(C)$ and for each $m_{kj} < 0$ we choose l such that $m_{k_l j}$ is smallest. Define

$$J = \{j \in \{1, ..., m\} : \exists k \in \{m+1, ..., n\} \text{ such that } j \sim k\}.$$

Then $\pi(C)$ is defined by

$$\varphi'(x_1, ..., x_m) \wedge \exists x_{k_1}, ..., x_{k_r} \left[\bigwedge_{j \in J} \bigwedge_{k \in K, k \sim j} x_j = f^{m_{jk}}(x_k) \right].$$

For each $j \in J$, let $l(j) \in K$ such that $m_{j,l(j)}$ is greatest. Let

$$J' = \{j \in J : m_{j,l(j)} > 0\}.$$

Then $\pi(C)$ is defined by

$$\varphi'(x_1, ..., x_m) \wedge \neg \left[\bigvee_{j \in J'} \bigvee_{q=0}^{m_{j,l(j)}-1} R_q(x_j) \right].$$

So we may take F to be the set defined by

$$\bigvee_{j \in J'} \bigvee_{q=0}^{m_{j,l(j)}-1} R_q(x_j)$$

which is closed. Hence, $\pi(C) = \overline{\pi(C)} \setminus F$.

Therefore, (Z1) holds in $\langle A, f \rangle$. □

Example 6.30. Let C be defined by the formula

$$x_1 = f(x_2) \wedge x_3 = f^2(x_4) \wedge x_5 = x_6$$
$$\wedge x_{11} = f^{-2}(x_1) \wedge x_{12} = f(x_{11})$$
$$\wedge x_{13} = f^{-1}(x_3) \wedge x_{14} = f^{-5}(x_{13})$$
$$\wedge x_{15} = f^2(x_{16}) \wedge x_{17} = f(x_7).$$

and take $m = 10$. So $\varphi'(x_1, ..., x_{10})$ is equivalent to

$$x_1 = f(x_2) \wedge x_3 = f^2(x_4) \wedge x_5 = x_6.$$

Then $\varphi(x_1, ..., x_{17})$ is equivalent to

$$\varphi'(x_1, ..., x_{10}) \wedge [x_1 = f^2(x_{11}) \wedge x_1 = f(x_{12}) \wedge x_2 = f^3(x_{11})$$
$$\wedge x_2 = f^2(x_{12}) \wedge x_3 = f(x_{13})$$
$$\wedge x_3 = f^6(x_{14}) \wedge x_4 = f^3(x_{13})$$
$$\wedge x_4 = f^8(x_{14}) \wedge x_7 = f^{-1}(x_{17})].$$

So F will be given by

$$\varphi'(x_1, ..., x_{10}) \wedge [R_0(x_1) \vee R_1(x_1) \vee R_0(x_2) \vee R_1(x_2) \vee R_2(x_2) \vee \bigvee_{j=0}^{5} R_j(x_3) \vee \bigvee_{j=0}^{7} R_j(x_4)]$$

Lemma 6.31. (Z2) holds in $\langle A, f \rangle$.

Proof. i) By Lemma 6.21, A is irreducible.

ii) Let $C \subseteq A^n \times A$ be closed and irreducible. Consider the projection $\pi : A^{n+1} \longrightarrow A^n$. For $\overline{a} \in A^n$, let $C(a) = \{x_{n+1} \in A | (a, x_{n+1}) \in C\}$. Let φ be a formula in special form defining C. If $a \notin \pi(C)$ then $C(a) = \varnothing$. Assume $a \in \pi(C)$. Then we will have the following cases:

Case 1: If $n + 1 \in Fix$ then $\pi \restriction_C : C \longrightarrow \pi(C)$ is a bijection. Therefore, $|C(a)| = 1$.

Case 2: If $n + 1 \notin Fix$ and there is $i < n + 1$ such that $i \sim n + 1$ then $C(a) = \{f^{-m_{i(n+1)}}(a_i)\}$. Therefore, $|C(a)| = 1$.

Case 3: If $n+1 \notin Fix$ and there is no $i < n+1$ such that $i \sim n+1$ then the same φ defines $\pi(C)$ we get $dim(\pi(C)) = dim(C) - 1$. Thus $C = \pi(C) \times A$ and $C(a) = A$ for all $a \in \pi(C)$.

Hence Z2 holds in $\langle A, f \rangle$. □

Lemma 6.32. (Z3) holds in $\langle A, f \rangle$.

Proof. Let $C \subseteq A^n$ be closed and irreducible. Let φ be a formula in special form defining C. Let $i, j \in \{1, ..., n\}$ and $i \neq j$. Let W be non empty irreducible component of $C \cap \Delta^n_{i,j}$ where $\Delta^n_{i,j} = \{x \in A^n : x_i = x_j\}$ and let θ be the formula $\varphi \wedge (x_i = x_j)$. We are examining the intersection of the diagonal with the irreducible closed sets. If $C = A^n$ then $W = A^n \cap \Delta^n_{i,j} = \Delta^n_{i,j}$. So $dim(W) = n - 1 = dim(C) - 1$. If $C = \varnothing$ then $W = \varnothing \cap \Delta^n_{i,j} = \varnothing$

but we assumed that W is not empty. If $C \neq A^n$ then we will have the following cases:

Case 1: If $i, j \in Fix_C$ then we will have the formula $x_i = a_i \wedge x_j = a_j$ in φ. If $a_i \neq a_j$ then $W = \varnothing$ but we assumed that $W \neq \varnothing$. So $a_i = a_j$. So $W = C$. So $dim(W) = dim(C)$.

Case 2: If $i \in Fix_C$ and $j \notin Fix_C$ then we have the formula $x_i = a_i$ in φ. So θ is equivalent to

$$\bigwedge_{k \in Fix_C} x_k = a_k \wedge \bigwedge_{k \sim_C j} x_k = f^{m_{kj}}(a_i) \wedge \bigwedge_{k \sim_C l, k \nsim_C j, k<l} x_k = f^{m_{kl}}(x_l).$$

So $i, j \in Fix_W$. So $Fix_W = Fix_C \cup \{the \sim_C - \text{equivalence class of } j\}$ and $C \cap \Delta^n_{i,j}$ is irreducible as θ in special form.
So $dim(\sim_W) = dim(\sim_C) - 1$. So $dim(W) = dim(C) - 1$.

Case 3: If $i, j \notin Fix_C$ and $i \sim_C j$ then φ will imply the formula $x_i = f^{m_{ij}}(x_j)$. So θ will imply the formula $x_i = f^{m_{ij}}(x_j) \wedge x_i = x_j$. If $m_{ij} = 0$ then $W = C$. If $m \neq 0$ then θ will imply the formula $x_j = f^{m_{ij}}(x_j)$. By the proof of proposition 6.12, $C \cap \Delta^n_{i,j} = \bigcup_{k=1}^{t} C_k$ where each C_k is irreducible and given by $\varphi \wedge x_j = a_{j,k}$ for some $a_{j,k} \in A$. Thus $j \in Fix_W$. So $Fix_W = Fix_C \cup \{the \sim_C - \text{equivalence class of } j\}$ which means that $dim(\sim_W) = dim(\sim_C) - 1$. So $dim(W) = dim(C) - 1$.

Case 4: If $i, j \notin Fix_C$ and $i \nsim_C j$ then θ is equivalent to

$$\bigwedge_{i \in Fix_C} x_i = a_i \wedge \bigwedge_{k \sim_C l, k<l} x_k = f^{m_{kl}}(x_l) \wedge \bigwedge_{\{(k,l): k \sim i, l \sim j, k<l\}} x_k = f^{m_{ki}+m_{jl}}(x_l).$$

So $i \sim_W j$. So i and j will become in one equivalence class in W. As the above formula is in special form, $C \cap \Delta^n_{i,j}$ is irreducible. So the number of equivalence classes of $\sim_W$ is equal to the number of equivalence classes of $\sim_C -1$. So $dim(W) = dim(C) - 1$.

Thus (Z3) holds in $\langle A^n, f \rangle$. □

7

Classification of all Strongly Minimal Unars

In this chapter we work towards the classification of strongly minimal unars (not necessarily injective ones). We will assume through this chapter that T is strongly minimal and $\mathcal{A}$ is a saturated model.

Definition 7.1. We say a is an infinite point if $\{x \in A : f(x) = a\}$ is infinite.

Lemma 7.2. Suppose $Th(\mathcal{A})$ is strongly minimal. Then there is at most one infinite point in $\mathcal{A}$.

Proof. Suppose $Th(\mathcal{A})$ is strongly minimal. Let $\varphi(x)$ be the formula $f(x) = y$. As $Th(\mathcal{A})$ is strongly minimal, then either $\{y \in A : f(y) = a\}$ is finite or cofinite. If it is infinite, then for any $b \in A$ where $b \neq a$, $\{y \in A : f(y) = b\} \cap \{y \in A : f(y) = a\} = \varnothing$ as f is a function. So $\{y \in A : f(y) = b\}$ is finite. □

Recall that a theory T is limited if $T \vdash \forall x[\bigvee_{n,m=1}^{N} (f^n(x) = f^{n+m}(x))]$ for some $N \in \mathbb{N}$, and T is not limited otherwise.

Lemma 7.3. Suppose $Th(\mathcal{A})$ is strongly minimal and limited and has no infinite point. Then every connected component of $\mathcal{A}$ is finite and all but finitely many connected components are cycles of the same length m.

Proof. Suppose the set $\{y \in A : f(y) = a\}$ is a finite set for all $a \in A$. As $Th(\mathcal{A})$ is strongly minimal, $\mathcal{A}$ has uniform finiteness. By uniform finiteness, there is $N_0 \in \mathbb{N}$ such that $Th(\mathcal{A}) \models \forall y \exists^{\leq N_0} x[f(x) = y]$. Let C be a connected component of $\mathcal{A}$. Since T is limited, for some N_1, $T \models \forall x[\bigvee_{n,m=1}^{N_1} f^n(x) = f^{m+n}(x)]$ and C contains at least one periodic point, say a. Then for every $b \in C$, there is $m \leq 2N_1 - 1$ such that $f^m(b) = a$. So $C = \bigcup_{m=0}^{2N_1-1} f^{-m}(a)$. So

$$|C| \leq 1 + N_0 + {N_0}^2 + ... + {N_0}^{2N_1-1} < \infty.$$

So each connected component has a finite size.

Recall that the formula $\psi_m(x)$ is equivalent to

$$f^m(x) = x \wedge \bigwedge_{s|m, s\neq m} f^s(x) \neq (x) \text{ for } m \in \mathbb{N}^+.$$

Now consider the formula $\theta_{m,r}(x)$ for $m, r \leq N_1$ which is equivalent to the formula $\psi_m(f^r(x)) \wedge \bigwedge_{s=0}^{r-1} \neg\psi_m(f^s(x))$. If $\theta_{m,r}(\mathcal{A})$ is infinite for some $m, r \in \mathbb{N}^+$, then $\theta_{m,0}(\mathcal{A})$ is infinite. Then by strong minimality, $r = 0$. So all but finitely many connected components are cycles of length m. □

Any point satisfies the formula $\theta_{m,r}(x)$ for some $m, r \in \mathbb{N}^+$ is called a pre-periodic point.

Lemma 7.4. Suppose $Th(\mathcal{A})$ is strongly minimal and not limited. Then there is no infinite point and not every point is pre-periodic.

Proof. Suppose there is $a \in A$ such that $|f^{-1}(a)|$ is infinite. So the set $\{x \in A : x = f^{-1}(a)\}$ is cofinite in A. So $A \setminus f^{-1}(a)$ is finite. As f is a function, for any $b \in (A \setminus f^{-1}(a))$ and $b \neq a$ we will have $\mathcal{A} \models \theta_{m,r}(b)$ for some $m, r \in \mathbb{N}^+$. So if a is a pre-periodic then $Th(\mathcal{A})$ is limited. But we assumed that $Th(\mathcal{A})$ is not limited. So a is not a pre-periodic point. Also any copy of $\mathbb{N}$ in A is infinite and can have at most one element in common with $f^{-1}(a)$, a contradiction to strong minimality. So no point has an infinite pre-image. As A is not limited then for each $N \in \mathbb{N}^+$, $Th(\mathcal{A}) \models \exists x[\bigwedge_{n,m=1}^{N} f^n(x) \neq f^{m+n}(x)]$. So by saturation of $\mathcal{A}$, not every point is pre-periodic. □

Definition 7.5. If $|f^{-1}(a)| = k$, we say that a is a k-branching point. We say that $\mathcal{A}$ is k-branching almost everywhere if every point in $\mathcal{A}$ is k-branching except for finitely many points.

Definition 7.6. For $k \in \mathbb{N}^+$, we define T_k to be a connected component in which no point is pre-periodic and every point is k-branching.

Proposition 7.7. Suppose $\mathcal{A}$ and $\mathcal{B}$ are connected unars which are both k-branching and have no pre-periodic points. Then $\mathcal{A} \cong \mathcal{B}$. So T_k is unique up to isomorphism.

Proof. We will define the isomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ inductively. Choose $a \in A$ and $b \in B$. Define $\pi_1(a) = b$ and $\pi_1(f^n(a)) = f^n(b)$ for $n \in \mathbb{N}$.

We will show that π_1 is a partial isomorphism. The $dom(\pi_1) = \{f^n(a) : n \in \mathbb{N}\}$. Since $\mathcal{A}$ has no pre-periodic point, a is not a pre-periodic point. So, if $f^n(a) \neq f^m(a)$ then $n \neq m$. Then as $\mathcal{B}$ has no pre-periodic point, b is not a pre-periodic point. So $f^n(b) \neq f^m(b)$. So $\pi_1(f^n(a)) \neq \pi_1(f^m(a))$. So π_1 is injective. Also π_1 respects f by construction. So π_1 is a partial isomorphism.

Assume we have a partial isomorphism $\pi_m : \mathcal{A} \longrightarrow \mathcal{B}$. Define π_{m+1} by $dom(\pi_{m+1}) = \{x \in \mathcal{A} | f(x) \in dom(\pi_m)\}$. For $x \in dom(\pi_m)$, define $\pi_{m+1}(x) = \pi_m(x)$. For each $z \in dom(\pi_m)$ enumerate the $x \in \mathcal{A}$ such that $f(x) = z$ as $x_1, ..., x_k$ where $x_1, ..., x_r \notin dom(\pi_m)$ and $x_{r+1}, ..., x_k \in dom(\pi_m)$. Enumerate the $y \in \mathcal{B}$ such that $f(y) = \pi_m(z)$ as $y_1, ..., y_k$ where $y_1, ..., y_s \notin Im(\pi_m)$ and $y_{s+1}, ..., y_k \in Im(\pi_m)$. As both $\mathcal{A}$ and $\mathcal{B}$ are k-branching and π_m is a partial isomorphism, $r = s$. Define $\pi_{m+1}(x_i) = y_i$ for $i = 1, ..., r$. We do this for each $z \in dom(\pi_m)$.

Now we will show that π_{m+1} is partial isomorphism. Suppose

$x, x' \in dom(\pi_{m+1})$ are such that $\pi_{m+1}(x) = \pi_{m+1}(x')$.We will have two cases:

Case 1: $x \in dom(\pi_m)$. By the construction of π_{m+1}, if $x' \notin dom(\pi_m)$ then $\pi_{m+1}(x') \notin Im(\pi_m)$. So $x' \in dom(\pi_m)$, but π_m is injective, so $x = x'$.

Case 2: $x \notin dom(\pi_m)$. By construction, $\pi_m(f(x)) = f(\pi_{m+1}(x)) = f(\pi_{m+1}(x')) = \pi(f(x'))$. Since π_m is injective, $f(x) = f(x')$. By construction, x, x' are two from the list $x_1, ..., x_r$ corresponding to $z = f(x)$, and π_{m+1} is injective on this list. So $x = x'$.

So π_{m+1} is injective, and it respects f by construction. So π_{m+1} is a partial isomorphism.
So by induction on m, π_m is a partial isomorphism for each $m \in \mathbb{N}$.
Define $\pi = \bigcup_{m \in \mathbb{N}} \pi_m$. Then π is also a partial isomorphism.
Since $\mathcal{A}$ is connected, π is defined on all of $\mathcal{A}$. Since $\mathcal{B}$ is connected, π is surjective. So π is an isomorphism. □

Proposition 7.8. Suppose that $\mathcal{A}$ is a connected unar such that one point $a \in A$ is pre-periodic. Then all points in $\mathcal{A}$ are pre-periodic.

Proof. Suppose a is a pre-periodic point of period $m \in \mathbb{N}^+$. Then $\mathcal{A} \models \theta_{m,r}(a)$ for some $r \in \mathbb{N}^+$. Let $b \in \mathcal{A}$. As $\mathcal{A}$ is a connected unar, there are $q, n \in \mathbb{N}$ such that $f^q(a) = f^n(b)$. Let $r' = r + q + n$. Then $f^{r'}(b) = f^{r+q}(a) = f^{r+q+m}(a) = f^{r'+m}(b)$. So $\mathcal{A} \models \theta_{m,r'}(f^q(a))$. So $\mathcal{A} \models \theta_{m,r'}(f^n(b))$. So every point in $\mathcal{A}$ is pre-periodic. □

Definition 7.9. For $k \in \mathbb{N}^+$ and $m \in \mathbb{N}^+$, we define $P_{m,k}$ to be a connected component which is pre-periodic of period length m and is k-branching everywhere. Note that $P_{m,1}$ is just a cycle of length m.

Proposition 7.10. Let $\mathcal{A}$ be a strongly minimal unar. Then there is $k \in \mathbb{N}$ such that $\mathcal{A}$ is k-branching except at finitely many points.

Proof. Let $\beta_k(x)$ be the formula $\exists^{=k}y[f(y) = x]$, which defines the set of k-branching points. We have three cases:

Case 1: There is an infinite point a. So a is infinitely-branching. Then a does not satisfy β_k for any $k \in \mathbb{N}$. However, $\{y : f(y) \neq a\}$ is finite. So only finitely many $x \in A$ have non-empty pre-image. So $\mathcal{A}$ is 0-branching almost everywhere.

Case 2: There is no infinite point, but $Th(\mathcal{A})$ is limited. Then, by Lemma 7.3, all but finitely many points are on cycles. So $\mathcal{A}$ is 1-branching almost everywhere.

Caes 3: $Th(\mathcal{A})$ is not limited. By Lemma 7.4, there is no infinite point, so every point is k-branching for some $k \in \mathbb{N}$. So the sets defined by $\beta_k(x)$ for $k \in \mathbb{N}$ give a partition of $\mathcal{A}$ into definable sets. As T is strongly minimal, $\mathcal{A}$ has uniform finiteness. By uniform finiteness, there is $N \in \mathbb{N}$ such that every point is at most N-branching. So we get a finite partition of $\mathcal{A}$ into $\beta_0(\mathcal{A})$, ..., $\beta_N(\mathcal{A})$, each of which is finite or cofinite. So exactly one, say $\beta_k(\mathcal{A})$, is cofinite.

□

Definition 7.11. If $\mathcal{A}$ is k-branching almost everywhere and $a \in A$ is not k-branching, we say that a is a *defective point.* If B is a connected component which has at least one defective point, we say that B is a defective component.

Corollary 7.12. Suppose B is a defective component. Then B has only finitely many defective points.

Proof. Immediate from Proposition 7.10. □

Now we classify the models in Case 3 in Proposition 7.10.

Lemma 7.13. Let $\mathcal{A}$ be a strongly minimal unar which is not limited. Let $k \in \mathbb{N}^+$ be such that $\mathcal{A}$ is almost everywhere k-branching. Then $\mathcal{A}$ consists of:

i) Finitely many defective components.(Possibly none)

ii) For each $m \in \mathbb{N}^+$, finitely many copies of $P_{m,k}$.

iii) Some number of copies of T_k. (Possibly none)

Proof. i) By Proposition 7.10, there are only finitely many defective points. So there are only finitely many defective components.

ii) If the formula $\psi_m(x)$ is cofinite then $\mathcal{A}$ is limited. But $\mathcal{A}$ is not limited, so each $\psi_m(x)$ defines a finite set. So there are only finitely many copies of $P_{m,k}$ for each $m \in \mathbb{N}^+$.

iii) Suppose $a \in A$ is not a pre-periodic point and it is not on a defective component. Then by Proposition 7.7, a is on a copy of T_k.

□

In the classification of strongly minimal unars, we can find a finite substructure which we will characterise it so that we can prove the completeness of the theory of strongly minimal unars.

Definition 7.14. Let $\mathcal{A}$ be a strongly minimal unar. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $\mathcal{A}_0$ is finite, say of size N for $N \in \mathbb{N}^+$. Enumerate $\mathcal{A}_0$ as $a_1, ..., a_N$. Define $\chi_{\mathcal{A}_0}(x_1, ..., x_N) = \bigwedge\{f(x_i) = x_j | i, j = 1, ..., N, \text{ and } \mathcal{A}_0 \models f(a_i) = a_j\} \wedge \bigwedge_{1 \leq i < j \leq N} x_i \neq x_j \wedge \bigwedge_{\{i: f(a_i) \notin A_0\}} \bigwedge_{j=1}^{N} (f(x_i) \neq x_j)$

We will axiomatize the theory of all strongly minimal unars and will use the Łos-Vaught test to prove that this axiomatization is complete.

Axiomatization of $\mathcal{A}$ in Case 1 where $\mathcal{A}$ is 0-branching almost everywhere:

As $\mathcal{A}$ is 0-branching, then almost all points x satisfy the formula $\beta_0(x)$. Also there is an infinite point $a_1 \in \mathcal{A}$. So, almost all points x satisfy the formula $f(x) = a_1 \wedge \beta_0(x)$. Let $\alpha(x)$ be the formula $f(x) = a_1 \wedge \beta_0(x)$. So $\alpha(\mathcal{A})$ is infinite. So cofinite. Let $\mathcal{A}_0$ be $\neg\alpha(\mathcal{A})$. So $\mathcal{A}_0$ is finite, say of size N. Enumerate $\mathcal{A}_0$ as $a_1, ..., a_N$. So $\mathcal{A}_0 \models \chi_{\mathcal{A}_0}(a_1, ..., a_N)$. As we are axiomatizing $\mathcal{A}$, we need to avoid the use of the parameter a_1. So let $\varphi(z)$ be the formula $\exists^{>N}y[f(y) = z]$ where $N = |\mathcal{A}_0|$. Now, $\varphi(\mathcal{A})$ gives $\{a_1\}$. So rewrite $\alpha(x)$ to be the formula $\exists z \left[\exists^{>N}y[f(y) = z] \wedge f(x) = z\right] \wedge \beta_0(x)$.
Take σ to be the axiom

$$\exists x_1, ..., x_N \left[\chi_{\mathcal{A}_0}(x_1, ..., x_N) \wedge \varphi(x_1) \wedge \forall y[\bigvee_{i=1}^{N} y = x_i \vee \alpha(y)]\right].$$

Proposition 7.15. If $\mathcal{A} \models \sigma$ and $\mathcal{B} \models \sigma$ and both have the same cardinality, then $\mathcal{A} \cong \mathcal{B}$.

Proof. Suppose $\mathcal{A}, \mathcal{B} \models \sigma$ and $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Let $a_1, ..., a_N \in A$ and $b_1, ..., b_N \in B$ be witnesses to σ. So $\mathcal{B} \models \chi_{\mathcal{A}_0}(b_1, ..., b_N) \wedge \varphi(b_1) \wedge \forall y[\bigvee_{i=1}^{N} y = b_i \vee \alpha(y)]$ So $|\alpha(\mathcal{B})| = \kappa$. Define $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ by $\pi(a_i) = b_i$ for $i = 1, ..., N$ and $\pi \restriction_{\alpha(\mathcal{A})}$ is any bijection from $\alpha(\mathcal{A})$ to $\alpha(\mathcal{B})$. Let $a \in \mathcal{A}$. If $a \in \mathcal{A}_0$, say $a = a_i$, then $f(a) = a_j \in \mathcal{A}_0$ for some $j \in \{1, ..., N\}$. So $\chi_{\mathcal{A}_0}(a_1, ..., a_N) \vdash f(a_i) = a_j$. So $\mathcal{B} \models f(b_i) = b_j$. So $\pi(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(\pi(a))$. If $a \in \mathcal{A} \setminus \mathcal{A}_0$ then $f^{\mathcal{A}}(a) = a_1$ and $\mathcal{A} \models \alpha(a)$. So $f^{\mathcal{B}}(\pi(a)) = b_1 = \pi(f(a_1))$ and $\mathcal{B} \models \alpha(\pi(a))$. So $\pi(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(\pi(a))$. So π is an isomorphism. $\square$

Axiomatization of $\mathcal{A}$ in Case 2 where $\mathcal{A}$ is 1-branching almost everywhere and every point is pre-periodic:

Let $\gamma_m(x)$ be the formula

$$\psi_m(x) \wedge \neg\exists y[\bigvee_{s=1}^{m} f^s(y) = x \wedge \bigwedge_{r=0}^{m-1} f^r(x) \neq y].$$

So $\mathcal{A} \vDash \gamma_m(x)$ if and only if the connected component of x is a cycle of length m. By Lemma 7.3, $\gamma_m(\mathcal{A})$ is cofinite. Let $\mathcal{A}_0$ be $\neg\gamma_m(\mathcal{A})$. So $\mathcal{A}_0$ is finite, say of size N. Enumerate $\mathcal{A}_0$ as $a_1, ..., a_N$. So $\mathcal{A}_0 \models \chi_{\mathcal{A}_0}(a_1, ..., a_N)$. Take σ to be the axiom

$$\exists x_1, ..., x_N \left[\chi_{\mathcal{A}_0}(x_1, ..., x_N) \wedge \forall y[\bigvee_{i=1}^{N} y = x_i \vee \gamma_m(y)]\right].$$

Proposition 7.16. If $\mathcal{A} \models \sigma$ and $\mathcal{B} \models \sigma$ and both have the same cardinality, then $\mathcal{A} \cong \mathcal{B}$.

Proof. Suppose $\mathcal{A}, \mathcal{B} \models \sigma$ and $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Let $a_1, ..., a_N \in A$ and $b_1, ..., b_N \in B$ be witnesses to σ. So $\mathcal{B} \models \chi_{\mathcal{A}_0}(b_1, ..., b_N) \wedge \forall y[\bigvee_{i=1}^{N} y = b_i \vee \gamma_m(y)]$. So $|\gamma_m(\mathcal{B})| = \kappa$. Define $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ by $\pi(a_i) = b_i$ for $i = 1, ..., N$ and $\pi \restriction_{\gamma_m(\mathcal{A})}$ is any isomorphism from $\gamma_m(\mathcal{A})$ to $\gamma_m(\mathcal{B})$. Let $a \in \mathcal{A}$. If $a \in \mathcal{A}_0$, say $a = a_i$, then $f(a) = a_j \in \mathcal{A}_0$ for some $j \in \{1, ..., N\}$. So $\chi_{\mathcal{A}_0}(a_1, ..., a_N) \vdash f(a_i) = a_j$. So $\mathcal{B} \models f(b_i) = b_j$. So $\pi(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(\pi(a))$. If $a \in \mathcal{A} \setminus \mathcal{A}_0$ then $\pi(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(\pi(a))$ by choice of π. So π is an isomorphism. $\square$

Axiomatizing Case 3 is more complicated as we have some defective components and $P_{m,k}$ and due to lack of time we have managed so far only to axiomatize Case 1 and Case 2 in Proposition 7.10. In future work, we are planning to axiomatize Case 3.

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