Linear stability or graphical analysis? Routines and visual mediation in students' responses to a stability of dynamical systems exam task

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Introduction

Dynamical systems are a crucial topic of differential equations, taught in Mathematics as well as in other, e.g. Engineering, departments. In the UK, where the study we report from was conducted, differential equations are usually introduced in the second year of study after first year courses in Calculus. Despite extensive research in elementary Calculus, at large, differential equations, and specifically dynamical systems, are not widely researched topics. Previous work on dynamical systems highlighted students' use of different procedures, analytical, graphical and numerical methods and the importance of graphical representations (Rasmussen, 2001). Rasmussen comments on the methods needed for the solution of differential equations and notes that:

> "With graphical (or qualitative) methods, one obtains overall information about solutions by viewing solutions to differential equations geometrically and by analyzing the differential equation itself" (Rasmussen, 2001, p. 56).

By analysing the work of six undergraduate students, Rasmussen offers a framework aimed at investigating students' understanding of the graphical and numerical methods used to solve differential equations. His results show that, while students use various methods, they do not make connections between these. Rasmussen notes the scarcity of studies in this area, as does Artigue (2016), referring to her own work done as early as the 1980s. Artigue also discusses the limited experiences that undergraduate students are offered in terms of differential equations with the main focus being on procedures of solving these equations (ibid., p. 13). Both studies stress that the focus of the students' experience in solving differential equations is on the procedures and that this experience is excessively compartmentalised.

The data analysis we present here – in which we take a discursive perspective, Sfard's (2008) theory of commognition – takes this discussion further by investigating the *how* (procedure) and the *when* (applicability and closing conditions) of a routine (ibid, pp. 208-209) as evident in students' written responses to an exam task that asked them to examine the stability of equilibrium points of two dynamical system via linear stability analysis or graphical analysis. We note that the commognitive perspective views mathematics as a discourse characterised by *word use* (e.g. dynamical systems, equilibrium points), *visual mediators* (e.g. sketches of functions), *endorsed narratives* (e.g. definition of equilibrium point) and *routines* (e.g. examining the stability of equilibrium points). The data and analysis we report here are part of a larger study, conducted in the UK, in a well-regarded mathematics department. The study investigates students' engagement with university mathematical discourses in the context of final year examinations (Thoma, 2018; Thoma & Nardi, 2017; 2018).

The task, the lecturer's aims and the students' responses

The module Differential Equations and Applied Methods was attended by ninety-seven students. The first author selected thirty-four student examination scripts for further analysis, to represent a variety of marks (we explain the process for another module in Thoma & Nardi, 2017, Figure 3 on p. 2269). Here, we focus on the responses to this task in a part of these thirty-four scripts:

(a) Consider the one-dimensional system $\dot{x} = f(x)$, where $x(t)$ is a real-valued function of time $t \geq 0$ with $x(0) = x_0$ given, and $f(x)$ is a smooth real-valued function of x. Define what it means to say that x^* is an hyperbolic equilibrium point for this dynamical system. [4 marks]

(b) Find the equilibrium points for each of the dynamical systems: $\dot{x} = x^3+x^2$, $\dot{x} = 1+\sin x$, and analyse their stability. You may use linear stability analysis or graphical analysis. [6 marks]

The task initially invites students to recall and provide the definition of a hyperbolic equilibrium point for a one-dimensional system (a). Then, the students are given two different one-dimensional dynamical systems (b), and they are asked to, first, find the equilibrium points for each dynamical system and, then, analyse their stability. There is also a direct instruction regarding the procedures that they can choose from in order to analyse the stability "You may use linear stability analysis or graphical analysis". It is this choice that this paper focuses on. The students are allowed to choose which procedure they want to use when examining the equilibrium points stability. In an interview with the exam-setting lecturer that took place after the final examination, the lecturer comments on providing the students with both of the procedures to analyse the stability as follows:

> "I give them a choice, there are two methods of studying stability (...) one of them is not appropriate for all of the stability. But the idea is for them..., is to recognize that one of them...to get the full marks they have to use the other one, the graphical analysis. (...) I could have asked only for graphical analysis without linear stability but I also want to test, to see if they understand linear stability in the part which can be used."

The lecturer's aim is dual: first, to give the students the option to use both procedures (linear stability and graphical analysis) in analysing stability; and, second, to examine whether the students can distinguish which procedure is suitable in the situation. In commognitive terms, the focus here is both on the procedure (*how*), and the applicability conditions and closing conditions (*when*) of the routine. The lecturer places value on the *when* of the routine. The dynamical systems are selected on purpose to provide an opportunity to examine the applicability of the two procedures. For one of the equilibrium points of the first dynamical system ($x^*=0$), if one uses linear stability analysis, the information from the first derivative is not sufficient to decide on the stability as it gives the value zero. This is the same for the infinite equilibrium points of the second dynamical system.

Of the thirty-four students, two did not attempt the task at all. The rest mainly used graphical analysis (e.g. Figures 1 and 2). For the first dynamical system, fourteen students used graphical visual mediators to discuss the stability of $x^* = -1$, and twenty-nine for the stability of $x^* = 0$. For the second dynamical system, twenty-seven students used graphs. Specifically, for the equilibrium point $x^* = -1$ of the first dynamical system, eighteen students used linear stability to characterise the stability by finding the value of the first derivative at that point, thirteen used graphical analysis and one provided the graph of the function without characterising the stability of the equilibrium point. For the second equilibrium point (x*=0), tweny-one used graphical analysis to discuss the stability of the point. Of these, nineteen plotted x^3+x^2 and two plotted the functions x^3 and x^2 separately; seven used graphical analysis but their sketches were not the correct function; one plotted the function but did not characterise the stability; and, three provided the characterisation without providing a sketch of the function basing their argument on linear stability arguments. Regarding the second dynamical system, twenty-four students used graphical analysis. Of these: twenty-two sketched 1+sin$x$ as one function and two plotted the two functions separately. Further, three students used graphs in their responses but they either did not provide characterisation regarding stability of the infinite equilibrium points or provided the sketch of a different function. Finally, five students provided only comments using linear stability analysis. In the following, we discuss in detail the responses of two students regarding the stability of the first dynamical system.

Student [03] initially identifies the two equilibrium points and examines both using linear stability analysis (Figure 1); identifies (with the linear stability analysis) that the information is not sufficient to determine the stability of one of the equilibrium points (x*=0); and, decides to provide the graphs of -x^2 and x^3 in order to analyse the stability. Additionally, in the narrative accompanying the graphical realisation of the function, [03] discusses the monotonicity of the function x^3+x^2 by examining what happens on the left and right hand side of the equilibrium point.

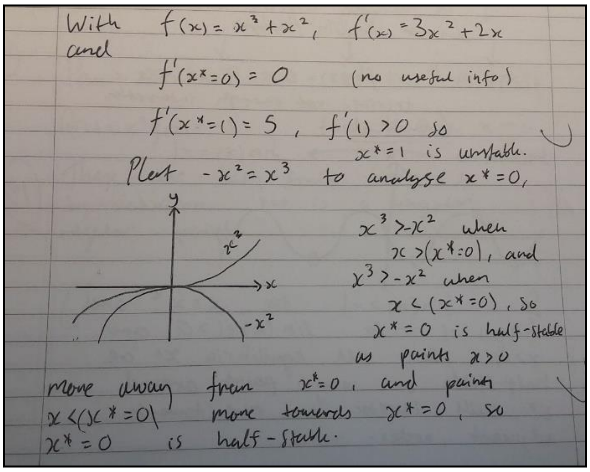

Figure 1: Student [03]' response to task (iib)

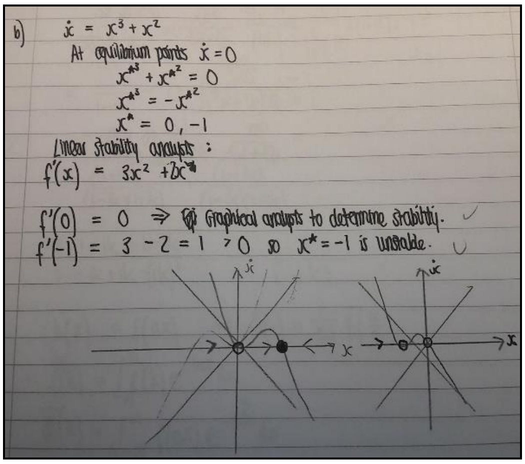

Figure 2: Student [10]'s response to task (iib)

In student [10]'s response (Figure 2), there are two graphs: both are attempts at the graphical realisation of the function x^3+x^2. However, both are crossed out (they are the graphs of $-x^3-x^2$ and $-x^3+x^2$ respectively, not x^3+x^2). Although student [10] realises that the procedure to analyse the stability of the dynamical system is graphical analysis, the graphs provided are from different functions and thus not suitable to decide the stability of the equilibrium point. Both of these students decide to use linear stability analysis for x* = -1 and graphical analysis to characterise the stability of x* = 0. The when of the routine is examined and both discuss what we classify as applicability conditions in order to use linear stability. Further, we note that, despite opting for graphical analysis, the how of the routine is different in these two responses. Student [03] provides the graphs of -x^2 and x^3 and student [10] attempts to provide the graph of x^3+x^2.

Conclusion

The analysis summarised here illustrates that the students are mainly using graphical analysis in their characterisation of the equilibrium points. However, we note that there were students who used linear stability analysis even though the applicability conditions of the routine were not satisfied, namely the value of the first derivative at the equilibrium points was zero; still, the students chose to continue with the linear stability analysis. Our analysis also highlights the different procedures that are discernible in the students' responses even when they opt for graphical analysis to discuss the stability of the equilibrium points on both dynamical systems. Some sketch the actual function and others examine the parts of the function in the same coordinate system and discuss the range where one of the parts of the function is above or below the other one.

Our results highlight the importance of investigating further students' engagement with the routines the exam setter is expecting them to engage with, especially in relation to whether they take into account the applicability conditions of each of these routines and the variety of the procedures that they could use while deciding on the stability of equilibrium points in dynamical systems. Previous work discussed the compartmentalization of these procedures (Rasmussen, 2001); our work offers further discussion of whether students take on board the applicability conditions (the *when* of a routine) as well as the various procedures (*how*) of the routines. We are currently analysing analogous data focusing on examination tasks from other mathematical areas as well.

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