

CRITICAL CARDINALS

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ABSTRACT. We introduce the notion of a *critical cardinal* as the critical point of sufficiently strong elementary embedding between transitive sets. Assuming the Axiom of Choice this is equivalent to measurability, but it is well-known that Choice is necessary for the equivalence. Oddly enough, this central notion was never investigated on its own before. We prove a technical criterion for lifting elementary embeddings to symmetric extensions, and we use this to show that it is consistent relative to a supercompact cardinal that there is a critical cardinal whose successor is singular.

1. INTRODUCTION

Elementary embeddings play a central role in modern set theory. An elementary embedding is nontrivial if it is not the identity on the ordinals, and the least ordinal moved by the embedding is called the *critical point*. We can prove that the critical point is a large cardinal, and requiring the target model be “more similar to V ” provides us with stronger notions of cardinals.

Assuming the Axiom of Choice we can exchange elementary embeddings for combinatorial or logical properties such as trees, ultrafilters, and compactness properties of infinitary languages. We can also prove there is an upper limit to the large cardinal hierarchy: there is no nontrivial embedding $j: V \rightarrow V$. However the use of the Axiom of Choice in the known proofs seems to be essential.

Without assuming the Axiom of Choice, combinatorial properties cannot provide us with elementary embeddings. And much of the choiceless work on large cardinals was in the context of obtaining combinatorial properties of large cardinals at “accessible” levels (e.g. ω_1 carrying a measure).

In recent years, however, there is a renewed interest in large cardinals without the Axiom of Choice. Woodin’s work on the HOD Conjecture and cardinals related to the existence of a nontrivial $j: V \rightarrow V$ are notable examples, especially since they focus on the largeness formulated in terms of embeddings.

These developments led us to ask what kind of properties we can prove on the structure of the set theoretic universe assuming there is a cardinal which is a critical point of an embedding. We present in this paper the results of this research: first by isolating the notion of a critical cardinal, then by proving a technical theorem which is used to prove that the successor of a critical cardinal can be singular.

1.1. The structure of this paper. The paper covers the basic technical preliminaries of supercompact Radin forcing and symmetric extensions in [Section 2](#). We define critical and weakly critical cardinals in [Section 3](#), and we prove some basic positive results about them. In [Section 4](#) we prove a Silver-like criterion for a lifting elementary embedding from the ground model to a symmetric extension.

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Finally, in [Section 5](#) we prove the main theorem. It is consistent, assuming the consistency of a supercompact cardinal, that a critical cardinal's successor is singular. This utilizes the Silver-like criterion with a symmetric extension via a supercompact Radin forcing.

There are open questions throughout the paper. Of course, there are many other naturally arising questions, and we encourage the reader to come up with their own questions, as well as answers to them.

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2. TECHNICAL PRELIMINARIES

Most of our notation and terminology are standard. We use V to denote the ground model in which we work, and if M is a transitive class (or set) and α is smaller than the height of M , then M_α will denote the subset of M of sets with rank $< \alpha$. We say that $A \subseteq M$ is *amenable* (to M) if for all α below the height of M , $A \cap M_\alpha \in M$.

If \mathbb{P} is a notion of forcing, then it always has a maximum denoted by $1_{\mathbb{P}}$, or 1 when there is no chance of confusion; we will also write $q \leq p$ if q is *stronger* than p . We denote \mathbb{P} -names by \dot{x} , and by \check{x} the canonical \mathbb{P} -name for x in the ground model. If \dot{x} is a \mathbb{P} -name, we say that a condition p or a name \dot{y} *appears* in \dot{x} if there is an ordered pair $\langle p, \dot{y} \rangle \in \dot{x}$.

When $\{\dot{x}_i \mid i \in I\}$ is a class of names (possibly a proper class), we denote by $\{\dot{x}_i \mid i \in I\}^\bullet$ the class-name $\{\langle 1, \dot{x}_i \rangle \mid i \in I\}$. Note that if I is a set, then this is a \mathbb{P} -name. We extend this notation to ordered pairs and sequences as needed. Note that using this notation $\check{x} = \{\check{y} \mid y \in x\}^\bullet$.

We use DC_κ to abbreviate the statement Dependent Choice for κ : Every κ -closed tree of height $\leq \kappa$ without maximal elements has a cofinal branch. DC denotes DC_ω , and $\text{DC}_{<\kappa}$ abbreviates $\forall \lambda < \kappa, \text{DC}_\lambda$.

Our main method for constructing models where the Axiom of Choice fails is symmetric extensions, and our forcing will be a supercompact Radin forcing. For the convenience of the reader, we have included a brief overview on both of these topics.

2.1. Symmetric extensions. Symmetric extensions are inner models of generic extensions, where the Axiom of Choice may fail. They are obtained by identifying a particular class of names using permutations of the forcing.

Let \mathbb{P} be a notion of forcing, and let π be an automorphism of \mathbb{P} . Then π extends to a permutation of \mathbb{P} -names, defined recursively

$$\pi \dot{x} = \{ \langle \pi p, \pi \dot{y} \rangle \mid \langle p, \dot{y} \rangle \in \dot{x} \}.$$

We say that \mathcal{F} is a *normal filter of subgroups*¹ over a group \mathcal{G} if it is closed under finite intersections and supergroups, and for any $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, $\pi H \pi^{-1} \in \mathcal{F}$.

If \mathbb{P} is a notion of forcing, \mathcal{G} is a group of automorphisms of \mathbb{P} , and \mathcal{F} is a normal filter of subgroups over \mathcal{G} , then we say that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a *symmetric system*.²

¹This is a terribly unfortunate overlap in the terminology of a normal filter. We will refer to normal ultrafilters in the large cardinals context as “measures” to avoid this confusion.

²We will often replace \mathcal{F} by a filter base which generates it, granted it satisfies the closure under conjugation.

Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. We say that \mathbb{P} -name \dot{x} is \mathcal{F} -*symmetric* if $\text{sym}_{\mathcal{G}}(\dot{x}) = \{\pi \in \mathcal{G} \mid \pi\dot{x} = \dot{x}\} \in \mathcal{F}$, and \dot{x} is *hereditarily \mathcal{F} -symmetric* if it is \mathcal{F} -symmetric and every \dot{y} which appears in \dot{x} is hereditarily \mathcal{F} -symmetric. We denote by $\text{HS}_{\mathcal{F}}$ the class of all hereditarily \mathcal{F} -symmetric names. If the symmetric system is clear from the context, we omit the subscripts and write $\text{sym}(\dot{x})$, HS , etc.

The following are standard in the study of symmetric extensions. The proofs can be found, for example, in [4] as Lemma 14.37 and 15.51, respectively.

Lemma (The Symmetry Lemma). *Suppose that $\pi \in \text{Aut}(\mathbb{P})$, \dot{x} is a \mathbb{P} -name, and φ is a formula in the language of set theory, then*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

Theorem. *Suppose that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and let G be a V -generic filter for \mathbb{P} . Then the class $M = \text{HS}^G = \{\dot{x}^G \mid \dot{x} \in \text{HS}\}$ is a model of ZF satisfying $V \subseteq M \subseteq V[G]$.*

The class M is called a *symmetric extension*. And we can define a forcing relation \Vdash^{HS} satisfying the usual forcing theorem, and even the Symmetry Lemma for $\pi \in \mathcal{G}$.

Definition 2.1. Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. We say that $D \subseteq \mathbb{P}$ is a *symmetrically dense* set if there is some $H \in \mathcal{F}$ such that for all $\pi \in H$, $\pi"D = D$. We say that a filter $G \subseteq \mathbb{P}$ is *symmetrically V -generic* if it is a filter and for all symmetrically dense open sets $D \in V$, $D \cap G \neq \emptyset$.

The notion of symmetrically generic filters is the one needed to interpret correctly the names in HS . This is reflected in the following theorem [6, Theorem 8.4].

Theorem. *Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system, and $\dot{x} \in \text{HS}$. Then the following are equivalent:*

- (1) $p \Vdash^{\text{HS}} \varphi(\dot{x})$.
- (2) For every symmetrically V -generic filter G such that $p \in G$, $\text{HS}^G \models \varphi(\dot{x}^G)$.
- (3) For every V -generic filter G such that $p \in G$, $\text{HS}^G \models \varphi(\dot{x}^G)$.

Finally, while we do not discuss iterations of symmetric extensions in full, not even in the case of a two-step iteration, it will be conceptually relevant to the Silver-like criterion for lifting elementary embeddings to symmetric extensions, so we urge the reader to glance through the second author's [6]. One definition from that context is relevant to this work, and that is the generic semi-direct product of groups.

Definition 2.2. Suppose that $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system and $\dot{\mathbb{Q}}, \dot{\mathcal{H}} \in \text{HS}$ such that

- (1) $\text{sym}(\dot{\mathbb{Q}}) = \text{sym}(\dot{\mathcal{H}}) = \mathcal{G}$, and
- (2) $\Vdash_{\mathbb{P}} \dot{\mathcal{H}}$ is a group of automorphisms of $\dot{\mathbb{Q}}$.

The *generic semi-direct product* $\mathcal{G} * \dot{\mathcal{H}}$ is the group of automorphisms of $\mathbb{P} * \dot{\mathbb{Q}}$ of the form $\langle \pi, \dot{\sigma} \rangle$,³ such that $\pi \in \mathcal{G}$ and $\Vdash \dot{\sigma} \in \dot{\mathcal{H}}$, and the action of $\mathcal{G} * \dot{\mathcal{H}}$ on $\mathbb{P} * \dot{\mathbb{Q}}$ is

$$\langle \pi, \dot{\sigma} \rangle(p, \dot{q}) = \langle \pi p, \pi(\dot{\sigma}\dot{q}) \rangle.$$

We remark that in the case of a product of two symmetric systems, considering \mathcal{H} and \mathbb{Q} with their canonical names, we actually obtain the product of $\mathcal{G} \times \mathcal{H}$ with the natural action on $\mathbb{P} \times \mathbb{Q}$.

³In [6] the notation is different, but since we will not be applying these automorphisms, this is less important.

2.2. Supercompact Radin forcing. Our other main technical forcing tool will be supercompact Radin forcing, we will generally follow [7] for the presentation of the forcing. For the completeness of this work, we provide the definition of the forcing and a few of its basic properties. For the rest of the discussion on supercompact Radin forcing, κ will be a fixed supercompact cardinal.⁴

If a is a set of ordinals, we will write π_a as the Mostowski collapse of a , which is the isomorphism between a and its order type. We will denote by $\mathcal{P}_\kappa(A)$ the set $\{a \subseteq A \mid |a| < \kappa\}$. For $x, y \in \mathcal{P}_\kappa(\lambda)$, we say that x is a *strong subset* of y , if $x \subseteq y$ and $|x| < y \cap \kappa$. We shall denote this by $x \subseteq_\kappa y$.

We follow Krueger [7] for the presentation of supercompact Radin forcing. We define a *coherent sequence of supercompact measures on $\mathcal{P}_\kappa(\lambda)$* as a sequence of measures $\mathcal{U} = \langle U(\alpha, i) \mid i < o^\mathcal{U}(\alpha), \alpha \leq \kappa \rangle$ where $o^\mathcal{U}: \kappa + 1 \rightarrow \text{Ord}$ is a partial function satisfying the following properties.

- (1) There is a function $\beta \mapsto \lambda_\beta$ defined on the the domain of $o^\mathcal{U}$ such that $\lambda_\kappa = \lambda$, $\lambda_\beta \geq \beta$ is a cardinal, and if $o^\mathcal{U}(\alpha)$ is defined, then $\lambda_\beta < \alpha$ for all $\beta < \alpha$. Moreover, for α in the domain of $o^\mathcal{U}$ and $i < o^\mathcal{U}(\alpha)$,

$$\{a \in \mathcal{P}_\alpha(\lambda_\alpha) \mid \text{otp}(a) = \lambda_{a \cap \alpha}\} \in U(\alpha, i).$$

- (2) Each $\langle U(\alpha, i) \mid i < o^\mathcal{U}(\alpha) \rangle$ is a sequence of normal measures on $\mathcal{P}_\alpha(\lambda_\alpha)$.
- (3) For each $\alpha \in \text{dom } o^\mathcal{U}$ and $i < o^\mathcal{U}(\alpha)$, the set $\{x \in \mathcal{P}_\alpha(\lambda_\alpha) \mid x \cap \alpha \in \text{dom } o^\mathcal{U}\}$ is in $U(\alpha, i)$.
- (4) For any $\beta < o^\mathcal{U}(\alpha)$, if $j: V \rightarrow \text{Ult}(V, U(\alpha, \beta))$ is the ultrapower embedding, then $j(\langle U(\beta, i) \mid i < o^\mathcal{U}(\beta), \beta < \alpha \rangle)(\alpha) = \langle U(\alpha, i) \mid i < \beta \rangle$.

If \mathcal{U} is a coherent sequence of supercompact measures, we say that ρ is a *repeat point* of \mathcal{U} if

$$\bigcap_{\zeta < \rho} U(\kappa, \zeta) = \bigcap_{\zeta \leq \rho} U(\kappa, \zeta),$$

or equivalently, if for every $X \in U(\kappa, \rho)$ there is $\zeta < \rho$ such that $X \in U(\kappa, \zeta)$. By counting arguments, if $o^\mathcal{U}(\kappa) = (2^{\lambda^{<\kappa}})^+$ then \mathcal{U} has repeat points.

Fix $\lambda > \kappa$, and let $\Lambda: \kappa \rightarrow \kappa$ be a function such that some $(2^{\lambda^{<\kappa}})^+$ -supercompact embedding j exists with $\text{crit}(j) = \kappa$ and $j(\Lambda)(\kappa) = \lambda$. By supercompactness of κ , there is a coherent sequence of measures $\mathcal{U} = \langle U(\alpha, i) \mid i < o^\mathcal{U}(\alpha), \alpha \leq \kappa \rangle$, where $U(\alpha, i)$ is a measure on $\mathcal{P}_\alpha(\Lambda(\alpha))$ (see [7, Section 2]).

Normally the Radin forcing derived from a coherent sequence of measures, \mathcal{U} , is denoted by $\mathbb{R}(\mathcal{U})$, but since in our case \mathcal{U} will always be clear from context we will simply write \mathbb{R} . Krueger also defines $A_\mathcal{U}$ to be a superset of all the “large sets” in the Radin forcing. $A_\mathcal{U}$ is the following set,

$$\left\{ a \in \mathcal{P}_\kappa(\lambda) \mid \begin{array}{l} a \cap \kappa \text{ is a cardinal,} \\ o^\mathcal{U}(a \cap \kappa) \text{ is defined, and} \\ \text{otp}(a) = \Lambda(a \cap \kappa) \end{array} \right\}.$$

To simplify our notation, especially as we omit \mathcal{U} , we will abuse the notation and write $\mathcal{P}_\kappa(\lambda)$ to mean $A_\mathcal{U}$.

We follow Krueger’s definition which appears in [7, Section 3] with a few inconsequential modifications. A condition p in \mathbb{R} is a finite sequence $\langle d_0, \dots, d_n \rangle$ such that for all $i \leq n$,

- (1) if $i < n$
 - (a) $d_i \in \mathcal{P}_\kappa(\lambda)$, or

⁴We can generally replace “supercompact” by “ λ -supercompact” for a sufficiently large λ in all of our proofs, which of course matters for exact consistency strength, but clutters the text. We leave the calculation of the exact λ for the interested reader.

- (b) $d_i = \langle x_i, A_i \rangle$ such that $x_i \in \mathcal{P}_\kappa(\lambda)$ and $\pi_{x_i} "A_i \in \bigcap_{\beta < o^{\mathcal{U}}(\kappa_i)} U(\kappa_i, \beta)$, where $\kappa_i = x_i \cap \kappa$,
- (2) $d_n = \langle \lambda, A_n \rangle$ where A_n is in $\bigcap_{\alpha < o^{\mathcal{U}}(\kappa)} U(\kappa, \alpha)$.
- (3) if $i < j \leq n$, then $x_i \subseteq x_j$, and if $d_j = \langle x_j, A_j \rangle$, then for all $a \in A_j$, $a_i \subseteq a$.

We define $\text{len}(p) = n$ to be the length of p (in particular a condition of length 0 has only $\langle \lambda, A_0 \rangle$ in it).

Given two conditions $p, q \in \mathbb{R}$, such that $p = \langle d_0, \dots, d_n \rangle$ with $d_i = x_i$ or $d_i = \langle x_i, A_i \rangle$, and $q = \langle e_0, \dots, e_m \rangle$ with $e_i = y_i$ or $e_i = \langle y_i, B_i \rangle$, we write $q \leq p$ if:

- (1) $n \leq m$,
- (2) $B_m \subseteq A_n$,
- (3) if $n > 0$, then there are $i_0 < \dots < i_{n-1} < m$ such that for all $k < n$,
 - (a) if $d_k \in \mathcal{P}_\kappa(\lambda)$, then $d_k = e_{i_k}$,
 - (b) if $d_k = \langle x_k, A_k \rangle$, then $e_{i_k} = \langle x_k, B_{i_k} \rangle$, where $B_{i_k} \subseteq A_k$,
- (4) for each $l < m$, such that l is not any i_k ,
 - (a) if $n = 0$, or if $l > i_n$, then either $e_l \in A_n$ or $e_l = \langle y_l, B_l \rangle$ with $y_l \in A_n$ and $B_l \subseteq A_n$,
 - (b) if $n > 0$ and k is the least such that $l < i_k$, then d_k has the form $\langle a_k, A_k \rangle$ and either $e_l \in A_k$ or $e_l = \langle y_l, B_l \rangle$ where $b_l \in A_k$ and $B_l \subseteq A_k$.

If $q \leq p$ and $m = n$, we write $q \leq^* p$ and we say that q is a *direct extension* of p .

We denote by $\text{stem } p$ the tuple $\langle x_0, \dots, x_{n-1} \rangle$. It is routine to verify that every two conditions with the same stem are compatible.

Proposition 2.3. *The forcing \mathbb{R} has several important combinatorial properties:*

- (1) If $p = \langle d_0, \dots, d_n \rangle$ and $\xi < \kappa$ such that whenever $d_i = \langle x_i, A_i \rangle$, we have that $\xi < x_i \cap \kappa$, then for any $\{p_i \mid i < \xi\}$ such that $p_i \leq^* p$ for all i , there is $q \leq^* p_i$ for all $i < \xi$ ([7, Lemma 3.1]).
- (2) It satisfies the $(\lambda^{<\kappa})^+$ -c.c. ([7, Proposition 3.2]).
- (3) If p is a condition and $m < \text{len}(p)$ such that $d_m = \langle x_m, A_m \rangle$, then $\mathbb{R} \upharpoonright p$ is isomorphic to the product $\mathbb{R} \upharpoonright p^{<m} \times \mathbb{R} \upharpoonright p^{\geq m}$, where $p^{\geq m}$ is a condition in \mathbb{R} , and $\mathbb{R} \upharpoonright p^{<m}$ is a condition in a Radin forcing defined on a suitable measure in the sequence \mathcal{U} . Moreover, the isomorphism also respects direct extensions. (See [7, Section 4] for the exact details.)
- (4) The Prikry property with respect to \leq^* ([7, Section 5]).

If G is a V -generic filter for \mathbb{R} , then the *Radin club*, C_G is the generic club in $\mathcal{P}_\kappa(\lambda)^V$ defined as

$$C_G = \{a \in \mathcal{P}_\kappa(\lambda) \mid \exists p \in G \ a \in \text{stem } p\}.$$

For every $a \in C_G$, let $\kappa_a = a \cap \kappa$, then $\{\kappa_a \mid a \in C_G\}$ is a club in κ which we will denote by C_κ .

The Radin forcing collapses all the cardinals in the interval $(\alpha, \Lambda(\alpha)^{<\alpha})$ for every $\alpha \in C_\kappa$, and depending on the length of \mathcal{U} , κ can remain regular, measurable, or even supercompact in the generic extension.⁵

To wrap the overview of Radin forcing, we will need to use a technique that lets us construct generic filters over inner models. The following theorem is a mild modification of an unpublished theorem of Woodin.

Theorem 2.4. *Let $p \in \mathbb{R}$ be a condition in the supercompact Radin forcing. There is an elementary embedding $k: V \rightarrow M$ such that in V there is an M -generic filter H for $k(\mathbb{R})$ which is compatible with $k(p)$.*

⁵We again leave the subtle question of determining the exact length needed for the interested reader.

Before proving the theorem, we will need the following technical lemma, which shows that \mathbb{R} satisfies a strong Prikry property.

Lemma 2.5. *Let $D \subseteq \mathbb{R}$ be a dense open set, and let $p \in \mathbb{R}$ be a condition. Denote by x_0, \dots, x_{n-1} the stem of p . There exists a condition $p^* \leq^* p$, a finite sequence of natural numbers m_0, \dots, m_n , and a finite sequence of trees T_0, \dots, T_n such that*

- (1) T_i is a tree of height m_i of candidates for extending the i th member of the stem of p^* ,
- (2) T_i is fat, in the sense that for any $\vec{t} = \langle t_0, \dots, t_{k-1} \rangle \in T_i$ with $k < m_i$, there is $\gamma_{\vec{t}} < \delta^{\mathcal{U}}(x_i \cap \kappa)$ such that the possible values for extending \vec{t} inside T_i is a set in $U(x_i \cap \kappa, \gamma_{\vec{t}})$.
- (3) Any extension of p^* using branches through the trees lies in D .

Proof. Let $p = \langle d_0, \dots, d_n \rangle$ be a condition in \mathbb{R} and let x_0, \dots, x_{n-1} be the stem of p . Let $x_n = \lambda$. Let us consider the name for the condition $\dot{r} \in D \cap \dot{G}$. Let y_0, \dots, y_m be names for the elements in the Radin club that r introduces. Let $\{i_0, \dots, i_n\}$ be the names for the indices, such that $y_{i_j} = x_j$. Let us assume that r has the minimal length in the sense that \dot{m} is minimal and $\langle i_{j+1} - i_j \mid j < n - 1 \rangle$ is minimal in the lexicographic order.

Using the Prikry property of \mathbb{R} and the closure of \leq^* , we can find a direct extension $p^* \leq^* p$ such that for each $j \leq n$, p^* decides the values i_j .

Note that if $d_{j+1} = x_{j+1}$ then $\Vdash i_j = i_{j+1} - 1$.

Let j_0 be minimal such that $d_{j_0} = \langle x_{j_0}, A_{j_0} \rangle$. For each $y \in A_{j_0}$, let us pick a direct extension of the extension of p^* by adding y to the stem, which again decides \dot{m} and i_j for all $j \leq n + 1$. Let p_y^* be this direct extension. Since p_y^* is compatible with p^* , the value of \dot{m} does not change and similarly, the values of i_j only change to reflect the addition of y to the stem. The only thing which is not automatically determined by p^* is the membership of y to the stem of \dot{r} - the minimal condition in the intersection of D and \dot{G} . The truth value of this statement is decided by p_y^* .

Using the closure of the measures on x_j for $j > j_0$ in the stem of p and the normality of the measure on x_{j_0} , we can construct a single condition $p^{**} \leq^* p^*$, such that for all $y \in A_{j_0}^{p^{**}}$, the extension of p^{**} by appending y to the stem is stronger than p_y^* and in particular, it decides the membership of y to stem \dot{r} .

Let $B_{j_0}^0$ be the collection of all $y \in A_{j_0}^{p^{**}}$ such that y is decided to be the minimal element of stem \dot{r} which is not in stem p , by the extension of the condition p^{**} with y . Clearly, $B_{j_0}^0$ is of measure one for $U(x_{j_0} \cap \kappa, \beta)$ for some $\beta < \delta^{\mathcal{U}}(x_{j_0} \cap \kappa)$ if and only if $p^* \Vdash i_{j_0} > j_0$. If $B_{j_0}^0$ is of measure zero with respect to all measures on $x_{j_0} \cap \kappa$ in the coherent sequence, then by taking an additional direct extension of p^{**} we may assume that it is empty.

Repeat the process and construct a tree of height $i_{j_0} - j_0$, such that above each element in the tree the set of successors is large with respect to one of the measures of $x_{j_0} \cap \kappa$. This tree is going to be T_0 .

For indices j above j_0 , we essentially repeat the same argument but with a minor difference: the diagonal intersection is defined differently. Note that when considering the collection of conditions p_y^* as before, we have no control over the different large sets which are attached to x_k for $k < j$. Those large sets belong to measures which are not sufficiently closed, so we cannot simply intersect them. Instead, we take diagonal intersection only above x_k and consider the sets

$$B_s = \{y \in A_k^{p^*} \mid p_y^* \restriction k = s, p_y^* \Vdash y \text{ is the next element in stem } \dot{r}\}$$

By the completeness of the measures, there is s such that B_s is large. Let direct extend p^* by taking the diagonal intersection above coordinate k and replace the lower k coordinates with s . From this point the rest of the argument is the same. \square

We are now ready to prove [Theorem 2.4](#).

Proof. Let us start by arguing that it is sufficient to prove the Theorem for the case when p has an empty stem. Since the forcing $\mathbb{R} \upharpoonright p$, when p has a nontrivial stem, splits into a product of $\text{len}(p)$ components, each of them being a Radin forcing using some coherent measure sequence below some condition with empty stem,⁶ by applying the theorem finitely many times over each of those components we obtain a generic filter for each component. Using [Lemma 2.5](#), we conclude that those filters are mutually generic and therefore their product is a generic filter for the image of \mathbb{R} that contains the image of p . So we can safely assume without loss of generality that p has an empty stem.

We define by recursion a sequence of models M_α and elementary embeddings $j_{\alpha,\beta}: M_\alpha \rightarrow M_\beta$ ($M_0 = V$ and $j_{\alpha,\alpha} = \text{id}$), as well as a sequence \vec{s} of seeds s_α . To improve readability, we denote by j_α the embedding $j_{0,\alpha}$, as well as $\mathcal{U}_\alpha = j_\alpha(\mathcal{U})$, $\kappa_\alpha = j_\alpha(\kappa)$ and $\lambda_\alpha = j_\alpha(\lambda)$.

Suppose that M_α , all the embeddings up to α , and the sequence $\vec{s} \upharpoonright \alpha$ was defined. Let $\gamma < j_\alpha(o^{\mathcal{U}}(\kappa))$ be the least ordinal such that there is no cofinal sequence $\langle \alpha_i \mid i < \text{cf}(\gamma) \rangle$ such that $\mathcal{U}_\alpha(\kappa_\alpha, \gamma)$ is the set

$$\left\{ X \subseteq \mathcal{P}_{\kappa_\alpha}(\lambda_\alpha)^{M_\alpha} \mid \{i < \text{cf}(\gamma) \mid s_{\alpha_i} \in X\} \text{ contains a tail of } \text{cf}(\gamma) \right\}.$$

If no such γ exists, then we halt.

Let $M_{\alpha+1}$ be the ultrapower $\text{Ult}(M_\alpha, \mathcal{U}_\alpha(\kappa_\alpha, \gamma))$, and define $j_{\gamma, \alpha+1}$ as the composition of $j_{\gamma, \alpha}$ with the ultrapower embedding. In particular, $j_{\alpha, \alpha+1}$ is itself the ultrapower embedding. Finally, let s_α be $j_{\alpha, \alpha+1} \lambda_\alpha$. Note that if α was a successor ordinal, then $\gamma = 0$.

If α is a limit ordinal, and M_β were defined for all $\beta < \alpha$, then M_α is the direct limit of these models, combined with the embeddings, and $j_{\beta, \alpha}$ is the canonical embedding from M_β to M_α . By Gaifman's theorem, M_α is well-founded.

We claim that the process halts. Otherwise, let μ be a regular cardinal larger than $2^{o^{\mathcal{U}}(\kappa)} + 2^{2^{\lambda < \kappa}}$. By the definition of the direct limit of ultrapowers, each element in M_μ is of the form $j_\mu(f)(j_{\alpha_0+1, \mu}(s_{\alpha_0}), \dots, j_{\alpha_{n-1}+1, \mu}(s_{\alpha_{n-1}}))$, where $f: \mathcal{P}_\kappa(\lambda)^n \rightarrow V$. Let γ_α denote the γ used in the α th ultrapower, and let $g_\alpha: \mathcal{P}_\kappa(\lambda) \rightarrow o^{\mathcal{U}}(\kappa)$ be a function such that

$$\gamma_\alpha = j_\alpha(g_\alpha)(j_{\alpha_0+1, \alpha}(s_{\alpha_0}), \dots, j_{\alpha_{n-1}+1, \alpha}(s_{\alpha_{n-1}})).$$

Since $\alpha_0, \dots, \alpha_{n-1} < \alpha$, there is a stationary $S \subseteq \mu$ such that all of these are the same. Similarly, since there are not many possible g_α 's either, we may assume that for all $\alpha \in S$, g and $\alpha_0, \dots, \alpha_{n-1}$ are the same.

Let $\langle \delta_n \mid n < \omega \rangle \subseteq S$ such that $\delta = \sup\{\delta_n \mid n < \omega\} \in S$ as well. We claim that for all $X \in \mathcal{U}_\delta(\kappa_\delta, \gamma_\delta)$ there is a natural number N , such that for all $n > N$, $j_{\delta_{n+1}, \delta}(s_{\delta_n}) \in X$. Let N be the least such that $X = j_{\delta_N, \delta}(Y)$ for some set Y . Since $\gamma_\delta = j_{\delta_N+1, \delta}(s_{\delta_N})$, it follows that $Y \in \mathcal{U}_{\delta_N}(\kappa_{\delta_N}, \gamma_{\delta_N})$. In particular, $s_{\delta_N} \in j_{\delta_N, \delta_{N+1}}(Y)$. Therefore $j_{\delta_N+1, \delta}(s_{\delta_N}) \in X$, and the argument is similar for any $n > N$ as wanted. But this is a contradiction to the choice of γ_δ , so the process had to halt at some limit ordinal.

Let δ denote the length of the recursion. We claim that $\langle j_{\alpha+1, \delta}(s_\alpha) \mid \alpha < \delta \rangle$ is an M_δ -generic Radin club for $j_\delta(\mathbb{R})$. By the definition of δ , for all $\gamma < j_\delta(o^{\mathcal{U}}(\kappa))$, there are cofinally many $\alpha < \delta$ for which $j_{\alpha, \alpha+1}$ is an ultrapower embedding using a measure U_* such that $j_{\alpha, \delta}(U_*) = U(\kappa_\delta, \gamma)$, thus the sequence enters any finite fat

⁶We only defined splitting for the case of $d_i = \langle x_i, A_i \rangle$ and not when $d_i = x_i$. To overcome this we can either treat this case as a Radin forcing with an empty sequence, which would be a trivial forcing.

tree, and is therefore M_δ -generic. Therefore taking $M = M_\delta$ and $k = j_\delta$ completes the proof. \square

3. CRITICAL CARDINALS

3.1. Weakly critical cardinals.

Definition 3.1. We say that κ is a *weakly critical cardinal* if for every $A \subseteq V_\kappa$ there exists an elementary embedding $j: X \rightarrow M$ with critical point κ , where X and M are transitive and $\kappa, V_\kappa, A \in X \cap M$.

Assuming Choice, this definition is equivalent to the statement that κ is weakly compact, as a consequence of the following proposition and [5, Theorem 4.5]. However, without Choice the term “weakly compact” is ambiguous in the sense that the many definitions need not be equivalent anymore.

Note that by a simple coding argument we can extend the requirement from just one A to $|V_\kappa|$ subsets at the same time.

Proposition 3.2. κ is weakly critical if and only if for every $A \subseteq V_\kappa$ there is a transitive, elementary end-extension of $\langle V_\kappa, \in, A \rangle$.

In the standard context of weak compactness this is known as the *Extension Property*, and it is due to Keisler (see also [5, Theorem 4.5]).

Proof. Suppose that κ is weakly critical, $A \subseteq V_\kappa$, and $j: X \rightarrow M$ is an elementary embedding witnessing the fact κ is weakly critical. Let $W = j(V_\kappa)$ and let $B = j(A)$, then W is transitive and $B \cap V_\kappa = A$. Easily, $\langle W, \in, B \rangle$ is an elementary end-extension of $\langle V_\kappa, \in, A \rangle$ as wanted.

In the other direction, suppose that $\langle W, \in, B \rangle$ is a transitive, elementary end-extension of $\langle V_\kappa, \in, A \rangle$, denote by $\kappa' = W \cap \text{Ord}$. Let $M = W \cup \{W, \kappa', B\}$, and let $X = V_\kappa \cup \{V_\kappa, \kappa, A\}$. Then both M and X are transitive sets, and defining $j: X \rightarrow M$ by $j \upharpoonright V_\kappa = \text{id}$ and $j(V_\kappa) = W$, $j(\kappa) = \kappa'$ and $j(A) = B$ is an elementary embedding between transitive sets with critical point κ . \square

Proposition 3.3. If κ is weakly critical cardinal, then it is Mahlo, it is in particular strongly inaccessible, and in particular regular.

This is a very similar proof to the proof in ZFC. It should be noted that without Choice “strongly inaccessible” could have different meanings (see [2]), and here we mean that V_κ satisfies second-order ZF, or that κ is a regular limit cardinal and for all $\alpha < \kappa$, V_α does not map onto κ .

Proof. If $\alpha < \kappa$ and there is a function $f: V_\alpha \rightarrow \kappa$ which is cofinal, let $j: X \rightarrow M$ witness that κ is weakly critical with $f \in X$. Then $j(f) = f$, as its domain is fixed by j , which would have range cofinal in $j(\kappa) > \kappa$, which is a contradiction. Therefore κ is strongly inaccessible.

Now let A be the set of strongly inaccessible cardinals below κ , and let $C \subseteq \kappa$ be a club. Let $j: X \rightarrow M$ witness that κ is weakly critical with $A, C \in X$. Then $M \models \kappa \in j(A) \cap j(C)$, therefore $X \models A \cap C \neq \emptyset$. Therefore A is indeed stationary. \square

The proof above generalizes to many other properties we have grown to expect from weakly compact cardinals in ZFC. For example a weakly critical cardinal is Mahlo to any degree up to κ^+ . Similarly, a weakly critical cardinal has the tree property.

The following question has been raised by Itay Kaplan.

Question 3.4. How bad can a weakly critical cardinal’s identity crisis be without Choice? Can the least weakly critical cardinal be the least measurable?

3.2. Critical cardinals.

Definition 3.5. We say that a cardinal κ is a *critical cardinal* if it is the critical point of an elementary embedding $j: V_{\kappa+1} \rightarrow M$, where M is a transitive set.

Easily, assuming Choice a cardinal is critical if and only if it is a measurable cardinal, and we can then assume that j is defined on V itself by taking an ultrapower of V by a normal measure. This argument uses Łoś' theorem which relies on the Axiom of Choice, so there is no reason to expect that in ZF we can replace j by an embedding defined on the whole universe, V . Therefore, we require only an initial segment to be the domain of j .

Clearly, every critical cardinal is weakly critical. But even more is true.

Proposition 3.6. *If κ is critical, then κ carries a normal measure on κ which concentrates on the set of weakly critical cardinals.*

Again the proof is quite similar to the proof in ZFC.

Proof. We define \mathcal{U} to be $\{A \subseteq \kappa \mid \kappa \in j(A)\}$. It is easy to check that \mathcal{U} is a normal measure on κ . Observe now that if $A \subseteq V_\kappa$, then $\langle j(V_\kappa), \in, j(A) \rangle$ is a transitive elementary end-extension of $\langle V_\kappa, \in, A \rangle$ inside M .

Therefore $M \models \kappa$ is weakly critical. Let A be $\{\lambda < \kappa \mid \lambda \text{ is weakly critical}\}$, then $\kappa \in j(A)$ and therefore $A \in \mathcal{U}$. \square

It is known that ω_1 can be measurable ([4, Theorem 21.16]), but it is not even a weakly critical cardinal, as follows from Proposition 3.3. And Eilon Bilinsky and Moti Gitik proved in [1] that it is consistent that there is a measurable cardinal with no normal measures. This shows that from a combinatorial point of view, being a critical cardinal is much stronger than being measurable.

Proposition 3.7. *Let κ be a critical cardinal. If $V_\kappa \models \text{ZFC}$, then $V_{\kappa+1}$ can be well-ordered and κ^+ is regular and not measurable.*

These consequences are somewhat similar in flavor to the theorems of Everett Bull in [3], where he proves a similar theorem for a measurable cardinal (albeit under the implicit assumption there are normal measures).

Proof. Let $j: V_{\kappa+1} \rightarrow M$ be an elementary embedding witnessing that κ is critical. By elementarity $M_{j(\kappa)} \models \text{ZFC}$ and since $V_{\kappa+1} \in M_{j(\kappa)}$, $V_{\kappa+1}$ can be well-ordered.

To see that κ^+ is regular, note that there is a definable surjection from $V_{\kappa+1}$ onto κ^+ , so it follows from the well-orderability of $V_{\kappa+1}$ that there is a sequence $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ such that f_α is an injective function from α to κ . This implies in ZF alone that κ^+ is regular and not measurable, using the standard Ulam matrix argument (see, for example, Corollary 2.4 in Kanamori's book [5]). \square

Note that we do not require anything about the closure of the target model M , and while ZFC proves that there is always such M satisfying $M^\kappa \subseteq M$, where M is an ultrapower, it might not be the case in ZF. This raises several questions.

Question 3.8. If κ is a critical cardinal, can we always find a transitive set M such that $j: V_{\kappa+1} \rightarrow M$ witnesses that κ is critical, and M is closed under ω -sequences? Under $< \kappa$ -sequences? Under V_κ -sequences?

Question 3.9. If κ is critical, and \mathcal{U} is a normal measure on κ , is it true that $V_{\kappa+1}^\kappa/\mathcal{U}$ is well-founded and extensional? What about V^κ/\mathcal{U} ?

By Mitchell Spector's work in [8], the above question has a positive answer if and only if for every family of non-empty sets of size κ , we can find a partial choice function whose domain is in \mathcal{U} .

We would also like to point out the obvious ways one can extend this definition by replacing $V_{\kappa+1}$ with larger initial segments, or requiring better closure properties of the target models. This leads us quite naturally to the next part.

3.3. Supercompact cardinals. The following formulation of supercompactness was identified by Woodin [9, Definition 220] as the appropriate one for ZF.

Definition 3.10. A cardinal κ is supercompact if for each $\alpha > \kappa$ there exist $\beta > \alpha$ and an elementary embedding, $j: V_\beta \rightarrow N$ such that

- (1) N is a transitive set and $N^{V_\alpha} \subseteq N$,
- (2) j has critical point κ ,
- (3) $\alpha < j(\kappa)$.

Clearly, supercompact cardinals are critical cardinals. But the existence of arbitrarily large and arbitrarily closed target models lends to a greater impact on the structure of the universe. For example, we have the following theorem, which is also due to Woodin [9, Lemma 225].

Theorem. *Suppose that κ is a regular cardinal and $\text{DC}_{<\kappa}$ holds, and δ is a supercompact cardinal such that $\delta > \kappa$. Then there is a forcing extension given by the forcing $\mathbb{P}_\kappa^\delta = \text{Col}(\kappa, < V_\delta)$ such that $\delta = \kappa^+$ and DC_κ holds.*

In the proof of the theorem, it is evident that δ^+ , and indeed all cardinals above δ are preserved. And since the extension satisfies DC_κ , it means that $\text{cf}(\delta^+) > \kappa$ there, which in turn implies that δ^+ could not have small cofinality to begin with (we can always take $\kappa = \omega$ for this since $\text{DC}_{<\omega}$ is a theorem of ZF). We also make the observation that if δ is supercompact and AC fails, then it fails in V_δ as well. This leads to the definition of a *nontrivial failure of choice* at a supercompact cardinal δ , which means that δ is supercompact, but \mathbb{P}_ω^δ does not force the Axiom of Choice. In other words, the failure of choice is *generated* by a specific set in V_δ .

Woodin suggested, in a private communication, that the only way currently known to obtain a nontrivial failure of choice with a supercompact starts by assuming the existence of a Reinhardt cardinal, making the above question more interesting from a consistency strength point of view.

Question 3.11. Can we construct a model in which there is a nontrivial failure of choice above a supercompact just by starting with a single supercompact in ZFC?

4. LIFTING EMBEDDINGS TO SYMMETRIC EXTENSIONS

If we want to prove theorems about critical cardinals in the absence of Choice, we need some technical machinery which allows us to create models of ZF where the Axiom of Choice is false, but there is a critical cardinal. In the case of ZFC we have Silver's theorem that lets us lift embeddings to the generic extension. Seeing how symmetric extensions are the basic tool for moving from models of ZFC to models of $\text{ZF} + \neg\text{AC}$, the relevant generalization seems almost necessary.

Let $\mathcal{S} = \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system and let $j: V \rightarrow M$ be an elementary embedding. We want to identify a sufficient condition for the embedding j to be amenably lifted to the symmetric extension given by \mathcal{S} . Namely, if W is the symmetric extension of V given by \mathcal{S} , and N is the symmetric extension of M given by $j(\mathcal{S})$, we are looking for a condition that lets j be extended to an embedding from W to N , such that $j \upharpoonright W_\alpha \in W$ for all α .

Definition 4.1. Let $\mathcal{S} = \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system and let $j: V \rightarrow M$ be an elementary embedding. We say that \mathcal{S} is *j-decomposable* if the following conditions hold.⁷

- (1) There is a condition $m \in j(\mathbb{P})$ and a name $\dot{Q} \in \text{HS}_{\mathcal{F}}$ such that:
 - (a) $\pi: j(\mathbb{P}) \upharpoonright m \cong \mathbb{P} * \dot{Q}$, π extends the function $j(p) \mapsto \langle p, 1_{\dot{Q}} \rangle$,
 - (b) with $\text{sym}(\dot{Q}) = \mathcal{G}$,
 - (c) there is $\dot{H} \in \text{HS}_{\mathcal{F}}$ such that $\Vdash_{\mathbb{P}} \dot{H}$ is symmetrically \check{M} -generic for \dot{Q} .
- (2) There is a name $\dot{\mathcal{H}}$ such that:
 - (a) $\dot{\mathcal{H}} \in \text{HS}_{\mathcal{F}}$ and $\Vdash_{\mathbb{P}} \dot{\mathcal{H}} \leq \text{Aut}(\dot{Q})$, with $\text{sym}(\dot{\mathcal{H}}) = \mathcal{G}$,
 - (b) there is an embedding $\tau: \mathcal{G} \rightarrow \mathcal{G} * \dot{\mathcal{H}}$, the generic semi-direct product, such that $\tau(\sigma)$ is given by applying π to $j(\sigma)$,
 - (c) $\tau(\sigma) = \langle \sigma, \dot{\rho} \rangle$, and $\dot{\rho}$ is a name such that $\Vdash_{\mathbb{P}} \dot{\rho} \dot{H} = \dot{H}$.
- (3) The family $j^{\ulcorner} \mathcal{F}$ is a basis for $j(\mathcal{F})$.

In the above definition, if $\pi(p^*) = \langle p, \dot{q} \rangle$ we write $\pi_0(p^*) = p$ and $\pi_1(p^*) = \dot{q}$.

Definition 4.2. Under the notation of the previous definition and the assumption that \mathcal{S} is indeed *j-decomposable*, let \dot{x} be a $j(\mathbb{P})$ -name. We recursively define the *partial interpretation of \dot{x} by \dot{H}* as the \mathbb{P} -name \dot{x}^H :⁸

$$\dot{x}^H = \{ \langle \pi_0(p^*), \dot{y}^H \rangle \mid \langle p^*, \dot{y} \rangle \in \dot{x}, \pi_0(p^*) \Vdash_{\mathbb{P}} \pi_1(p^*) \in \dot{H} \}.$$

In the rest of this section, we will use the above notation implicitly.

Proposition 4.3. *Suppose that \mathcal{S} is a *j-decomposable symmetric system*. Then for any $\sigma \in \mathcal{G}$ and $(j(\mathbb{P}) \upharpoonright m)$ -name, \dot{x} , we have $\sigma(\dot{x}^H) = (j(\sigma)\dot{x})^{\sigma(H)}$.*

Proof. First we do a short analysis of the interactions of various automorphisms with j and \dot{H} . If $p^* \in j(\mathbb{P}) \upharpoonright m$, let $\pi(p^*)$ be $\langle p, \dot{q} \rangle$. We observe the following fact: $\pi(j(\sigma)p^*) = \langle \sigma p, \sigma(\dot{\rho}(\dot{q})) \rangle$, where $\tau(\sigma) = \langle \sigma, \dot{\rho} \rangle$. Suppose now that $p \Vdash \dot{q} \in \dot{H}$, then by 2(c) in the definition of decomposability, $p \Vdash \dot{\rho}(\dot{q}) \in \dot{H}$ as well. Therefore, $\sigma p \Vdash \sigma(\dot{\rho}(\dot{q})) \in \sigma\dot{H}$, which is the same as saying $\pi_0(j(\sigma)p^*) \Vdash \pi_1(j(\sigma)p^*) \in \sigma\dot{H}$.

We can now prove by induction on the rank of \dot{x} that $\sigma(\dot{x}^H) = (j(\sigma)\dot{x})^{\sigma(H)}$:

$$\begin{aligned} \sigma(\dot{x}^H) &= \left\{ \langle \sigma(\pi_0(p^*)), \sigma(\dot{y}^H) \rangle \mid \begin{array}{l} \langle p^*, \dot{y} \rangle \in \dot{x}, \text{ and} \\ \pi_0(p^*) \Vdash \pi_1(p^*) \in \dot{H} \end{array} \right\} \\ &= \left\{ \langle \sigma(\pi_0(p^*)), \sigma(\dot{y}^H) \rangle \mid \begin{array}{l} \langle j(\sigma)p^*, j(\sigma)\dot{y} \rangle \in j(\sigma)\dot{x}, \text{ and} \\ \sigma(\pi_0(p^*)) \Vdash \sigma(\pi_1(p^*)) \in \sigma(\dot{H}) \end{array} \right\} \\ &= \left\{ \langle \pi_0(j(\sigma)p^*), (j(\sigma)\dot{y})^{\sigma(H)} \rangle \mid \begin{array}{l} \langle j(\sigma)p^*, j(\sigma)\dot{y} \rangle \in j(\sigma)\dot{x}, \text{ and} \\ \sigma(\pi_0(p^*)) \Vdash \sigma(\pi_1(p^*)) \in \sigma(\dot{H}) \end{array} \right\} \\ &= \left\{ \langle \pi_0(j(\sigma)p^*), (j(\sigma)\dot{y})^{\sigma(H)} \rangle \mid \begin{array}{l} \langle j(\sigma)p^*, j(\sigma)\dot{y} \rangle \in j(\sigma)\dot{x}, \text{ and} \\ \pi_0(j(\sigma)p^*) \Vdash \pi_1(j(\sigma)p^*) \in \sigma(\dot{H}) \end{array} \right\} \\ &= (j(\sigma)\dot{x})^{\sigma(H)}. \quad \square \end{aligned}$$

Under the notation and assumptions of the previous proposition, we have these corollaries.

Corollary 4.4. *If $\sigma\dot{H} = \dot{H}$, then $\sigma(j(\dot{x})^H) = (j(\sigma)(j(\dot{x})))^H$.* □

Corollary 4.5. *If $\dot{x} \in j(\text{HS})$, then $\dot{x}^H \in \text{HS}$.*

⁷This means that $j(\mathcal{S})$ is essentially a two-step iteration of symmetric systems with an M -generic for the second iterand.

⁸In order to be fully compatible, this would be $\dot{x}^{\dot{H}}$, but this just adds clutter to the page, so we chose to omit that dot.

Proof. Let K be such that $K \subseteq \text{sym}(\dot{H})$ and $j(K)$ is a subgroup of $\text{sym}_{j(\mathcal{G})}(\dot{x})$, which exists due to the fact that $j^{\ast}\mathcal{F}$ is cofinal in $j(\mathcal{F})$. Then if $\sigma \in K$, by [Proposition 4.3](#) we get that

$$\sigma(\dot{x}^H) = (j(\sigma)\dot{x})^H = \dot{x}^H.$$

By induction on the rank of \dot{x} the conclusion follows. \square

Corollary 4.6. *If $\dot{x} \in \text{HS}$, then $j(\dot{x})^H \in \text{HS}$.* \square

Theorem 4.7 (The Basic Lifting Theorem). *If $j: V \rightarrow M$ is an elementary embedding and \mathcal{S} is j -decomposable symmetric system, then j can be amenably lifted to the symmetric extension defined by \mathcal{S} .*

Proof. The class $j^{\ast} = \{\langle \dot{x}, j(\dot{x})^H \rangle \mid \dot{x} \in \text{HS}\}^{\bullet}$ is stable under all automorphisms in $\text{sym}(\dot{H})$, since if $\sigma \in \text{sym}(\dot{H})$, then

$$\sigma(\langle \dot{x}, j(\dot{x})^H \rangle) = \langle \sigma\dot{x}, (j(\sigma)j(\dot{x}))^H \rangle = \langle \sigma\dot{x}, (j(\sigma\dot{x}))^H \rangle.$$

Therefore $j^{\ast} \upharpoonright \text{HS}_{\alpha} \in \text{HS}$ for all α , where HS_{α} denotes all the names in HS with rank $< \alpha$. The elementarity of j^{\ast} follows from the elementarity of j and the forcing theorem. \square

Remark 4.8. It is worth noting that j^{\ast} itself might not be a class of the symmetric extension. It is unclear whether or not a subclass of HS which is stable under a large group of automorphisms is itself a class of the extension, and there is no reason to believe that it is. This is why we can only prove that the lifting is amenable, and not definable.

A typical case would be where $j(\mathbb{P}) \upharpoonright m$ is particularly nice and \mathcal{G} fixes $\dot{\mathbb{Q}}$ pointwise, for example, when $j(\mathbb{P}) \upharpoonright m \cong \mathbb{P} \times \dot{\mathbb{Q}}$.

The following is a trivial generalization of the Levy–Solovay theorem.

Theorem 4.9. *Let κ be a critical cardinal and let $\mathcal{S} \in V_{\kappa}$ be a symmetric system, then \mathcal{S} is j -decomposable to any j such that $\text{crit}(j) = \kappa$.* \square

Question 4.10. [Theorem 4.7](#) tells us that an embedding can be lifted. But it tells us nothing about the closure of the target model, M . What sort of closure properties can we get from the amenable lifting?

The requirements in the definition of j -decomposable seem almost necessary. Of the three items in [Definition 4.1](#), item (3) is the odd duck. It seems to be somewhat limiting: if \mathcal{F} is κ -complete, then $j(\mathcal{F})$ is $j(\kappa)$ -complete, which will often render the requirement as blatantly false. It seems to be sufficient for this proof, especially for the proof of [Corollary 4.5](#). This leads to these three questions.

- Question 4.11.**
- (a) What is the exact requirements needed for lifting an embedding to a symmetric extension?
 - (b) Moreover, since [Theorem 4.7](#) gives us that the entire embedding was lifted, what if we only want to lift an initial segment of it?
 - (c) What can we weaken in that case? What if we only want to lift a weakly compact embedding (to obtain a weakly critical cardinal)?

And of course, iterations. While the iteration of forcing extensions can be realized as a forcing extension, the question is subsumed into the Silver criterion when considering the ZFC case. This therefore raises the following question.

Question 4.12. How do we formulate the generalization of [Theorem 4.7](#) to iterations of symmetric extensions?

5. SUCCESSORS OF CRITICAL CARDINALS

In this section we show how very little ZF has to say about successors of a critical cardinal. We assume ZFC + GCH for this proof.⁹

Theorem 5.1. *If κ is a supercompact cardinal, then there is a symmetric extension in which κ remains a critical cardinal, and κ^+ is singular such that $\text{cf}(\kappa^+) < \kappa$.*

5.1. The symmetric Radin system. Let $\lambda > \kappa$ be a limit cardinal, and let us assume that κ is at least $(2^{\lambda < \kappa})^+$ -supercompact. By [7], there is a coherent sequence over $\mathcal{P}_\kappa(\lambda)$, $\tilde{\mathcal{U}}$ of length $(2^{\lambda < \kappa})^+$. In particular, there is a repeat point, ρ , in $\tilde{\mathcal{U}}$. Let $\mathcal{U} = \tilde{\mathcal{U}} \upharpoonright \rho + 1$. Let \mathbb{R} be the supercompact Radin forcing which is defined from \mathcal{U} .

If $h: \lambda \rightarrow \lambda$ is a permutation, then $h_*(x) = h \circ x$ defines a permutation of $\mathcal{P}_\kappa(\lambda)$, and thus a permutation of $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$ defined in a similar way: $h_{**}(A) = h_* \circ A$.

For every permutation h , the set $\{x \in \mathcal{P}_\kappa(\lambda) \mid h_*(x) = x\}$ is a club, so for all $A \subseteq \mathcal{P}_\kappa(\lambda)$, $A \triangle h_{**}(A)$ is non-stationary, and in particular has measure zero in any normal measure on $\mathcal{P}_\kappa(\lambda)$. It follows that the natural action of h_* and h_{**} on \mathbb{R} , defines an automorphism of \mathbb{R} .¹⁰ We will use σ_h to denote this automorphism.

Let \mathcal{G} be the group of all automorphisms σ_h , induced by a permutation of λ , h , such that $h \upharpoonright \kappa = \text{id}$ and h preserves cardinality, namely $|h(\alpha)| = |\alpha|$.

Define F_α to be the subgroup of \mathcal{G} of those σ_h for which $h \upharpoonright \alpha = \text{id}$. And let \mathcal{F} be the filter of subgroups generated by $\{F_\alpha \mid \alpha < \lambda \text{ is a cardinal}\}$.

Proposition 5.2. *For every $\sigma_h \in \mathcal{G}$ and a cardinal $\alpha < \lambda$, $\sigma_h F_\alpha \sigma_h^{-1} = F_\alpha$. Consequently, \mathcal{F} is a normal filter of subgroups.*

Proof. Note that if $\sigma_g \in F_\alpha$, then $g \upharpoonright \alpha = \text{id}$. By the fact that h preserves cardinality, if $\xi < \alpha$, then $|h(\xi)| = |\xi| < \alpha$ and therefore, $g(h^{-1}(\xi)) = h^{-1}(\xi)$.

Therefore $\sigma_h \circ \sigma_g \circ \sigma_h^{-1} = \sigma_{h \circ g \circ h^{-1}} \in F_\alpha$. \square

Let \mathcal{S} denote the symmetric system $\langle \mathbb{R}, \mathcal{G}, \mathcal{F} \rangle$, and let HS denote the class of hereditarily symmetric names. Since the F_α 's generate \mathcal{F} , we say that F_α is a support for $\dot{x} \in \text{HS}$ if F_α is a subgroup of $\text{sym}(\dot{x})$.

Our goal is to show that \mathcal{S} is i -decomposable for a suitable elementary embedding i , and that we can control the subsets of λ which are symmetric—in particular, we can ensure that it is not collapsed.

In many symmetric extensions we control the sets of ordinals added by using homogeneity. This is one of the main reasons why so many examples of symmetric extensions use homogeneous forcings such as Levy collapses. However, the Radin forcing is far from homogeneous. Instead, we use the following lemma to get a modicum of homogeneity which will be sufficient for our proof.

Fix $\alpha \in (\kappa, \lambda)$. For a given condition $p \in \mathbb{R}$, recall that stem p is $\langle x_0^p, \dots, x_{n-1}^p \rangle$. Define $\pi_\alpha(p)$ to be the sequence

$$\langle x_0^p \cap \alpha, \dots, x_{n-1}^p \cap \alpha \rangle \wedge \langle \text{otp}(x_i^p \cap \beta) \mid i < n, \beta = \text{cf } \beta \leq \lambda \rangle.$$

Namely, $\pi_\alpha(p)$ contains the information about the value of the projection of the stem of p below α as well as the order types of the intersection of each x_i^p with regular cardinals below λ .

Lemma 5.3. *If $\pi_\alpha(p) = \pi_\alpha(q)$, then there is an automorphism $\sigma_h \in F_\alpha$ such that $\sigma_h(p)$ is compatible with q .*

⁹As usual in these cases, GCH is only used to simplify cardinal arithmetic calculations and we can omit it by paying the price of a slightly less readable proof.

¹⁰We are being slightly inaccurate: h induces an automorphism of a dense subset of \mathbb{R} . But we ignore this in favor of readability.

Proof. Recall that the members of the Radin club are strongly increasing, we define a permutation h recursively in n steps, such that $h_*(x_i^p) = x_i^q$ and $h \upharpoonright \alpha = \text{id}$. Suppose that we defined h_{k-1} such that $h_{k-1}(x_i^p) = x_i^q$ for all $i < k$, with $h_{-1} = \text{id}$. As $|x_i^p| < x_k^p \cap \kappa$ for all $i < k$, there is a bijection g between $x_k^p \setminus \bigcup_{i < k} x_i^p$ and $x_k^q \setminus \bigcup_{i < k} x_i^q$. Moreover, since $x_k^p \cap \alpha = x_k^q \cap \alpha$, we can assume that g does not move any point below α and since $|x_k^p \cap \beta| = |x_k^q \cap \beta|$ for all regular cardinal $\beta \leq \lambda$, we may assume that g preserves cardinality. Therefore h_k can be taken as $g \circ h_{k-1}$. \square

Let $\bar{\lambda}$ be $|\text{Reg} \cap [\kappa, \lambda]|$, i.e. the cardinality of the set of regular cardinals between κ and λ .

Corollary 5.4. *Let $\dot{f} \in \text{HS}$ be such that $\Vdash \dot{f}: \check{\tau} \rightarrow \check{\text{Ord}}$ for $\tau \in \text{Ord}$, and suppose that F_α is a support for \dot{f} . Then there exists a sequence of sets $\langle B_\rho \mid \rho < \tau \rangle$ such that $|B_\rho| \leq \alpha^{<\kappa} \cdot \kappa^{\bar{\lambda}}$ and $\Vdash \forall \rho, \dot{f}(\rho) \in \check{B}_\rho$.*

Proof. Let B_ρ be the set $\{\xi \mid \exists p, p \Vdash \dot{f}(\check{\rho}) = \check{\xi}\}$. Then by Lemma 5.3, if $\pi_\alpha(p) = \pi_\alpha(q)$, then there is some $\sigma_h \in F_\alpha$ such that $\sigma_h(p)$ is compatible with q . But since $\sigma_h(\dot{f}) = \dot{f}$, q and p cannot force different values for $\dot{f}(\check{\rho})$. Since there are only $\alpha^{<\kappa} \cdot \kappa^{\bar{\lambda}}$ possible values for $\pi_\alpha(p)$, it has to be the case that $|B_\rho| \leq \alpha^{<\kappa}$. \square

Corollary 5.5. *Assume that $\lambda < \kappa^{+\kappa}$. Then $\Vdash^{\text{HS}} \check{\kappa}^+ = \lambda$.*

Proof. If $\alpha \in (\kappa, \lambda)$, then F_α is a support for the canonical collapse of α , given by the Radin club. But if $f: \kappa \rightarrow \lambda$ is a function in the symmetric extension, it has a name with support F_α , and the lemma tells us that f can only obtain $\alpha^{<\kappa} \leq \alpha^+ < \lambda$ values, and in particular it is not surjective. \square

Lemma 5.6. *If \mathcal{F} has a normal basis of size $< \kappa$, then there is an embedding $i: V \rightarrow N$ such that \mathcal{S} is i -decomposable. In particular, κ is critical in the symmetric extension.*

Proof. Recall that \mathcal{U} is a coherent sequence of supercompact measures on $\mathcal{P}_\kappa(\lambda)$ of length $\rho + 1$, where ρ is a repeat point of \mathcal{U} . Let $j: V \rightarrow M$ be the ultrapower embedding given by $U(\kappa, \rho)$. Thus, $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j^{\llbracket \lambda \in M$.

Let $p = \langle \lambda, A_* \rangle \in \mathbb{R}$ be any condition with an empty stem. Define m_* to be $\langle j^{\llbracket \lambda, j^{\llbracket A_*} \rangle \wedge \langle j(\lambda), j(A_*) \rangle \rangle$, this is a condition in $j(\mathbb{R})$ which is stronger than $j(p)$. In M , the forcing $j(\mathbb{R})$ below the condition m_* decomposes into the product $\mathbb{P} \times \mathbb{Q}$. Since ρ is a repeat point, the map that sends a condition $p = \langle d_0, \dots, d_n \rangle \in \mathbb{R}$ to $p' = \pi_{j^{\llbracket \lambda}^{-1}}(p)$ is an isomorphism between \mathbb{R} and \mathbb{P} .

Inside M , apply Theorem 2.4 to obtain some elementary $k: M \rightarrow N$, such that in M there is an N -generic filter for $k(j(\mathbb{R}) \upharpoonright m_*)$ which is compatible with $m = k(m_*)$. Let i denote the composition of these embeddings, $k \circ j: V \rightarrow N$, and let H be the N -generic filter for $k(j(\mathbb{R}))$.

We claim that \mathcal{S} is i -decomposable.

- $i(\mathbb{R}) \upharpoonright m = k(\mathbb{P}) \times k(\mathbb{Q}) \cong \mathbb{P} \times k(\mathbb{Q}) \cong \mathbb{R} \times k(\mathbb{Q})$.
- The isomorphism extends $j(p) \mapsto \langle p, 1_{k(\mathbb{Q})} \rangle$.
- There is an N -generic filter for $k(\mathbb{Q})$ in M , and therefore there is one in V .
- If $\sigma_h \in \mathcal{G}$, then $i(\sigma_h) = \sigma_{i(h)}$ acts on $k(\mathbb{P})$ pointwise, and has a remainder which is in $i(\mathcal{G})$. In particular, this allows us to decompose $i(\sigma_h)$ into $(\sigma_h, \sigma_{h'})$ where $\sigma_{h'}$ is an automorphism of $k(\mathbb{Q})$.
- Moreover, $\sigma_{h'} \upharpoonright H = H$. To see that, recall $B_h = \{x \in \mathcal{P}_\kappa(\lambda) \mid h_*(x) = x\}$ is a club in $\mathcal{P}_\kappa(\lambda)$, and so it must appear in $U(\kappa, \alpha)$ for all $\alpha < o^{\mathcal{U}}(\kappa)$. Now let $C_H = \langle x_\eta \mid \eta < i(\kappa) \rangle$ be the Radin club determined by H , then we claim that $h'_*(x_\eta) = x_\eta$ for all $\eta < i(\kappa)$. By the construction of H in the proof of Theorem 2.4, x_η is a seed for a normal measure on the image

of $(i_\eta \circ j)(B_h)$, where i_η is the η th embedding of the construction. Since $(i_\eta \circ j)(B_h)$ is a club, it is the case that $x_\eta \in i(B_h)$.

- Finally, since \mathcal{F} has a small basis, its pointwise image is a basis for $j(\mathcal{F})$ by elementarity, so $j''\mathcal{F}$ is a basis for $j(\mathcal{F})$.

Therefore i lifts to the symmetric extension, and thus κ is critical there. \square

Proof of Theorem 5.1. For any regular $\mu < \kappa$, take $\lambda = \kappa^{+\mu}$ and with the symmetric Radin system, force below a condition that forces the Radin club to only start above μ . Then in the symmetric extension $\text{cf}(\lambda) = \mu$. Since $\{F_\alpha \mid \alpha < \mu\}$ is a basis for \mathcal{F} , the conditions of Lemma 5.6 hold and κ remains critical with a singular successor of cofinality μ . \square

The construction, however, does not fit very well with the assumption that $\text{cf}(\kappa^+) = \kappa$ (starting with $\lambda = \kappa^{+\kappa}$), or with a more ambitious assumption that κ^+ will be measurable (starting with some $\lambda > \kappa$ which is measurable). In both cases we get that $\sup j''\lambda < j(\lambda)$, a fact which is then carried over the i obtained from Theorem 2.4, and since \mathcal{F} is a λ -complete filter, $\bigcap i''\mathcal{F} \in i(\mathcal{F})$. This means that the symmetric Radin system is not i -decomposable, so the embedding does not lift. Which leaves us wide open with the following questions.

Question 5.7. Suppose that κ is a critical cardinal. Can κ^+ be of cofinality κ ? Can it be measurable? Is there an embedding that can be always lifted when forcing with the symmetric Radin system for any limit cardinal λ ?

We strongly suspect that the answer to the first question is positive, which hints that the answer to the second question could be positive as well. If, however, the answer to the first question is negative this implies that a successor of a supercompact cardinal is always regular, since its cofinality cannot be smaller than κ itself. In either case, however, the results of this section point out the obvious gap between just having a critical cardinal to having a supercompact cardinal. We also note that Jech's Lemma about the preservation of measurability in symmetric extensions [4, Lemma 21.17] can be used to show that the symmetric Radin system would preserve the measurability of λ . So finding an embedding to be lifted is really the only missing ingredient in this construction.

Finally, an obligatory question about the structure of the cardinals above a critical cardinal which seems natural.

Question 5.8. Can a critical cardinal be the last regular cardinal?

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