Abstract

Numerous studies from developmental psychology have suggested that human symbolic representation of numbers is built upon the evolutionally old capacity for representing quantities that is shared with other species. Substantial research from mathematics education also supports the idea that mathematical concepts are best learned through their corresponding physical representations. We argue for an independent pathway to learning "big" multi-digit symbolic numbers that focuses on the symbol system itself. Across five experiments using both between- and within-subject designs, we asked preschoolers to identify written multi-digit numbers with their spoken names in a two-alternative-choice-test or to indicate the larger quantity between two written numbers. Results showed that preschoolers could reliably map spoken number names to written forms and compare the magnitudes of two written multi-digit numbers. Importantly, these abilities were not related to their non-symbolic representation of quantities. These findings have important implications for numerical cognition, symbolic development, teaching, and education.

Keywords: symbolic numbers; non-symbolic quantities; early learning; relational learning; education

There are many reasons to believe that human mathematics begins with the ability to perceive and discriminate physical quantities. Humans, including human infants, possess an approximate quantification system, sometimes referred to as the approximate number (ANS) or approximate magnitude (AMS) system, for perceiving and discriminating quantities that are not exact. Considerable evidence indicates that this system adheres to Weber's Law and is shared by nonhuman primates, pigeons and rats (Brannon & Terrace, 1998; Cordes, Gelman, Gallistel, & Whalen, 2001; Fornaciai, Brannon, Woldorff, & Park, 2017; Libertus & Brannon, 2010; Meck & Church, 1983). Many studies using various methods have shown a predictive relation between the perceptual discrimination of quantities and mathematics achievement (e.g., Barth et al., 2006; Dehaene, 2011; Gallistel & Gelman, 1992; Gilmore, McCarthy, & Spelke, 2010; Libertus, Feigenson, & Halberda, 2013; Piazza et al., 2010). These include correlational studies linking children's perceptual discriminations with current and later mathematics performance (Bonny & Lourenco, 2013; Chen & Li, 2014; Feigenson, Libertus, & Halberda, 2013; Gilmore et al., 2010; Halberda, Mazzocco, & Feigenson, 2008; Inglis, Attridge, Batchelor, & Gilmore, 2011), analyses of the perceptual discrimination skills of children with mathematics disabilities (Mazzocco, Feigenson, & Halberda, 2011; Piazza et al., 2010), as well as demonstrations that perceptual training benefits performance in mathematics tasks (Hyde, Khanum, & Spelke, 2014; Park & Brannon, 2013; Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009; Wilson, Dehaene, Dubois, & Fayol, 2009). Therefore, there have been multiple suggestions that perceptual representations of larger physical quantities are foundational to mathematics learning (Barth, Starr, & Sullivan, 2009; Park & Brannon, 2013; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). Although there are points of disagreement about the nature of these representations as well as the causal direction of the relation to mathematics learning (Fuhs & Mcneil, 2013;

Leibovich, Katzin, Harel, & Henik, 2017; Lyons, Bugden, Zheng, De Jesus, & Ansari, 2018; Matthews, Lewis, & Hubbard, 2016; McCrink & Spelke, 2016; Mix, Levine, & Newcombe, 2016; Noël & Rousselle, 2011; Price, Palmer, Battista, & Ansari, 2012; Sasanguie, Defever, Maertens, & Reynvoet, 2014), the weight of the evidence points to an important role for perceptual judgements about physical quantities in children's entrance to mathematics.

Here, we ask: Could learning about the symbol system itself offer a second, independent, entrance to mathematics? Mathematics at its core is not about specific quantities but rather is about the systems of relations among quantities as variables. The notational and naming system we use to represent specific quantities is founded on a system of relations, representing and naming quantities as counts within a multiplicative hierarchy of sets of 10. Thus, the symbol "342" is named "three-hundred and forty-two" and counts three sets of one hundred, four sets of ten, and two sets of one, with 10 sets of one equal to 1 set of ten, and 10 sets of ten equal to 100 sets of one. Likewise, the symbol "546" is named "five-hundred and forty-six" and counts 5 sets of one hundred, 4 sets of ten and 6 sets of one. Although 342 and 546 refer to different specific quantities, the written forms and spoken names reflect the same relational structure. Computational models have shown that learners could, in principle, capitalize on these regularities within and across spoken and written names to learn the underlying relational structure—enabling such a learner to understand never-before-seen multi-digit numbers—without any grounding of the represented number to a specific perceived quantity (Grossberg & Repin, 2003; Rule, Dechter, & Tenenbaum, 2015). These kinds of models derive the underlying structure from mappings of spoken numbers to written numbers and thus demonstrate that learning about the surface properties of the notational system and their names could be a path into understanding place value notation and as an introduction to mathematics as a relational system.

However, the consensus view from research on children's learning of place value concepts provides little support for this idea. The difficulty of place value concepts for children—evident late into elementary school—is well-documented (S. Ross, 1986; Fuson & Briars, 1990; Fuson, 1990; Gervasoni et al., 2011; S. Ross & Sunflower, 1995). The irregularities in number names that characterize many languages (e.g., "eleven", "fifteen" in English) are known to cause children considerable difficulty and have led some researchers to conclude that children cannot discover place value principles from the surface structure of names and written numbers alone (Fuson & Kwon, 1992; Miura & Okamoto, 1989; Saxton & Towse, 1998). Further, many theorists have argued that curricula designed to ground place value notation in discrete counts and physical models benefit learning about place value, although there is mixed evidence in support of this conclusion (Mix, 2010; Mix, Smith, & Crespo, in press.; Mix, Smith, Stockton, Cheng, & Barterian, 2017). Recently, researchers have further suggested that large number meanings might be grounded in approximate perceptual representation of ungrouped quantities (Barth et al., 2009; Piazza et al., 2004), although recent evidence suggests that these links may not be easily formed by children (Sullivan & Barner, 2011). All this would seem to suggest that an understanding of the relational underpinnings of the place value system is difficult to learning and unlikely to be achieved through learning only about the symbols.

We believe that this conclusion may be premature and missing a possibly important role for early informal learning about multi-digit numbers and their names. This hypothesis is suggested by several findings showing that preschool children, prior to formal instruction, know more about multi-digit numbers than one would expect given the difficulties of school-age children (Byrge, Smith, & Mix, 2014; Mix, Prather, Smith, & Stockton, 2014). One study (Mix et al., 2014) presented 4 and 5-year-old children with 2-, 3- and 4-place multi-digit numbers in a 2-alternative-forced-choice task and asked them to indicate the one that matched a spoken number name ("Which is N?") or was of greater magnitude ("Which is more?"). The children performed well above chance. These preschool children's performances certainly do not indicate an explicit understanding of base-10 notation in the sense of knowing that the 6 in 642 is 6 sets of 100, that the 4 is 4 sets of 10, and that the 2 is 2 sets of one—the goal of formal training about place value in school-age children. However, the likelihood that these children had encountered the name and written form of any individual 3- or 4-digit numbers used in the study (e.g., 836) is vanishingly small given the sparsity of all number names in talk to preschool children (Dehaene, 1992; Dehaene & Mehler, 1992; Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010). Yet these children showed implicit knowledge about how number names and written forms work. Evidently, preschool children, without formal instruction, are deriving generalizable knowledge about the notational system and number names based on very limited exposure.

The five experiments were designed to further document this emerging knowledge and to test the hypothesis that this early understanding of multi-digit notation is distinct from children's ability to map these number names to physical quantities or to make comparative magnitude judgments about the physical quantities, and thus offers an independent and potentially critical early pathway into mathematics. The experiments sought two kinds of evidence of independence. First, in large between-subject cross-sectional studies of children from 3 to 6 years of age, we asked whether the developmental trajectory for growing knowledge about the written forms and their names was different from the developmental trajectories of comparable tasks involving physical quantities. Second, in smaller within-subject cross-sectional studies, we measured how individual children's developing abilities in judgments of symbols and judgments of physical quantities were related. Figure 1 provides an overview of the five experiments. There were two experimental tasks. In the "Which is N?" task children were presented with two choices and asked to indicate the one referred to by the spoken name. In the Digits conditions, the choices were written symbols; in the Dots conditions, the choices were arrays of dots of the same quantities used in the Digits conditions. In the "Which is more?" task, children were also shown two choices—written symbols or dot arrays—and were asked to indicate which choice was more.

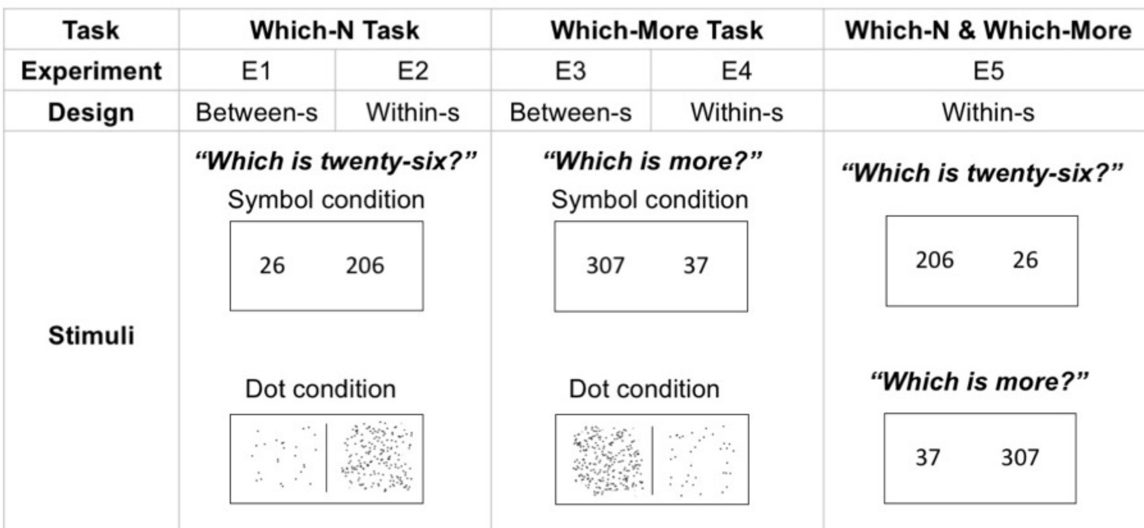

Figure 1. Overview of the five experiments. Experiment 1 used a between-subject design and Experiment 2 used a within-subject design to compare children's ability to map names for large (2 and 3 digits) numbers to written digits and to dot arrays. Experiment 3 and 4 used between and within subject designs to examine children's ability to compare the relative magnitudes of large quantities again given written multi-digit numbers or dot arrays of those same quantities. Experiment 5 provides evidence on the link between children's ability to map number names to written digits and their ability to make magnitude judgements given written representations of quantities.

Experiment 1

Because research on developing place-value knowledge suggests a special role for zeros (Byrge et al., 2014; Zuber, Pixner, Moeller, & Nuerk, 2009), this first experiment focused on written representations that differed only in the presence or location of zero (e.g., 64 versus 604, 305 versus 350).

Method

Participants. The participants were 176 children from 3 to 6 years of age, with half male and half female in each of 4 broad age groups: 44 3-year-olds (mean = 43.4 mo, range = 38 to 47 mo); 43 4-year-olds (mean = 54.2, range = 48 to 59 mo); 45 5-year-olds (mean = 65.3 mo, range 60 to 71 mo), and 44 6-year-olds (mean =79.4 mo, range = 72 to 84 mo). The sample of children was broadly representative of the local population: 84% European American, 5% African American, 5% Asian American, 2% Latino, 4% Other) and consisted of predominantly working- and middle-class families. Children were recruited through community organizations (e.g., museums, child outreach events, boys' and girls' clubs) and at 12 different preschools and daycares selected to serve a diverse income population. Eighteen percent of the children attended daycares or lived in neighborhoods serving schools with over a 50% participation in the free-lunch program. Most of the 5- and 6-year-olds were in some form of half-day kindergarten (at a public school or in daycare); kindergarten is not required by the local state and the curriculum varies considerably across different schools. Counting to 100 and exposure to the corresponding written digits were part of some children's kindergarten experiences. Equal numbers of children from each participating school or school district were assigned to the Dots or Digit Conditions.

Stimuli. To accommodate the goal of testing a broad sample of young children in a variety of contexts, each child was tested on only 10 two-alternative-forced-choice trials: 3 v 7, 11 v 24, 15 v 105, 21 v 201, 36 v 306, 42 v 402, 64 v 604, 78 v 807, 206 v 260 and 305 v 350.

For the Dots condition, the comparison sets were arrays of happy-face dots as shown in Figure 2. The to-be-compared arrays for each trial were presented on an 11-inch by 8-inch card with each array centered in its half of the card. The dots were randomly placed within an irregular region of the same area such that the density (or inter-dot distance) co-varied with set size while the area was comparable. Two unique sets of arrays were constructed for each trial that differed in the random arrangement of the dots and the side of the correct choice. Half the children in each age group in the Dots condition were presented one of these two sets of arrays.

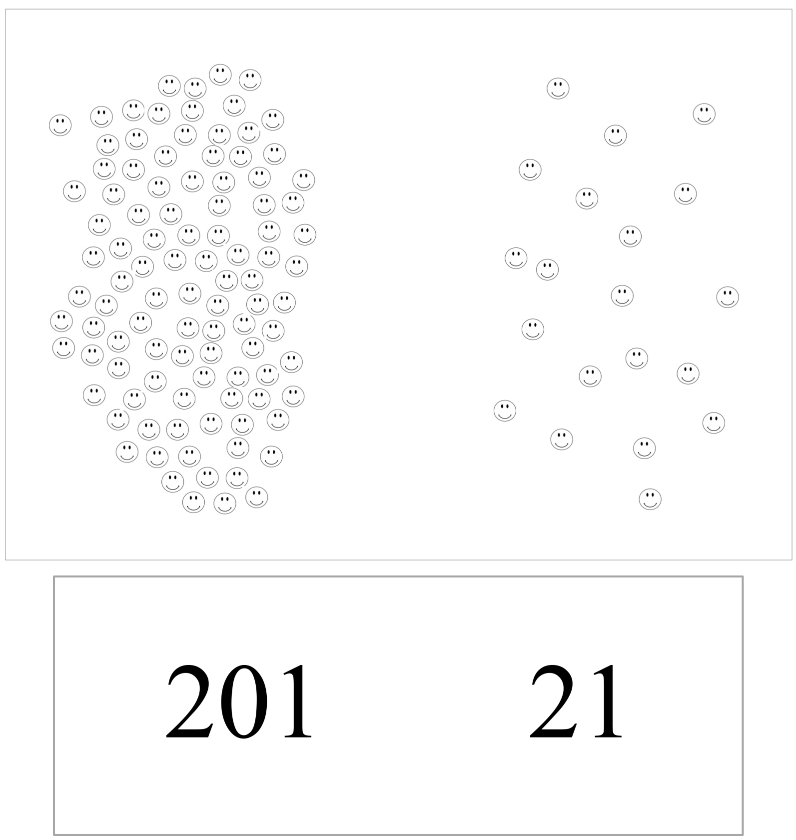

Figure 2. Top: sample stimuli from the Dots condition, depicting the 201(left) and 21 (right) smiley faces. Bottom: sample stimuli from the Digits condition.

For the Digits condition, the digits were printed in 100 pt font (Times New Roman) and presented on 8.5-inch X 3-inch cards with each number centered in its half of the card as shown in Figure 2. Two versions of these cards were also constructed with the left-right position of the to-be-compared numbers counterbalanced across the two sets and with half the children in each age group in the Digits condition receiving one of these sets.

In both conditions, for all card sets, half of the correct choices were on the left and half were on the right and the order of test trials was randomly determined for each subject. All cards were laminated in plastic.

Procedure. The children were tested in a quiet room. There were two warm-up trials using cards that showed two objects (dog, cup) located on separate sides of the cards. The experimenter asked the child to indicate the named object, saying "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is ____ ?" Children were asked to indicate by pointing to the labeled side. All children correctly did so on the two warm-up trials. For the immediately following 10 test trials, the experimenter said: "now I am going to say a number and I want you to tell me which picture shows that number." She presented the two choices and said "Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is _____." The experimenter offered no feedback of any kind as she proceeded through the 10 trials. Pilot studies indicated a bias among very young children to always choose the larger dot array. Accordingly, to be conservative in measuring children's ability to map number names to dots, children were asked—in both the Digits and the Dots conditions—for the smaller of the two numbers or amounts.

Results and discussion. Because gender differences in mathematics abilities are typically not observed in preschool children (Geary, 1994; Jordan, Kaplan, Ramineni, &

Locuniak, 2009; Lachance & Mazzocco, 2006; Lummis & Stevenson, 1990) but have been reported in a few studies (Byrge et al., 2014; Ginsburg & Russell, 1981; Robinson, Abbott, Berninger, & Busse, 1996), in all experiments we applied a highly sensitive measure of possible gender differences, comparing the male versus female performance for all children within each condition by a simple t-test. Across the 5 experiments, for which is N, which is more, both with digits and with dots, we observed no gender differences, p >.50 for all comparisons. We will not consider this factor further.

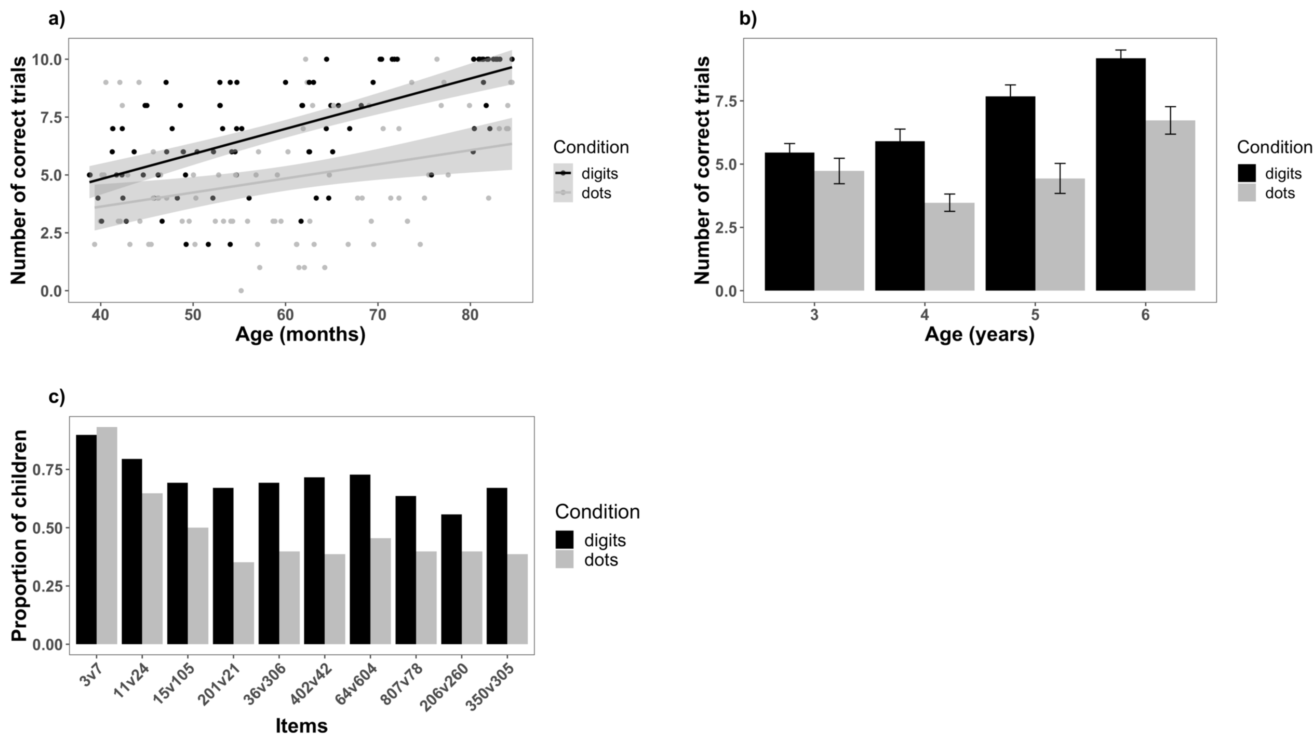

Figure 3. Results from Experiment 1: a) The number of trials with the correct answer for the digits and dots tasks as a function of age. b) The average number of correct trials for the digits and dots task in each age group. d) The proportion of children who answer correctly for each of the test items in the digits and dots tasks.

Figure 3 a) shows each child's number correct on the 10 trials as a function of continuous age in the two conditions. Performance increased consistently with age in the Dots condition, R^2

= .10, F (1, 86) = 9.01, p = .003, and in the Digits condition, R^2 = .43, F (1, 86) = 66, p < .0001. As is also clear, many children, including some quite young children, performed very well in the Digits version of the task whereas many children, including many older children, performed well below chance in the Dots task. As is also apparent, age-related growth was steeper in the Digits than Dots task (B = .66 and .31 respectively). Figure 3 b) shows the mean performances in the Dots and Digits conditions as a function of the four age groups. A 4 (Age group) by 2 (Stimulus condition) analysis of variance yielded only a reliable main effect of Stimulus condition, F (1, 168) = 46.4, p < .0001, partial η^2 = .27, and main effect of Age group, F (3, 168) = 19.58, p < .0001, partial η^2 = .22. The interaction between Stimulus condition and Age group approached conventional significance, F (3, 168) = 2.67, p = .049, partial η^2 = .04. As shown in Figure 3 b), as a group, 5- and 6-year old children performed very well in mapping number names to Digits, and did so at levels greater chance, $t_{5yearolds}$ (21) = 6, p < .0001; $t_{6yearolds}$ (21) = 13, p < .0001, two-tailed. In contrast, no age group in the Dots task performed above chance, and indeed, all age groups other than the 6-year-olds, performed at chance level or reliably below chance, $t_{3yearolds}$ (21) = -.54, p = .6; $t_{4yearolds}$ (20) = -4.5, p = .0002 and $t_{5yearolds}$(22)= -.95, p = .4, two-tailed, reflecting a bias on some children's part to simply choose the array of dots with the larger set size. Figure 3 c) shows the proportion of children getting each test item correct in the Dots and Digits condition, and with the exception of 3 versus 7, reliably more children responded correctly on all individual item types in the Digit than Dots condition, Chi-squares (1) > 3.8, ps < .05.

Altogether the pattern of results indicates the following: Many preschool children performed quite well in matching number names to written digits for 2- and 3- digit numbers. Preschool children showed incremental growth as a function of age in matching both number names to written digits and to arrays of dots; however, performance was much better at all ages for Digits than for Dots. Overall, children's ability to map number names to dot arrays lagged far behind their ability to map number names to the written versions of those names. This makes sense if children are building intuitive knowledge about the relational principles underlying the symbol system itself. Although children may never have been asked to map a heard number name such as "three hundred and five" to either the written form of this number or to a dot array, both number names and written digits are based on the same base-10 principles and thus have corresponding relational structures. Clouds of dots, in contrast, do not have any structure that aligns with that of the symbols through which we represent those quantities.

Experiment 2

Experiment 2 replicated Experiment 1 using a within-subject design to determine whether the age-related advances in the Digits task and the Dots task are correlated.

Method

Participants. Fifty-four children (26 male) were recruited from the same population as in Experiment 1, but none of the children participated in both experiments. There were 15 3-year-olds (mean age 41.3 months, range 36- 48 months), 13 4-year-olds (mean age 51.6 months, range 48 - 57 months), 13 5-year-olds (mean age 64.02 months, range 60-71 months), and 13 6-year-olds, (mean age 75.6 months, range 72- 81 months).

Stimuli and Procedure. All aspects of the stimuli and procedure were identical to Experiment 1 except that each child was tested twice, on separate days (separated by at least one day but no more than 10 days) in the Digits and Dots conditions. Across the entire sample, half the children were tested in the Digits task first and half in the Dots task first and within each age group, at least 6 children at each age level were tested with Digits first or with Dots first. The order of test trials was randomly determined for each child.

Results and discussion. As shown in Figure 4 the overall pattern of performance for the four age groups and for the individual items was the same as in Experiment 1. Regression analyses showed that performance in the Digits condition was strongly related to continuous age, $R^2 = .26$, $F(1,52) = 18.5$, $p < .001$ but performance in the Dots condition was not related to age, $R^2 = .02$, $F(1, 52) = 1.02$, $p = .32$. The lack of relation between Dots condition and age was not due to a lack of power as a priori power analysis indicated that a minimum of 53 participants is adequate to detect a significant result at 0.05 level with a medium effect size of f2 = 0.15. The little change in the Dots condition across age was also not due to ceiling, as accuracy averaged around 50% for all age groups. A 4 (Age group) by 2 (Order of tasks) by 2 (Stimulus condition) analysis of variance for a mixed design yielded a main effect of Age group, $F(3, 46) = 6.92$, $p < .0007$, partial $\eta^2 = .21$ and a main effect of Stimulus condition, $F(1, 46) = 33.92$, $p < .001$, partial $\eta^2 = .23$. The interaction between Age group and Condition approached conventional significance levels as the performance in the Digits task increased with age group more than did the performance in the Dots task, $F(3, 46) = 2.85$, $p = .048$, partial $\eta^2 = .06$.

The new question for this smaller sample within-subject replication of Experiment 1 was the relation between performance in the two tasks. Although knowledge of the mapping of number names to multi-digit numbers grows more steadily and rapidly during this developmental time frame than does knowledge of how these same number names map to dot arrays, it may still be the case that children who are better in one task are better in the other. As shown in Figure 4 d), this appears to be weakly the case, $r(52) = .27$, $p = .05$. However, if continuous age is entered first in a stepwise regression predicting performance in the Digit condition (F (1, 53) = 18.467, p < .001) , there is no variance explained by performance in the Dot condition (t = 1.749, p < .05).

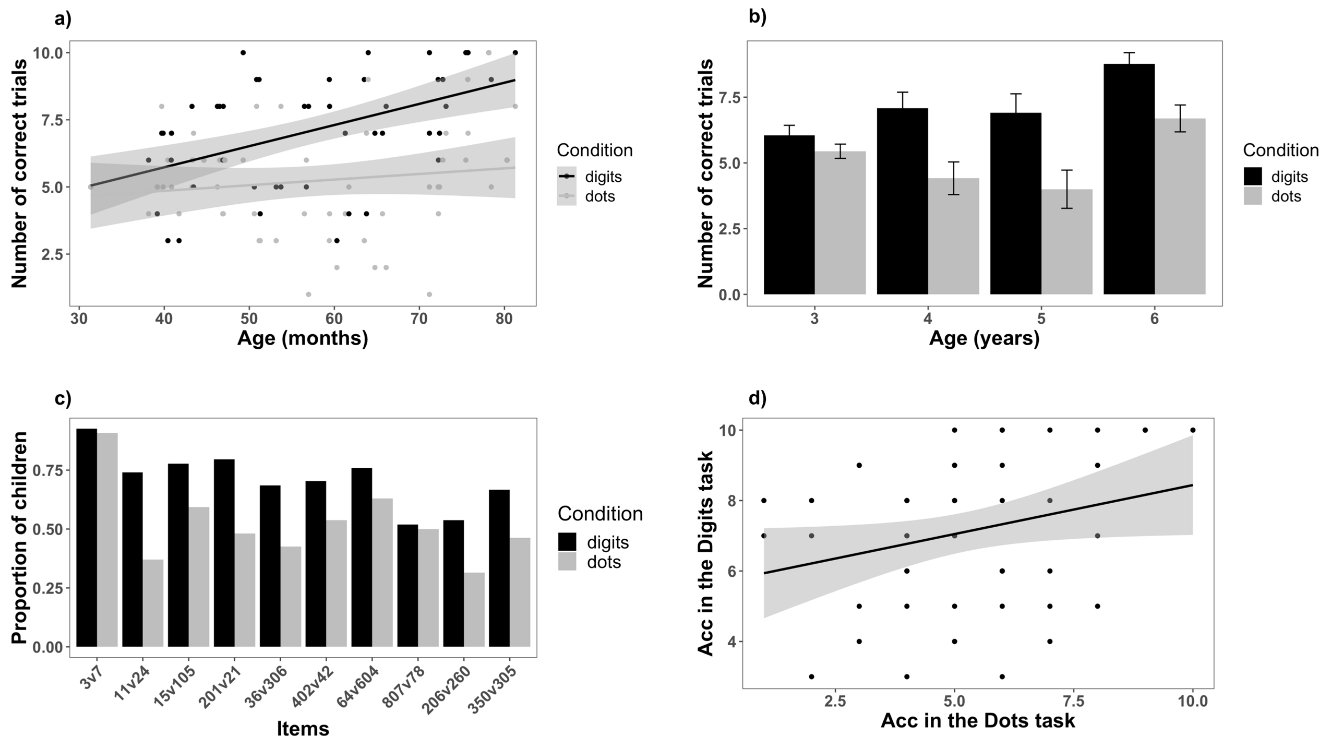

Figure 4. Results from Experiment 2: a) The number of trials with the correct answer for the digits and dots tasks as a function of age. b) The average number of correct trials for the digits and dots task in each age group. c) The proportion of children who answer correctly for each of the test items in the digits and dots tasks. d) The correlation between the number of correct trials in the dots task and that in the digits task.

In sum, the overall pattern of children's performances in mapping spoken number names to multi-digit numbers and to dots arrays indicates: (1) During the preschool years, children's understanding of how spoken number names map to written multi-digit numbers increases systematically. (2) During this same period, children perform much more poorly and show much less systematic growth in their ability to map the same number names to dot arrays. (3) Children's performances in these two mapping tasks are not related beyond what can be explained by age alone. Overall, the results of Experiments 1 and 2 indicate that preschool children are building knowledge of how names and written numbers map to each other and that this emerges earlier than and is independent of the ability to map these same number names to physical quantities.

Experiment 3

Preschoolers' ability to map number names to written forms in Experiments 1 and 2 implies at least an implicit knowledge about the relational principles behind the written notational system and names for multi-digit numbers but does not necessarily signal any understanding of the meanings of those symbols. In Experiment 3, children were asked to make relative magnitude judgments, given digits or dots, in a between-subjects design. Given the large literature on children's competence in making relative magnitude judgments for perceived quantities (Halberda & Feigenson, 2008; Libertus, Feigenson, & Halberda, 2011), we expected children to perform well in the Dots condition. The main question was whether they would also perform well in the Digits condition.

Method

Participants. The participants were 129 (64 male) children recruited from the same population as in Experiment 1. Some children (36) had participated in Experiment 1 at least 9 months prior to Experiment 3. Separate analyses of these children's performances indicated no differences in the pattern of results. There were 32 3-year-olds (mean =41.6 mo, range 36 to 47 mo), 36 4-year-olds (mean = 53.6 mo, range 50 to 59 mo), 31 5-year-olds (mean = 64.1 mo, range 60 to 71 mo), and 31 6-year-olds (mean = 76.7 mo range 74 to 82 mo). Children at each age level were randomly assigned to either the Digits (n=63) or Dots condition (N=66) with roughly equal numbers of children within each age group assigned to each condition.

Stimuli and Procedure. The 10 trials were identical to those in Experiments 1 and 2 in both the Digits and Dots condition; the only difference was the question asked. The experimenter said "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is more?" There were no warm-up trials. The procedure in both conditions began with the presentation of the two choice stimuli for the first trial. The order of the 10 trials was randomly determined for each child.

Results and discussion. Children performed quite well in both conditions, albeit performance was better in the Dots than Digit condition (88.79% vs 79.02%). As shown in Figure 5 a), correct responses were reliably related to continuous age for both the Dots condition, $R^2 = .19$, $F(1,64) = 14.5$, $p < .0004$, and Digits condition, $R^2 = .19$, $F(1, 62) = 14.1$, $p < .0004$. A 4 (Age group) X 2 (Stimulus condition) analysis of variance yielded only the two main effects of Age group, $F(3, 121) = 4.73$, $p <. 004$, partial $\eta^2 =.12$ and Stimulus condition, $F(1, 121) = 13.89$, $p < .0003$, partial $\eta^2 =.1$, with older children performing better than younger children and with performance in the Dots condition superior to performance in the Digits condition (see Figure 5 b). As shown in Figure 5 c), children performed well (>75%) on all items except the comparisons of the two 3-digit-numbers. Overall, quite young children performed very strongly in the Dots condition, which is consistent with what is known about the early development of the approximate number system. The new result is that children also responded well in the Digit condition. Apparently, their emerging knowledge of the written multi-digit numbers includes at least partial knowledge of how these symbols are ordered by relative magnitude.

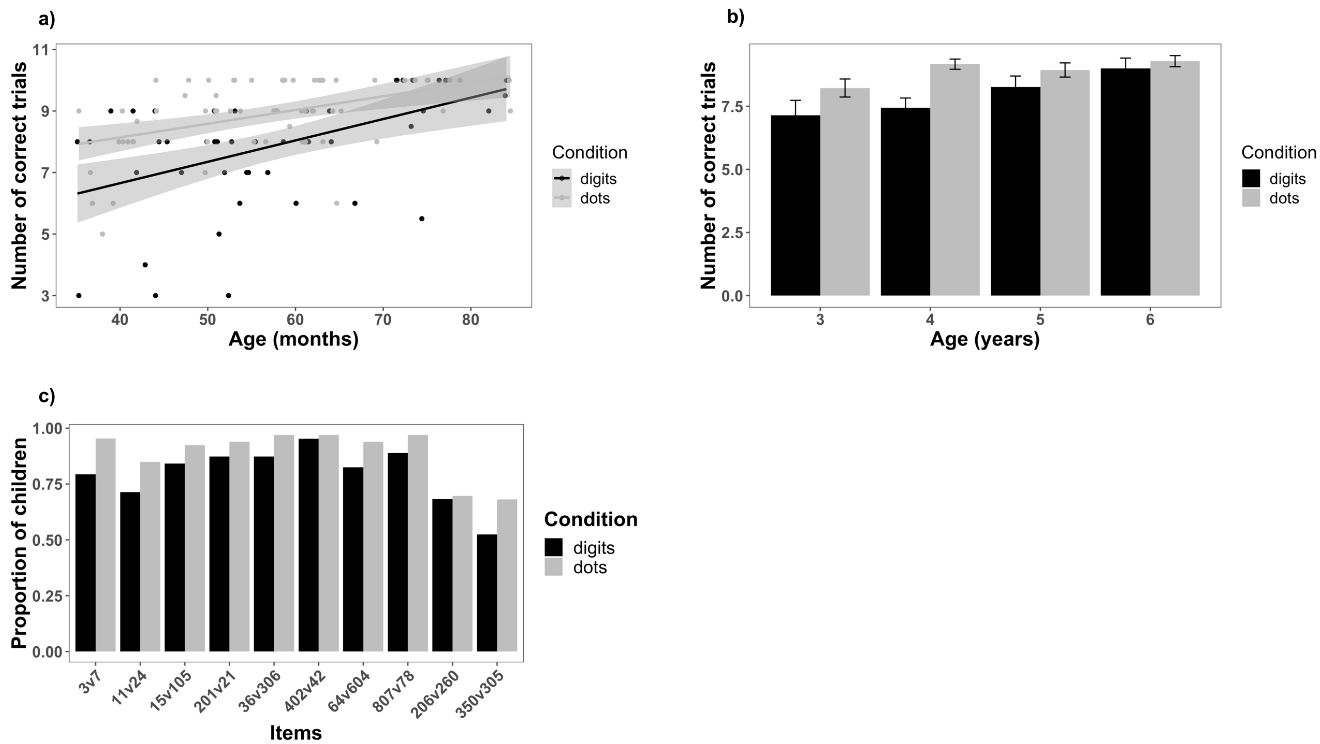

Figure 5. Results from Experiment 3: a) The number of trials with the correct answer for the digits and dots tasks as a function of age. b) The average number of correct trials for the digits and dots task in each age group. c) The proportion of children who answer correctly for each of the test items in the digits and dots task.

One limitation of this experiment, however, is that these were easy comparisons as shown in Figure 5c: For the dot arrays, there was a nearly 9-fold difference in quantity for 6 of the items; for the digits, the same comparisons were between a 2-digit and a 3-digit number such that children might just know that more digits mean more, a perhaps first but not very sophisticated step in understanding how the symbol system works. Accordingly, in Experiment 4, we used more challenging comparisons in a within-subject design comparing children's magnitude comparisons of multi-digit numbers and the same quantities in dot arrays.

Experiment 4

Method

Participants. The participants were 62 (31 male) children recruited from the same population as the previous experiments and none had participated in any of the prior experiments. There were 13 3-year-olds (mean =40.7 mo, range 36 to 47 mo), 19 4-year-olds (mean = 53.7 mo, range 48 to 58 mo), 16 5-year-olds (mean = 65.1 mo, range 60 to 70 mo), and 14 6-year-olds (mean = 75.8 mo range 72 to 81 mo). Children in each age group were randomly assigned to either the Digits or Dots condition as the first tested condition. Children were tested in the two conditions on separate days at their daycares or nursery schools with at least one day but no more than 10 days separating the two testing sessions.

Stimuli. The 10 comparison trials were constructed as in Experiment 3 and consisted of the following three kinds: (1) 2-digit number comparisons—11 v 24, 16 v 23, 30 v 60; (2) 2- v 3-digit number comparisons—15 v 105, 21 v 201, and (3) 3-digit number comparisons—220 v 223, 321 v 323, 525 v 585, 305 v 350, 206 v 260. The 3-digit comparisons differ in one-place—the tens or ones—or are transpositions of the ten's and one's place. These 3-digit number comparisons provide a strong test of children's ability to make magnitude judgments given the written forms

Procedure. There were no warm-up trials. The procedure in both conditions began with the presentation of the two choice stimuli for the first trial. The experimenter said "Look at these. Look at this one. Now, look at this one. Look at them both before you make your choice. Which one is more?" The order of the 10 trials in each condition was randomly determined for each child.

Results and discussion. Figure 6a shows individual performance in the Dots and Digits conditions as a function of continuous age. Performance in the Dots condition was only weakly related to continuous age, $R^2 = .09$, $F(1, 60) = 5.57$, $p = .02$, as most children performed above chance ($t(61) = 15$, $p < .0001$. Performance in the Digits condition, in contrast, was strongly related to age, $R^2 = .41$, $F(1, 60) = 41.69$, $p < .0001$. In this harder test of the relative magnitude meaning of written digits, younger preschoolers performed quite poorly but older preschoolers performed quite competently. An analysis of variance for 4 (Age group) X 2 (Stimulus condition) X 2 (Order of Tasks) mixed repeated measure design revealed a main effect of Age group, $F(3, 54) = 13.46$, $p < .001$, partial $\eta^2 = .31$, and a main effect of Stimulus condition, $F(1.54) = 4.58$, $p < .04$, partial $\eta^2 = .22$. The interaction between Age and Condition as also significant, $F(3,54) = 3.01$, $p < .04$, partial $\eta^2 = .09$; as shown in Figure 6c, the difference between performance in the Dots and Digits condition declined with age. No other main effects or interactions approached significance. Figure 5c also shows that children's overall performances in both conditions were quite strong after 5 years of age, including in the 3-digit number comparisons which require an understanding of the magnitude implications of places in the representational system. However, performance in the two tasks were unrelated, $R^2 = .03$, $p = .8$. As is apparent in Figure 6b, children with the same level of performance in the Dots condition varied in their performance in the Digits conditions from quite poor to perfect. These results indicate that preschool children's emerging knowledge about the symbol system ultimately goes beyond mapping names to written numbers (certainly by 5 years of age) to include knowledge about relative magnitudes and that this emerging knowledge is not predicted by their ability to make relative magnitude comparisons of the physical quantities.

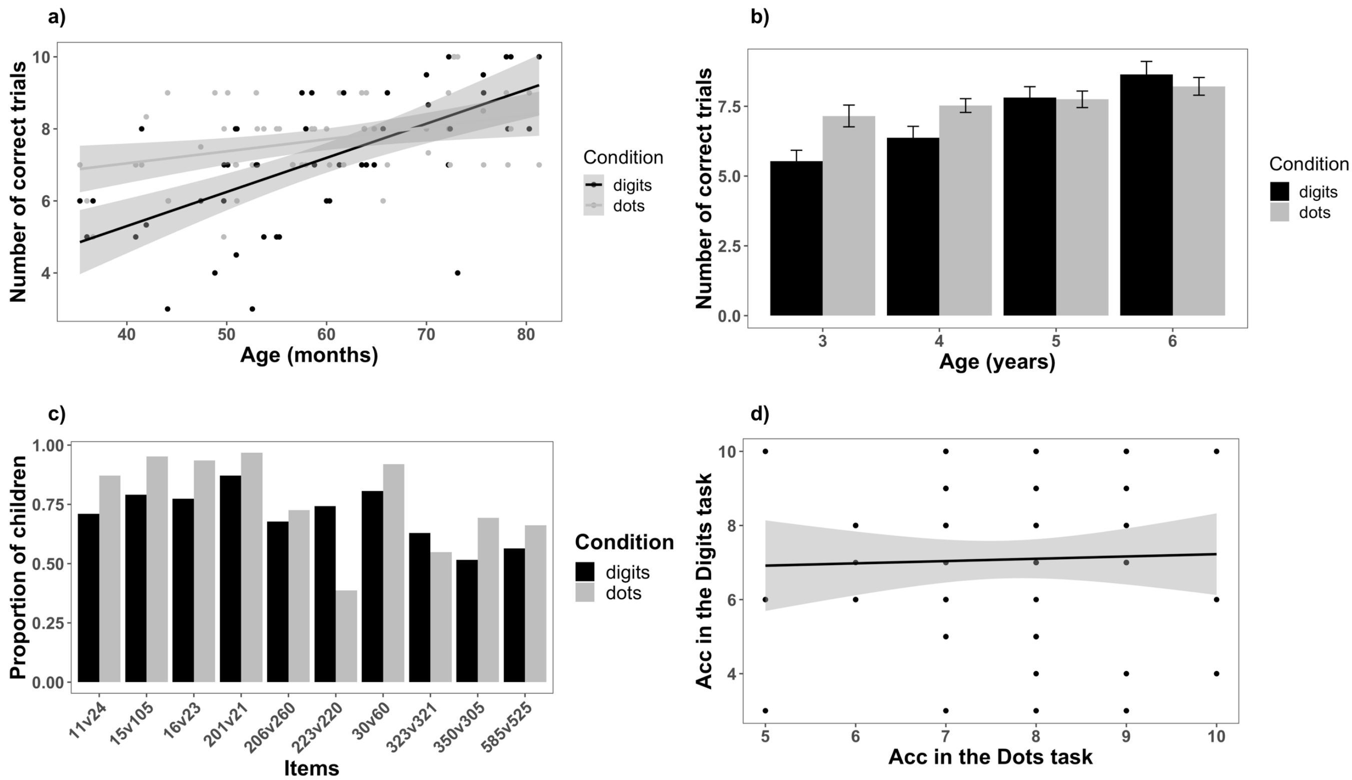

Figure 6. Results from Experiment 4: a) The number of trials with the correct answer for the digits and dots tasks as a function of age. b) The correlation between the number of correct trials in the dots task and that in the digits task. c) The average number of correct trials for the digits and dots task in each age group. d) The proportion of children who answer correctly for each of the test items in the digits and dots tasks.

The results of Experiments 1 to 4 indicate that preschool children are developing knowledge of how written multi-digit numbers represent large quantities and that these developments appear unrelated to their abilities to directly compare the perceptual quantities or to map number names to those quantities. In all four experiments, performance in the Digits task was strongly related to continuous age whereas performance in the Dots task was not. This fact makes sense if this these are two independent paths to understanding large numbers. Knowledge about multi-digit notation must emerge from exposure to heard number names and written digits; it is knowledge about a cultural artifact and thus time in the world is a driver of this knowledge. Perceiving and comparing large sets of objects is, in contrast, a core perceptual ability (Dehaene, 2011; Halberda et al., 2008) that could be more strongly influenced by intrinsic individual differences than by experience. The main conclusion from Experiments 1 – 4 is this: There is a route to understanding large numbers that begins early and concerns knowledge about the symbol system, and is not strongly related to the ability to perceive and judge those same quantities as sets of things.

Experiment 5

Experiment 5 examined the relation between preschool's children ability to map heard number names to written multi-digit numbers and their ability to make relative magnitude judgements given just the written form, making the final point that emerging knowledge of the symbol system—names, written forms, and relative magnitudes—are tightly connected, robust when tested with a variety of numbers, and strengthening steadily over the preschool period. Toward that end, we presented the same children with the two Digits tasks—Which is more and Which is N—to determine whether their performances were related across age.

Method

Participants. The participants were 54 preschoolers (28 male) recruited from the same population as Experiments 1 to 4:14 3-year-olds (mean 41.5, range 36 – 47 months), 13 4-year-olds (mean 51.9, range 48 to 58 months), 13 5-year-olds (mean 64.1, range 60 to 71 months), and 14 6-year-olds (mean 75.6, range 72 to 75 months). Children were tested in the laboratory or at preschools. None had participated in the previous experiments.

Stimuli. The stimuli were written numbers**.** The 15 comparisons for the "Which is more" task all involved digits with equal numbers of places: 3 v 7, 6 v 8, 11 v 19, 14 v 41, 16 v 62, 26 v

73, 30 v 60, 72 v 27, 100 v 10, 101 v 99, 123 v 321, 220 v 223, 525 v 585, 670 v 270 and 1620 v1520. The "Which is N" trials included some of the comparisons used in the previous experiments but also added several new items to add converging evidence on preschool children's competence in mapping number names to written. The items were: 2 v 8, 11 v 24, 12 v 22, 15 v 5, 21 v 201, 36 v 306, 42 v 402, 64 v 604, 85 v 850, 105 v 125, 305 v 350, 206 v 260, 670 v 67, 100 v 1000, 807 v 78, 1002 v 1020.

Procedure. The design and procedures for the "Which is N" and "Which is more" task was identical to those used in the previous experiments. Children were tested on the same day with a break between tasks. Half were tested on "Which is N" first and half were tested on "Which is more" first.

Results and discussion

Figure 7a grey dots show scatterplots of children's performances on the which is N task as a function of age and Figure 7a black dots shows performance on the which is more task as a function of age. Performance on both the which is N and which is more tasks were strongly related to age (R^2 (52) = .30, $p < 0.001$; R^2 (52) = .31, $p < 0.001$) and to each other (R^2 (52) = .31, $p < 0.001$).

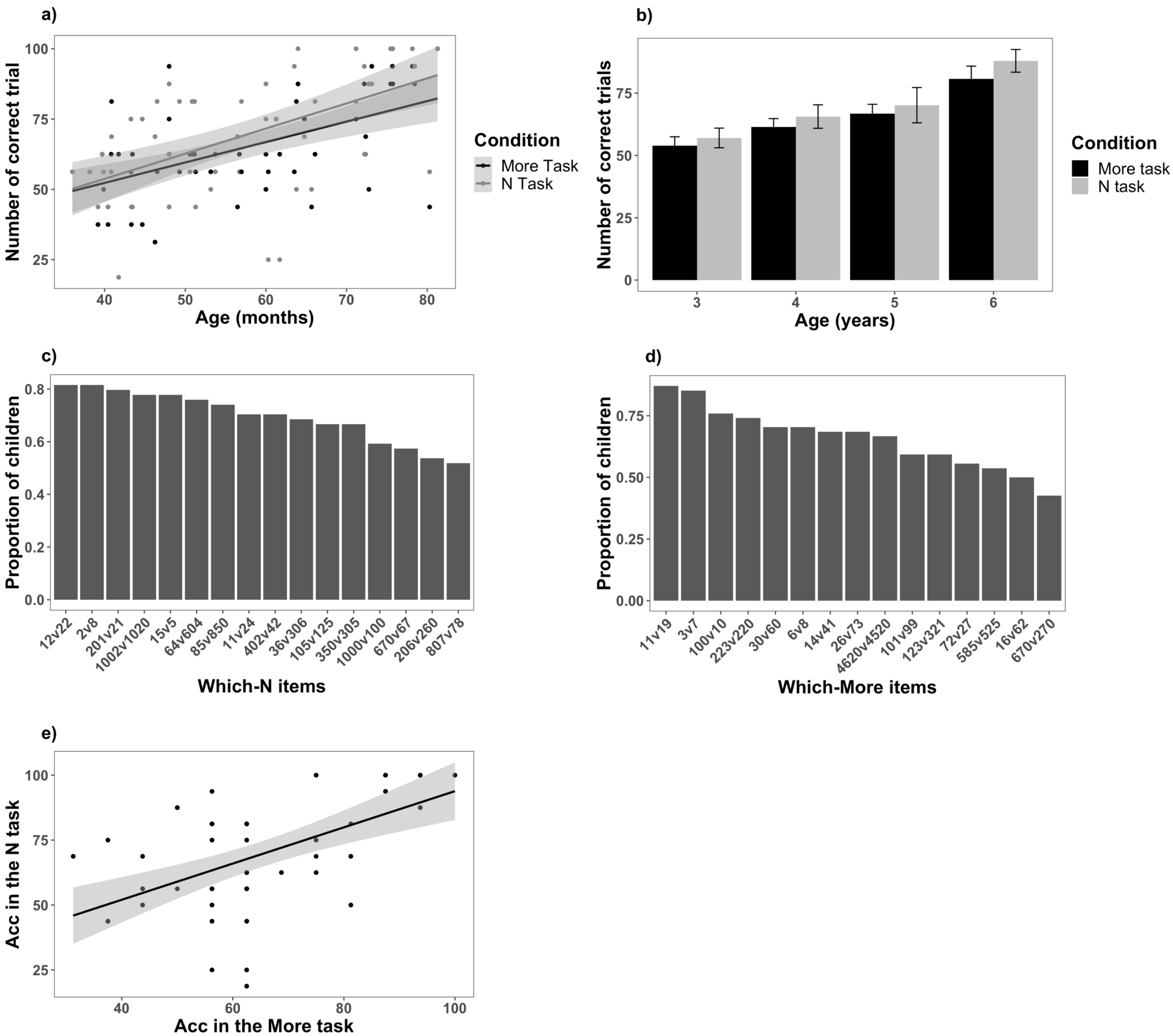

Figure 7. Results from Experiment 5: a) The number of trials with the correct answer for the digits and dots tasks as a function of age. b) The average number of correct trials for the digits and dots task in each age group. c) shows the proportion of children who answer correctly for each of the test items in the which-N task. c) The proportion of children who answer correctly for each of the test items in the which-M task. e) The correlation between the number of correct trials in the dots task and that in the digits task.

Table 1 shows the results of two multiple regression models: a) Age and performance in the which is N task predicting performance in the which is more task and b) Age and performance in the in the which is More is task predicting performance in the which is N task. Both models indicate that Age is the primary predictor in each task but performance in the other task also contributes reliably. A 4 (Age group) X 2 (Task) X 2 (Task order) ANOVA revealed a significant main effect of age, $F(3, 46) = 11.66$, $p < .0001$. partial $\eta^2 = .33$. There was no main effect of task, $F(1, 46) = 2.76$, $p = .1$ or task order $F(1, 46) = .01$, $p = .9$. Although these analyses cannot tell us precisely how performance in the two tasks develop, they suggest that they are developing as an early unified system during the preschool years: children's knowledge about the names of and relative magnitudes represented by written multi-digit numbers grow steadily and jointly in 3- to 6-year-olds as shown in Figure7 b.

	B	*SE B*	β
(a)			
Model 1:			
Dependent variable:			
Accuracy at the More task			
Predictor variables:			
Constant	17.962	8.511	0.001*
Age	0.466	0.169	0.359**
Accuracy at the N task	0.292	0.105	0.362**
(b)			
Model 2:			
Dependent variable:			
Accuracy at the N task			
Predictor variables:			
Constant	7.476	11.009	0.001
Age	0.564	0.211	0.35*
Accuracy at the More task	0.454	0.162	0.365**

Table 1. Two multiple regression models: a) Age and Accuracy at the N Task predicting Accuracy at the More Task; b) Age and Accuracy at the More Task predicting Accuracy at the N Task

General Discussion

Preschool children's knowledge of the notational system through which large quantities are represented grows incrementally throughout the preschool years and appears to have little relation to their judgments of the corresponding physical quantities. Some children as young as 3 years of age and most (but not all) children by the time they are 5 and 6 years of age succeed in mapping names to written numbers and making magnitude judgments of 3- and 4-digit numbers, even when the choices differ in a single digit or involve transpositions. They do this all before formal training on the place value system. These findings and the implicated pathway to mathematics through the surface properties of the symbol system open new opportunities for understanding children's entrance to mathematics, for supporting early and sustained success in school, and for understanding why some children have difficulties in learning place value concepts.

Considerable evidence indicates that understanding the notational system is central to success in elementary school mathematics (Anderson, 2013; Mix et al., in press; Wai, Chan, Au, & Tang, 2014; Zuber et al., 2009). Considerable research also indicates that place value concepts are difficult to master, not just for a few children, but for many (S. Ross, 1986; Fuson & Briars, 1990; Fuson, 1990; Gervasoni et al., 2011; S. Ross & Sunflower, 1995). Accordingly, much previous work on place value used conceptual analyses to understand children's difficulties (e.g., Fuson, 1990; Mix et al., in press), including the irregularities of the English number names from 11 to 19, and have (because of these challenges) focused their research efforts on older children's misunderstandings (2nd - 5th grade) (Cobb, 1988; Kamii, 1988; Ross, 1986; Ross & Sunflower, 1995). But, preschool children are clearly learning the regularities inherent in the surface properties of the notational system. There is a great deal that we do not yet know about how young children formed this knowledge nor its exact character. Nonetheless, it seems likely that an early understanding of the symbols used to represent big numbers plays a positive role in subsequent mathematics learning and may be an independent—and malleable—factor in early mathematics learning. With these larger goals in mind, we consider the findings with respect to four open questions.

What is being learned?

Many of the children in the five experiments clearly know how number names map to written forms and have—in some manner—induced the general principles through which these mappings take place. Because it is unlikely that they have had much if any experience with any particular 3- or 4-digit number, what they have learned must be generative principles that they can apply to any number name and written form. Many of the children are also able to make relative magnitude judgments again of 3- and sometimes even 4-digit numbers with which they are unlikely to have had much experience. Thus, they have also linked the structural properties of the notational system to general ideas about relative magnitude. The present experiments do not provide a detailed determination of just what preschool children know about the symbol system; there is much more to be done on this topic. But, by the time they are five years of age, most children appear to know one or more structural regularities, such as the first-mentioned number name corresponds to the left-most written digit, the magnitude indicated by each place decreases from left to right, and the 5 in “five hundred”, “fifty”, and “five” are all represented by the same symbol, but it has different names and signals different magnitudes depending on the place. These are *not* the core concepts necessary for later success in multi-digit calculation, which requires an explicit understanding of the multiplicative hierarchy underlying base-10 notation—that the hundreds place counts sets of tens, the tens place counts sets of ones, and that 100 is 10 tens, and 10 is 10 ones, and so forth. Given the difficulty of these explicit concepts for elementary school children, it is highly unlikely that any of the children in the present experiments have such an explicit understanding of base-10 principles. However, children who can already map number names to multi-digit numbers and judge which written number indicates a greater magnitude may be ready to learn these more explicit concepts.

How might this early learning matter for later learning?

Competence with the surface forms may be essential to benefit from formal training in the classroom because the fluent mapping heard names to visual forms supports the real-time comprehension of instruction. Formal instruction includes talk about numbers and this talk often takes place in highly cluttered contexts—many numbers (as in number lines, or charts of numbers from 1 to 100) along with physical models of sticks or blocks being grouped and ungrouped. Facility in visually attending to and encoding the structure in the spoken number names and written representations may be essential to understanding all else that is going on during classroom instruction and in doing so may prevent the formation of wrong ideas that characterize some children even as late as sixth grade (Gervasoni et al., 2011; Ross & Sunflower, 1995). Thus, early learning about the surface properties of the naming and notational system for place value may support later learning in the same way that building perceptual fluency in recognizing the structure of an equation or how a function relates to a graph has been shown to advance learning in higher mathematics (Goldstone, 1998; Kahnt, Grueschow, Speck, & Haynes, 2011; Kellman, 2002; Kellman, Massey, & Son, 2010; Landy & Goldstone, 2007).

If so, then the individual differences evident across the five studies become of critical importance. Although many school-age children have difficulty learning about place value given formal instruction, many children—learning from the same teachers and instructional methods—

do not. Some children as young as three years of age in the present study already knew a lot about how number names are mapped to written forms; some children as old as 5 and 6 years, however, performed quite poorly. The origins of these considerable individual differences and the likely associated experiential difference in the early learning environment of children are of significant importance and merit further investigation. These individual differences also raise the possibility that the key predictive factor for success in learning from classroom instruction may not be passing the "which is N?" or " which is more?" test by the start of school but rather the longer-term experiences that might yield rapid and fluent comprehension, for example, such that the direction of a gaze to a named number in a visually cluttered classroom is accurate and a rapid (a potentially more meaningful measure than untimed choice among alternatives).

It is also possible that early learning about number names and written forms is teaching deeper latent knowledge about base-10 principles, an implicit knowledge that may be prerequisite to an explicit understanding of base-10 just as an implicit understanding of syntax in spoken language precedes an explicit, linguistic understanding of syntactic categories and structures in one's native language. From the language in their everyday environments, children develop an intuitive understanding of the relational structure of their language and of word classes such as nouns and verbs that are then made explicit in formal instruction in school. Critically, the deeper intuitive knowledge about these syntactic categories holds the meaning to which explicit categories such as "noun phrase" or "verb" refer. Mathematical knowledge about base-10 principles is arguably similar in that the meaning does not lie in bundled sticks arranged in sets of 10 nor in base-10 blocks but in *the relational structure* between that is base-10 notation (and that exercises with concrete models may help to make it explicit). If this idea is right, then successful instruction may depend not just on the quality of classroom instruction but on the hidden latent knowledge about place value that children bring to school from their early exposures to spoken number names and their written forms. Are differences in this latent knowledge the reason why some children in the same classrooms and schools succeed while others do not?

What are the origins of this learning?

How did these children acquire the knowledge that enables them to make judgments about rarely (maybe never) experienced individual numbers such as “three-hundred and twenty-one?” Past research provides two pieces of information relevant to answering this question. First, we know that preschool children whose parents talk about numbers—in a variety of contexts—have greater success in formal learning about numbers (Levine et al., 2010). Second, we know that the amount of talk about multi-digit numbers in everyday conversations with children is extremely sparse (Dehaene, 1992; Dehaene & Mehler, 1992; Levine et al., 2010). Thus, children with more number talk in their environments are likely to acquire knowledge about the symbol system earlier than those without such talk, but even the children in the richest number environments may not be hearing multi-digit numbers or seeing their written forms with great frequency. These facts raise the possibility that preschool children are learning the principles behind number names and written forms—not through explicit instruction—but incrementally through casual encounters with names and numbers that occur with relatively low frequency compared to other forms of talk. How could children find the generalizable structure that links number names to written forms (and their relative magnitudes) through such kinds of experiences?

Number names and written digits comprise two parallel relational structures of the kind studied within the framework of Structure Mapping (Gentner, 1983, 2010; Gentner & Colhoun, 2010). Given varying but relationally alignable surface forms—for example, models of the solar system and atoms or relational series such as big-little-big—learners can discover the relational structure and apply that structure to new instances (Gick & Holyoak, 1983; Goldwater, Bainbridge, & Murphy, 2016; Loewenstein, Thompson, & Gentner, 1999). Moreover, research with children as well as adults indicates that relational structures may be learned from exposure to two alignable series, without explicit teaching or feedback (Christie & Gentner, 2010; Fisher, 1996; Gentner et al., 2016; Namy & Gentner, 2002). We conjecture that the alignable structure of number names and written forms may be key to supporting learning from casual and potentially infrequent encounters with the number names and written forms of multi-digit numbers. Although many researchers have pointed to the non-alignability of names and numbers across the teens (Fuson & Kwon, 1991, 1992; Geary, Bow-Thomas, Liu, & Siegler, 1996; Ho & Fuson, 1998, Bussi, 2011; Saxton & Towse, 1998), there apparently is sufficient evidence in casual encounters beyond these numbers for children to induce the underlying principles. Clearly, we need a systematic study of both learning environments and mechanisms that might underlie this learning.

What is the relation of emerging knowledge about the symbol system to perceived quantities?

The present results suggest that children's emerging knowledge about multi-digit numbers—including their ability to make relative magnitude judgments—does not include the mapping of these symbols to representations in physical quantities. If the knowledge demonstrated by the children in the "Which is N?" and "Which is more?" tasks with the written forms was in some way linked to physical representations for the judged quantities, one would have expected similar developmental trends in the two sets of tasks and correlated performances within individual children. The first four experiments provide no evidence for such a relation. However, children's knowledge of the quantities 1 through 10, through direct perception (Huang, Spelke, & Snedeker, 2010; Odic, Le Corre, & Halberda, 2015; Pinhas, Donohue, Woldorff, & Brannon, 2014; Wagner & Johnson, 2011) or counting (Carey, 2001; Gentner, 2010; Le Corre & Carey, 2007) seem essential to even intuitive understanding of multi-digit notation. For example, children's ability to determine the exact quantity of small sets and the names of sets—through counting (Gallistel & Gelman, 1992; Wynn, 1992) or by linking names to perceptual representations via processes such as subitizing, pattern recognition or object files (Mix, Sandhofer, & Baroody, 2005; Spelke, 2003; Spelke & Tsivkin, 2001) —have been shown to be related to children's developing understanding of basic numeracy principles, including cardinal concepts of number as well as addition and subtraction concepts (Carey, 2001, 2010; Le Corre & Carey, 2007; Libertus et al., 2011; Mazzocco et al., 2011; Park & Brannon, 2014; Sullivan & Barner, 2014). Further, other evidence implicates potential two-way influences as training in solving math problems benefits judgments of physical quantities and training perceptual judgements of physical quantities benefits mathematics problem solving (Landy, Charlesworth, & Ottmar, 2016; Lyons, Ansari, & Beilock, 2012; Mussolin et al., 2014; Park & Brannon, 2014; Sullivan & Barner, 2014; Thompson & Opfer, 2010).

Nonetheless, perceptual judgments of physical quantities and mathematics may be fundamentally different skill sets. Elementary school mathematics—arithmetic—is about determining exact quantities. But, mathematics proper is not; it is instead about systems of relations among quantities. Base-10 notation is based on a system of relations among quantities. There are also some empirical indicators that symbolic skills may be the more critical factor in later achievements. Meta-analyses of the relation between older children's performances in symbolic and nonsymbolic number tasks with later mathematics achievement (Fazio, Bailey, Thompson, & Siegler, 2014; Chen & Li, 2014; Schneider et al., 2016) indicate that symbolic knowledge is strongly related to later mathematics achievement but that nonsymbolic knowledge is only weakly related. Further, in some analyses of these predictive relations to later mathematics achievement, performance in symbolic and non-symbolic tasks have been found to load on different factors, and there are limited or no correlation between symbolic and non-symbolic magnitude knowledge, just as observed here (Lyons, Ansari, & Beilock, 2012; Sasanguie et al., 2014). The new contribution of the present study is that it shows this same non-relation in preschool children who show early competence in judgments about symbolic representations of numbers larger than 100. The findings bring us to new questions and emphasize the importance of understanding what preschool children know about the notational system, the learning environments that support this early knowledge, and its consequences for later mathematics learning.

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