Transition to university mathematical discourses: A commognitive analysis of first year examination tasks, lecturers' perspectives on assessment and students' examination scripts

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Abstract

This thesis addresses the transition from secondary school to university mathematics at the setting of closed book examinations. I chose a first-year undergraduate module on Sets, Numbers and Probability, given its transitional nature from school to university mathematics; and, its coverage of a variety of mathematical topics. The data of the study includes lecture notes; worksheets and exercise sheets; six examination tasks; interviews with the two lecturers who posed the tasks and their solutions to these; and, twenty-two students' examination scripts. I use Sfard's (2008) theory of commognition to analyse the examination tasks, lecturers' interviews, and students' scripts. Specifically, an adaptation of Morgan and Sfard's (2016) analytical framework enriched by a category regarding students' solutions is used. The adapted analytical framework is applied to the tasks and student data focusing on: word and visual mediator use; engagement with routines; and, participation in varying mathematical discourses. In the lecturer data, I concentrate on lecturers' assessment practices aimed at students' engagement with the university mathematics discourse.

The analysis of the lecturers' interviews and examination tasks revealed: directions on the procedure of mathematical routines and the expectations of what constitutes a sufficient student response; the gradual structure of tasks; students' enculturation in the mathematical community (e.g., defining, proving, justifying). Findings suggest that lecturers, through their experience in marking students' scripts in coursework and examinations, seem to design the tasks with awareness of students' difficulties and aim to assist them in a smooth transition between the different mathematical discourses. However, the analysis of students' scripts shows evidence of commognitive conflicts between school and university mathematical discourses or different mathematical discourses at the university level. Especially, I report cases concerning conflation of different discourses (e.g., Set Theory and Probability or algebra) which is visible in students' use of suitable visual mediators and their engagement with the routines. This insight into students' transitions suggests that explicit attention needs to be given in the transitions between these discourses during the teaching period.

To my parents

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Chapter 1. Introduction

1.1 The rationale and research questions of the study

Research at the University of Mathematics Education (UME) community is a fast-growing field. Recent studies report developments that take place on research at tertiary level (Nardi, Biza, González-Martín, Gueudet & Winsløw, 2014; Winsløw, Gueudet, Hochmuth & Nardi, 2018; Biza, Giraldo, Hochmuth, Khakbaz, and Rasmussen, 2016; Nardi & Winsløw, 2018). These studies illustrate how research is now turning to specific aspects of the teaching practice and offer in-depth insight into the teaching and learning at university level. My study is part of this rise and narrowing down of UME research. I investigate assessment practices and specifically closed-book examinations, which are an aspect of the teaching and learning at university. I focus on assessment by analysing closed-book examination tasks; studying lecturers' perspectives on these tasks and their expectations from students' responses and then exploring how these reflect in students' written responses.

Assessment illustrates to students what their lecturers deem important (Smith, Wood, Coupland, Stephenson, Crawford & Ball, 1996; Van de Watering, Gijbels, Dochy & van der Rijt, 2008) and shows them how their lecturers expect them to use their time and what to engage with during their studies (Smith and Wood, 2000). In the context of the United Kingdom (UK), where this study takes place, the most common assessment method is closed-book examinations (Iannone & Simpson, 2011). Closed-book examinations are the examinations usually given at the end of the academic year, in which the students are required to engage with specific tasks without being able to have access to their textbooks or lecture notes. The students are allotted a specific time within which they need to solve compulsory and optional tasks. This method of assessment seems to be preferred by mathematics undergraduate students (Iannone & Simpson, 2015). Iannone and Simpson also note that the mathematics undergraduates believe that closed-book exams are the best way to distinguish mathematical ability (Iannone & Simpson, 2015).

The tasks used in closed-book examinations at university level have been studied by various researchers (e.g., Griffiths & McLone, 1984, Smith et al., 1996; Galbraith & Haines, 2000; Pointon & Sangwin, 2003; Bergqvist, 2007; Tallman & Carlson, 2012; White & Mesa, 2014; Darlington, 2014; Capaldi, 2015; Tallman et al., 2016). All the above studies are focusing on tasks from closed-book examinations. However, two studies also examine other material of the undergraduate modules. Specifically, tasks from textbooks and coursework are also analysed by White and Mesa (2014) and are used as background to analyse the tasks by Bergqvist (2007), who takes into consideration students' familiarity with the task by taking into account the material of the module in her analysis.

Lecturers' practices are gaining more and more attention from researchers (Nardi & Winsløw, 2018) with studies focusing on lectures and small group tutorials (Jaworski, Mali, & Petropoulou, 2017), lecturers' messages to undergraduate students (Kouvela, Hernandez-Martinez, & Croft, 2017). However, lecturers' assessment practices are still under-researched. Some of the researchers who examined the assessment tasks have also included lecturers in their study, aiming to gain insight into their perspectives regarding the examination tasks either via a survey (Tallman & Carlson, 2012; Capaldi, 2015; Tallman et al. 2016) or using interviews Bergqvist (2012).

Although, most of the studies investigating the examination tasks focus on the first year of undergraduate studies (e.g., Bergqvist, 2007; Tallman & Carlson, 2012; White and Mesa, 2014; Capaldi, 2015; Tallman et al., 2016) the issue of transition between secondary school and university is not explicitly examined. Darlington's (2014) offers insight into this transition by analysing tasks in the university and secondary assessment. However, her study does not take into consideration lecturers' perspectives on these tasks and students' solutions.

The issue of transition from school to university mathematics has been the focus of a wealth of studies (Gueudet, 2008; Gueudet, Bosch, diSessa, Kwon, & Verschaffel, 2016). However, as there are quite a few variables in this transition, the phenomenon becomes quite complicated with various aspects needing to be examined in depth. The examination tasks from first-year examinations are offering insight as to the expectations that the lectures have from their students in the first year of their studies. Furthermore, by examining the students' actual scripts on these examination tasks we gain insight into difficulties that they face after a year of engagement with university mathematics. These difficulties are highlighting instances where the secondary and university mathematics are different, as the first-year students are new comers to the university mathematics community.

In my study, I investigate closed-book examination tasks taking into account lecturers' perspectives on these and the lecturers' expectations regarding students' engagement with the tasks. I, then, explore students' actual engagement with the tasks by focusing on their written solutions. I do so through a discursive perspective developed by Anna Sfard (2008). Specifically, I adopt an analytical framework developed by Morgan and Sfard (2016) and adapt it to examine undergraduate students' expected and actual participation to the mathematical discourses at university. The expected participation is examined using the examination tasks and lecturers' perspectives on these tasks. Students' actual participation is then examined by viewing their solutions to the examination tasks. I focus on a first-year module offered in a well-regarded mathematics department in the United Kingdom. I chose this module as I am interested in observing the differences between secondary school and university mathematical discourses, and in examining the students' scripts for engagement with various mathematical discourses. Specifically, the research questions guiding my study are:

> **R.Q.1** What are the discursive characteristics of the examination tasks?
>
> **R.Q.2** What are mathematics lecturers' perspectives on the examination tasks and their expectation from students' engagement with the university discourse in the closed-book examination setting, and how are these perspectives enacted in the formulation of the examination tasks?
>
> **R.Q.3** How different are university mathematical discourses from the secondary school mathematical discourses and what commognitive conflicts can be observed as result of those differences in students' scripts?

In the section that follows, I describe the structure of my thesis.

1.2 The structure of the thesis

In chapter 2, I offer a review of the relevant literature. I present studies which characterise examination tasks, some of which were originally designed for university level and others which were introduced at secondary level and then adapted for university level. I, then, offer a description and discussion of studies investigating lecturers' assessment practices. Specifically, I focus, on lecturers' perceptions of examination tasks. Finally, I report studies addressing students' transition from school to university, as my study aims to particularly examine the differences in the discourses by analysing examination tasks and students' responses to those. Then, I identify the gap in the literature and explain the aims of my study.

In chapter 3, I present the theoretical framework of the study, Sfard's theory of commognition (Sfard, 2008), and associated studies that elaborate and inform my use of the framework. This theory takes a sociocultural and discursive perspective. I describe the main tenets and elements of this theory and discuss the concepts related to my study. The commognitive theory has gained wide interest in the university mathematics education community (Nardi et al., 2014). I present an overview of the studies that are using it and explore how they have employed the basic tenets of the theory to describe the discursive practices at university level.

The methodology of the study is described in chapter 4. I first discuss my research design which is a naturalistic qualitative paradigm. I, then, review the context where my study took place, namely a well-regarded mathematics department in a United Kingdom (UK) university. The general context of the study and the participants of my research are presented afterwards followed by the data collection methods, namely observations, semi-structured interviews, and document analysis. I, then, focus on the specific module of this study, a module taught in the first-year of the undergraduate studies. The module is called Sets, Numbers and Probability and is split in the two terms. In the autumn term, the focus is on Sets, Numbers and Proofs and in the spring term on Probability. Additionally, I describe the process of data analysis of the interview, and document data from Sets, Numbers and Probability. There, I discuss the use of the analytical framework by Morgan and Sfard (2016); my adaptation of the framework to university mathematics examinations; and, the addition of a component to investigate students' solutions to the tasks. I end this chapter with considerations regarding the ethical issues of my study, my role as a researcher and comments on the triangulation and validity of the data analysis.

This chapter is followed by the four analysis chapters. The first two chapters, chapters 5 and 6, deal with the Sets, Numbers and Proofs part of the module. Specifically, in chapter 5, I present the three tasks, corresponding to this part of the module, and their commognitive analysis. I, then, offer interview excerpts, corresponding to the discussion of these three tasks with the lecturer who posed them (L1). I end with a discussion on the results from the analysis from the lecturer's data and analysis of the three tasks. In chapter 6, I analyse students' responses to the three tasks on Sets, Numbers and Proofs part of the module. My analysis of students' solutions is informed by the analysis presented in chapter 5. I examine the word and visual mediator use in students' responses and students' engagement with routines. Specifically, I focus on incidents where this use and engagement signal conflation of discourses, either between secondary school and university discourses or between different university mathematical discourses.

The other two analysis chapters (chapters 7 and 8) focus on the Probability part of the module. In chapter 7, I present the commognitive analysis of the three Probability tasks and the lecturer's perspectives on students' expected participation in university mathematics. I, then, comment on the combined analysis of the tasks and lecturers' perspectives on these. This is followed by chapter 8, where I focus on students' written responses to the three Probability tasks. The analysis in chapter 7 contributes to the analysis in chapter 8, where I delve in students' actual participation in university discourses. I investigate in detail students' use of word and visual mediators, and students' engagement with routines, highlighting cases where there is a conflation between mathematical discourses.

In the concluding chapter, chapter 9, I start by providing answers to the research questions (section 9.1). I, then, present the contribution of my study in the field of University Mathematics Education, the implications of my findings to practice (section 9.2.1), the theoretical contributions of my study (section 9.2.2), but also the advantages and challenges in using commognition (section 9.3). I, then, consider the limitations of my study (section 9.4). Finally, I discuss ideas for future research (section 9.5) and close with thoughts on my journey as a commognitive researcher (section 9.6).

Chapter 2. Literature Review

In this chapter, I present studies that are relevant to my research on assessment practices and students' participation in university mathematics. In the first section, I discuss studies on examination tasks, highlight the variety of frameworks used by the researchers and report on studies analysing tasks used in different modules. Then, I report studies regarding lecturers' practices and specifically their assessment practices. Finally, I discuss the issues regarding the transition from secondary to university level.

Here, I would like to note that different studies use different terminology to refer to the mathematical tasks used in closed-book examinations, on some occasions the words problems or questions are being used. The same issue occurs when researchers refer to the lecturers who pose the tasks at the examinations, they use lecturers, teachers, professors or instructors. Similarly, they use the terms course or module to refer to a unit taught in a mathematics department. In my thesis, I use the terms module, tasks and lecturers to mean a unit taught in a mathematics department, the mathematical tasks used in the examinations and the teachers at the university level who posed these tasks.

2.1 Studies of examination tasks: various classifications and findings from empirical studies

Researchers have focused on the analysis of the examination tasks from different modules, using a variety of frameworks. Each of the frameworks focuses on different aspects of the tasks. In this section, I present studies reporting frameworks aimed at analysing examination tasks and studies using these frameworks at the university context.

Concentrating on the qualities the tasks should have to assist the students in achieving the qualities desired by the employers Griffiths and McLone, created a list of the qualities (Griffiths & McLone, 1984a) and they examined the extent of these qualities in 1404 tasks from examination given in ten British universities (Griffiths & McLone, 1984b). The qualities were:

- Procedure: the presence of indications as to how this task can be solved,
- Objectives: the degree to which the task illustrated the conclusions that the student needed to arrive to,
- Jargon: the presence of specialised language,
- Mathematical content,
- Definition, bookwork, stock example: the relation of the task to memory and understanding,
- Abstraction: the examination of theory instead of the application of the theory,
- Mathematical manipulation: the manipulation of symbols and calculations which were needed to answer the question,
- Logical manipulation: the reasoning that is required to solve the question based on the researchers' experience
- Sustained thinking: the ability to merge and connect ideas to produce a solution
- And open solution: where the solution is not determined.

The 1404 tasks were from different modules: Pure, Applied, Numerical methods, Computing, and Statistics. The analysis of the tasks presented evidence of differences between the modules. In the Statistics the tasks focused more on mathematical content and understanding, required sustained thinking to arrive at the final solution. However, the tasks were not open, and the students did not have flexibility regarding their approach when solving them. The tasks from applied mathematics and numerical methods were very similar to Statistics. The Applied tasks focused more on mathematical manipulations rather than logical manipulations. In the Pure maths modules, the tasks concentrated less on the mathematical content and did not indicate the way that the solution should develop. Finally, in tasks from Computing modules, the students were not directed regarding the procedure, there was not much mathematical manipulation, but there was great use of jargon.

Focusing on the nature of the examination tasks and specifically whether the tasks required students to act as competent practitioners or as experts, Pointon and Sangwin (2003). They analysed 82 examinations with 489 tasks from two first-year modules: one core Algebra and Calculus and the other one a foundation module for pure mathematics. They focused on whether the tasks required the students to act as competent practitioners asking to recall facts, carry out a routine calculation or algorithm, classify some mathematical objects, or interpret a situation or answer; or to act as experts: requiring them to prove, show, justify a general argument, extend a concept, construct an example/instance and criticise a fallacy. The results of the analysis showed that three out of five tasks only require calculations. Whereas the percentage of the tasks that require higher-level skills were 3.4%. Specifically, 71.2% of the examination tasks either required recalling facts or carrying out a routine calculation or algorithm. Regarding the analysis and the differences between the two subjects: the tasks from the core algebra and calculus course were mostly routine calculations, interpret and construct examples. The ones from the foundation for pure mathematics course were more proof and factual recall tasks.

Focusing on examinations from different mathematical modules across the four years of an undergraduate degree, Maciejewski and Merchant (2016), use Bloom's taxonomy (1956) aiming to find the relationship between study approaches and module grades. The focus of this taxonomy are the educational objectives set by the lecturer, and the purpose of the taxonomy is to assist lecturers to develop balanced assessments. The six categories of the taxonomy are knowledge, comprehension, application, analysis, synthesis, and evaluation. Their results highlight the differences between the modules. In first-year modules, the focus is on calculations and procedures. In the modules offered in the last years, the nature of the tasks changes and the emphasis is on evaluation and creativity, and many tasks involve remembering and understanding theorems and definitions. However, the upper year modules also had many tasks demanding recall and understanding, but the focus was given in recalling statements of definitions and theorems. In the first years, the tasks involved recalling and applying a procedure that the students had learned during the year. The results show that the tasks in the upper years are representing the upper and lower part of the taxonomy. Also, the researchers conclude by saying that: "This leads us to think about the values mathematics instructors have for their students" (Maciejewski and Merchant, 2016, p.384)

Tallman and Calrson (2012) use an adaptation of Bloom's taxonomy also reported in Tallman, Carlson, Bressoud, and Pearson (2016). In the US context, Tallman et al. (2016), focused on the nature and level of student learning in the final examination of Calculus 1. The researchers introduce an Exam Characterisation Framework (ECF), which categorises the task according to three dimensions: orientation, representation, and format. They analysed 3735 examination tasks from 150 calculus examination papers. The task orientation is based on Anderson *et al.* (2001) adaptation of Bloom's taxonomy. The subcategories are the following: Remember, Recall and Apply procedure, Understand, Apply understanding, Analyse, Evaluate and Create. The representation dimension is based on the nature of the statement and the solution of the task, and it is distinguished in Applied/Modelling, Symbolic, Tabular, Graphical, Definition/Theorem, Proof, Example/Counterexample, and Explanation. Finally, the format of the tasks is examining whether the task is multiple choice, short answer, broad open-ended or word problem. The results of the study regarding the orientation of the tasks showed that 85.21% of the examination tasks could be solved by Remember or Recall and Apply procedure (with Remember being 6.51% and Recall and Apply procedure being 78.7%). Only 14.83% asked students to show understanding. The authors also coded the exams as procedural and conceptual. A procedural examination was an examination that more than 70% of the examination task were coded as belonging to the first two categories of the adaptation of Bloom's taxonomy. Of the 150 examinations that the researchers analysed, 90% were coded as procedural. The representation of the majority of the examination tasks was Symbolic, and 73.7% and 89.4% of the solutions were asked to be given in symbolic statement. The results of the analysis regarding the format of the task illustrated that only 3.05% required an explanation from the students. These researchers also surveyed lecturers' views, as mentioned in the previous section. Almost 70% of the lecturers claimed that they frequently require students to explain their solutions in the examinations. However, as seen above only 3.05% of the tasks asked explicitly for an explanation. Also, there was a difference in the results from the survey regarding the focus of the tasks on the skills and the methods needed for the solution. The lecturers' survey showed that the lecturers believed that 50% of the points are given to skills and methods for calculations. However, the tasks analysis presented evidence that 78.7% asked for recall and application of procedure.

White and Mesa (2014) propose a framework for task analysis aiming to examine the lecturer’s goals and the students' opportunities to learn. The framework consists of two dimensions: cognitive orientation and knowledge type. The cognitive orientation follows Tallman’s and Carlson’s (2012) framework. However, White and Mesa (2014) add another category of cognitive orientation, Recognize and Apply procedures. The tasks belonging to this category could be solved using conceptual understanding however depending on how the students were taught they could solve it using procedural understanding. The knowledge type dimension is further distinguished in factual, procedural, conceptual and meta-cognitive. In their analysis, the researchers extend their analysis and include tasks from textbooks, worksheets, and examinations. The analysis of 4,954 Calculus tasks, of which 475 were examination tasks, allowed the categorisation in a simple procedure, complex procedure, and rich tasks. The tasks from the simple procedure category require students to Remember and Recall and Apply procedures. Recognizing and applying procedures are needed in the tasks from the complex procedure category. Finally, the rich tasks were tasks requiring engagement with Understanding, Apply understanding, Analyse, Evaluate and Create. However, in this framework, the subjectivity of the researcher is a factor that determines the classification of a task, as there are assumptions regarding students’ familiarity with certain type of tasks.

The results showed that there were differences regarding the cognitive orientation depending on whether the task was a textbook, worksheet or an examination task. They also found that there were differences between the lecturers who posed the tasks. There are 11% of the tasks belonging in the complex procedures’ category. However, the percentage of rich and simple procedures depends on the lecturers. There were more rich tasks in the examinations than in the bookwork or web-work, with 49% and 25% respectively. The authors argue that this could be because of the time limitation, which does not allow the lecturers to put many tasks. Having this in mind the lecturers might choose to examine more complex work, which includes the simple procedures. Finally, the authors comment that rich tasks could be demanding less cognitive demand if the students are familiar with the tasks and call for further research into the “nature of assessment, on how instructors conceptualise them and use them” (White and Mesa, 2014, p.688)

Smith, Wood, Coupland, and Stephenson (1996), a team of mathematicians and mathematics educators, designed an adaptation specifically for undergraduate mathematics closed-book examinations. The aim of the introduction of the Mathematical Assessment Task Hierarchy taxonomy (also known as the MATH taxonomy) is to assist lecturers in constructing balanced examinations assessing a range of knowledge and skills. Smith et al. group eight classifications of knowledge and skills in three groups: Group A, Group B and Group C (Table 2.1).

Table 2.1: The MATH taxonomy

Group A	Group B	Group C
Factual knowledge and fact systems	Information transfer	Justification and interpretation
Comprehension of factual knowledge	Application to new situations	Implications, conjectures, and comparisons
Routine use of procedures		Evaluation

Group A tasks consist of tasks requiring students to recall factual knowledge and fact systems, comprehend factual knowledge and be able to use basic procedures. In solving tasks belonging to Group B, students have to be able to transfer information and apply information or methods in new situations. Moreover, in tasks from Group C students are asked to justify and interpret a result; offer conjectures and comparisons; and evaluate results. Smith et al. argue that tasks should be from all three groups if the aim is to achieve a balanced examination. The researchers also note that if the examination tasks were from Group A then this might result into students adopting a surface learning approaches (Ramsden, 1992) whereas deep learning approaches (Ramsden, 1992) could be achieved through tasks from Group B and C.

Using the MATH taxonomy, Darlington (2014) investigates the distribution of the marks of the examination tasks from first-year undergraduate modules, Pure Mathematics I and Pure Mathematics II, in Algebra and Real Analysis accordingly. The results show that 49.9% from the Algebra tasks and 58.3% from the Real Analysis tasks belong in Group C. And the analysis of the marks, of these tasks, showed that the majority of the marks, were given to tasks requiring justifications, interpretations, implications, conjectures, and comparisons. Also, findings show 29.1% from the Algebra tasks, and 34% from the Real Analysis are in Group A. The marks' analysis shows that 90% of these Group A tasks, required Factual Knowledge and Fact systems. Comparing the results from the two modules, it seems that in Real Analysis the percentages regarding Group A and Group C tasks are much higher. Whereas, in Algebra there is a higher percentage regarding Group B tasks. There are 21.2 % in Algebra and 7.6% in Real Analysis.

In her study, Darlington also investigates A-Level (Advanced-Level) exams. The A-Level examinations are school leaving qualifications available in the United Kingdom to students who are completing their secondary studies and are preparing for undergraduate studies. Darlington's results suggest that there is a big difference between secondary and university mathematics regarding the type of tasks being asked. Darlington concludes that "The large increase in the proportion of Group C questions between school and university is indicative of the changing nature of mathematics between these two points" (Darlington, 2014, p. 13).

Similar to the MATH taxonomy is a framework introduced by Galbraith and Haines (2000) focused on the mathematical demand of the tasks. The researchers distinguish the following categories: mechanical, interpretive and constructive tasks. Mechanical tasks are tasks requiring a standard procedure indicated in the wording of the task. Tasks requiring recall and the application of conceptual knowledge were categorised as interpretive. Moreover, constructive tasks required the use of both conceptual and procedural knowledge, and the introduction of necessary mathematical procedures. By examining students' performance on the tasks, the researchers claim that the categories are in increasing order of difficulty. They also note the similarity between their framework and the MATH taxonomy. The researchers comment that the tasks belonging to the mechanical group are analogous to the tasks from the Group A. The interpretative tasks are similar to the information transfer one of the skills in

Group B, and finally the constructive tasks are similar to the tasks requiring justification and interpretation; and implication, conjectures and comparisons two of the skills from Group C.

In all the above-mentioned frameworks some of the tasks are categorised as requiring knowledge, either as remember and recall (e.g., Tallman et al., (2016)), factual knowledge and fact systems (e.g., Smith et al. 1996), recalling facts (e.g., Pointon and Sangwin (2003)), mechanical (e.g., Galbraith & Haines, 2000). However, the extent to which the students are familiar with this knowledge is not being explicitly taken into account. A framework that investigates the students' familiarity with the examination tasks explicitly is developed by Lithner (2008). The framework focuses on the reasoning demanded by the student, and the analysis of the tasks takes into account students' familiarity through the content of the module. The reasoning is distinguished in creative and imitative. Imitative reasoning is required when the students are asked to recall something from memory (memory reasoning) either a mathematical fact or an algorithm (algorithmic reasoning). Whereas, creative reasoning is the reasoning that requires the student to act in an innovative and logical way.

Using Lithner's (2008) framework on creative and imitative reasoning, Bergqvist (2007) analysed tasks from 16 Calculus examination papers in the Swedish university context. The tasks were analysed and classified according to the reasoning they demanded from the student, taking into account the content of the module and the textbook used. Tasks requiring imitative reasoning were tasks asking students to state a theoretical statement, which they were informed that might be present in the examination or the task occurred in the textbook at least three times. The tasks demanding memorised reasoning are further categorised in definitions, theorems, and proofs. Basic algorithms, complex algorithms, choice-dependent algorithms, and proving algorithms are the tasks requiring algorithmic reasoning. Finally, tasks demanding creative reasoning are distinguished in those requiring creative reasoning in one step of the algorithm called local creative reasoning; and the ones requiring global creative reasoning further categorised in the construction of an example, the proof of something new and modelling. The results of the analysis showed that 70% of the tasks could be solved using imitative reasoning. Moreover, 15 out of the 16 exam papers could be passed by using only imitative reasoning.

Boesen, Lithner, and Palm (2010) offer another categorisation based on Lithner's (2008). The focus is on tasks present from secondary Swedish national tests. The researchers propose the following categorisation: High relatedness (answer) tasks requiring memory reasoning; high relatedness (algorithm) requiring algorithmic reasoning; local low relatedness demanding local creative reasoning; and global low relatedness where the students are asked to use global creative reasoning. A different categorisation results from the analysis of the reasoning required to solve tasks in the Swedish national tests and teacher-made tests, again in the secondary school context. The tasks are categorised as requiring familiar algorithmic reasoning if there are at least three instances in the textbook where the same solution algorithm is applied. If the task is similar at least to one other task in the textbook where the same solution algorithm is applied, the task is classified as requiring guided algorithmic reasoning. If the task occurs in three instances in the textbooks and the answer required is the same one as presented in the textbook, it is distinguished as memory reasoning. Finally, if the task requires creative reasoning, it is classified as creative mathematically founded reasoning.

Mac an Bhaird, Nolan, O'Shea, and Pfeiffer (2017), using Lithner's framework of creative and imitative reasoning (Lithner, 2008) investigate first-year undergraduate assessment focusing on analysing the opportunities, given to the students, for creative reasoning. The study took part in two Irish universities. The researchers focused on three first-year calculus courses offered to undergraduate mathematics, business and science students. The data they collected were lecture notes, recommended textbooks, assignment and examination tasks. The data were coded following the same system as Lithner (2008) and Bergqvist (2007) to tasks requiring imitative or creative reasoning. The findings showed that there were differences in the distributions of imitative reasoning and creative reasoning tasks. The most opportunities for creative reasoning were in the module offered to undergraduate mathematics students. However, more tasks were demanding creative reasoning in all the modules in the tasks either from the practice, submitted and optional tasks (which were 640) compared to the 50 examination tasks. The results of the analysis of the 50 examination tasks, showed that there was a proportion of 36.4% of tasks with creative reasoning (either local creative reasoning or global creative reasoning). In the other two modules offered to non-mathematics undergraduate students, there was a smaller percentage (10.3%), and these were all classified as local creative reasoning.

Capaldi (2015) collected 243 examination tasks from 18 lecturers of proof-based modules. The lecturers were asked to comment on the students' familiarity with the specific examination tasks. The lectures commented that the students had not previously seen these examination tasks. However, a similar style of tasks was presented to them during the module. Capaldi analysed the examination tasks according to three dimensions: item format, representation, and orientation. This framework is very similar to the one proposed by Tallman et al. (2016). Capaldi in aiming to examine students' familiarity categorised the lower levels of the Bloom's taxonomy as imitative reasoning and the higher level as requiring creative reasoning. The results regarding the representation of the tasks showed 21.4% of tasks and 21.2% of solutions represented theorems and proofs. However, the results from the lecturers' survey showed that the lecturers believed that 77.8% of tasks asked students to evaluate a statement or a conjecture. The lecturers' perceptions are not in accordance with the analysis of the tasks regarding the applied problems. Only four tasks were coded as applied whereas 44.4% of the instructors said that the ability to solve applied problems is important. However, this difference could be explained as the researcher and the instructors could have used the description applied differently. Regarding the requirement to explain their thinking, the lecturers' perceptions were corresponding with the results of the tasks analysis as 58.02% of tasks required explanation and all the lecturers agreed that they frequently ask their students to explain their solutions. Capaldi's framework aims to bridge Anderson et al. (2001) adaptation of Bloom's (1956) taxonomy and Lithner's framework of creative and imitative reasoning.

Focusing on the mathematical aspects of the tasks and the students' position in relation to the tasks, Morgan and Sfard (2016) introduce a framework for task analysis. The framework draws on Systemic Functional Linguistics (Halliday, 1978; Morgan, 2006) and Sfard's theory (2008) of *commognition*.

The goal of this framework is to investigate the changes in the nature of students' participation in the mathematical discourse in the last thirty years, focusing on the public examinations in the UK (GCSE- General Certificate of Secondary Education). The framework (see appendices section 12.5) consists of two components examining the *mathematising* and the *subjectifying.* The mathematising refers to the mathematical objects and the narratives about those objects. Whereas, subjectifying refers to the students and their expected participation in the mathematical discourse. Specifically, different aspects of the discourse are examined in the mathematising forming four categories corresponding to Sfard's (2008) characteristics of the discourse (Morgan and Sfard, 2016, pp. 106-107):

- Vocabulary and Syntax: The specialisation, objectification and the logical complexity of the discourse are examined.
- Visual mediators: The presence of multiple visual mediators and the transition between visual mediators are investigated.
- Routines: The types of actions required by the students regarding the areas of mathematics involved and the characteristics of the procedures of the routines
- Endorsed narratives: The origin and the status of mathematical knowledge are examined.

Regarding the 'subjectifying' aspects of the discourse, the framework examines the student-author relationship and the student's autonomy regarding the path, form, and mode of the solution and the complexity of the solution (Morgan and Sfard, 2016, pp. 108).

This literature review on studies investigating examination tasks illustrates the use of various analytical frameworks with many of them using an adaptation of Bloom's taxonomy (White & Mesa, 2014; Capaldi, 2015; Maciejewski & Merchant, 2016; Tallman et al., 2016) or the MATH (Mathematical Hierarchy Task Hierarchy) taxonomy (Smith et al., 1996; Darlington, 2014). These studies focus mainly on Calculus (Pointon & Sangwin, 2003; Bergqvist, 2007; White & Mesa, 2014; Tallman et al., 2016; Mac an Bhaird et al., 2017) but also look at Algebra (Pointon & Sangwin, 2003; Darlington, 2014) and other mathematical areas (Griffiths & McLone, 1984b; Darlington, 2014; Capaldi, 2015; Maciejewski & Merchant, 2016) which studied a variety of different mathematical areas (e.g., pure, applied and more). The context of the examination tasks is taken to account in some cases (Bergqvist, 2007; Boesen et al., 2010) aiming to provide insights regarding students' familiarity with the examination tasks.

Similar to Griffiths and McLone (1984b), Morgan and Sfard (2016) examine the directions as to the procedure to be followed in the tasks. The former authors examine this focusing on the procedure to be followed in the task. However, Morgan and Sfard (2016) see this as part of the student autonomy in the task solution. There are also other similarities between the qualities jargon from Griffiths and McLone (1984b), and the categories mathematical and logical manipulation, and the vocabulary and syntax aspects of the mathematising component (Morgan & Sfard, 2016).

All the taxonomies take into account tasks that are asking for definitions or theorems and give various names to these: remember and recall (e.g., Tallman et al. (2016)), factual knowledge and fact systems (e.g., Smith et al. 1996), recalling facts (e.g., Pointon & Sangwin (2003)), mechanical (e.g. Galbraith & Haines, 2000), imitative reasoning (e.g., Bergqvist, 2007). In other taxonomies the focus is more on the algorithms and on the level to which the students are asked to engage with a procedure that they are not familiar with (Bergqvist, 2007; Boesen et al., 2010). In my study I am interested in both the students' engagement with mathematical practices (e.g., defining, proving and justifying) and in the way that the tasks designed by the lecturers shape this engagement (e.g., in terms of the directions). I use an adaptation of the Morgan and Sfard (2016) framework, to be presented in section 4.5.4, to analyse the examination tasks of a module which focuses on a variety of mathematical areas. Furthermore, in this adaptation. I provide a dimension which focuses on students' actual engagement with the mathematical discourses and is used to analyse students' scripts.

2.2 Studies of lecturers' perspectives on assessment

The studies that analysed tasks, presented in the section above, rely on the researchers' classification of the tasks based on their own experience of the tasks (e.g., Smith et al., 1996) and the supporting materials in some cases (e.g., the textbooks being used in the analysis) (Bergqvist, 2017). Seeking the lecturers' perspectives on the examination tasks is crucial as they are the ones who design the tasks, taking into account the materials being taught, the previous examinations and the experience they have from students' performance on similar tasks. The lecturers' perspectives on the tasks and their rationale on the way they posed them, provide insight into lecturers' assessment practices but also illustrate which are their expectations from students' engagement with these tasks. I turn now to studies that examine lecturers' perspectives on assessment and specifically on assessment tasks.

A study examining the classification of tasks by lecturers is Schoenfeld and Herrmann's (1982). They investigated the way that undergraduate students and lecturers categorised mathematical tasks, aiming to examine how the different mathematical backgrounds may affect the classification. The researchers analysed the way that 9 mathematics lecturers and 19 undergraduates categorised 32 tasks. The researchers had done an a priori analysis of the tasks regarding their structure into surface or deep. The surface structure involved noticing the mathematical items described in the tasks. Whereas, the deep structure referred to the mathematical principles which are necessary for the solution of the task. Their analysis showed that the two groups had different criteria for the classification of the tasks. The lecturers categorised the tasks in a more consistent way than the students. Also, the criteria for the categorisations were different. The lecturers categorised the tasks according to the deep structure namely the mathematical principles necessary for the solution of the task, whereas the students classified the tasks according to the items described focusing on the surface structure.

In recent years, researchers focused on lecturers' views on examination tasks also taking into account the tasks themselves. Bergqvist (2012), Capaldi (2015) and Tallman et al. (2016) provide insight into the posing of the examination tasks and the lecturers' perceptions. In this section, I focus on the lecturers' perspectives on the tasks and in a later section, I comment on the relationship between the lecturers' perspectives and the classification of the tasks.

Bergqvist (2012) examines six lecturers' views on the factors that they take into account when designing examination tasks. The results show that the lecturers take into account time, student proficiency, prior knowledge, course content, perceived degree of difficulty and students' familiarity with the task, when designing the examination tasks. Also, she investigated lecturers' views on the reasoning that is expected from the students when solving examination tasks, using Lithner's (2008) distinction in imitative and creative reasoning. Lithner's distinction was presented to the lecturers, and then they were asked to engage with the classifications of the tasks. The lecturers agreed that the majority of the exam tasks demanded mostly imitative reasoning and argued that otherwise, the exams would result in large failing rates.

Following Bergqvist's results (2012) are the findings from Iannone and Simpson's study (2015). Iannone and Simpson (2015) explore lecturers' views on assessment, in the UK context. Using an online survey, they asked participants to comment on the different assessment methods and their preferences regarding the method that students' achievements are assessed; and whether the method allowed them to make a distinction between the students in good and poor mathematicians. Fourteen lecturers from two different UK universities completed the questionnaire. The results showed that 86% of the lecturers preferred closed-book examinations and 79% perceived them as the method that allows a distinction between good and poor mathematicians. The researchers followed the online-survey with interviews with some of the participants, where they examined in more detail the reasons behind the preferences observed in the questionnaire. Here, I focus only on the lecturers' views on the closed-book examinations. The lecturers who were interviewed commented on the potentiality of the closed-book examinations regarding the assessment of understanding. However, this potential depends on the tasks that are posed in the examination. The lecturers commented on the examinations currently assessing more memory and routine skills and conceptual skills, as the weaker students might not be able to answer the tasks. The results of this study illustrate the need to inquire in more depth about the lecturers' views on closed-book examinations and specifically on the specific tasks that they use in those examinations.

Using questionnaires Capaldi (2015) and Tallman et al. (2016) focus on lecturers' perspectives on examination tasks. Capaldi (2015) asked 18 lecturers to complete a survey where they ranked the importance of specific skills and commented on the frequency that those skills are assessed in the exams. The skills were knowledge of definitions or theorems; fluency in mathematical symbols; demonstration of understanding through explanations or providing examples; understanding proof structure and logic; ability to solve applied problems (p.114-115). The results from the survey show that 77.8% of the lecturers thought that they frequently ask students to evaluate a statement or a conjecture in an examination. Regarding the solution of applied problems, 44.4% of lecturers comment that the ability to solve applied problems is important. Finally, all the lecturers believed that students are asked to explain their thinking during the examinations. However, as seen in section 2.1 the results from the task analysis illustrate that this is not the case (Capaldi, 2015)

Similarly, Tallman et al. (2016) examined lecturers' views regarding the focus of the task on a concept or a procedure and whether the task required students to explain their answers. The lecturers claim that they usually require their students to explain their thinking and think that the proportion of tasks focusing on concepts is the same as the tasks focusing on procedures. However, the results of the analysis from the tasks show that there is a discrepancy between what the lecturers believe and what happens in the tasks.

The literature reported in this section examines lecturers' perspectives on assessment either in general terms of assessment (Iannone & Simpson, 2015), the focus of the task (Schoenfeld & Herrmann, 1982; Bergqvist, 2012; Capaldi, 2015; Tallman et al., 2016), the justifications required from the students (Tallman et al., 2016) and the aspects they take into account when designing the tasks (Bergqvist, 2012). The methods that the researchers have used in these studies are either categorisation of tasks (Schoenfeld and Herrmann, 1982), interviews (Bergqvist, 2012) or surveys (Iannone &

Simpson, 2015; Capaldi, 2015; Tallman et al., 2016). In my study, I am interested in finding out the lecturers' perspectives on the tasks taking also into account their expectations regarding the engagement with university mathematics, and thus I used in-depth interviews which were triggered by the tasks they designed for the final year examination of a first-year module.

2.3 Studies dealing with the transition from secondary to university mathematics

In the first year of their studies, students are faced for the first time with university mathematics, which is different from secondary mathematics. The difference between secondary school mathematics and university mathematics varies between countries. However, there are aspects of this transition which seem to be common in many different contexts (De Guzmán, Hodgson, Robert & Villani, 1998; Hoyles, Newman & Noss, 2001; Gueudet, Bosch, DiSessa, Kwon, & Verschaffel, 2016). Students' transition from secondary school to university has attracted researchers' attention over the years. Various angles and perspectives are utilized to shed light and increase the community's insight in the complex nature of the transition from school to university but also within the university years (Gueudet et al., 2016).

Klein, also, discusses the discontinuity between university and school mathematics (Klein 1908/1932 as cited in Winsløw & Grønbæk, 2014). Specifically, Klein discusses the 'double discontinuity' that teachers face. The first discontinuity occurs when moving from school to university and the second when moving from university to teaching practice. The studies reported in this section and of my work focus on the first discontinuity faced by students when they initially enter university. Many researchers report this first discontinuity (e.g., Gueudet, 2008; Nardi 1999; 2008; Clark & Lovric, 2009) and discuss the difficulties that students face when they move from the secondary school context to university.

A seminal work on transition by De Guzmán et al. (1998) involved three universities in three different countries: namely, France, Spain, and Canada. The researchers provide three categories of difficulties in the transition from secondary to university: sociological/cultural, epistemological/cognitive and didactical. In my review of their work, I focus on the epistemological/cognitive and didactical difficulties. Researchers report that the criteria for what counts as mathematical activity in secondary and university are different. In university, mathematical activity is more rigorous, abstract and formalised. Students are introduced to new abstract notions, and they are asked to revisit other notions, as noted in the following quote: "other concepts are acquiring a different status when passing from one education to another" (De Guzmán et al. 1998, p. 753)

Gueudet (2008) offers a similar categorisation concerning the discontinuities between secondary and university. In her review on studies focusing on transition, she categorises studies examining the thinking mode and organization of knowledge, proofs and mathematical communication, didactical transposition and didactical contract. The issues reported are initially mainly focused on the individual and they further expand to concerns regarding the institution.

The studies which I draw upon in this section, discuss specific issues regarding aspects of the enculturation to university mathematics. I focus on studies examining proof and definitions (Moore, 1994; Nardi, 1996; Alcock and Simpson, 2017), rigour and symbolism in university mathematical texts (Chellougui, 2004a, 2004b; Mamolo, 2010; Epp, 2011; Corriveau & Bednarz, 2017) and lecturers' perspectives on transition (Nardi, 1999; Iannone & Nardi, 2007; Hong, Kerr, Klymchuk, McHardy, Murphy, Thomas & Watson, 2009; Corriveau, 2017; Jablonka, Ashjari & Bergsten, 2017; Ní Shé et al., 2017).

Proof and formal definitions of mathematical concepts is an essential aspect of university mathematics (Gueudet, 2008) and is one of the main shifts that students are asked to make. Students' difficulties with the practices of defining and proving are reported by Moore (1994). Specifically, he mentions seven difficulties the students are faced with while engaging with proof.

> "*D1. The students did not know the definitions, that is, they were unable to state the definitions.*
>
> *D2. The students had little intuitive understanding of the concepts.*
>
> *D3. The students' concept images were inadequate for doing the proofs.*
>
> *D4. The students were unable, or unwilling, to generate and use their own examples.*
>
> *D5. The students did not know how to use definitions to obtain the overall structure of proofs.*

D6. The students were unable to understand and use mathematical language and notation.

D7. The students did not know how to begin proofs." (Moore, 1994, pp. 251-252).

Adding to Moore's results, Gueudet (2008) notes that proving is something new for the students coming to university in many countries and highlights the different role that proofs take in university mathematics "they are central in the building of the university mathematical culture, because they indicate methods, and also what requires justification or what does not. (Gueudet, 2008, p. 247).

Mathematics at university is presented in the structure of definition, theorem, and proof. This structure illustrates the focus on the abstractness of the nature of university mathematics but also the rigorous nature of it (Engelbrecht, 2010). The difference of proofs at the tertiary level is also discussed by Selden (2011). Particularly, she notes the precision, conciseness, and complexity of the structure of proof at university level compared to the ones that students are asked to engage with in secondary school. The use of definitions and other theorems in proofs, selecting relevant representations and interpreting the goal of the proof, the statement to be proven, are also reported as challenges that students are facing. Lithner (2011) also comments on the level of abstraction as a crucial difference between school and university mathematics

"A general qualitative step in this transition is with respect to an increased level of abstraction, a difficult transition from intuitively-based concepts to formal definitions." (Lithner, 2011, p. 297)

These shifts are needed in the transition between school and university regarding abstractness. These shifts are being explored in the setting of small group tutorials (Nardi, 1996) and interviews with lecturers (Nardi, 1999; Iannone & Nardi, 2005; Nardi, 2008). Nardi (1996) studies twenty first-year mathematics undergraduates in small group tutorials from a prestigious university in the United Kingdom. Focusing on a variety of mathematical areas such as Analysis, Calculus, Linear Algebra and Group theory the students were interviewed regarding the new concepts they encountered in these areas. Results showed that students faced difficulties with the abstract and formal nature of the mathematics. Nardi (1999) interviewed three lecturers examining their perspectives on students' difficulties in the first year namely: students' difficulty with writing and the influences from secondary school writing and the writing they are exposed at university; and, difficulties with between school and university mathematics but also with inter (between different) and intra (within the same) university course conflict.

Furinghetti, Maggiani, and Morselli (2013) study students' perspectives on the transition with a specific focus on proving. They used a variety of methods including questionnaires, interviews, problem-solving, and proving activities. They administered a questionnaire to 50 first-year students in a Mathematics department and invited students to participate in the interviews and the activities. The results show that students believe that at university, the focus is less on procedures compared to secondary school. Additionally, they recognized that concepts which have been presented and used at secondary school are revisited in a more precise and abstract way.

Aiming to further our insight into the defining routine of the mathematical community, Alcock and Simpson (2017) focus on increasing and decreasing infinite sequences. They studied the relationships between mathematical concepts, justifying their meanings and classifying consistently with formal definitions. The participants of the study were 132 first-year undergraduate students in a UK university, either studying mathematics or natural sciences with an important component of mathematics. The participants of the study were given two tasks. One of them asked for a classification of fifteen sequences as increasing, decreasing, both or neither and the other was focused on defining. However, this task varied from group to group. In one group, the students were given the definitions of increasing and decreasing sequences, and they were asked to examine these, in the other groups they were asked either to define these mathematical concepts or to explain their meanings. The results show that there is a significant correlation between the scores of the students between the two different tasks. However, this correlation is weak. The results showed that the groups that were asked to classify second had better results regardless as to whether these were asked to explain or define. The results seem to indicate that the students who were asked to explain had better results than the ones who were asked to study or define.

> "[D]efinitions are often not well remembered and not spontaneously invoked, and that prompts to consider definitions do not result in ideal classifications." (Alcock & Simpson, 2017, p. 17)

In her study Chellougui (2004a, b as cited in Gueudet, 2008, pp. 244-245) investigates the use of logical quantifiers "for all " and "there exists" in textbooks and teachers' practices in the first year of university. The observed formulations illustrate a variety of uses by the same lecturer or textbook. The quantified relations either remain implicit, are strictly used or used incorrectly as abbreviations.

Mamolo (2010) examines cases where a symbol has multiple meanings in different mathematical areas. Specifically, the different meaning of the symbols other mathematical areas is investigated and how this new use can be inconsistent with their use in the other context. The author posits the importance of the awareness of the meaning of the symbol, the meanings that this symbol may take in other mathematical areas and how it is crucial to know when to use this symbolism by examining the context. This article focuses on the symbol of addition "+" and how its use is very different in the context of modular arithmetic and transfinite arithmetic. Similar to this study the different word use "quotient" and "divisor" are examined by Zazkis (1998) and Epp's study (2011) which observes that variables have various uses in mathematics which could potentially create difficulties for students' transition to advanced topics.

The use of symbolism in the university is also studied by Corriveau & Bednarz (2017). Using Hall's (1959) theory regarding culture, the researchers explore the transition as a change in the mathematical culture and focus on symbolism and its use by examining the ways of doing mathematics from three lecturers and three secondary school teachers. The analysis revealed that in secondary school, teachers' ways of doing maths could be described in three characteristics: progressive symbolism, transparent symbolism, and chosen symbolism. Progressive symbolism describes the use of an intermediate symbolism by the teachers and their work with this symbolism or work with a familiar symbolism which is then later transformed. Transparent symbolism notes the consistent use of specific symbols linked to the graphic register or choosing symbolism relevant to the context of the task. Finally, chosen symbolism illustrates that the teachers chose the symbolism used in each case. Three themes emerged from the analysis of the ways of doing mathematics of the lecturers: an explicated symbolism and a determined exterior symbolism. Explicated symbolism describes the translation into symbolism and the translation of symbolism, essentially the lecturers explain and specify the meaning of symbols. Determined exterior symbolism shows use of pre-given symbolism, this symbolism varies according to the circumstances, and there is flexibility in the use of the symbols.

Transitions between different mathematical areas are also discussed by Campbell (2006). The focus is on the transition from arithmetic from whole numbers and arithmetic on rational numbers.

> *"relatively recent development in the history of mathematics that has logically subsumed whole (and integer) numbers as a formal subset of rational (and real) numbers. This development appears to have motivated and encouraged some serious pedagogical mismatches between the historical, psychological, and formal development of mathematical understanding"* (Campbell, 2006, p. 34).

This change between different mathematical areas is also noted by Niss (1999) which discusses the role of the mathematical domain.

> *"For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student."* (Niss, 1999, p. 15)

Focusing on the teacher and lecturer perspectives regarding the transition from secondary to university Hong, Kerr, Klymchuk, MchArdy, Murphy, Thomas and Watson (2009) study lecturers' and teachers' responses to differences between school and university focusing on Calculus. A questionnaire was used, and data collected included responses from 178 teachers and 26 lecturers in New Zealand. The teaching approach is more formal at the tertiary level compared to secondary. The procedures are more emphasized at school whereas at university the focus is more on concepts, mathematical thinking and problem solving. The authors claim that the results from the study show that there is a lack of communication between teachers and lecturers. There are some similarities and differences in the results from the teacher and lecturer data but also some differences illustrating that there is this lack of communication between the two groups.

The different focus between secondary and university mathematics is also noted by Winsløw (2008) (as cited in Gueudet, 2008). Using Chevallard's (1991) Anthropological Theory of Didactics, Winsløw discusses the secondary to university transition concerning two specific shifts. The focus of the mathematical activity in secondary school is mostly on the practico-technical block whereas in university the technologico-theoretical block is deemed more important. The other shift concerns the tasks at the university. The elements which were the technologico-theoretical block of a mathematical organization become the practico-technical block of a new mathematical organization.

Ní Shé et al. (2017) examine lecturers and first-year students' views on mathematical concepts or procedures that the first-year students find difficult. The results from the student and the lecturer survey do not agree. This paper reports the results from two surveys. One was given to 460 first-year undergraduate students. These were students attending undergraduate mathematics modules from four universities in Ireland and another one given to 32 lecturers in all the universities in Ireland. The students were asked to list the topics that they found most difficult but also to comment on whether they found the concept or the procedures more difficult. The results of the lecturer and student survey did not agree on all the topics. The majority of students and lecturers reported that Calculus (integration and differentiation) and manipulating and using logarithms where the topics that they found most difficult. Most of the lecturers, 25 out of the 32, also believed that students have difficulties with basic algebra (e.g., manipulating a formula and solving equations) whereas the students did not feel that they faced difficulties with this topic with only ten students from the 460 mentioning these types of difficulties.

Aiming to examine what is considered as a legitimate mathematical activity, Jablonka, Ashjari, Bergsten (2017) interviewed sixty engineering students from two Swedish universities and eight of their lecturers. The participants of the interview were given excerpts from textbooks. These were from four mathematical textbooks, and the excerpts included proofs or applications of mathematics. The participants of the study were asked to classify these as more or less mathematical and why. The theoretical perspective adopted by the researchers is Bernstein's theory regarding pedagogic discourse. Specifically they focus on utilising the idea of recognition rules

> "outcomes of principles of knowledge classification that reflect dominant power relations, which are described in terms of the nature of relations between categories, whether these categories are between agencies, between agents, between discourses, between practices (Bernstein 1996, p. 20)".

The results regarding the ranking of the texts by lecturers and students are roughly the same. The results show students' awareness of the criteria for mathematical rigour. The content of the text influenced students' choices, and they did not consider the application side of mathematics. Also, the texts with many "different mathematical representations, technical terms and specialised mathematical symbols" (Jablonka, Ashjari & Bergsten, 2017, p. 90) were ranked as more mathematical. The students also talked about the way that the argument was presented and the accessibility of the text.

Gueudet and colleagues (2016) offer a review of the literature in transitions. One of the transitions that they examine is the secondary to university mathematics. They comment on the difference between university and secondary school mathematics saying that:

> "[I]nvestigations characterize university mathematics as being more focused on the theoretical organization of mathematical content, the foundations of knowledge, and presenting proofs and theorems as tools to approach problems. In contrast, secondary mathematics stresses the production of results and the practical aspect of mathematical activities, assigning a more "decorative" role to axioms, definitions, and proofs" (Gueudet et al., 2016, pp. 19-20).

Transitions from secondary to university involve quite a few shifts in the practices of mathematics. These involve revisiting mathematical objects, defining them and proving them, and the use of formal mathematical language and symbolism (Chellougui, 2004a, 2004b; Mamolo, 2010; Epp, 2011; Corriveau & Bednarz, 2017). Additionally, shifts between different mathematical contexts are required (Nardi, 1999; Niss, 1999; Campbell, 2006). As these shifts are various and research has shown that students experience difficulties, I aim to examine these in students' responses in closed-book examination tasks aiming to observe students' actual participation to university mathematical discourses.

My research questions are:

> **R.Q.1** What are the discursive characteristics of the examination tasks?
>
> **R.Q.2** What are mathematics lecturers' perspectives on the examination tasks and their expectation from the students' engagement with the university discourse in the closed-book examination setting and how are these perspectives enacted in the formulation of the examination tasks?
>
> **R.Q.3** How different are university mathematical discourses from secondary school mathematical discourses and what commognitive conflicts can be observed as a result of those differences in students' scripts?

Chapter 3. Theoretical Framework

In this chapter, I discuss the theoretical framework of my study. I initially discuss the sociocultural perspectives which are recently prominent in mathematics education research. Then, I present the discursive approach that my study adopts: Sfard's (2008) theory of commognition and the uses of this framework in university mathematics education studies. I then delve into more details in the aspects of the framework that I am using in my research; and, offer examples related to university mathematics and the shift in the discourses from the secondary to university mathematics, drawing on studies using commognition. Finally, I discuss the potential of this framework used in the context of my study.

3.1 Sociocultural perspectives and Sfard's theory of Commognition

In recent years, sociocultural perspectives are widely used in mathematics education. This shift is also illustrated in the breadth of the use of sociocultural frameworks in university mathematics education research (i.e., Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016; Nardi, Biza, González-Martín, Gueudet & Winsløw, 2014). The basic tenet of the sociocultural perspectives is that "*patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual*" (Sfard, 2008, p. 78, italics in the original). The individual is seen "as a participant in established historically evolving cultural practices" (Cobb, 2006, p. 151).

One of the sociocultural frameworks are the discursive approaches. In the discursive approach, the focus is on language and communication. Learning is viewed as communicating in a discourse of a specific community. Mathematics are seen as a discourse about mathematical objects (e.g., Moschkovich, 2002; Sfard, 2007). Thus, the investigation focuses on students' discourse and the shifts that this discourse undergoes in order to become mathematical. This means "becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors." (Kieran, Forman & Sfard, 2001, p. 5). The research focus is on "the change of one's ways of communicating with others" (Kieran *et al.*, 2001, p.6). This change in the practices of the discursants is influencing the process of communication itself. By embracing the discursive approach, researchers can discuss the changes in the communication of the individual learner at the same time as the changes that arise in the practice itself. Discursive approaches are being used by many researchers (Ryve, 2011) and Sfard's (2008) theory of commognition is widely used in a variety of contexts (Tabach & Nachlieli, 2016) and increasingly in the university level (Nardi *et al.*, 2014). In the next part of the chapter, I present the basic tenets of the commognitive framework.

3.2 Main tenets of Commognition

Sfard's theory adopts the participationist perspective which deals with the evolution of human activities and their growth in complexity (Sfard, 2015, p. 130). Influenced by Wittgenstein, Vygotsky (1978) and Lave and Wenger's (1991) works, Sfard defines thinking as "an individualised version of *interpersonal communication"* (Sfard, 2008, p. 81 italics in the original) and connects cognition and communication noting that they are "different manifestations of basically the same phenomenon" (p. 83). This connection of *comm*unication and co*gnition* creates the neologism *commognition*.

The "[d]ifferent types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (p. 93) are the discourses, one of the basic definitions of this theory. The rules of the discourse are classified in *object-level* rules, "narratives about regularities in the behavior of objects of the discourse" (p. 201) and *metarules* which "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (p. 201). Learning is considered as the evolution of the learner's discourse and can occur either at *object-level* or at *meta-level*. In the former, the existing discourse is developed and in the later a change occurs in the *meta-rules* of the discourse. The *meta-level* learning can be *horizontal* in which case separate discourses are combined "into a single discourse by subsuming them to a new discourse" (Tabach & Nachlieli 2016, p. 302) or *vertical* where "the existing discourse [is combined] with its own meta discourse" (ibid.). *Meta-level* learning occurs primarily through *commognitive* conflict: "the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules" (Sfard, 2008, p. 161). I explore this theoretical construct in more detail in the following section (3.2.1).

In commognition doing mathematics is the active engagement with mathematical discourse. The mathematical community comprises from people practising the mathematical discourse. Consequently, learning mathematics is becoming a member of a mathematical community and thus becoming fluent in the discourse of this community. However, the learner is faced with a paradox: to become familiar with the discourse they should participate in the discourse. However, this familiarity can only be accomplished with their participation in the discourse (Sfard, 2008, p. 130).

Mathematical discourse has four interrelated characteristic features: *keywords*, *visual mediators*, *routines* and *endorsed narratives*. *Keywords* include mathematical terminology and words used in everyday discourse with a special meaning for mathematics, for example function, proof, set, probability and equivalence relation. *Visual mediators* are the symbols (numerical or algebraic) and graphs created specifically for mathematical communication. *Routines*, the set of *metarules* "that describe a repetitive discursive action" (Sfard, 2008, p. 208) such as defining and proving. Finally, *narratives* are "any sequence of utterances, spoken or written, framed as a description of objects of relations between objects, or of activities with or by objects" (ibid. p.223). A *narrative* is called *endorsed narrative* "if it can be derived according to general accepted rules from other endorsed narratives" (ibid., p. 223). Theorems and definitions are examples of *endorsed narratives*.

The *metarules* that define the *routines* are the *applicability conditions,* the *procedure* of the routine and the *closing conditions*. The *procedure* of the routine constructs the *how* of a routine "a set of metarules that determine, or just constrain, the course of the patterned discursive performance (the course of action or procedure, from now on)" (Sfard, 2008, p. 208). The *applicability* and *closing* conditions establish the *when* of a routine "a collection of metarules that determine, or just constrain, those situations in which the discursant would deem this performance as appropriate" (ibid., p. 208). The *how* of a routine is easily learned and retained, but the *applicability* and *closing* conditions of a routine "may be a lifelong endeavor" (ibid).

The routines of discourse are categorised in *explorations*, *deeds* and *rituals. Explorations* are routines that produce or substantiate an endorsable narrative (p. 224). *Deeds* are "a set of rules for a patterned sequence of actions that ... produce or change objects" (p. 237) and *rituals* are "sequences of discursive actions whose primary goal is ... creating and sustaining a bond with other people" (p. 241). The *closing* conditions are the crucial difference between explorations, deeds, and rituals. A ritualistic participation in the discourse is "a matter of rote implementation of memorized routines" (Sfard, 2016, p. 44) whereas an explorative participation involves construction and substantiation of the narratives about mathematical objects.

The exploration routines are further divided in "construction, which is a discursive process resulting in new endorsable narratives; substantiation, the action that helps doers of mathematics decide whether to endorse previously constructed narratives; and recall, the process one performs to be able to summon a narrative that was endorsed in the past" (Sfard, 2008, p. 225).

The objects that are involved in the mathematical discourse are the primary, simple and compound discursive objects. Primary objects are "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p. 169). Examples of primary objects include the geometrical shapes 3D and 2D. The simple discursive objects "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artifact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization." (Sfard p.169). And the compound discursive objects "arise by according a noun or pronoun to extant objects, either discursive or primary" (p. 170) by saming, encapsulating or by reifying (ibid.). Saming occurs by connecting one signifier to many realizations. Realizations are "perceptually accessible object[s] that may be operated upon in the attempt to produce or substantiate narratives" (p. 154) about a signifier (a word or symbol). In the process of encapsulation, the narratives about multiple objects are replaced with narratives about one signifier which encompasses the multiple objects. The association of a noun with a discursive process is called reification. Essentially, narratives about the process are replaced with narratives about the object.

Commognition is a learning theory that explains the human development of discourses and is very specific for the mathematical discourse. This provides the researcher with a toolbox for analysis. The oral or written communications are the focus of the commognitive research and the characteristics of different discourses are highlighted (the discourses on mathematical topics and also students' and lecturers' discourses). This discursive framework also puts forward the opportunities that the differences between the various discourses bring. In the next section, I delve into more details on the theoretical constructs of commognitive conflict and the discontinuity between discourses.

3.2.1 Commognitive conflict and discursive discontinuity

Key ideas in my study are the notions of commognitive conflict and discursive discontinuity. In this section, I discuss these and their relation. Commognitive conflict is defined by Sfard as

> "the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules" (Sfard, 2008, p. 161).

As noted above, commognitive conflict can occur when a symbol or a word is being used by a participant in the discourse and it is incompatible with the discourse that is being expected. In the case of responding to a question on discrete functions, a student might be using words or symbols that relate to the discourse of continuous functions (e.g. continuous, differentiable). This use creates a conflict between the student who responds to the task and the lecturer who reads the solution and expects the answer to the task to have narratives with words relating to discrete functions. The commognitive conflict here lies in incompatible word use. The expected student engagement is with discrete functions, and the actual engagement is with the discourse of continuous functions. The student here conflates the two discourses illustrating a problematic meaning-making between the discursive object of continuous and discrete function.

A similar situation could occur with the use of logical symbols, namely implications ($\Rightarrow$), equivalences ($\Leftrightarrow$) and quantifiers ($\exists, \forall$). These symbols indicate relationships between mathematical objects involved in the narratives. The meaning of these symbols in the university mathematical discourses are very specific, and the students who are now becoming members of this community are learning to use the symbols and the students could be using them ritualistically. This could be observed in their written responses in instances where the use of the symbol illustrates problematic meaning-making of the relationships between mathematical objects. For example, a student might be using an implication instead of equivalence in the narrative they produce. This illustrates conflation between the logical statements “if … then”, which is the implication ($\Rightarrow$), and “if and only if”, which is the equivalences ($\Leftrightarrow$).

Another instance, where commognitive conflict can occur is in the rules of the discourses. In this case, the students could be using routines that are incompatible with the mathematical discourses that they are asked to use in that question. For example, when proving a statement, initially the students might be asked to examine whether the statement is correct by providing an example and in some cases at secondary school they would stop there. However, at the university level, proving is much more rigorous and providing just an example is not accepted by the lecturers as endorsable narrative.

Tabach and Nachlieli (2016) discuss meta-level learning. “Meta-level development of a discourse may be of a horizontal or vertical nature” (Tabach & Nachlieli, 2016, p. 302). Manifestations of commognitive conflict can occur when two separate discourse are combined in a new discourse (e.g., discourses about functions and discourses about integers combined to form discourses about discrete functions) signalling a commognitive conflict at a horizontal level. Commognitive conflict occurs at the vertical level when a discourse is combined with its own meta-discourse (e.g., the discourse on discrete and continuous random variables combined with the discourse on random variables). In my analysis, I explore both manifestations through discussing the various mathematical discourses that are being combined at a horizontal level or the ones that are subsumed at the vertical level.

For commognitive conflict to occur, two discourses should be present which are incompatible either regarding the rules or the mediators used. This incompatibility signals that there is a discontinuity between the discourse that the learner is asked to engage with and the one he or she engages with. The discontinuity between discourses can occur between the secondary school and university mathematical discourses, where the rules of the discourses change, and the use of word and symbol becomes more specialised. This discontinuity can also occur between various mathematical discourses within university mathematics. For example, at university mathematics, the students are exposed to multiple mathematical discourses and they are asked to engage with multiple ones at the same time (e.g., discourse of integers and discourse of reals). Also, various mathematical areas which to the students might seem unrelated are connected between them (e.g., the discourse of continuous random variables in Probability with the discourse of Calculus). These connections between the discourses occur both at a horizontal and vertical level depending on the nature of the discourses. However, for this meta-level learning to occur students have to go through commognitive conflicts.

Students should be experiencing commognitive conflicts during their studies and their lecturers with their teaching practices are aiming to help them to overcome these. Initially, the students, when coming to university or generally when being faced with a new mathematical discourse they imitate their lecturers' actions when engaging with the discourse. Thus, they present a ritualistic use of procedures and visual mediators of discursive objects. However, this ritualistic use changes when the students become more experienced in the discourse and becomes an exploration. It is important to note that some rituals might remain rituals until a commognitive conflict occurs which would illustrate to the discursant that the *how* or the *when* of the routine is no longer accepted by the other participants of the discourse.

In the next section, I discuss the characteristics of the school and university mathematical discourses.

3.2.2 University and school discourses in the context of this study.

Prior to discussing the university and school discourses, I need to note that these differences exist between institutions and countries. However, studies regarding transition have pointed out before that there are similarities in the aspects of the transition in many contexts (e.g., De Guzmán et al., 1998; Hoyles, Newman & Noss, 2001; Gueudet et al., 2016). De Guzmán and colleagues (1998) examined transition in France, Spain, and Canada. A few years later Hoyles, Newman, and Noss (2001) discuss transition in the United Kingdom. More recently, Gueudet and colleagues (2016) survey international studies that examine transitions generally, but also secondary to university mathematics.

The commognitive conflicts that I present in my study are embedded in the particular context of my study. However, such commognitive conflicts may also occur in other institutions. Additionally, other commognitive conflicts might arise in other institutions which are inherent to the particular educational context.

In reviewing articles using the commognitive framework at the university level, Sfard comments that the university mathematical discourses are extremely objectified, rely on rules that promote analytical thinking and is rigorous (Sfard, 2014, p.200). Similar observations are made by De Guzmán et al. (1998) regarding the mathematical activity that is expected at the university level. The students are asked to engage with a mathematical discourse that is more rigorous than the one they are used to engage with, and it is more formalised and abstract. The mathematical objects are revisited at university level and they are presented in a more abstract and rigorous way than the way that they were presented in secondary school. These characteristics described above have to do both with the rules and the mediators of the discourses.

The studies reported in section 2.3 deal with the transition and focus on various aspects of the transition from secondary school to university. Each one of them can be viewed as providing characteristics regarding the discourses of the secondary and university discourses. Some of them describe the differences with particular practices of the mathematical community which are proofs and definitions (Moore, 1994; Nardi, 1996; Alcock and Simpson, 2017) and others focus on both the visual mediators and the rules regarding the construction of narratives mainly examining the symbol use and the precise nature of the narratives being used at university level (Chellougui, 2004a, 2004b; Mamolo, 2010; Epp, 2011; Corriveau & Bednarz, 2017).

From the above studies, the characteristics of the university mathematics discourse are narratives that are mainly objectified and use more symbolism compared to word use. Now, regarding the rules of the discourses, a common theme is that the discourse is more rigorous than the secondary school. This illustrates the discontinuity between the university and secondary school mathematical discourses. At school the students are producing narratives that are not so reliant on symbolism, their narratives are not necessarily logically connected with the logical symbols used at the university level, and the word use might be routine driven rather than object driven. When entering the university, the endorsement rules of the mathematical discourses change. The lecturers now expect students to engage with the rigour and abstractness of the university mathematical discourses. That is also visible in the tasks that students are being asked to engage with. For example, the results from the analysis between the A-level secondary (Advanced level) examination tasks and the first-year university tasks in Darlington's work (2014) illustrate that at university students are being asked to engage with more tasks requiring them to justify, interpret, imply, conjecture, and evaluate.

In the next section, I refer to studies utilising the commognitive framework at the university level, which also highlight the relevance and importance of its use in the university mathematics education research.

3.3 The commognitive framework in university mathematics education research

Sfard's theory of commognition is gaining more and more attention from the mathematics education community (Sfard, 2012; Morgan & Sfard, 2016) and at the university level (Nardi et al., 2014; Biza et al., 2016)). As mentioned in the previous section the focus of the commognitive framework is thinking as communicating. Thus, the communication and the participants of the communication are the central focus of studies adopting this approach. The communication is not restricted to verbal communication, but it also includes written form. The studies that follow either focus on verbal communication or written communication. Another focus that they have is that they examine the discourse of students (Ioannou, 2012; 2016; Kjeldsen and Blomhøj, 2012; Park, 2013; Ryve *et al*., 2013; Remillard, 2014; Biza, 2017); the differences between discourses of students and lecturers (Bar-Tikva, 2009; Stadler, 2011; Güçler, 2013; 2016; Nardi, 2014; Cooper and Karsenty; 2016) and the discourses of lecturers (Viirman, 2014; 2015; Park, 2015; 2016).

Students' *object-level* and *meta-level* learning of group theory is examined by Ioannou (2012). He reports the *commognitive* conflicts the students experience in their study and the changes in their learning approaches. Focusing on students' engagement with a mathematical task on commutativity Ioannou (2016) forms two categories. In the first, students' engagement illustrates problematic meaning-making at the object level whereas in the second category at the meta-level.

Students' meta-level learning is the focus also of Kjeldsen and Blomhøj study (2012), who, using historical sources, aim to promote meta-level learning. The researchers report results from two projects and analyse them examining the undergraduate students' reflection about *meta-rules* of the discourse. As "meta-discursive rules affect how participants in the discourse interpret the content of the discourse" (Kjeldsen & Blomhøj, 2012, p. 330), the researchers use historical sources illustrating the changes in the mathematical discourse and the creation of new meta-rules. Through the examination and study of the history of mathematics the authors that students notice that "first of all, there are meta-rules that govern the narratives of mathematical texts; second, that such meta-rules have characteristic properties; and third, that rules of the discourse of the sources are different from those that govern the narratives of contemporary textbook versions of mathematical analysis" (Kjeldsen & Blomhøj, 2012, p. 346). Sfard also illustrates the potential of the mathematical discourse in the written text, in the following quote:

> "Mathematical discourse, especially when frozen in the form of a written text, can be seen as a multilevel structure, any layer of which may give rise to, and become the object of, yet another discursive stratum" (Sfard, 2008, p. 129)

The object-level learning of the students is the focus of Park (2013). Her analysis of the students' discourse on the derivative of a function focuses on the keywords and the visual mediators. Ryve, Nilsson & Pettersson (2013) have a similar focus. They investigate the communication that takes place between university students working with proof by induction. Their data analysis shows the importance of visual mediators and keywords for effective communication. Effective communication occurs when "the different utterances of the interlocutors evoke responses that are in tune with the speakers' met discursive expectations" (Sfard & Kieran, 2001, p. 49). Students' engagement with the proof construction is the focus of Remillard (2014) who studies nine undergraduate students' discourse. She notes issues that hindered the development of the interlocutors' discourses in the small group discussion. Furthermore, she notes the discursive entry points; potential moments where the students' discourses could develop upon being challenged by the intervention of an expert. These discursive entry points and their potential are also noted in the analysis of a low-lecture observation in Nardi (2014). In the low lecture content, the lecturer, an expert participant of the mathematical discourse, has to choose the discursive entry points carefully.

The students' discourse is also studied by Bar-Tikva's (2009). The focus is on students' engagement with the discourse in a teaching episode where they are asked to comment on the validity of a proof. The students' discourse is closely connected with the secondary mathematical discourse and it is not rigorous enough. Bar-Tikva proposes that the teacher's position in this development is to make explicit the change of the meta-rules of the previous discourse to meta-rules of the new discourse. And the learner's role is to accept these changes and from a "discourse-for-others", the discourse becomes a “discourse-for-oneself”. The students should accept these rules of the discourse prior to being able to use them, themselves. This shift is also studied by Stadler (2011) focusing on students solving a task with the help of their lecturer. The presence of two different mathematical discourses is highlighted, the *school* mathematical discourse drawn upon by the students, and the *scientific* mathematical discourse used by the lecturer. The difficulties in the shift are discussed in Güçler (2013). Her analysis notes that the difficulties with students’ engagement with the university discourse occurred in moments where the shifts in the lecturer’s discourse were implicit.

As mentioned earlier, learning occurs as discourses shift. Undergraduate students come to the university being participants of the school discourse and aiming to become participants of the mathematical discourse at the university level. The distinction between the two discourses depends on the context of the study. There are differences between school discourse in the UK context and the US context for example. My study is based on the UK context and I try to illustrate in the next parts the differences between secondary school and university discourses in this context. As Sfard notes “there are important differences between construction and substantiation routines practised in colloquial and literary mathematical discourses, and these routines change again in the transition from school discourse to the scholarly discourse of mathematicians.” (Sfard, 2008, p. 225). In her later writing she underlines characteristics of the university mathematical discourse: “first, this discourse’s extreme objectification; secondly, its reliance on rules of endorsement that privilege analytic thinking and leave little space for empirical evidence; and thirdly, the unprecedented level of rigour that is to be attained by following a set of well-defined formal rules.” (Sfard 2014, p. 200).

In their first year of study at university in the UK, students are asked to shift their discourse into the university mathematics discourse. The latter discourse is particularly different in terms of keywords, visual mediators and meta-rules. For this shift to occur the students may experience *commognitive conflicts*. This shift of discourses from secondary school to university is the focus of some of the studies mentioned in the previous section. Specifically, the mathematical discourse used by undergraduate students while solving a task with the help of their lecturer is examined by Stadler (2011). The analysis emphasizes the presence of the *school* mathematical discourse, which the students draw from, and the *scientific* mathematical discourse used by the lecturer. This difference between students' discourses and lecturers' discourses is also emphasized by Güçler (2013). Her analysis emphasizes that the students face difficulties when the shifts in the lecturer's discourse are not made explicit. The communication between lecturer and students is crucial for this shift of discourses. In their study, Ben-Zvi and Sfard (2007) provide a discussion of the nature of the learning-teaching agreement which should exist to overcome commognitive conflicts and support meta-level learning. The discursants should agree on a common discourse for their communication, the lecturers should be responsible for the changes in students' discourse, and the students should show a willingness to follow the lecturer's guidance in this shift even if at the start their engagement in ritualistic. This ritualistic engagement is to be expected from the students' initial participation in the discourse. Aiming to shift from ritualistic participation to explorative participation in the discourse can be supported by the lecturers. Sfard (2016) illustrates the two ways this support can take place as follows:

> "First they can model such discourse by demonstrating the type of explorative discourse they would like their students to develop. Second, they can explicitly encourage the desired kind discourse by appropriate pedagogical moves" (Sfard, 2016, p. 44)

In my study, I examine students' participation in the university mathematical discourse and lecturers' guidance for this shift through the tasks used in one first-year module, focusing specifically on the examination tasks. I also, present analysis of the unresolved commognitive conflicts which are observed in students' responses to the examination tasks; stemming from the use of visual mediators and rules from the school discourse in examination tasks at the university level.

Chapter 4. Methodology

In this chapter, I first present the methodology of this study following the naturalistic qualitative inquiry paradigm (4.1). I discuss the general context of my study, a well-regarded mathematics department at a United Kingdom based university (4.2). Then I present the participants of the study, students and lecturers from different modules (4.3). In section (4.4), I discuss the process and the methods of data collection that I use namely interviews, observations, and document analysis. Next, I concentrate on the module taught in the first year of the undergraduate studies which is the focus of the analysis chapters and the process of data analysis. I end this chapter with a discussion on the ethical considerations (4.5); my role as a researcher (4.6) and issues on validity and reliability of the qualitative study (4.7).

4.1 Research Design

This study aims to investigate students' engagement with the university mathematical discourse in the context of the examinations, also considering lecturers' pedagogical and epistemological considerations. Specifically, my research addresses the following questions, also presented in section (1.1)

> **R.Q.1** What are the discursive characteristics of the examination tasks?
>
> **R.Q.2** What are mathematics lecturers' perspectives on the examination tasks and their expectation from the students' engagement with the university discourse in the closed-book examination setting and how are these perspectives enacted in the formulation of the examination tasks?
>
> **R.Q.3** How different are university mathematical discourses from the secondary school mathematical discourses and what commognitive conflicts can be observed as a result of those differences in students' scripts?

To answer the above questions, different variables should be studied to allow in-depth information on the examination setting focusing on the engagement with university mathematical discourse as communicated by the lecturers, in the tasks and the students in their scripts. My research follows the qualitative paradigm as the aim is to provide a description and analysis of the students' engagement with the university mathematical discourse and lecturers' perceptions and their decisions regarding the design of the examinations.

Qualitative research "describes and analyzes people's individual and collective social actions, beliefs, thoughts, and perceptions." (Mc Millan & Schumacher, 1997, p. 391). Specifically, "qualitative researchers study things in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them" (Denzin & Lincoln, 2013, p. 7). In this sense, my study is a naturalistic inquiry as the focus is on the "natural flow of events and processes and how participants interpret them." (Mc Millan and Schumacher, p. 391). My research studies students' engagement with university mathematical discourse in the context of the examinations, in the natural setting of the mathematics department. Moreover, it is also interpretive as "relies heavily on observers defining and redefining the meanings of what they see and hear" (Stake, 2010, p.26).

4.2 Context of the Study

My study aims to describe the students' engagement with the university mathematical discourse in the setting of examinations considering also lecturers' considerations on the examination tasks. The context of my study is a well-recognized mathematics department in the UK. The entry requirements to the Bachelor of Science course offered in this department is an A-Level in Mathematics and the equivalent entry requirements for European and international students. The focus of my study are modules from the first years of the undergraduate course offered at that department. Specifically, I concentrate on two compulsory modules, one offered in the first year and the other one in the second year; and one optional module offered at either the second or third year of study. The first-year compulsory module is named Sets, Numbers and Probability, and the second-year compulsory module is Differential Equations and Applied Methods. Furthermore, the optional module is on Combinatorics and Mathematical Modelling. I selected these modules having in mind the diversity of the mathematical topics that each of them dealt with and considering the year of study that these were taught. In the next part I present details for each of these modules starting with the compulsory from the first year, then the one from the second year and finally the optional module.

4.2.1 Sets, Numbers and Probability Module

This first-year module on Sets, Numbers and Probability comprised of two parts and was taught by two different lecturers (with aliases L1, L2). The first part, taught in the Autumn Semester, is the Sets, Numbers and Proofs part. The information presented on the syllabus about the first part of the module is the following:

> *"The unit provides a thorough introduction to some systems of numbers commonly found in Mathematics: natural numbers, integers, rational numbers, modular arithmetic. It also introduces common set theoretic notation and terminology and a precise language in which to talk about functions. There is emphasis on precise definitions of concepts and careful proofs of results. Styles of mathematical proofs discussed include: proof by induction, direct proofs, proof by*

> *contradiction, contrapositive statements, equivalent statements and the role of examples and counterexamples."*

The mathematical areas that are covered in twenty lectures are Set Theory, with a focus on the notation used of sets; Venn diagrams, union and intersection, distributivity, the difference of 2 sets, complement, de Morgan's laws; inclusion-exclusion principle and applications; power set and ordered pairs; cardinality and countability. The next topic presented are the basics of functions; injectivity and surjectivity of a function with examples of bijective functions and functions that are injective and not surjective and similarly surjective and not injective. Different approaches in proofs were also a part of this module, specifically direct proof; proof by induction; proof by contraction; and a discussion about examples and counterexamples. This part of the module also dealt with Number theory topics, namely Euclidean Algorithm; greatest common divisors; discussions about prime numbers; the fundamental theorem of arithmetic; rational and irrational numbers: irrationality of root 2; basics of modular arithmetic and equivalence relations.

The other part of the module, taught in the Spring Semester, is Probability. The information provided on the syllabus of the course is given below:

> *"The term **probability** refers to the study of randomness and uncertainty. In any situation in which one of a number of possible outcomes may occur, the theory of probability provides methods for quantifying the chances or likelihood associated with the various outcomes. The study of probability as a branch of mathematics goes back over 300 Years and it is now a fundamental prerequisite for the study of statistics."* (bold in the original)

The areas, covered in this part of the module in eighteen lectures, are the following: Classical and modern definition of Probability; Kolmogorov's axioms; basic properties proved from the Kolmogorov's axioms; permutations; combinations; conditional probability; Binomial and Bayes' theorem; independent events. The rest of the module focused on Discrete and Continuous samples. Specifically, after the presentation of the probability mass function and the cumulative distribution function; expectation and variance of different variables samples following the binomial, geometric, hypergeometric and Poisson distributions were presented. Similarly, in the Continuous samples after the discussion on the expectation and variance a presentation samples from uniform, Gaussian and exponential distributions followed. Additionally, there were some topics on Reliability and Markov Chains. However, these were not examined at the final examination at the end of the module, but reliability was examined in the coursework context.

The module was worth twenty credits, and it was compulsory for the first-year students. The distribution was forty percent from coursework and sixty percent of the grade from the final examination at the end of the year. The first twenty percent of the marks were given from two coursework sheets in the first semester from the Sets, Numbers and Proofs part of the course with questions focusing on sets and set operations; proof by induction; direct proof; and constructing examples; composition of functions; images of functions; injective, surjective and bijective functions; properties of divisors; reflexive; symmetric and transitive relations. The other twenty percent were given from a coursework sheet on Probability in the second semester. This coursework sheet had tasks on Kolmogorov's axioms; the definition of disjoint events; combinations; probability mass and cumulative distribution function for discrete samples; proofs of relationships between expectation and variance of independent random variables; probability density and cumulative density functions of continuous random variables; Gaussian samples; Reliability functions; and parallel and series systems.

The students had access to a variety of materials. For the Sets, Numbers and Proofs part of the module the students were given six handouts, lecture notes covering the range of topics mentioned above; three exercise sheets and the solutions to those; two coursework sheets, their solutions and feedback on their solutions. They also had formative coursework and the solutions to that. For the Probability part of the module, the students had access to the old lecture notes, the new lecture notes, statistical tables, three exercise sheets, and three problem sheets and their solutions; a coursework sheet and the solutions produced by the lecturer.

The rest of the sixty percent of the grade was given from the final examination at the end of the academic year for both parts of the module. The examination had two compulsory and four optional tasks. Both parts of the module had one compulsory and two optional tasks. The examination lasted for two hours, and the students were asked to solve five tasks. These five tasks included both compulsory and three of the optional tasks. Each task was worth twenty marks and the pass grade of the examination is forty marks. The three tasks from the Sets, Numbers and Probability part of the module focused on: proof by induction; divisors; Euclidean Algorithm and Greatest Common Divisor; Unions and intersections of sets; reflexive, symmetric and transitive relations; injective and surjective functions; basics of modular arithmetic. The three tasks from the Probability part of the module were on Kolmogorov's axioms and propositions following those axioms; probabilities of the union, intersection and conditional probability; Poisson random variable; expectation of discrete samples; expectation and variance of continuous random variables; probability density and cumulative distribution function and variables following the normal distribution.

4.2.2 Differential Equations and Applied Methods Module

This module was also split into two parts. One taught in the autumn by one lecturer (with alias D1) and one in the spring semester by two lecturers (with aliases D2, D3). The content of the part taught in the autumn semester was split into four. Part 1 dealt with definitions and general theory. Part 2 presented solutions to Ordinary Differential equations with a specific form $y''+p(x)y'+q(x)y=r(x)$ also Bessel functions and Legendre Polynomials. In part 3 Fourier series were introduced, and part 4 dealt with elementary partial differential equations. Specifically, in the partial differential equations part of the module the focus was on Laplace's equation in 2D, wave and heat equations; method of separation of variables in different coordinates systems: Cartesian, cylindrical and spherical.

There were two pieces of coursework for this part of the module. The topics that were examined were: second order differential equations; reduction order method; method of variation of parameters; solutions using power series; regular singular points; Fourier series; and method of separation of variables. The students had access to the following material for this module: lecture notes, three problem sheets used in seminars and their solutions, four problem class sheets used in workshops and their solutions; the coursework and their solutions produced by the lecturer.

In the spring semester taught by two other lecturers (D2 and D3) the focus was on further methods for Partial differential equations and Dynamical systems. In the first part, the topics that the module focused on were Fourier Transforms and Method of Characteristics. Specifically, this part started with an introduction to Fourier transforms; examples of transforms theorems; techniques for inverse transforms; application to integral equations; application to linear ordinary differential equations; application to partial differential equations and interpretation of the Fourier Transforms. Then focusing on methods of characteristics, the module focused on first order partial differential equations; application of the method of characteristics to first order linear and semi linear partial differential equations; application to first order quasi-linear partial differential equations (traffic flow problems) and theory and application for second order quasi linear partial differential equations. In the second part of the spring semester, the module dealt with Dynamical systems. Specifically, the following topics were one dimension dynamical systems (stability and phase portrait); Bifurcation (saddle node, transcritical, pitchfork, hysteresis, fold); two dimension (planar) continuous dynamical systems (classifications of fixed points, conservative systems, periodic orbits); Limit cycles (gradient systems, Liapunov function, Poincare-Bendixson); Bifurcation in two dimensions (saddle node, transcritical, pitchfork, Hopf, Poincare maps); three dimension continuous dynamical systems (Lorenz equations, strange attractor, chaos); discrete dynamical systems (1 Dimension recurrence relation, linear and non-linear recurrence, fixed point, stability, cobwebbing, convergence); and Logistic map (period-doubling, transition to chaos, Feigenbaum number).

For this part of the module, the students also had two pieces of coursework. The first focused on Fourier Transforms and method of characteristics and the second on Dynamical systems. Specifically, the first coursework for the spring part of the Differential Equations and Applied methods module had questions on Fourier transform; characteristics. The second coursework was split into three parts and had questions on linear stability analysis; equilibrium points; bifurcations in dynamical systems; phase portrait; and gradient systems.

The students had access to a variety of materials for this part of the module: lecture notes; handouts; four coursework and the solutions produced by the lecturers, ten problem sheets and their solutions used in the context of seminars and workshops.

The twenty percent of the grade was coming from the four pieces of coursework from the different parts of the module as mentioned above. The final examination lasted for three hours, and the paper accounted for rest eighty percent of the grade. The examination paper had six questions. The first two were compulsory, and the remaining four were optional. The students were asked to answer the compulsory and three of the four optional questions. There was one compulsory and two optional tasks (tasks 1, 3 and 4) on topics taught in the autumn semester. Half compulsory (task 2a) and one optional (task 5) was on Fourier Transformation and method of characteristics. Finally, the other part of the compulsory task (2b) and the last optional (task 6) were on dynamical systems.

4.2.3 Combinatorics and Mathematical Modelling Module

This module was part of a larger optional module for year two and year three students, named Topics in Mathematics. This part of the module was taught only in the Spring Semester and focused on Combinatorics the first weeks and then on Mathematical Modelling. The combinatorics part of the module, taught by the lecturer C1, introduced the following areas of Combinatorics: enumerative combinatorics, looking at binomial coefficients, Stirling's formula, inclusion and exclusion formula and properties of partitions. Then, it continued with colourings and particularly Ramsey's theory for finite and infinite sets.

The lecturer for the Mathematical Modelling part was the same lecturer that taught the Autumn Semester in Differential Equations and Applied Methods (D1). In the lectures of this part of the module the following topics were presented: The modelling process, units, dimensions and dimensional analysis, traffic flow and populations dynamics.

For this module, the students were given lecture notes; two problem sheets, one coursework sheet and the solutions to these produced by the lecturer C1, for the Combinatorics part; three seminar sheets; one coursework sheet and the solutions to these produced by the lecturer D1, for the Mathematical

Modelling part. In the coursework sheet, the topics examined were: enumerative combinatorics; colourings of graphs for the Combinatorics and traffic flow and populations dynamics for the other part of the module.

The examination of these two topics took place the same day as the examination of the module Topics in Mathematics. The students that chose these two topics only had an examination with four questions: two compulsory questions and two optional with one compulsory question and one optional question from each of the parts of the module. The duration of the examination was two hours, and the students had to answer both compulsory tasks and one of the optional ones. In the Combinatorics tasks, the topics examined were: enumerative combinatorics; inclusion-exclusion formula; colourings of graphs; theorems about Ramsey theory. The Mathematical modelling tasks were on population dynamics and traffic flow modelling.

4.3 Participants of the study

The participants of my study are undergraduate students studying Sets, Numbers and Probability, Differential Equations and Applied Methods and Combinatorics and Mathematical Modelling in the mathematics department and the lecturers teaching these modules. As Patton notes “qualitative inquiry typically focuses in depth on relatively small samples” (Patton, 1990, p. 169). In order to gain access to my participants, I used convenience and network techniques. Specifically, I first gained permission about my study from the Head of School of Mathematics, and then I started approaching the lecturers who were teaching the modules that my study focuses on. After getting in touch with one of the lecturers of the module they put me in touch with the rest of the lecturers teaching in the same module. Similarly, for my student participants, I approached students who were willing to participate in my study using convenience or opportunistic sampling (Onwuegbuzie and Leech, 2007, p. 114; Tracy, 2012, p. 134) and I then asked them to invite their friends and peers, essentially other students “who fit the profile” (McMillan and Schumacher, 1997, p. 398). This last method is called snowball sampling (Tracy, 2012, p.136). In the next part I present the participants of my study in more detail.

4.3.1 Lecturer participants

The mathematics lecturers that participated in my study are six lecturers involved teaching the modules mentioned above and in designing, implementing and correcting the closed-book examinations of these. Specifically, there were two lecturers involved in the teaching of the module of the Sets, Numbers and Probability (L1 and L2). L1, taught the materials on Sets, Numbers and Proofs in the autumn semester and L2 taught Probability in the spring semester. In the module, Differential Equations and Applied methods there were three lecturers (D1, D2, and D3). D1 taught the material in the autumn semester and the other two in the spring semester. D1 was also lecturing the Mathematical Modelling part of the module Combinatorics and Mathematical Modelling. Different lecturers also taught this module. C1 taught Combinatorics, and D1 taught the Mathematical Modelling.

4.3.2 Student Participants

The student participants of my study are fifteen undergraduate students studying the optional and the compulsory modules. Regarding the mathematical background of all the undergraduate students of the mathematics department, the requirements are that the students have to have an A in A-level mathematics or an equivalent level of qualification in mathematics. The fifteen students that participated in the interviews of my study included five students from year 1, studying Sets, Numbers and Probability (A1, A2, A3, A4, A5), five students from year 2 and five students from year three (B1, B2, B3, B4, B5, B6, B7, B8, B9, B10) studying either Differential Equations and Applied Methods or Combinatorics and Mathematical Modelling. I approached these students by inviting them to participate in my study using the Blackboard of the modules and by having a 5-minute introduction of my study and myself in the lectures of the modules. The detailed process of the data collection will be explained in detail in section (4.4).

4.4 Process and Methods for data collection

My doctoral study focuses on the students' engagement with the mathematical discourse. Specifically, I examine the last instance of mathematical writing that these students have, their participation in examinations. The data collection methods that I use in my study are: semi-structured interviews with the students; students using think-aloud protocol while solving past examination tasks; semi-structured interviews with the students after the completion of the think-aloud protocol, semi-structured interviews with the lecturers; observations of lectures; document analysis. In the following section I first discuss the process of data collection, then, I present each one of the methods.

4.4.1 Process of data collection

I used many different data collection methods: observations, interviews and document analysis to gain in-depth information to the students' engagement with the university mathematical discourse as Denzin and Lincoln state "the use of multiple methods, or triangulation, reflects an attempt to secure an in-depth understanding of the phenomenon in question. Objective reality can never be captured" (Denzin and Lincoln, 2013, p. 9).

Prior to collecting any data, I applied for the ethics application from the EDU Ethics committee. After obtaining the consent from the ethics committee, I first approached the Head of School of Mathematics informed him of the nature and the aims of my study and asked whether I could contact the lecturers of different modules to participate in my study. I also asked whether I could approach students studying these modules.

Then, I asked the lecturers of the modules to grant me access to the materials of the modules via Blackboard and to inform their students about their potential involvement to my study. Essentially, this was an invitation to a first interview which was focused on their ways of studying; their views on pure and applied maths. This initial interview aimed was to invite the students to participate in a further interview where they would solve past examinations papers using think-aloud protocol, more details in section (4.4.3). All fifteen students presented in (4.3.2) took part in the first interview. However, not all the students who participated in the first interview were willing to take part in the next interview. More particularly, student B2 decided that she did not want to participate in the next part of the study and student B8 even though initially expressed interest and willingness to participate in the second part of the interview he did not respond to the invitation for the second interview. In the table below (Table 4.1), I present the information from the second stage of the interview and the students' solutions from different past papers.

Table 4.1: Student data from the second interview

Data from the second interview with the students			
Sets, Numbers and Probability	**Differential Equations and Applied Methods**	**Combinatorics**	**Mathematical Modelling**
Examination tasks from the previous year: **Task 1**: A1, A5 **Task 2**: A1, A3, A5 **Task 3**: A1, A5 **Task 4**: A1 **Task 5**: A1 **Task 6**: A1 **Examination tasks from two years before:** **Task 1**: A2, A4	**Examination tasks from the previous year:** **Task 1:** B1, B3, B6, B9 **Task 2**: B3, B6, B9 **Task 3:** B9 **Task 4**: B9 **Task 5:** B9	**Examination tasks from the previous year:** **Task 1:** B5, B7, B10 **Task 2:** B3, B4, B5, B10 **Task 3**: B3, B4, B5, B10 **Task 4**: B5	**Examination tasks from the previous year:** **Task 2:** B4, B6

Task 2: A3 **Task 3**: A2 **Task 4**: A2, A4			

After this stage with the students' interviews, the examinations took place. I contacted the lecturers to gain access to the examination tasks and the marked students' examination scripts. I anonymised the students' scripts and photocopied several scripts from each of the modules. The way that I decided to select these was based on the lecturers' marks of the scripts. I chose scripts to illustrate a variety of different marks for each of these modules. I present in detail the marks from the Sets, Numbers and Probability module in section (4.5.3). In table 4.2, I present the total data that I collected from the students' scripts.

Table 4.2: Selected student data from the final examinations

Data from the students' scripts at the final examinations	
Sets, Numbers and Probability	22 scripts out of 54
Differential Equations and Applied Methods	34 scripts out of 97
Combinatorics	45 scripts out of 90
Mathematical Modelling	48 scripts out of 103

Prior to conducting the interviews with the lecturers, I conducted an initial analysis of the examination tasks in order to create more focused and in-depth interview questions. I also created a selection of anonymised snapshots from the students' scripts to discuss with the lecturers. Most of the interviews with the lecturers took part as close to the examination as possible. However, one of them due to the lecturer's annual leave had to take place two months later.

4.4.2 Observation of the lectures and note taking

I also observed the lectures of the optional module and the revision lecture for the first-year compulsory module. My aim in observing these lectures was to note down the written and most of the spoken communication from the lecturer to the students. This process is called participant observation which according to Lofland et al. (2006)

> "refers to the process in which an investigator establishes and sustains a many-sided and situationally appropriate relationship with a human association in tis natural setting for the purpose of developing a social scientific understanding of that association (Lofland et al., 2006, p. 17)"

I attended the revision lecture of Sets, Numbers and Proofs where both lecturers (L1 and L2) presented and discussed solutions to past examination tasks. Also, I attended 12 out of 15 lectures on Combinatorics and 15 out of 15 lectures on Mathematical Modelling. I noted down what the lecturer wrote on the board. I, also, recorded as much as possible from the lecturers' talk while they were writing. I did not audio record the sessions as the attention was on the tasks presented and the written solutions.

4.4.3 Semi-structured interviews with the lecturers and the students

As Tracy notes "Qualitative interviews provide opportunities for mutual discovery, understanding, reflection, and explanation via a path that is organic, adaptive, and sometimes energizing" (Tracy, 2012). In my study, I am interested in understanding lecturers' perspectives on assessment and students' perspectives on assessment. I use semi-structured interviews both with my student and lecturer participants. Before the interviews occurred, I compiled a list of themes and suggested questions to be covered in the interview (Kvale, 1996, p.124). For the first semi-structured interviews with the students these questions were informed by the literature on student approaches to learning and their views of mathematics; and by the materials of the modules that I had access through the Blackboard site of the university namely the lecture notes, the observation notes where possible, the coursework tasks. The second semi-structured interview questions were based on my reflections on the students' solutions of the past examination tasks and their utterances during the Think-Aloud protocol. In the lecturers' interviews, the questions were based on the tasks used in the final examination and on images from the students' scripts.

This type of interview allowed me to adjust the sequence and format of questions and ask additional follow up questions to acquire more information on what the participant was saying (Denscombe, 2010, p. 175). The interviews with the students took place during the spring semester whereas the interviews with the lecturers took place after the examinations. The duration of the first interview with the students was twenty to thirty minutes, but the room used for the interview was booked for longer, considering that the interviewees might be willing to discuss with me a bit more than that. This willingness to discuss for longer was the case in most of the interviews especially the second interview with the students and the interviews with the lecturers.

The interviewees were sent the information sheet about my study before the interview, and I provided for them a printed copy during the first few minutes of our discussion. I explained the aims of my study and their involvement in it and made clear that participation is voluntary and that they can decide to withdraw from the study at any point. After the information of the study was explained to them, I then asked for written consent to audio record the discussion and keep the materials that would be produced during the discussion. These materials involved the students' solutions to the past examination papers.

The first interview with the students focused on their ways of studying and their views on pure and applied mathematics. The primary purpose of this interview was to invite them to participate in the second interview where they could solve past examinations papers, and they would use thinking aloud protocol. Through this initial conversation, the students were more willing to participate in the second interview. Only two from the fifteen did not take part in the second one.

The second interview was structured into two parts. In the first part, I asked students to engage with one or more tasks, of their choice, from a past examination paper which was available to all the students of the university. All the interviews took place in a quiet room but special care was taken for the first part of this in order to replicate the conditions of closed-book examinations. Specifically, lecture notes and other material from the module was not allowed. The students were asked to report their thoughts and approaches using think-aloud protocol. The purpose for this was to examine in detail students' engagement with the mathematical discourse, in the written form (their solutions) and the verbal form (from their reports of their thinking).

Then, I asked them some follow up questions to clarify instances of their verbal or written engagement with the mathematical discourse. It was made clear to the participants to this interview that I will not offer any assistance in the process of engagement with the tasks and that I will not correct or mark their response. The duration of this interview depended on the participants' willingness to solve one or more tasks from the past examination papers. The interview questions were informed by a preliminary analysis of the past examination task, and by the observations made during the first part of the interview. The aim for this interview was to understand students' perceptions on the discursive characteristics of the task they are solving, ask them to reflect on their participation in the mathematical discourse at the university level and their views of mathematics.

Finally, the lecturers' interviews were based on the examination tasks they posed in their corresponding part of the module. After an explanation about my study, I shared with them the examination tasks they posed. This method has also been used by other researchers to elicit fruitful discussions with the lecturers either by sharing students' work (e.g., Iannone and Nardi (2005) and Nardi (2008)) or by sharing the examination tasks (Bergqvist, 2012).

I, first, asked them about the whole examination and then focused on the different parts of the tasks. The interviews with the lecturers ranged from forty minutes to more than one hour. The questions of the semi-structured interview were based, as mentioned above, on the tasks, the initial analysis of the tasks, students' responses to these and the materials used in the module. These interviews took place in the lecturers' offices. This environment was quiet and allowed us to stay there as long as the lecturers wished.

4.4.4 Document data

The main data of my study are the documents that I collected from the modules and the examinations. Specifically, the teaching material, comprised of coursework tasks, past examination tasks, and lecture notes. This data was mainly used as the background information in the analysis of the final examination tasks. And the main data of my study which are the tasks from the closed-book examination tasks, the model solutions produced by the lecturers for departmental use and the students' scripts. This data was anonymised and photocopied.

Here, I focus on the data that I am analysing and presenting in my thesis. This data corresponds to the final examination of the first-year module on Sets, Numbers and Probability. The examination had six tasks that I present in the analysis chapters (chapter 5 and chapter 7). The model solutions to these tasks were also collected. The model solutions are created by the lecturers who pose the tasks for department use. Finally, I analyse students' examination scripts. As mentioned in (4.4.1), I selected twenty-two of the marked scripts. These purposefully illustrated the breadth of the marks that students received. In the following table, I present the tasks of the examination on the Sets, Numbers and Probability module, the mathematical content, the number of students who engaged with each task, and the average, maximum and minimum mark they received for their work.

Table 4.3: Summary of the tasks from the closed-book examination. Also presented in Thoma & Nardi (2018, p. 7)

Task	**Mathematical content**	**Number of students**	**Students' marks**		
			Average Mark	**Maximum**	**Minimum**
1	Proof by induction, Greatest common divisors, Proof by contradiction	54	16.85	20	4
2	Kolmogorov's axioms, Propositions, Conditional probability	54	14.17	20	5
3	Operations on sets	53	13.23	20	0

	Equivalence relations				
4	Injective and surjective functions Modular arithmetic	54	14.31	20	0
5	Discrete random variable Probability density function Expectation	46	9.98	20	0
6	Continuous random variable Expectation and Variance Probability density function Cumulative distribution function	12	6	14	0

4.4.5 Methodological limitations

In this section, I discuss the limitations of my study in terms of the methodology. My study is a small-scale qualitative study and focuses on one first-year module. The results cannot be generalised for all the modules or other universities as they are context specific. However, preliminary analysis of the examination tasks and the students' scripts in the other modules illustrate that this commognitive analysis can offer insight into the transitions the students are going through in further years of their studies. Additionally, parts of the analysis have been applied to examination tasks in other contexts and showed that the framework allows a similar discussion in that context (Thoma & Nardi, 2015). This analysis enables researchers to examine the different discourses and discuss the discourses behind the mistakes the students make in the exams. A further refinement might be needed for analysis in other mathematical contexts based on the various mathematical discourses involved.

Another limitation of the data collection process was the focus on 22 of the 54 student scripts. Although this selection was made carefully aiming to illustrate a variety of marks in the examinations, it did not allow a presentation of all the unresolved commognitive conflicts in the students' scripts. However, the aim of the study was not to characterise all the unresolved commognitive conflicts in students' scripts but to offer insight into why the students are faced with these conflicts.

In some occasions, during the interviews with the lecturers, I felt that the participants were feeling slightly uncomfortable sharing some thoughts, I was on purpose becoming compassionate and agreeing with what they were saying. This situation did not hinder the data collection in any way, as the participants continued to feel comfortable to discuss with me further topics.

Also, the study did not include interviews with the students after their examinations. This was due to the timing of the examinations and the students' inability to participate in an interview afterward. The justifications, which characterise an instance in the students' scripts as commognitive conflict, are based on the written data only. However, the aim of the study was to characterise the lecturers' intended assessment practices and the actual students' engagement in the discourse focusing on their written response. Currently, the closed-book examinations are the predominant assessment method in the UK (Iannone & Simpson, 2011) and the lecturers are asked to make judgements based only on the written work of the students. Similarly, in my study, I decided to focus on analysing only the written work. However, an interview with the students on their solutions in the exams would strengthen the results of the study. In section 9.5, I discuss how this could also be taken into consideration in further research.

The student data which is analysed in chapter 6 and chapter 8, mainly consists of photocopied students' scripts with the markers' comments on. Both markers have different writing from the students, and one of them writes in capital letters which makes the comments from him easily identifiable.

However, when the marker decides to complete the students' writing, it is not easy to tell from the scanned script. In publications of preliminary findings of my analysis, colleagues and reviewers commented on this issue. Following up on these comments, in the thesis and in further publications, every time an image from a students' script is presented, the caption of the image explains the part of the writing added by the lecturer to avoid confusion.

4.5 Data analysis

I collected data from three modules, but the analysis presented in this thesis focuses on the first-year Sets, Numbers and Probability module. This decision to present that analysis is due to the transitional aspect of the module from secondary to university and the variety of the topics it included. This data comprises of six examination tasks, their model solutions, interviews with the lecturers of the module (L1 and L2), and 22 students' examination scripts from the fifty-four that participated in that module. I selected these 22 students' scripts to represent a variety of marks as can be shown in the figures in the analysis chapters (6.1 and 8.1).

Merriam (1998) mentions the two types of interpretative analysis the cross-case and the within case analysis. In my analysis of the examination tasks, lecturers' interviews and students' scripts I performed first the within case analysis which was focusing on each of the tasks and the corresponding data from the students' scripts and lecturers' perspectives on assessment. I, then, performed cross-case analysis to examine for similarities between tasks coming from the two parts of the module. In the next sections, I present the analysis of each of the data sets in detail.

4.5.1 Examination tasks and their model solutions

The examination task and their model solutions they were initially organized according to the mathematical topic examined. The analysis of the tasks aimed to offer answers to the first research question.

> **R.Q.1** What are the discursive characteristics of the examination tasks?

To understand and select a framework to analyse the tasks, I trialled an analysis of tasks using different frameworks. Specifically, I analysed some examination tasks collected for a previous study (Ioannou, 2012) using the MATH taxonomy (Smith et al., 1996), Lithner's framework as used in Bergqvist (2007); and the framework introduced by Tang, Morgan, and Sfard (2012). Results of this analysis have been submitted for a conference paper (Thoma & Iannone, 2015). This pilot analysis was a beneficial experience as it highlighted potential limitations of the frameworks, the need for more background information (teaching material, coursework, past examinations) and the solutions offered by the students and the lecturers (model solutions); and finally, it emphasised the importance of the lecturers' perspectives on the examination tasks.

After conducting the pilot analysis, I decided to inform my analysis using parts of the framework introduced by Morgan and Sfard (2016), based on Sfard's theory of commognition. Specifically, from the *Analytic scheme for subjectifying aspects of examination discourse*, I focused on the student autonomy aspect. I examined the extent to which students received directions regarding the degree of accuracy and use of specific procedures or specific narratives in their response. From the *Analytic framework for mathematising aspects of examination discourse*, I examined the “types of action demanded of students”. This aspect examined the procedures of the routines involved.

In my analysis of the tasks, I, initially considered the background materials, namely the; lecture notes, coursework and past examination papers. Also, to avoid subjectivity, I used the model solution produced by the lecturer to guide my analysis. I analysed, sub-task by sub-task and the corresponding part from the model solutions. I then examined the lecturers’ interviews, which I present in detail in the next section, on either the directions given or the routines the students were asked to engage in.

4.5.2 Lecturers’ interviews

The focus of the semi-structured interviews with the lecturers was on the examination tasks and lecturers’ perspectives on assessment. The duration of the interview was 110 minutes with L1, teaching the Sets, Numbers and Proofs part of the module, and 83 minutes with L2, who taught the Probability part of the module. I, first, transcribed all the data and took out all the information that could identify the participants.

This analysis aimed to provide answers to the second question:

> **R.Q.2** What are mathematics lecturers’ perspectives on the examination tasks and their expectation from the students’ engagement with the university discourse in the closed-book examination setting and how are these perspectives enacted in the formulation of the examination tasks?

Essentially, in my analysis of the lecturers' interviews, I examined the steps the lectures take in order to facilitate the students' participation in the mathematical discourse at the university level. This was done by investigating the lecturers' perspectives on students' engagement with visual mediators, word use, routines through the wording of the examination tasks.

4.5.3 Students' examination scripts

Students' examination scripts consisted of their responses to the two compulsory tasks and three of the optional tasks. In order to address my research questions, I first examined carefully the answers of the students to the same task focusing on the use of visual mediators and their word use. In the analysis, I also considered the comments from the markers mainly the circles or the underlined parts of the students' responses as these illustrated a part where the marker was not accepting the narrative produced by the student. Essentially, the use of words and visual mediators and students' engagement with the meta-rules of the discourses, were examined for cases where this use would indicate conflating discourses. These conflating discourses were either different university mathematical discourses or the school and university discourse. I first, identified these instances according to sub-task, then according to the task. I, then, examined all the data looking for these instances across tasks from the same part of the module and then from both parts of the module. My analysis aimed at providing answers to the third research question:

> **R.Q.3** How different are university mathematical discourses from the secondary school mathematical discourses and what commognitive conflicts can be observed as a result of those differences in students' scripts?

4.5.4 Analytical framework for tasks and students' examination scripts.

In the following table, I present the categories from Morgan and Sfard's (2016) analytical framework (for the full framework see Appendix 12.5) which I am examining in my data namely: student-author relationship, student autonomy, specialisation, logical complexity, the presence of multiple visual mediators, transitions between visual mediators, the types of actions demanded of students. I have added two columns indicating how these aspects of the discourse are examined in my data.

For the analysis of the tasks, I chose to focus on specific parts of the analytical framework aiming mainly to examine the routines the students are asked to engage in and the mathematical areas involved. As these provided insights into the discourses expected from the students. I did not examine the aspects of “objectification of the discourse”, the degree of specialisation and logical complexity in the tasks’ analysis. In the objectification of the discourse: The questions guiding the analysis is “To what extent does the discourse speak of properties of objects and relations between them rather than of processes” and the textual indicators are the following: “Nominalisation: use of a ‘grammatical metaphor’, converting a process (verb, e.g., *rotate*) into an object (noun, e.g., *rotation*); the use of specialised mathematical nouns such as *function*, *sequence* which encapsulate processes into an object; complexity of compound nominal groups”. I chose not to examine these aspects in the tasks but decided to examine them in students’ scripts. These aspects, of course, provide useful insight into the objectification, specialisation and logical complexity expected by the students at this first-year of their studies. Moreover, an analysis on these aspects on tasks coming from various universities can illustrate the different expectations that lecturers from various institutions can have from their students. However, for the current study, this will not be examined as these are tasks coming from one module and one institution.

Aspects of the discourse	Morgan and Sfard – Questions guiding the analysis (Q) and textual indicators (TI) as reported in the tables in Morgan & Sfard (2016, pp. 106-108)	Tasks	Solutions of tasks
Student-author relationship	Q: What kind of relationship is constructed between the student and a mathematical community? TI: Use of personal pronouns (inclusive or exclusive we, other personal pronouns)	Not examined	The pronoun “we” being used in the students’ solutions

Student Autonomy	Q: In responding to an examination question, how many independent decisions is the student allowed/required to make in: - Designing the path to follow? - Interpreting the tasks? TI: Complexity of utterances: length of a sentence; grammatical complexity: the depth of the “nesting” of subordinate clauses and phrases; logical complexity. - Choosing the form of the “answer” TI: The layout: the physical size of the answer; the space provided for the work to be done on the way toward solution; format of the answer (units, precision, no. of solutions); modality of the	How many decisions is the student required to make when designing the path to follow? (The procedure of the routine; the endorsed narratives) Instructions/directions given (or not given) to the students regarding the procedure of a routine (either given the name of the routine, a hint in brackets, implicit connection with other parts of the task)? Instructions/directions given (or not given) to the students regarding the justification required.	

	answer (graph? algebraic expression?) - Choosing/constructing the mode of the response? TI: Visual mediators: Verbal; symbolic; or graphic: supplied or to be produced?		
Specialisation	Q: To what extend is specialised mathematical language used?	Not examined	Use of mathematical terminology (words) which is not compatible with the mathematical discourses required to be used in the task. - **Commognitive conflict**
Logical complexity	Q: What kinds of logical relationships are present and how explicit are they? TI: the types and frequencies of conjunctions, implications, negations and quantifiers	Not examined	Use of implications, equivalences and quantifiers illustrating problematic meaning making of the logical relationships. - **Commognitive conflict**

The presence of multiple visual mediators	Q: To what extent does the discourse make use of specialised mathematical modes? TI: presence of tables, diagrams, algebraic notation etc. Q: How are multiple visual mediators incorporated into the discourse? TI: Provided in the text or to be produced by the student; Linguistic, visual and/or spatial relationships between modes	Visual mediators (algebraic notation)	Visual mediators (diagrams, graphs, algebraic notation) present in the students' solution

Transitions between visual mediators	Q: What transformations need to be made between different modes? TI: The presence of or demand for two or more modes of communicating "equivalent" information, e.g. an equation formed from a word problem; a unit of text that involves table, graph and algebraic expressions corresponding to the same function Q: How are transformation indicated in the discourse? TI: provided in the text or to be produced by the student; explicit linguistic or visual links between modes	Not applicable for the tasks	Visual mediators used in the solutions. Examining the links between the modes (text and graphs) Use of visual mediators (graphs and algebraic notation) which are not compatible with the mathematical discourses required to be used in the task. - **Commognitive conflict**

The types of actions demanded of students	Q: What areas of mathematics are involved? TI: topics Q: What are the characteristics of the routine procedures? TI: algorithmic or heuristic? Complexity, explicitly hinted at?	Determining the mathematical discourses involved in the task Examining the routines: Characterising them as rituals, recall, substantiation or construction (based on the lecturer's solutions and comments) Explicit directions on the procedures of the routines (this is also examined at the Student Autonomy)	Determining the mathematical discourses involved in the student's solution. Use of procedures which are not compatible with the mathematical discourses required to be used in the task. - **Commognitive conflict**

4.6 Ethical Considerations

Prior to collecting any data, I applied for ethical approval from my department's ethics committee. Then I gained permission from the Head of the Mathematics department. To gain this permission, I send an e-mail including an information sheet for my study. As Boeije comments the researcher has the

> "obligation to outline fully the nature of the data collection and the purpose for which the data will be used to the people or community being studied in a style and a language that they can understand" (Boeije, 2009, p. 45).

This was done through the information sheets given to all the participants of the study. These included the: aims of my study, participants involvement in the various stages of data collection, and the use of data for the thesis and publications (see information sheet in Appendix 12.1). Additionally, I used consent forms to receive written consent from the participants to audio-record the interviews and provide the document data (see consent forms in Appendix 12.2). The lecturers of the three modules agreed for me to make a short introduction to my research at the end of a lecture and they forwarded an invitation electronically from me to their students regarding my research. It was always made clear that participation in the study is completely voluntary and that it would not have any effect with their studies. Also, the students and lecturers who participated in the study were informed that they could drop out of the study if they wished to (Boeije, 2009, p. 45). Moreover, they had the right to contact my supervisor or the Head of the department in case they did not feel comfortable with their involvement in the study. Two of the student participants decided that they only wished to take part in the first interview and thus a second interview was not conducted (B2 and B8). Particularly for the participation in the second interview, where students were asked to solve past papers, I made clear that I would not offer any support to them. All the interviews took place on the university premises, and the participants were offered juice and cakes during the interview. There was no other form of payment to the participants of the study. Prior to observing any lectures, the lecturer had signed a consent form and informed the students about my research and my presence in the lectures.

The raw data (interviews, interview transcripts, observation notes and document data) was locked up physically, in a locked cabinet, and electronically, in a password secured folder. Adhering to confidentiality and anonymity, I anonymised the data using aliases. For the students who participated in the interviews, I used a letter and a number. Similarly, for the lecturers of each of the module and for the students' examination scripts I used numbers. Finally, the university where the study was conducted was not named and when the name of the university appeared in a task, I concealed this information. As Stake points out

> "In social research the dangers are almost never physical. They are mental. They are the dangers of exposure, humiliation, embarrassment, loss of respect and self-respect, loss of standing at work or in the group" (Stake, 2010, p. 206)

All the forms of data prior to analysis were only shared with the supervisors. The analysed data was presented in conferences of the Mathematics Education community: British Society for Research into Learning Mathematics (BSRLM), European Congress on Research in Mathematics Education (CERME) and International Network for Didactic Research on University Mathematics (INDRUM). It was also shared in meetings with the Research in Mathematics Education (RME) group of the university.

4.7 Role of Researcher

In this section, I discuss my role and involvement as a participant observer in the observations of the lectures and my role in the interviews with the lecturers and students. As, a graduate from a Mathematics department myself, I am familiar with the university mathematical discourse. I also have knowledge of all the topics that were covered in these modules, either from my own undergraduate studies or through personal reading. This allowed me to have meaningful conversations with the students and lecturers in the interviews and was particularly important in my analysis of the document data.

Regarding, my role in the observations of the lectures. My presence in the lectures did interfere with the course of the session as I did not participate at all. I was taking notes from what the lecturer was writing on the board, saying to the students, students' questions to the lecturer and their responses.

I mentioned in section (4.5) that it was made clear to all the students, that participation in the study would have no effect in their studies. However, the interviews both the first one and the second one gave them a chance to reflect on their studying, the assessment practices and their solutions to the examination tasks. Especially, the second interview provided the students with a chance to revise, explain and explore further the examination tasks from the previous years. However, my position in these interviews was clearly not to provide feedback about any of these but to ask questions and follow up their engagement with university mathematical discourses.

During the interviews with both lecturers and students, I avoided using mathematics education research terminology as well as asking leading questions. During the second interview with the students, I was supportive, but I did not provide any feedback regarding their solutions. Also, during the interviews, I took a sympathizing and empathizing position towards the difficulties and challenges my participants shared with me.

Finally, at the start of section (4.6), I mention my familiarity with the mathematical topics of the modules that my study focused on. However, as Sfard posits one of the challenges of the commognitive researcher is to be an outsider of the discourse. This can be achieved "[…] by putting herself in the position of a perfect beginner that the research may hope to get useful insights into processes of learning" (Sfard, 2008, p.130). My familiarity with the mathematical discourse in my mother tongue positioned me as an insider of the discourse. However, as all these modules were in English and the terminology was new to me, I could also take the position of a beginner when analysing my data.

4.8 Validity and Reliability

In the following section, I discuss the validity and reliability of my study. Both aspects “can be approached through careful attention to a study’s conceptualization and the way in which the data were collected, analyzed and interpreted, and the way in which the findings are presented” (Merriam, 1998, p. 199-200). There are strategies with which trustworthiness and credibility of the results of qualitative studies can be ensured (e.g., Lincoln & Cuba, 1985; Merriam, 1998). I discuss these strategies in relation to my study.

- “Peer examination – asking colleagues to comment on the findings as they emerge” (Merriam, 1998, p. 204). I have discussed the emergent findings from my analysis with colleagues. Early findings of my analysis have been presented in national and international conferences (e.g., BSRLM, CERME, INDRUM, and ICME). Furthermore, I have also had lengthy discussions with my supervisor, colleagues in the Research in Mathematics Education (RME) group to reduce research bias. During these discussions, alternate perspectives or interpretations of the data were suggested.
- Making clear the researcher’s position. In chapter 3, I discuss the connection between my study and the theory. Additionally, my role as a researcher is discussed in section 4.7, regarding the data collection and the data analysis stages.
- Data triangulation. I ensured triangulation in my study by using various material in my analysis of the tasks and the students’ examination scripts. These included: lecture notes, coursework, past examinations, and interviews with the lecturers.
- Providing a rich and thick description of the data. The analysis chapters (chapters 5, 6, 7 and 8) include lengthy quotes from the interviews and scanned images from the examination tasks and students’ scripts. The reader can have the opportunity to determine whether they agree or disagree with my interpretations of the data.

Chapter 5. Sets, Numbers and Proofs: Tasks and lecturer's perspectives

As mentioned in chapter 4, the examination in the module Sets, Numbers and Probability has six tasks three of which correspond to Sets, Numbers and Proofs. The first two tasks (1 and 2) are compulsory, and the other four (3, 4, 5, and 6) are optional. In this chapter, I focus on the tasks from the Sets, Numbers and Proofs part of the module, namely tasks 1, 3 and 4. For each task, I first introduce the task and provide its commognitive analysis, and then I explain the context of the task by taking into account similar tasks from the worksheets and the solution of the task as given by the lecturer of the module. Excerpts from the interview with the lecturer of the module follow this presentation. These are also analysed through the commognitive lens. Finally, I end each section of the chapter (5.x.4) with a summary of the analysis of these three tasks, and in section (5.4) I highlight issues that cut across the data analysis for each task.

5.1 Examination task 1 (Compulsory)

5.1.1 Task and commognitive analysis

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

[6 marks]

(ii)

(a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

[14 marks]

Figure 5.1: Compulsory task from Sets, Numbers and Proofs – Task 1

The first compulsory task (Figure 5.1) of the examination is focusing on the content of the first part of Sets, Numbers and Proofs. More particularly it is concentrated in proof by induction, divisors, Euclidean Algorithm and proof by contradiction. L1 produced a model solution to the task (Figure 5.3) for departmental use. The students did not have access to this solution; however, solutions to exercise sheets and coursework tasks were made available to students during the term. In the next section, I present the analysis of the task using the theory of commognition (Sfard, 2008).

Task 1 (Figure 5.1) has two subtasks which are testing different parts of the module: proof by induction; divisors and greatest common divisors; direct proof; and, proof by contradiction. In sub-task (1i) the students are given a summation, and they are expected to prove that this is equal to 2^{n+1}-2 for all natural numbers *n*. The students are asked to prove that this statement is true for all natural numbers *n*. The method that they are expected to follow to prove this statement is explicitly stated in the task "Prove by induction". In commognitive terms, the students are given a narrative, and they are asked to substantiate it for all natural numbers. The focus of the task is the engagement with a substantiation routine. The *procedure* of the routine is provided, as the type of proof that the students are expected to use is visible in the statement of the task.

The subtask (1iia) has two parts. For the first part of (1iia), the students are expected to engage with another routine, which is characteristic in the mathematical community: defining. This is a recall routine, as the students are asked to recall the definition of the divisor. In the next part of (1iia), they are asked to engage in a substantiation routine specifically in a constructive proof (or direct proof). There is an instruction regarding the *procedure* of the routine the students should use in proving that *d* is a divisor of the linear combination of *a* and *b* ("using this ... prove that").

For (1iib), the students are asked to compute the common divisor, using the Euclidean algorithm ("use the Euclidean algorithm"). The focus is to examine whether the students can follow this algorithm and find the *g.c.d* (greatest common divisor). After finding the *g.c.d* they are expected to find the integers for which the equality is true. Using the phrase "Hence (or otherwise)" they are instructed to use the narrative they produced while performing the

Euclidean algorithm or use another *procedure*. The routines that the students are asked to engage here are rituals and substantiation routines.

In (1iic), students have to combine the endorsed narrative of part (1iia) and (1iib) to produce the narrative for this part. They are asked to engage in a substantiation routine as they are asked to consider whether a linear combination which is a multiple of 3 is equal to 7. The students are instructed by the phrase "Explain your answer carefully" to provide justification for their response.

From the analysis of the task, it is important to note that the structure of the subtasks in (1ii) leads to each of the parts and finally completes with the combination of (1iia) and (1iib) in (1iic). Also, there are directions regarding the *procedure* of the routines, and finally in the last part, the students are also instructed regarding the construction of their narrative. They are asked to "Explain your answer carefully". These are directions regarding the degree of accuracy or specifically about the justification required.

This commognitive account of the task provides insights into the pedagogical considerations of the lecturer and their expectations of the students' engagement with this task and with these topics. Excerpts from the interview with the lecturer L1 presented and analysed in section (5.1.3) provide similar insights. Prior to the comments and the analysis from the lecturer's part, I present tasks similar to task 1 (Figure 5.1) which were part of the students' seminar and coursework worksheets in order to provide evidence of the university mathematics discourse that the students engaged with during the year and to locate the task in the context of the module. Furthermore, I present the model solution, produced by the lecturer, and a brief commognitive analysis of the solution.

5.1.2 Context and the lecturer's model solution

Tasks like (1i) were present in the exercise sheets and the coursework sheets. In these, the students were asked to prove specific statements by induction. The *procedure* of the routine was specified in the wording of the task as in part (1i). Concerning (1ii), students were asked to engage with the object of the divisor. One of the tasks had slightly different wording from the one in the examination. Specifically instead of "Give the definition of what is meant by saying that *d* is a divisor of *a*" in this task the lecturer used the following wording "Write out in full the definition of the statement "*d* divides both *a* and *b*."" (Figure 5.2). Here, I note the routine-driven use of words in the task used in the exercise sheet compared to the object-driven use of words in the examination task. Another important difference illustrated between the tasks that the students were asked to engage in the duration of the year (Figure 5.2) and the one at the final examination (Figure 5.1) is the amount of directions both about the *procedure* of the routines and the hints provided. The above illustrates the transition from the mathematical discourse that the students are first exposed to during the year and then the one that they are expected to be able to engage in by the end of the academic year.

6. Suppose a, b, d, m, n are integers.

(i) Write out in full the *definition* of the statement "d divides both a and b."

(ii) Using your answer to (i), prove that if d divides both a and b, then d divides $am + bn$.
[HINT: You may find it useful first to write down the definition of "d divides $am + bn$" – this should be the conclusion of what you are proving.]

(iii) Show that there do not exist integers x, y with $51x - 621y = 49$.
[HINT: find a natural number dividing both 51 and 621 and use (ii).]

Figure 5.2: Task from the first exercise sheet in Sets, Numbers and Proofs

Concerning the Euclidean algorithm, there were many tasks both in the exercise sheets and in the coursework requesting its use. The *procedure* of the routine is mentioned in the wording of the task, as is in the task (1iib).

Students were expected to engage in substantiation routines during the year. However, this engagement was more directed compared to the one in the examinations. More evidence about students' expected engagement with these tasks and lecturers' expectations about their engagement with the mathematical discourse can be traced in the model solution that the lecturer produced for departmental use (Figure 5.3).

In the solution produced by the lecturer for (1i), the statement is given a name, and its symbolic realization is repeated. Then the process of the inductive proof is divided into two parts: the base and the inductive step. The base step involves the substantiation of the statement for 1. First, the statement is written and then the substantiation of the narrative. Then, for the inductive step, a random natural number (*k*) is selected for which the statement holds. Using that the statement for *k* holds, the same number 2^{k+1} is added to both sides. Then by rearranging, the statement for *k*+1 is achieved. The lecturer ends the solution by saying that, because of this process, the statement is true for all natural numbers. Apart from the evident structure of the two steps and the concluding sentence of the substantiation routine, I note that this solution explains the steps of the *procedure* of the routine and whenever a new variable is appearing the numerical context of this variable is defined (this is the case for both *k* and *n*). Also, the statement is named and used as an object in the narrative.

Similarly, in (1iia) every time notation (*d*, *a*, *b*, *m*, *n*, *k*, *l*) is introduced, the numerical context in which the variables belong are clarified in the text. There are operators (*if*, *then*, *for all*) that are linking narratives about the discursive objects. Initially, the definition of the *divisor* is given. Then this is used to produce narratives linking the *divisor d* with the numbers *a* and *b.* In creating these narratives, the integer variables (*k, l*) are introduced and their numerical context is defined. Then, the linear combination of *a* and *b* is written, and the narratives produced earlier about *a* and *b* are used and factorised to show that the linear combination is a product of *d* and an integer. The summation of products of integers is an integer and so the linear combination is divisible by the *divisor d*.

Solution 1: (i) Let $P(n)$ be the statement

$$P(n): \qquad 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

We proceed by induction on n:

Base Step $P(1)$ says

$$P(1): \qquad 2^1 = 2^{1+1} - 2$$

which is certainly true since both sides are 2.

[2 marks]

Inductive Step Suppose k is a natural number for which $P(k)$ is true– so

$$2^1 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 2 \qquad (\dagger)$$

We want to deduce that $P(k+1)$ is true. Adding 2^{k+1} to both sides of ($\dagger$), and doing some rearranging, we get:

$$\begin{aligned} 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2.2^{k+1} - 2 \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2, \end{aligned}$$

so $P(k+1)$ is also true, completing the inductive step. Hence $P(n)$ is true for all $n \in \mathbb{N}$, by induction. **[4 marks]**

(ii)(a) d is a *divisor* of a means that there is $k \in \mathbb{Z}$ with $a = kd$. **[2 marks]**

If d is a divisor of a and of b then there exist $k, l \in \mathbb{Z}$ with $a = kd$ and $b = ld$. Then for all $m, n \in \mathbb{Z}$ we have

$$ma + nb = m(kd) + n(ld) = (mk + nl)d.$$

As $(mk + nl) \in \mathbb{Z}$, it follows that d divides $ma + nb$. **[2 marks]**

(ii)(b) Following the method in lectures, let $a = 123$ and $b = 45$. Carrying out the Euclidean algorithm we obtain:

$$\begin{array}{rcr|rcl} a & = & 123 & & & \\ 2b & = & 90 & 45 & = & b \\ \hline a-2b & = & 33 & 33 & = & a-2b \\ \cline{4-6} 6b-2a & = & 24 & 12 & = & 3b-a \\ \hline 3a-8b & = & 9 & 9 & = & 3a-8b \\ \cline{4-6} & & 9 & 3 & = & 11b-4a \\ \cline{1-3} & & 0 & & & \end{array}$$

We conclude that $\gcd(123, 45) = 3$ and that

$$3 = 11b - 4a = 11 \cdot 45 + (-4) \cdot 123 = 123m + 45n$$

where $m = -4$ and $n = 11$. **[8 marks]**

(ii)(c) No. Since 3 divides both 123 and 45 it follows from (ii)(a) that 3 divides $123s + 45t$ for all $s, t \in \mathbb{Z}$, but 3 does not divide 7. **[2 marks]**

Figure 5.3: Model solution to compulsory task in Sets, Numbers and Proofs

In the answer for (1iib), prior to writing the Euclidean algorithm, the lecturer mentions that the answer is “Following the method in lectures”. This procedure of performing, structuring and writing the Euclidean algorithm was presented to the students in the lectures. In the interview, the lecturer comments on the choice of this representation of the algorithm (section 5.1.3). After the algorithm, the lecturer ends the solution of this part with the conclusion that the *g.c.d* of 123 and 45 is 3 and provides the linear combination of the two numbers that results in 3. Then, he writes the linear combination given in the wording of the task 123*m*+45*n* and provides the values -4 and 11 for *m* and *n* accordingly.

Finally, in (1iic) the answer provided is brief and illustrates the connection with the previous narratives constructed to answer the other parts of the task. Specifically, connecting with (1iia), the linear combination $123s+45t$ with the two variables s and t belonging to the integers is divisible by 3, but the other side of the given equality is 7 (which is not divisible by 3). As in the previous parts of the task, the numerical context that the variables belong in is mentioned in the solution provided by the lecturer.

At the end of the solution, the lecturer provides characterisations for the tasks. (1i) is considered easy as a basic induction proof. (1iia) is considered easy as seen in the tutorials and (1iib) as computation. Then (1iic), is considered as moderate and the similarity with the tutorial sheet tasks is noted. In the next section, I discuss excerpts of the interview with the lecturer illustrating the expectations he has from his students about their engagement with the task, and with university mathematics discourse at large.

5.1.3 Lecturer's perspectives: a commognitive account

During the interview, L1 talks about specific elements of the task, aimed at assisting students' engagement and comments on expectations about students' engagement with the mathematical discourse. In the next parts, I first present L1's comments on the specific elements of the task and then his expectations on students' engagement with the mathematical discourse, namely their familiarity with the discursive objects and routines that the task asks them to engage in; their engagement with word use and visual mediators; and their engagement with recall and substantiation routines.

(i) The wording and structure of the task which aims to assist students' engagement

L1 talks about specific elements of the examination task, namely the use of specific word use (e.g., "(<u>or otherwise</u>)", "<u>Explain your answer carefully</u>"); visual mediators and the structure of the task. These elements are assisting students in their engagement with the mathematical discourse and are presented by L1 as an integral part of the practices of the community.

In (1iib) and (1iic) accordingly there are two instructions the first one "(<u>or otherwise</u>)" signalling that there could be another *procedure* that the students could follow in order to answer that part of the task and the second one, "Explain your answer carefully", regarding the depth of the expected response. In the following, I discuss the excerpts from the interview where L1 talked about these instructions.

L1 talks about how the phrase "(or otherwise)" in (1iib), allows him to give full marks to students using a different *procedure* in this substantiation routine and reward also those who take an alternative *procedure* in finding the *g.c.d*. He then talks about the creativity in the *procedure* of the mathematical routines and how this phrase allows students to be creative. L1 talks about the creativity of the *procedure* of the routines and mentions that this is a common practice in the mathematical discourse.

> "*in mathematics generally, solving some mathematical problem usually there is not a unique way to do that, and that is a good thing, that is a nice thing about mathematics. So, a very bright student might be able to solve some mathematical problem in some, in some completely interesting different way that you don't expect and that sometimes happens, and it is really fantastic when it happens, and they should get credit for it*"

In (1iic) the students are expected to combine the narratives they substantiated in the previous parts (1iia) and (1iib) to decide whether the equality with the linear combination of 123 and 45 equal to 7 can be substantiated. There is an explicit instruction regarding the justification they should produce "Explain your answer carefully". L1 explains this choice of words in the following:

> *"[W]e use these a lot in mathematics. Justify your answer, explain your answer, give reasons (…) the danger would be that the student would write yes or no and then write nothing else (…) I want them to explain why they are answering what they are saying what they are saying (…)"*

L1 with the above comment refers to another practice of the mathematical community. In the previous excerpt, he discusses the importance of creativity in the *procedure* of a routine and in this excerpt, he comments on the justifications, which are usually used in mathematics "*we use these a lot in mathematics*". This is another shift that is required from the students' engagement with the university mathematics discourse; they have to justify their response. He also uses the pronoun "we" to speak about the mathematical community. Earlier on he also referred to this community using the phrase "*in mathematics generally*".

L1 also comments on the importance of the visual mediators present in the wording of the task. He provides the algebraic mediators aiming to assist students in using them in further parts of the task and in seeing the connections between these. In the following, L1 highlights the importance of visual mediators in the whole sub-task 1ii). In a commognitive sense, the lecturer talks about the relationship between the "smaller" narratives which are the sub-tasks.

> *"So, I think what is important in this sentence is that I mention d and I mention a. Because it helps them (…) by writing this way, using d and a and the fact that later in the question d is a divisor of a, they already have the, some of the symbols they need right? They'll say d divides a if there exists whatever x such that and then they can then use that somehow that's already written down. And then the next sentence then d is a divisor of a and they look at the previous line and go okay, I've written down what that means. d is a divisor of b, I write down what that means as well, hopefully. And then the last, right, and then the last part hopefully, they can- they can then figure it out."*

The structure of the task also serves at assisting students' engagement with the discourse. L1 talks about the gradual structure of the task. First the students are asked to recall the definition, assisting them to position themselves in using the discourse of the integers and not the real numbers, so then they can be aware of which discourse they have to engage with, in the next stages of the task. This gradual structure can assist in the production of solutions that are worth more marks.

> *"asking them first to write down formally what it means for one thing to divide another is remin- it's a reminder- it's a reminder to them that a theme of the course was to being careful about definitions being precise about things and using, then using those definitions (…) what's being tested here is their ability to write down something*

> *formally and correct. (…) in the pressure of the exam and so on then their answers could start looking very creative at the second part. And they might start writing down fractions (…) without this I think actually they would have been more incorrect answers on this part. Or with, things that aren't quite right and quite what should have been written down."*

(ii) The lecturer talks about the task and the expectations about students' engagement with the mathematical discourse

In this part of the chapter, I present the interview excerpts and their commognitive analysis that correspond to the expectations that the lecturer has on students' engagement with the mathematical discourse in the context of examinations. L1 commented on students' engagement with task 1, specifically on their expected familiarity with the task (in terms of routines and objects); their engagement with word use and visual mediators; and their engagement with the recalling and substantiating routine.

"They all have seen this before or seen things very similar to this"

L1 explains the expectations he has about students' engagement with the university discourse. As the module Numbers, Sets and Probability, is a first-year module it serves as the first chance that students have in engaging with different discursive objects and routines of the university mathematical discourse. The students are asked to engage with different routines these routines are mostly explorations (Recall, Substantiation and Construction).

> "So, in a sense question 1 differs maybe slightly from 3 and 4 in the sense that this question is meant to be let's say maybe easier or more accessible than these."

In the excerpt above, the lecturer speaks about the difficulty of task 1 taking into account the students' familiarity. The first and the second tasks as they are compulsory they are designed to be more accessible and easier than the rest. In this way, the lecturers of both parts of the module, illustrate which are supposed to be the basic engagement that the students should have with the mathematical discourse of this module.

> “really all students that put some effort and studied they should be able to do these questions (…) you would be aiming for an average mark of 17 something like that out of 20 for a question like this.”

Based both on the experience that the lecturer has but also on the students’ engagement with these concepts from the seminars and the coursework, the expectations for the average mark in this task are high. The expectation is based on the amount of times students should have engaged with this mathematical routine in exercise sheets and tutorial sheets.

> “(…) this covers things from early in the module, so these are some of the very first things that they see which is proof by induction. (…) They all have either seen this before or seen things very similar to this before on exercise sheets or tutorial sheets.”

“They came to university thinking that they knew what that meant”

He then speaks about the commognitive conflict the students face as they are asked to define divisor restricting themselves in the discourse of the integers, which is a subsumed discourse in the discourse of real numbers. In the school mathematical discourse, the students were engaging with the discourse of real numbers, and in this module, they are asked to engage with different numbers sets and restrict their engagement in discourses which are subsumed in discourse of real numbers.

> *“they came to university thinking they knew what that meant [the divisor] but in this situation it really matters that they are restricting themselves to the-to the ring of integers and they can only, and all the symbols represent integers so what it means to divide is very different than if they were working with fractional numbers or something were they could write a over b and things like this. I mean somehow the danger here is that a student before they – before they took the module writing something like this, division for them is to write a symbol and then a line and a symbol underneath they are writing a fraction. But they are not really allowed to that here, they have to write everything in terms of integers, so fractions aren’t really meant to appear anywhere in the things they are doing.”*

He then comments on the importance of algebraic mediators. He talks explicitly about the algebraic mediators in (1ii) and the importance of clarifying what each of the algebraic mediators means in the context of the specific sub-task. In doing so, he talks about the importance of clearly defining the algebraic mediators and using different algebraic mediators to avoid potential commognitive conflicts with the lecturer who is correcting the students' solutions.

> *"And so, for, and so writing this, there exists some k such that this. Typical mistake might be that-that they get this correct more or less correct al-right. They say that there exists some k such that this. Then they get on to the b) part but they use the same k (…) And then the rest of the proof, I mean they get some kind of proof that makes sense but of course they need some other symbol and, ahm well okay yeah, and again that's the kind of mis- that's the kind of mistake where someone can get the definition right so technically the definition is right exists a k such that. But then when they write, you know, that. Then they, for b, they also write k. b is kd which they shouldn't and then the rest of the proof kind of works out, but they are going to lose marks because they shouldn't have used the same k and what they are misunderstanding is the logic."*

L1's perceptions of students' engagement with recalling and substantiation routines

L1 discusses the routine of defining both in terms of defining a mathematical concept (e.g., (1iia) the divisor) but also defining the numerical context of a variable in the different parts of (1ii). He compares the university mathematical discourse with the school mathematical discourse. In the school discourse the focus, according to L1, was more on the rituals than explorations routines, specifically the defining routine which is a combination of recall and construction routines.

> *"this is the first module that these students are doing in pure mathematics and em, it is a new…. My guess is in school, they are not asked much to write down formal definitions of things. I guess, I don't know – I don't know so much about the A-Levels but my memory*

> *of school mathematics is that there was a lot of doing things but not necessarily a lot of formally defining things"*

L1 here talks about the word use in students' engagement with the mathematical discourse. Specifically, in recalling the definition of a divisor, he mentions that students struggle with defining the numerical context of the variables and with quantifiers such as "there exists" and "for all" which mainly used in the university mathematics discourse.

> *"the weaker answers [meaning answers produced by weaker students] they tend to write things like, they tend to mess up things like quantifiers so they-they… (…) I mean here they meant to write em you know d is a divisor of a means there exists an integer such that k such that a is kd right? Now, the common answer would be something like em they might not even write any of these words they'll just write something like. Ahm okay they might not even write d is a divisor of a means they might just write something like a equals kd something like that (…)"*

In the excerpt above, L1 comments on his perspectives on the solutions that the weaker students are producing. These would not have the definition of the numerical context of the variables involved in the narrative, and they would also not use the quantifiers.

> *"In particular where they sometimes struggle is this idea of you know for all and there exists quantifiers is something that students struggle with, some students struggle with. (…) they are able to manipulate the symbols and they are very comfortable with symbol manipulation which is something they do a lot of in school and they are very good at that. But just the idea of there exists* ***some*** *integer such that, that part for the weaker students is sometimes the-the hard thing for them to get their head around, actually."*

Comparing with the nature of the routines that the students are used to from secondary school, the lecturer comments that the use of quantifiers is not something that they have seen before. Continuing with the comparison, L1, comments on the nature of the mathematical routine of the Euclidean algorithm and says that students comfortable with this. He remarks that they do not necessarily need to understand the theory to be able to carry out the process. In commognitive terms, I see this as a distinction between a ritualistic engagement and an explorative. The engagement with the Euclidean algorithm can be explorative and ritualistic, as the *procedure* is mentioned in the wording of the task. The students' actual engagement with this routine is shown in chapter 6.

> *"(…) so, the students tend to find processes once they know how to carry them out, they tend to do well on those things. If it's something, even if they don't understand the theory that's behind, ah that's making this process work for example the Euclidean algorithm in this case."*

L1 comments on the familiarity the students should have with the *procedure* of the Euclidean Algorithm based on the tasks they have seen in the exercise sheets and the lecturer's teaching experience.

> *"(…) Again, it's a process that they need know how to carry out its designed-its designed to be something that all the students know they need to know how to do and that they can carry it out these numbers are different from the numbers they've seen in the exercise sheets. (…) Every student is capable of learning to carry out the process and so I would mark that as easy because it's the kind of thing I would expect them all to be able to do and it tends to be the case and that's based on I guess from teaching last year from teaching previous courses in the other university before I came here and from coursework"*

The (1iic) is the only part of the task that is seen by him as potentially challenging for the students as the *procedure* of the routine is not specified and the students could potentially struggle in seeing the connection with the sub-tasks (1iia) and (1iib). In this part, the engagement with the discourse is different compared to the Euclidean algorithm. The students are asked to engage with a substantiation routine in which they should combine narratives that they have constructed in the previous parts. As the quote below shows, L1 says that in this part "some thought" is required and "understand or remember that somehow it relates to what happened up here (showing the other two parts of (1ii)". Using the commognitive theory, the lecturer talks about the differences between the rituals, the Euclidean Algorithm, and the explorations that this latter part belongs to.

> *“the only challenging maybe part would be the last part, the part that requires some thought and they need to-to sort of understand or remember that somehow it relates to what happened up here”*

5.1.4 In summary

Task 1		
Commognitive analysis		Lecturer's perspective on assessment
Mathematical discourses involved in the task	Discourse of natural numbers and integers	Possible conflation between the discourse of reals and the discourse of integers
Visual mediators	Variables which take values from the natural numbers or the integers	Aiming to assist the students in their production of narratives Possible conflation between the discourses if the algebraic notation is not defined clearly
Routines (rituals, recall, substantiation, construction)	Substantiation: (1i), (1iia), (1iib), (1iic) Recall: (1iia) Ritual: (1iib)	The students should be familiar with all these routines Students face difficulties with the logical complexities in the recall and substantiation routines

Instructions are given regarding the procedure of the routine	Instruction are given explicitly in the wording of the task (1i), (1iia), (1iib) Students are allowed to choose the procedure of the routine (as either the instruction is implicit or non-existent): (1iib), (1iic)	Allowing creativity in the procedures of routines – a practice of the mathematical community
Instructions are given regarding the justification required	(1iic)	Justifying – a practice of the mathematical community
Structure of the task	Gradual structure of task (1ii)	Assisting in the production of the solution. However, as this is implicit, the students might not be able to see it.

5.2 Examination task 2 (Optional)

5.2.1 Task and commognitive analysis

(i)	Prove carefully that if A, B and C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
	Give an example of sets A, B and C such that $A \cap (B \cup C) \neq (A \cap B) \cup C$.
	[10 marks]
(ii)	Suppose that A is a non-empty set and $\sim$ is a relation on A. Give the definitions of what is meant by saying that $\sim$ is *reflexive, symmetric* and *transitive*. In each of the following cases, decide which (if any) of these properties the given relation has. Give reasons for your answers. (a) $A = \mathbb{Z}$ and $a \sim b \Leftrightarrow \lvert a - b \rvert \leq 10$ (for a, $b \in \mathbb{Z}$). (b) $A = \mathbb{R}$ and $a \sim b \Leftrightarrow a - b \in \mathbb{Q}$ (for a, $b \in \mathbb{R}$).
	[10 marks]

Figure 5.4: Task 3 from the Sets, Numbers and Proofs part of the module

Task 3 (Figure 5.4), thereafter known as task 2, focuses on sets and relations. The task has two equally marked parts. In (2i), the students are asked to engage in a substantiation routine as they are asked to prove the equality with the sets *A*, *B*, *C*. There is an instruction regarding the expected justification that the students should produce (“Prove carefully”). There is no instruction regarding the *procedure* of the substantiation routine and the sets involved in the equality do not have a specific nature. In the same sub-task, the students are asked to construct three sets that would satisfy the second relation where the resulting sets are not equal. This is a construction routine where the students are asked to find sets and they can decide on the nature of the elements of the three sets. The choice of the two sets to be shown that are not equal in (2i) links to illustrating that the operations in Set Theory are not following the associative property. The order of the operations results in very different sets. The agency of the student, in this case, is not restricted regarding the *procedure* of the routine. In the first part of this sub-task the students are asked to engage with the discursive objects of sets without being given any information or restrictions regarding their nature. Finally, in the last part of this sub-task, the students have to decide on the nature of these discursive objects and select three sets that would satisfy the given non-equality.

In (2ii), initially, the students are asked to engage in a recall routine in order to produce the definitions of properties of the relations: reflexive, symmetric and transitive. Then, they are given some examples of relations, and they are asked to engage in substantiation routines for the three properties of the relations. "Decide which (if any) of these properties the given relation has. Give reasons for your answers". In the wording of the task, there is a prompt about justifying their responses and also a hint that some of these properties might not necessarily be true for these relations. In addition, the students are asked to engage with different sets, as the relations are defined on a different set in each case (integers in (2iia) and reals in (2iib)).

5.2.2 Context and the lecturer's model solution

During the term time, the students were asked to engage with the substantiation routine of the equality between sets in two tasks, one in the exercise sheets (Figure 5.5) and the other in the coursework (Figure 5.6). The wording used in those is again "Prove carefully" and this substantiation routine is followed by a sub-task that asked for an example of sets *A*, *B*, *C* such that two sets resulting from operations on them are not equal, just like the one used in the examination, challenging the associative property in the operations of Set Theory.

2.(i) Prove carefully that if A, B, C are sets then $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(ii) Give an example of sets A, B, C such that $A \setminus (B \cap C) \neq (A \setminus B) \cap C$.

Figure 5.5: Task from exercise sheets on Set Theory

2.(i) Prove carefully that if A, B, C are sets then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.	**[12 marks]**
(ii) Give an example of sets A, B, C such that $A \cup (B \cap C) \neq (A \cup B) \cap C$.	**[4 marks]**

Figure 5.6: Task from coursework on Set Theory

Similar to (2ii), were two tasks one in the exercise sheets (Figure 5.7) and the other from the coursework (Figure 5.8). In these, the students are not asked to define a reflexive, symmetric or transitive operation, but they are asked to substantiate whether the given relations have these properties.

Also, as in the (2ii) the sets that these relations are defined on different numbers sets, namely real numbers and integers.

3. For each of the following relations $\sim$ on the given set X, decide whether it is reflexive, symmetric or transitive. Explain your answers briefly.

(i) Let $X = \mathbb{Z}$ and $a \sim b \Leftrightarrow 5|(2a+b)$.

(ii) Let $X = \mathbb{Z}$ and $a \sim b \Leftrightarrow (a-b) \in \mathbb{N}$.

(iii) Let $X = \mathbb{R}$ and $a \sim b \Leftrightarrow (\exists x \in \mathbb{Q})((a-b) = x^2)$.

Figure 5.7: Task from exercise sheets on relations

3. For each of the following relations $\sim$ on the given set X, decide whether it is reflexive, symmetric or transitive. Explain your answers briefly.

(i) Let $X = \mathbb{Z}$ and $a \sim b \Leftrightarrow ab \neq 0$. **[5 marks]**

(ii) Let $X = \mathbb{R}$ and $a \sim b \Leftrightarrow |a-b| \leq 13$. **[5 marks]**

(iii) Let $X = \mathbb{Z}$ and $a \sim b \Leftrightarrow 11|(5a+6b)$. **[6 marks]**

Figure 5.8: Task from coursework on relations

In the model solution (Figure 5.9), the lecturer's expectation about students' engagement with the university mathematical discourse can be seen. As mentioned earlier, these responses are not given to the students, but they are created for departmental purposes.

The structure of the solution of (2i) shows that the lecturer signals from the start about the two steps that consist the substantiation routine of the equality of two sets. The equality of the two sets often consists by examining whether one set is a subset of the other and then the opposite way. The substantiation routine of the equality of the two sets corresponds to two sub-routines. The substantiation routine that an element of the first set belongs to the second set making this way the first set a subset of the second set and similarly the other way. Finally, since this is the case for both the closing conditions of the substantiation routine are true and the sets are equal. In the solution, produced by the lecturer, the narratives involve both the discursive objects of sets and their elements. First, an element *x* is taken from the first set and shown that it belongs in the second. In the solutions, there is a transition from words to symbols and the other way around. Specifically, the symbols for union and intersection are used to start with which are then transformed into words (“*or”* and “*and”*) and then these words are further transformed into symbols again to produce the second set. Similarly, that is the case for the second set. To examine the element for the second set the lecturer introduces a different notation for the element of the second set, which is then shown belonging to the first set. Two different symbols for the elements of the different sets to assist in the distinction between both the elements but also the sets that these elements are coming from. In addition, as noted earlier, there is a flexible use of both words and symbols. Finally, the narratives of the two sub-routines are starting from the *goal* (the *closing condition* of the routine) which is to show that one set is a subset of the other and then move to the choice of one element in one of the sets.

Solution 3:

(i) First we show that $A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$. So suppose $x \in A\cap(B\cup C)$; that is, $x \in A$ and $x \in B\cup C$. The latter means that $x \in B$ or $x \in C$.

- If $x \in B$, then $x \in A$ and $x \in B$ so $x \in A\cap B$;
- If $x \in C$, then $x \in A$ and $x \in C$ so $x \in A\cap C$.

So $x \in A\cap B$ or $x \in A\cap C$ – that is, $x \in (A\cap B)\cup(A\cap C)$. **[4 marks]**

Now we show that $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. So suppose $y \in (A\cap B)\cup(A\cap C)$; that is, $y \in A\cap B$ or $y \in A\cap C$.

- If $y \in A\cap B$ then $y \in A$ and $y \in B$, so $y \in B\cup C$;
- If $y \in A\cap C$ then $y \in A$ and $y \in C$, so $y \in B\cup C$.

Hence in both cases $y \in A$ and $y \in B\cup C$ – that is, $y \in A\cap(B\cup C)$. **[4 marks]**

For the last part, we can take $A = \varnothing$, $B = C = \{1\}$. Then

$$A\cap(B\cup C) = \varnothing\cap\{1\} = \varnothing,$$

while

$$(A\cap B)\cup C = \varnothing\cup\{1\} = \{1\}.$$

[2 marks]

(ii) Let A be a set and $\sim$ a relation on A. The relation $\sim$ is *reflexive* if for all $a \in A$ we have $a \sim a$. The relation $\sim$ is *symmetric* if for all $a, b \in A$, whenever $a \sim b$ then $b \sim a$. The relation $\sim$ is *transitive* if for all $a, b, c \in A$, whenever $a \sim b$ and $b \sim c$ then $a \sim c$. **[4 marks]**

(ii)(a) This is reflexive as $|a - a| = 0 \leq 10$ for all $a \in \mathbb{Z}$. It is symmetric as $|a - b| = |b - a|$. It is not transitive: for example $1 \sim 9$ and $9 \sim 18$ but $|1 - 18| = 17 > 10$, so $1 \not\sim 18$. **[3 marks]**

(ii)(b) The relation is reflexive as $(a - a) = 0 \in \mathbb{Q}$. It is symmetric: if $a \sim b$ then $a - b \in \mathbb{Q}$, so $(b - a) = -(a - b) \in \mathbb{Q}$. It is transitive, because if $a \sim b$ and $b \sim c$ then $(a - b), (b - c) \in \mathbb{Q}$ so their sum $(a - c)$ is in $\mathbb{Q}$. **[3 marks]**

Comments: 3(i) Proof seen in lectures. Moderately difficult. 3(ii) Standard definitions. Easy. 3(ii)(a) and (b). Two standard relations questions. Moderate.

Figure 5.9: Solution to task 3 from Sets, Numbers and Proofs part of the module

For the next part of the task, the lecturer provides an example of two sets *B* and *C* being the same and *A* being the empty set. Then, illustrates that $A \cap (B \cup C)$ is the empty set and that $(A \cap B) \cup C$ is the set with the element 1. Initially, the sets are defined by providing their elements and then the operations between the three sets *A*, *B* and *C* are performed and it is shown that the two sets $A \cap (B \cup C)$ and $(A \cap B) \cup C$ are not equal, showing that the order in which the intersection and the union creates different resulting sets.

For (2ii) the lecturer first provides the definitions of reflective, symmetric and transitive. Prior, to providing the definitions of the properties of the relations, the lecturer sets the scene by defining the elements that he will use in the properties. L1 introduces the set *A* and the relation on *A*, then the three definitions are given using quantifiers (*if*, *whenever*, *then*, *for all*). Distinct symbols are used for the elements of the sets and the nature of each element is written every time. Then in the last part of this task, the lecturer comments on the two properties for the two given relations. To check whether this is reflexive the opposite of the definition is used. It is interesting here to note the quantifiers role between the substantiation routines and the defining routines.

Underneath the solution, the lecturer has characterised these tasks according to students' familiarity and his experience with the students over the years. The proof of the two equal sets and the counterexample is noted down as moderately difficult. The definitions as easy and the identification of whether the two relations are reflexive, symmetric and transitive is noted as moderate.

5.2.3 Lecturer's perspectives: a commognitive account

During the interview, L1 commented on two main ideas regarding students' engagement with this task. He commented on his perceptions on students' engagement with specific objects of the discourse that is taking an abstract form for the first time at the university mathematics discourse; and, his perceptions on the students' engagement with substantiation routine. In the following, I first present his comments and their commognitive analysis on the objects of the discourse (the sets and their elements, and the realization trees) and then his comments on the substantiation routine (proving the equality between two sets).

Lecturer's perceptions of students' engagement with objects

L1 in the following excerpt talks about expectations he has about students' engagement with task 3.

> *"it's something that I get the impression students find difficult (...) they have three abstract sets and they are trying to show some equality between them with intersections and unions and they need to show that this is a subset of this and this is a subset of this (…)"*

L1 in the following excerpt talks about the routine of defining and the realisation trees of the mathematical object of a reflexive/symmetric/transitive relation. He mentions that he is not asking the definition of a relation, which is the first stage of the realisation tree, or the definition of a Cartesian product, which is the second stage of the realisation tree. The definitions of these mathematical objects which are included in the reflexive relation were mentioned in the lectures formally, but then they were used as a base for the mathematical object of a reflexive, symmetric and transitive relation.

> *"I don't ask them to define what a relation is. (…) we do formally define it, and it's a subset of AxB, fine. But then we very quickly go on to just thinking about the notion"*

In the next excerpt, L1 illustrates the development of the realisation trees and the development of discourses.

> *"[W]e take time to formally define these things but then I also want them to be able (...) to think about the ideas and relax a little bit and not get bunked down in, in certain situations formality stands in the way of understanding. (…) You say we can formalise it and now that we are comfortable that we can formalise it we go back to thinking intuitively always knowing that we can go back there if we needed to or if we start to get confused or things seemed ambiguous we can always back here."*

L1 in the two excerpts above, using commognitive terms, talks about the flexible moves between the formal mathematical narratives and the engagement with routines that involve these discursive objects. Sometimes the abstract nature can hinder the engagement with routines. The excerpts illustrate the *endogenous* nature of the mathematical discourse and the difficulty that the learners encounter while engaging with the discourse. L1 talks about a characteristic of the mathematical discourse, which is concerned with the ability to use these objects and not necessarily be concerned with the nature of those objects.

"[T]hey would find this a lot more difficult than for example the Euclidean algorithm"

Below, the lecturer comments on students' difficulty with the abstract objects that are involved in task 2.

> *"But my experience is that they find this, that they would find this a lot more difficult than for example the Euclidean algorithm or something (…)."*

The first part of (3ii) is asking students to engage with abstract sets. During the school years, the students worked with sets, but the sets had specific elements. In (2i), they are asked to engage with abstract sets, operate on them with the intersection and the union and then show that these two sets, resulting from the operations, are equal to each other. The lecturer, in the excerpt above, comments on this process being more difficult than the one that the students are doing with the Euclidean algorithm. The discursive objects involved in these two routines are of completely different natures. The Euclidean algorithm involves numbers whereas the sets are a completely different object. He also mentions the steps of the proof, where the students have to show that the first set is a subset of the second one. Here the lecturer is referring specifically to the *procedure* of the substantiation routine that two sets are equal when their elements are not known. This part of substantiation is also something new to the students as usually the substantiation of an equality would be showing that one side of the equality is the same as the other side. Alternatively, starting from one side ending up to a specific point and then illustrating that the other side is equal to that. However, in the case of Set Theory that is not the case. Two sets, which elements are unknown are equal when the first set is a subset of the second and when the second is a subset of the first. This is a very different procedure from the ones that the students are used to and the lecturer notices that this is something that they are finding difficulty in doing.

Even though the students have seen something similar to the *procedure* of the substantiation routine and they should be familiar with it, they seem to have more problems here compared to engaging with the Euclidean algorithm. He speculates that this is because of the nature of the mathematical discursive objects involved. In this substantiation routine, the students have to deal with abstract objects whereas in the Euclidean algorithm they were dealing with numbers. So, in commognitive terms I can argue that the difficulty here stems from the objects in the substantiation routine being discursive whereas in the Euclidean algorithm the objects are primary objects. Then, he continues talking about students' familiarity with the proof that two sets are equal and his expectation of their engagement with this proof.

> *"But it's just somehow standard it's a variation of something they've seen in coursework it's something they've seen proved before and they would get marks for at least saying that they need to... understanding the method that they've seen in lectures for this, it's to show that each is a subset for the other so I am looking for that in their answer, at least. And then trying to make some assessment of how their argument looks when they…, what their justification for each of those parts is."*

In the above, L1 says he is more interested in seeing the *procedure* of the substantiation routine, in commognitive terms. The *procedure* of the substantiation routine should be familiar to the students, as they have seen it during the module in lectures, exercise sheets and coursework, and they should be able to say that for two sets to be equal one has to be the subset of the other. L1 says that he is expecting to see at least the *procedure* of the routine described in the students' written answers. Then he examines the relationship between the narratives and the justification given.

As mentioned above the lecturer points out that the importance of this task is on the substantiation routine and on the fact that they have to go through the procedure that is not something that they have been doing so far in their school years. Moreover, the students have seen this before, and the lecturer is expecting to see at least that this is the *procedure* that they should follow. This is the first thing that he will care about and then, later on, he is going to care about the justification they give and the arguments they make. The important part of the task is the *procedure* and this procedure is something new to them and something that is very different from their school years. In his solution, L1 signals this from the start, this two-step approach.

5.2.4 In summary

Task 2		
Commognitive analysis		Lecturer's perspective on assessment
Mathematical discourses involved in the task	Discourse of set theory, integers and real numbers	-
Visual mediators	Symbols indicating sets Variables which take values from the integers or real numbers	Students face difficulties when dealing with sets
Routines (rituals, recall, substantiation, construction)	Substantiation: (2i), (2iia), (2iib) Construction: (2i) Recall: (2ii)	Students face difficulties when proving that one set is equal to another, as this is something very different to the school mathematics.
Instructions given regarding the procedure of the routine	Students are allowed to choose the procedure of the routine (as either the	-

	instruction is implicit or non-existent): (2i), (2ii) Hint regarding the relations not having all the properties (2ii)	
Instructions given regarding the justification required	(2i), (2ii)	-
Structure of the task	Gradual structure of the task (2ii)	-

5.3 Examination task 3 (Optional)

5.3.1 Task and commognitive analysis

(i)	Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*. For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief reasons for your answers. (a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$. (b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$. **[10 marks]**
(ii)	(a) State (but do not prove) Fermat's Little Theorem. (b) Compute the remainder when 27^{313} is divided by 11. (c) Find an integer $x \in \mathbb{Z}$ such that $19x \equiv 1\ (mod\ 36)$ **[10 marks]**

Figure 5.10: Second optional task Sets, Numbers and Proofs – Task 3

Task 3 (Figure 5.10) is split into two sub-tasks, the first one focusing on surjective and injective functions and the next one on modular arithmetic. In (3i), the students are asked to engage with the recall routine, as they have to provide the definitions of surjective and injective functions. Then they are asked to engage with a substantiation routine, by examining whether the given functions are injective or surjective or both. They are instructed regarding justification to be provided in their response by the prompt "<u>Give brief reasons for your answers</u>". During the revision lecture, the lecturer explained the difference between the phrase <u>"Give reasons"</u> and <u>"Give brief reasons for your answers"</u> and said that they could provide a sketch of the function to prove their claim. The gradual structure of the task is guiding the students to use the recalled narratives to examine whether the two functions given are satisfying them. Additionally, the wording of the task hints that the functions do not necessarily have the properties of surjectivity and injectivity with the phrase "<u>(or both, or neither)</u>". Similarly, with the task 3, the students are asked to engage with functions that have domains and codomains that are different numerical sets (reals in (3ia) and integers in (3ib)).

The next part of task 3 is about modular arithmetic. It is split into three sub-tasks. The first one is asking the students to engage in a recall routine as they have to recall Fermat's Little Theorem and they are explicitly not to prove it ("State (but do not prove)"). In the next part, the students are asked to compute a remainder of a number divided by 11. This is essentially an application of Fermat's Little Theorem. However, this connection is not explicitly stated in the wording of the task. This can be either an exploration routine or a ritual that the students engage in. For the last part of the task, the students are asked to engage in a substantiation routine as they are asked to find an integer which satisfies the given relation. As in (3iib), the *procedure* of the routine is not given to the students, the students' agency is not restricted, and there are no instructions regarding the justification of their response in the last two parts of (3ii).

5.3.2 Context and the lecturer's model solution

The students have been engaged with the properties of injectivity and surjectivity during their course both in the exercise sheets (Figure 5.11) and in the coursework (Figure 5.12). In the one given in the exercise sheets (Figure 5.11), the students are given more hints regarding the properties of functions that feature in the task "You may use properties of the sine, cosine function which you know from (say) Calculus XXXXX [the name of a module offered at this university] or 'A' level". Also, they were told that "your solution should not depend on a curve sketch, nor on differentiation". In the wording of this task, the students were given more hints regarding the *procedure* of the substantiation routine. Specifically, they were told that they could use narratives that they have endorsed about these functions in other modules from the same course or narratives from the secondary school. However, their answer should not depend on the visual mediator of a "curve sketch" or on "differentiation". The hints here do not only attend to the *how* that the students could engage but also with the *how* that the students should not engage. This level of guidance regarding the expected narrative is not given in the tasks in coursework (Figure 5.12) or the examination task (Figure 5.10).

5. For each of the following functions decide whether it is injective, surjective (or both, or neither). You may use properties of the sine, cosine functions which you know from (say) Calculus MTHA4005Y or 'A' level, but your solution should not depend on a curve sketch, nor on differentiation.

(i) $f_1 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = 2\sin(5x) - 3\cos x$;

(ii) $f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_2(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Figure 5.11: Task from exercise sheets on surjective and injective functions

5. For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief reasons for your answers.

(i) $f_1 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = x/(1+x^2)$; **[6 marks]**

(ii) $f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_2(x) = \exp(x)$; **[6 marks]**

(iii) $f_3 : \mathbb{N} \to \mathbb{N}$ given by $f_3(x) = 5x$. **[6 marks]**

Figure 5.12: Coursework task on injective and surjective functions

Regarding (4ii), the students have been asked in two tasks from the exercise sheets to engage with similar routines (Figure 5.13 and Figure 5.14). In these as well as the ones in the examination the students are not given instructions regarding the *how* of the substantiation routines.

5.(i) What is the last digit of 97^{17}?

(ii) Calculate the remainder when 37^{253} is divided by 29.

(iii) Show that there is no $x \in \mathbb{Z}$ for which $x^2 - 3$ is divisible by 8.

Figure 5.13: Exercise sheet task on modular arithmetic

6. Find $x, y \in \mathbb{Z}$ with $19x \equiv 1 \pmod{45}$ and $19y \equiv 15 \pmod{45}$.

Figure 5.14: Exercise sheet task on modular arithmetic

The solution (Figure 5.15) produced by the lecturer for task (3i) starts with the definition of a surjective and an injective function. Prior to producing that definition, the lecturer reiterates the function's domain and codomain. Then he defines the properties of a surjective and injective function by using the quantifiers and appropriate small letter symbols corresponding to the elements of the corresponding set *A* and *B*. Making use the notation of the function and illustrating the connection between the elements of *B* and the image of the function. In the injective definition, the lecturer also provides another narrative illustrating what injective means in simpler terms "(<u>so *f* sends distinct elements of *A* to distinct elements of *B*</u>)".

In (3ia) lecturer chooses two numbers 0 and π is that with the specific function result both in 1, however as the two elements are distinct this example shows that the function is not injective. This is a substantiation routine following the *procedure* of a counterexample. Then by using that the sine function has an image between -1 and 1 and its square is between 0 and 1 for all the elements of the function, he shows that the function is bounded. Thus, not able to have the whole set of real numbers as an image so it is not surjective. Also, the lecturer gives an example of why 2 which is in the reals does not belong in the image of the function.

In the next function, the domain and codomain are the integers. A counterexample is given where a number belonging to the integers is chosen but then the lecturer shows that there is no *n* belonging to the integers that can be multiplied with 3 to give 1. Then the substantiation of the function being injective is given. This is a directed proof, by setting the images the same and showing that the elements from the domain are also the same.

In the next part, the lecturer writes Fermat's little theorem. The quantifiers and the definition of the nature of the symbols is also present here. Then in (3iib), he shows the connection with Fermat's Little Theorem and the numbers given. The applicability conditions of the theorem are checked and since they are fine the lecturer continues by writing the result of the theorem. He then writes the connection between the number given and the number given in the theorem. Here it is important to note that the lecturer uses both the equality symbol and the equivalency symbol. Illustrating where the operation of divisibility has been used.

Solution 4:

(i) A function $f : A \to B$ is

- *surjective* if, for every $b \in B$ there exists $a \in A$ with $f(a) = b$;
- *injective* if, whenever $a, a' \in A$ and $a \neq a'$, then $f(a) \neq f(a')$ (so f sends distinct elements of A to distinct elements of B).

[2 marks]

(i)(a) g is not injective since $g(0) = 1/(1+0) = g(\pi)$ with $0 \neq \pi$. **[2 marks]**

g is not surjective. Indeed, since $0 \leq \sin^2(x) \leq 1$ for all $x \in \mathbb{R}$ it follows that

$$1/2 \leq 1/(1 + \sin^2(x)) \leq 1$$

for all $x \in \mathbb{R}$. Thus, for example, $2 \in \mathbb{R}$ but $g(x) \neq 2$ for all $x \in \mathbb{R}$. **[2 marks]**

(i)(b) h is not surjective. For example, $1 \in \mathbb{Z}$ but there is no integer $n \in \mathbb{Z}$ such that $1 = h(n) = 3n$. **[2 marks]**

h is injective, since for all $a, b \in \mathbb{Z}$ we have $h(a) = h(b) \Rightarrow 3a = 3b \Rightarrow a = b$. **[2 marks]**

(ii)(a) FLT: "If p is prime and $a \in \mathbb{Z}$ is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$". **[2 marks]**

(ii)(b) By FLT since 11 is a prime and 27 is not divisible by 11 it follows that $27^{10} \equiv 1 \pmod{11}$. Therefore $27^{313} = 27^{310}27^3 \equiv 27^3 \equiv 5^3 \pmod{11}$. Now $5^2 = 25 \equiv 3 \pmod{11}$. So $5^3 = 5.5^2 \equiv 5.3 \equiv 15 \equiv 4 \pmod{11}$. So the remainder on dividing 27^{313} by 11 is 4. **[4 marks]**

(ii)(c) Applying the Euclidean Algorithm with $a = 36$ and $b = 19$ gives:

$$\begin{array}{rcr|rcl}
a & = & 36 & & & \\
b & = & 19 & 19 & = & b \\
\hline
a - b & = & 17 & 17 & = & a - b \\
\hline
16b - 8a & = & 16 & 2 & = & 2b - a \\
\hline
9a - 17b & = & 1 & 2 & & \\
 & & & 0 & &
\end{array}$$

We conclude that $\gcd(36, 19) = 1$ and that

$$1 = 9a - 17b = 9 \cdot 36 + (-17) \cdot 19.$$

Reducing modulo 36 then gives

$$1 \equiv (-17) \cdot 19 \pmod{36}.$$

So $x = -17$ would do. (Equivalently we could take $x = 19 \equiv -17 \pmod{36}$. So in fact, as it turns out, $19 \cdot 19 \equiv 1 \pmod{36}$.) **[4 marks]**

Comments: 4(i) Standard definitions. Easy. 4(i)(a) Similar to tutorial question. Easy. 4(i)(b) Similar to coursework question. Moderate. 4(ii)(a) Standard result from lectures. Easy. 4(ii)(b) Computation. Moderate. 4(ii)(c) Computation, similar to tutorial sheet questions. Moderate.

Figure 5.15: Solution to Task 3 from Sets, Numbers and Proofs

In (3iic), the Euclidean algorithm is used. As mentioned in the analysis of the task, the *procedure* of the routine is not given in the wording of the task. Using the Euclidean algorithm, the greatest common divisor of 19 and 36 is identified and given as a linear combination of 19 and 36. Then the expression is rewritten with the modulo 36 and then the two numbers -17 and 19 are identified as equivalent solutions to this part.

In the end of Figure 5.15 after the solution, as in the previous tasks, the lecturer comments that (3i) is easy as standard definitions, (3ia) is also easy due to similarities with tutorial tasks. (3ib) is characterised as moderate and like coursework tasks. Then (3iia) is classified as easy as the students have seen this in the lectures. Finally, the last two parts are considered moderate (3iib) as a computation and (3iic) as similar to tutorial tasks.

5.3.3 Lecturer's perspectives: a commognitive account

During the interview, L1 commented on three main themes regarding students' engagement with this task. He comments on the difficulty that students have with the definition of an injective function and his perceptions as to why this is the case. He also explains some of the instructions, which either are present or absent from the wording of the task on the *procedure* of the routine and the justification of the expected solution.

"[T]hey know what injective means they just don't know how to write down the definition"

In the interview, the lecturer mentions the difficulty the students face when they learn a definition by rote. He mentions that there is a possibility for a student to decide correctly whether the functions are injective or surjective but provide a wrong definition.

> *"[S]omehow they know what injective means they just don't know how to write down the definition (...) it is a very strange experience to see that a student knows what injective means but can't write down what it means, it's something about maybe not even about mathematics it's about language and about logic"*

The lecturer talks about the routine of recalling a definition. Recalling a definition and constructing a narrative that can be endorsed by the community of the mathematicians and as such their lecturer requires engagement with visual mediators and specific word use as well as recall and construction routines. In order to construct the narrative, there should be a logical connection between the parts of the narrative. The lecturer comments on the fact that the students seem to understand the object of an injective function, but they are unable to give the definition. He thinks that the reasons behind this are difficulties with language and logic. University mathematical discourse as Sfard comments on has specific characteristics and relies a lot on abstraction (Sfard, 2014). The ability to recall and construct a mathematical definition is part of the practices of the mathematical community that the students should be able to engage in. However, having the sense and the meaning-making of what is an injective function is not necessarily meaning that the students would be able to give the definition of an injective function. In the definition, they have to engage with the domain of the function and the codomain and different elements belonging to the domain. Being able to engage with the elements of a definition and being able to apply the definition is something different in nature. Specifically, defining means engaging in the abstract nature of the discourse. Whereas examining whether a function is injective is asking to engage in a substantiation routine.

In commognitive terms, the lecturer talks about the importance of the meta-rules in constructing narratives that can be endorsed by the mathematics community. The students seem to struggle in connecting logically the written words and algebraic mediators. Essentially, they struggle in understanding the meta-rules of the definitions.

> *"[T]he big problem with this kind of definition is the students that just try to memorise definitions by rote to just try to memorise the sequence of symbols and words (…) the concept of injective is not hard and they would get full mark if they'd just explain in words with no symbols at all that injective means that distinct things map to distinct things. (...) some of them when they are preparing for exams seem to think that what they'll do is try and memorise the definition and this definition is easy to memorise it and then write it down incorrectly."*

The lecturer also mentions that the problem is that the students are trying to just recall the definitions of the objects. Without actually understanding what these objects, are and what they mean. He is also saying that they do not necessarily have to use the word use and the appropriate visual mediators but what he wants them to be able to do is to explain that distinct elements are mapped on to distinct elements. However, the students according to L1 seem to try to memorize the words without understanding them and thus end up with a narrative that is not actually the definition of the injective function.

He explains further the meta-rules governing the definition of an injective function. Essentially, the logic behind the definition, which has to do with the conditional statement of logic. In commognitive terms, the definition (an endorsed narrative by the general mathematical community) is based on the routines of the logical discourse. The students in order to be able to provide a narrative that can be endorsed have to use the visual mediators from the mathematical discourse and express their relationships using routines from the logical discourse. The students of this module have engaged with these routines. The lecturer continues arguing that this difficulty regarding the meta-rules of the logic could be because these meta-rules are still new for them. However, this shift needs to happen, and it could be difficult to achieve this by just engaging in the logic discourse.

> *"[S]ome people seem to struggle with these logical, logical ideas actually, implications, counter-positive of the statement, the fact that to show something is not true you just need to find one instance where it fails. Part of the module is about trying to teach them these things, again these things are new compared to what they are doing in school. And of course many of them do understand or do start to understand and the more pure maths they-they do at university the, my hope is the more familiar they become because they see more and more, because they need to see examples you know formally teaching logic with no examples is kind of useless as well because then that's just more symbols and more rules and a bit abstract."*

Commenting on the difference between the mathematical discourse that they were using in school and the one that they are asked to engage in now, the lecturer says that the defining routine is something new to them. Which is also the case for the numerical context of the variables mentioned in task 1. This module serves as an introduction to these practices of the mathematical community. Also, the routine of defining is a new routine for them. He then goes on saying that as much as they get to see these new practices of the community the more accustomed to this they will become. He comments on the familiarity that the students have with this routine. It starts as a ritualistic engagement and then it becomes explorative. Using commognitive terms, L1 says that the students become used to this engagement with the new discourse. Moreover, he is hopeful that students become more familiar with this as they see more and more examples of this new discourse being used. He also mentions the importance of seeing these examples where logic is part. However, teaching logic without the application might not be so useful. Therefore, even though he said earlier that the difficulty could be caused due to the logic that is behind those definitions. He then says that getting familiar with logic should be in an applied context for the students. As the absence of an application could possibly cause more problems as this is more abstract and has more symbols and rules compared to the definition of the injective function.

According to L1, this shift in their discourses will happen if they see this transition between the word use of the mathematical discourse they engage with (in this case the discourse of Sets, Numbers and Proofs) and the meta-rules of the logic. He also mentions that this is the case in students' engagement with the discourse of Analysis and the meta-rules of logic.

> *“They need to see this transition between (…) the symbols and the meaning and the logic of things and its one of the most important things and it is one of the hardest things to teach and not just here in analysis as well in particular in analysis getting the students to write down the definition of convergence is a real challenge because there are quantifiers and you know for all ε here exists n such that , that that that and again there are two types of students there are students that try to memorise the sequence of quantifiers and symbols and just mix them up because they are memorising symbols and it’s very easy to mix them up and what they write is meaningless or incorrect”*

In the excerpt above the lecturer comments generally in the routine of defining. He comments on the three parts of the defining process. The engagement with the symbols, the meaning and logical structure of the narrative. He is saying that this is one of the most important but also hard parts to teach to the students. According to him, as he mentioned above, the logical structure is better illustrated through examples rather than a pure engagement with the logic discourse, at this stage. He talks about how this is the case in other mathematical areas too. This is illustrating that the engagement with the symbols and the word use is not necessarily followed by a logical and meaningful meaning-making about the object. This causes problems when the students attempt to reproduce a definition that they have seen before as they tend to memorise things and when the definition involves many quantifiers and symbols it could create difficulties and end up with a definition that is not making sense. L1 comments that it is not that the students can not engage with the discursive object of an injective function but the difficulty is with the logic. Thus highlighting the connection between word use and meta-rules of the logic discourse again.

L1 also mentions that this discourse is new to them compared to school. The university discourse is based on rules and justifications and logical connections between word use.

> *"But it's new to them and also like I say compared with school they are used to, in school they are used to filling pages and pages with symbols they write solutions where there are almost no words. School mathematics is this equals this, equals this, equals this, equals this, equals and then they proved that 3 equals 3 or something and then they are happy and then they carry on. Right? What's missing is reasoning and logic and and writing sentences and writing arguments."*

In the above the lecturer comments in the differences of the university mathematics discourse and the school discourse. Many of the routines and word use are new to the students who as newcomers to the discourse they have to shift their discourse to the one used at university level. The lecturer acknowledges that this is different from the one that the students are used and that this is new (as this is a first-year module). He compares the narratives that they were asked to produce at school level and the ones they are asked to produce at university level. At university level, the students are asked to engage with abstract objects and they have to create narratives with words and not just sequences of symbols. As he mentions in the ones that they were asked to do at school the students were asked to produce narratives that were symbols only and they did not involve words or they rarely involved word use.

Lecturer's comments on engagement with the *procedure* of a routine

He then talks about the routine in (3iic), the *procedure* is not given to the students in the wording of the task and the students should decide on this. The *procedure* they have seen in the duration of the module is using the Euclidean algorithm. However, they could also try by trial and error.

"The way they've been shown how to do this is to use the Euclidean algorithm and they kind of have to remember that to get this right. (...) It just says find this. So again, if they just did some trial and error. I mean they technically they can try every number from zero to thirty… no from one to thirty five and then they'd eventually find one that works and that will be fine. It is not what I am looking for all right. But they could."

The *procedure* that the students should follow is not given for this part of the task, and the students are given the freedom to decide on the *procedure*. Essentially, the students who did not use the Euclidean algorithm but figured out the solution of the task are able to get full marks for their response, similar to (1iib). As the L1 mentions the students could try via trial and error, in chapter 6 I will present in detail the students' responses. However, the lecturer is not expecting them to do this.

Lecturer's comments on justifications and the different prompts

Finally, L1 talks about providing directions regarding the justifications in the students' solutions. He explains his meaning of the phrase "<u>Give brief reasons for your answers</u>". He elaborates that he means that they could provide a sketch of the function to prove their claim. (The lecturer, in the revision lecture, explained the difference between the phrase <u>"Give reasons"</u> and <u>"Give brief reasons for your answers"</u>.) In the following excerpt, the differences between the justification routine in different areas of mathematics is highlighted. The justification routine in the Analysis module is a long and detailed process whereas in this module a sketch and a short justification are enough.

"[T]here are situations in mathematics where you look at something and you go clearly this is true, right? We do it all the time. And there are situations where you really can do that, you just look at the thing and say yes it is clear and you don't want to waste your time writing a page of ... especially if it breaks to cases you don't really want to write a page of cases, case by case arguments to verify that something is true when it really is clearly true. And that's sort of what is going on here. (…) I try to indicate for questions like this they can be more sketchy with their solutions. And as long as they are correct

> *and as long as it's clear from what they are sketching that they understand then I am happy with that."*

In the excerpt above the lecturer comments on the justification that he expects the students to produce in this part of the task. In the previous excerpt, he talked about the fluency to shift the discourse from abstract to having an idea. I see this as talking about the realization trees. As Sfard notes there are moves upwards and downwards and also sideways in the realisation trees (Sfard, 2008, p. 191). Apart from the lecturer talking about situations where the narrative produce is endorsed without going into details. Here I note that he talks about how the same function given in a different mathematical area and being asked the same definition about injectivity or surjectivity it could involve a different *procedure.* He also comments here on the importance of the visual mediators.

5.3.4 In summary

Task 3		
Commognitive analysis		Lecturer's perspective on assessment
Mathematical discourses involved in the task	Discourse of functions, real numbers, integers	-
Visual mediators	Variables which values from integers, real numbers Symbols indicating functions and elements of functions	-
Routines (rituals, recall, substantiation, construction)	Substantiation: (3ia), (3ib), (3iic) Recall: (3i), (3iia) Ritual: (3iib)	Students face difficulties when writing the definition of injective
Instructions given regarding the procedure of the routine	Students are allowed to choose the procedure of the routine (as either the	Allowing creativity in the procedures of routines – a practice of the mathematical community

	instruction is implicit or non-existent): (3i), (3iib), (3iic) Hint regarding the functions not having all the properties (3i)	Specifically, in (3iic), students could either recall that the Euclidean algorithm can be used here or take examples
Instructions given regarding the justification required	(3i), (3iia)	Comparing the justifications that would be required for (3i) in the analysis discourse and the one asked in this part of the task
Structure of the task	Gradual structure of the task (3i) and (3iia), (3iib)	-

5.4 Summary and conclusion on Sets, Numbers and Proofs tasks

Analysis in this chapter focuses on closed-book examinations tasks from the first half of the Numbers, Sets and Probability part of the module, the Sets, Numbers and Proofs. The analysis highlights differences between the school mathematical discourse and the university mathematical discourse and at the same time the pedagogical actions in the context of assessment, that the lecturer implements aiming to assist students in their engagement.

The majority of the visual mediators are symbols and defining the numerical context of these symbols is an important routine of the university mathematics discourse, as can be seen from the interview excerpts but also the model solutions produced by the lecturer. The use of specific words and visual mediators is part of the engagement with the university mathematics discourse. In university, much attention is given to the numerical context of the variables whereas that is not necessarily the case for secondary school as usually, the numerical context is the context of real numbers. Within these three tasks, the importance of the clarification of the numerical context is visible in (1ii), (2ii) and (3). For example, the word divisor (1ii) is based on all the variables being part of the numerical context of the integers.

During the interview, L1 uses the pronoun “we” to refer to two groups. In the occasions reported in sections (5.2.3.1 and 5.3.3.1) he refers to the students and him as a group and in sections (5.1.3.1 and 5.3.3.3) he refers to the mathematical community of which he is a participant. In the latter cases and section (5.3.3.1), L1 speaks about practices of the mathematical community that the students with their entrance to university would become familiar such as engaging with routines (e.g., defining and justifying). The routines of the mathematical discourse that are present in these tasks are mostly substantiation and recall routines. The recall routines are routines of defining an object, a theorem or recalling the steps of a ritual or a substantiation routine. Regarding, the substantiation routines, in substantiating that an object has a certain property (e.g., 2iia, 2iib, 3ia, 3ib), in substantiating an equality relationship between objects (e.g., 2i, first part) or illustrating that these objects are different (e.g., 2i, second part). The substantiation and the defining routines are not something that the students are used to from the secondary school.

In section (5.3.3.1) L1, comments on students' difficulty recalling the definition of an injective function. He comments on how by trying to recall the definition the students find difficulties in the logical structure and the use of the symbolic visual mediators. The symbolic mediators, as shown in the quote below, due to their nature are forming the baseline for an abstract and autonomous mathematical discourse:

> "The process-object duality of symbolic mediators is a basis for compression and the subsequent extension of mathematical discourse, and it renders this discourse independent of external, situation-specific visual means. All this ensures a very wide applicability of the discourse." (Sfard, 2008, p. 162).

The definitions, in the university mathematics discourse, are using many symbolic mediators. So the definition of an injective function is an example of a definition that involves symbolic mediators, which the students have to have a clear meaning-making in order to be able to recall the definition using the appropriate symbolism when requested.

L1, in an excerpt in section (5.3.3.3) compares the engagement with the mathematical discourse of this module with the engagement with other modules. Specifically, this comparison concerns the justification in a substantiation routine (4ia, 4ib) in different mathematical areas namely this module and Analysis. The endorsement routines, as illustrated in Sfard's quote below differ between discourses.

> "Terms and criteria of endorsement may vary considerably from discourse to discourse, and more often than not, the issues of power relations between interlocutors may in fact play a considerable role." (Sfard, 2008, p. 134)

In the case of the module Sets, Numbers and Proofs a sketch accompanied with a short argument can be endorsed. Whereas, in the case of Analysis for a classification of a function as injective or surjective a much more detailed narrative would be required.

Having in mind the differences between the discourses, L1 is assisting the students in their engagement with the university mathematics discourse. There are prompts about the *procedure* of the routines, the justification in the expected solution and the absence of directions about *procedures* of routines. The latter one highlights the creativity of the *procedure* of the routines as a characteristic of the practices of the community. According to Sfard's theory of commognition a routine has a procedure or a routine course of action which is defined as a “set of metarules that determine (e.g., in numerical calculations) or just constrain (e.g., in proving or writing a poem) the way the routine sequence of actions can be executed” (Sfard, 2008, p. 302). There are three instances (1iib), (2i), (3ii), (3iii) where the *procedure* of the routine is not specified, and the students’ agency is not restricted. L1 comments on the beauty of following different *procedures* and how that allows him to give full marks to a response that does not follow the expected *procedure*.

I also note that the compulsory task (task 1) is more structured and with more directions on the *procedures* of the routines, compared to the other two tasks (task 2 and 3). More, specifically, as mentioned earlier in this section, there are instances where there is a direction regarding the *procedure* of a routine (1i, 1iib) or that specific narratives can be used to assist in the *procedure* of the routine (1iia, 1iib) also instructions regarding the justifications (1iic, 2ii, 3i).

The directions regarding the *procedures* could guide students to a ritualistic engagement with the routines. As these are routines that the students are not yet familiar with, as this is a first-year module, this engagement with rituals can be seen as a base towards building an explorative engagement with the routines of the university mathematics discourse. In the next chapter, I will be turning to students’ written responses to the same tasks in order to examine their actual engagement with the mathematical discourse. Also, I will be examining for differences between what the lecturer’s intended practice and the students’ actual engagement with the university mathematics discourse.

Chapter 6. Sets, Numbers and Proofs: Students' scripts.

In this chapter, I present the analysis of students' scripts from the three examination tasks in the final examination, corresponding to the content of Sets, Numbers and Proofs part of the module. I start by presenting the marks initially that the 22 selected students received in these tasks in relation to the whole cohort. Then, I continue with a presentation of the themes that emerged from the analysis of the scripts. I conclude with remarks about the students' scripts; I connect with the task analysis and the lecturer's perspectives presented in chapter 5; and, I link with the relevant literature.

6.1 Overview of student marks in the three tasks

Prior to presenting the analysis of students' scripts on the three tasks from Sets, Numbers and Proofs, from the examination of Sets, Numbers and Probability module, I provide information about the students' marks on each of these tasks. Task 1 (Figure 6.1) was worth 20 marks and the students had to achieve at least 40 marks to be able to pass the examination. Considering the analysis in chapter 5 (section 5.1), I examined in detail the students' scripts for the following: engagement with proof by induction; the *procedure* of the routines; the variables used and the numerical context of the variables.

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

[6 marks]

(ii)

(a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

[14 marks]

Figure 6.1: Compulsory task from Sets, Numbers and Proofs – Task 1

Fifty-four students took part in the final examination. In the graph (Figure 6.2), the marks of the student scripts are given. The marks of the selected 22 scripts are illustrated in grey. The students' marks ranged from 4 to 20, with the mean being around 16.85 marks.

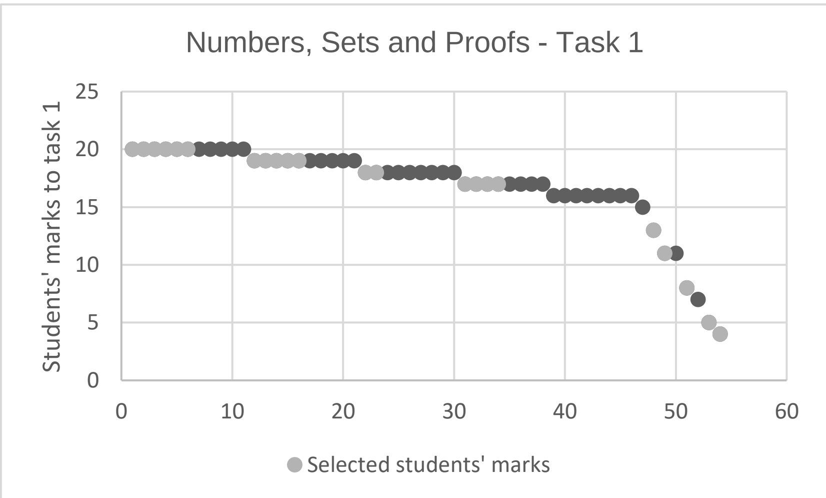


Figure 6.2: Marks to Sets, Numbers and Proofs compulsory task – Task 1

The first optional task, task 2 (Figure 6.3) is divided into two subtasks each one of them worth ten marks. The first one from Set Theory and the second one focuses on relations and the reflexive, symmetric and transitive properties. Considering the analysis in chapter 5 (section 5.2.1) and the corresponding lecturers' data (section 5.2.3), I examined in detail the students' scripts for the following: engagement with the substantiation routine proving that two sets are equal; use of Venn diagrams; engagement with the routine of first defining and then substantiating the reflexive, symmetric and transitive properties for the given relations; and, identifying variables and their numerical context.

(i) Prove carefully that if A, B and C are sets then
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Give an example of sets A, B and C such that
$A \cap (B \cup C) \neq (A \cap B) \cup C$.

[10 marks]

(ii) Suppose that A is a non-empty set and $\sim$ is a relation on A. Give the definitions of what is meant by saying that $\sim$ is *reflexive, symmetric* and *transitive*. In each of the following cases, decide which (if any) of these properties the given relation has. Give reasons for your answers.
(a) $A = \mathbb{Z}$ and $a \sim b \Leftrightarrow |a - b| \leq 10$ (for $a, b \in \mathbb{Z}$).
(b) $A = \mathbb{R}$ and $a \sim b \Leftrightarrow a - b \in \mathbb{Q}$ (for $a, b \in \mathbb{R}$).

[10 marks]

Figure 6.3: First optional task from Sets, Numbers and Proofs – Task 2

The students' marks for this task are in Figure 6.4. The marks ranged from 0 to 20 marks with an average of 13.23.

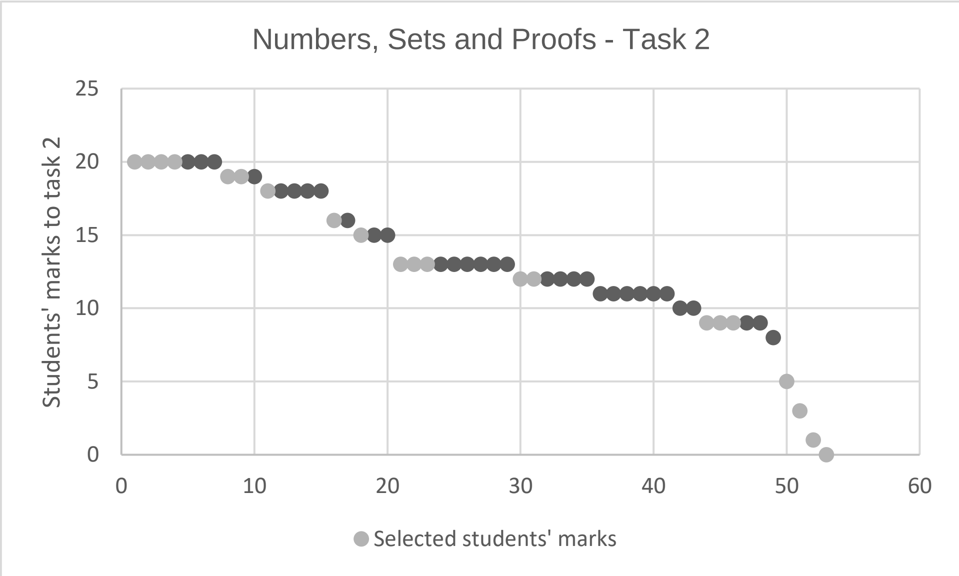


Figure 6.4: Marks to Sets, Numbers and Proofs first optional task - Task 2

The second optional task (Figure 6.5), was worth 20 marks and the first ten marks are on defining and examining the injectivity and surjectivity of functions and the other ten on modular arithmetic. Considering the analysis for task 3 in chapter 5 (section 5.3), I analyse students' scripts according to the following: engagement with the routine of defining and then the routine of substantiating that the given functions are (or are not) surjective and injective; identifying the variables used in their solutions and their numerical context; engagement with routines where the wording of the task does not specify the procedure.

(i) Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*.

For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief definitions for your answers.

(a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$.

(b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

[10 marks]

(ii)

(a) State (but do not prove) Fermat's Little Theorem.

(b) Compute the remainder when 27^{313} is divided by 11.

(c) Find an integer $x \in \mathbb{Z}$ such that $19x \equiv 1\ (mod\ 36)$

[10 marks]

Figure 6.5: Second optional task from Sets, Numbers and Proofs – Task 3

The students' marks to this task ranged from 0 to 20 with an average of 14.31 (Figure 6.6).

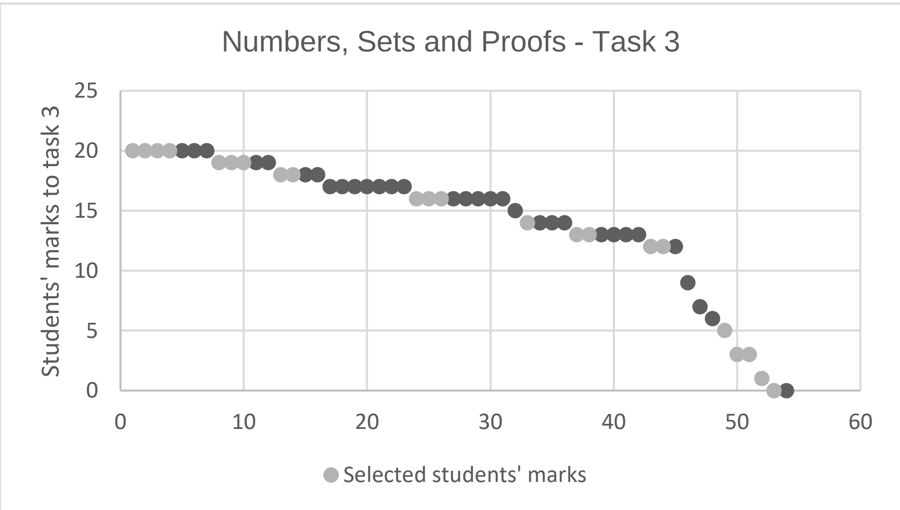


Figure 6.6: Marks to Sets, Numbers and Proofs second optional task - Task 3

6.2 Students' scripts: Commognitive analysis (word use, visual mediators, narratives)

The analysis of students' scripts focuses on the characteristics of the discourse as described in chapter 3 (section 3.3), namely word use, visual mediators, routines and narratives. The students' scripts are the narratives constructed by the students to answer the tasks. In the analysis, I focus on instances where the written word use, and the presence of visual mediators signal the evidence of unresolved commognitive conflicts in students' word use or the engagement with the routines of the university mathematics discourse.

In this first section of the chapter, I focus on word use and visual mediators. Specifically, I start by examining the variables introduced or used in the narrative and their numerical context. Then, I focus on the consistency in the naming of the variables or the objects involved in the narratives; I next turn to the logical symbols used by the students to help their narratives; and, I investigate the use of graphs and diagrams in the students' scripts. Finally, I comment on students' word use in these Sets, Numbers and Proofs tasks which draw on different mathematical discourses (e.g., Linear Algebra), or signal students' position in the university mathematics community.

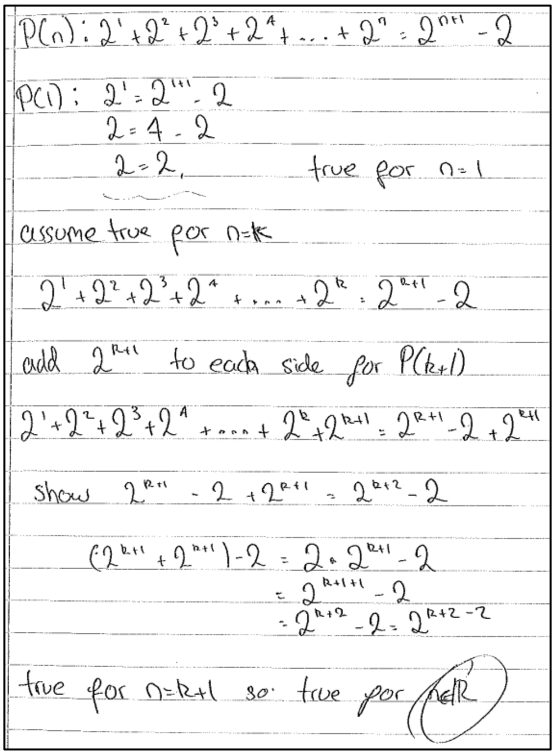
In the sections that follow, I mention the part of the task, which corresponds to the sampled students' scripts. Then, I discuss the students' scripts, especially, in relation to unresolved commognitive conflicts.

6.2.1 Specifying the set that a variable belongs to

In different parts of the tasks (Figures 6.1, 6.5) the students are asked to engage with different numerical sets. This implies the need to engage with different discourses: the discourse of integers and the discourse of real numbers. The analysis of the students' responses to these tasks showed errors, due to students' not being able to retain their narratives within a specific numerical context. Results from this category are also presented in Thoma and Nardi (2017, 2018a)

(i) The numerical context of the variables in proof by induction [Task (1i)]

In (1i) the students are performing proof by induction. The numerical context of the proof is the natural numbers. Five students ([01], [03], [04], [15], [17]) do not comment on the numerical context of variable *k*. In figure 7, when [17] introduces the variables *n* and *k*, the student does not define their numerical context. At the concluding section [17] writes that the variable *n* belongs to the real numbers instead of the natural numbers. However, this is not the case. The statement is true for *n* belonging to the natural numbers, and the proof by induction shows that the statement is true only for natural numbers. By not commenting on the numerical context of the variables (*n* and *k*), [17]'s script presents evidence of problematic sense making of the routine of proof by induction.


$P(n): 2^1+2^2+2^3+2^4+\ldots+2^n = 2^{n+1}-2$

$P(1): 2^1 = 2^{1+1}-2$

$2 = 4-2$

$2 = 2,$ true for $n=1$

assume true for $n=k$

$2^1+2^2+2^3+2^4+\ldots+2^k = 2^{k+1}-2$

add 2^{k+1} to each side for $P(k+1)$

$2^1+2^2+2^3+2^4+\ldots+2^k+2^{k+1} = 2^{k+1}-2+2^{k+1}$

show $2^{k+1}-2+2^{k+1} = 2^{k+2}-2$

$(2^{k+1}+2^{k+1})-2 = 2\cdot 2^{k+1}-2$

$= 2^{k+1+1}-2$

$= 2^{k+2}-2 = 2^{k+2}-2$

true for $n=k+1$ so true for $n\in\mathbb{R}$

Figure 6.7: Student [17]'s response to (1i) – the marker circled "$\mathbb{R}$"

(ii) The numerical context of the divisor and the variables from the linear combination [Task (1ii)]

(ii)

(a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

[14 marks]

Figure 6.8: Snapshot of the compulsory task from Sets, Numbers and Proofs – Task 1(ii)

In task (1ii) the students are asked to engage in different routines in the numerical context of the integers (Figure 6.8). The set of integers is closed under addition, subtraction, and multiplication but not under division. Students' scripts ([01], [03], [04], [06], [07], [08], [11], [13], [16], [17]) present evidence that there is a conflation between the discourses of integers and reals. Specifically, in five students' responses ([01], [04], [07], [08], [13]), students either introduce variables belonging to a different numerical context (e.g., the natural numbers) (Figure 6.9) or they talk about the variables as constants (Figure 6.10)

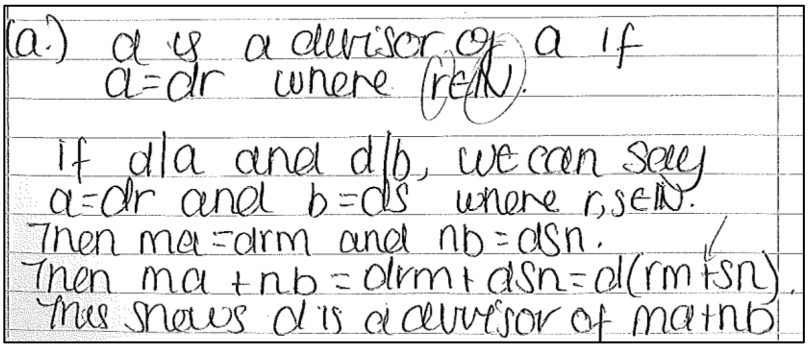
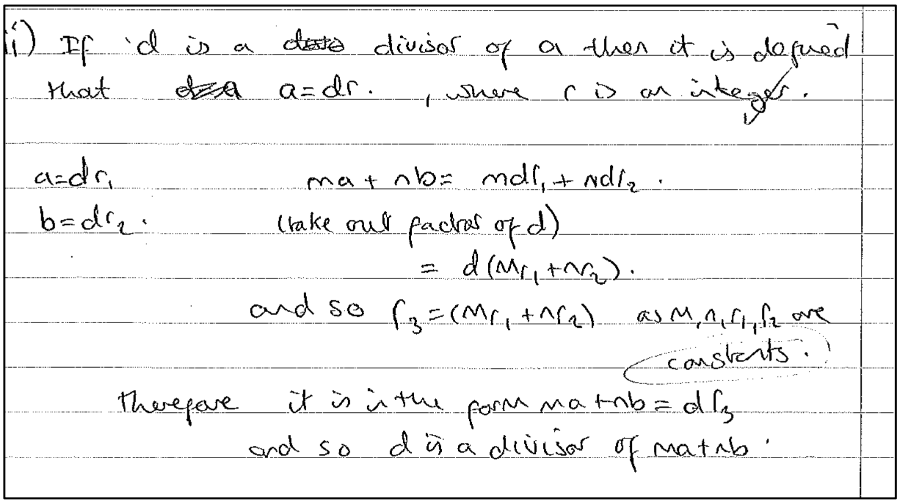

(a.) d is a divisor of a if
$a=dr$ where $(r \in \mathbb{N})$.
If $d|a$ and $d|b$, we can say
$a=dr$ and $b=ds$ where $r,s \in \mathbb{N}$.
Then $ma=drm$ and $nb=dsn$.
Then $ma+nb=drm+dsn=d(rm+sn)$.
This shows d is a divisor of $ma+nb$

Figure 6.9: Student [07]'s response to (1ii) – the marker added the two circles and the tick

Student [07], in the definition of the divisor and then later in the substantiation routine, defines variables *r* and *s* as natural numbers instead of integers. This illustrates a conflation between the numerical contexts that the new variables introduced by the student belong to. The other variables that the student uses in her response are defined in the wording of the task as being integers. However, the ones that the student introduces, instead of belonging to the integers, are defined as natural numbers.

Similarly, student [04], when defining the new variables and including some of the ones used in the wording of the tasks, says that they are constants (Figure 6.10). The constants take a specific value, and their value cannot be changed. However, a variable can change its value. By defining the integers *m*, *n*, r_1, r_2 as constants, the student does not consider that these variables do not take one specific value but, as *a* and *b* are not defined with a specific value, their values will also vary.



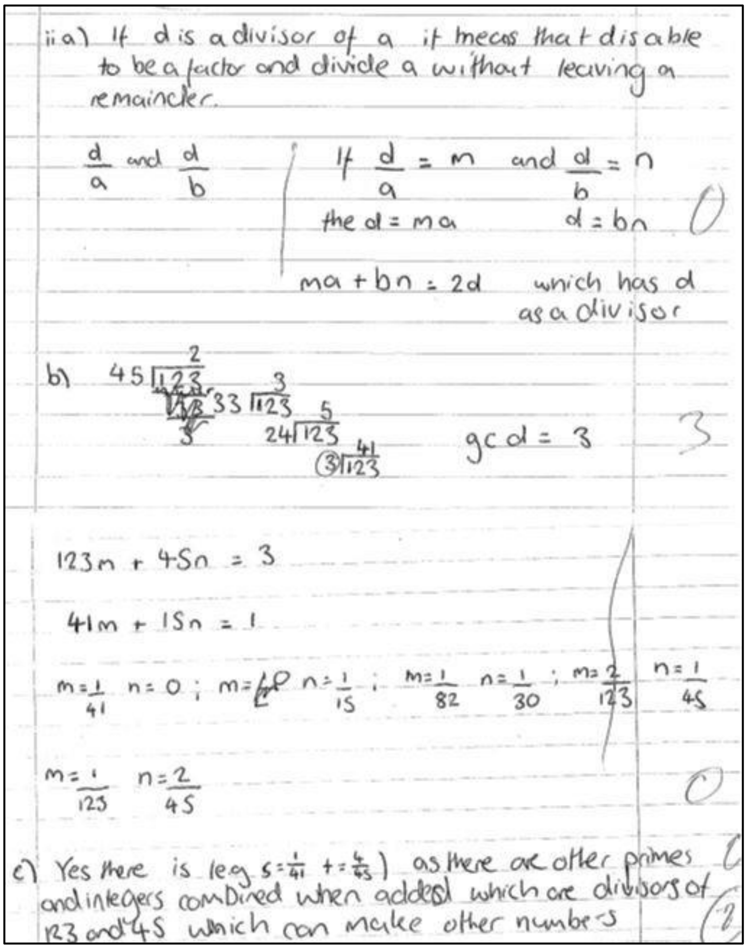
ii) If 'd is a ~~d~~ divisor of a then it is defined that ~~d~~ $a=dr$. , where r is an integer.

$a=dr_1$ $\quad ma+nb= mdr_1+ndr_2$.
$b=dr_2$. $\quad$ (take out factor of d)
$= d(mr_1+nr_2)$.
and so $r_3=(mr_1+nr_2)$ as m,n,r_1,r_2 are constants.
Therefore it is in the form $ma+nb=dr_3$
and so d is a divisor of ma+nb.

Figure 6.10: Student [04]'s response to (1ii) – the marker circled "constants" and added the tick

In figure (11), student [03] starts by writing the relationship between the divisor *d* and *a* using verbal written mediators and then, in the second part of (1iia), the student uses symbolic mediation to show what happens when *d* is the divisor of *a* and *d* is divisor of *b*. The symbolic realisation of the divisor involves fractions with *d* being the numerator and *a* and *b* being the denominators. If [03] had written the fractions the other way around, the result of the division would be an integer. However, as the divisor is smaller or equal to a, by definition, the fractions $\frac{d}{a}$ and $\frac{d}{b}$ are non-integers. The variables *m* and *n* seem to be taken as non-integers, conflicting with the introduction of the variables *m* and *n* in the wording of the task as integers. The symbolic mediators used by the student can be seen as a translation of the verbal written mediators, without taking into account that the fraction line means that the denominator *a* divides the numerator *d*. [03]'s response is evidence of unclear meaning making regarding the object of divisor as the student initially explains that *d* is a factor of *a,* then concludes that $d=ma$, using the discourse of reals and then saying, correctly, that the product $2d$ has *d* as a divisor.

In the next part of the task, [03], having found that the greatest common divisor is 3, writes $123m + 45n = 3$. Then, dividing the equality by 3 without considering that the integers are not closed under division, [03] takes different rational numbers for which the new equality stands. Apart from the symbolic mediation, the word use also suggests evidence of unresolved commognitive conflict. The terms "primes" and "integers" are used to describe the fractions $\frac{1}{41}$ and $\frac{4}{45}$ calculated in the next part of the task, signalling a ritualised use of the word "integer" by using the words provided in the wording of the task without examining their numerical context. Both the word use and the symbolic mediation show that [03]'s work is not embedded within the numerical context of the integers.



ii a) If d is a divisor of a it means that d is able to be a factor and divide a without leaving a remainder.

$\frac{d}{a}$ and $\frac{d}{b}$ | If $\frac{d}{a} = m$ and $\frac{d}{b} = n$

the $d = ma$ $d = bn$ 0

$ma + bn = 2d$ which has d as a divisor

b) $45\overline{)123}$ 2 ... $33\overline{)123}$ 3 ... $24\overline{)123}$ 5 ... $3\overline{)123}$ 41 gcd = 3 3

$123m + 45n = 3$

$41m + 15n = 1$

$m = \frac{1}{41}$ $n = 0$; $m = 0$ $n = \frac{1}{15}$; $m = \frac{1}{82}$ $n = \frac{1}{30}$; $m = \frac{2}{123}$ $n = \frac{1}{45}$

$m = \frac{1}{123}$ $n = \frac{2}{45}$ 0

c) Yes there is (e.g. $s = \frac{1}{41}$ $t = \frac{1}{45}$) as there are other primes and integers combined when added which are divisors of 123 and 45 which can make other numbers 1/2

Figure 6.11: Student [03]'s response to 1(ii)

(iii) The numerical context of the variables in the injective and surjective functions [Task (3i)]

(i) Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*.

For each of the following functions decide whether it is injective, surjective (or both, or neither).

Give brief reasons for your answers.

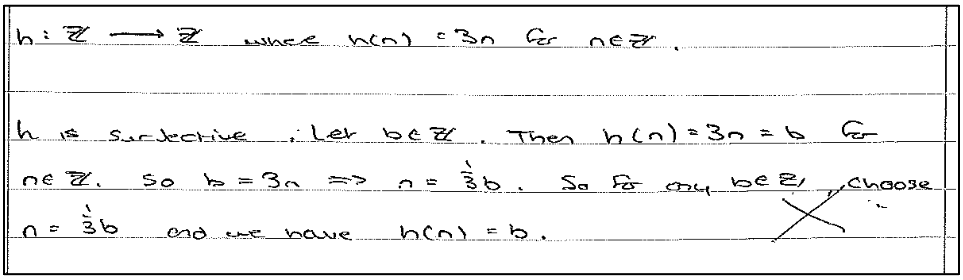
(a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$.

(b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

[10 marks]

Figure 6.12: Snapshot of the second optional task from Sets, Numbers and Proofs – Task 3

The students are asked to engage in examining whether the two given functions are injective or surjective (Figure 6.12). In their response, the students are dealing with elements of both the domain and the codomain and the different functions are defined in different numerical contexts (3ia) is in the reals (ℝ) and (3ib) is in the integers (ℤ). Illustrating which element belongs to the domain and codomain is an important part in showing that the function is injective or surjective. Five scripts ([01], [02], [08], [10], [13]) evidence that they do not examine the numerical context of the variables used. This is problematic in the case of (3ib) as the numerical context is the integers which are not closed under division and the students have to contain their response within this context. In the case of (3ia) the students are using the numerical context of ℝ which is the largest numerical context they have been using in school as well.


$h: \mathbb{Z} \longrightarrow \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

h is surjective : Let $b \in \mathbb{Z}$. Then $h(n) = 3n = b$ for $n \in \mathbb{Z}$. So $b = 3n \Rightarrow n = \frac{1}{3}b$. So for any $b \in \mathbb{Z}$, choose $n = \frac{1}{3}b$ and we have $h(n) = b$.

Figure 6.13: Student [13]'s response to (3iib)

Student [13] copies the definition of the function *h* as given in the wording of the task and defines *b* and *n* as integers (Figure 6.13). However, when performing a division, [13] does not examine the numerical context of *n* and check whether the variable still belongs in the integers. Thus, their solution evidences a commognitive conflict. The numerical context of the variable *n* is not examined again, and it is taken for granted that this is an integer.

Student [08] produced a graph for function *h* from the ℤ to the ℤ (Figure 6.14). The graph is a straight line, which is the way that this graph would be if the domain and codomain of this function were the reals. However, the function is discrete as the domain and codomain are the integers. This confusion with the numerical context is also visible in the narrative that is underneath the graph. The variables *a* and *b* are used without their numerical context being defined.

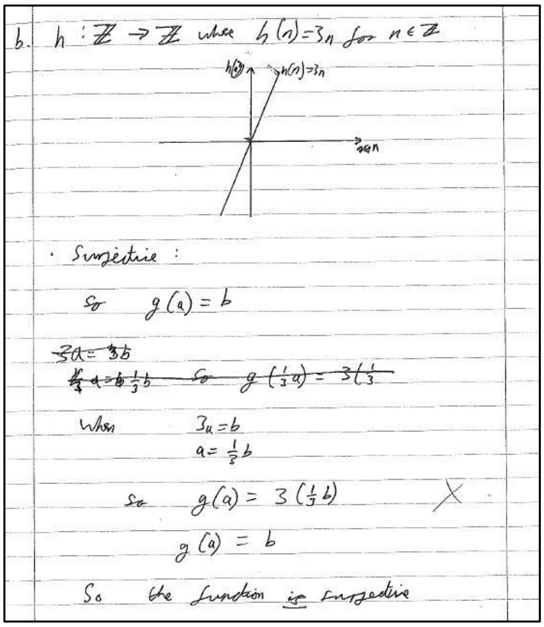


Figure 6.14: Student [08]'s response to (3ib). The marker added the cross symbol at the right-hand side of the image.

Attempting to examine the surjectivity of the function, [08] finds the inverse function of *h* by solving 3*a*=*b* for *a* and dividing the equality by 3. The numerical context of the numbers involved in the equality does not change, regardless of whether the numbers belong to the real or the rational numbers. However, in this case, the variables should be integers. [08] does not define the numerical context of the variables prior to engaging with the substantiation routine. This causes an error, later, as the division of an integer *b* by 3 does not ensure that the quotient would also be an integer.

In student [10]'s response to (3ib) (Figure 6.15), the elements of the domain and the codomain are mixed up. At the start, set *A* is defined as the domain and *B* as the codomain. In the scribbled-out part, *A* is appearing as equal to *a*/3, and, this way, *A* becomes an element of the codomain. The elements of set *A* using lower case letters and the actual set *A* are conflated. Similarly, *B* is defined as an element of the codomain, and then *A*=*a* becomes an element of the domain. These all are deleted. The student then defines *B*=*b* as an element from the codomain and *A* is *a*/3 so *a*=*b*. Probably what the student wanted to write was that there are two elements in the codomain and these would be in capital letters *A* and *B* and these would be equal to *a*/3 and *b*/3 accordingly. Thus, this would mean that *a*=*b* if the *A* and *B* are equal.

However, the use of the upper case and lower-case letters illustrates conflation between the sets and the elements of these sets, which confuses the marker of the script.

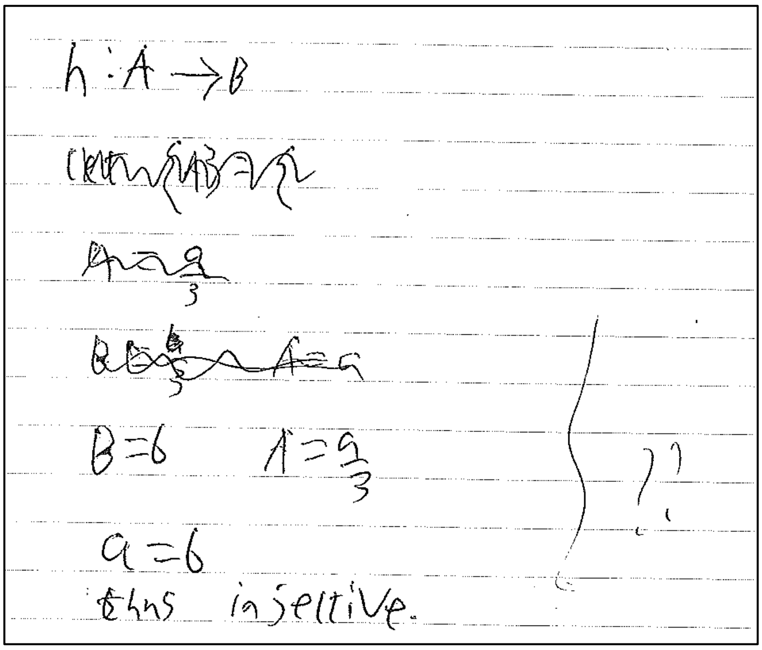


Figure 6.15: Student [10]'s response to (3ib) – The marker added the question marks

(iv) The numerical context of the variables in modular arithmetic [Task (3ii)]

There are cases where the students do not specify the numerical context of a symbol being used in a narrative they produce. In task (3ii) (Figure 6.16) the students are asked to recall Fermat's Little Theorem.

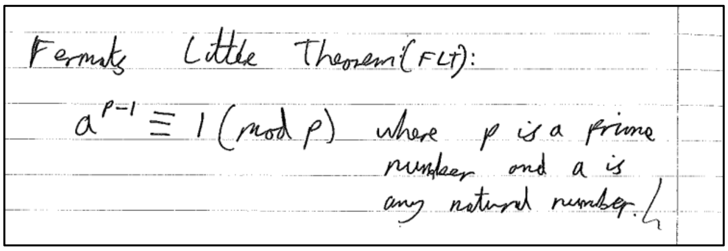
(ii)

(a) State (but do not prove) Fermat's Little Theorem.
(b) Compute the remainder when 27^{313} is divided by 11.
(c) Find an integer $x \in \mathbb{Z}$ such that $19x \equiv 1\ (mod\ 36)$

[10 marks]

Figure 6.16: Snapshot from second optional task from Sets, Numbers and Proofs – Task (3ii)

Ten students ([01], [02], [03], [04], [07], [10], [14], [16], [20], [22]), when writing the theorem, do not comment on the numerical context of the variables that they introduce in their narratives either for both of the variables or just for one. Student [08] does talk about the numerical context of variables *a* and *p*. However, instead of *a* being integer, [08] defines *a* as natural (Figure 6.17). Also, as I illustrate in sections 6.3.1 and 6.3.3, not defining the numerical context of the variables used causes problems in the application of the theorem in the next section of the task.


Fermats Little Theorem (FLT):

$a^{p-1} \equiv 1 \pmod{p}$ where p is a prime number and a is any natural number.

Figure 6.17: Student [08]'s response to task (3iia)

Student [03] defines the unknown *x* as a rational number in an attempt to find a number that, multiplied by 19, would provide the right response (Figure 6.18). However, [03] does not question the numerical context that *x* should belong to and thus the rational number resulting is not challenged, leading [03] to the wrong answer.

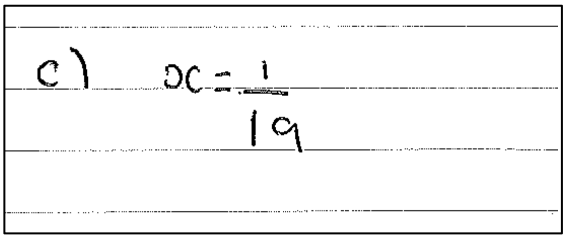

c) $x = \frac{1}{19}$

Figure 6.18: Student [03]'s response to (3iic)

6.2.2 Inconsistency in the naming of variables

Here, I focus on the visual mediators present in the students' scripts. Specifically, I focus on the consistency of the symbolic mediators. Symbols are used to illustrate different objects involved in the students' narratives. In the following sections, I comment on the scripts that illustrated the inconsistency in the use of visual mediators.

(i) Inconsistency in the naming of variables involved in the proof of induction [Task (1i)]

The first group of scripts corresponds to task (1i) (Figure 6.19)

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

[6 marks]

Figure 6.19: Snapshot of the compulsory task from Sets, Numbers and Proofs – Task (1i)

In the wording of the task, the students are not given a name for the statement. They are given the realisation of the statement using the equality. In the previous tasks that they have applied the proof by induction the statement usually had a name (e.g., *P*(*n*)). However, in this case, the students are expected to name the statement. There are two students ([02], [07]) that provide a name that could signal a conflict. [07] and [02] use the name *P*(*x*) instead of *P*(*n*) to signal the statement (Figure 6.20) or used *f*(*n*) (Figure 6.21). For the first, [07] uses *x* instead of using the symbol *n* from the equality. The variable *x* is usually used to signal an unknown variable. [07] uses this *x* in the naming of the statement, in the first line of writing. However, when providing a different realisation of the statement, [07] does not use *x* but uses *n.*

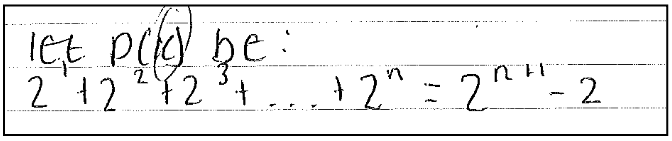


Figure 6.20: Student [07]'s response to (1i)

The latter naming is not problematic, but it does signal conflation with the functions with integer or natural values, and this is also symbolised with the equality connecting the name of the statement and the realisation of the statement. Usually, the letter *f* is used for functions. Being consistent with the symbols used is something new for the students as in school they did not have so many different objects to deal with, and they were usually working within one numerical context. The symbol *f*(*n*) is typically used to signal a function with the domain being the natural or the integer numbers. In [02]'s response, the statement *P*(*n*) and the symbol of the function, *f*(*n*), are conflated.

1 $2^1+2^2+2^3+2^4+\ldots+2^n = 2^{n+1}-2 = f(n)$

Figure 6.21: Student [02]'s response to (1i)

(ii) Inconsistency in the naming of variables involved in substantiation: the case of equality between sets

The students are asked to engage in proving that two sets are equal in (2i) (Figure 6.22). In the next section of this chapter, I examine in detail students' engagement with this proving routine. In proving that two sets are equal, they have to take an element of one of the sets and show that this is an element of the other set, thus proving that the first set is a subset of the second and then showing that the second set is a subset of the first one.

(i) Prove carefully that if A, B and C are sets then
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Give an example of sets A, B and C such that
$A \cap (B \cup C) \neq (A \cap B) \cup C$.

[10 marks]

Figure 6.22: Snapshot of the first optional task from Sets, Numbers and Proofs – Task (2i)

In engaging with this routine, three students ([11], [19], [20]), use this procedure to show that the sets are equal. However, in their responses, they use the same variable to mean an element coming from different sets.

Student [19] (Figure 6.23) starts by writing an equivalence that needs to be proven. This had also been given to them in their lecture notes. Then, [19] shows that the element *x* of $A \cap (B \cup C)$ belongs to $(A \cap B) \cup (A \cap C)$ and the other way around. However, [19] uses *x* to signal a random element of $A \cap (B \cup C)$ and then uses *x* again to indicate an element from $(A \cap B) \cup (A \cap C)$. Being able to manipulate the variables, and give different names in order distinguish which variable is illustrating an element belonging to one set and which variable is belonging to the other set, is an essential skill. By giving the same name to the element, there could be an assumption from [19] that the element is the same in both sets since the sets are equal. However, the assumption to start with should be that these sets are not equal and the students are aiming to prove that they are. By taking different elements and distinguishing that these are different, then the students consider the sets as different objects and, later, prove that they are the same one. This though could also be explained: as [19], instead of writing that they have to show the equality, translates the equality into an equivalence relation with the realisation of the relationship between the sets becoming a realisation between the elements of sets. However, that stands in the way of distinguishing that the element of one set could be different from the other set.

Need to show that

$x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)$

If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in (B \cup C)$

Since $x \in B \cup C$, then either $x \in B$ or $x \in C$.

If $x \in B$, then $x \in A$ and $x \in A \cap B$

If $x \in C$, then $x \in A \cap C$.

Then we must have that

$x \in (A \cap B) \cup (A \cap C)$.

Now we show that if $x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap (B \cup C)$.

If $x \in (A \cap B) \cup (A \cap C)$ then either $x \in A \cap B$ or $x \in A \cap C$.

If $x \in A \cap B$ then $x \in A$, and if $x \in A \cap C$ then $x \in A$. So we have $x \in A$.

Then $x \in A \cap (B \cup C)$.

Hence,

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

Figure 6.23: Student [19]'s response to task (2i)

(iii) Inconsistency in the naming of variables involved in the recall and substantiation routines: the case of injective and surjective functions

In (3i), there are three different functions accompanied by their notation *f*, *g*, and *h* (Figure 6.24). Similarly, the visual mediators used to signal the independent variable in each is different. For *f*, there is no indication regarding the independent or dependent variable, apart from giving *A* as the domain and *B* as the codomain of the function. For function *g,* the domain is $\mathbb{R}$ and the symbol used to show the elements of the domain is *x*. The domain of *h* is $\mathbb{Z}$, and the symbol *n* is used to signal the independent variable. In the scripts of eleven students ([01], [02], [04], [05], [06], [07], [08], [13], [20], [21], [22]), I observed inconsistency in the naming of the functions or the independent variables or the visual mediators signalling elements of the domain or the codomain of the function.

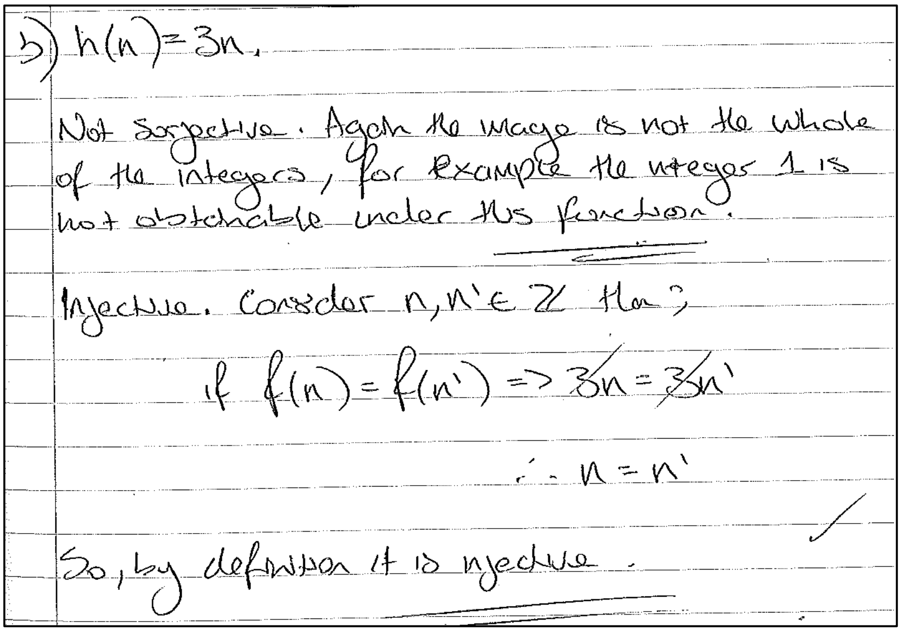
(i) Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*.

For each of the following functions decide whether it is injective, surjective (or both, or neither).
Give brief reasons for your answers.
(a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$.
(b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

[10 marks]

Figure 6.24: Snapshot from the second optional task from Sets, Numbers and Proofs – Task 3(i)

Student [21] uses *f* instead of *h* to signal the function (Figure 6.25). The symbol *f* is being used in the definition of injective and surjective, but in this case the function h is being examined. There is a formula that connects the dependent and the independent variable for the h function. However, there is no formula for the *f* function. [21] starts by rewriting the given about the *h* function. However, when later is asked to show that this is injective, the symbol *f* appears.


b) $h(n)=3n$.

Not surjective. Again the image is not the whole of the integers, for example the integer 1 is not obtainable under this function.

Injective. Consider $n, n' \in \mathbb{Z}$ then;

if $f(n)=f(n') \Rightarrow \cancel{3}n=\cancel{3}n'$

$\therefore n=n'$

So, by definition it is injective.

Figure 6.25: Student [21]'s response to task (3ib)

This inconsistency with the symbols is also visible in [01]'s response to the same task (Figure 6.26). However, in this case, the conflation of the symbols regards the independent variables. The function is defined in the integers, and the variable n is used. However, in trying to show that the function is injective and surjective, [01] changes the independent variable to *x*. Usually, *x* is used to represent real numbers, and this was the case in task (3ia). However, in this case, the symbol *n* signals that the independent variable takes values from a different numerical set. The conflation between the two symbols could also be attributed to the fact that the student does not consider that, due to the numerical context of the independent variable, the function is not surjective. However, if the function was defined in the reals (ℝ), it would be both injective and surjective.

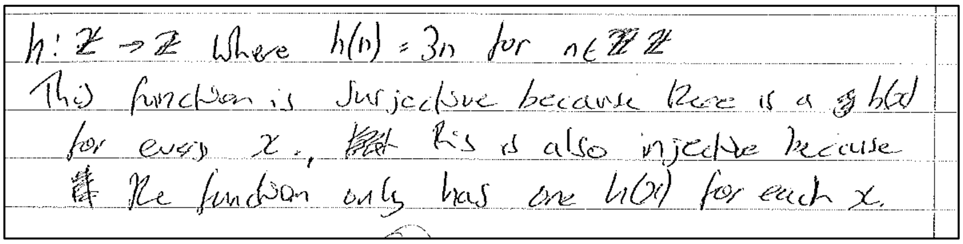

$h: \mathbb{Z} \to \mathbb{Z}$ Where $h(n) = 3n$ for $n \in \mathbb{Z}$
This function is Surjective because there is a $h(x)$
for every x, ~~But~~ this is also injective because
the function only has one $h(x)$ for each x.

Figure 6.26: Student [01]'s response to task (3ib)

Another case where I observed inconsistency between the symbols used is the following (Figure 6.27).

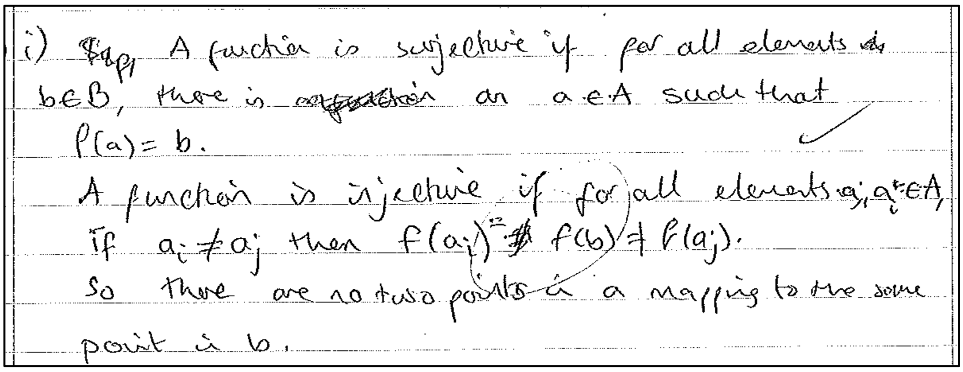
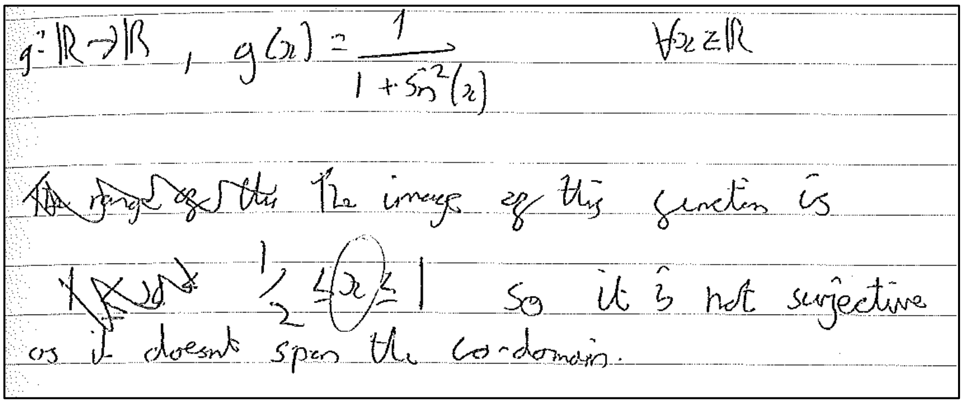

i) ~~Sup~~ A function is surjective if for all elements ~~in~~
$b \in B$, there is ~~[illegible]~~ an $a \in A$ such that
$f(a) = b$.
A function is injective if for all elements $a_i, a_j \in A$,
if $a_i \neq a_j$ then $f(a_i) = f(b) \neq f(a_j)$.
So there are no two points in a mapping to the same
point in b.

Figure 6.27: Student [04]'s response to task (3i)

In the definition of a surjective function, [04] uses the symbol *b* to be *f*(*a*) (Figure 6.27). However, in the definition of an injective function, [04] says that $f(a_i)$ is equal to *f*(*b*). But *b* was defined earlier as an element of the codomain and not as an element of the domain. This conflation between symbols and elements of the domain and the codomain could signal an underlying conflict regarding the relationship between the elements of the domain and the codomain. Also, there is an inconsistency between the symbols *a* and *A* and similarly *b* and *B*. In the mathematical community, lower-case letters are used to signal elements of sets, whereas the upper-case letters are used to signal the sets themselves. In [04]'s response, there is problematic use of these two symbols as the student uses lower case *a* but writes about points in "*a*" meaning that there are elements in "*a*". This signals a conflation between the elements of *a* and the set *A* itself.

Finally, another case of conflation between symbols is illustrated in Figure 6.28. [20] wants to show that the function can only take value between ½ and 1. However, instead of writing that $g(x)$ is between ½ and 1, the student writes that x is between ½ and 1. [20], then writes that the function does not span the codomain. However, what the inequality signals is a constraint in the values that x takes, not $g(x)$.


$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{1+\sin^2(x)}$ $\forall x \in \mathbb{R}$

~~The range of this~~ The image of this function is

$\frac{1}{2} \leq x \leq 1$ So it is not surjective

as it doesn't span the co-domain.

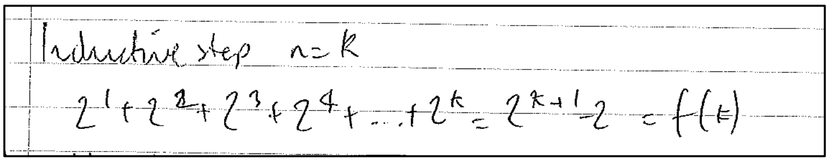
Figure 6.28: Student [20]' response to task (3ia) – the marker circled the "*x*".

6.2.3 Use of logical symbols: The structure of students' narratives

Here, I focus on a specific type of visual mediators present in the students' scripts, the logical symbols. Specifically, I present the analysis of students' scripts which evidences conflating use of the equality and quantifiers.

(i) Conflating use of the equality symbol

In five students' scripts ([02], [03], [08], [11], [16]) the students use the symbol of the equality to denote that an object is defined as another object. This occurs four times in the students' scripts to (1i). The students signal that there is a conflation between the object of the statement and a function (Figures 6.29, 6.30). I have discussed the notation of the function instead of the usual *P*(*n*) in the previous section (6.2.2). The focus now is on the equality sign between the statement and the *f*(*k*).

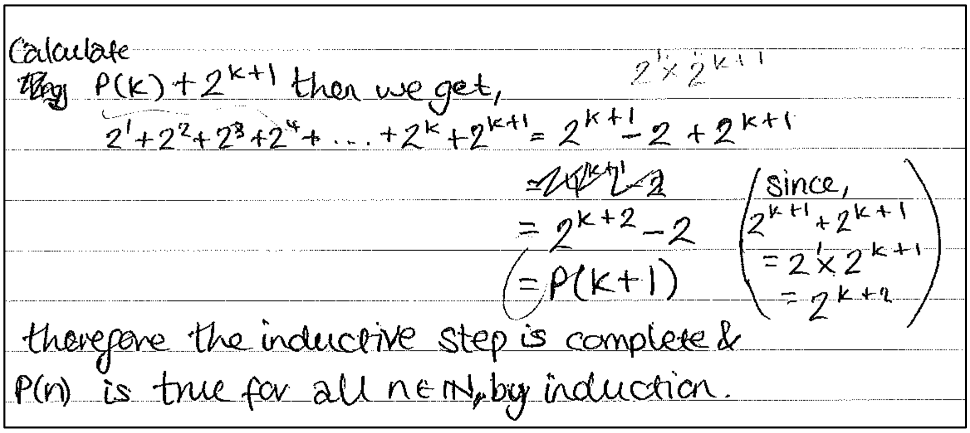


Inductive step $n=k$

$2^1+2^2+2^3+2^4+\dots+2^k = 2^{k+1}-2 = f(k)$

Figure 6.29: Student [02]'s response to (1i)

Similar to [02]'s script, there are two instances in [11]'s script (Figure 6.30) where conflation between the statement and the object of a function is visible. The first one is when [11] writes $P(k) = 2^{k+1}$ signalling an operation between the statement $P(k)$ and the 2^{k+1}. Here, the student seems to have written that, to indicate that they will add 2^{k+1} on both sides of the equality of the statement $P(k)$. This conflation is, also, visible in the bottom of the figure where $2^{k+2} + 2 = P(k+1)$. It seems that the student conflates statement $P(n)$ with a function.



Calculate

~~they~~ $P(k) + 2^{k+1}$ then we get, $\quad 2^1 \times 2^{k+1}$

$2^1+2^2+2^3+2^4+\dots+2^k+2^{k+1} = 2^{k+1}-2+2^{k+1}$

$= 2^{k+2}-2$ $\quad$ (since, $2^{k+1}+2^{k+1} = 2^1 \times 2^{k+1} = 2^{k+2}$)

$(= P(k+1))$

therefore the inductive step is complete & $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

Figure 6.30: Student [11]'s response to (1i)

Finally, the symbol of equality is used by [16] (Figure 6.31) to denote that the set resulting from operations between the three sets A, B and C can be depicted in Venn diagrams. Here, the student equates the different realisations of the union and the intersection or a combination of unions and intersections. These realisations are in their symbolic visual register and Venn diagrams.

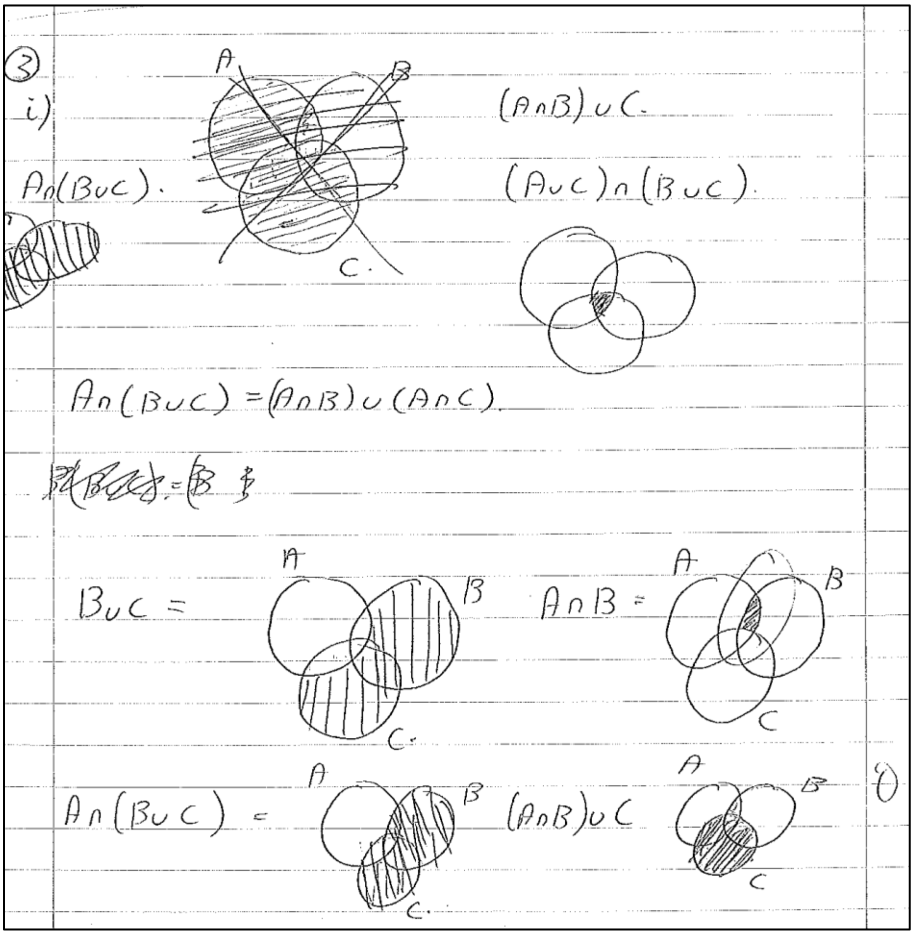


Figure 6.31: Student [16]'s response to task (2i)

In (1i) the students are asked to prove by induction a statement for natural numbers (Figure 6.32). In doing so, they have to engage in two substantiation routines: first, to prove that the statement stands for $P(1)$; then that if the statement stands for a natural k, then it stands for $k+1$. Four student scripts' ([01], [05], [11], [15]) evidence conflation of the use of the equality instead of the symbol for "implies" or "equivalence" in these parts of their proof by induction.

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

[6 marks]

Figure 6.32: Snapshot from the compulsory task from Sets, Numbers and Proofs – Task 1(i)

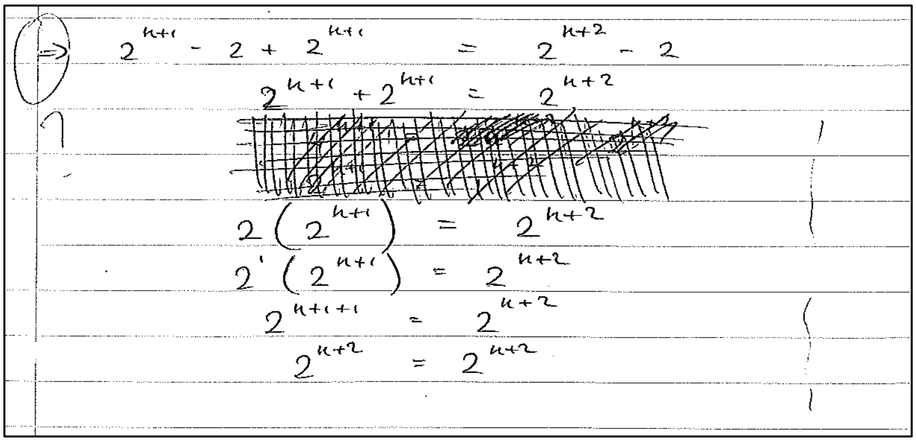
In response to (1i), [11] aims to prove that the base step *P*(1) holds (Figure 6.33). To do that, the student starts by writing the statement for *n*=1, continues working on one side and shows that 2=2. However, to show that the two sides of the statement are equal and conclude that the statement holds for *n*=1, the student should have written that $2^1 = 2 = 4 - 2 = 2^2 - 2 = 2^{1+1} - 2$. [11] shows that the right-hand side of the equality can be reached if they work with the left-hand side. However, in this case, the student starts from the given and works on one side using the equalities and proves that 2=2 but not that the base step holds.

BASE STEP: let n=1, then P(1) $2^1 = 2^{1+1} - 2 = 4 - 2 = 2$

therefore P(1) holds.

Figure 6.33: Part of student [11]'s response to (1i)

Student [15] wants to show that the two sides of the equality are the same (Figure 6.34) and ends by saying that $2^{k+2} = 2^{k+2}$. However, [15] does not connect the equalities written in the different lines with a logical connection (e.g., the equivalence symbol). [15] ends the narrative by saying $2^{k+2} = 2^{k+2}$ which is true. However, this last equality is not connected with the equivalence symbol to the initial equality written on the first line, making the written text illustrating a problematic meaning making with the logical symbols and the connections between the equalities.



$$\Rightarrow 2^{k+1} - 2 + 2^{k+1} = 2^{k+2} - 2$$
$$2^{k+1} + 2^{k+1} = 2^{k+2}$$
$$2\left(2^{k+1}\right) = 2^{k+2}$$
$$2^{1}\left(2^{k+1}\right) = 2^{k+2}$$
$$2^{k+1+1} = 2^{k+2}$$
$$2^{k+2} = 2^{k+2}$$

Figure 6.34: Part of student [15]'s response to (1i)

As mentioned in all the above cases of this section, the symbol of equality sometimes is used without considering what it means in the context of this task. Generally, when asked to prove an equality in algebra one can start from one side and prove that they can end with the other side. Alternatively, they can work on both sides of the equality to show that this can become something true such as $2^{k+2} = 2^{k+2}$ using equivalence (add the symbol of equivalence) to illustrate the connection between the equalities. And, so, since the final equality is true, this means that the original equality is true.

In (3ii), the students are asked to engage with modular arithmetic (Figure 34). Modular arithmetic is about the division between integers. The congruency symbol shows that the number is congruent to the remainder of the division.

(ii)

(a) State (but do not prove) Fermat's Little Theorem.
(b) Compute the remainder when 27^{313} is divided by 11.
(c) Find an integer $x \in \mathbb{Z}$ such that $19x \equiv 1\ (mod\ 36)$

[10 marks]

Figure 6.35: Snapshot from the second optional task from Sets, Numbers and Proofs – Task (3ii)

Twelve students, of the 22, in their narratives, conflate the use of equality and congruency mostly in (3iib). For that part of the task, the students are asked to find the remainder between two numbers. This involves several operations using both congruency and equality symbols. In Figure (6.36), student [14]'s response illustrates this conflation. From the first line, instead of using congruency, [14] uses equality and then later uses equality correctly to signal that $27^{(10\times31+3)} = \left(27^{10}\right)^{31} \times 27^{3}$, then conflates again the equality with the congruency in $\left(27^{10}\right)^{31} \times 27^{3} = 27^{3}(\mathrm{mod}11)$. This conflation may indicate unclear meaning making regarding the congruency between two numbers within the discourse of Modular Arithmetic.

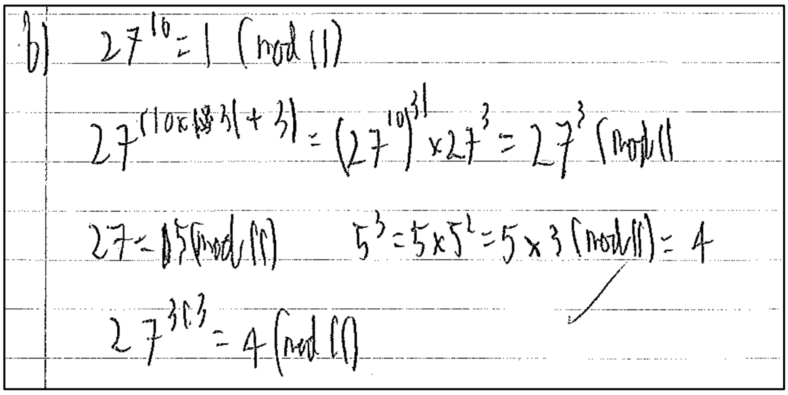
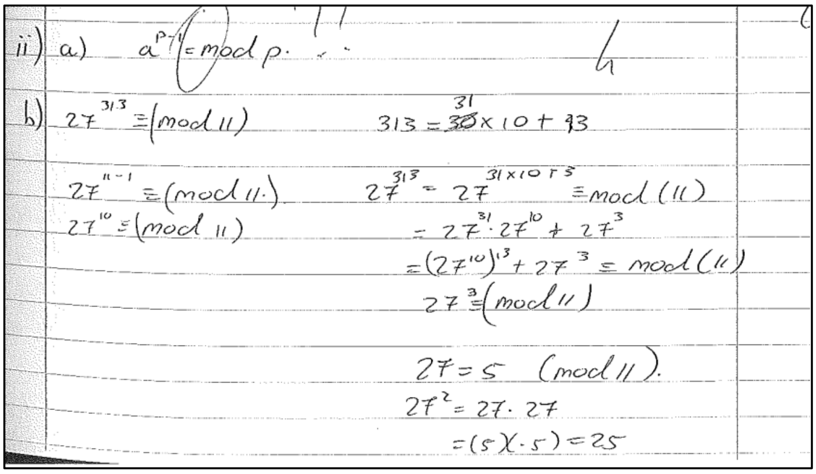


Figure 6.36: Student [14]'s response to task (3iib)

This conflation is also evidenced in student [16]'s script (Figure 6.37). From the first part of (3iia) with the recall of Fermat's Little Theorem, [16] writes that $a^{p-1} = (modp)$. Apart from the missing 1 in that relationship, there is also the equality instead of the congruency. This conflation is also visible in the next part of the task (3iib), with "$27 = 5(mod11)$" and later with "$27 \cdot 27 = (5)(5) = 25$". It seems that the process of congruency is not transparent for student [16]. There are nine instances, in this script, where there is $a^{p-1} = (modp)$ or something like $27^{31\times10+3} = mod11$. In these realisations of the congruency, it seems that the process of the modular arithmetic and the product of the modular arithmetic are conflated.



ii) a) $a^{p-1} \equiv \bmod p$.

b) $27^{313} \equiv (\bmod 11)$ $\quad 313 = \cancel{30}\,31 \times 10 + \cancel{1}3$

$27^{11-1} \equiv (\bmod 11)$ $\quad 27^{313} = 27^{31\times 10 + 3} \equiv \bmod (11)$

$27^{10} \equiv (\bmod 11)$ $\quad = 27^{31} \cdot 27^{10} + 27^{3}$

$= (27^{10})^{13} + 27^{3} \equiv \bmod (11)$

$27^{3} \equiv (\bmod 11)$

$27 = 5 \pmod{11}$.

$27^{2} = 27 \cdot 27$

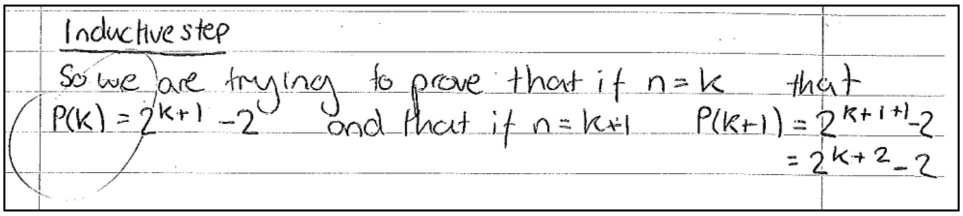
$= (5)(\cdot 5) = 25$

Figure 6.37: Part of student [16]'s response to (3iia) and (3iib)

(i) Conflating use of quantifiers

In this section, I present nine students' scripts that illustrate conflation in their use of quantifiers ([01], [03], [06], [08], [09], [10], [11], [15], [18]). In some cases, this conflation occurs more than once. These scripts correspond to responses to (1i), (2i), (2ii) and (3i).

In response to (1i), [03] assumes that, if *n* takes the value *k* the statement would be equal to 2^{k+1}-2 and, if the *n* takes the value *k*+1, then the statement would equal to 2^{k+2}-2 (Figure 6.38). In the inductive step of the proof by induction, the assumption is that the statement is valid for a value *k* and the goal of the inductive step is to show that for *k*+1 the statement is also true.



Inductive step

So we are trying to prove that if $n = k$ that $P(k) = 2^{k+1} - 2$ and that if $n = k+1$ $P(k+1) = 2^{k+1+1} - 2$

$= 2^{k+2} - 2$

Figure 6.38: Part of student [03]'s response to (1i)

Another case that evidence confusion between what is assumed and what is to be proven is shown in figure 6.39. Student [15] assumes that "if it true for *P*(*n*) it is true for *P*(*k*)". This signals difficulties with the routine of proving and the quantifiers used.

If it is true for $p(n)$, it is true for $p(k)$

$p(k)$: $2^1 + 2^2 + 2^3 + \dots + 2^k = 2^{k+1} - 2$

Figure 6.39: Student [15]'s response to (1i)

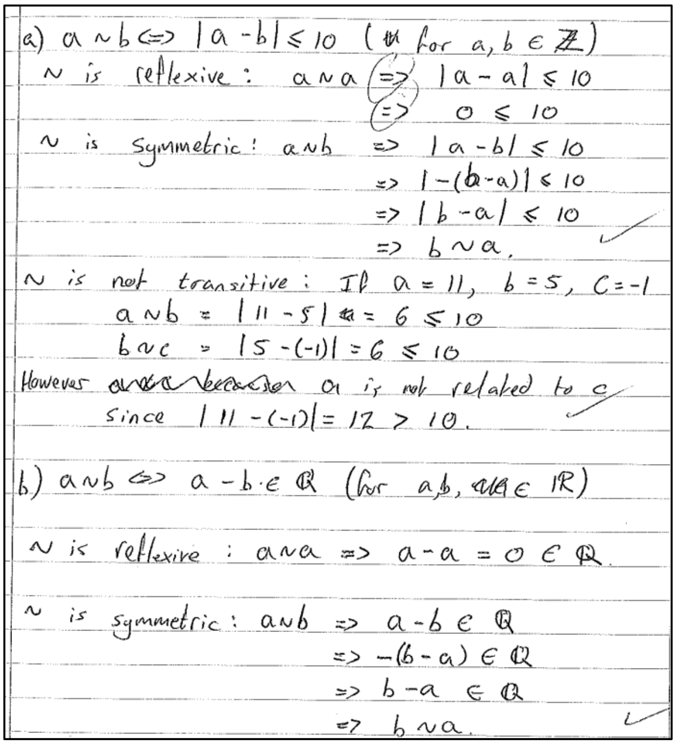
The scripts of three students ([06], [10], [11]) presented difficulties in the use of quantifiers in the definition of injective and surjective functions (task (3i)). [06], in the definition of the surjective and injective function, provides a sequence of variable symbols connected with logical symbols (Figure 6.40). However, in [06]'s definition of injective function, there is an equivalence relation instead of an implication, illustrating the problematic meaning-making of the logical symbol of equivalence and confusion with the object of the injective function.

(i) Surjective $(\forall b \in B)(\exists a \in A): f(a) = b.$

Injective $(\forall a, a' \in A)[a \neq a' \Leftrightarrow f(a) \neq f(a')]$.

Figure 6.40: Student [06]'s response to (3i)

There are also scripts that illustrate problematic meaning making regarding the "implies" symbol (⇒). In student [18]'s script (Figure 6.41), the implication symbol is used in proving, for the given relations, the reflexive property. In proving that the relation has the reflexive property, the implication symbol shows that [18] takes into account that it is already reflexive and uses that fact. However, it should be an equivalence instead of an implication or [18] should have started the other way. This use of the implication symbol shows conflict regarding the use of the symbol "implies" and raises questions whether this could signal a ritualised use of the symbol. The symbol (⇒) is used in the symmetrical and transitive property. In both properties, there are assumptions for the relationship between different elements. Using these assumptions, the aim is to prove that the relationship exists between other elements.


a) $a \sim b \Leftrightarrow |a-b| \leq 10$ (for $a, b \in \mathbb{Z}$)

$\sim$ is reflexive: $a \sim a \Rightarrow |a-a| \leq 10$
$\Rightarrow 0 \leq 10$

$\sim$ is symmetric: $a \sim b \Rightarrow |a-b| \leq 10$
$\Rightarrow |-(b-a)| \leq 10$
$\Rightarrow |b-a| \leq 10$
$\Rightarrow b \sim a$.

$\sim$ is not transitive: If $a = 11$, $b = 5$, $C = -1$
$a \sim b = |11-5| = 6 \leq 10$
$b \sim c = |5-(-1)| = 6 \leq 10$
However a is not related to c
since $|11-(-1)| = 12 > 10$.

b) $a \sim b \Leftrightarrow a - b \in \mathbb{Q}$ (for $a, b \in \mathbb{R}$)

$\sim$ is reflexive: $a \sim a \Rightarrow a - a = 0 \in \mathbb{Q}$.

$\sim$ is symmetric: $a \sim b \Rightarrow a - b \in \mathbb{Q}$
$\Rightarrow -(b-a) \in \mathbb{Q}$
$\Rightarrow b - a \in \mathbb{Q}$
$\Rightarrow b \sim a$.

Figure 6.41: Student [18]'s response to (2iia)

6.2.4 Use of visual mediators: The case of graphs and Venn diagrams

In this section, I present the analysis of scripts from students who use graphs and Venn diagrams in their narratives. Specifically, in the responses to task (2i) five students ([01], [02], [12], [16], [22]) used Venn diagrams and, in the responses to task (3i), six students ([07], [08], [11], [12], [18], [22]) used graphs as another realisation of the given functions.

(i) Using Venn diagrams in the solution to (2i)

In (2i), the students are asked to engage with Set Theory. The use of Venn diagrams for sets is quite familiar to the students as this was the realisation that they were using for sets in secondary school. In this task (Figure 6.42), the students are, first, expected to engage with three abstract sets *A, B*, and *C*; then, they are asked to construct an example of sets that satisfy a certain relation. In the script of five students ([01], [02], [12], [16], [22]) there is use of Venn diagrams.

(i)	Prove carefully that if A, B and C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Give an example of sets A, B and C such that $A \cap (B \cup C) \neq (A \cap B) \cup C$.	
		[10 marks]

Figure 6.42: Snapshot from the first task from Sets, Numbers and Proofs – Task 2(i))

Student [16] attempts to solve this part of the task using Venn diagrams and trying to identify which parts of the set the given relations (e.g., $A \cap (B \cup C)$) correspond to (Figure 6.43). However, there is a conflation between what the symbols of intersection and union mean. [16] writes $(A \cup B) \cap (B \cup C)$ which is equal to the whole set *C* but the first Venn diagram, located on the right-hand side of Figure 6.47, signals the intersection of the three sets. Later, to show that $A \cap (B \cup C) \neq (A \cap B) \cup C$, the student first constructs $B \cup C$ but does not illustrate this in the Venn diagram provided beside this union. The same occurs when constructing the Venn diagram corresponding to $A \cap B$. Then, in the next line, [16] creates a Venn diagram for $A \cap (B \cup C)$ but, in the corresponding Venn diagram, [16] shades just the $B \cup C$. The Venn diagram for $A \cap (B \cup C)$ is correct. However, considering the Venn diagram produced for $A \cap B$, this shows problematic meaning-making between the intersection and the union of sets and the correspondence with the Venn diagrams.

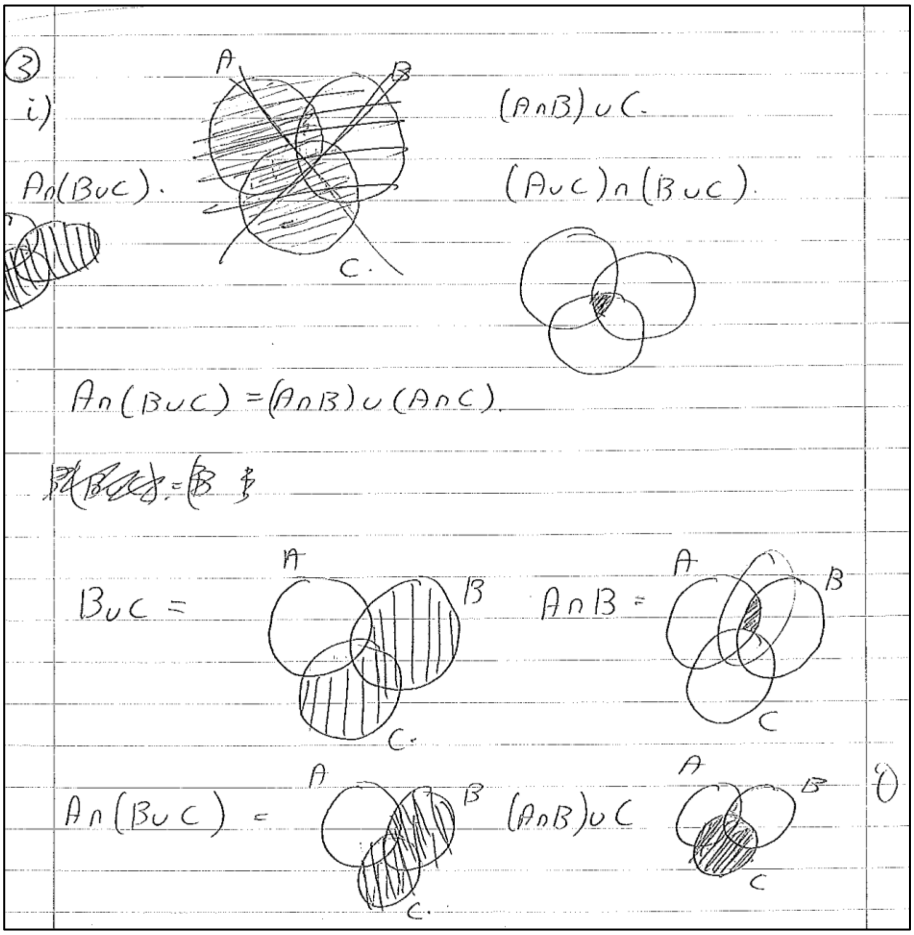


Figure 6.43: Student [16]'s response to task (2i) – The marker has circled the incorrect intersection in the Venn diagram for $A \cap B$

Venn diagrams are also used by [02] (Figure 6.44). [02] uses a Venn diagram that does not highlight any intersection or union between the sets and provides a narrative that is not accepted as the expected proof.

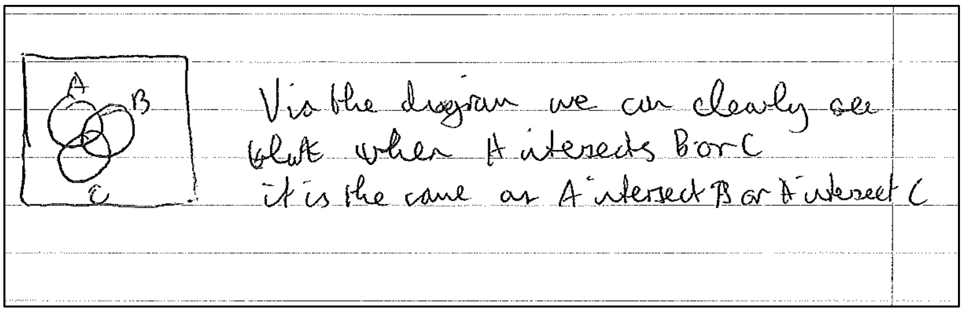


Figure 6.44: Student [02]'s response to task (2i)

In contrast to the above two figures from students [16] and [02], student [22] illustrates the corresponding Venn diagram and, also, provides a narrative demonstrating that $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$ (Figure 6.45).

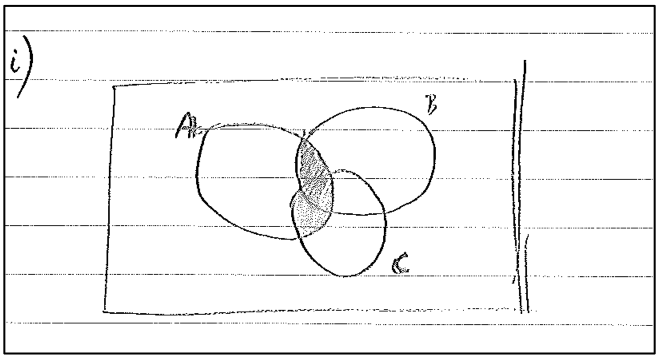


Figure 6.45: Student [22]'s response to task (2i)

Also, student [12] (Figure 6.46) uses Venn diagrams to show that $A \cap (B \cup C) \neq (A \cap B) \cup C$.

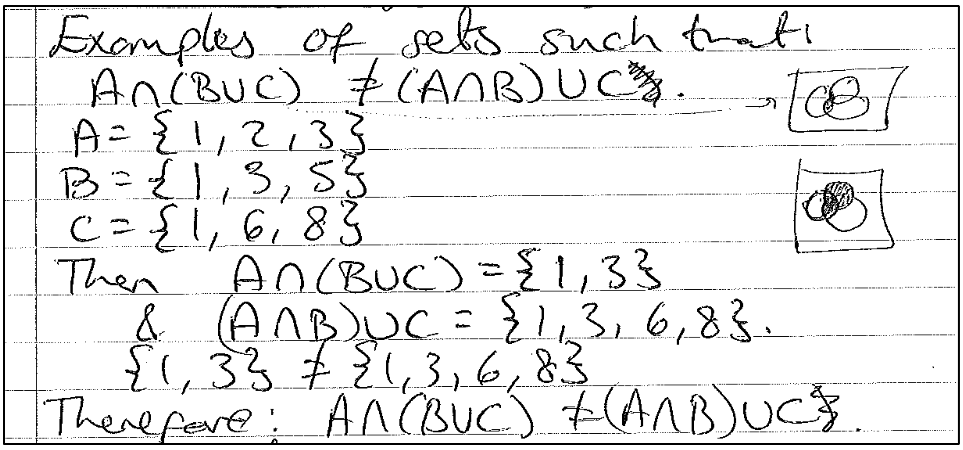


Figure 6.46: Student [12]'s response to task (2i)

Finally, [01] also uses Venn diagrams (Figure 6.47). In this case, though, the Venn diagrams also include the elements of sets. However, [01] does not write anything else either to prove that the two sets are equal or to show that these sets are the example to show that $A \cap (B \cup C) \neq (A \cap B) \cup C$.

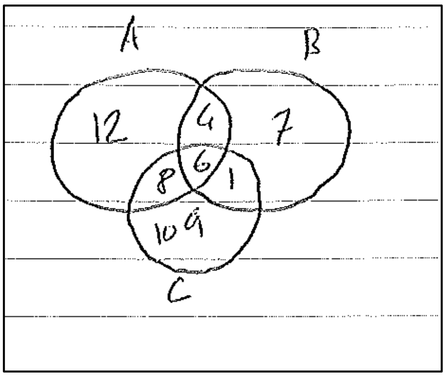


Figure 6.47: Student [01]'s response to task (2i)

(ii) Using graphs to show that the function is injective and surjective

In (3i) the students are asked to show whether the given functions are injective and surjective (Figure 6.48). In the scripts of six students ([07], [08], [11], [12], [18], [22]) I observed use of function graphs. The graphs of three students evidence conflation of discourses between functions with real domain and functions with the domain being integers. I first discuss these three scripts.

> (i) Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*.
>
> For each of the following functions decide whether it is injective, surjective (or both, or neither).
> Give brief reasons for your answers.
> (a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$.
> (b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.
>
> **[10 marks]**

Figure 6.48: Snapshot from task 3 from Sets, Numbers and Proofs

The responses from three students use a graph for the function *h*(*n*). However, when sketching this graph, they do not take into account that the domain and the codomain of the function are $\mathbb{Z}$. Student [11] is one of those students (Figure 6.49). In this response, the arguments are based on the graph of the function and, even though there is a symbol *n* at the *x*-axis, the values in the *x*-axis, *y*-axis and the line showing the function are straight lines without gaps. This suggests that the student does not consider the integers but thinks of the function as a function in the reals.

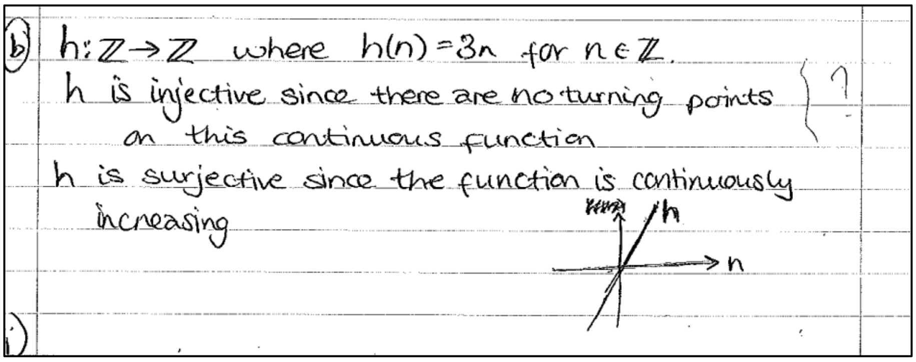

b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.
h is injective since there are no turning points on this continuous function ?
h is surjective since the function is continuously increasing

Figure 6.49: Student [11]'s response to task (3ib) – the lecturer added the question mark

On the other hand, even though student [12] uses a very similar graph for *h*(*n*) (Figure 6.50), in the argument about the function being injective and not surjective is not based on the graph. Also, the variables *n* and *n*' are defined as members of ℤ.

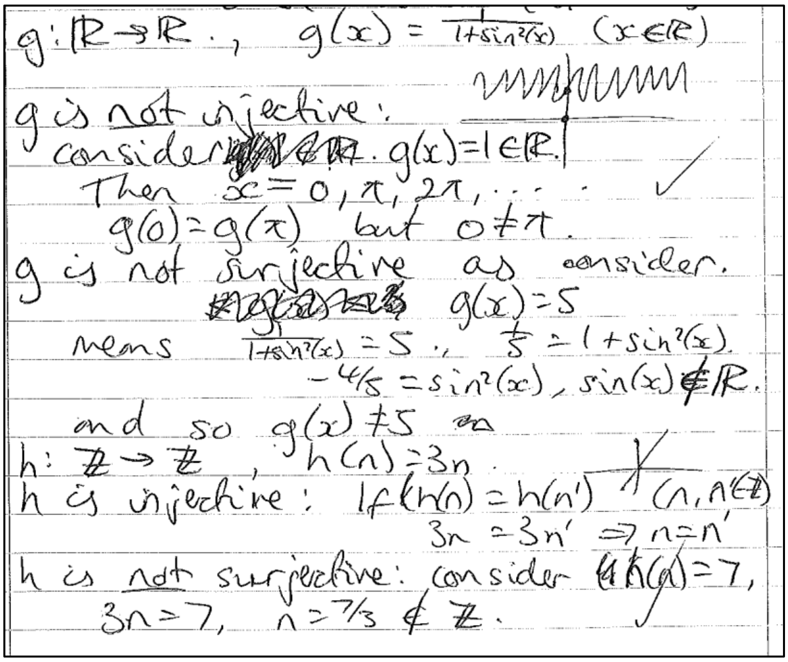

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{1+\sin^2(x)}$ $(x \in \mathbb{R})$
g is not injective:
consider $g(x) = 1 \in \mathbb{R}$.
Then $x = 0, \pi, 2\pi, \ldots$
$g(0) = g(\pi)$ but $0 \neq \pi$.
g is not surjective as consider. $g(x) = 5$
means $\frac{1}{1+\sin^2(x)} = 5$, $\frac{1}{5} = 1 + \sin^2(x)$.
$-4/5 = \sin^2(x)$, $\sin(x) \notin \mathbb{R}$.
and so $g(x) \neq 5$
$h: \mathbb{Z} \to \mathbb{Z}$, $h(n) = 3n$.
h is injective: If $h(n) = h(n')$ $(n, n' \in \mathbb{Z})$
$3n = 3n' \Rightarrow n = n'$
h is not surjective: consider $h(n) = 7$,
$3n = 7$, $n = 7/3 \notin \mathbb{Z}$.

Figure 6.50: Student [12]'s response to task (3i) – the lecturer added the ticks

Student [18] initially used a graph but crossed it out (Figure 6.51). It seems that [18] started with a graph and used the variable *x* instead of *n* both in the function but in the *x*-axis too. It seems though that the student realised that the function only takes integer values and thus it cannot be depicted by a line. The student scribbles these and rewrites the function by using the variable *n* and showing that the domain and the codomain are integers. This is where the graph of the function could have signalled to the student that the approach using the graph is not suitable for the case of a function from ℤ to ℤ.

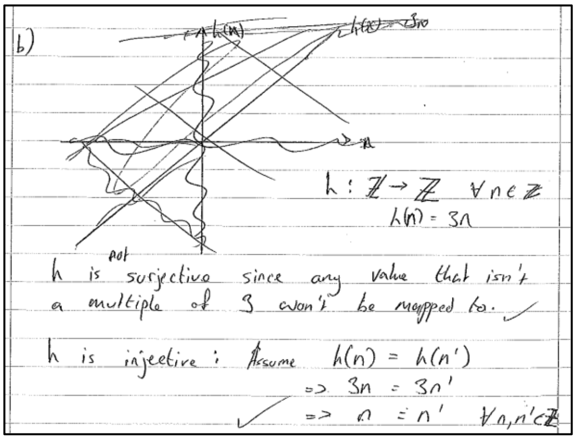


Figure 6.51: Student [18]'s response to task (3ib)

6.2.5 Word use (miscellaneous)

In the following, I present examples from the students' scripts that illustrate conflating word use from different mathematical discourses. I, also, comment on students' use of personal pronouns.

(i) Conflating word use from different mathematical discourses

In four scripts ([03], [11], [15], [20]), I observed word use evidencing commognitive conflict as the students use words coming from different discourses. This occurs in the solutions to task (2ii) (Figure 6.52) and 3i) (Figure 6.53)

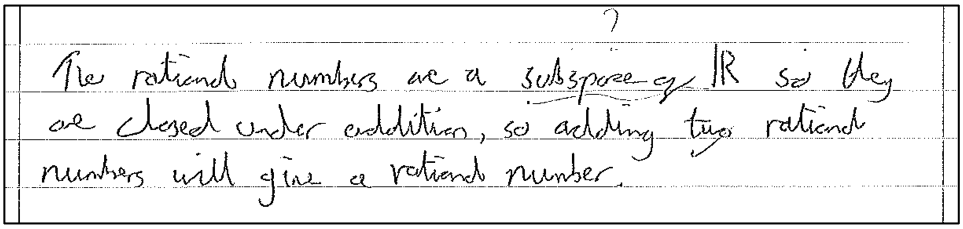
(ii)	Suppose that A is a non-empty set and $\sim$ is a relation on A. Give the definitions of what is meant by saying that $\sim$ is *reflexive, symmetric* and *transitive*. In each of the following cases, decide which (if any) of these properties the given relation has. Give reasons for your answers. (a) $A = \mathbb{Z}$ and $a \sim b \Leftrightarrow \|a - b\| \leq 10$ (for $a, b \in \mathbb{Z}$). (b) $A = \mathbb{R}$ and $a \sim b \Leftrightarrow a - b \in \mathbb{Q}$ (for $a, b \in \mathbb{R}$). **[10 marks]**

Figure 6.52: Task (2ii) part of the second task to Numbers, Sets and Proofs as illustrated in Figure presented in 5.3

(i)	Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*. For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief reasons for your answers. (a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$. (b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$. **[10 marks]**

Figure 6.53: Task (3i) part of the third task to Numbers, Sets and Proofs as illustrated in Figure (add figure number from chapter 5)

Student [20], in (2ii), attempts to show that the rational numbers are closed under addition and thus indicating that the relation is transitive. [20] writes that "The rational numbers are a subspace" of the real numbers (Figure 6.54). However, the term "subspace" is from the Linear Algebra terminology. This term is used to talk about the subspace topology and to show that the set of rational numbers are disconnected.

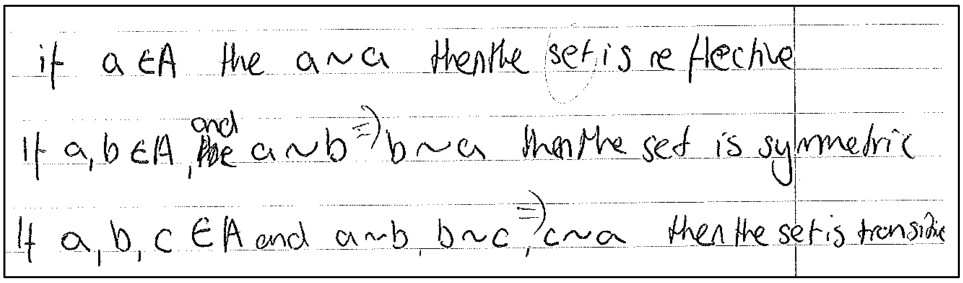


?

The rational numbers are a subspace of $\mathbb{R}$ so they are closed under addition, so adding two rational numbers will give a rational number.

Figure 6.54: Student [20]'s response to the last part of (2iib) – The marker added the question-mark

In solving the same task (2ii), [03] uses the term "set" instead of the term "relation" (Figure 6.55). This word use signals the possibility of a commognitive conflict between the two objects: a relation is an operation on a set, and a relation can have the characteristics of being reflexive, symmetric and transitive.



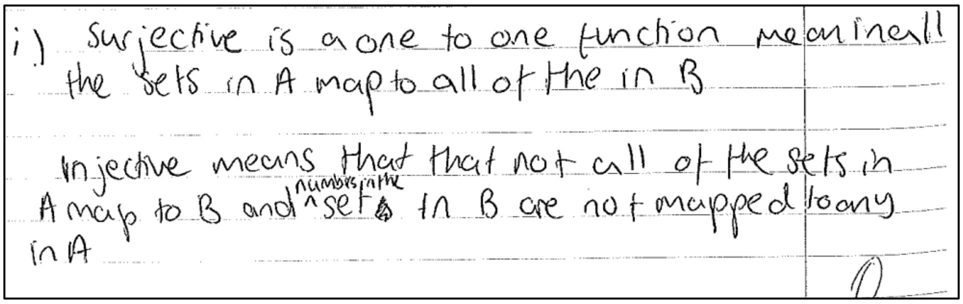
if $a \in A$ the $a \sim a$ thenthe set is reflective

If $a, b \in A$, and ~~the~~ $a \sim b \Rightarrow b \sim a$ thenthe set is symmetric

If $a, b, c \in A$ and $a \sim b$, $b \sim c$, $\Rightarrow c \sim a$ then the set is transitive

Figure 6.55: Student [03]'s response to task (2ii) – The marker has added the two implies symbols in the definition of symmetric and transitive.

In the same task (2ii), when engaging with substantiation routines, [03] keeps using the term "set" instead of "relation". This, as mentioned before, signals problematic meaning-making on the object of relation but also on the characteristics of a relation. The same student (Figure 6.56) in response to (3i) uses the word "sets" to refer to elements of set *A* in writing about a surjective function. In the student's definition of an injective function, there is, again conflating word use between "sets" and "elements of sets". *A* and *B* are defined as sets from the definition of a function but, [03]'s meaning-making seems to suggest that *A* is a set of sets and *B* is a set of numbers.

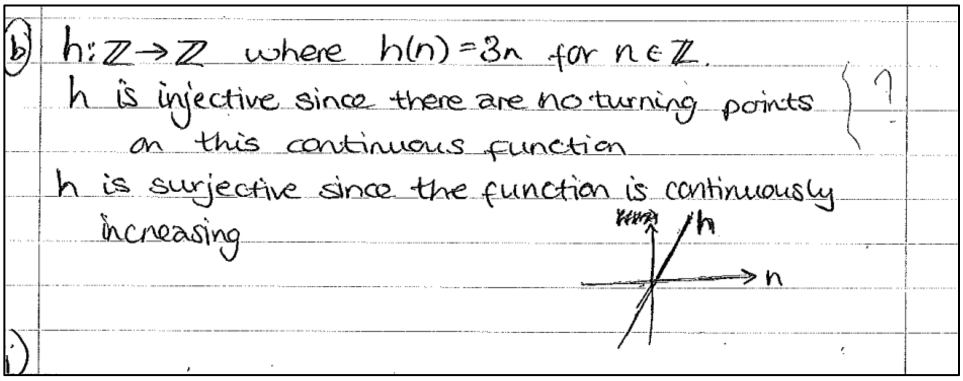

i) Surjective is a one to one function meaning all the sets in A map to all of the in B

Injective means that that not all of the sets in A map to B and numbers in the sets in B are not mapped to any in A

0

Figure 6.56: Student [03]'s response to task (3i) – the marker added the 0 at the right bottom corner of the script.

In response to the next part of (3ii), two students ([11], [15]) are using words from different mathematical discourses (Figures 6.57 and 6.58)


b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

h is injective since there are no turning points on this continuous function

h is surjective since the function is continuously increasing

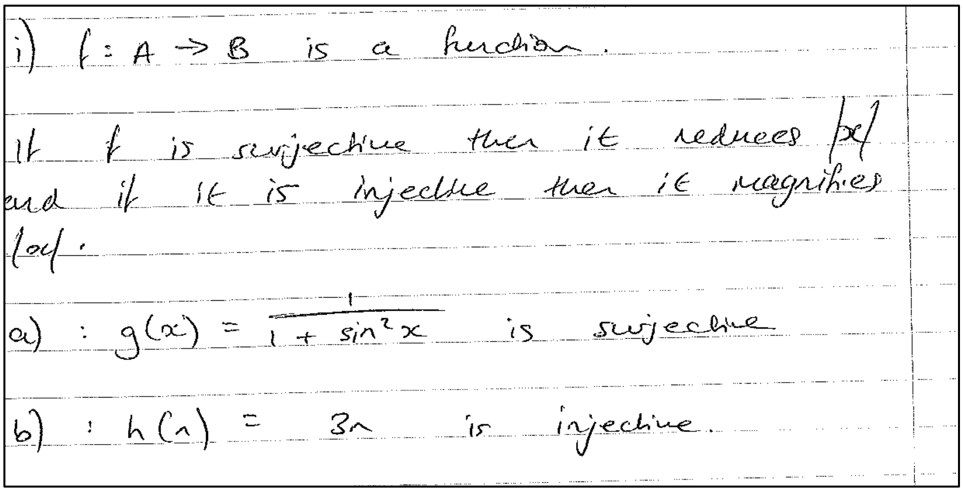
?

i)

Figure 6.57: Student [11]' response to task (3ib) – the marker added the question mark

[11] tries to show that the function *h*(*n*) is not an injective function (Figure 6.60). If a function is defined in ℝ, its injectivity can be shown if the function is continuous and does not have turning points. However, a function from ℤ to ℤ takes integer values. The word use "turning points", "continuous", "continuously increasing" signals use of the discourse on functions defined in ℝ and not ℤ. This is also illustrated in the function of the graph. The function is sketched as a straight line taking all the values from the domain and codomain. The label on the *x*-axis has the symbol *n*, which is used in the wording of the task and defined as an integer. [11] conflates the discourse of functions defined in ℝ with the functions defined in ℤ.

Another script that signals similar commognitive conflict is [15]'s response to (3i) which involves the use of the modulus of x and the word use "reduces" and “magnifies” (Figure 6.58). The student, then, writes that the first function is surjective and the second one is injective without providing any reasons. Apart from the deficiency in justification of [15]’s response, the word and visual mediator use signals conflict between the definition of surjective and injective and the composition of the given function *f* and the modulus function.



i) $f : A \to B$ is a function.

If f is surjective then it reduces $|x|$ and if it is injective then it magnifies $|x|$.

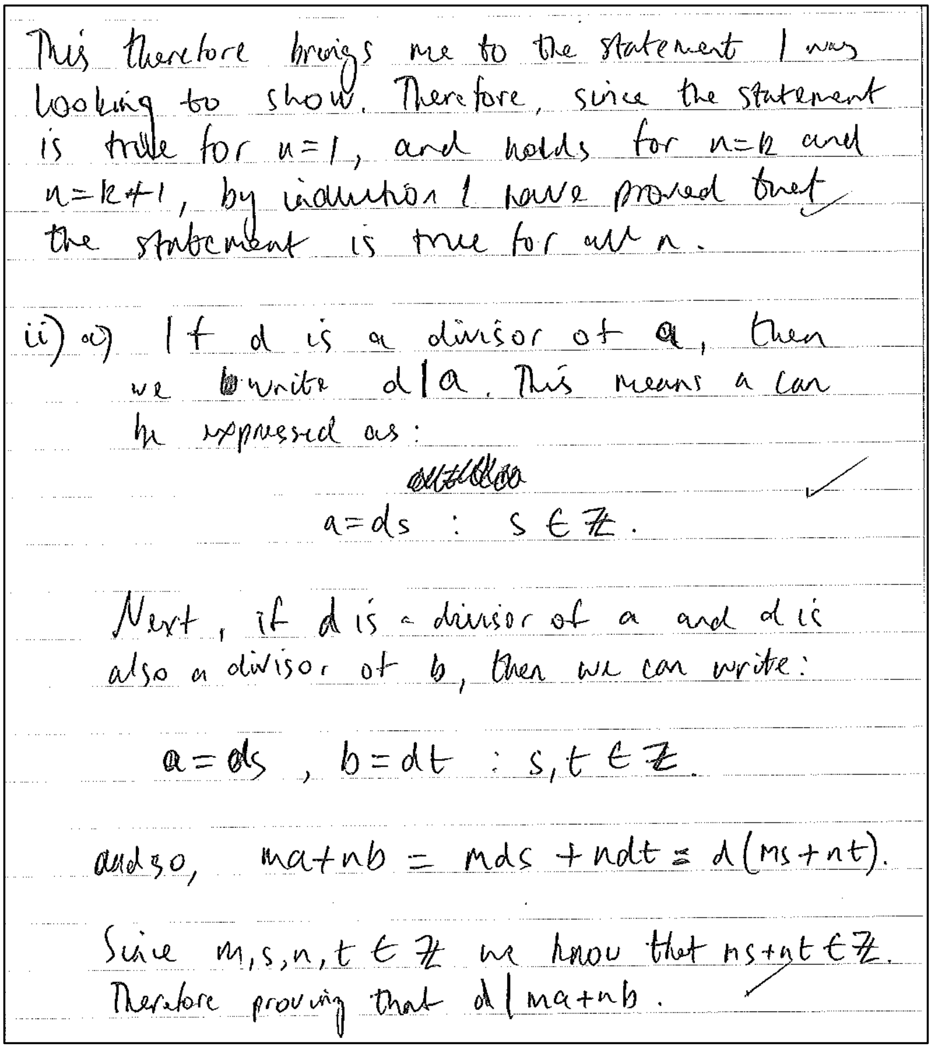
a) : $g(x) = \frac{1}{1 + \sin^2 x}$ is surjective

b) : $h(x) = 3x$ is injective.

Figure 6.58: Student [15]’s response to (3i)

(ii) Using the pronouns “we” and “I” in the produced narratives

In the narratives of seventeen students, I observed the use of the pronoun “we” either when signalling the procedure that they should follow, or a specific step of the procedure, or the introduction of a new variable in their narrative. The use of the pronoun “we” is used typically in the mathematical discourse at the university level. During the interview, L1 used “we” to mean in different cases, the community of mathematicians, or to the students and himself. The use of “we” is also visible in the lecturer’s solution to the task. I assume that the students are using the “we” to mean the community of mathematicians. Only one of the students, [22], uses the personal pronoun “I” and “me” but then also, [22] switches to using “we” (Figure 6.59)



This therefore brings me to the statement I was looking to show. Therefore, since the statement is true for $n=1$, and holds for $n=k$ and $n=k+1$, by induction I have proved that the statement is true for all n.

ii) a) If d is a divisor of a, then we write $d|a$. This means a can be expressed as:

$$a=ds \ : \ s \in \mathbb{Z}.$$

Next, if d is a divisor of a and d is also a divisor of b, then we can write:

$$a=ds \ , \ b=dt \ : \ s,t \in \mathbb{Z}.$$

and so, $ma+nb = mds + ndt = d(ms+nt)$.

Since $m,s,n,t \in \mathbb{Z}$ we know that $ms+nt \in \mathbb{Z}$. Therefore proving that $d|ma+nb$.

Figure 6.59: Student [22]' response to the last part of (1i) and (1iia) – the marker has added the three ticks.

6.3 Students' scripts: Commognitive analysis (routines)

In section 6.3, I present the analysis regarding students' engagement with the routines in the Sets, Numbers and Proofs tasks. I focus on the procedure of the routine and on the closing conditions. Specifically, I examine students' engagement: first with recall and then substantiation routines; with a substantiation routine with a given procedure; with a substantiation routine with the procedure not given. Finally, I comment on cases where the extent of justification provided by the student was not sufficient for the marker signalling that for him the closing conditions of the routine were not met.

6.3.1 Recall and substantiation routines

There are instances in the tasks, where the definition of an object is asked and, then, in the next part, the object is used in a substantiation routine. The students' scripts revealed difficulties in students' engagement with recall and substantiation routines. There were three categories, one regarding relations, another one regarding the injective and surjective functions and one regarding Fermat's Little Theorem. I now present each with illustrative examples from students' scripts.

(i) Recall and substantiation: Relations [Task (2ii)]

In (2ii) the students are asked to recall the definition of reflexive, symmetric and transitive relation. The scripts of four students ([03], [07], [15], [16]) evidence unclear meaning-making, also evidenced later in the substantiation part of the task.

Student [03] provides the definitions of reflexive, symmetric and transitive (Figure 6.60). However, the relationships between the objects involved in the definition are not clear. In (2iia), [03] checks only the symmetric property and in (2iib) s/he checks only the reflexive property conflating the characteristics of the two properties (reflexive and symmetric). As noted earlier, there are no connections in the definitions regarding the assumption of the definition and the implication. This might be the reason that [03] was not able to check whether the given relations have the characteristics: reflexive, symmetric and transitive.

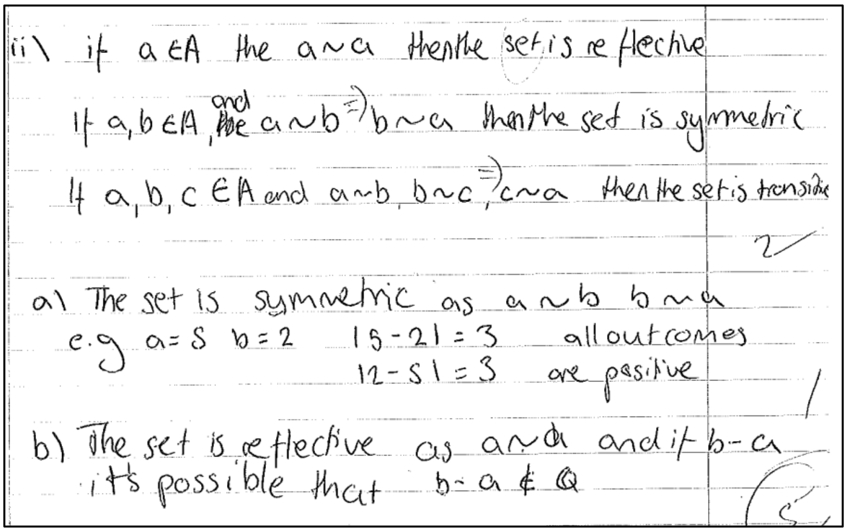

ii) if a ∈A the a~a then the set is reflective
If a, b ∈A, and the a~b ⇒ b~a then the set is symmetric
If a, b, c ∈A and a~b, b~c, ⇒ c~a then the set is transitive
2
a) The set is symmetric as a~b b~a
e.g a=5 b=2 |5-2|=3 all outcomes
|2-5|=3 are positive
1
b) The set is reflective as a~a and if b-a
it's possible that b-a ∉ Q
5

Figure 6.60: Student [03]'s response to (2ii) – The marker has added the two implies, the 2, 1 and the circled 5.

Another example is in figure (6.61). [15] gives definitions that do not involve at all elements of the relation. The definitions given illustrate unclear meaning-making and this is also evidenced in the lack of arguments regarding the characterisation of the relation (a) as reflexive and the relation (b) as transitive.

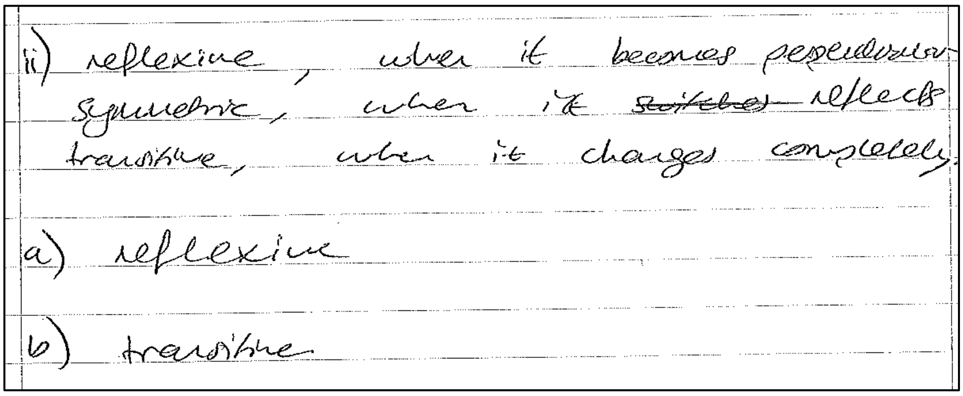
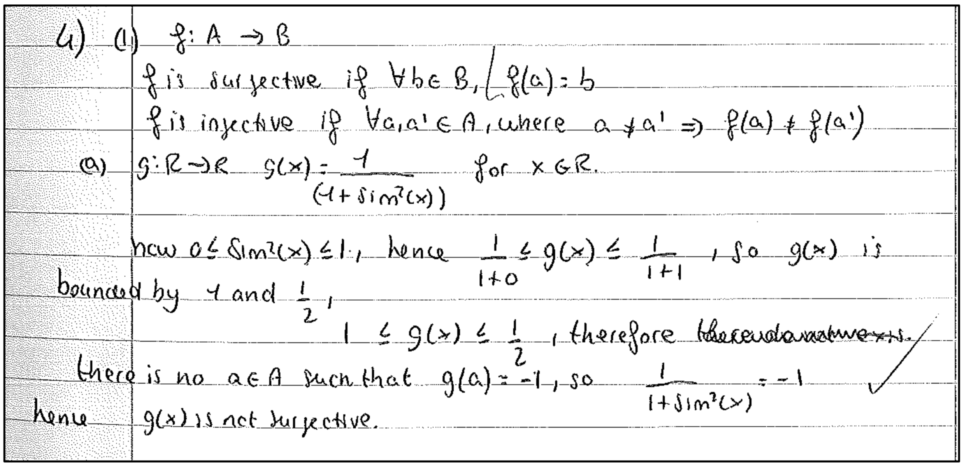

ii) reflexive, when it becomes perpendicular
symmetric, when it ~~switches~~ reflects
transitive, when it changes completely
a) reflexive
b) transitive

Figure 6.61: Student [15]'s response to (2ii)

(ii) Engagement with first recall and then substantiation routines – injective and surjective functions [Task (3i)]

In (3i), the students are required to, first, recall the definitions of injective and surjective function and then engage in substantiation routines to examine whether two functions are injective or surjective. The analysis of the scripts illustrated difficulties in recalling the definitions and cases where the unclear meaning-making of the definitions also resulted in problematic engagement with the substantiation routines. The scripts of nine students ([01], [02], [03], [05], [08], [11], [15], [16], [20]) belong here.

In [05]'s definition for surjective the phrase "there exists *a* in *A* such that" is missing (Figure 6.62). As the lecturer mentioned in the interview (section 5.3.3) the students do not have a clear meaning-making of the objects; they recall some elements of the definition and they write without making the connections between their elements specific.



4) (i) $f: A \to B$

f is surjective if $\forall b \in B$, $f(a) = b$

f is injective if $\forall a, a' \in A$, where $a \neq a' \Rightarrow f(a) \neq f(a')$

(a) $g: \mathbb{R} \to \mathbb{R}$ $g(x) = \frac{1}{(1+\sin^2(x))}$ for $x \in \mathbb{R}$.

now $0 \leq \sin^2(x) \leq 1$, hence $\frac{1}{1+0} \leq g(x) \leq \frac{1}{1+1}$, so $g(x)$ is bounded by 1 and $\frac{1}{2}$,

$1 \leq g(x) \leq \frac{1}{2}$, therefore

there is no $a \in A$ such that $g(a) = -1$, so $\frac{1}{1+\sin^2(x)} = -1$

hence $g(x)$ is not surjective.

Figure 6.62: Student [05]'s response to (3i)

There are also two cases ([02], [11]) where the students recall the definition of an injective function. However, there is a confusion with the definition of injective function and the definition of a function (Figure 6.63). This unclear meaning making leads to difficulties also in the substantiation part of the task (Figure 6.64). In student [02]'s definition for an injective function, there is the phrase "one to one relationship" commenting on the relationship between the elements of the domain and the codomain of the function. However, later, the student, in trying to clarify what this "one to one relationship" is, writes "f(a)=b and vice versa".

Injective describes a one to one relationship
Thus there is only one value of a that $f(a) = b$
and vice versa. ?

Figure 6.63: Student [02]'s response to (3i) – The marker added the question-mark

This phrase is not further explained but is also used when the student examines whether the given function *h* is injective (Figure 6.67). This phrase signals that there is unclear meaning making between the definition of a function and the definition of an injective function.

$h(x)$ is Injective but not surjective this is because
it is impossible to represent all integers such as
1 via $3x$ as you would need $x = \frac{1}{3}$ which is
not an integer. It is injective however as
for all $f(x)$ there is a distinct value of y and
vice versa. ?

Figure 6.64: Student [02]'s response to (3ib)

Student [15] gives definitions for a surjective and an injective function that involve the modulus function (Figure 6.65). In the substantiation part of the task, [15] does not give any explanation regarding why the specific claims about the functions *g* and *h* are made.

4 i) $f: A \to B$ is a function.

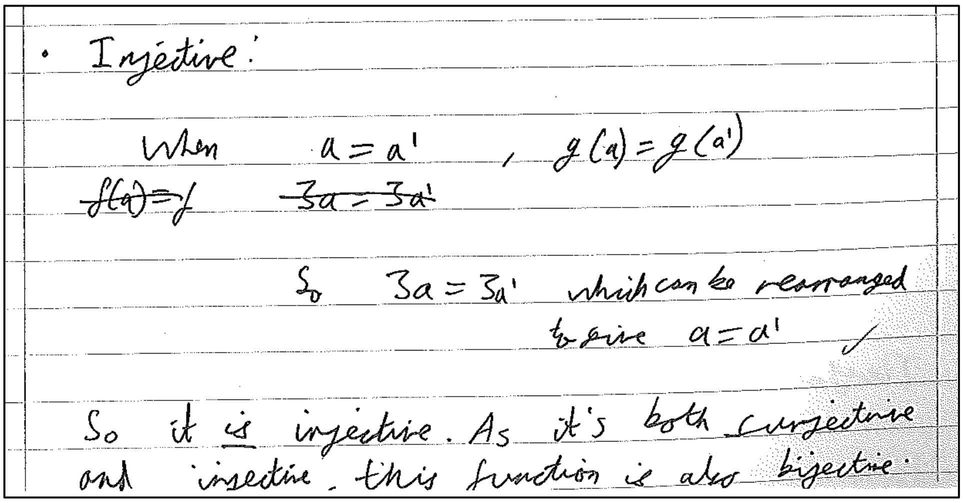
If f is surjective then it reduces $|x|$
and if it is injective then it magnifies
$|x|$.

a) : $g(x) = \frac{1}{1 + \sin^2 x}$ is surjective

b) : $h(x) = 3x$ is injective.

Figure 6.65: Student [15]'s response to (3i)

Although [08] gives the correct definition for an injective function, when examining whether function *h* is injective the student draws on the definition of a function instead of the definition of injective function (Figure 6.66). The student starts by assuming that two elements of the domain are equal and, then, writes that their corresponding elements in the codomain should be equal too. However, this is a property of a function not the definition of an injective function. In the next part, [08] writes and then scribbles out the two images of the elements *a* and *a'*. [08] concludes by writing that the images are the same and, thus, the elements should also be the same. The confusion between the first part of the narrative, the scribbled-out bit and the concluding part signal that there is unclear meaning making regarding what an injective function is.


• Injective:

When $a = a'$, $g(a) = g(a')$

~~f(a)=f~~ ~~3a = 3a'~~

So $3a = 3a'$ which can be rearranged to give $a = a'$ ✓

So it is injective. As it's both surjective and injective, this function is also bijective.

Figure 6.66: Student [08]'s response to (3ib)

The procedure of substantiation that a function is injective, is also problematic in [11]'s response (Figure 6.67). [11] gives a definition which, also, conflates the property of a function and the injective function in the definition part of the task. When it comes to substantiating, the student does not use the definition but relies on other procedures. [11] talks about "turning points on this continuous function" and "continuously increasing". The procedures implemented here are from Calculus.

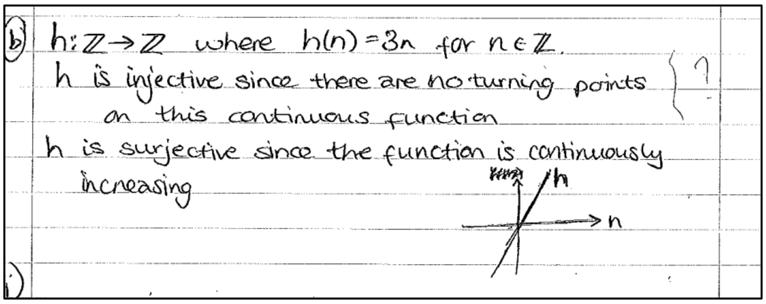

b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.
h is injective since there are no turning points on this continuous function } ?
h is surjective since the function is continuously increasing

Figure 6.67: Student [11]'s response to (3ib)

(iii) Recall and substantiation in the case of Fermat's Little Theorem [Task (3ii)]

In (3iia), the students are asked to recall Fermat's Little Theorem. In this case, the students have to define the modular arithmetic of *p* as a prime number. As I illustrate in section (6.2.1), some students did not specify the numerical context of the variables involved in the theorem and did not comment on *p* being prime. Six student scripts ([01], [02], [03], [04], [15], [17]) evidence difficulties with the recall routine, either for particular elements of the theorem (e.g., *p* is prime) or the theorem itself.

a) Fermat's little theorem: $a^p \equiv a \bmod (p)$ 4

b) 27^{313} divided by 11.

$a^{p-1} \equiv 1 \bmod (p)$.

$27^{10} \equiv 1 \bmod (11)$.

$27^{313} = (27^{10})^{31} \cdot (27)^3 = 1 \bmod \cdot (27)^3 \bmod (11)$.

$11 \times 1789 = 19679$.

$27^3 - 19679 = 4$.

$\therefore \; 27^{313} = 4 \bmod (11)$

So the remainder = 4.

c) $a^{p-1} = 1 \bmod (p)$.

$19x = 1 \bmod (36)$

$p = 36$, $\therefore$

$19x = x^{35}$ $\quad x^{34} = 19$.

$x =$

Figure 6.68: Student [04]'s response to (3ii)

Student [04] recalls Fermat's Little Theorem but does not comment on *p* being prime (Figure 6.68). Then, [04] applies the theorem in part (b) and (c) of task (3ii). In the first case, the application of the theorem is fine as the divisor, 11, is prime. However, in the next part of the task, the divisor is 36, which is not prime. [04]'s response to (3iib) shows that the student does not take into account the property of the variable *p* being prime and thus applies the theorem also for 36 without questioning whether 36 is prime or not.

Another example, illustrating students' difficulties, with recalling Fermat's Little theorem, that also hinders students' engagement with the next parts of the task is in figure (Figure 6.69)

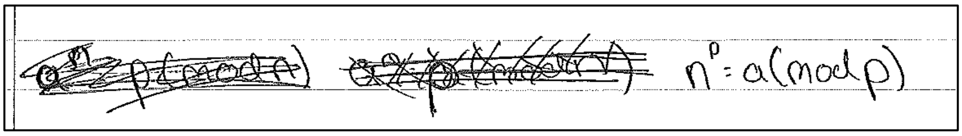


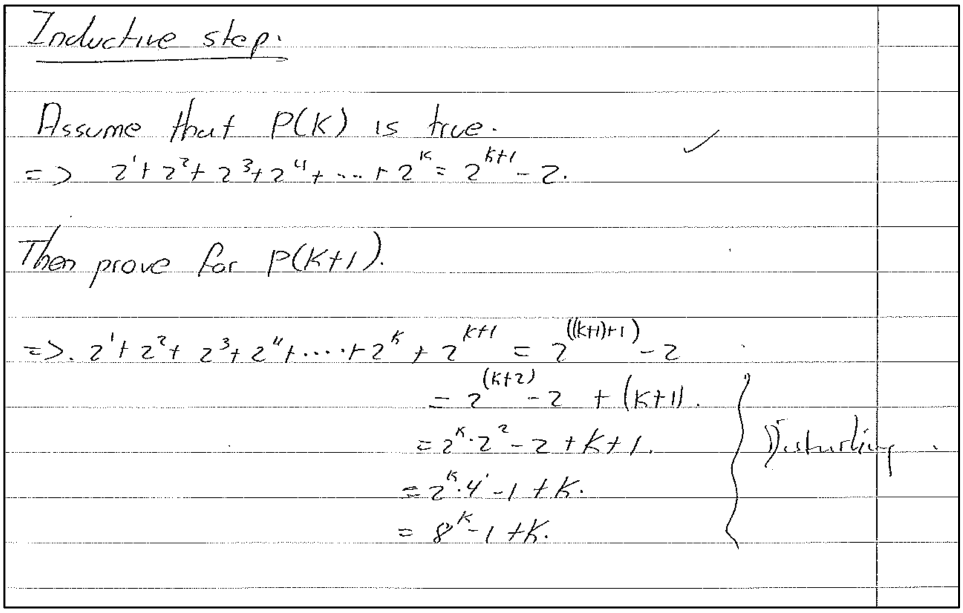
Figure 6.69: Student [17]'s response to (3iia).

In the first two attempts, [17] tries to connect the power a^n and the modular of n then changes to n^p and says that this is "a(mod$p$)". The scribbled out, and the ending narrative are involving symbols that are used in Fermat's Little Theorem. However, the way that these are recalled, illustrate that there is unclear meaning making regarding Fermat's Little Theorem.

6.3.2 The procedure of a substantiation routine

(i) Procedure of a substantiation routine: Proof by induction [Task (1i)]

Students' [01] and [16] scripts show evidence of conflicts in the procedure of the proof by induction. [16] assumes that P(k) is true and tries to prove that P(k+1) is true (Figure 6.70). In engaging with the substantiation routine, [16] writes what is to be proven. Then, [16], adds k+1 on the right-hand side of the equality. Here, there is a conflation between the assumptions of the proof and the result, as [16] writes as an index k+1 +1 instead of rewriting what was true for P(k) and then adding the 2^{k+1}. Also, the rest of the narrative illustrates unclear meaning making regarding the rules of indices.



Inductive step:

Assume that $P(K)$ is true.

$\Rightarrow 2^1+2^2+2^3+2^4+\cdots+2^K = 2^{K+1}-2.$

Then prove for $P(K+1)$.

$\Rightarrow 2^1+2^2+2^3+2^4+\cdots+2^K+2^{K+1} = 2^{((K+1)+1)}-2$

$= 2^{(K+2)}-2+(K+1).$

$= 2^K\cdot 2^2-2+K+1.$

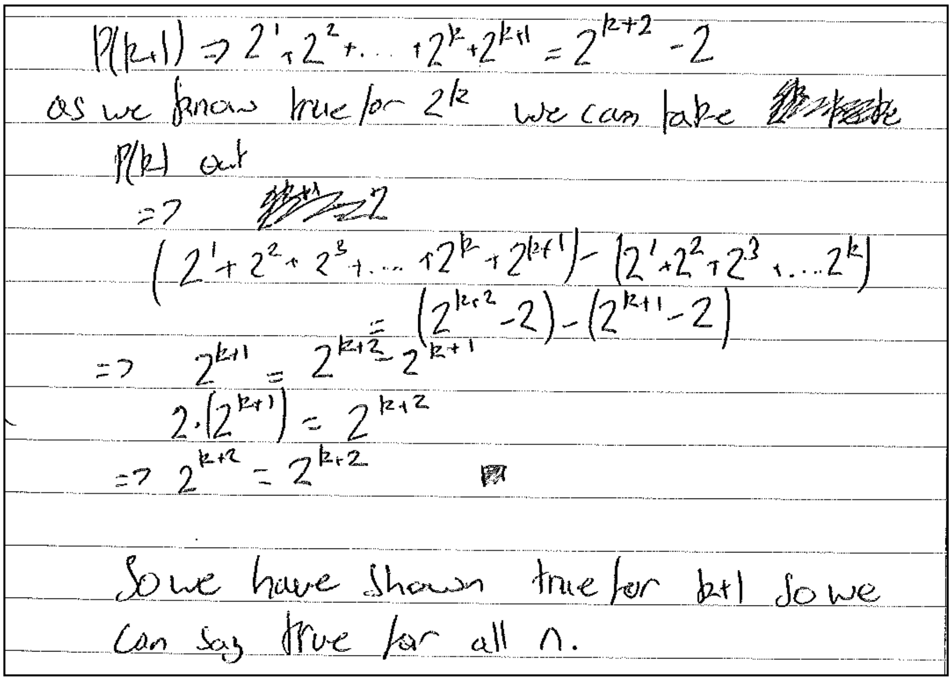
$= 2^K\cdot 4^1-1+K.$

$= 8^K-1+K.$

Disturbing.

Figure 6.70: Student [16]'s response to (1i) – the marker wrote "Disturbing".

Similar to [16], [01] also uses in proof what is to be proven (Figure 6.71). [01] uses the difference of the two statements $P(k+1)$-$P(k)$ in order to prove that the statement holds for $P(k+1)$. This procedure assumes that both statements hold, whereas the proof by induction assumes only that one of the statements hold and then, based on that assumption, [01] proves that the statement for $k+1$ holds. Furthermore, the assumption, as described by the student, refers to "true for 2^k" signalling unclear meaning-making regarding the variable used in the induction.



$P(k+1) \Rightarrow 2^1 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 2$

as we know true for 2^k we can take

$P(k)$ out

$\Rightarrow$ ~~$2^{k+1} - 2$~~

$\left(2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1}\right) - \left(2^1 + 2^2 + 2^3 + \ldots 2^k\right)$

$= \left(2^{k+2} - 2\right) - \left(2^{k+1} - 2\right)$

$\Rightarrow 2^{k+1} = 2^{k+2} - 2^{k+1}$

$2 \cdot \left(2^{k+1}\right) = 2^{k+2}$

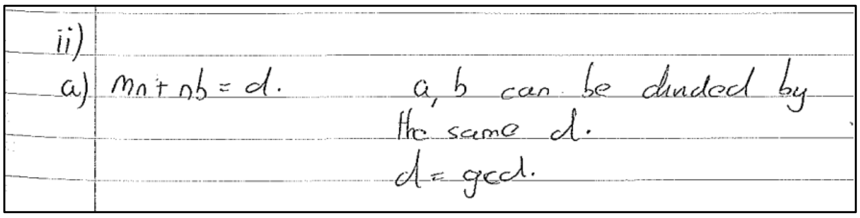
$\Rightarrow 2^{k+2} = 2^{k+2}$

So we have shown true for $k+1$ so we can say true for all n.

Figure 6.71: Student [01]'s response to (1i).

(ii) The procedure of a substantiation routine: The linear combination of *a* and *b* [Task (1ii)]

Apart from student [03]'s response, which I have talked about in detail in section (6.2.1), the responses of two students ([16], [17]) illustrate difficulties with the engagement with the substantiation routine of the linear combination of *a*, *b* and their divisor *d*. Student [16] writes that the linear combination of *a* and *b* is equal to their divisor *d* (Figure 6.72). This is a special case of the relationship that needed to be proven in this task and [16] adds that *d* is equal to the gcd (greatest common divisor). Also, the student does not define the divisor and thus does not introduce crucial variables which illustrate the connection between the divisor and a and b.



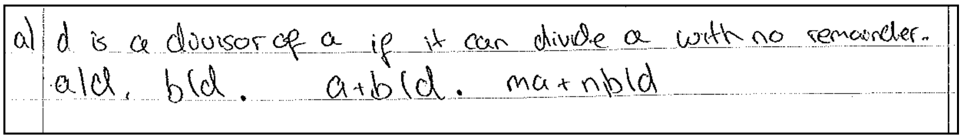
ii)

a) $ma + nb = d$. a, b can be divided by the same d.

$d = $ gcd.

Figure 6.72: Student [16]'s response to (1ii).

[17], says that *a* divides *d*, *b* divides *d*, the summation divides *d*, and the linear combination divides *d* (Figure 6.73). Here, as in Figure 6.76, the student does not define the relationship between the divisor *d* and *a* and *b* using symbolic mediation and thus, [17] does not introduce the variables that would help towards the proof.



a) d is a divisor of a if it can divide a with no remainder.

$a|d$, $b|d$. $a+b|d$. $ma+nb|d$

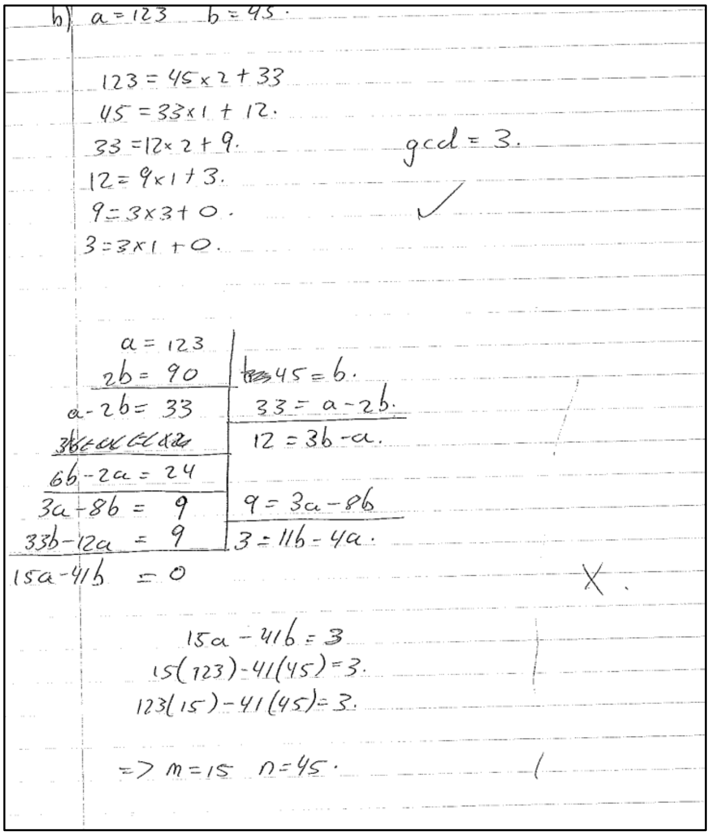
Figure 6.73: Student [17]'s response to (1ii)

6.3.3 The procedure is not given in the wording of the task

In parts of the three tasks (1iib), (2i), (3ii) the procedure of the routine is not given, and the students have to decide what procedure to follow. In the following, I illustrate examples of cases where the students followed a different procedure, from the one that the lecturer expected (as we can see in the model solutions in sections (5.1.2), (5.2.2) and (5.3.2)).

(i) The procedure of finding the greatest common divisor (Task (1iib)]

In (1iib), the students are asked to find the greatest common divisor of 123 and 45 and then find the specific integers (*m* and *n*) that make the linear combination of 123 and 45 equal to their gcd. The students are instructed in the wording of the task to use the Euclidean Algorithm and "Hence (or otherwise)" find the integers *m* and *n*. Five students ([01], [03], [11], [15], [16]) use a different procedure than the one presented in the lecture notes. The lecturer had presented a specific way of writing the Euclidean algorithm that helped the students also to find the linear combination of 123 and 45 that was equal to their gcd. However, some students used different procedures to find the gcd.


b) $a = 123$ $b = 45$.

$123 = 45 \times 2 + 33$
$45 = 33 \times 1 + 12$.
$33 = 12 \times 2 + 9$. gcd = 3.
$12 = 9 \times 1 + 3$.
$9 = 3 \times 3 + 0$.
$3 = 3 \times 1 + 0$.

$a = 123$ | $45 = b$.
$2b = 90$ | $33 = a - 2b$.
$a - 2b = 33$ | $12 = 3b - a$.
$6b - 2a = 24$
$3a - 8b = 9$ | $9 = 3a - 8b$
$33b - 12a = 9$ | $3 = 11b - 4a$.
$15a - 41b = 0$

$15a - 41b = 3$
$15(123) - 41(45) = 3$.
$123(15) - 41(45) = 3$.

$\Rightarrow m = 15$ $n = 45$.

Figure 6.74: Student [16]' response to (1iib) – the marker added the tick and the cross

[16] finds the gcd using a different procedure of writing the Euclidean algorithm and then uses the procedure shown in the lectures to find the linear combination (Figure 6.74). However, in doing that, instead of taking the linear combination that resulted in the gcd which was $3 = 11b - 4a$, [16] takes $15a - 41b$ and makes that equal to the gcd. This could signal a ritualised use of the procedure as the connection between the linear combination equal to zero that results at the end of the procedure of the Euclidean algorithm, and the student does not explicitly state that the linear combination equals to the gcd.

Another procedure that was used by [03] and [01] was long division (Figure 6.75).

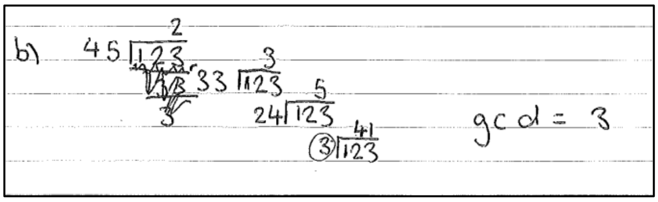


Figure 6.75: Student [03]'s response to (1iib).

(ii) The procedure of the substantiation routine in Set Theory: Proving that two sets are equal [Task (2i)]

Students ([02], [16]) use Venn Diagrams in their narratives, and the substantiation routine is based on these Venn Diagrams. Specifically, [02] makes a Venn diagram and using the phrase "as we can clearly see" says that the two sets are the same (Figure 6.76)

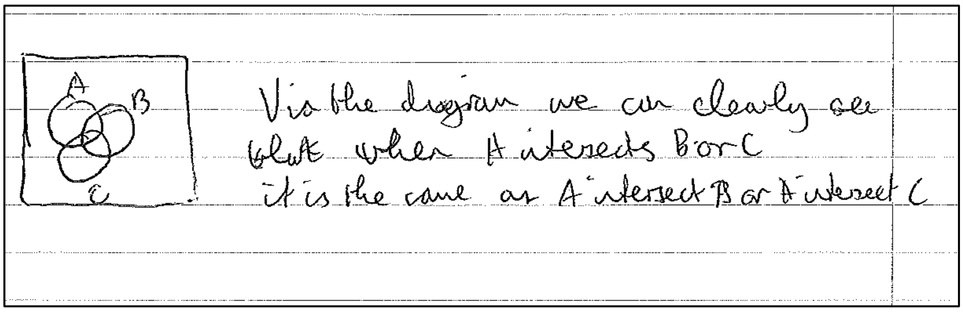


Figure 6.76: Student [02]' response to task (2i).

Student [16] also uses Venn diagrams (Figure 6.77) but does not conclude the equality between the sets. Also, their use of Venn diagrams signals unclear meaning making between the different realisations of the union and intersection of sets (discussed in section 6.2.4).

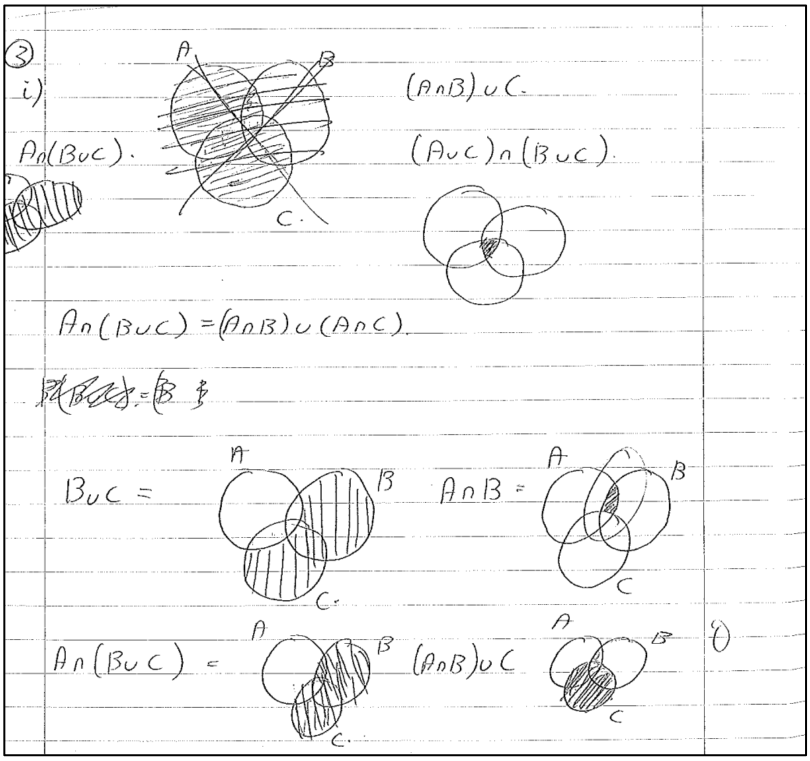
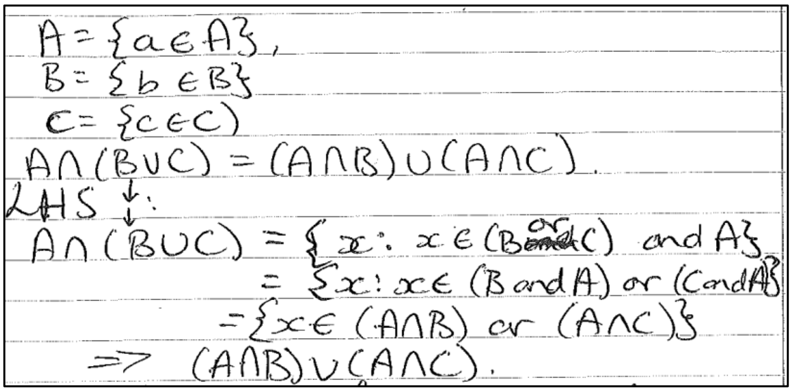


Figure 6.77: Student [16]'s response to (2i).

Students ([06], [10], [12], [22]) only solve one side of the equality. These students seem to engage in rules from the algebra discourse to show the equality between two algebraic expressions without considering that the rules and objects in the discourse of Set Theory are different. The substantiation routine of an equality in the algebra discourse involves starting from the left or the right-hand side of the equality and by proving that the expression in one side of the equality can be reached by rearranging the other side. In the Set Theory discourse, the substantiation routine of the equality between two sets involves two steps, proving that the first set is a subset of the second and vice versa. [12] (Figure 6.78) takes an element in the set from the left-hand side of the equality (LHS) and shows that this element belongs to the second set. The student then stops.



$A = \{a \in A\}$,
$B = \{b \in B\}$
$C = \{c \in C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
LHS $\downarrow$:
$A \cap (B \cup C) = \{x : x \in (B \cancel{\text{and}} \text{ or } C) \text{ and } A\}$
$= \{x : x \in (B \text{ and } A) \text{ or } (C \text{ and } A)\}$
$= \{x \in (A \cap B) \text{ or } (A \cap C)\}$
$\Rightarrow (A \cap B) \cup (A \cap C)$.

Figure 6.78: Student [12]'s response to (2i) – The acronym LHS means "Left-Hand Side".

The responses from [08] and [15] evidence similar commognitive conflict. Student [08]'s response (Figure 6.79) shows a conflict between the rules of algebra and Set Theory discourses. [08] examines both sides of the equality. However, when starting to prove that the left-hand side is equal to the right-hand side, there is also a conflation between the signifier of the union with addition. Specifically, the union of the two sets *B* and C becomes their sum. Moreover, [08] concludes that this implies the set in the right-hand side. Turning to the other side of the equality, [08]' script shows more instances of the conflation between the rules of the two discourses. [08] uses factorisation to substantiate that the right-hand side equals the left-hand side.

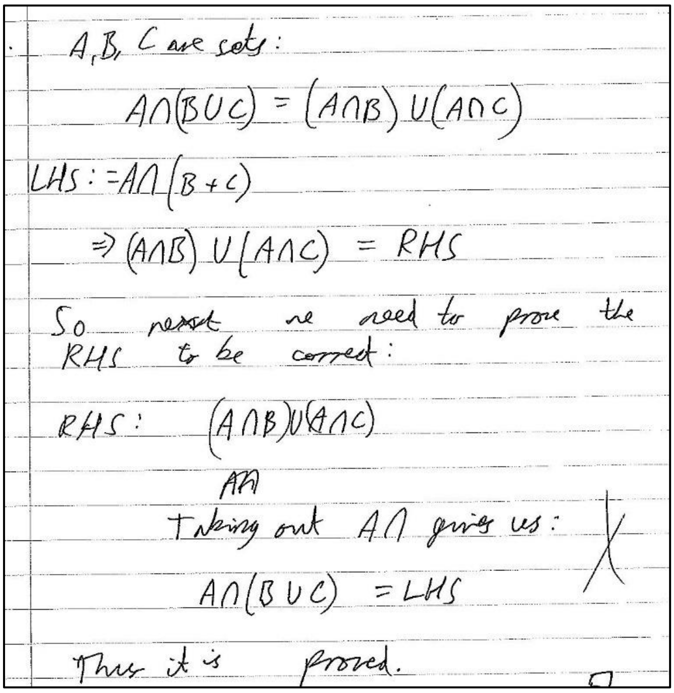


Figure 6.79: Student [08]'s response to (2i). LHS means "Left-Hand Side", RHS means "Right-Hand Side". The marker added the cross symbol.

Student [15] reproduces the equality to be proven and starts from the set on the right-hand side (Figure 6.80). The scribbled-out part suggests that the student tried first to start from the left-hand side but gave up. The student conflates the signifiers (+) and (-) and uses an identity used in Probability, $P(D \cup E) = P(D) + P(E) - P(D \cap E)$. The presence of the signifiers for intersection and union of sets may have made the student recall this identity. Also, this could contribute to the module addressing Probability in the other half of the tasks.

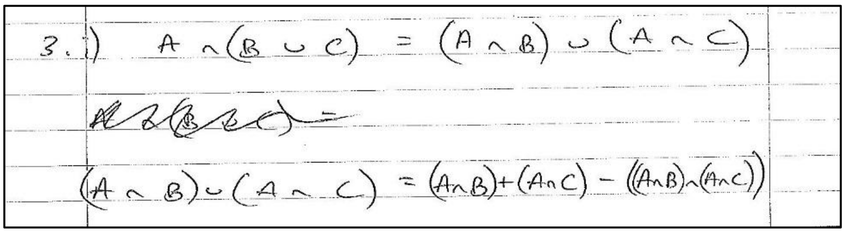


Figure 6.80: Student [15]'s response to (2i)

(iii) The procedure of the substantiation routine: An example from Modular arithmetic

[03] and [15] translate modular arithmetic into fractions (Figure 6.81). [03] writes that the fraction $\frac{27}{11}$ leaves a remainder of 5 and, similarly, uses this notation to write the remainder of the different powers of 27. However, the wrong remainder is calculated for the power of 27. Before engaging with this procedure, [03] attempts to solve this task using natural logarithms (see scribbled out text in Figure 6.81)

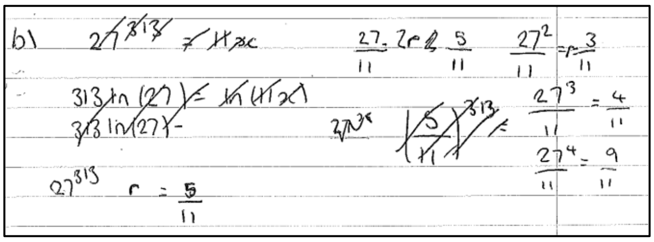


Figure 6.81: Student [03]'s response to (3iib)

Four students ([02], [04], [11], [13]) use either a variation of Fermat's Little Theorem (Figure 6.82) or another theorem (Figure 6.83). Specifically, [02] writes the substantiation of Fermat's Little theorem without considering that 36 is not prime (as also discussed in section (6.2.1)) and concludes that x $=19^{34}$ (Figure 6.82).

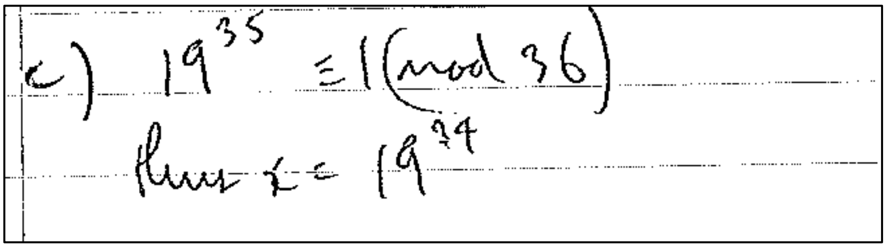


Figure 6.82: Student [02]' response to (3iic)

[12] introduces another theorem that connects three integers *a*, *b* and *d*. If *d* is not the divisor of *a* then there is another integer, *b*, for which the product *ab* results in 1 mod*d* (Figure 6.83). After introducing this, the student applies the theorem using the specific numbers 36 and 19. [12] checks that the assumptions of the theorem are met and says that 36 should not divide *x*. Then. [12] concludes by saying that x is a set where *x* is an integer that is not divisible by 36. Apart from the student not concluding with a specific value for *x*, I note, also, that there is a conflation between the object of set "*x*" and

its element “*x”.* Usually, for sets the upper-case letters are used and not the lower-case ones.

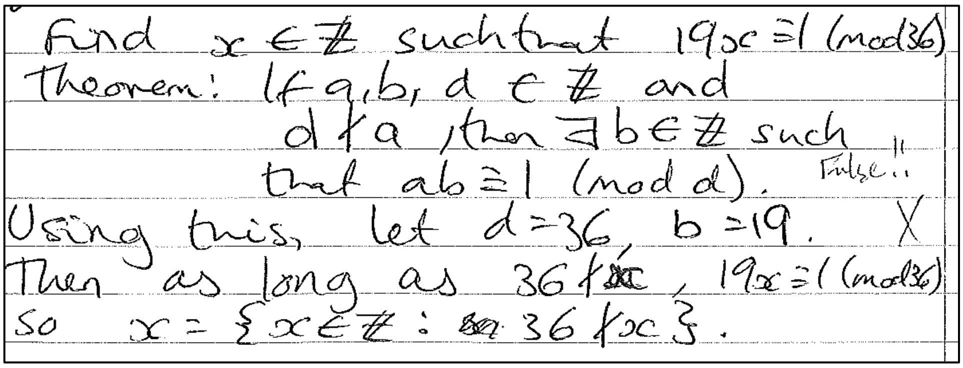

Find $x \in \mathbb{Z}$ such that $19x \equiv 1 \pmod{36}$
Theorem: If $a, b, d \in \mathbb{Z}$ and $d \nmid a$, then $\exists b \in \mathbb{Z}$ such that $ab \equiv 1 \pmod{d}$. False!!
Using this, let $d = 36$, $b = 19$. X
Then as long as $36 \nmid x$, $19x \equiv 1 \pmod{36}$
So $x = \{x \in \mathbb{Z} : 36 \nmid x\}$.

Figure 6.83: Student [12]’s response to (3iic) – The marker added the “False!!” and “x”.

Two students ([07], [14]) write that 19 is a value for *x*. However, they do not show the procedure they followed to find this value (Figure 6.84)

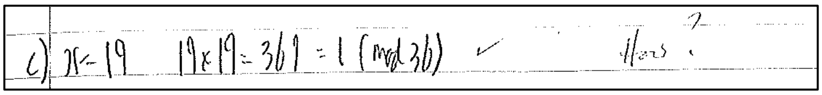
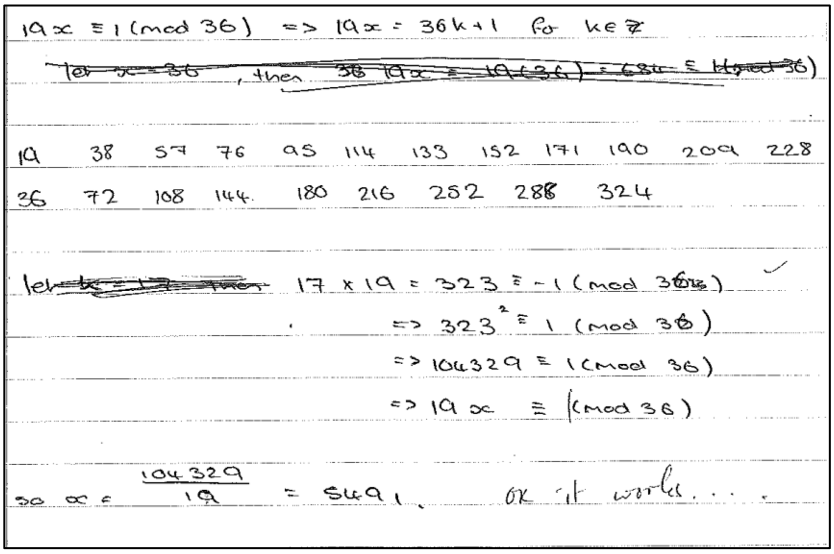

c) $x = 19$ $19 \times 19 = 361 \equiv 1 \pmod{36}$ How?

Figure 6.84: Student [14]’s response to (3iic) – The marker added the tick and the “How?”

[06] and [13] examine different cases by considering that $19x = 36k + 1$ and, then, find a value for which the equality is true. [13] first translates the modular arithmetic into an equality, examines the multiples of 19 and then stops that procedure and starts by finding the multiples of 36 (Figure 6.85). [13] finds that 324 is a multiple of 36. This is very close to 323 which is a multiple of 19. Then writes the connection of 323 with the modular of 36 and uses properties of the modular arithmetic to conclude with a value for *x*.



$19x \equiv 1 \pmod{36} \Rightarrow 19x = 36k+1$ for $k \in \mathbb{Z}$

~~let x = 36, then 36 19x = 19(36) = 684 ≡ (mod 36)~~

19 38 57 76 95 114 133 152 171 190 209 228

36 72 108 144 180 216 252 288 324

~~let~~ $17 \times 19 = 323 \equiv -1 \pmod{36}$ ✓

$\Rightarrow 323^2 \equiv 1 \pmod{36}$

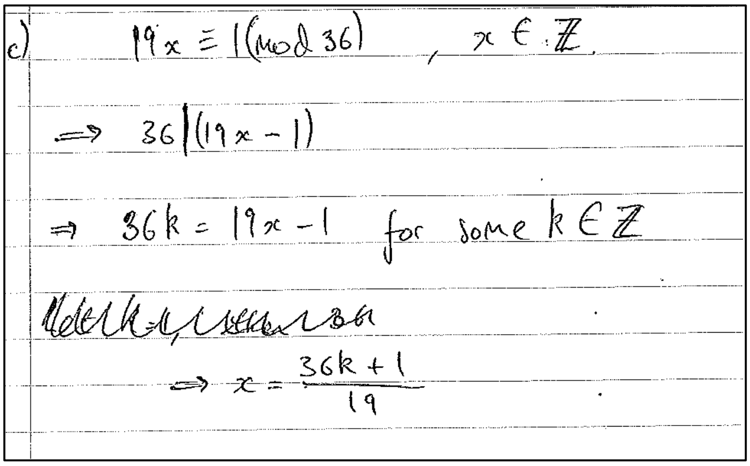
$\Rightarrow 104329 \equiv 1 \pmod{36}$

$\Rightarrow 19x \equiv 1 \pmod{36}$

so $x = \frac{104329}{19} = 5491$, OK it works . . .

Figure 6.85: Student [13]'s response to (3iic) – The marker added the ticks and wrote: "OK it works…"

The translation of the modular arithmetic to multiples is also visible in [19]'s solution (Figure 6.86). [19] connects modular arithmetic with multiples. However, [19] does not reach the closing condition and does not find a value for *x*. From the scribbled-out part, it seems that [19] tried to give different values to *k* aiming to find a value for *x*. However, the value *k*=1 results in a non-integer value for *x.* This, potentially, may be the reason that [19] stopped*.*



c) $19x \equiv 1 \pmod{36}$, $x \in \mathbb{Z}$

$\Rightarrow 36 \mid (19x - 1)$

$\Rightarrow 36k = 19x - 1$ for some $k \in \mathbb{Z}$

~~[illegible]~~

$\Rightarrow x = \frac{36k+1}{19}$

Figure 6.86: Student [19]'s response to (3iic)

6.3.4 Are the closing conditions of the routine met?

In the previous sections, I discussed the applicability conditions and the procedure of the routines and what the students' scripts evidence when these are specified explicitly or not in the wording of the task. I now turn to the closing conditions, and I report cases where the closing conditions were not met resulting in ineffective communication between the marker and the student.

(i) Is the linear combination equal to 7? [Task (1iic)]

The responses of five students ([04], [11], [12], [15], [22]) illustrate a difficulty regarding the closing conditions of the substantiation routine regarding the linear combination of 123 and 45 is equal to 7.

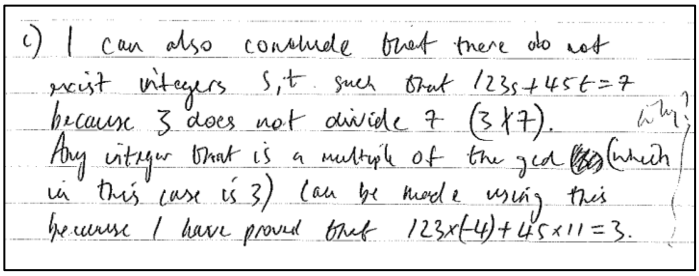
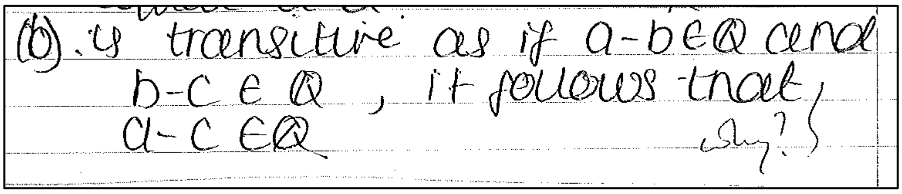

c) I can also conclude that there do not exist integers s, t such that $123s+45t=7$ because 3 does not divide 7 ($3 \nmid 7$). why?
Any integer that is a multiple of the gcd (which in this case is 3) can be made using this because I have proved that $123\times(-4)+45\times 11=3$.

Figure 6.87: Student [22]'s response to (1iic) – The marker wrote: "why?".

Student [22], tries to prove that there are no integers *s* and *t* such that make the linear combination is equal to 7 (Figure 6.87). However, [22] is not explicit about the connection with the linear combination and the greatest common divisor of 123 and 45. After claiming that such integers do not exist, [22] attempts to show that any multiple of 3 can be written in the form of the linear combination. However, as the connection between the linear combinations is not clear, the closing conditions of the substantiation routine are not achieved.

(ii) Is the relation transitive? [Task (2ii)]

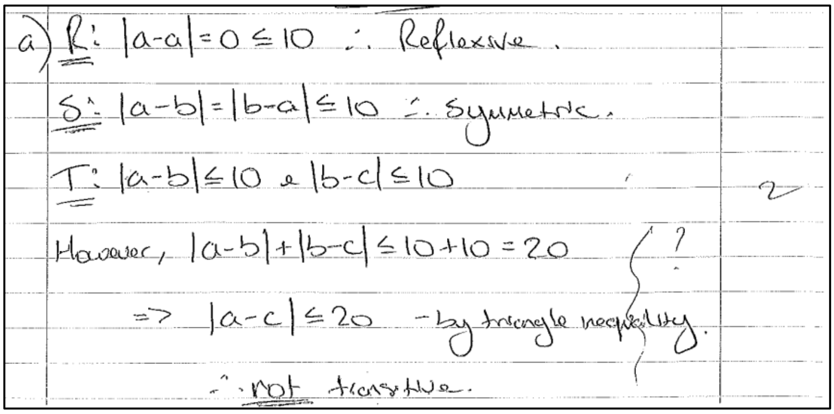
The closing conditions are not achieved in the narratives of three students ([04], [07], [21]) in (2ii). [07] writes that the relation is transitive and explains that, if the two differences are in $\mathbb{Q}$, then "it follows" that *a-c* is also in $\mathbb{Q}$ (Figure 6.88). However, the justification is deficit, as [07] does not provide an explicit explanation regarding why this is the case, showing clearly that the sum of two rational numbers is still a rational number and that *a*-*c*=(*a*-*b*)+(*b*-*c*).


(b). is transitive as if $a-b \in \mathbb{Q}$ and $b-c \in \mathbb{Q}$, it follows that $a-c \in \mathbb{Q}$

why?

Figure 6.88: Student [07]'s response to (2iib) – The marker wrote: "why?".

Similarly, the closing conditions regarding the relation in (2iia) not being transitive are not met in the narrative of [21] (Figure 6.89). Attempting to prove that the given relation is not transitive, [21] uses the triangle inequality and finds an upper limit of 20 for $|a-c|$. However, doing so is not sufficient explanation for showing that the relation is not transitive.


a) R: $|a-a| = 0 \le 10$ ∴ Reflexive.

S: $|a-b| = |b-a| \le 10$ ∴ Symmetric.

T: $|a-b| \le 10$ & $|b-c| \le 10$

However, $|a-b|+|b-c| \le 10+10 = 20$?

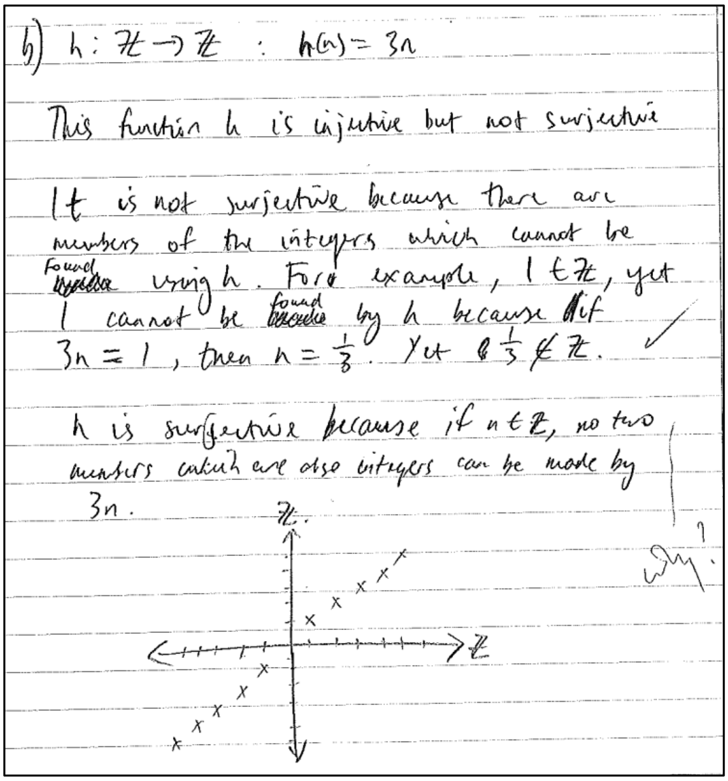
$\Rightarrow |a-c| \le 20$ - by triangle inequality.

∴ not transitive.

Figure 6.89: Student [21]'s response to (2iia) the marker added the question mark and did the line

(iii) Is the function surjective and injective? [Task (3i)]

The script of [22] illustrates that the closing conditions for proving the surjectivity of function *h* are not met (Figure 6.90). [22] starts from the concluding narrative saying that "this function *h* is injective but not surjective". Then, illustrates that *h*(*n*) is not surjective and tries to prove that it is an injective function. In that part of the script, [22] writes "surjective" instead of "injective". [22] says that "no two numbers which are also integers can be made by 3*n*” and provides a graph to support this claim. The graph of the function illustrates that this is a discrete function. However, the argument and the graph are not sufficient, as there is not a clear indication regarding the relationship between the elements of the codomain and the corresponding elements of the domain.


b) $h: \mathbb{Z} \to \mathbb{Z}$: $h(n) = 3n$

This function h is injective but not surjective

It is not surjective because there are numbers of the integers which cannot be found ~~made~~ using h. For example, $1 \in \mathbb{Z}$, yet 1 cannot be found ~~made~~ by h because if $3n = 1$, then $n = \frac{1}{3}$. Yet $\frac{1}{3} \notin \mathbb{Z}$.

h is surjective because if $n \in \mathbb{Z}$, no two numbers which are also integers can be made by 3n.

Figure 6.90: Student [22] question 3ib – the marker added the “why?”

6.4 Concluding remarks on the analysis of students' scripts in Sets, Numbers and Proofs

In this section, I first present the summary of the findings from the students' scripts, then I connect the analysis of the scripts to the analysis of the task and the lecturer's perspectives (chapter 5) and, finally, I relate the results of the analysis to the relevant literature.

6.4.1 Summarising the findings

Studies have shown that secondary students face difficulties when using rational and natural numbers (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). The results of my analysis in (6.2.1) show similar results, illustrating the difficulty in engaging with various numerical contexts. Specifically, the results show that undergraduate students are finding it difficult to identify and work consistently within the appropriate numerical context. In the university discourse, specific attention is given to the numerical context of the task whereas this is not the case at the school level. Many mathematical discourses involve different numerical domains and engagement within those numerical domains is something new for the students. The latest numerical domain introduced, at school, are the real numbers and usually, it is assumed that the students work within that numerical context. However, in the university, the students revisit the numerical domains, namely natural, integer, rational and real numbers and have to work within and alternate between different ones in varying modules and sometimes in the same task.

Another theme that emerged in the analysis is the inconsistency in the naming of the variables used in narratives (discussed in section 6.2.2). This theme is in accordance with the results discussed in Epp's (2011) study. This inconsistency could be the result of negligence, but it could be evidence of unclear meaning making regarding the objects of statements of propositions $P(n)$ and functions $f(x)$, elements of the domain and the codomain. In some cases, these are not seen as creating a reason for conflict in the communication between the marker and the student, thus not hugely impacting the effectiveness of the communication. However, evidence of the analysis shows that there are cases that this inconsistency hinders the effectiveness of the communication.

The structure of the narratives, and specifically students' use of the logical symbols illustrating relationships between objects is another theme that emerged in the analysis presented in (6.2.3). This is also due to the differences between the school and university discourse. This difference is reported also by Gueudet et al. (2016). Specifically, these researchers review literature on transition between secondary and university and note that "secondary mathematics stresses the production of results and the practical aspect of mathematical activities, assigning a more "decorative" role to axioms, definitions, and proofs" (Gueudet et al., 2016, pp. 19-20).

Similarly, not much attention is given in school mathematics regarding the structure of students' narratives. In the incidents reported in (6.2.3), the students influenced by their use of the equality symbol in school, use this symbol either to signal implication or to name an object. However, different symbols are used for this purpose at the university. This difficulty with the equality symbol is also observed by Stadler's study (2011) on transition. Moreover, the analysis suggests that students conflate newly introduced symbols (e.g., equivalence and congruency) with the ones that they are familiar with (e.g., equality).

Other types of visual mediator that featured in students' scripts are Venn diagrams and graphs of functions. The students used these mediators to support their narratives. However, there were students who used graphs and Venn diagrams and their use illustrated either conflation of discourses or unclear meaning-making. Specifically, students' scripts with graphs showed evidence of conflation between the discourses of integers and reals regarding the realisation of a function with domain and codomain integers. The use of the Venn diagrams signalled conflation of rules, regarding the extent and sufficiency of substantiation narratives. At school sometimes showing a graph or an example of a set that satisfies some conditions is enough. However, at university the substantiation process requirements are different.

In the mathematical community, the personal pronoun “we” is typically used. I observed that the majority of the students (17 out of 22) also used the personal pronoun “we” and only one of them alternated between “I” and “we”. Use of the personal pronouns positions the students closer to the community but also suggests endorsement and adds authority.

Apart from the word use mentioned in the above paragraphs, I also observed four students’ scripts used terminology that belonged to other mathematical discourses. This signals another commognitive conflict between the university and school discourses. The students at the university are exposed to different mathematical discourses which are interlinked with each other. Using appropriate terminology, visual mediators and rules from different discourses is a part of the complexity of the university mathematical discourse.

Students’ engagement with a recall routine first, had an impact on the substantiation routine based on that same object. This was visible both concerning recalling objects (6.3.1) and procedures (6.3.2) and (6.3.3). Specifically, there were cases where students did not recall either a definition of reflexive, symmetric and transitive relation; injective and surjective function or a theorem (e.g., Fermat’s Little Theorem) and this had an impact in their engagement with substantiation routines.

Similarly, difficulties in recalling procedures of routines are illustrated in the *applicability* (changing the when of a routine and keeping the *how* constant) and flexibility (changing the *how* keeping the *when* constant) of using routines. Regarding the *applicability* of the Euclidean algorithm, students used it when asked in Task 1 but when this was not explicitly asked in Task 3 eight of them used another procedure illustrating the *flexibility* of the routines.

Students’ engagement with the routines where the procedure was not given in the wording of the task, illustrate unresolved commognitive conflicts in the case of Set theory and school algebra. In Section (6.3.3ii), I report scripts that conflate rules from the school algebra and set theory. Similarly, in (6.3.3iiii) there are scripts in which, instead of using the discourse of modular arithmetic, the students resort to using fractions, a topic with which they are more familiar. The above is evidence of the turbulent shift between the discourses of university and school. When faced with a discourse that they are not familiar with, students apply rules that they are familiar with from their school discourse. This results in commognitive conflict.

Finally, in (6.3.4), I examine scripts where the extent and sufficiency of the narratives produced by the students do not meet the closing conditions of the routine. The narratives, produced, do not illustrate clearly the connections between the objects involved. Being explicit about the relationships between the objects and describing the procedure followed to show the relationships is another difference between the school and university discourse that the students are asked to become accustomed to. This difference is also visible in Darlington's (2014) analysis of A-level and first-year undergraduate examination tasks. She notes that "The findings here suggest that A-level Mathematics and Further Mathematics' main focus is on assessing students' abilities to repeat procedures, rather than to develop mathematical skills" (Darlington, 2014, p. 226).

In the above, I discussed findings from the analysis of the students' scripts. I now turn to connect with lecturer's expectations about students' engagement with these tasks and the analysis of the task itself, as presented in chapter 5.

6.4.2 Connecting with task analysis and the lecturer's perspectives

In section (5.4), I summarised the results of the analysis regarding the tasks and the lecturer interview. The analysis showed that students are asked to engage mostly in recall and substantiation routines which are different from the routines they were familiar with in school. In analysing the students' scripts (6.3.1), I highlight difficulties when the engagement with recall (both for routines and objects) impacts in the engagement with substantiation routines but also when the procedure of substantiation routines shows conflation of different discourses (6.3.2).

The analysis of tasks (1ii) and (3ii) (sections (5.1.1) and (5.3.1) respectively) showed that the stepped structure of the task signalled that parts of the narratives produced earlier could be used in later stages of the task. However, the *when* is not specified in the wording of the latter parts of the task. Students did use the narratives they produced. However, the closing conditions were not achieved, as the explicit connection between the parts of the task was not made (section 6.3.4). The analysis of the scripts (section 6.3.3) showed that students used the theorem also at a later stage of the task (3iic) without taking into account the conditions of applicability.

The existence of directions regarding the procedures, and the extent and sufficiency of the routines in the task (3i) lead to some students using a graph to give a quick response regarding the injectivity of the function or using procedures without questioning whether these are appropriate for the numerical context of the task, the set of integers. Students' difficulties with the injective function have also been reported in the literature (Vinner & Dreyfus, 1989; Thompson, 1994). As the lecturer comments (section 5.3.3), the engagement with the same task (3i) in a different module (e.g., Analysis) would involve a different level of justification. The absence of directions regarding the procedure ((2i), (3i), (3ii)) or by giving the flexibility to choose (1iib) showed in the scripts that students were creative regarding the procedures of the routines (section 6.3.3). Also, apart from the different procedure, the abstractness of sets and the difference in substantiating an equality in Set Theory is noted by the lecturer (5.2.3). The students' scripts illustrate this difficulty as they resort to using rules (6.3.3) that they are familiar with or using Venn diagrams (6.2.4).

In (5.1.3), L1 talks about the importance of the sub-task regarding the definition of the divisor and the visual mediators used. Students' solutions show that they had difficulty embedding their discourse in the numerical context of the integers (6.2.1). The value of the symbols (variable or logical symbols) is discussed by the lecturer in sections (5.1.3), (5.2.3) and (5.3.3). The analysis of the scripts illustrates students' difficulties with the use of symbolic mediation in the form of variables, in sections (6.2.1) and (6.2.2), and the logical symbols (6.2.3) as stemming from the differences between the school and the university discourse.

In chapters 5 and 6, the examination tasks, lecturer's perspectives on assessment, and students' scripts from the Sets, Numbers and Proofs part of the module are analysed. In the next chapter, I analyse the examination tasks from the Probability part of the module (chapter 7). This is followed by analysis of the students' scripts to these tasks (chapter 8).

Chapter 7. Probability: Tasks and lecturer's perspectives

As mentioned in chapter 4 (4.2.1), the examination in the module Sets, Numbers and Probability has six tasks, three of which correspond to the Probability part of the module. The first two tasks (1, 2) are compulsory, and the other four (3, 4, 5 and 6) are optional. In this chapter, I focus on the tasks from the Probability part of the module, namely tasks 2 (7.1), 5 (7.2) and 6 (7.3). For each, first I introduce the task and a commognitive analysis of the task. I then explain the context of the task by considering the solution of the task given by the lecturer of the module and similar tasks from the worksheets. Analysed excerpts from the interview with the lecturer follow. Finally, I conclude by highlighting issues that cut across the data analysis for each task (7.4).

7.1 Examination task 1 (Compulsory)

7.1.1 Task and commognitive analysis of the task

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 7.1: Compulsory task from the Probability part of the module

The second compulsory task (Figure 7.1), thereafter known as task 1, is focusing on Kolmogorov's axioms, Propositions, and conditional probability. The lecturer, L2, produced a model solution to the task (Figure 7.3) for departmental use, which was not made available to the students. In this next section, I present a commognitive analysis of the task.

Task 1 (Figure 7.1) has two subtasks. The first part focuses on the theoretical part of Probability: Kolmogorov's axioms and two propositions and the second part asks for the application of the theory towards calculating specific probabilities. Sub-task (1i) starts with the phrase "In the framework of modern probability" situating the students to a context described the historical background of the probability as a subject, starting from the 16th century up to the modern axiomatic definition of probability given by Kolmogorov. The students are asked to engage in recall routines. Initially, giving the definition of disjoint events and another by stating the Kolmogorov's axioms. In the next parts (1ia) and (1ib) the students are asked to engage in two substantiation routines by proving two propositions. There is an instruction regarding the *procedure* that they should follow as; first, there is a prompt to use Kolmogorov's axioms, and then the guideline to '<u> use them to demonstrate"</u>. This prompt signals the connection between the endorsed narratives, that the students should have produced in (1i) and the substantiation routines the students are expected to engage in while proving the two propositions (1ia) and (1ib). Also, the lecturer provides another prompt ("You may assume Proposition 2, that is

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.") (I note that "2" corresponds to the numbering of the propositions in the lecture notes). The various parts of (1i) are interdependent, as the students have to recall Kolmogorov's axioms in order to use them in their responses for (1ia) and (1ib). Finally, in (1ia) and (1ib) the students are asked to use interchangeable events and their probabilities. The objects of events come from Set Theory discourse, and they are connected with their probabilities in the Probability discourse, where students are asked to find the probabilities of specific sets.

In the wording of (1ii), the probabilities of specific events – namely of event *A,* of event *B*, considering that event A has occurred and of event *A* or *B* – are given. Sub-task (1ii) has three subtasks and the way that these are structured aims to assist the students in solving them. In (1iia), the students are asked to engage in a substantiation routine to prove that the probability of the simultaneous occurrence of events *A* and *B* is equal to ¼. To engage in this substantiation routine, the students need to also engage in a recall routine as they have to recall and use the multiplication rule ($P(A \cap B) = P(A|B)P(B)$) or the definition of conditional probability ($P(A|B) = \frac{P(A \cap B)}{P(B)}$). However, the *procedure* is not defined in the wording of the subtask. The students are expected to choose this *procedure*.

For (1iib), the students are asked to find P(B), the probability of event *B* occurring. In solving this part, the students are expected to recall proposition 6 ($P(A \cup B) = P(A) + P(B) - P(A \cap B)$, the number 6 is from the lecture notes). Then, in the second part of (1iib), they have to determine the range of possible values for *p*, where *p* is *P*(*A*U*B*), the probability of event *A* or *B* occurring. In calculating the range of possible values for *p*, the students need to consider Kolmogorov's axioms. Here, I note that apart from the values given in the wording of the task, the value of $P(A \cap B)$ is also needed in order to answer (1iib) and this value is given in (1iia). With the phrase "Show that" instead of "Calculate" or "Find", the lecturer provides the value of $P(A \cap B)$, which is needed later on.

For (1iic), the students are asked to calculate two probabilities. To do so, they are expected to recall proposition 3 ($P(A^c) = 1 - P(A)$). The numbering of the propositions is the one used in the lecture notes. Furthermore, they also have to recall the multiplication rule ($P(A \cap B) = P(A|B)P(B)$). First, they need to find the probability of the complement of B, given that event A has occurred; and, then they need to find the probability of event A and the complement of B happening at the same time. If, as asked, the students calculate the probability $P\left(B^c|A\right)$ first, then knowing this value can assist in calculating the probability $P(A \cap B^c)$. However, a different *procedure* could also be followed by calculating first the second probability and then using that to find the first one.

Finally, I note the structure of the task is split into two parts: a theoretical part (1i) with recall and substantiation routines and an application of the theory (1ii) with mostly recall routines and engagement with rituals. I also note that the structure of (1i) and (1ii) signals that the initially produced narratives can be used for the latter parts of the task. However, this also shows the dependency between the parts of the task: if a student cannot produce a correct response in the first parts of the task, then they are unlikely to continue with the following parts. Finally, there are directions regarding the *procedure* of the substantiation routines in (1i) as there are specific prompts regarding the narratives that the substantiation routines should be based on.

7.1.2 Context and the lecturer's model solution

Task 1 is similar to the compulsory task used in the examination the previous year (Figure 7.3). However, a clear difference with task 1 (Figure 7.1) is that in the last year the lecturer did not ask for the definition of disjoint events. In the interview (section 7.1.3), he comments as to why he decided to make this addition this year.

2. (i) State the three Kolmogorov axioms of modern probability and make use of them to demonstrate the following two properties.

(a) For any two events A and B prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,.$$

(b) For any event A prove that

$$P(A^c) = 1 - P(A)\,.$$

[8 marks]

(ii) Let A and B be two events, with $P(A) = \frac{1}{4}$, $P(B|A) = \frac{1}{3}$ and $P(A \cup B) = p$.

(a) Find $P(A \cap B)$ and $P(A \cap B^c)$.

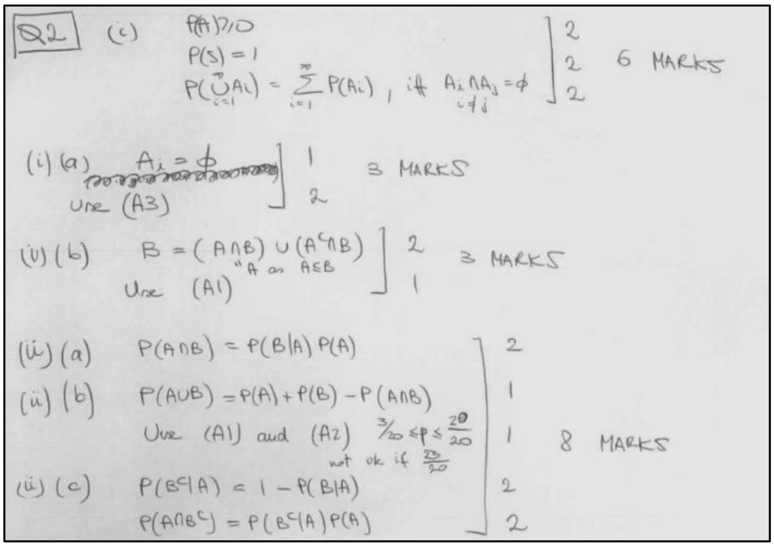
(b) Show that $P(B) = p - \frac{1}{6}$.

(c) For what value of p are A and B independent?

[12 marks]

Figure 7.2: Compulsory task in the Probability part of the module from the final examination of the previous year.

The model solution produced by the lecturer for departmental use is in Figure 7.3. The commentary on the model solution provides more evidence about students' expected engagement with the tasks as well as lecturer's expectations about their engagement.



Q2 (c) $P(A) \geq 0$
$P(S) = 1$
$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, if $A_i \cap A_j = \phi$, $i \neq j$ — 2, 2, 2 — 6 MARKS

(i) (a) $A_i = \phi$
Use (A3) — 1, 2 — 3 MARKS

(i) (b) $B = (A \cap B) \cup (A^c \cap B)$
" A as $A \subseteq B$
Use (A1) — 2, 1 — 3 MARKS

(ii) (a) $P(A \cap B) = P(B|A)P(A)$ — 2
(ii) (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ — 1
Use (A1) and (A2) $3/20 \leq p \leq \frac{20}{20}$ — 1 — 8 MARKS
not ok if $\frac{23}{20}$
(ii) (c) $P(B^c|A) = 1 - P(B|A)$ — 2
$P(A \cap B^c) = P(B^c|A)P(A)$ — 2

Figure 7.3: Model solution of the compulsory task in Probability (For another version of the model solution see Appendix 12.4.2).

In the solution produced by the lecturer, Kolmogorov's axioms are stated. The definition of the disjoint events is not given separately at the beginning of the task. This, also, signals that the addition of recalling the term disjoint events was to assist students in recalling that they should engage with the object of disjoint events in the third Kolmogorov axiom.

For the substantiation of the propositions (1ia) and (1ib) different *procedures* can be followed. For (1ia) the lecturer in the model solution uses (A3) which refers to the third proposition from the lecture notes (($P(A^c) = 1 - P(A)$). However, in the substantiation of this proposition, in the lecture notes a different *procedure* is used. For the solution of (1ib), the lecturer gives another realisation of the event *B* as $(A \cap B) \cup (A^c \cap B)$ and uses (A1) which refers to the first proposition from the lecture notes ($P(\emptyset) = 0$). The students are able to make their own choices regarding the *procedure* they want to follow while engaging in a routine.

For the substantiation routine in (1iia), L2 uses the multiplication rule. In (1iib), the lecturer uses the equality connecting the probabilities of the union of *A* and *B*, their intersection and the probabilities of *A* and *B*, in order to find *P*(*B*) in relation to *p*. Also, he uses the first, (A1), and the second (A2) propositions from the lecture notes concludes that the range of values for *p* is between $\frac{3}{20}$ and 1. He also notes that the upper limit $\frac{23}{20}$ is not acceptable.

This upper limit would result if the students tried to find the range for *p* without taking into account that it is also a probability. In the last part (1iic), the lecturer uses the relationship between the probability of an event occurring and its complement occurring. Finally, for the last part, the lecturer uses the multiplication rule to find $P(A \cap B^c)$.

7.1.3 Lecturer's perspectives

During the interview, L2 comments on the structure of the task, the creation of independent parts of sub-tasks and the existence of the prompts in order to assist the students. He then comments on specific engagement with defining or proving routines and the word use of independent and disjoint events. In what follows I first provide the interview excerpts and then I discuss these in terms of the commognitive theory.

"Usually students especially in the first year they are not used to give proper definitions".

L2 describes his intentions in relation to task (1) as follows:

> *"Okay so the second question is, which is the first question on my part it is usually designed to test the student on the basic material of the... my part of the course. So, it is usually divided in two parts the first part is more on the theory. In that case, they were let's say the main theoretical background of the course which are the three Kolmogorov's axioms of probability. And then the second part is usually some exercise which, where I ask students to use the basic properties of all the axioms of probability and the following corollaries and properties. So, it's usually a standard question. So, all the years I just make some combinations of the previous questions of the exams, so it is something that students usually practise a lot, and they should be able to do, and it is also designed to be easier. "*

In the above excerpt, the lecturer considers the compulsory task, testing the "basic material" of the Probability part of the module. Then, he comments on the structure of his tasks, where the first part asks students to engage in recall routines of narratives describing objects (disjoint events) or relationships between objects (Kolmogorov's axioms). In the second part, students' engagement with substantiation routines of different propositions involving events and their probabilities. The lecturer describes the task as "a standard question" and then he describes his assessment practice, in terms of designing the task, "some combinations of the previous question, of the exams". In designing this task, the lecturer refers to tasks from previous examination papers and how in designing this year's tasks he takes them into account. I note that the past examination papers are made available to students and they can use them to practise. Before the examination period, the lecturers of this module did a revision lecture. They used past examination papers and went through them with the students that attended that session. It is common practice for the students to use the previous examinations papers to revise for their exams. Later, the lecturer also refers to this student practice. The lecturer expects students to be able to solve this type of tasks. This expectation is based on the previous examinations being accessible to students and the lecturer's expectation that the students will use them to revise for the examination and is a tacit element of the learning-teaching agreement.

> *"Usually students especially in the first year they are not used to give proper definitions, or if they give the definitions, or they write a part of the theorems they don't specify what are the objects they are talking about. Okay? So in other words is like if you are speaking a language but you, you are speaking to somebody which [sic] is able to understand you but what I want the students is to make an effort to try to explain something in a most complete way and that is why I sometimes try to guide them in giving the right definition or writing the right axioms because in the third axiom they need to speak about disjoint event, pairwise disjoint event. I've asked them to give first the definition of disjoint even just to see if they really know what is a disjoint event, and they are able to explain it in the axiom."*

In this excerpt, the lecturer shares his perspective about students' previous engagement with the routine of defining. According to the lecturer they have not been engaging with these routine prior to coming to the university as he says, "*they are not used to give proper definitions*". Then he elaborates, "or if they give the definitions, or they write a part of the theorems, they don't specify what are the objects they are talking about". With this comment, he highlights the accuracy and completeness that should characterise the students' written narratives so that he can endorse them. He also provides a metaphor to illustrate his point “So in other words is like if you are speaking a language but you, you are speaking to somebody which is able to understand you”. He compares the communication happening between him and the students as “speaking a language to someone who already knows what you are talking about”. He can to understand them, but they shouldn’t be writing to him as he would be able to understand them, they have to produce a response that would be complete for somebody who doesn’t know the language. The metaphor of “speaking the language” illustrates the lecturer’s ways of seeing his students’ needs. Knowing that first-year students face this difficulty with the routine of defining, L2 tries to assist students in providing the response he is expecting from them by adding the request regarding the definition of disjoint events.

One of the goals of the lecturer is to make students write complete definitions, or write theorems defining all objects. The lecturer seems aware of the school mathematics discourse which did not necessarily required them to engage with recall the definitions of abstract objects. In aiming to assist this recall, he provides students with a complete definition. Specifically, L2, first, asks them to define what are disjoint events, aiming to examine whether they know what this object is and if then they would be able to use it in Kolmogorov’s third axiom.

> *“this is a typical pitfall. I don’t know why maybe because we think about...well maybe in the common language we have this, we use maybe the idea of disjoint being two separate things and so we also think about two independent things which are also separate somehow, but this is not at all the same in the theory of probability. So, I really stressed that during the course. And the definition of disjoint even comes in the first lecture of the course, the definition of independence comes much later, well much later, I mean it is still on the first part of the course but maybe is on the fourth or fifth lecture I would say, if I remember well, and I always stress that there are two different things and usually I put much more attention to the fact, to defining what is a disjoint event because it's much more important to know what is a disjoint event rather than an independent event. (…)*

but students always make this, yes, this error. I don't know why, I really stress that in the lecture and also in the coursework."

L2 talks, in the above excerpt, about the objects of disjoint and independent events. This has been brought up by me during the interview, noting the additional request for the definition of disjoint events compared to the task from the previous year (Figure 7.2). L2 says that this is "typical pitfall" and elaborates that the use of the word in the colloquial discourse and the mathematical discourse is not the same. L2 bases his perceptions on students' difficulties with the objects of independent and disjoint in the word use in the colloquial and mathematical discourse. In the colloquial discourse, these words could be considered synonyms. However, in the mathematical discourse they are not. There seems to be a commognitive conflict between the word use independence and disjoint between the discourse of probability and the colloquial one. Having noted this difference, L2 tries to draw students' attention to this "I really stressed that during the course". He also says that disjoint events appear first, and the independent events are introduced much later. Having noted this difficulty, L2 purposefully adjusts his teaching practice to draw attention to this difference between these objects during his lectures, he introduces the words with a time difference and he places more attention to the definition of disjoint events.

"[Y]ou ask them to prove things, some of them maybe find an example and prove the example"

L2 also speaks about another difficulty that he has observed in students' engagement with university mathematics, involving their engagement in the routine of proving.

> "So, this is again a common problem in first-year students. When you ask them to prove things ah some of them maybe find an example and prove the example which is not really a proof (…) so, proof is something much more general so like they need to be able to use ah for example letters or-or-or general concepts to-to prove things and this is-is a common problem that we are trying to face with first-year students yes."

In (2ia) and (2ib) students are asked to substantiate specific propositions using Kolmogorov's axioms. The lecturer comments that proving is difficult for the first-year students. Some of them when they are asked to prove they find an example and they show that the endorsed narrative is substantiated for the specific example. However, the lecturer comments that engaging in the proving routine is actually “something more general”, the students will have to engage in using “for example letters or general concepts”. So here he speaks about using the visual mediators to represent possibly general events and other discursive objects. By distinguishing the “general concepts” and the “examples” he is referring to the difference between the discourses at university and the school level. L2 could be referring to the difficulty the students have in grasping the abstract nature of the discursive objects at this stage. L2 ends by saying that “this is-is a common problem that we are trying to face with first-year students” uses the personal pronoun “we” to refer to the community of the lecturers teaching mathematics.

> “Just to guide them and to help them to give them a little help more. Because I’ve, well that's a good point so wh- usually I ask them to state the three Kolmogorov axioms and then prove two properties. And the first one, so a) in this case ah is not a property that, let's say that prove that property is not something that we've seen many times, so we just saw it once when I explained the axioms but then we didn’t revise it. While we use a lot for example in exercises proposition 2 or we use a lot for e(-example) the proposition b) but we don’t use it too much in proposition a) so I wanted, I put this sentence also to help them to think that they could use the proposition to prove the first case because maybe they didn't see this proof (...) a lot, yes, and I guess it was the first time that I was asking to prove this proposition to show the proposition a)”

The lecturer comments on his aim in adding proposition 2. His aim is “to guide them and to help them”. He then says that he added that having considered the familiarity that students have with this proposition “is not something that, we've seen many times”. Here the pronoun “we” seems to be used to depict the students and the lecturer together. Since the students were not very familiar with this “we just saw it once when I explained the axioms but then we didn't revise it”. He compares then the proposition a) with the propositions he asks students to prove in b) and proposition 2. He concludes “we don’t use it too much in proposition a) so I wanted, I put this sentence also to help them to think that they could use the proposition to prove the first case because maybe they didn't see this proof (...) a lot, yes, and I guess it was the first time that I was asking to prove this proposition to show the proposition a)”. The aim of providing the proposition 2 was to help the students in using it to substantiate the other propositions, as they are not as familiar with this specific proposition as they are with the others. He also comments that this is the first time that he is “asking to prove this proposition, to show the proposition A”.

> *“So, as they need the value ah of the probability of A intersecting B to solve the second part of the question, I always try to, well in that case, I always put show that this is equal to ¼ in this case because they need to use this value in the second part. So, I don't want them to penalise if they are not able to get the first solution. So, in that case the solution for point a). I don’t want them to be penalised because they then would not be able to solve also part b). While for part c) they don’t need this value to solve any other question, any other part of that question so I can just ask them find. Of course, it would be better to ask them to find everything but it's just to again to help them in order to do to let’s say separate all subsections of exercise so that they can do it, they can do them separately without need of any other values, I mean, (from) before”*

The lecturer, knowing that the students will need the value of the probability from (1iia) in their attempt to solve (1iib) he purposefully chose to phrase (1iia) as a “<u>show that</u>” task so it can help the students to continue with the rest of the task. He says that “I always try to, well, in that case I always put show that this is equal to ¼ in this case because they need to use this value in the second part” Here the lecturer highlights one of his assessment practices breaking the task in parts; and, if the parts are depending on each other, deciding to phrase them as “<u>show that</u>” to provide the answers needed for later stages of the task. “I don't want them to be penalised because they then would not be able to solve also part B”. He then continues in saying that this is his practice only when the answer is connected to a following subtask which is the case between a) and b) but not the case between b) and c). He also acknowledges that "it would be better to ask them to find everything". However, under the current circumstances, he also recognises that this is how students can achieve better results as the subtasks become independent.

Creating independent sub-tasks and the numerical context of probability

He then talks about sub-task c)

> *"So, okay so, it seems quite a nonsense but these, well of course for part C the two values are interlinked, but they all come in the same sub-question which is C. So, it's somehow the same idea and the same topic. Ah and if I remember well em when I've started to-to run this course and look at the previous examinations ah I think that there was just the first part of the question so find the probability of the complement of B given A. And then I thou-(thought), I also asked them to find the probability of A intersecting the complement of B because actually this is just the guideline to find the first value (...) It's just, it's again a hint without saying it. (…)* it's just again to guide them to force that they, to get them both values to-to solve that question. (...) *So, again it's something that is designed to help them"*

First, the lecturer reinforces what he had said before about making sub-tasks independent only if the answer is connected to the following subtask. In subtask c) the values needed are in the same subtask. However, the sub-task itself assists the students. L2 talks about how this specific sub-task different from the previous examinations' tasks "there was just the first part of the question, so find the probability of the complement of B given A". He added the request for finding the other probability. During the interview, I asked him to elaborate on the part of the task that he expects students might have difficulties with. He says:

> *"So, based on my experience I would have said part a) of the first part. (...) And, then so usually in the second part I-I ask something about independence but in-in this year I didn't. And so again part c). Because I know that also conditional probability it's a little difficult for them. I didn't realise that there was also an issue with the second part*

of part B so defining the interval. This is something that I didn't realise well I was not expecting let's say that and then I-I just looking at scripts I just noticed that"

In the above excerpt, the lecturer comments come after he has seen the students' responses to the tasks as well as his expectations prior to seeing the scripts. According to the lecturer the students face difficulties in providing the proofs of the propositions. Then, in the last subtask (1iic), where he is asking students to engage with the discursive object of the conditional probability, he says *"I know that (also) conditional probability it's a little difficult for them".* So, these are the parts of the tasks that the lecturer expects students to have difficulties with. Here I note that there are either explicit or implicit directions to assist the students in those specific parts of the task.

However, there was a part of the task that had unexpected results. While marking students' scripts, the lecturer realised that the students in finding the range of possible values for the parameter p they were having some difficulties.

> *"(…) finding probability of B but then they didn't realise that p was actually also the probability of A union B. So, also this should have been, so this-this probability should have been less, equal or less than one and so this-this would set the limit to 20 yes 20/20 not 23/20. (…) I usually try not to-to-to ah to make such-such ah such let's say not difficult questions but maybe some-some questions which are not so straightforward for the second exercise. But in any case, it was just at the end, it was just one mark of the total, so it was not-not a big mistake of my part, or mistake I mean..."*

In the above excerpt, the lecturer talks about the students' approach to the tasks. They seem to be focused at engaging in the ritual and thus finding the value of the probability P(B) and the range without reflecting whether their answer can be endorsed. In this case for the numbers, describing the range of p, to be endorsed the students should also take into account that p is a probability itself. He then says that this omission from the students' part would probably be avoided if students had more time. He also talks about the students who achieve better, and he noticed that those also made this mistake so if given more time they might be able to notice it.

7.1.4 In summary

Task 1		
Commognitive analysis		Lecturer's perspective on assessment
Mathematical discourses involved in the task	Discourse of set theory, probability	
Visual mediators	Variables, parameter *p*, which takes values between -1 and 1. Symbols indicating sets and probabilities, unions and intersections.	Students face difficulties with being able to see that the parameter is *p* is the probability of the intersection.
Routines (rituals, recall, substantiation, construction)	Substantiation: (1ia), (1ib), (1iia) Recall: (1i) Rituals: (1iib), (1iic)	Students face difficulties when defining mathematical objects and proving
Instructions given regarding the procedure of the routine	Instructions are given explicitly in the wording of the task (1i)	Aiming to assist the students in their engagement with the substantiation routines

	Students are allowed to choose the procedure of the routine (as either the instruction is implicit or non-existent): (1ii) Hint: to use a proposition (1i)	He added the proposition to assist the students as the students might not be very familiar with proposition (1i)
Instructions given regarding the justification required	-	-
Structure of the task	Split between the theoretical and application part and gradual structure of the task • (1i) recalling the axioms to be used in (1ia) and (1ib) • (1ii)	Aiming to assist the students in their engagement

7.2 Examination task 2 (Optional)

7.2.1 Task and commognitive analysis of the task

(i) Let X be a Poisson random variable with parameter λ having probability mass function $P(X = x) = \frac{\lambda^k e^{-\lambda}}{k!}$

(a) Show that

$$\sum_{k=0}^{\infty} P(X = x) = 1$$

(b) By assuming the validity of the relation in (a), calculate $E(X)$.

[8 marks]

(ii) Students travelling to the city centre arrive at the (name of the university) bus stop according to a Poisson process of intensity 15 per 10 minutes between 5pm and 7 pm, and of intensity 4 per 15 minutes during the rest of the day.

(a) What is the probability that at least 15 students arrive at the bus stop between 5pm and 5.10pm?

(b) What is the probability that at most 10 students arrive at the bus stop between 9am and 9.30am?

(c) Suppose that no students are at the bus stop at 10.30am. What is the probability that the bus stop will remain empty for a further 6 minutes?

(d) What is the most probable event between: the event A describing 15 students arriving between 5.30pm and 5.40pm; and the event B describing 4 students arriving between 10am and 10.15am?

[12 marks]

Figure 7.4: Task 5 from the Probability part of the module

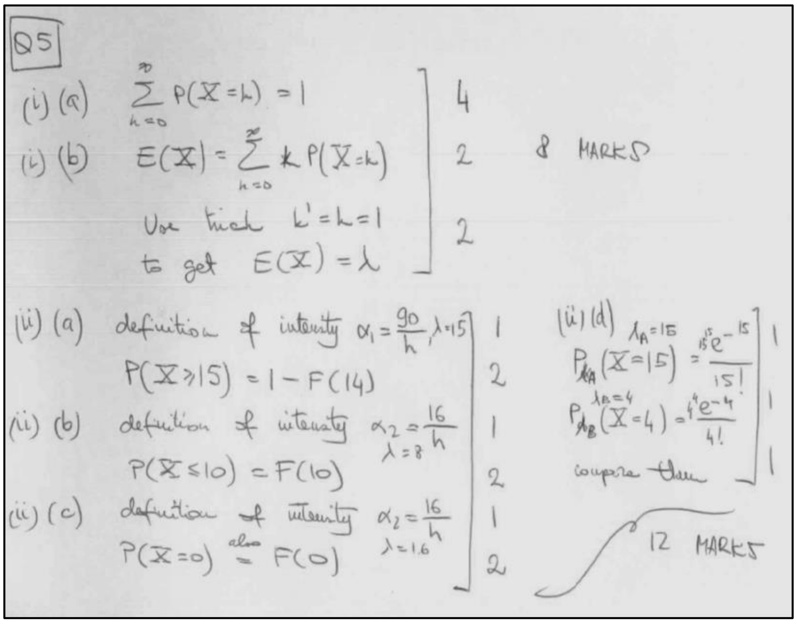
Task 5 (Figure 7.4), thereafter known as task 2, deals with a discrete random variable, namely a Poisson random variable; the expectation and variance of a continuous random variable; and the probability mass function. The task has two parts, as task 1. The first part is focusing on the theoretical part and the second part on the application of the theory.

In (2i), the students are asked to engage with the object of a Poisson random variable. In the wording of the task the probability mass function is given to the students, and the variables involved in it are defined. The students in (2ia) are asked to engage in a substantiation routine by showing that the given relation holds. This is a substantiation routine as the students are asked to prove that this equality holds for the probability mass function of the Poisson random variable. This equality is an application of the second Kolmogorov's axiom for Poisson’s random variable. For (2ib), the students are asked to engage in another substantiation routine as they have to find the expectation of the Poisson’s variable $E(X)$. However, the word “expectation” is not visible in the wording of the task only the symbolic mediation $E(X)$. The *procedure* of (2ib) is signalled using the prompt “By assuming the validity of the relation in (a)”.

The second part of task 2 is an application of the theory regarding discrete random variables. The wording of (2ii) starts by setting the scene of the application. The students are asked to engage in rituals to find three probabilities (2iia), (2iib) and (2iic). The assumptions needed for their engagement with the ritual are given in the wording of the task (2ii). However, to use the assumptions, the students should make the connections between the translation of the word use and the visual mediators of the probability of a Poisson random variable as those are given in (2i). Finally, for (2iid) the students are asked to find two probabilities and then decide which of the two is more probable. The *procedure* of the routine is not given to the students. The structure of the task might lead them to using the equality given in (2i). However, the students can also choose to use the statistical tables provided at the examination.

The analysis of task 2, like task 1, illustrates the structure of the task in a theoretical part (2i); with substantiation routines and an application of the theory; mostly by engaging with rituals (2ii). Also, there is an instruction regarding the *procedure* of the routine for (2ib). Instructions regarding the *procedures* of the substantiation routine for (2ia) and the rituals in (2ii) are not given. Finally, in (2ii), the students are asked to engage in the translation between word use symbolic mediators.

7.2.2 Context and the lecturer's model solution



Q5

(i) (a) $\sum_{k=0}^{\infty} P(X=k) = 1$ — 4

(i) (b) $E(X) = \sum_{k=0}^{\infty} k\,P(X=k)$ — 2 — 8 MARKS

Use trick $k' = k = 1$ to get $E(X) = \lambda$ — 2

(ii) (a) definition of intensity $\alpha_1 = \frac{90}{h}$, $\lambda = 15$ — 1

$P(X \geq 15) = 1 - F(14)$ — 2

(ii) (b) definition of intensity $\alpha_2 = \frac{16}{h}$, $\lambda = 8$ — 1

$P(X \leq 10) = F(10)$ — 2

(ii) (c) definition of intensity $\alpha_2 = \frac{16}{h}$, $\lambda = 1.6$ — 1

$P(X=0) \overset{\text{also}}{=} F(0)$ — 2

(ii) (d) $\lambda_A = 15$ $P_{\lambda_A}(X=15) = \frac{15^{15} e^{-15}}{15!}$ — 1

$\lambda_B = 4$ $P_{\lambda_B}(X=4) = \frac{4^4 e^{-4}}{4!}$ — 1

compare them — 1

12 MARKS

Figure 7.5: Model solution of task 5 (in the text I refer to it as task 2) from the Probability part of the module (For another version of the model solution see Appendix 12.4.5)

In the lecturer's solution to task 2, the *procedure* for the substantiation procedure is not given. In the substantiation of the expectation of the Poisson random variable, the lecturer shows that the equality proved in the previous part is used and also there is the need for a definition of a new variable $k' = k - 1$ (there is a typo in the model solution – the lecturer meant to write $k' = k - 1$ instead of $k' = k = 1$). By using the narratives mentioned, the lecturer concludes that the expectation is equal to *λ*.

The solution of the second part shows that the students have first to define the intensity and the parameter, and then find the probability using the values of Poisson's random variable from the statistical tables. It also signals that the students should understand how the probabilities of the *X* being higher, less than or equal to a numerical value connects with the probabilities given in the statistical tables. Another way that could be used to engage in this ritual would be the calculation of the probability using the formula given in the wording of (2i). This *procedure* is followed by the lecturer in the calculation of the probabilities for (2iid).

7.2.3 Lecturer's perspectives

During the interview, L2 comments on the existence of the prompts that will be used in the *procedures* of the tasks (the definitions but also instructions regarding the *procedures*) and the introduction of (2ii) as a new task. In the first part of this section, I provide excerpts from the interview regarding the prompts used in the task (2i) and in the latter part of this section, I present excerpts and their analysis regarding the design of the new subtask (2ii).

Providing narratives (formulae and definitions) to assist the students

L2, in the following excerpts, comments on providing the narratives regarding the definition of the random variable and the probability mass function.

> *"(...) as I don't want them to really memorize the definition of the random variable and the definition of the probability mass function, which is associated to that, I usually write it as an introduction to the exercise (…). And I also write it because they also have it in the statistical tables. So, it's some information that they have and I think that is just stupid to ask them to memorize it because they just make a silly mistake and then they got the exercise completely wrong, which is not the scope of the exercise I just want them to practice on that and be able to show me that they've understood what is this object for instance"*

As illustrated above, L2 aims to focus more on students being able to engage in the *procedure* by providing them key narratives needed in their responses. According to L2, recalling specific narratives (such as the definition of the probability mass function) is not part of the scope of the module. The focus is more on the *procedure* of the substantiation routines and the rituals. The definition of a random variable and the probability mass function are usually also provided at statistical tables, and the lecturer does not consider the recall routine of these objects as valuable. These specific narratives are usually provided and if they are not when the students try to recall them they "*make a silly mistake"*. The lecturer focuses on the students' engagement with the objects in the substantiation and rituals rather than the recall routines. Also, L2, comments on the existence of the prompt in (2ib) "<u>By assuming the validity …</u>".

> *"I was trying to help them ah by saying "By assuming the validity of relation in a)" because it's something that you can use to speed up your calculation for the expectation (...) so at some point, you should obtain something that looks like the first part. So even if you are not able to prove the first part maybe you can try to prove the second and use the value of the first part."*

From the excerpt above, it is visible that L2 aims to help the students by providing the prompts that assist in the substantiation of the *procedure* of (2ib).

Designing new tasks – changing the practice

Also, L2 the lecturer talks about designing a different type of task for this year (2ii) and how this will enable him to change the examination practices.

> *"For the first time to design an exercise which was telling that actually what I've been studying could be also applied to something which is a real example I was really putting the XXX* [L2 uses the university's name for confidentiality reasons this is omitted] *bus stop and trying to give real numbers, so that was designed to give real number and a real feeling of that"*

In the excerpt above, L2 comments on the importance of illustrating the application of Probability to the real world. L2 designed a task modelling the probabilities of different events occurring. These events were situated in a setting of a university bus stop, giving the probabilities of a specific number of students being at the bus stop at a different time. The lecturer's purpose of adding this task, is to illustrate the applications of Probability in real life. Also, by making the name of the university part of the task, the lecturer aimed to create a connection with the students and show them the usefulness of the abstract objects they were dealing with.

> *"when I designed it, I thought it was ah yes more difficult- more difficult than-than the usual one on continuous random variable. But I really wanted to put that ah that question ah because I well also for future*

> *students and future examinations I want them also to practice on that type of question. So, this-this was relatively new."*

After designing it, the lecturer realised that students might find it more difficult to engage with this than "*the usual one*" here the lecturer is referring again to the tasks from the previous examinations. As mentioned in the methodology, the students have access to the previous exam papers and when they are preparing for the examinations they use the previous exam papers to practice. By practising with those, the expectation is that they would see in their final examinations' tasks similar to ones from the previous years. In the excerpt, the lecturer realises that (2ii) is "more difficult" but decides to use it to introduce a new task to the pool of previous tasks. This way he enables himself to put a similar one the year after and slowly change the application part of the tasks to more contextualised and personalised for the students.

7.2.4 In summary

Task 1		
Commognitive analysis		Lecturer’s perspective on assessment
Mathematical discourses involved in the task	Discourse of discrete variables and probability	-
Visual mediators	Symbols indicating probabilities	-
Routines (rituals, recall, substantiation, construction)	Substantiation: (2ia), (2ib) Ritual: (2ii)	New part of the task using “real context” aiming to show where the probabilities can be applied
Instructions given regarding the procedure of the routine	Instruction are given explicitly in the wording of the task (2ib) Students are allowed to choose the procedure of the routine (as either the instruction is implicit or non-existent): (2ia), (2ii)	Assisting the students in the production of their narratives

Instructions given regarding the justification required	-	-
Structure of the task	Gradual structure and split in theoretical and application part	-

7.3 Examination task 3 (Optional)

7.3.1 Task and commognitive analysis of the task

(i) Define expectation $E(X)$ and variance $V(X)$ of a continuous random variable X.

(ii) A random variable X is said to have a normal $N(\mu, \sigma^2)$ distribution with mean μ and variance σ^2 if its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(a) Show that the probability density function $f(x)$ satisfies the first Kolmogorov axiom of modern probability.

(b) By rigorously evaluating the expectation $E(X)$, prove that it is equal to the mean μ.

You may use the result $\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}$.

[10 marks]

(iii) The *standard normal* random variable Z is a particular case of *normal* random variable having mean $\mu = 0$ and variance $\sigma^2 = 1$. Its cumulative density function is defined by

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$$

and its values are computed numerically and tabulated in the statistical tables.

(a) Give the definition of cumulative distribution function $F(X)$ of the *normal* random variable X having $f(x)$ as a probability density function. Then, explain why the following relation holds

$$F(X) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \text{ that is } z \equiv \frac{x-\mu}{\sigma}.$$

(b) Consider the *normal* random variable T which has mean $\mu = 50$ and variance $\sigma^2 = 64$. Find $P(T \leq 26)$ and $P(\mathrm{T} < 130 | \mathrm{T} > 90)$.

[10 marks]

Figure 7.6: Task 6 from the Probability part of the module.

Task 6 (Figure 7.6), thereafter known as task 3, deals with a continuous random variable; the expectation and variance of a continuous random variable; the normal distribution; the probability density function; the cumulative density function; and Kolmogorov's first axiom. The task has three parts. The first part is focusing on the theoretical part of continuous random variables. The second part on the theoretical parts of a normal random variable. The third part on "a particular case of a *normal* random variable".

In (3i), the students are asked to engage in a recall routine as they have to define the expectation and the variance of a continuous random variable.

The second part of task 3 is split into two subtasks. The focus of this task is on a normal random variable. The probability density function of the random variable is given in the wording of the task, and the visual mediators involved in that function are defined. In (3iia), the students are asked to engage in a substantiation routine as they have to show that the probability density function satisfies the first Kolmogorov axiom. This part of the task depends on students recalling the Kolmogorov's axioms. They were asked to recall them also in (1i), so this part of the task depends on (1i). In the next part (3iib) the students are asked to engage in a substantiation routine as they have to prove that the expectation is equal to the mean. Here, there is a prompt regarding the *procedure* of the substantiation "You may use the result ...) which indicates to the students that they should find the given integral in their calculations.

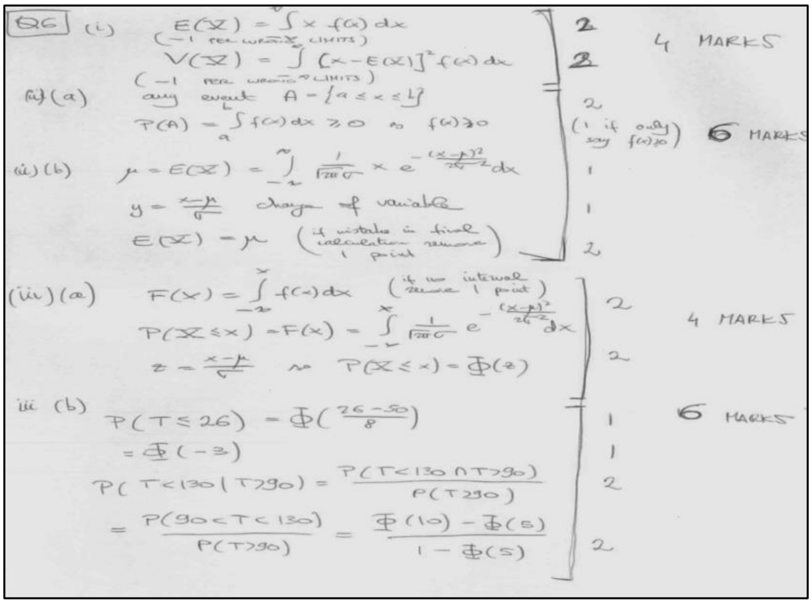
In (3iii), the focus is on the standard normal random variable. The wording of the task gives information regarding the values for the mean and the variance of the standard normal random variable, and the cumulative density function is also given. There is also a hint as to the *procedure* that the students should follow when finding the values of this cumulative density function "its values are computed numerically and tabulated in the statistical tables". There are two subtasks in (3iii) the first one is a theoretical one and the second one an application. Specifically, the students in (3iiia) engage in a recall routine as they are asked to give the definition of the cumulative distribution function of the normal variable. Then, they are asked to explain why the relation that relates the cumulative distribution function of the normal variable and the cumulative density function of $(x\text{-}\mu)/\sigma$ holds. In this part of the task there are prompts regarding the *procedure* that the students should follow for both parts of (3iii). In the recall routine, the students are prompted to use the probability density function in their narrative "having $f(x)$ as a probability density function". And in the second part, the visual mediators in the equality between the cumulative distribution function $F(x)$ and the cumulative density function signal that the realisation of the cumulative density function given in the wording of (3iii) are connected as *Φ* appears in both.

For (3iiib), the students are asked to engage in a ritual by calculating two probabilities: the probability of the value of the normal random variable *T* is less than or equal to 26 and a conditional probability, the probability of the value *T* is less than 130 given that the value of *T* is more than 90. Instructions regarding the *procedures* of the rituals are not given explicitly. However, there is the general hint regarding the values of the cumulative distribution function *Φ* in the wording of (3iii) and the visual mediators used in (3iiia) can assist in finding the connection between the values of the probabilities of the random variable *T* and the values of the cumulative density function *Φ* for the standard normal variable *Z*.

The analysis of task 3, highlights that in this task the students are mostly asked to engage in substantiation and recall routines (3i), (3ii) and (3iiia) rather than rituals (3iiib). There are instructions regarding the *procedures* of all the routines. However, some of them are explicit (3iib) and (3iiia) and others are implicit (3iiib). Finally, in this task, the students would have to engage in the discourses of both Probability and Calculus as the probability density function of continuous random variables involves integration.

7.3.2 Context and the lecturer's model solution

In the model solution produced by the lecturer (Figure 7.7), the expectation and the variance are defined. There is also a note that says that a mark would be taken from the student's response if the limits of the integration are wrong both in the case of variance and expectation. Then in (3iia) the probability of any event *A*, with *x* taking values between *a* and *b*, is defined and the first Kolmogorov's axiom is substantiated as the probability density function is a positive function. In (3iib) the substantiation of the equality between the expectation and the mean is given. The key feature in the *procedure* of this substantiation routine is the definition of a different variable *y* such that $y = \frac{x-\mu}{\sigma}$ for the integral. There is another note here talking about the marking of the students' scripts saying that if there is a mistake in the calculation of the integral, this would result in one mark being deducted.



Q6 (i) $E(X) = \int x f(x)\,dx$ (−1 per wrong limits) 2

$V(X) = \int [x - E(X)]^2 f(x)\,dx$ (−1 per wrong limits) 2 — 4 MARKS

(ii)(a) any event $A = \{a \le x \le b\}$ 2

$P(A) = \int_a^b f(x)\,dx \ge 0$ as $f(x) \ge 0$ (1 if only say $f(x) \ge 0$) — 6 MARKS

(ii)(b) $\mu = E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \times e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$ 1

$y = \frac{x-\mu}{\sigma}$ change of variable 1

$E(X) = \mu$ (if mistake in final calculation remove 1 point) 2

(iii)(a) $F(x) = \int_{-\infty}^{x} f(x)\,dx$ (if no interval remove 1 point) 2

$P(X \le x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$ — 4 MARKS

$z = \frac{x-\mu}{\sigma}$ so $P(X \le x) = \Phi(z)$ 2

iii (b) $P(T \le 26) = \Phi\left(\frac{26-50}{8}\right)$ 1 — 6 MARKS

$= \Phi(-3)$ 1

$P(T < 130 \mid T > 90) = \frac{P(T<130 \cap T>90)}{P(T>90)}$ 2

$= \frac{P(90<T<130)}{P(T>90)} = \frac{\Phi(10) - \Phi(5)}{1 - \Phi(5)}$ 2

Figure 7.7: Model solution of task 6 (in the text I refer to it as task 3) from the Probability part of the module (For another version of the model solution see Appendix 12.4.6)

For (3iiia), the lecturer defines the cumulative distribution function $F(x)$, and there is a note regarding the deduction of one mark if the limits of integration are absent from the student's response. In the next part, the lecturer illustrates the relation between $F(x)$ and $\Phi(z)$ by taking the cumulative density function and making the substitution $z = \frac{x-\mu}{\sigma}$ in the integral. In the solution for (3iiib) L2, writes the relationship between the probability of the normal random variable T and $\Phi(z)$ for the specific values given in the wording of the task, using the relation shown in (3iiia). For the conditional probability, the definition of conditional probability is used first and then the relationship between the probability of T and $\Phi(z)$.

From the analysis of the model solution, it is evidenced that engagement with two discourses; namely, the discourse of the probability for continuous variables and the discourse of integral calculus is needed. Specifically, the students will need to engage with the properties of definite integrals of positive and odd functions.

7.3.3 Lecturer's perspectives

In commenting on task 3, L2 talks about the expectations on students' engagement with the theoretical part of task 3; the discourse of Calculus and the usual difficulties for the students and the application of the definition of conditional in the context of a continuous random variable.

Students' difficulties with Probability and Integral Calculus discourses

> *"(…) this is pretty standard let's say a part of-of question 6 which is with continuous random variable, and this was a way again to give them ah points in this question. So, to help them and then there were-there was the part on the more theory on the normal distribution ah which was ah difficult especially ah so part a is-is something that we do for any ah probability density function so even if it is Gaussian I mean it is something that they are used to do for any pdf and there were-there were indeed succeeding in it. But question b was, so part b was really difficult I guess."*

The lecturer talks about students' familiarity with the first part of the task (3i). He uses the phrases "pretty standard", "a way again to give them points in this question". Then, he continues talking about the theoretical part of (3iia) saying that this is "something we do for any probability density function" and "something that they are used to do for any pdf". Here the lecturer refers again to students' familiarity with this part of Probability and how his expectations were met as "*there were indeed succeeding in it*". However, he then turns to (3iib) and discusses how this is different from the other parts of the task.

> *"So, if I use z instead of x, at the beginning they're-they're suspicious let's say okay? Which is again strange because they should know that, well actually this is well any dummy variable well you can name the dummy variable inside an integral as you want but again they-they-they maybe they don't appreciate that in the first year of-of university so because maybe in high school or in A-level they-they're not taught in that way, maybe"*

The lecturer compares earlier the parts of the task and says that this is more difficult. According to him, the difficulty lies in the use of Calculus. Using commognitive terms, this is due to the interwoven nature of the Probability discourse with the Calculus discourse. The discourse surrounding the Probability of a continuous variable is based on Calculus. Specifically, the background knowledge, needed for the solution of this task, is integrals. Essentially, that the definite integral of a positive defined function is equal or bigger than zero and then for b) and that the definite integral $(-\infty,+\infty)$ of an odd function is zero. The lecturer assumes that the students should be familiar with the procedures of the dummy variable in the integral and be able to use this procedure in the Probability discourse flexibly. However, as he continues, he starts questioning the way that this is taught in the school discourse. The lecturer comments on the other background knowledge needed for this task during the interview:

> *"(...) our students I don't know why they are not really able to..., when I explained that in the lecture they..., so none of them knew what was an odd function and an even function and how they could use that property to compute faster integrals (...) maybe because they have to integrate this exponential function they, they just think this is something difficult"*

Further on in the interview, the lecturer talks about how the students are not able to use some of the procedures of the integral calculus. Specifically, he speaks about the integral of an odd or an even function. He finds from his experience that students face difficulties when asked to use this and similarly they are finding integration of an exponential function difficult. The lecturer observes that students have difficulty with these, even though he presented these *procedures* during the lectures.

Engaging with the conditional probability definition in a continuous random variable task

In (3iiib) the students are asked to engage in a ritual calculating two probabilities, the second one is a conditional probability. During the interview, L2 says:

> *"for the second part is usually..., I know that this is standard problem when you ask to compute the probability of a conditional event ah sometimes they are not able to do it. I mean they are not able - which is strange because some of the students are able to, for example, to define conditional probability in the first exercise, so in exercise two - to write the definition, to compute ah to compute the definition of conditional probability to this new topic which is continuous random variable. So, they see this object and they are somehow lost okay? While if they just apply the definition of conditional probability then they are able to actually get that. But they are not sometimes able to make [the] link."*

L2 comments on students' difficulty regarding the flexibility of their discourse, when discussing the last part of the task. In this part of the task, the students are asked to calculate a conditional probability, similar to the one asked in task 1. However, the lecturer discusses how the application of what they did in the first task is not easily transferable as they are unable to use the definition of the conditional probability when talking about the continuous random variable. They see this conditional probability as something new and "they are not sometimes able to make [the] link".

7.3.4 In summary

Task 1		
Commognitive analysis		Lecturer's perspective on assessment
Mathematical discourses involved in the task	Discourse of continuous variables, integrals, probabilities	Students face difficulties with the application of properties with integrals
Visual mediators	Symbols indicating probabilities	-
Routines (rituals, recall, substantiation, construction)	Substantiation: (3iia), (3iib) Recall: (3i), (3iiia) Ritual: (3iiib)	-
Instructions given regarding the procedure of the routine	Instruction are given explicitly in the wording of the task (3iib) Students can choose the procedure of the routine (as either the instruction is implicit or non-existent): (3iia), (3iiib)	-

Instructions given regarding the justification required	(1iic)	-

7.4 Summary and conclusion on Probability tasks

The analysis from the lecturer data and the examination tasks showed that students' responses are guided in terms of procedure of routines. They are also given statements they should use (e.g., the hints are given in task 1 and 3) and explicit directions regarding the required justification are also given in the wording of the task.

Comparing these tasks with the ones from the Sets, Numbers and Proofs part of the module, I observe that the lecturer is introducing a new type of task (2ii). This task serves, according to the lecturer, the purpose of illustrating to the students that the mathematical objects they have been dealing with during the module can be applied in real contexts. Other lecturers have also tried to incorporate a new task, as one of the optional ones, in this year examinations aiming to use it as a template for the revision with the students in the next year. Specifically, this is done by the lecturer of the Combinatorics module.

During the interview, the lecturer refers to his attempts at assisting the students. Apart, from the hints and guidance regarding the procedure to be used and the theory which would be useful in progressing with the tasks. The lecturer talks about creating subtasks that are independent of each other. This way, the students who are not able to reach the correct solution to a probability (1iia) are able to continue with the next parts of the task (1iib) and (1iic). This is done specifically for numerical values which are crucial for later on in the task. However, this is not the case for the tasks involving the realisation of conditional probability either in (1iia); (1iic) and (3iiib). In these occasions, the students are asked to recall the definition of conditional probability and other propositions to be used.

Similarly, the students in (3iia) are asked to prove that the first Kolmogorov's axiom holds for the specific probability density function. Some students might not be able to recall which is the first Kolmogorov axiom and when trying to engage in this part of the task, they might find themselves proving something else. In the case of (1iia) the dependence between (1iia) and the rest of the subtasks is removed, but in other cases, this is not the case. I consider these as evidence of the interwoven nature of the routines which is very characteristic of university mathematics. According to Sfard's categorisation of exploration routines, there are three types: recall, substantiation and construction routines. These are linked with each other. In order to be able to engage in exploration routines the discursant has to have a certain level of discursive fluency. The substantiation routines are based on recalling narratives and constructing narratives. Similarly, in order to construct a narrative, the discursant might engage in substantiation and recall routines. Finally, a similar dependency to recalling narratives appears in the first part of task 1. In this task, the students are asked initially to provide the Kolmogorov's axioms and then to continue by engaging in substantiation routines proving two propositions.

During the interviews, L2 also mentioned differences between the school mathematical discourse and the university mathematics discourse in terms of engagement with routines of proving and defining but also in terms of the complicated nature due to the constant shifts between discourses. Several tasks are structured in the form: recall theory– application (substantiation of theory) in both topics of the module. This structure helps the students recall the theory they have to use in the following parts of the task. Also, it introduces them to another routine, the routine of defining and stating the theory they use. Furthermore, the structure of the questions in the application part is also assisting the students to answer the questions.

In the interview, the lecturer shifts his discourse from speaking about him and his students to speaking about "we" where "we" here could be interpreted as the people in general. In other occasions, the "we" means the lecturers or him and his students.

Additionally, the students' responses are guided in terms of method, statements they should use and in the extent of justification. In many tasks, the students are given prompts to use specific statements in order to solve them. This practice together with the directions regarding the justification required is also aimed to help the students to shift their mathematical discourse to the university mathematical discourse which is extremely concise and precise.

Chapter 8. Probability: Students' scripts

In chapter 7, I discuss the Probability tasks and the lecturer's perspectives on these. Here, I present the analysis of the students' responses to these tasks. I initially report the marks of the 22 selected students' scripts in relation to the whole cohort. I, then, discuss the themes that emerged from the analysis of the scripts in relation to the characteristics of the discourse (as described in section 3.2). Finally, I provide a summary of the analysis of the students' scripts, and I connect with the task analysis and the lecturer' perspectives presented in chapter 7.

8.1 Overview of student marks in the three tasks

In the final examination on Sets, Numbers and Probability, three tasks correspond to the Probability part of the module. The compulsory task (Figure 8.1) is worth 20 marks. In analysing the students' scripts, I consider the analysis of the task and the lecturer's perspectives from chapter 7 (Section 7.1). Specifically, I examined the students' scripts for the following: their engagement in recall routines regarding the definition of disjoint events and Kolmogorov's axioms; the procedure of the substantiation routines in (1ia) and (1ib); the ritual for (1iic); and, the notation specific to Set Theory and Probability. Finally, I also examine whether the applicability conditions and the closing conditions of routines (substantiation (1ia), (1ib), (1iia) or rituals (1iib), (1iic)) are considered in the students' responses.

(i)	In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions: (a) For any event $A = \emptyset$, prove that $P(A) = 0$. You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events. (b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$. **[12 marks]**	
(ii)	Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B\|A) = \frac{5}{8}$ and $P(A \cup B) = p$. (a) Show that $P(A \cap B) = \frac{1}{4}$. (b) Find $P(B)$ and the range of possible values for the parameter p. (c) Find $P(B^c\|A)$ and $P(A \cap B^c)$. **[8 marks]**	

Figure 8.1: Compulsory task from the Probability part of the module – Task 1.

Fifty-four students took part in the final examination. In Figure 8.2, the marks of the students' scripts are given. The marks of the selected 22 scripts are illustrated in grey. Students' marks to task 1 ranged from 5 to 20, with the mean being around 14.17 marks.

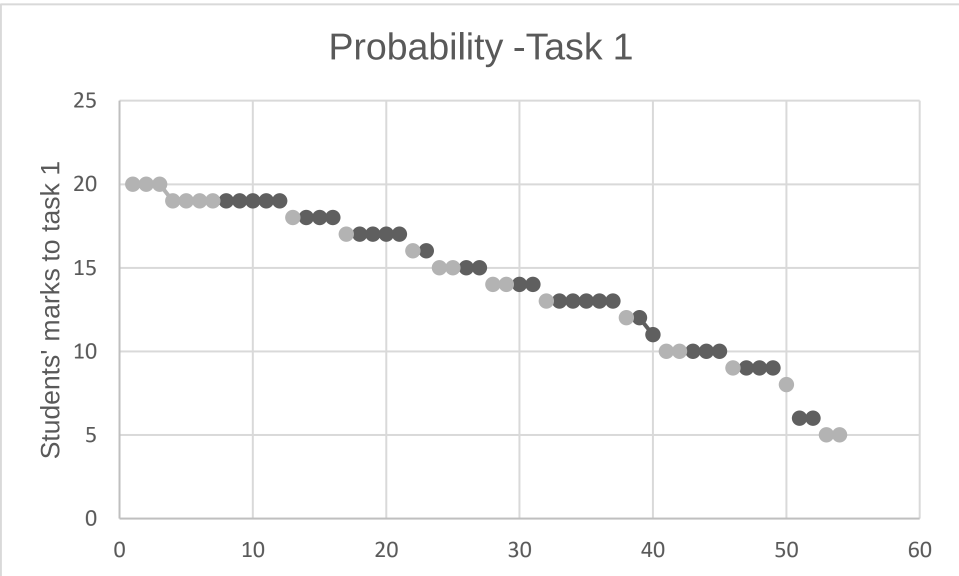


Figure 8.2: Marks to Probability compulsory task – Task 1

The first optional task focuses on a discrete random variable (Figure 8.3). In analysing students' responses to task 2, I consider the analysis in section 7.2. I focus on the *procedures* regarding the substantiation of the summation of the probability mass function (2ia), the calculation of the expectation (2ib) and the probabilities in (2ii). I also examine whether the closing conditions of the routine regarding the summation of the probability mass function are satisfied according to the marker.

(i) Let X be a Poisson random variable with parameter λ having probability mass function $P(X = x) = \frac{\lambda^k e^{-\lambda}}{k!}$

(a) Show that

$$\sum_{k=0}^{\infty} P(X = x) = 1$$

(b) By assuming the validity of the relation in (a), calculate $E(X)$.

[8 marks]

(ii) Students travelling to the city centre arrive at the (name of the university) bus stop according to a Poisson process of intensity 15 per 10 minutes between 5pm and 7 pm, and of intensity 4 per 15 minutes during the rest of the day.

(a) What is the probability that at least 15 students arrive at the bus stop between 5pm and 5.10pm?

(b) What is the probability that at most 10 students arrive at the bus stop between 9am and 9.30am?

(c) Suppose that no students are at the bus stop at 10.30am. What is the probability that the bus stop will remain empty for a further 6 minutes?

(d) What is the most probable event between: the event A describing 15 students arriving between 5.30pm and 5.40pm; and the event B describing 4 students arriving between 10am and 10.15am?

[12 marks]

Figure 8.3: The first optional task from the Probability part of the module – Task 2

The marks of the students from the whole cohort are shown in the chart below (Figure 8.4), with the selected ones shown in grey. The higher mark achieved was 20 and the lower 0 with the mean being 10.07 marks. I note here that from the twenty-two selected students' scripts, sixteen attempted 2(i) with only seven achieving a mark higher than zero and thirteen students attempted 2(ii) with seven of these attaining higher than zero marks.

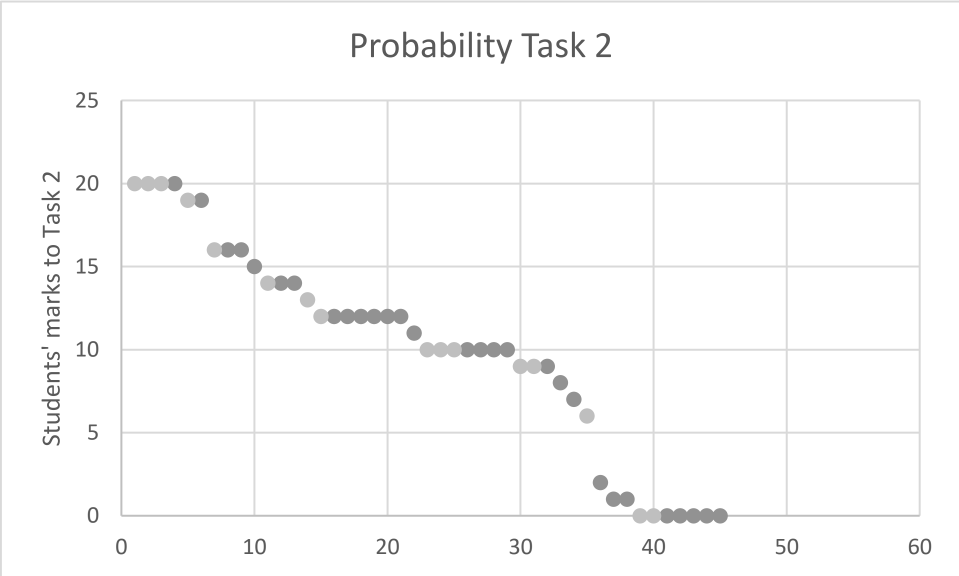


Figure 8.4: Marks to the fist optional task from Probability– Task 2

The second optional task deals with continuous random variables (Figure 8.5). The analysis I present in section 7.3 guided the analysis of the students' scripts. However, as the majority of the selected scripts answer only the first part of the task, I focus on students' engagement mostly in recall routines.

(i)	Define expectation $E(X)$ and variance $V(X)$ of a continuous random variable X.	
(ii)	A random variable X is said to have a normal $N(\mu, \sigma^2)$ distribution with mean μ and variance σ^2 if its probability density function is $$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ (a) Show that the probability density function $f(x)$ satisfies the first <u>Kolmogorov</u> axiom of modern probability. (b) By rigorously evaluating the expectation $E(X)$, prove that it is equal to the mean μ. You may use the result $\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}$.	**[10 marks]**
(iii)	The *standard normal* random variable Z is a particular case of *normal* random variable having mean $\mu = 0$ and variance $\sigma^2 = 1$. Its cumulative density function is defined by $$\Phi(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$$ and its values are computed numerically and tabulated in the statistical tables. (a) Give the definition of cumulative distribution function $F(X)$ of the *normal* random variable X having $f(x)$ as a probability density function. Then, explain why the following relation holds $$F(X) = P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \text{ that is } z \equiv \frac{x-\mu}{\sigma}.$$ (b) Consider the *normal* random variable T which has mean $\mu = 50$ and variance $\sigma^2 = 64$. Find $P(T \le 26)$ and $P(\mathrm{T} < 130 \mid \mathrm{T} > 90)$.	**[10 marks]**

Figure 8.5: The second optional task from the Probability part of the module – Task 3

Regarding task (3i) only seven students, from the 22 selected, attempted it and only four of them achieved marks higher than zero. The highest mark achieved was 14 and the lowest 0 with the mean being 6 marks (Figure 8.6).

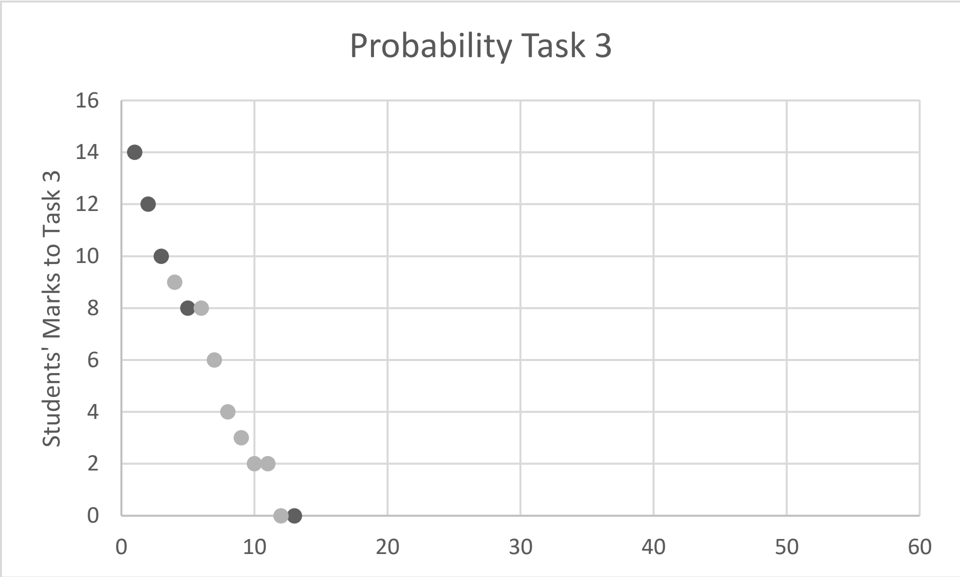


Figure 8.6: Marks to the second optional Probability task – Task 3.

8.2 Students' scripts: Commognitive analysis (word use, visual mediators, narratives)

The analysis of students' scripts focuses on the characteristics of the discourse as described in chapter 3 (section 3.2), namely word use specific to Probability, visual mediators (Venn diagrams and symbols specific to Probability and Set Theory), routines and narratives. In analysing students' scripts, I examine the occasions where the written word use, and the presence of visual mediators signals unresolved commognitive conflicts in students' engagement with the university mathematics discourse.

In 8.2.1, I focus on students' word use and their use of visual mediators. Specifically, I start by examining cases where there is conflation between the notation used in the discourses of Set Theory and Probability. I, then, turn to the symbols used in Kolmogorov's axioms and the use of Venn diagrams. Finally, I discuss commognitive conflict regarding the terms "disjoint" and "independent events"; the use of words from other mathematical discourses than probability and Set Theory; and, the use of personal pronouns in students' scripts.

In analysing the students' scripts, I focus on identifying instances where unresolved commognitive conflict occurs. In the following sections, I first present the part of the task corresponding to the students' scripts that I sample, and then I discuss the students' scripts.

8.2.1 Conflations in visual mediators

In this first section, I present instances where the students' scripts evidence conflation of visual mediators coming from different discourses. This evidence is seen in their use of visual mediators either between the discourse of Set Theory and Probability or within the discourse of Probability regarding the discourse of discrete and continuous variables. Similar analysis is presented in Thoma and Nardi (2017, 2018a)

(i) Conflating visual mediators in Set Theory and Probability

In task 1 (Figure 8.7) the students are asked to engage with two objects: events and their probabilities. In commognitive terms, this implies an engagement with the discourse of Set Theory and Probability. The analysis of students' responses to different parts of this task revealed errors, due to a conflation between the two discourses.

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 8.7: Compulsory task from the Probability part of the module – Task 1

This conflation between notation that is specific to the discourses of Probability and Set Theory is present in the responses of five students ([01]. [02], [11], [15], [20]). In the students' responses, there are equalities signalling the relationship between two objects, one being a set and the other one being a probability, a number. However, in these five cases, on the one side of the equality there is a set, and on the other side, there is a number or the symbols used are for relationships between sets and are used to signal relationships between numbers.

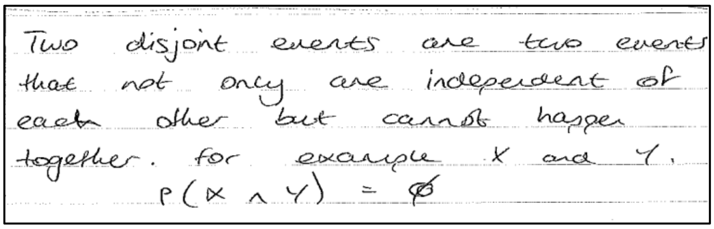
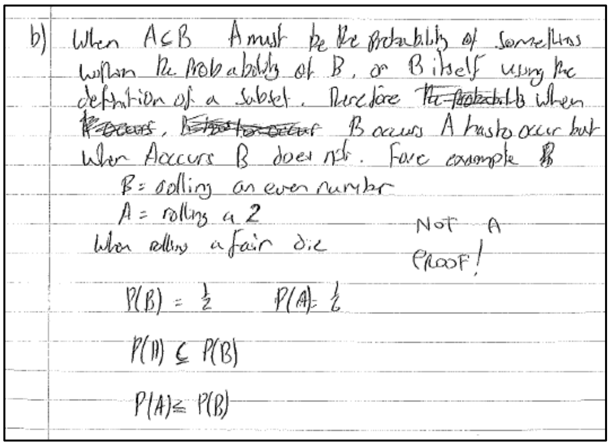

Two disjoint events are two events that not only are independent of each other but cannot happen together. for example X and Y.
$P(X \wedge Y) = \emptyset$

Figure 8.8: Student [15]'s response to (1i) – the definition of disjoint events

In the definition of disjoint events, [15] ends by giving the following equality: $P(X \cap Y) = \emptyset$. This equality connects a number, which is the probability of the intersection of the events *X* and *Y*, with the symbol for an empty set, not a number (the student may have meant to write "0"). I discuss the student's use of the word "independent" in section 8.2.3.



b) When $A \subseteq B$ A must be the probability of something within the probability of B, or B itself using the definition of a subset. Therefore ~~the probability~~ when ~~A occurs~~, ~~B has to occur~~ B occurs A has to occur but when A occurs B does not. Fore example ~~B~~

B = rolling an even number

A = rolling a 2

When rolling a fair die

Not A PROOF!

$P(B) = \frac{1}{2}$ $P(A) = \frac{1}{6}$

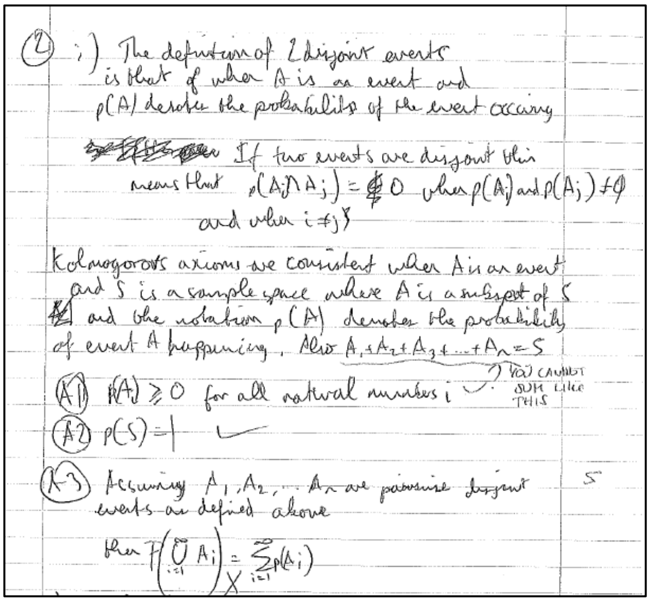
$P(A) \subseteq P(B)$

$P(A) \leq P(B)$

Figure 8.9: Student [01]'s response to (1ib)

[01] (Figure 8.9) is trying to substantiate that the probability of event *A* is less than the probability of event *B*, taking into account that *A* is a subset of *B*. In this response, the student provides an example as proof of this. I discuss this *procedure* of the substantiation routine in section 8.3.2. The focus here is on the last two lines in the student's response. The first line shows the two probabilities with the visual mediator signalling that the probability of *A* is a subset of the probability of *B*. Then, the next line arrives at the desired response, the inequality between probabilities. The discursive object of probability is a number, and thus the relationship between two probabilities can be described using the signifier of inequality. Events *A* and *B* are sets and the relationship between them can be described using the signifier of the subset. The student uses the signifier of subset at the start of the narrative to illustrate the relationship between the two events. Then [01] uses it again to illustrate the relationship between probabilities. This use is signalling unclear meaning-making between the two objects: events and probabilities and also shows the difficulty when engaging with two different mathematical discourses in the same task.

Another example of this can be seen in Figures 8.10 and 8.11 which show the response of [02] to task 1. Here, I note the conflation of the discursive objects of probabilities and events in four instances. First, in the definition of disjoint sets, the probability of the intersection of the events is taken into account, and the student comments that the two events must have probabilities different from the empty set. However, it should have been that the probabilities are different from zero (e.g., $P(A_i) \neq 0$, $P\left(A_j\right) \neq 0$) and not the empty set (e.g., $P(A_i) \neq \emptyset$, $P\left(A_j\right) \neq \emptyset$). In the lines that follow, the student connects the sample space with the events *A_1, A_2*. The events are added (using the symbol of addition) and then presented as equal to the sample space S. However, this is not consistent with the symbolic use in Set Theory. Then, for the third of Kolmogorov's axioms, the student writes that the union of the pairwise disjoint events A_i, where *i* takes values from 0 until infinity is equal to the probability of $P(A_i)$. The lecturer has completed the student's script and added the probability in the left-hand side of the equality and the sum from *i* =1 to infinity.


2 i) The definition of 2 disjoint events is that of when A is an event and $p(A)$ denotes the probability of the event occuring

If two events are disjoint this means that $p(A_i \cap A_j) = \emptyset\ 0$ when $p(A_i)$ and $p(A_j) \neq \emptyset$ and when $i \neq j$ }

Kolmogorovs axioms are consistent when A is an event and S is a sample space where A is a subset of S and the notation $p(A)$ denotes the probability of event A happening. Also $A_1 + A_2 + A_3 + \dots + A_n = S$

YOU CANNOT SUM LIKE THIS

A1 $p(A_i) \geq 0$ for all natural numbers i

A2 $p(S) = 1$

A3 Assuming $A_1, A_2, \dots A_n$ are pairwise disjoint events as defined above

then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} p(A_i)$

5

Figure 8.10: Student [02]'s response to (1i) – the third Kolmogorov's axiom is completed by the lecturer who added the probability in the left-hand side of the equality and the sum from *i*=1 to infinity in the right-hand side of the equality. Also, the lecturer underlined the summation, wrote "YOU CAN NOT SUM LIKE THIS" and added the check symbols.

Finally, in response to (1ib) (Figure 8.11), [02] writes that the probability "*P*(*A*) is an event in the sample space that may occur via the Kolmogorov's axioms". In response to (1ia), the student uses the word "event" to talk about event *A*. However, in this instance, the probability of an event is confused with the event itself.

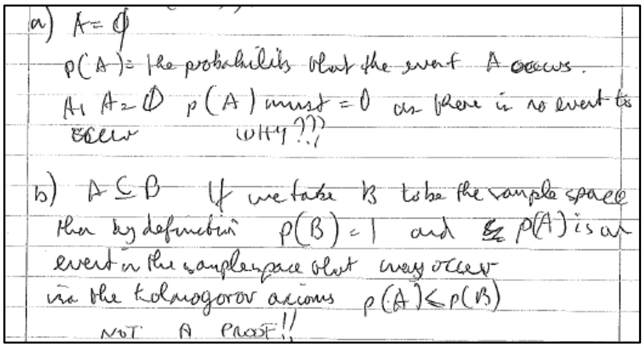

a) $A = \emptyset$
$p(A)$ = the probability that the event A occurs.
$A_1\ A_2 = \emptyset$ $p(A)$ must $= 0$ as there is no event to occur
WHY??
b) $A \subseteq B$ If we take B to be the sample space
then by definition $p(B) = 1$ and $p(A)$ is an
event in the samplespace that may occur
via the kolmogorov axioms $p(A) \leq p(B)$
NOT A PROOF!!

Figure 8.11: Student [02]'s response to (1ii) – The marker added "WHY??" and "NOT A PROOF!!".

(ii) Conflation in visual mediators between discrete and continuous random variables

The conflation of the objects of discrete and continuous variables is seen in four students' scripts ([02], [04], [08], [10]). The visual mediators used by the students is evidence of confusion with regard to the different sample spaces: the finite for the discrete and the infinite for the continuous random variable. This conflation occurs within the discourse of Probability and between the discourses of discrete and continuous variables. These are visible in the responses to task (2i) (Figure 8.12) and task (3i) (Figure 8.13).

(i) Let X be a Poisson random variable with parameter λ having probability mass function $P(X = x) = \frac{\lambda^k e^{-\lambda}}{k!}$

(a) Show that

$$\sum_{k=0}^{\infty} P(X = x) = 1$$

(b) By assuming the validity of the relation in (a), calculate $E(X)$.

[8 marks]

Figure 8.12: Snapshot of the first optional task from the Probability part of the module – Task 2i

(i)	Define expectation $E(X)$ and variance $V(X)$ of a continuous random variable X.

Figure 8.13: Snapshot of the second optional task from the Probability part of the module – Task 3i

Student [04] in response to (2ia) (Figure 8.12) adds an integral (Figure 8.14). The integral sign, just before the probability mass function, is evidence that [04] confuses the discourse on discrete random variables and the discourse on continuous random variables.

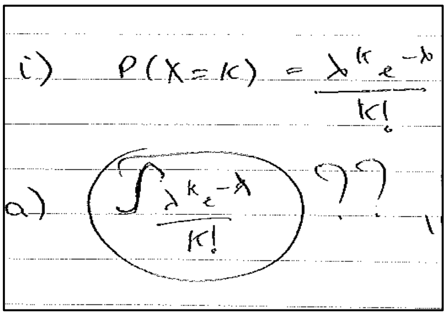


Figure 8.14: Student [04]' response to (2ia) – the marker added the circle around the integral and the two question-marks

The other three students ([04], [08], [10]) show this conflation in their responses to (3i) (Figure 8.13). In contrast to task 2, where variable *X* is defined as a "Poisson random variable" without explicitly saying that this is a discrete random variable, the wording in task 3 uses the phrase "continuous random variable". As noted in the overview (8.1) this was the least attempted task, with only thirteen students out of the fifty-four attempting this task. Three of these scripts show evidence of this commognitive conflict and receive zero marks for this part of the task. The signifiers illustrate an unclear distinction between these two discursive objects. This results in written texts with visual mediators from both discourses on discrete and continuous random variables. [08]'s response in Figure 8.15 is one such example. [08] defines the expectation using the binomial random variable which is discrete and the continues in stating a relation regarding the variance, which I discuss in detail in section 8.3.1

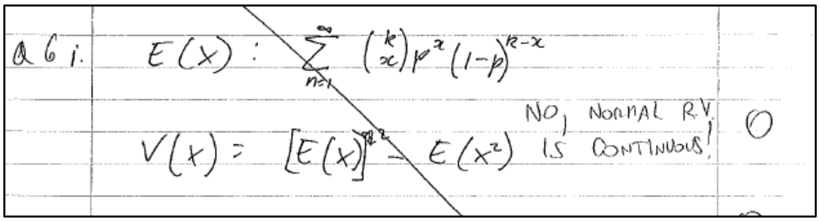


Figure 8.15: Student [08]'s response to (3i) – the marker added the line, the zero and wrote: "NO, NORMAL RV IS CONTINUOUS!".

8.2.2 Naming the objects involved in the narratives: The case of Kolmogorov's axioms

Here, I comment on the visual mediators, particularly the symbolic mediators, present in the students' scripts. In task 1, the students are asked to engage in a recall routine (state Kolmogorov's axioms, Figure 8.16).

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 8.16: Compulsory task from the Probability part of the module – Task 1

The students are asked to introduce specific objects, the event, the sample space and the probability of an event. These objects appear in their symbolic realisation in the three axioms. The students are asked to weave in their writing the symbols *A*, *S*, and *P*(*A*). There are twelve students who either partially explain the symbols they use in their narratives ([07], [12], [14], [20],

[21], [22]); or, they do not provide an explanation regarding these symbols being used at a later stage of their response ([03], [04], [12], [15], [16], [17]).

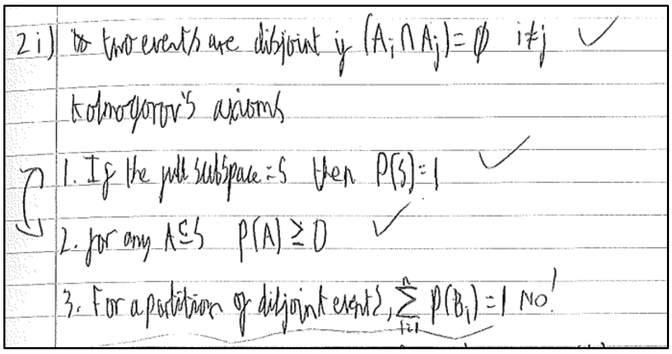

2i) two events are disjoint if $(A_i \cap A_j) = \emptyset$ $i \neq j$ ✓
Kolmogorov's axioms
1. If the full subspace = S then $P(S) = 1$ ✓
2. for any $A \subseteq S$ $P(A) \geq 0$ ✓
3. For a partition of disjoint events, $\sum_{i=1} P(B_i) = 1$ No!

Figure 8.17: Student [14]'s response to (1i)

Student [14] starts by defining disjoint events (Figure 8.17). The symbols A_i and A_j are introduced but these are not defined explicitly as events prior to their appearance in the intersection. Then, when giving the three axioms, [14] defines the sample space S as a "subspace". I discuss this word use in section 8.2.5 and the third Kolmogorov's axiom in section 8.3.1. The sample space and the event are defined in some sense, but the probability of an event is not defined at all. This is also the case in [22]'s response (Figure 8.18).

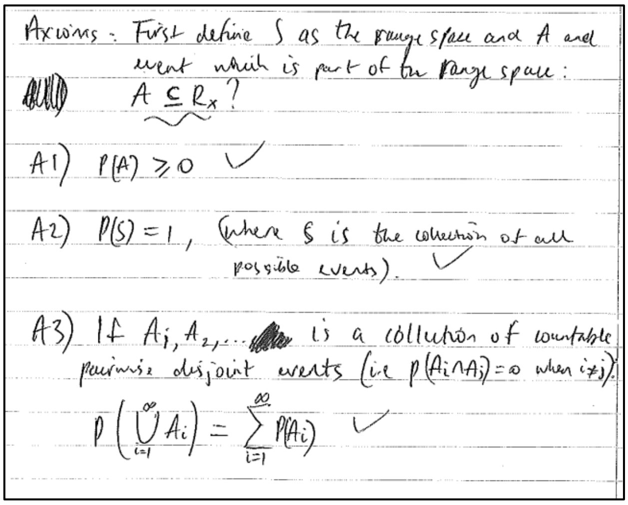

Axioms: First define S as the range space and A an event which is part of the range space:
$A \subseteq R_x$?
A1) $P(A) \geq 0$ ✓
A2) $P(S) = 1$, (where S is the collection of all possible events). ✓
A3) If $A_i, A_2, \ldots$ is a collection of countable pairwise disjoint events (i.e. $P(A_i \cap A_j) = 0$ when $i \neq j$)
$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ ✓

Figure 8.18: Student [22]' response to (1i) – the marker added the underline, the question-mark, and the three check symbols

Prior to writing the axioms, [22] provides information regarding the symbols that will appear in this writing, namely *S* and *A.* Apart from not introducing what the probability $P(A)$ stands for, [22] also uses the symbol R_x instead of S in the third line of the text, when defining the event as a subset of the sample space. The symbol R_x is being used for the range in the discrete random variable. This latter symbol shows conflation between the discourse of discrete random variables and the discourse of continuous random variables. I discuss this in section 8.3.1.

As mentioned above, there are students who do not define any of the symbolic mediators that they use in the axioms. [04]'s script is one such example (Figure 8.19). [04] states the three axioms without introducing what the symbols mean. However, [04] provides an explanation beside the second axiom to say that the "sum of all probabilities = 1". Instead of providing the third axiom, the student writes the application of disjoint events in terms of the probability of their union. However, in doing that, [04] takes a very specific example of disjoint events. These appear to be an event, and its complement as the "sum of two probabilities is 1". As the focus here is on the symbolic mediators and the objects that these realise, I discuss the third axiom further in section 8.3.1.

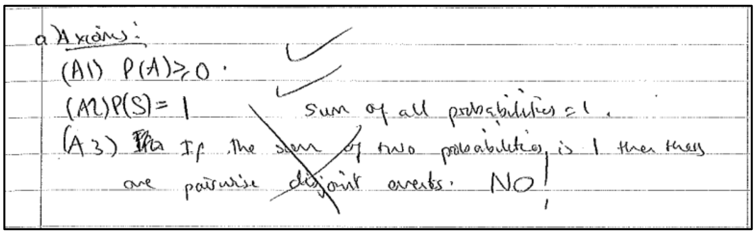


Figure 8.19: Student [04]'s response to (1i) – the marker added the two checks, the cross and wrote "NO!".

8.2.3 Conflating word use: The case of independent and disjoint events

In the interview, the lecturer noted that students have difficulty with the terms "independent" and "disjoint". He suggested that there is a conflation between the colloquial use of the word independent and the word independent in the probability discourse. This commognitive conflict appeared in the scripts of two students ([15] and [17]) in their response to (1i) (Figure 8.20) either in the definition of disjoint or in the third Kolmogorov axiom, where the applicability condition is that the events are pairwise disjoint.

> (i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

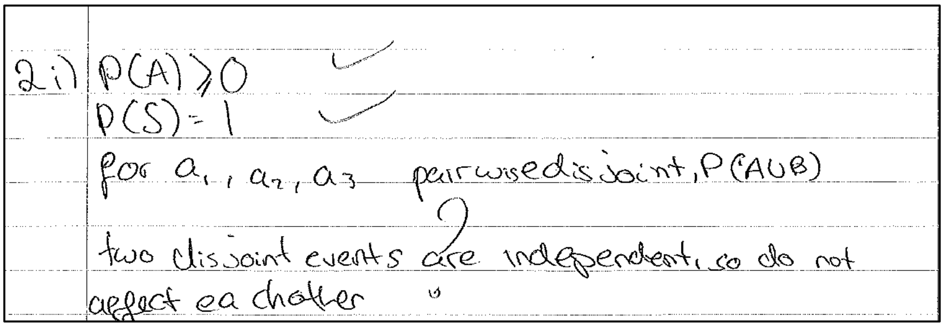
Figure 8.20: Snapshot of the compulsory task from the Probability part of the module – Task 1.

[15]'s script (Figure 8.21) was also discussed in section 8.2.1 regarding the conflation between the discourses of Probability and Set Theory. Here, I focus on the definition of disjoint events. [15] says that disjoint events have to satisfy two conditions: first, they have to be "independent of each other" and, second, they "cannot happen together". In Probability, two events are called independent if the occurrence of one does not affect the potentiality of occurrence of the other. The word use, from [15], signals conflation between the colloquial use of independent and the probability use of the word. However, the second condition given by the student ("cannot happen together") illustrates the student's take on what disjoint events are. Sometimes the word "mutually exclusive" is being used instead of disjoint in Probability (If *S* is a sample space and *A*, *B* are events in *S* then *A* and *B* are called disjoint or mutually exclusive if $A \cap B = \emptyset$). However, L2 chose to use the term "disjoint" throughout his lectures and lecture notes.

> Two disjoint events are two events that not only are independent of each other but cannot happen together. for example X and Y.
> $P(X \wedge Y) = \emptyset$

Figure 8.21: Student [15]'s response to (1i)

Similarly, in [17]'s writing about the third Kolmogorov's axiom, there is a conflation of independent and disjoint events (Figure 8.22). [17] writes that "two disjoint events are independent so do not affect each other". Two events are independent when the probability of one occurring does not affect the probability of the other occurring. However, disjoint events cannot happen at the same time which is very different from “do not affect each other”. The probability of the union of two disjoint events is zero whereas the probability of the union of two independent events is the product of the probability of the first event happening and the probability of the second event happening.



2i) $P(A) \geq 0$

$P(S) = 1$

for a_1, a_2, a_3 pairwise disjoint, $P(A \cup B)$

?

two disjoint events are independent, so do not affect ea chother

Figure 8.22: Student [17]’s response to (1i) – the marker added the two checks and the question-mark

8.2.4 Use of visual mediators: The case of Venn diagrams

In this section, I present the analysis of scripts from seven students who use Venn diagrams in their responses to task 1 (Figure 8.23): [02], [04], [05], [12], [13], [16], [21].

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov’s axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 8.23: Compulsory task from the Probability part of the module – Task 1

[16]'s response to (1i) provides the definition of disjoint and the Venn diagrams are used to show that the two sets *A* and *B* “do not have an intersection” (Figure 8.24). The visual mediator is used to support the definition of disjoint events. The use of Venn diagrams in the responses of the other students is similar.

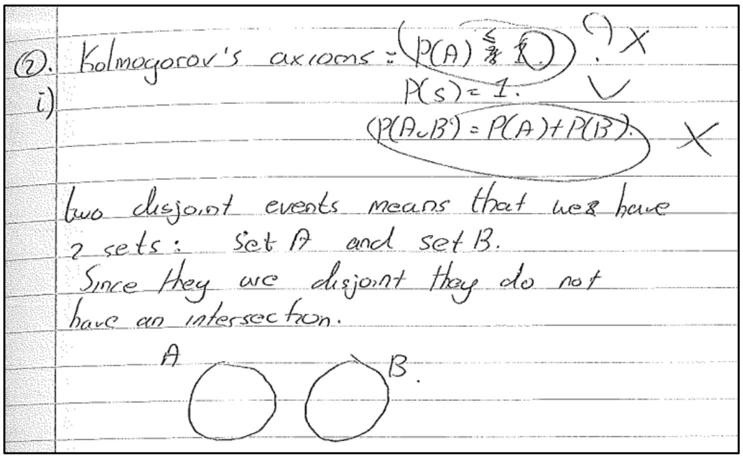


Figure 8.24: Student [16]'s response to (1i) – the marker added the two circles, the crosses, the check symbol and wrote zero in the first line

In the substantiation routine regarding the probability of an event which is a subset of another event, [05] uses Venn diagrams to illustrate this connection between the sets (Figure 8.25). Here, I note that originally the student drew a different Venn diagram where the two events *A* and *B* where disjoint, this is then scribbled out and a new diagram with *A* being subset of *B* is drawn. This is possibly used to help [07] visualise the two disjoint events that are used later in the substantiation routine.

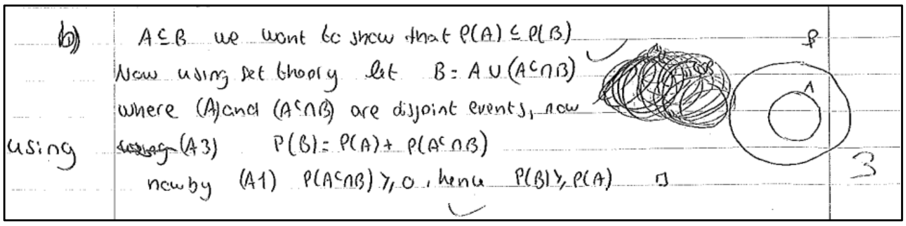
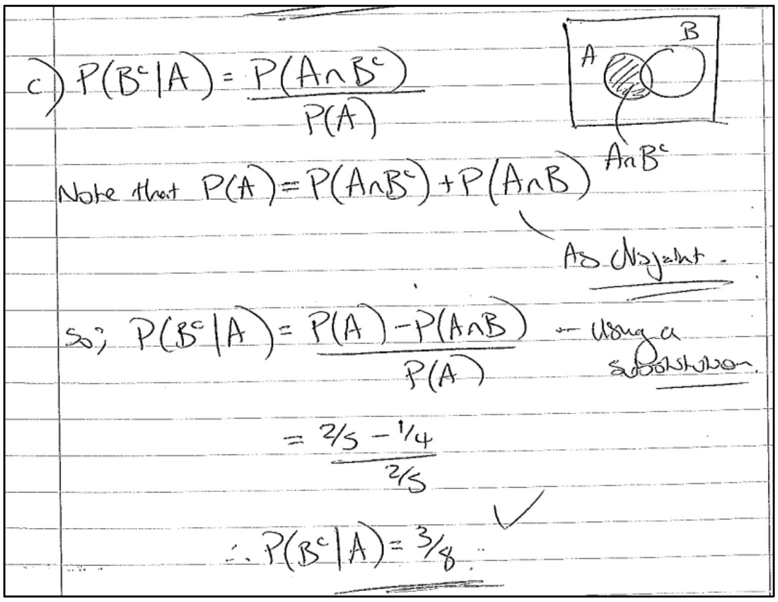


Figure 8.25: Student [05]'s response to (1ib) – the marker added the two check symbols

In trying to find $P(A \cap B^c)$ to answer (1iic), [21] realises $P(A)$ in terms of $P(A \cap B^c)$ and $P(A \cap B)$. (Figure 8.26). To support this realisation, [21] provides the Venn diagram where the $A \cap B^c$ is shaded. Then the student writes that $P(A) = P(A \cap B^c) + P(A \cap B)$ and says that the events are disjoint.



c) $P(B^c|A) = \dfrac{P(A \cap B^c)}{P(A)}$

A B

$A \cap B^c$

Note that $P(A) = P(A \cap B^c) + P(A \cap B)$

As Disjoint.

So; $P(B^c|A) = \dfrac{P(A) - P(A \cap B)}{P(A)}$ — Using a substitution.

$= \dfrac{2/5 - 1/4}{2/5}$

$\therefore P(B^c|A) = 3/8$

Figure 8.26: Student [21]' response to (1iic) – the marker added the check symbol

8.2.5 Word use (miscellaneous)

Here, I present examples from the analysed students' scripts that signal conflating word use from different mathematical discourses. I also comment on the use of personal pronouns in students' scripts.

(i) Conflating word use from other mathematical discourses

The responses to tasks (1i) (Figure 8.27) and (2i) (Figure 8.28) from three students ([03], [12], [14]) show evidence of conflation between different mathematical discourses, apart from Set Theory and Probability.

(i)	In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions: (a) For any event $A = \emptyset$, prove that $P(A) = 0$. You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events. (b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$. **[12 marks]**

Figure 8.27: Snapshot from the compulsory task from the Probability part of the module – Task 1i

(i)	Let X be a Poisson random variable with parameter λ having probability mass function $P(X = x) = \frac{\lambda^k e^{-\lambda}}{k!}$ (a) Show that $$\sum_{k=0}^{\infty} P(X = x) = 1$$ (b) By assuming the validity of the relation in (a), calculate $E(X)$. **[8 marks]**

Figure 8.28: The first optional task from the Probability part of the module – Task 2i

In response to (1i), [14] says that "If the (unclear) subspace =S" (Figure 8.29). The word "subspace" signals the use of linear algebra discourse, with the linear subspace defined as a subset of a vector space closed under addition and scalar multiplication; or topology discourse, with a subspace being a subset of a topological space with a subspace topology. However, the student here was referring to the sample space in the context of Probability.

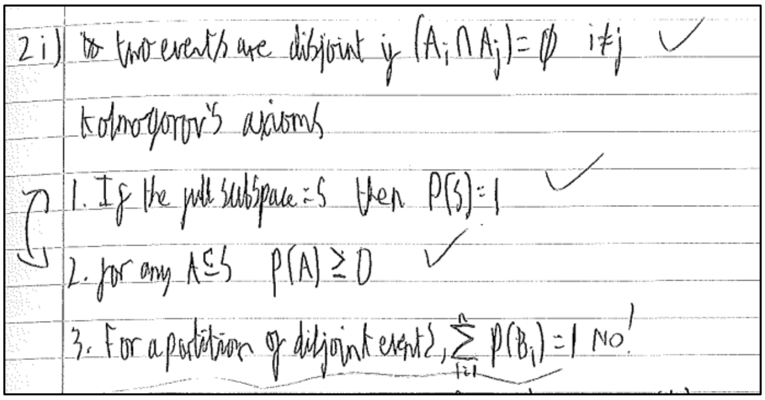


Figure 8.29: Student [14]'s response to (1i) – the marker added the three check symbols, underlined the response to the third Kolmogorov's axiom and wrote "NO!"

In response to (1ib), [03] connects probability with functions (Figure 8.30). Attempting to prove that $P(A) \leq P(B)$, the student takes $P(A) = 0$, and then tries to connect the possible values that the probability $P(B)$ takes with trigonometric functions.

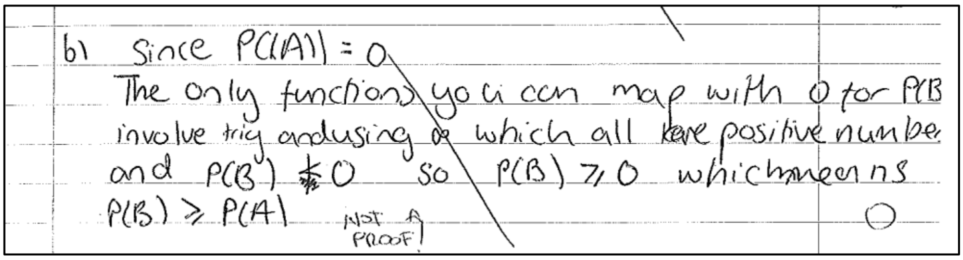


Figure 8.30: Student [03]' response to (1ib) – the marker added the "NOT A PROOF!" the line and the zero

Finally, [12]'s response to (2ia) makes reference to geometric series (Figure 8.31). In engaging with this part of the task, the students are asked to recall the McLaurin expansion of e^{λ}. This expansion is a series but not a geometric series. [12] attempts to substantiate the realisation that the sum of the probabilities is equal to 1 and, in this attempt, breaks the summation into $e^{-\lambda}$ and the sum from k=0 up to infinity. In the second line, [12] breaks the sum one more time attempting to relate the parts of the sum to something familiar. [12] recognizes that this sum is a geometric series and writes it underneath to find the value of the original sum. However, it seems that this attempt was fruitless as the student does not continue and scribbles out the next attempt to work with the product of $e^{-\lambda}$ and the two sums. (Note: I do not focus on the algebraic manipulation signalling unclear meaning making regarding the sums, I only focus on the explanation written underneath the first sum that says “geometric”).

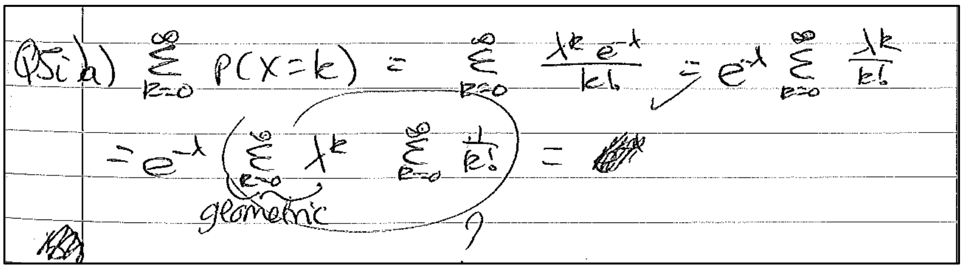


Figure 8.31: Student [12]’s response to (2ia) – the marker added the circle and the question-mark

(ii) Using the pronouns “we” and “I” in the produced narratives

In the scripts of eleven students ([05], [07], [09], [11], [12], [13], [16], [18], [19], [21], [22]) I note the use of the pronoun “we”. This pronoun is used in the introduction of either a new symbol in their response; or, a step of the procedure they followed (Figure 8.32). This is also observed in section 6.2.5 in students’ responses to the Numbers Sets and Proofs tasks. Additionally, the pronoun “we” is used by L2, in the interview, to signal either himself and his students or people in general (7.3.3).

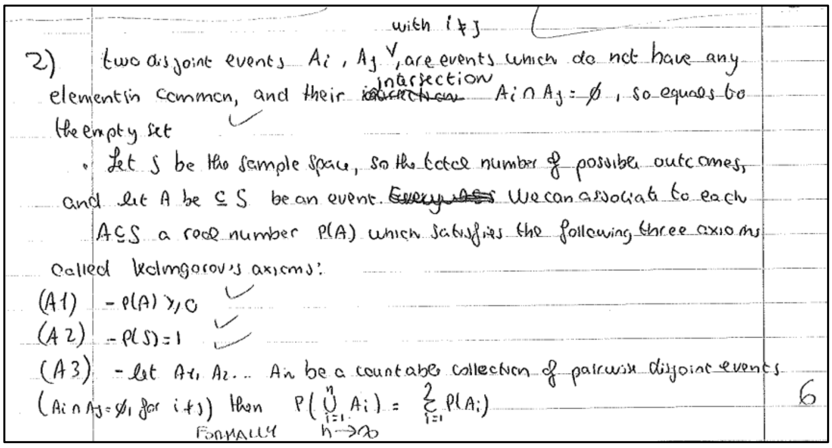


Figure 8.32: Student [05]’ response to (1i) – the marker added the “FORMALLY $n \to \infty$” and the check symbols

There is also one student [03] (Figure 8.33) that uses the pronoun “you”. [03]’s uses “you” when signalling the relationship between the value 0, the probabilities, and the trigonometric functions. I, also, discuss this example earlier (8.2.5i) regarding the conflating discourses.

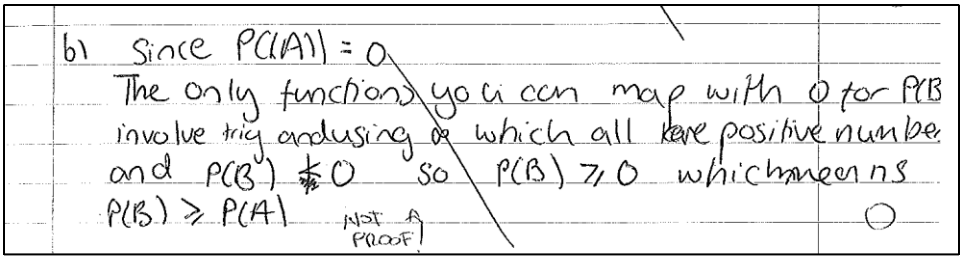

b) Since $P(A) = 0$
The only functions you can map with 0 for P(B
involve trig and using ~~or~~ which all have positive numbe
and $P(B') \not\leq 0$ so $P(B) \geq 0$ which means
$P(B) \geq P(A)$ NOT A PROOF!
0

Figure 8.33: Student [03]’ response to (1ib) – the marker added the “NOT A PROOF!” the line and the zero

8.3 Students' scripts: Commognitive analysis (routines)

In this section, I present results from the analysis regarding students' engagement with routines. Specifically, in (8.3.1) I examine evidence of recall routines in the scripts regarding the definition of disjoint events, conditional probability and the expectation and variance. Also, I examine the substantiation routines regarding the propositions (e.g., $P(\emptyset) = 0$ and If $A \subseteq B$ then $P(A) \leq P(B)$), and the *procedures* that the students followed, particularly those not directly indicated in the wording of the task. Finally, in (8.3.4), I report cases where the applicability and the closing conditions of a routine are not met. As in section 8.2, I first mention the part of the task, which corresponds to the sampled students' scripts, and then discuss the students' scripts.

8.3.1 Recall routines: disjoint events, conditional probability, expectation, and variance

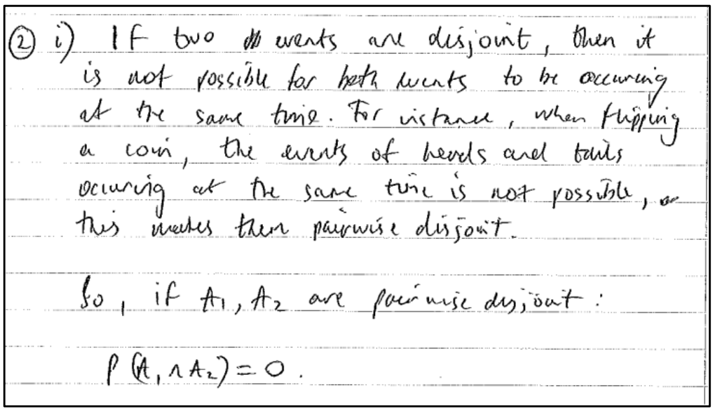
In the three probability tasks, there are several cases where recalling the definition of an object is asked explicitly. These are the definitions of disjoint events in (1i), the expectation and variance in (2i) and (3i). Also, implicitly, the students are asked to recall the conditional probability in (1i). The analysis of the students' scripts revealed difficulties in recalling these three definitions. In the next section, I present each by giving characteristic examples from the students' scripts.

(i) Recall routine: the case of disjoint events

The students are asked to engage in recall routines in (1i) where they first to define disjoints events. Two events are characterised as disjoint if their intersection is the empty set and this definition is particularly crucial in the third of the Kolmogorov's axioms. The scripts of nine students ([01], [02], [04], [06], [08], [09], [15], [17], [22]) showed difficulties in the definition of disjoint events.

Instead of stating that the intersection of the disjoint events is the empty set, six students mention the probability of the intersection is zero ([01], [02], [04],

[06], [09], [22]). This is the result of the definition of disjoint events also taking into account what probability means. [22], initially, provides the definition of disjoint saying that "it is not possible for both events to be occurring at the same time" (Figure 8.33). Then, [22] continues by providing the example of flipping coins and ends the response by saying that the probability of the intersection of two disjoint events is zero. As mentioned above, this is based on the definition of the disjoint events and the proposition that $P(\emptyset) = 0$ which is not proven at this stage.



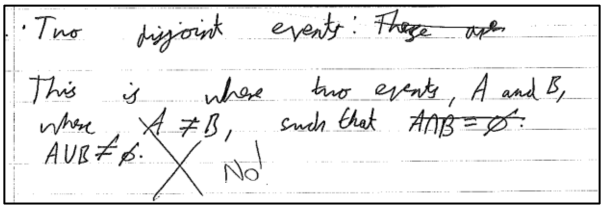
② i) If two ~~ev~~ events are disjoint, then it is not possible for both events to be occurring at the same time. For instance, when flipping a coin, the events of heads and tails occurring at the same time is not possible, so this makes them pairwise disjoint.

So, if A_1, A_2 are pairwise disjoint:

$P(A_1 \cap A_2) = 0$.

Figure 8.33: Student [22]'s response to (1i)

[08] conflates the definition of disjoint events (Figure 8.34). Initially, [08], states that the intersection is the empty set but then [08] scribbles this out and states that the union is the non-empty set. This is also seen in the third Kolmogorov's axiom and [08]'s response to (1ii).



· Two disjoint events: ~~These are~~

This is where two events, A and B, where $A \neq B$, such that ~~$A \cap B = \emptyset$~~.

$A \cup B \neq \emptyset$.

No!

Figure 8.34: Student [08]'s response to (1i) – the marker added the cross and wrote "NO!"

Finally, two students use the word "independent" to define disjoint events ([15], [17]). I also discuss this in section 8.2.5 regarding the commognitive conflict between the word independent in colloquial discourse and Probability discourse. Student [17] (Figure 8.35), defines disjoint events using the term independent. It seems that the student uses the word in its colloquial sense. However, in the discourse of probability, this term signals something different. This is evidence of commognitive conflict regarding the word use "independent".

2 i) $P(A) \geq 0$
$P(S) = 1$
for a_1, a_2, a_3 pairwise disjoint, $P(A \cup B)$
two disjoint events are independent, so do not affect each other

Figure 8.35: Student [17]'s response to (1i) – the marker added the two check symbols and the question-mark

(ii) Recall routine: The case of conditional probability

The object of conditional probability appears in different parts of the tasks mostly in (1ii) but also in (3iiib). There are two students ([01], [15]) who do not recall the definition of conditional probability correctly. Both instances occur in their response to (1ii) (Figure 8.36)

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.
(a) Show that $P(A \cap B) = \frac{1}{4}$.
(b) Find $P(B)$ and the range of possible values for the parameter p.
(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

Figure 8.36: Snapshot of the compulsory task from the Probability part of the module – Task 1ii

[01] (Figure 8.37) writes that the conditional probability is $\frac{P(B)-P(A)}{P(A \cup B)}$. This does not assist in finding $P(B)$. The student starts by providing the above-mentioned definition, then tries to find $P(A \cup B)$ and concludes with considering $P(A \cap B)$, which depends on the value of *p.* The student then gives up.

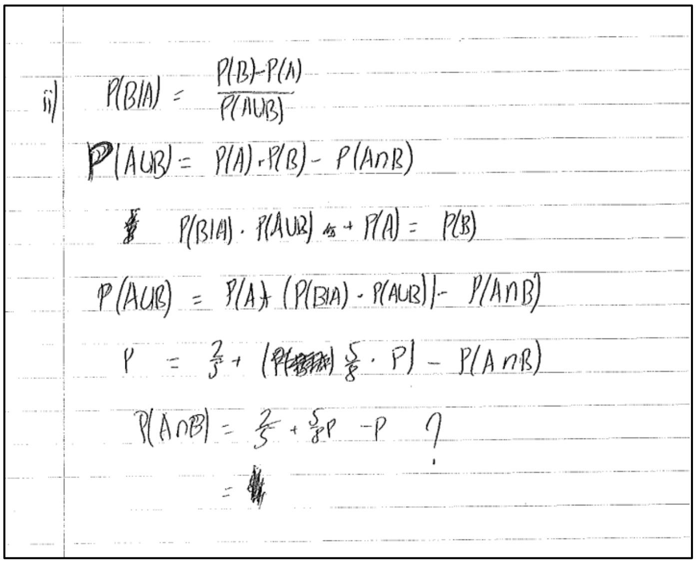


Figure 8.37: Student [01]'s response to (1ii) – the marker added the question-mark

[15] defines $P(B^c|A)$ as $\frac{P(A\cap B^c)}{P(B^c)}$ instead of $\frac{P(A\cap B^c)}{P(A)}$ (Figure 8.38). Then, [15] attempts to find $P(B^c)$ fails to eliminate *p* from the result and stops.

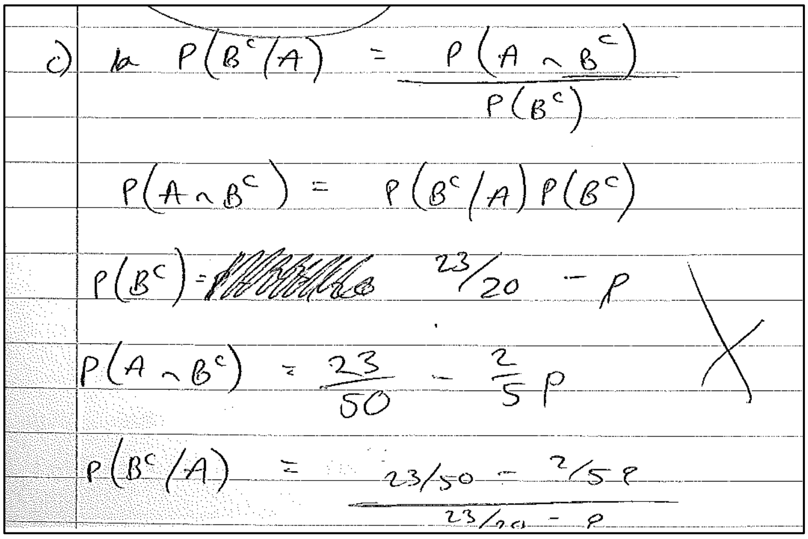


Figure 8.38: Student [15]'s response to (1iic) – the marker added the cross

(iii) Recall routine: The case of expectation and variance

The students are asked in (2i) and (3i) to provide the expectation for a discrete variable, Poisson's random variable and the expectation and variance for a continuous random variable accordingly. The scripts of eight students ([02], [04], [08], [09], [10], [16], [17], [20]) signal difficulties with recalling the definitions of expectation and variance.

Specifically, when writing the expectation for Poisson's random variable [04] uses the symbolic mediators for the Poisson's random variable but uses an integral sign as well (Figure 8.39). I see the presence of the latter is evidence of a commognitive conflict within the discourse of Probability concerning the objects of a random and discrete variable.

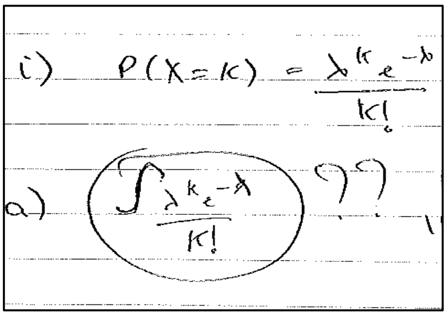
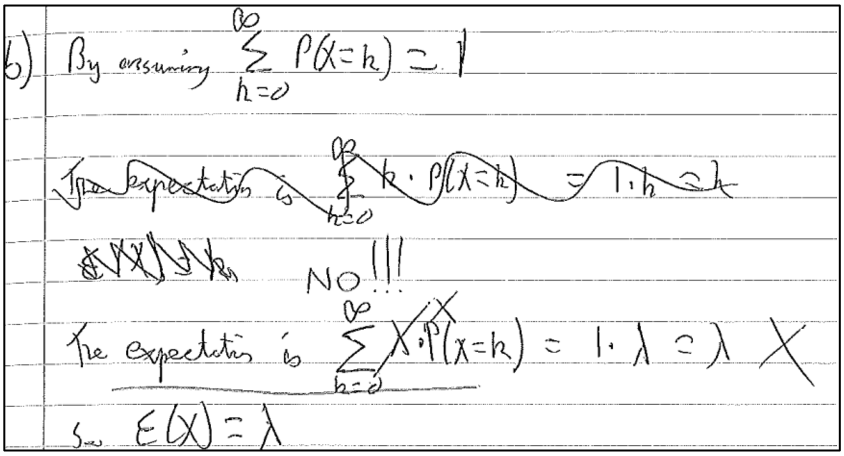


Figure 8.39: Student [04]' response to (2ia) – the lecturer circled the integral and added the two question-marks

Another attempt to recall the expectation of a discrete random variable is in Figure 8.40. [20] initially writes that the expectation is the sum of the product *k* and *P*(*X*=*k*) which is then written as equal to *k.* It seems though that the student recalls that this is not the case and that the expectation is equal to *λ.* The student scribbles out the first attempt to find the expectation and uses a different definition involving the parameter *λ* of the variable. This leads to the expectation being equal to *λ* and the student concludes. However, even though the end of the response is correct, by recalling the definition of the expectation incorrectly, the student does not produce writing that can be acceptable to the marker.



b) By assuming $\sum_{k=0}^{\infty} P(X=k) = 1$

~~The expectation is~~ $\cancel{\sum_{k=0}^{\infty} k \cdot P(X=k) = 1 \cdot k = k}$

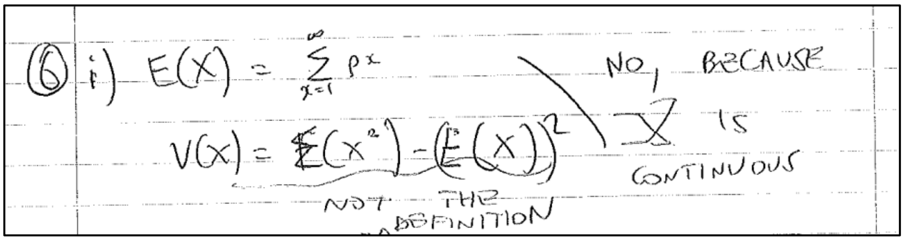
NO!!!

The expectation is $\sum_{k=0}^{\infty} X \cdot P(X=k) = 1 \cdot \lambda = \lambda$

So $E(X) = \lambda$

Figure 8.40: Student [20] response to (2ib) – the lecturer underlined the line about the expectation, wrote "NO!!!" and added the cross

The responses to (3i) of three students ([02], [08], [10]) show conflation between the discourses of discrete and continuous variables, even though the wording of the task mentions a "continuous random variable" explicitly. [02]'s response to (3i) (Figure 8.41) shows that, instead of the probability density function of the continuous variable, the student uses a probability mass function of a discrete variable.



6 i) $E(X) = \sum_{x=1}^{\infty} p^x$

$V(X) = E(X^2) - (E(X))^2$

NO, BECAUSE X IS CONTINUOUS

NOT THE DEFINITION

Figure 8.41: Student [02]'s response to (3i) – the lecturer underlined the variance, added "NO, BECAUSE X IS CONTINUOUS" and "NOT THE DEFINITION"

Finally, regarding variance, four students ([09] [02] [08] [10]), instead of giving the definition for variance for the continuous variable, attempt to write the relation connecting Variance with expectation. [09] initially provides the expectation of a continuous random variable using the probability density function $f(x)$. Then, [09] gives the variance as $E[X^2] - (E[X])^2$ (Figure 8.42). This relationship between variance and expectation is true for both discrete and continuous random variables. However, what is being asked here is the definition of variance of a continuous variable and not any relationship that gives the variance with respect to the expectation.

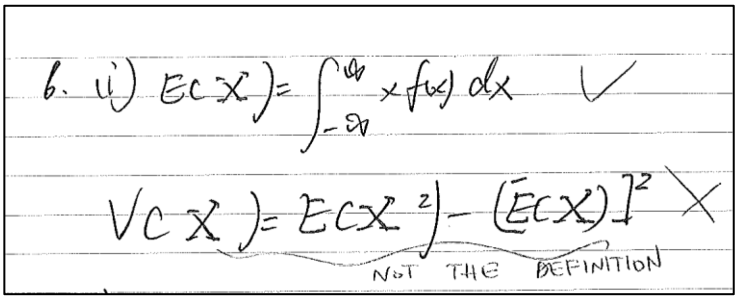


Figure 8.42: Student [09]'s response to (3i) – the marker added the check, the cross, underlined the variance and wrote "NOT THE DEFINITION"

8.3.2 The *procedure* is not given in the wording of the task

The *procedures* of routines are not given in (1), (2ii) and the students are asked to decide which *procedure* to follow. In the following, I illustrate examples where the students followed a different *procedure* from the one given in the model solutions (7.1.2, 7.2.2) or where they followed this *procedure* but problematically.

(i) *The procedure* of substantiation: Proving that $P(\emptyset) = 0$

The responses of ten students ([01], [02], [03], [04], [08], [11], [13], [14], [15], [16]) to (1ia) illustrate difficulties with proving. Specifically, there were two students ([01], [03]) who restated what was supposed to be proven, three that used what was asked to be proven in their text ([08], [11], [13]) and the other five ([02], [04], [14], [15], [16]) that did not provide sufficient explanation regarding the various steps of the *procedure.*

In attempting to substantiate the proposition $P(\emptyset) = 0$, [01] and [03] restate what is asked to be proven (Figure 8.43)

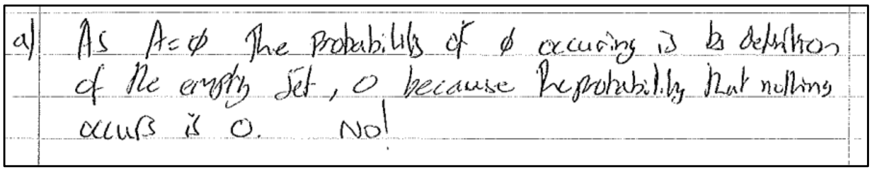
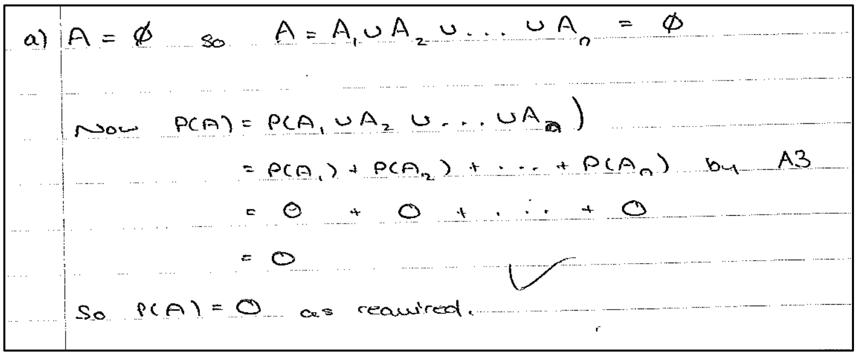

a) As $A=\emptyset$ the probability of $\emptyset$ occuring is by definition of the empty set, 0 because the probability that nothing occurs is 0. No!

Figure 8.43: Student [01]'s response to (1ia) – the marker added "NO!"

[01] does not use of Kolmogorov's axioms (the task asks that the students prove this proposition using Kolmogorov's axioms). (Figure 8.43). [01] refers to the definition of the empty set and says, "the probability that nothing occurs is 0". This illustrates difficulties with the routine of proving as the student just restates what needs to be proven.

A commognitive conflict stemming from the difference in the rules of school and university discourse is seen in the responses of the three students ([08], [11], [13]). [13] uses this realisation in trying to prove that $P(\emptyset) = 0$ (Figure 8.44). Initially, the student takes an event *A* and a countable collection of sets with their union being *A*. Then without mentioning that this is due to the sets being empty and thus being disjoint, [13] uses the third Kolmogorov's Axiom and writes the third line. At this point, commognitive conflict occurs as the student uses the fact that is to be proven to say that all these probabilities $(P(A_1), P(A_2), \ldots, P(A_n))$ are equal to zero. Essentially, the student uses what is to be proven within the proof.


a) $A = \emptyset$ so $A = A_1 \cup A_2 \cup \ldots \cup A_n = \emptyset$

Now $P(A) = P(A_1 \cup A_2 \cup \ldots \cup A_n)$

$= P(A_1) + P(A_2) + \ldots + P(A_n)$ by A3

$= 0 + 0 + \ldots + 0$

$= 0$

So $P(A) = 0$ as required.

Figure 8.44: Student [13]'s response to (1ia) – the marker added the check symbol

Scripts of five other students ([02], [04], [14], [15], [16]) illustrate that, although the *procedure* being used is correct, some of the steps are not explained. This results in an incomplete proof (this is also discussed in section 8.3.4). For example, [16] uses the proposition given by the lecturer in the wording of the task instead of using Kolmogorov's axioms (Figure 8.45). The two sets A_1, A_2 used are not defined and their relationship with *A* is also not explained. [16] says that the probability of their union is zero. Then, [16] uses the given proposition and writes that the sum of their probabilities is also zero. Initially, the student wrote that $P(A_1) = 0$ and $P(A_2) = 0$ but then [16] scribbled these out and wrote $P(A) = 0$. However, the student does not seem satisfied with this and tries to find a different realisation for $P(A)$ in the last three lines of the writing. This does not result in the substantiation of $P(\emptyset) = 0$.

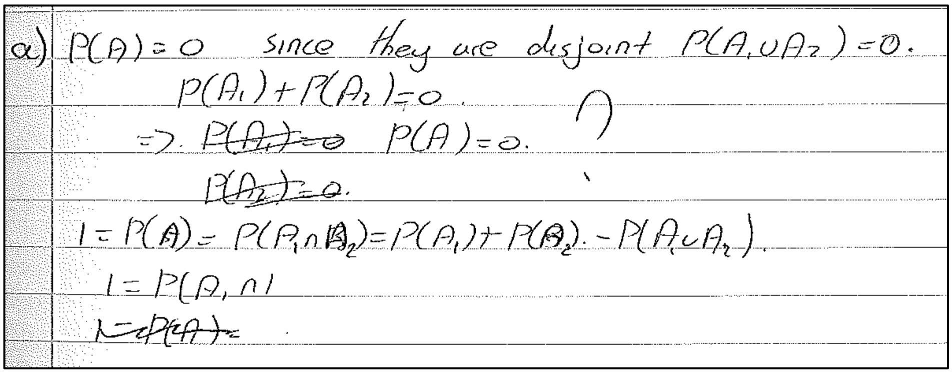
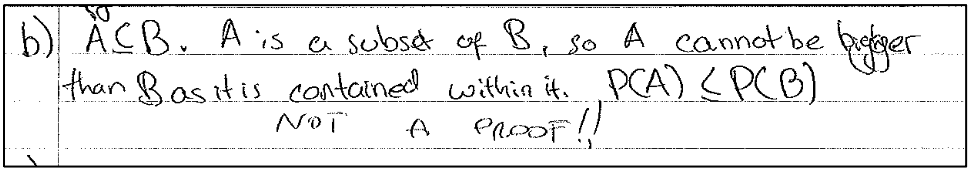

a) $P(A) = 0$ since they are disjoint $P(A_1 \cup A_2) = 0$.
$P(A_1) + P(A_2) = 0$. ?
$\Rightarrow$ $\cancel{P(A_1)=0}$ $P(A) = 0$.
$\cancel{P(A_2)=0}$.
$1 = P(A) = P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$.
$1 = P(A_1 \cap$
$\cancel{1 = P(A) =}$

Figure 8.45: Student [16]'s response to (1ia) – the marker added the question-mark

(ii) The procedure of substantiation: Proving that if $A \subseteq B$ then $P(A) \leq P(B)$

The responses of six students ([01], [02], [06], [07], [15], [17]), illustrate difficulties with the substantiation routine. Specifically, three students ([07], [15], [17]) provide a restatement of the proposition, one ([06]) describes a proof which is not based on Kolmogorov's axioms, one ([01]) gives an example that illustrates that the proposition stands and one student ([02]) shows that the proposition is true by taking specific sets.

Student [17], attempts to prove the proposition using the relationship between the two sets *A* and *B* and says, "*A* cannot be bigger than *B* as it is contained within it" (Figure 8.46). The student mostly engages in the discourse of Set Theory as the arguments towards proving the proposition are based on relationships between the two sets *A* and *B*.

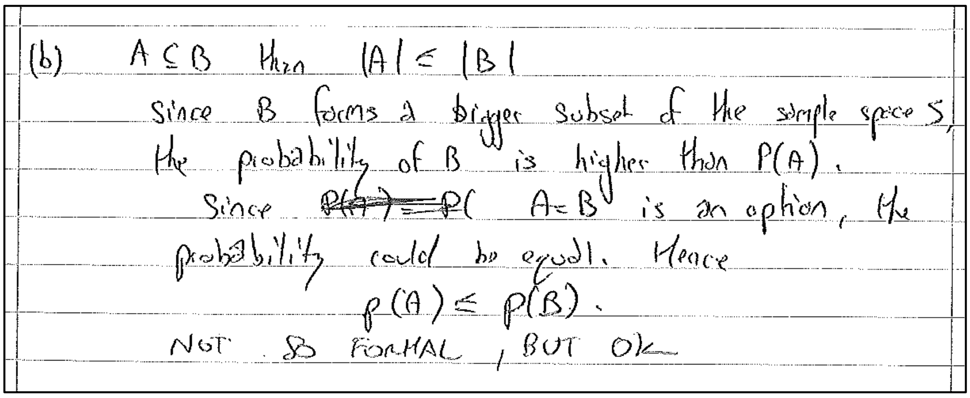


b) $A \subseteq B$. A is a subset of B, so A cannot be bigger than B as it is contained within it. $P(A) \leq P(B)$

NOT A PROOF!!

Figure 8.46: Student [17]'s response to (1ib) – the marker added: "NOT A PROOF!!"

Similar to the above response is [06]'s attempt to substantiate the proposition (Figure 8.47). [06] talks about the cardinality of the sets and expresses how – since one is the subset of the other – the cardinality would be smaller. S/he then resorts to the definition of a probability being the quotient of the event occurring over the sample space.



(b) $A \subseteq B$ then $|A| \leq |B|$

Since B forms a bigger subset of the sample space S, the probability of B is higher than $P(A)$.

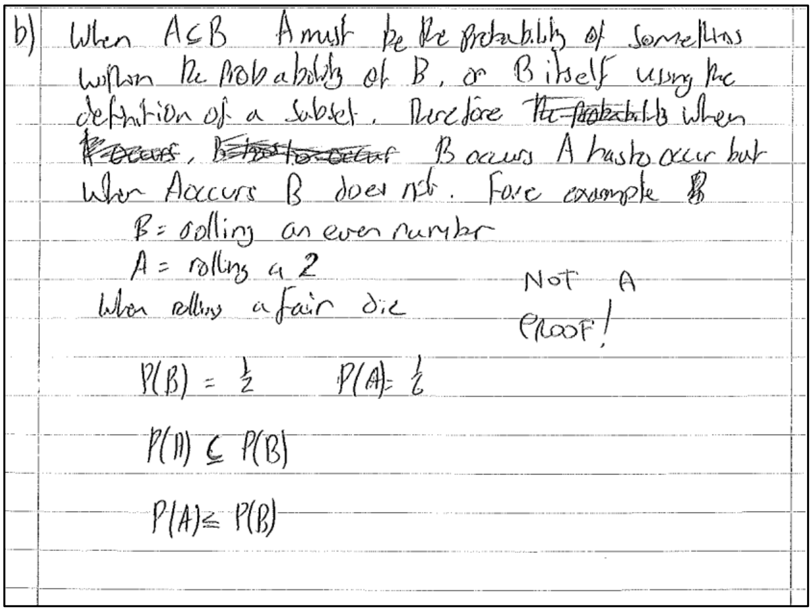
Since ~~P(A) = P(~~ $A = B$ is an option, the probability could be equal. Hence

$P(A) \leq P(B)$.

NOT SO FORMAL, BUT OK

Figure 8.47: Student [06]'s response to (1ib) – the marker added: "NOT SO FORMAL, BUT OK"

[01], attempts to substantiate the proposition by providing an example where the proposition is true (Figure 8.48). Initially [01] uses the discourse of Set Theory to substantiate the proposition and then resorts to providing an example of the proposition. However, substantiation routines at university level require rigour and arguments based only on the initial conditions. This is not the case for [01]'s response or [02]'s response (Figure 8.49).



b) When $A \subseteq B$ A must be the probability of something within the probability of B, or B itself using the definition of a subset. Therefore ~~the probability when~~ ~~A occurs~~, ~~B has to occur~~ B occurs A has to occur but when A occurs B does not. For example ~~B~~

B = rolling an even number

A = rolling a 2

When rolling a fair die

NOT A PROOF!

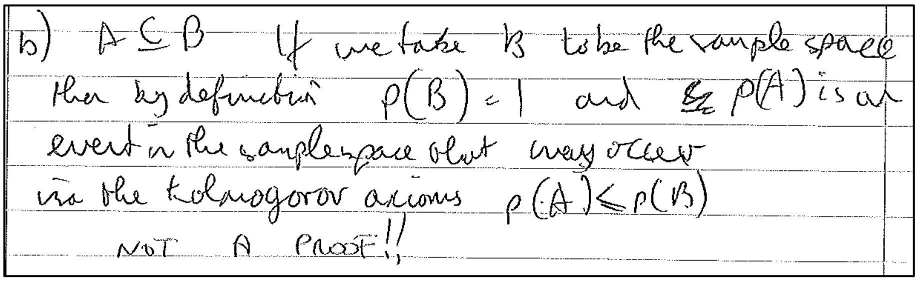
$P(B) = \frac{1}{2}$ $P(A) = \frac{1}{6}$

$P(A) \subseteq P(B)$

$P(A) \leq P(B)$

Figure 8.48: Student [01]'s response to (1ib) – the marker added: "NOT A PROOF!"

Instead of a proof [02] provides an example of specific sets where *B*=*S* and *A* is a subset of the sample space *S* (Figure 8.49). Based on the first and second of Kolmogorov's axioms, [02] says that this results in $P(A) \leq P(B)$.



b) $A \subseteq B$ If we take B to be the sample space then by definition $P(B) = 1$ and ~~E~~ $P(A)$ is an event in the samplespace that may occur via the Kolmogorov axioms $P(A) \leq P(B)$

NOT A PROOF!!

Figure 8.49: Student [02]'s response to (1ib) – the marker added: "NOT A PROOF!!"

(iii) The procedure of rituals: Calculating the probabilities in (1iic)

In (1iic) students are asked to engage in rituals to calculate different probabilities. The *procedure* of the ritual is not specified in the wording of the task and the students either use the relationship between an event and its complement to first find $P(B^c|A)$ and then $P(B^c \cap A)$ or use the definition of conditional probability to find $P(B^c \cap A)$, using the fact that $(A \cap B)$ and $B^c \cap A$ are disjoint events and their union is *A.* In this section, I focus on two students ([16], [20]) who used different *procedures* (Bayes theorem) and did not find the requested probabilities.

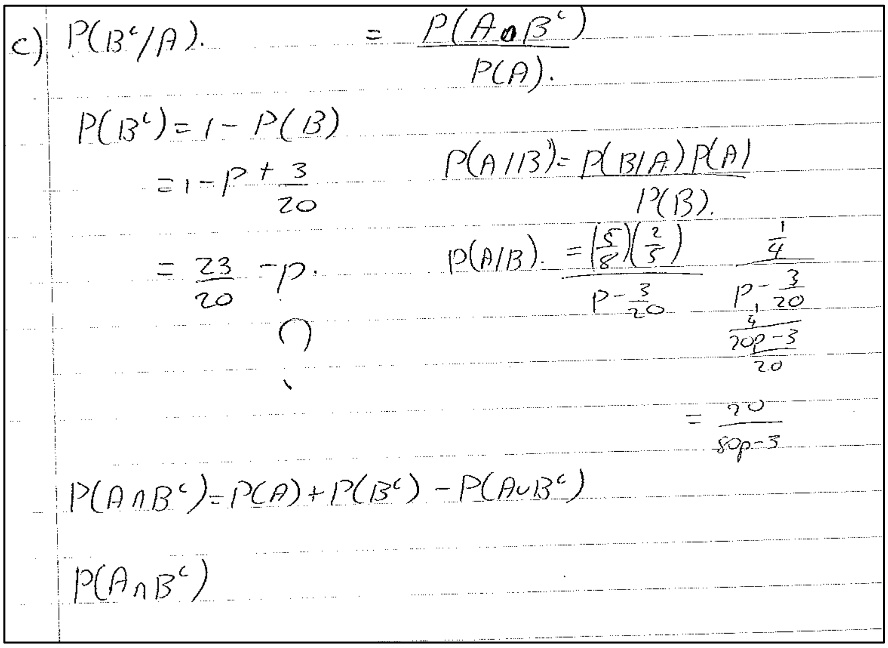


Figure 8.50: Student [16]'s response to (1iic) – The marker added the question-mark

[16] writes the definition of conditional probability and, to find $P(B^c \cap A)$, attempts to find $P(B^c)$ (Figure 8.50). However, as the result involves *p,* the student seeks a different *procedure* in finding $P(B^c)$. In the right-hand side of the image is the second attempt, using Bayes theorem, which also does not result in finding a value for *p*. The student attempts one more time to find a realisation that involves $P(B^c \cap A)$. However, as the values of the probabilities $P(A)$, $P(B^c)$ and find $P(A \cup B^c)$ are not all known, the student gives up and stops writing.

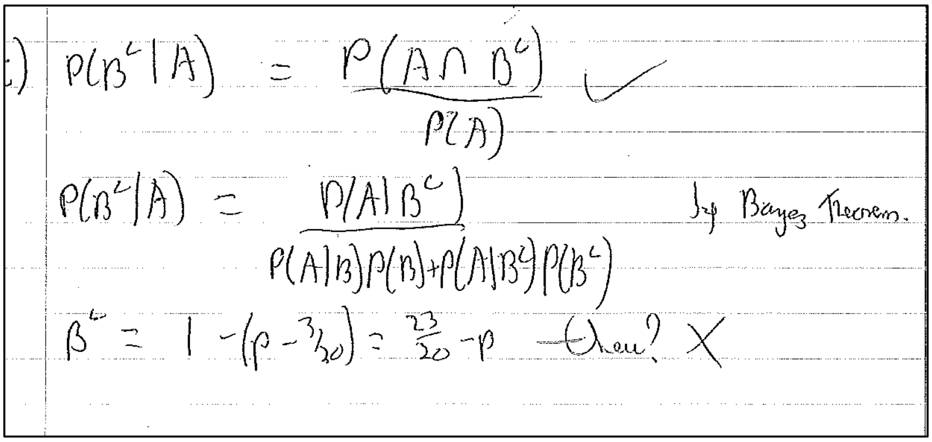


Figure 8.51: Student [20]'s response to (1iic)

[20] also starts using the definition of the conditional probability, but, instead of trying to find $P(B^c \cap A)$, the student attempts to find $P(B^c|A)$, using Bayes theorem (Figure 8.51). The student tries to find it but as the value depends on *p* does not continue this effort. Also, I note that [20] did not recall the extended version for Bayes theorem correctly as there is a $P(B^c)$ missing from the numerator of the fraction.

(ii) Procedures of rituals: Calculating the probabilities in (2ii)

In the second part of task 2 (Figure 8.52), the students are asked to engage in rituals in order to find the probabilities. Students can use either the statistical tables given to them or the probability mass function for Poisson's random variable.

(ii) Students travelling to the city centre arrive at the (name of the university) bus stop according to a Poisson process of intensity 15 per 10 minutes between 5pm and 7 pm, and of intensity 4 per 15 minutes during the rest of the day.

(a) What is the probability that at least 15 students arrive at the bus stop between 5pm and 5.10pm?

(b) What is the probability that at most 10 students arrive at the bus stop between 9am and 9.30am?

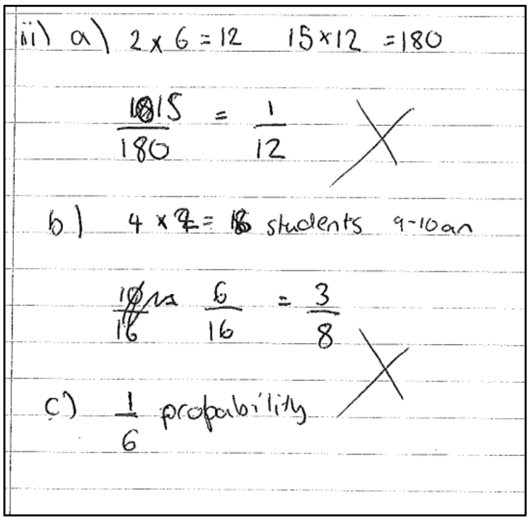
(c) Suppose that no students are at the bus stop at 10.30am. What is the probability that the bus stop will remain empty for a further 6 minutes?

(d) What is the most probable event between: the event A describing 15 students arriving between 5.30pm and 5.40pm; and the event B describing 4 students arriving between 10am and 10.15am?

[12 marks]

Figure 8.52: Snapshot of the first optional task from the Probability part of the module – Task 2ii

The *procedure* is not defined in the wording of the task and the students are expected to make their own choice. In the scripts of three students ([03] [13], [20]) there were *procedures* followed which signalled either conflation between the discourses of probability (Figures 8.53, 8.54) or problematic use of the statistical tables (Figures 8.55, 8.56).



ii) a) $2 \times 6 = 12$ $15 \times 12 = 180$

$\frac{15}{180} = \frac{1}{12}$

b) $4 \times 4 = 16$ students 9-10am

$\frac{6}{16} = \frac{3}{8}$

c) $\frac{1}{6}$ probability

Figure 8.53: Student [03]'s response to (2ii) part 1 – the marker added the two crosses

In the response of [03] to (2ii), there is a conflation between the definition of probability as the quotient of the event occurring over the sample space and the probability mass function of Poisson's variable (Figure 8.53). I view this as conflation between the discourses of continuous and discrete probability. Specifically, [03] starts by calculating how many ten-minute intervals exist in the two hours and then finds how many students would come in those 12 (ten-minute intervals). As the wording of the task asks for only one ten-minute interval which results in 15 students arriving, [03] writes $\frac{15}{180}$. For (2iib), a similar *procedure* is followed but, when writing the quotient, there is confusion as the time frame for the specific intensity is the rest of the day except the two hours (5 pm to 7 pm). [03]'s response to (2iid) is similar (Figure 8.54).

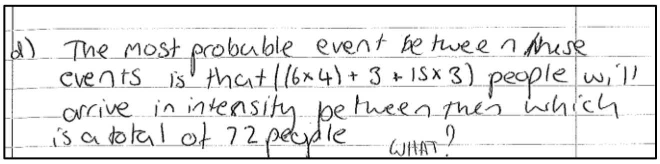
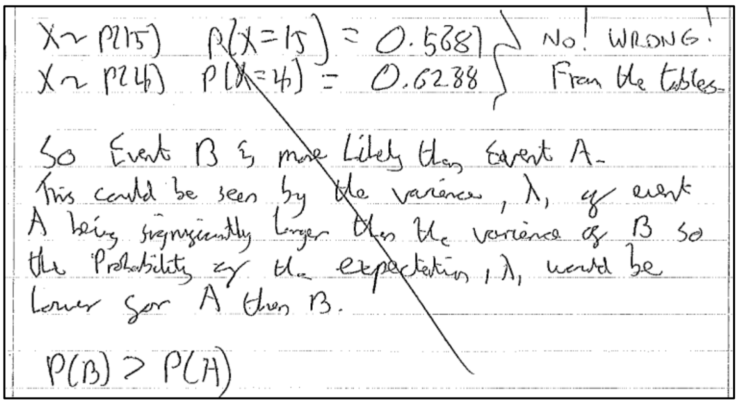

d) The most probable event between these events is that ((6×4) + 3 + 15×3) people will arrive in intensity between them which is a total of 72 people WHAT?

Figure 8.54: Student [03]'s response to (2ii) part 2 – the marker added "WHAT?"

[03] calculates the number of people that would arrive. However, it is not clear how the number resulting to 72 is related to the intensities and the timeframes are given.

The other two scripts ([13], [20]) correspond to students' problematic use of the statistical tables which present the cumulative distribution function instead of the values for the Poisson random variable.

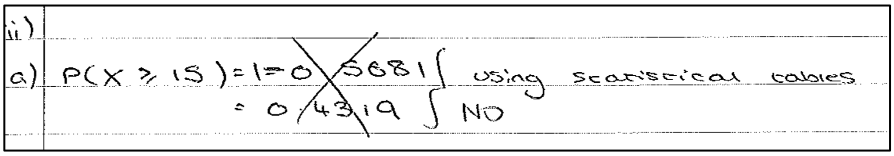

$X \sim P(15)$ $P(X=15) = 0.568$ NO! WRONG!
$X \sim P(4)$ $P(X=4) = 0.6288$ From the tables.

So Event B is more likely than Event A.
This could be seen by the variance, λ, of event A being significantly larger than the variance of B so the probability of the expectation, λ, would be lower for A than B.

$P(B) > P(A)$

Figure 8.55: Student [20]'s response to (2ii) – the lecturer wrote "NO! WRONG!" and made the two lines

In attempting to decide which of the two events is most probable, [20] uses the tables with the cumulative distribution function, without taking into account that these tables provide the sum up to the probability being 15 and 4 respectively (Figure 8.55). This unclear connection between the values asked and the ones given at the statistical tables is also visible in Figure 8.56.



ii)

a) $P(X \geq 15) = 1 - 0.5681$ } using statistical tables

$= 0.4319$ } NO

Figure 8.56: Student [13]'s response to (2ii) – The marker added the cross and wrote "NO"

[13] calculates the probability of at least 15 students arriving at the bus stop. [13] makes the connection between the values given in the statistical table which correspond to probabilities being less than or equal to a specific value, which are complement events and their probabilities add up to 1. However, instead of finding the probability of the event of fewer than 15 students arriving, which is exactly the same as less or equal to 14 students arriving (*F*(14)), [13] chooses the value *F*(15) from the table.

8.3.3 Are the applicability conditions of the routine met?

Here I note cases where the students did either not specify the applicability conditions of the routine they followed or cases where the students assumed different conditions and both cases resulted in them receiving no marks or low marks for their response. The responses of eight students ([01], [02], [03], [04], [09], [08], [17], [20]) to parts of task 1 (Figure 8.57) are illustrating difficulty with the applicability conditions of the routines.

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 8.57: Compulsory task from the Probability part of the module – Task 1

In recalling the first Kolmogorov's axiom, [01] says that $P(A) \geq 0$ with *A* not being the empty set (Figure 8.58). However, by making this assumption, the realisation made earlier ($P(A) \geq 0$) no longer stands, as the probability of the empty set is the one that takes the value zero. In doing so, [01]'s writing signals unclear meaning making regarding the relationship between the values that the probabilities take and the different events which are a subset of the sample space.

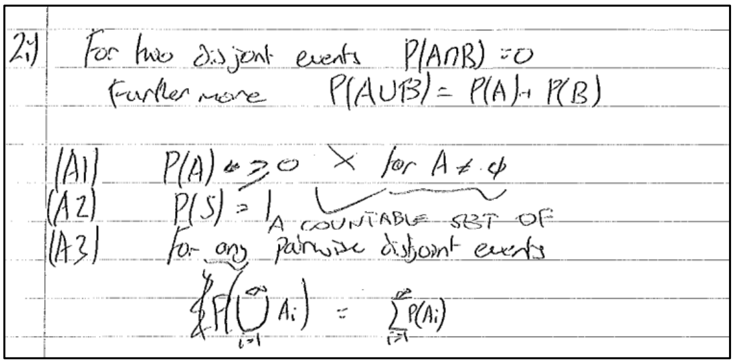

2i) For two disjoint events $P(A \cap B) = 0$
Furthermore $P(A \cup B) = P(A) + P(B)$
(A1) $P(A) \geq 0$ × for $A \neq \phi$
(A2) $P(S) = 1$ A COUNTABLE SET OF
(A3) for any pairwise disjoint events
$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Figure 8.58: Student [01]'s response to (1i) – the marker added the cross, the check symbol and wrote "A COUNTABLE SET OF"

There are also students who use the third Kolmogorov's axiom without examining whether the conditions are satisfied. [20] expresses B as a union of two sets and then uses the third Kolmogorov's axiom (Figure 8.59). However, [20] does not examine whether the sets are disjoint prior to using the axiom.

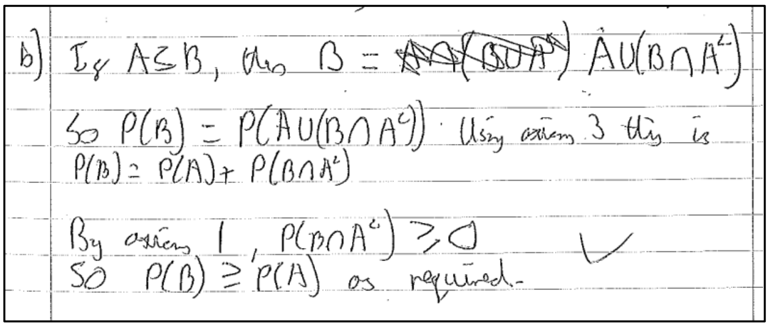


Figure 8.59: Student [20]'s response to (1ib) – the marker added the check symbol

In responding to the same part of the task (1ib), [08] expresses *B* as a union and then, without examining whether the sets are disjoint, uses the axiom (Figure 8.60). Here I note that [08]'s definition of disjoint events, as mentioned in 8.3.1, are two sets that have a non-empty union. The realisation of *B* (e.g., find $B = A \cup (A \cap B)$) that [08] gave is the realisation of *A* as $A = A \cap B$.

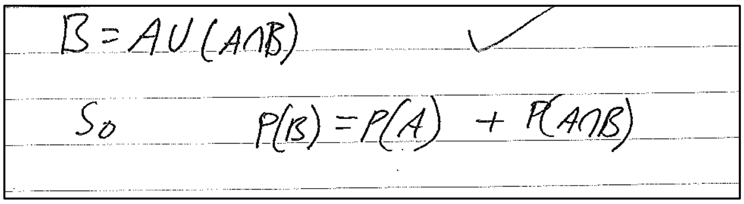
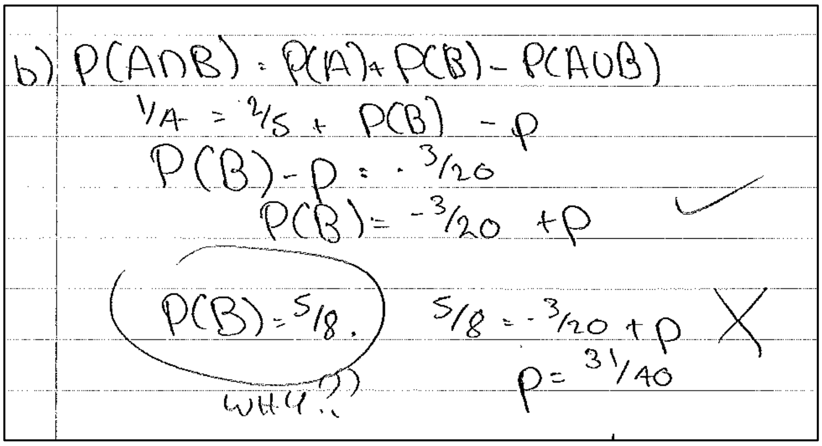


Figure 8.60: Student [08]'s response to (1ib) – the marker added the check symbol.

In the following images from students' responses to (1iib), I report cases where the students ([02], [03], [04], [09], [17]) made assumptions regarding the given sets and then used a routine without examining whether the applicability conditions were satisfied.



b) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$1/4 = 2/5 + P(B) - p$

$P(B) - p = -3/20$

$P(B) = -3/20 + p$

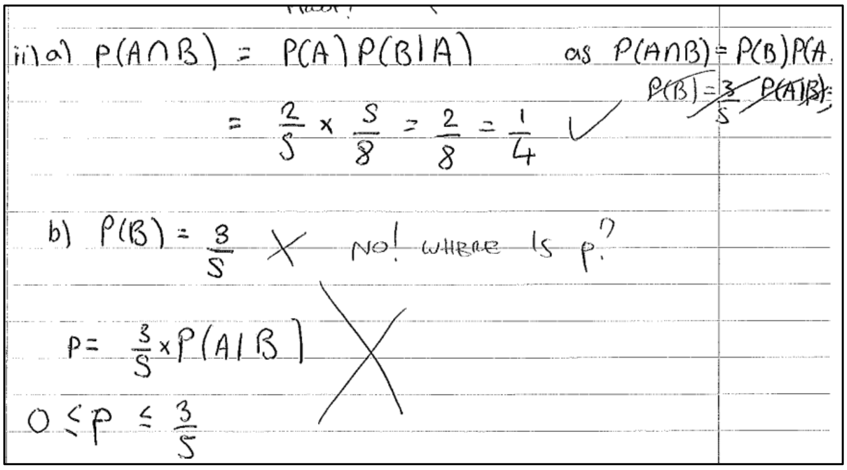
$P(B) = 5/8$. $\quad 5/8 = -3/20 + p$

WHY?? $\quad p = 31/40$

Figure 8.61: Student [17]'s response to (1iib) – the marker added the check symbol, the cross, the circle and has written: "WHY??"

In attempting to find the value for *p,* [17] takes $P(B) = \frac{5}{8}$, which is the case when the events *A* and *B* are independent and thus $P(B|A) = P(B)$ (Figure 8.61). The value of $P(B|A)$ is given in the wording of the task. To be able to use the equality, $P(B|A) = P(B)$ the condition of independency between the two events should have been first fulfilled by examining whether $P(A \cap B) = P(A)P(B)$. [17] does not examine this and takes the value $P(B) = \frac{5}{8}$. This leads to a wrong value for *p.*

In a similar attempt to find $P(B)$ and the range for *p,* [03] takes the events *A* and *B* as complements of each other: $P(B) = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$. However, for the two events to be complements, they would need to be disjoint and therefore the probability of their intersections should be 0. However, this is not the case as seen from the solution to (1iia) which asks to show that $P(A \cap B) = \frac{1}{4}$.



ii) a) $P(A \cap B) = P(A)P(B|A)$ as $P(A \cap B) = P(B)P(A$

$\cancel{P(B) = \frac{3}{5}\ P(A|B)}$

$= \frac{2}{5} \times \frac{5}{8} = \frac{2}{8} = \frac{1}{4}$ ✓

b) $P(B) = \frac{3}{5}$ ✗ NO! WHERE IS p?

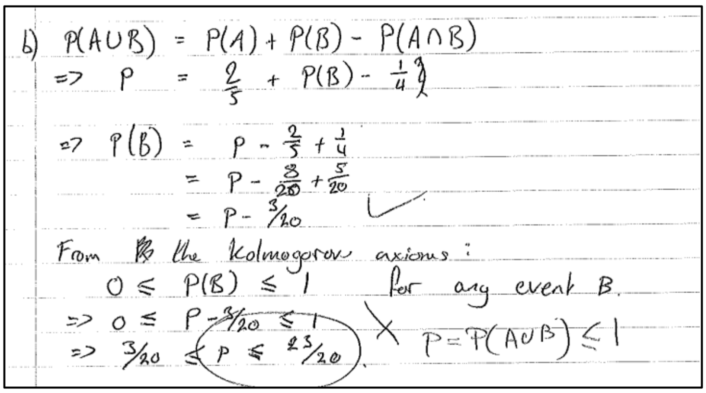
$p = \frac{3}{5} \times P(A|B)$ ✗

$0 \leq p \leq \frac{3}{5}$

Figure 8.62: Student [03]'s response to (1iia) – The lecturer wrote "NO! WHERE IS *p*?" and the two crosses

In the first line of the response shown in Figure 8.62, the multiplication rule was used correctly. However, in the next lines, there is evidence of unclear meaning making regarding the conditional probability as now *p* which is $P(A \cup B)$ is given as $P(B)P(A|B)$.

Another instance where the applicability conditions are not examined is evidenced in six responses to (1iib), where the range for values of *p* is asked. Six students ([05], [08], [11], [13], [15], [18]) do not take into account that *p* is a probability and thus $\frac{3}{20} \leq p \leq 1$. [18] first finds $P(B) = p - \frac{3}{20}$ and, then, uses Kolmogorov's axioms to find the range for *p* (Figure 8.63). However, in this attempt [18] forgets that the way that *p* is defined as $P(A \cup B)$, so the upper limit of the range has to be at most 1.



b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow p = \frac{2}{5} + P(B) - \frac{1}{4}$

$\Rightarrow P(B) = p - \frac{2}{5} + \frac{1}{4}$

$= p - \frac{8}{20} + \frac{5}{20}$

$= p - 3/20$ ✓

From the Kolmogorov axioms:

$0 \leq P(B) \leq 1$ for any event B.

$\Rightarrow 0 \leq p - 3/20 \leq 1$ ✗ $p = P(A \cup B) \leq 1$

$\Rightarrow 3/20 \leq p \leq 23/20$

Figure 8.63: Student [18]' response to (1iib) – the lecturer wrote $p = P(A \cup B) \leq 1$ and added the check symbol and the circle

8.3.4 Are the closing conditions of the routine met?

In this section, I report cases where the closing conditions of the routines are not met and there is a justification deficit in the students' responses. This occurs in nine student responses ([01], [02], ([03], [04], [07], [12], [14], [15], [16]) to tasks (1) (Figure 8.64) and (2) (Figure 8.65).

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $P(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

Figure 8.64: Compulsory task from the Probability part of the module – Task 1

(i)	Let X be a Poisson random variable with parameter λ having probability mass function $P(X=x)=\frac{\lambda^k e^{-\lambda}}{k!}$	

(a) Show that

$$\sum_{k=0}^{\infty} P(X=x)=1$$

(b) By assuming the validity of the relation in (a), calculate $E(X)$.

[8 marks]

(ii) Students travelling to the city centre arrive at the (name of the university) bus stop according to a Poisson process of intensity 15 per 10 minutes between 5pm and 7 pm, and of intensity 4 per 15 minutes during the rest of the day.

(a) What is the probability that at least 15 students arrive at the bus stop between 5pm and 5.10pm?

(b) What is the probability that at most 10 students arrive at the bus stop between 9am and 9.30am?

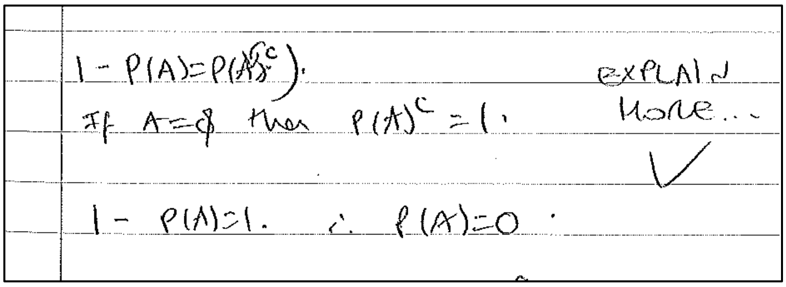
(c) Suppose that no students are at the bus stop at 10.30am. What is the probability that the bus stop will remain empty for a further 6 minutes?

(d) What is the most probable event between: the event A describing 15 students arriving between 5.30pm and 5.40pm; and the event B describing 4 students arriving between 10am and 10.15am?

[12 marks]

Figure 8.65: The first optional task from the Probability part of the module – Task 2

In proving that $P(\emptyset)=0$, five students ([02], [04], [14], [15], [16]) do not provide sufficient justification. For example, [04] (Figure 8.66) writes that the probability of the complement of *A* is $1-P(A)$, then takes the case where *A* is the empty set, making the complement of *A* the whole sample space *S*. However, the student does not explicitly write this, and the marker writes "EXPLAIN MORE…".


$1-P(A)=P(A^c)$.

If $A=\emptyset$ then $P(A)^c=1$.

$1-P(A)=1$. $\therefore P(A)=0$.

EXPLAIN MORE...

Figure 8.66: Student [04]'s response to (1ia) – the lecturer wrote "EXPLAIN MORE…" and added the check symbol

The responses of two students ([03], [07]) do not show their working towards finding the values of specific probabilities. In responding to (1iic), [03] writes the correct values of the probabilities without illustrating how these values were achieved (Figure 8.67).

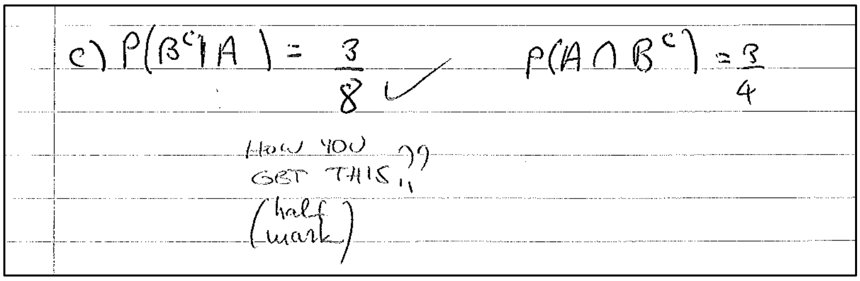


Figure 8.67: Student [03]'s response to (1iic) – The lecturer wrote "HOW DID YOU GET THIS?? (half mark)" and added the check symbol

Also, in responding to (2ii), [07] does not provide explanation regarding the first probability (Figure 8.68). It seems that the student rewrote part of the task for the two first subtasks of (2ii) and then placed an equality after the phrase. For (2iia), the student results in the probability being 1. However, there is no writing that shows how this value has emerged.

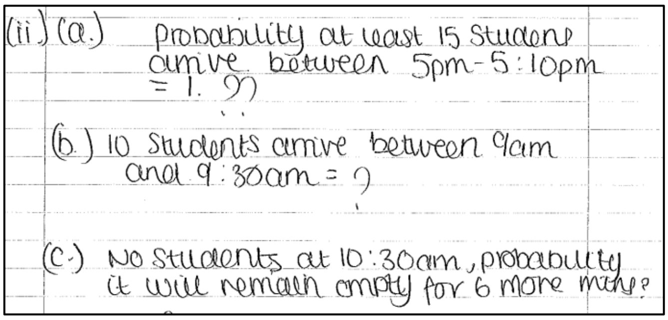


Figure 8.68: Student [07]' response to (2i) – The lecturer added the question-marks in (2iia) and (2iib)

In (2i), students are asked to engage in a substantiation routine. However, the *procedure* is not given in the wording of the task. In attempting to prove that the given sum is equal to one, six students ([01], [03], [07], [12], [14], [15]) either do not provide sufficient justification or, in trying to show that this is 1, they find the limits of the values and show that the sum tends to 1.

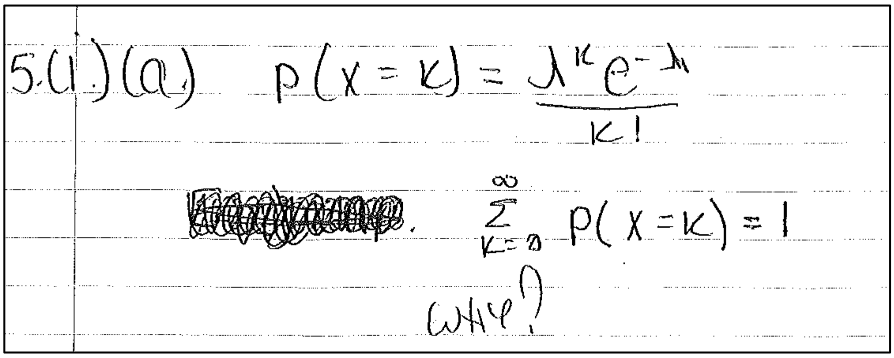


Figure 8.69: Student [07]'s response to (2ia) – The lecturer wrote: "WHY?"

[07] writes the probability mass function, then writes something which is scribbled out and ends with the sum equal to 1 (Figure 8.69). This is not considered as a proof by the marker, and the script receives zero marks. The focus of this part of the task is on the *procedure* of the substantiation routine and, as [07] did not demonstrate this, the closing conditions are not met.

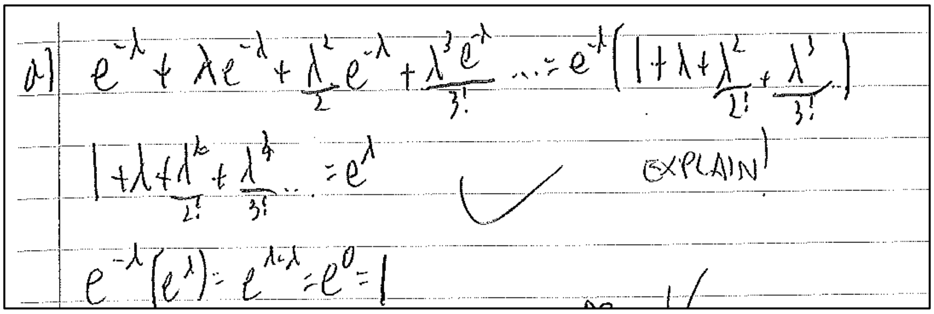
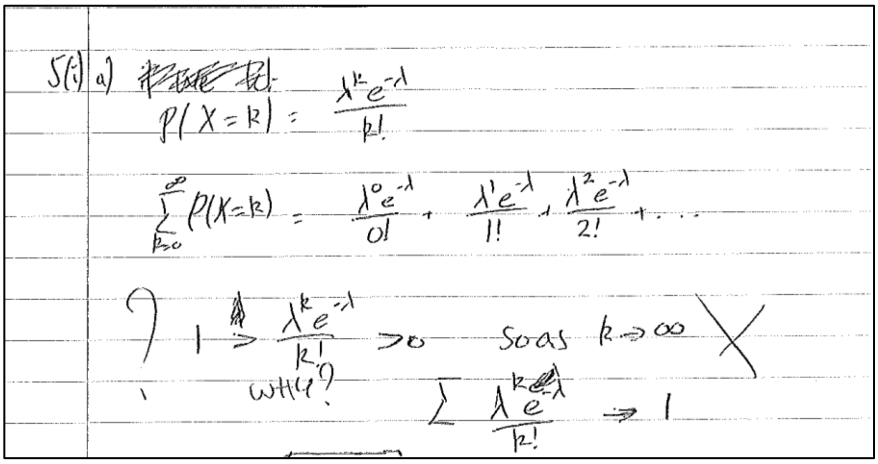


Figure 8.70: Student [14]'s response to (2ia)- The marker added the word "EXPLAIN!!" and the check symbol

[14] uses McLaurin's expansion for e^{λ} in the second line of the response (Figure 8.70). However, [14] does not explain how the connection between e^{λ} and the summation is achieved. The lecturer writes "EXPLAIN!!" to signal the justification deficit.

Finally, [01]'s response to (2ia) shows that, initially, the terms of the sum are written, and the k^{th} term of the sum is bounded between 0 and 1 (Figure 8.71). However, [01]'s writing does not provide an explanation as to why this is the case. After this assumption is made without any explanation, the student writes that "as k $\rightarrow \infty$" the sum tends to 1. The marker does not accept the response, and the student receives zero. Similar cases of attempting to find the limit of the summation are visible in the responses of other students ([03], [12], [15]).


5(i) a) $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$\sum_{k=0}^{\infty} P(X=k) = \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \dots$

? $1 \geq \frac{\lambda^k e^{-\lambda}}{k!} \geq 0$ So as $k \to \infty$

why? $\sum \frac{\lambda^k e^{-\lambda}}{k!} \to 1$

Figure 8.71: Student [01]'s response to (2ia) – the marker added the question-mark and the cross

8.4 Concluding remarks on the analysis of students' scripts in Probability

Here, I summarise the findings from the analysis of the students' Probability scripts. Then, I discuss these findings in relation to those discussed in chapter 7.

8.4.1 Summarising the findings

In this chapter, I examined students' engagement with the mathematical discourse in the three probability tasks. My analysis focuses on the different characteristics of the discourse: use of Venn diagrams, symbols from the discourse of Set Theory and Probability, word use specific to Probability discourse and students' engagement with routines and rituals. The evidence shows difficulties the students have when engaging with the university mathematics discourse as they sometimes resort to rules or word use from the school discourse. The close connection of students' personal mathematical discourses to secondary school mathematical discourses is also observed by Bar-Tikva (2009). Also, the analysis highlighted difficulties when engaging in the discourse of Probability and Set Theory and within the discourse of Probability.

The analysis of the students' scripts illustrated the difficulties that students had when working between the discourses of Set Theory or between the discourse of discrete and continuous random variables (8.2.1). Specifically, this occurs in the responses of eight students (one of them [02] illustrates this conflation in both cases) as mentioned, also, in chapter 6, working within a specific discourse and moving between discourses swiftly and accurately is one of the differences between the school and university discourses.

The difficulty in the naming of the objects involved in the Kolmogorov's axioms is also due to the difference in school and university discourse. In the latter, the students are asked to provide narratives that are rigorous and explain all the symbols used. This is also seen in the students' scripts in chapter 6 (6.2.5) and it is visible in the model solutions provided by the lecturers and in the way that the tasks are set up.

Another signalling of the difference between school and university mathematical discourses is the commognitive conflict regarding the word use independent events (8.2.3) and words from other mathematical discourses (8.2.5). In university mathematics discourse, engagement with different discourses is demanded in different tasks and from their first year in university students are given a chance to see the rules, visual mediators and word use of each of these discourses. In some cases, as in the case of the continuous variable, the engagement with the discourse of Calculus is also asked. However, use of words or visual mediators signals unclear meaning making regarding the connection of the objects of one discourse with the objects of the other. This transition between multiple mathematical discourses was also reported in students' scripts in chapter 6, and is discussed by Mamolo (2010) and Campbell (2006)

Apart from the use of words, visual mediators (symbols and Venn diagrams), the analysis also highlights difficulties with students' engagement with routines (recall, substantiation, and rituals). The analysis of students' narratives regarding disjoint events, conditional probability and the expectation (in both discrete and continuous) and variance (in the case of the continuous random variable only) showed the difficulties students had defining these (8.3.1). These objects are needed in later parts of the tasks and students who are not able to recall these have difficulties in the substantiation routines.

The analysis of students' narratives to substantiation routines, showed evidence of conflation of the meta-rules of the substantiation routine in the university mathematics. Specifically, ten students (8.3.2) either used what was asked to be proven in their proof, or restated the proposition, or provided an example which shows that the proposition works for that one or two cases. However, the proof at the university level is a narrative that can be endorsed by the mathematical community and is governed by meta-rules of the university discourse.

To be able to use a routine, the students have to examine whether the *applicability* conditions of the routine are met prior to engaging in it. In 8.3.3, I report students' narratives where the applicability conditions of either a routine or a relationship between objects are not examined and this has an impact in their solutions.

As shown in section 8.3.4 there are nine students whose narratives did not provide enough justification either when engaging in proving routines (e.g., proving the propositions in (1i) or in (2i)) or when calculating a probability (e.g., in (1ii) or in (2ii)). The closing conditions of the routine are not met according to the marker, and thus the students' narratives are not accepted as complete. These results are in accordance with the analysis in section (6.3.4) and evidence the difficulty the students have when trying to engage with the university mathematical discourse.

Looking at the marks of the tasks, there are visible differences between students' marks in the compulsory and the optional tasks. Generally, the numbers show that the students seem to prefer the tasks from the Sets, Numbers and Proofs. The lecturer of the Probability part of the module has also noted this and discusses it in the interview.

Having discussed the findings from the students' scripts, I now turn to examine the relationship between lecturer's expectations on students' engagement and the analysis of the task, as presented in chapter 7, and the actual students' engagement as presented in this chapter.

8.4.2 Connecting with task analysis and lecturer's perspectives

In section 7.4, I presented the results of the analysis regarding the Probability tasks and L2's interview. The evidence showed that, similar to the tasks from the Sets, Numbers and Proofs part the students are mainly asked to engage in recall and substantiation routines, which are new routines for them. The analysis of the students' scripts showed difficulties with the engagement of both. Specifically, in section 8.3.1, I report cases where the engagement of the recall routines has an effect in the engagement with substantiation routines. In section 8.3.2, the students' narratives illustrate how when the *procedure* of the routine is not given in the wording of the task the students showed difficulties with their engagement with the routines of the university discourse.

The analysis of the tasks and the lecturer's perspectives highlighted that the aim of the lecturer is to create tasks which offer some assistance to the students, either with the stepped structure of the task, the creation of independent sub-tasks or the addition of an extra question regarding the object of disjoint events. However, the analysis of the students' scripts shows that still students report these unresolved commognitive conflicts. Specifically, in the case of the object of mutually exclusive events and independent events which is also reported by Kelly and Zwiers (1986).

Similarly, with section (6.3.1) where students are asked to recall an object or a *procedure* and then use it in a later stage in the task, the students' narratives in the Probability tasks illustrate the same difficulty (section 8.3.1). The routines recall, substantiation and rituals are interwoven. Although L2 tried to make some of the sub-tasks independent (7.1.3), engaging in university discourse requires students to efficiently move not only between different mathematical discourses but also between routines.

Student engagement in tasks in Probability require moves between the discourses of Set Theory and Probability discourse but also between different discourses within the Probability discourse. This is highlighted in chapter 7, and although there are attempts from the lecturer to assist the students in the wording of the task, the results reported in sections 8.2.1, 8.2.3, 8.2.5, and 8.3.1 show that students have difficulty in achieving this shift between and within discourses effectively. Similar results are also reported for the Sets, Numbers and Proofs part of the module.

Compared to the tasks from the Sets, Numbers and Proofs, the tasks from the Probability part of the module do not have directions regarding the extent of justification needed from the students' narratives. However, the analysis of the students' responses show that the students are not yet used to providing narratives where their every step is explained (section 8.3.4) and the symbols they use are introduced (section 8.2.2).

In the next chapter, chapter 9, I discuss the major findings of the thesis from the four analysis chapters corresponding to the research questions, I connect with the literature, reported in chapter 2. I then present the implications regarding the theoretical framework used, the potential use of the findings in professional development and ideas for further research.

Chapter 9. Conclusion

In this chapter, I first present the answers to the research questions and synthesize the results of my study (Section 9.1). Then, I discuss the contribution to university mathematics education research and implications to practice, but, also, concerning the commognitive theoretical framework commenting on my contribution to the theory (Section 9.2) and discussing the advantages and challenges I faced while using the theory (Section 9.3). I, also, report the limitations of my study (Section 9.4), discuss further ideas for research (Section 9.5) and finish with reflections on the journey as a commognitive researcher (Section 9.6).

9.1 Answering the research questions

My study examines students' participation to university mathematics in the closed-book examination setting and offers insight regarding the expected and the actual student participation in university mathematical discourses. Specifically, I aim to answer the following research questions, also described in section (1.1):

R.Q.1 What are the discursive characteristics of the examination tasks?

R.Q.2 What are mathematics lecturers' perspectives on the examination tasks and their expectation from the students' engagement with the university discourse in the closed-book examination setting and how are these perspectives enacted in the formulation of the examination tasks?

R.Q.3 How different are university mathematical discourses from the secondary school mathematical discourses and what commognitive conflicts can be observed as a result of those differences in students' scripts?

In section (9.1.1) I summarise the findings from the two analysis chapters (chapter 5 and 7) to discuss the characteristics of the examination tasks and answer **R.Q.1**. This section is followed by (9.1.2) where I consider lecturers' perspectives on the tasks and their expectation from students' engagement, reported in chapters 5 and 7 to answer **R.Q.2**. Finally, in section (9.1.3), I review the student data (chapter 6 and 8) to discuss the differences between university mathematics and secondary school discourses and to highlight the unresolved commognitive conflicts as presented in the student data aiming to answer **R.Q.3**.

9.1.1 Discursive characteristics of the examination tasks

In this section, I discuss the results of the first research question:

R.Q.1 What are the discursive characteristics of the examination tasks?

The results are discussed considering the themes from the analytical framework as described in section 4.5.4. Initially, I comment on student autonomy, namely instructions regarding the procedure of the routine to be followed and directions regarding the justification expected from the students. I, also, discuss: the presence of visual mediators (algebraic notation) and the types of actions demanded of students, which focuses on determining the mathematical discourses involved in the task; and, examining the routines (rituals, recall, substantiation, construction). In this section, I refer to results from both parts of the module and provide overall comments on the findings from chapters 5 and 7.

The analysis revealed that there are explicit or implicit instructions given to the students in the form of the gradual structure of the task. The explicit instructions involve guidance concerning the procedure to be followed and the justification expected from students' response which restricted students' agency. These instructions are mainly at substantiation routines and they are more present in the compulsory parts of the task. Showing the distinction between the compulsory and the optional tasks in both components of the module. Regarding optional tasks from both parts, only the ones from Probability have an instruction regarding the procedure of one substantiation routine.

Apart from the explicit instructions in the wording of the tasks, there are also implicit directions via the structure of the task. From the analysis of the Probability tasks, there is an apparent distinction between the theoretical and the application part of the task. The first part is always about recalling a definition, propositions, and substantiations whereas the second part of the task is asking students to engage in rituals or substantiations with the mathematical objects involved in the first part of the task. The theoretical part is focused more on the family of the mathematical objects (e.g., Poisson variable or continuous variables) and then in the application part there is focus on one of those objects (e.g., Poisson with different intensities specifying the parameter λ or standard normal variable). Another way that the structure is providing directions is the relationship between the dependent or independent subtasks. As mentioned above regarding the recall and substantiation routines, in some cases the students are asked to recall first mathematical objects or relationships between mathematical objects (e.g., expectation and variance of a continuous random variable or Fermat's Little Theorem) and then engage in other types of routines (substantiation or rituals) where these objects are operated upon. These type of structure makes the second part of the task dependent on the first part of the task, as providing an incorrect or incomplete definition hinders students' engagement in the next stages of the task. However, in some cases the tasks are formulated in a specific way making the subtasks independent (e.g., showing that the probability $P(A \cap B)=1/4$ which is then needed in the next parts of the task or similarly showing that the sum of probabilities in the Poisson random variable is one and then using that to calculate the expectation). This independence between the subtasks is observed only in the Probability tasks.

There are no instructions (or implicit instructions) regarding the procedure of some routines in both parts of the module, as I mentioned previously this is mainly in the optional tasks. This absence of instructions allows students to be creative in terms of the procedure they choose to follow in the specific routine. This creativity in tasks is different from the one mentioned in Bergqvist (2007), Boesen, Lithner, and Palm (2010), Capaldi (2015) and Mac an Bhaird et al. (2017). The creative reasoning operationalised by these researchers and defined by Lithner (2008) discusses students' familiarity with the reasoning demanded from the task. The creativity in the procedure has to do with students' autonomy to decide whether they wish to use the procedure outlined for them in the wording of the task or a different one, familiar to them from their studies either in this module or other modules.

The analysis of the tasks illustrated that the expectations from the students in each one of them were to engage with more than one mathematical discourses at a time. In the Sets, Numbers and Theory tasks the discourses were the following: discourse of natural numbers, integers, real numbers, set theory, and functions. Whereas in the Probability tasks, set theory, probability, integrals, discrete and continuous variables. There is an expectation to not only engage with all these discourses but also be able to flexibly shift between one and another within the response to one task or even to one sub-task. The students are required to change their discourse between different numerical domains but also between different mathematical areas.

In the next part, I focus on the visual mediators and the routines in the tasks. The visual mediators featuring in the tasks are algebraic symbols. These symbols indicate variables that take values from naturals, integers, reals depending on the task. They also have the constraint of being in the range of 0 and 1 when the symbol describes a probability. Regarding the routines, in both sets of tasks, there are mainly substantiation routines, recall, and rituals with one construction routine in Sets, Numbers, and Proofs. I should note here that engaging in a substantiation routine, for example, investigating whether the given functions are surjective or injective or both is dependent on the engagement with the recall routine of those mathematical objects.

Similarly, in the Probability part of the exam, proving the given propositions using the three Kolmogorov's axioms requires engagement with the recall routine initially. In the cases, I just described the structure of the task assists students in providing the definitions first and then using these definitions in the substantiation part. However, there are other parts of the task where this structure does not exist. For example in the parts where students are asked to calculate the conditional probability or use a procedure, as the Euclidean Algorithm. This structure signals the interwoven nature of the routines and the close connection between recalling a realisation of a mathematical object or a procedure and then using it in a substantiation routine or a ritual.

The students are asked to engage in mainly the same routines in the tasks from both parts of the module. However, the instructions either implicit or explicit change the nature of students' involvement in the routines. There are more directions regarding the substantiation routines in the compulsory, and optional Probability tasks and the gradual structure of the tasks make them independent. In contrast, the instructions are explicit in the Sets, Numbers and Proofs compulsory task but in the optional tasks, there are no explicit instructions given.

Examining the compulsory and the optional tasks, the analysis points out a difference in the amount of instructions and guidance given to the students. The optional tasks seem to have less or no instructions both in terms of the procedure, but also concerning the justification expected from the students. This difference is also visible in the different parts of the module. The optional tasks in the Sets, Numbers and Proofs part of the module have no directions whereas in the formulation of two subtasks, in the optional tasks from the Probability part of the module, engaging students in substantiation routines there are still instructions regarding the procedure of the routine. In the next section, I examine the data from the lecturers and discuss how they aim to assist students with the transition from secondary to university mathematics.

9.1.2 Lecturers' expectations from students' engagement with university mathematical discourses and their enactment in the formulation of the examination tasks.

Having discussed the results emerging from the commognitive analysis of the tasks, I now focus on the lecturer data. During the interviews with the lectures, the discussion was initiated using the tasks they formulated for the examination, following the methodology used by other researchers (e.g., Iannone & Nardi, 2005; Nardi, 2008). While discussing the wording of the tasks and their expectations from the students' solutions, the lecturers also commented on the transition of students from secondary school to university. In this section, I answer the following research question:

R.Q.2 What are mathematics lecturers' perspectives on the examination tasks and their expectation from the students' engagement with the university discourse in the closed-book examination setting and how are these perspectives enacted in the formulation of the examination tasks?

This question is answered based on the analysed data, reported in chapters 5 and 7. I also, consider the transitional nature of this first-year module when examining the lecturers' perspectives as enacted in the formulation of the examination tasks. I claim that this analysis provides insight into the lecturers' pedagogical rationale for what they include in the examination task and what they expect from students' responses.

Researchers (e.g., Moore, 1994; Nardi, 1996; Gueudet, 2008; Alcock and Simpson, 2017) have acknowledged the discontinuity between secondary school and university in terms of the mathematical activities (e.g., proving, justifying, defining and proving). The results from the lecturers' perspectives on the way the tasks are posed and their expectation from students' scripts confirm that the lecturers are aware of this discontinuity and they are trying to address this even at the last stage of the first-year, which are the examinations. Also, the interview data illustrates lecturers' awareness of students' difficulties with specific definitions or shifts between different mathematical discourses.

There are specific instructions in the formulation of the tasks and the lecturers' interviews, presenting evidence that the lecturers aim to enculturate their students into the practices of the mathematical community. These include occasions in the tasks where creativity in the procedure of the routine is rewarded and occasions where justification is encouraged and asked by the lecturers. In the first case, L1 talks about the creativity and how this is something valued by the mathematical community and how it is really rewarding to see occasions where the students are using a different procedure than the one expected. Similarly, L2 allows flexibility in the procedure of the routines by not specifying the procedure to be followed in the wording of the task.

Both lecturers also note students' difficulties with proofs. L1 discusses the incident about proving that two sets are equal. This procedure is different from the one they are used to from secondary school. Similarly, distinguishing what is to be proven and what can be used in the proving procedure is another difficulty reported also acknowledged by Moore (1994).

Difficulties with engagement in formal language is another issue that the lecturers discussed. This included the use of logical symbols and logical structure (e.g., Chellougui, 2004a; 2004b) but also use of symbolism in general (Mamolo, 2010; Epp, 2011, Kontorovich, 2018). Especially, this was visible in L1's view regarding students reproducing the definition of injective and L2's additional request in the compulsory task for the definition of exclusive events aiming to assist students in using that definition later in their writing of Kolmogorov's axioms.

The differences between school and university also included students' difficulties with constraining their discourse within a specific mathematics discourse. In all the tasks, as mentioned earlier, students are asked to shift between different mathematical discourses. The lecturers' data present evidence that they are expecting some students to conflate these discourses and thus they are aiming either with the directions mentioned above or with the structure of the task or the introduction of key visual mediators (algebraic symbols) to avoid this conflation. For example, from the Numbers, Sets and Proofs this is visible in the compulsory task where the lecturer provides the symbols to be used in the proof about the divisor where there is a possible conflation between the discourse of real and integer numbers. In the Probability part of the exam, this is visible in the hints being given regarding the integration in the second optional task which connects Probability and Integral calculus.

Another practice which is new for the first-year students, if not entirely new at least to the formality being requested in this case, is justifying. The instructions call the students to justify their choices. However, apart from the justification required, L1 talks about how the extent of justification varies according to the mathematics discourse (e.g., the case of Sets, Numbers and Proofs extent of justifying that a function is injective, and the same task being asked in an Analysis module).

There is also the expectation that some procedures (e.g., calculating the greatest common divisor or finding the conditional probability) are very familiar to the students compared to the rest of the routines. These procedures are not explicitly given to the students. The students have hints that they could follow, but the lecturers decide to leave it up to them to decide which procedure they want to use.

Another practice visible from the interviews is that L2 mainly discusses the creation of independent subtasks on purpose, with the aim to assist students in continuing with their solutions in case they were not able to answer the first part of the task. The analysis from chapters 5 and 7 show that this is different for the two parts of the module, with the Probability part having more tasks that are independent and the Numbers, Sets and proofs having more which are dependent.

Finally, L2 seems to want to start changing the assessment practices slightly by his addition of the subtask in the first optional task. This subtask is a task closely connected to the students' everyday life, as it involves a bust stop at their university. The lecturer mentions that he wants to show to his student where these objects that they think are abstract and formal can be applied in an everyday example. He wants to connect mathematics with an application in the real world and illustrate the applicability of the objects. In the next part of the chapter, I turn to the students' solution to summarize the results from the students' engagement in university mathematics discourse.

9.1.3 Unresolved commognitive conflicts in students' scripts

Having discussed the formulation of the tasks (section 9.1.1) and lecturers' perspectives on these focusing also on the transition from school to university and their expectations from students' engagement with the discourse (9.1.2), I now turn to the students' actual participation in the university mathematical discourses and answer the final research question.

R.Q.3 How different are university mathematical discourses from the secondary school mathematical discourses and what commognitive conflicts can be observed as a result of those differences in students' scripts?

Prior to discussing the results to this research question, I want to emphasize the importance of the context and how these results might be different in varying contexts as routines like proving, recalling definitions are encouraged and fostered in secondary schools in other countries. I now turn to the first part of the research question, the discursive characteristics in the students' responses. In the analysis chapters, concerning students' scripts (chapter 6 and chapter 8), I examined the characteristics of the mathematical discourse in terms of word use, visual mediators, routines and production of narratives. I then focused on occasions where the use of words, visual mediators and routines signalled unresolved commognitive conflicts. This was examined specifically in occasions where the use mathematical terminology, logical symbols, algebraic symbols, graphs, and Venn diagrams signalled the incompatibility of discourses.

I see, the use of mathematical terminology (words) which is not compatible with the mathematical discourses required to be used in the task, as commognitive conflict. Similar to word use, use of visual mediators (both algebraic symbols, plots, and Venn diagrams) which illustrates conflation of discourses are visible in the students' responses. This unresolved commognitive conflict concerns students' ability to first identify and then work consistently within the relevant numerical context or mathematics discourse (e.g., Set Theory, modular arithmetic, Probability of continuous random variable).

Another manifestation of unresolved commognitive conflict occurred in the use of procedures which were not compatible with the discourse that the students were being asked to use. This commognitive conflict occurred when students used procedures from the secondary school discourse. For example, resorting to proving using an example or proving without using the definition.

Another aspect is the interplay between different mathematical discourses. This difficulty is also reported in the literature (Niss, 1999; Campbell, 2006). Within this difficulty, there is another one which has to do with aspects of the procedure of the routines. These aspects are the closing and applicability conditions of routines. The students' scripts illustrate that students choose to apply a procedure of a routine without always examining the applicability of the routine or examining the satisfaction of the closing conditions of the routine. Engagement in these practices is not something that the students are used to from the secondary school. These difficulties which appeared in students' responses are occurring either between the combinations of several discourses in horizontal level or in vertical level (Tabach & Nachlieli, 2016) or using the terminology from Nardi (1999) between inter and intra university courses.

From the analysis, some characteristics of the university mathematics discourse are made more evident. I now discuss these both in terms of the routines but also the word and visual mediator use. In university mathematics discourse, the routines are interwoven. Specifically, from the students' scripts it showed how by not recalling the narrative needed either in terms of procedure or in terms of definition, the students were not then able to engage in the substantiation part of the task. Similarly, there were some tasks which only required substantiation by first glance. However, upon engaging with it, the complexity of the university mathematics is shown as there is a constant interplay between recall and substantiation.

Finally, regarding the word and visual mediator use, students' scripts illustrated another difference between school and university mathematics. This is precision in engagement with formal language. Specifically, this includes consistency in the naming of the variables and clarifying the numerical context and using logical expressions illustrating the relationships between various mathematical objects.

9.2 Contribution to knowledge

9.2.1 Contribution to university mathematics education research and implications to practice

My study investigates students' participation in university mathematical discourses focusing on the first-year final examination tasks. The results of my study contribute in three main aspects to the university mathematics education research, namely: regarding analysing examination tasks and contributing to literature that characterises tasks; in terms of providing further insight into lecturers' practices and finally in terms of examining the transition from secondary school to university. In this section, I first discuss the contribution of my study to each of these three areas of UME research, and then I consider how the results of the study can be used to inform practice.

My study focuses on a module that involves a variety of mathematical discourses (e.g., Probability, Modular Arithmetic, Number Theory and Set Theory). The majority of the current literature on task analysis focuses in investigating examination tasks from Calculus modules (e.g., Pointon & Sangwin, 2003; Bergqvist, 2007; White & Mesa, 2014; Tallman et al., 2016; Mac an Bhaird et al., 2017). The results of the task analysis from this module, illustrate that there are differences between the tasks of the different parts of the module. These results are in accordance with the results reported in Griffiths and McLone (1984b) who show that there are differences between the modules that they analysed.

The studies focusing on investigating mathematical tasks take into account only the examination tasks and the module material (e.g., Griffiths & McLone, 1948b; Smith et al. 1996; Pointon & Sangwin, 2003; Boesen et al., 2010; Darlington, 2014; White & Mesa, 2014; Mac an Bhaird et al., 2017) or take into account lecturers' perspectives using mainly a survey (e.g., Capaldi, 2015; Tallman et al., 2016). These studies report lecturers' perspectives without providing them with a specific frame of discussion (e.g., Capaldi, 2015; Tallman et al., 2016). As in Bergqvist's (2012) work, my study contextualises the lecturers' interviews further by asking specific questions on the examination tasks that these lecturers formulated and chose to include in the final first-year examination. Thus, allowing insight into the lecturers' perspectives regarding students' expected engagement with the mathematical discourse. Furthermore, I also investigated the students' scripts to these examination tasks, and I analysed them to examine students' actual engagement with university mathematics discourse. In summary, my analysis offers a characterisation of students' expected engagement in the university mathematical discourse which is given by examining both the tasks and lecturers' perspectives on these, but also students' actual participation by investigating students' written answers to the tasks. This adds to the existing literature on examination tasks which mainly focus on the characterisation of the tasks without considering students' responses to these or lecturers' perspectives on these tasks (e.g., Griffiths & McLone, 1948b; Smith et al. 1996; Pointon & Sangwin, 2003; Boesen et al., 2010; Darlington, 2014; White & Mesa, 2014; Mac an Bhaird et al., 2017).

The results concerning the expected participation, show that by providing instructions regarding the procedures of the routines in the wording of the tasks, the lecturers are restricting students' agency. The lecturers also commented on how the structure of the task itself, the presence of symbolic mediators and the instruction regarding the justifications expected from the students are aimed at assisting students to transition from secondary to university mathematics smoothly. These insights allow characterisation of lecturers' pedagogical practices mainly regarding the closed-book examination setting.

These insights contribute to the growing literature regarding lecturers' practices (e.g., Biza et al., 2017; Nardi & Winsløw, 2018). The analysis of the lecturers' interviews offers insight into their assessment practices and pedagogical rationale showing why they chose to include specific instructions or visual mediators, why the formulated the tasks in this way, and what they may expect from students' responses. These results agree with Bergqvist's study (2012). Bergqvist also observed that lecturers take into account students' familiarity with the task, the course content and prior knowledge (Bergqvist, 2012). However, my study also discusses how these are addressed in the wording and structure of the task aiming to assist students.

The findings of my analysis also highlighted difficulties regarding students' transition from secondary school to university mathematics. These results deepen our insights into teaching and learning during the first year of university mathematics. The study of the transition from secondary to university has been examined by researchers investigating a variety of aspects of teaching and learning (e.g., Gueudet et al., 2016). However, apart from Darlington's work (2014) which examines both university examination tasks, and secondary tasks, no other studies are looking at the transition using examination tasks. My study adds to the existing literature regarding the secondary to university transition considering both the expected student participation (investigating the tasks and lecturers' perspectives) but also at the students' actual participation (examining students' written responses).

As mentioned above, the results of my study could be used to inform further research that examines the transition between secondary and university mathematics. However, my adaptation of the Morgan and Sfard (2016) framework to examine students' engagement as well as examination tasks, could also be used to examine the transitions between other mathematical areas with which undergraduate students are asked to engage with in further years of their study.

Additionally, the results of my study show that the students are faced with unresolved commognitive conflicts at the stage of the final year examinations. This calls for rigorous and explicit attention to the differences between the secondary and university discourses, during term time. This attention could also be accompanied by alerting the students regarding the importance of identifying and consistently using a mathematical discourse but also being able to move between varying mathematical discourses easily.

The commognitive analysis of the tasks and also the students' solutions highlight the different mathematical discourses and allow their characterisation. Having this break down of the different mathematical discourses can be a useful tool for the lecturers to illustrate the emphasis they provide, or they might want to provide while teaching. Commognition "highlight(s) details of mathematical discourse (taken broadly) that have significant explanatory value" (Tabach & Nachlieli, 2016, p. 429). Having this tool to distinguish between the rules of the discourses and the different terminology and symbolism used by students can prove extremely useful.

Another way that the results of my study can be used could be in creating teaching material or creating resources to be used in professional development sessions for lecturers. Advantages of using this analysis would be that the identification of the shifts in discourses can provoke ideas about designing teaching material (either in the form of changes in lectures or in the form of different coursework and formative or summative assessment practices) that might assist in the awareness of the differences between these discourses and facilitate a smooth transition to these. This methodology is being used in research projects at secondary school (Nardi, Biza & Zachariades, 2012) and at the university level (Nardi, 2008).

Additionally, the results can be used at development sessions with the undergraduate students. Studies have shown that asking students to assess the work of their peers, helps them to improve (Jones & Alcock, 2014). By illustrating the analysed data to the students and getting them to consider the differences between various mathematical discourses in the form of a solution provided by one of their peers, during the term time, might be useful for them. The selected solution could illustrate a case where the discourses are not appropriately connected. A discussion regarding why this particular written communication reached a breach can follow the presentation of the selected solution. This will allow discussion with the students about these shifts prior to the examination stage. Güçler (2013) highlighted that students faced difficulties when there were implicit shifts in the lecturers' discourse. By using these solutions, the shifts can become more explicit.

Another way that the results of the study can be used is to request from the lecturers to code the examination scripts produced by the students. This is a methodological approach used by Iannone and Nardi (2005) and Nardi (2008) where the interviews with the lecturers and the discussion was triggered from students' responses. However, in this case, it would be a combination of this methodological approach and the approach used by Schoenfeld and Herrmann (1982) and Bergqvist (2007). These researchers asked lecturers to categorise the examination tasks (Schoenfeld and Herrmann's, 1982) and provided them with a framework and asked them to classify the reasoning, imitative and creative in the specific tasks (Bergqvist, 2007).

9.2.2 Contribution to the commognitive theory

Having discussed the contribution of my study to the UME field, I now turn to the contribution to the theoretical framework. Sfard's theory of commognition (2007), is about 10 years old and has been used in a variety of settings (primary, secondary and tertiary) and different countries (Nardi et al., 2014; Tabach & Nachlieli, 2016).

Recent publications of special issues focus on the use and elaboration of the framework. This elaboration focuses in examining the application of the theoretical framework to investigate various aspects of the mathematical discourse and explore the development of this discourse (Sfard, 2012). Also, other studies investigate the use of the framework at university level with a focus on the discursive shifts in Calculus discourse at the early years of university both from the lecturers and students' points of view (Nardi et al., 2014). Furthermore, research examines further the potential of the framework in different aspects of teaching and learning mathematics (Tabach & Nachlieli, 2016). Moreover, focusing specifically on the assessment practices, a recent issue reports the creation of an analytical framework to examine students' participation in the mathematical discourse looking at GCSE examinations over 30 years (Morgan & Sfard, 2016). Finally, in an issue that is currently under publication, the notions exploration and ritual are further elaborated (e.g., Heyd-Metzuyanim et al., 2018; Lavie et al., 2018; Nachlieli & Tabach, 2018; Viirman & Nardi, 2018).

The latest publications using the framework illustrate the insight that can be given using this nuanced way of analysing the mathematical discourse but at the same time call for further elaboration of the various theoretical constructs of the framework in different settings. This is where the contribution of my study lies. My study contributes to the further discussions and applications, at the university level, of the notion of commognitive conflict in the university mathematical discourses (Thoma & Nardi, 2017; Thoma & Nardi, 2018a; 2018b). This way the analysis of transition can be observed as an analysis of the distinct discourses that come into play in first-year students' responses. Apart from an analytical tool this notion also can be used to contribute to intervention studies, I explore this idea further in section 9.5. Furthermore, I also provide another version of the analytical framework as described in Morgan and Sfard (2016) which I adapted to the university setting. Finally, based on that analytical framework for examination tasks, I created an analytical tool to investigate unresolved commognitive conflicts in the students' responses to closed-book examinations tasks.

The nuanced theoretical framework of commognition allows a characterisation of the different mathematical discourses and thus offers insight into occasions where the incompatibility of these discourses generate conflicts. These commognitive conflicts are core moments in students' participation in the mathematical discourse. However, if these conflicts have not received enough attention at a stage prior to the examination then they remain unresolved and can occur again in the next parts of the students' studies. I discuss the importance of this in the previous section (9.2.1). Here, I want to focus more on how the framework allows both researchers but also lecturers to characterise these discourses and then examine the shifts between discourses maybe at an earlier stage prior to the examinations.

In the literature review section (2.2), I discuss studies which are analysing examination tasks using different frameworks. As reported in that section, most of the studies are mainly looking at Calculus modules. In my study, I focused on a module that employed a variety of mathematical discourses. This choice of module highlighted further the contribution of the theoretical framework. This is not only to say that the results would not be providing further insight if the module was only focusing on one mathematical area. One of the intricacies of mathematics is that the mathematical discourse expands both vertically and horizontally and looking at the discourses which are subsumed in the discourse of Calculus would surely illustrate findings of similar interest. Equally the investigation of students' participation in the Calculus discourse using the solutions they produce to Calculus examinations would additionally provide insights into either the transition from secondary to university, if the module was a first-year Calculus module or insights in the next stages of the transitions occurring in further years of study during the mathematics degree.

A further contribution of my study both in the theoretical framework but also in the methodological aspects of studies investigating examinations tasks was the use of lecturers' solutions in deciding whether a routine could be considered a ritual or an exploration routine. This categorisation of the tasks usually relies on the researchers' opinion of the task (e.g., Griffiths and McLone, 1984, Smith et al., 1996; Galbraith & Haines, 2000; Pointon and Sangwin, 2003; Bergqvist, 2007; Tallman and Carlson, 2012; White and Mesa, 2014; Darlington, 2014; Tallman et al., 2016). However, Lithner's framework of imitative and creative reasoning (Lithner, 2008), utilised in Bergqvist (2007) study takes into account the occurrence of each of the tasks in the textbooks used in the Swedish universities. In the context of this study, the resources mainly used by both lecturers and students were the lecture notes either provided in an electronic format by the lecturer or the ones that the students were keeping while attending the lecturers. In my analysis, I took the electronic notes and the exercises given to the students during the course of the module. However, I mainly focused on the lecturer's solution created for departmental use. These solutions though are highly contextualised. This practice of producing the solutions for departmental use is widely used in the UK context, but this is not necessarily the case in other countries. Having access to the solution produced by the lecturer gave me the chance of understanding further the lecturer's expectations on students' participation to the university discourses in the setting of closed book examination tasks. It also allowed me to characterise the routines as rituals or explorative routines.

After extensive analysis of my data, focusing mainly on tasks and students' scripts, I adapted the framework proposed by Morgan and Sfard (2016) aimed at investigating changes in students' participation in the mathematical discourse. I focused mainly on some of the themes of this analytical framework (as reported in 4.5.4) the others are examined but presented briefly and further discussed at the summaries (6.4 and 8.4). For example, regarding the student-author relationship, I show evidence that students use the personal pronoun "we" in their responses (6.2.5, 8.2.5), which is also observed by Morgan (2006). Similarly, the lecturers are using this personal pronoun in the interviews, and they refer to a variety of groups: the community of mathematicians, or the lecturer and the students (section 5.1.3 and 7.1.3). These results are in accordance with Rowland (1999; 2003) who examined the role of pronouns in the classroom discourse.

Due to the nature of the examination tasks at the university level, I did not examine aspects of the Morgan and Sfard (2016) framework in the task analysis. The presence of other human beings was not relevant in this context as the only instance where this was the case was in the Probability first optional task (task 2), where the lecturer with the formulation of the task aims at providing a “real world” application of Poisson’s random variable. The specialisation and objectification are two more themes that even though I did not focus on the analysis of the examination tasks, I used them in my analysis of the students’ written scripts. My analysis of these is in accordance to the literature which discusses the level of abstraction, objectification and specialisation at the entrance to the university mathematics (De Guzman et al. 1998; Gueudet, 2008; Sfard, 2014).

Furthermore, having adapted the Morgan and Sfard (2016) framework to also look at students’ responses provided a way to gain more insight into students’ transition as this adapted analytical tool assisted in examining both the tasks, lecturers’ perspectives on the tasks and their expectations on students’ responses. This adaptation allowed for a discussion both of the needed transitions between secondary and university mathematics which happen at vertical level but also between different university mathematical discourses with shifts occurring either at a vertical or a horizontal level.

Additionally, my study provides a characterisation of the university mathematical discourses. Sfard elaborating on the features of the university mathematical discourse as presented in the special issue (Nardi et al., 2014) reveals that this discourse is extremely objectified, relies on rules promoting analytic thinking and is exceptionally rigorous (Sfard, 2014, p. 200). My study adds on to the literature regarding university mathematical discourses as it elaborates further these characteristics and examines how and whether first-year students are able to shift from the secondary to the university mathematical discourses smoothly. The mistakes that are being made at the students’ scripts are viewed as unresolved commognitive conflicts which are due to the incompatibilities between the secondary and university discourses.

Another important aspect that emerged from the findings is the role of the context at the micro level of the context surrounding the task but also at the macro level the institution in which these examination tasks were used. Previous researchers (e.g., Smith et al., 1996; Tallman et al., 2016) analyse the tasks without explicitly taking into account the context in which these were posed. However, studies like Bergqvist's (2007) and the analysis of the tasks presented here take into account the context. Regarding the micro level, the previous tasks that these students have seen are playing a huge part in their engagement with the current tasks. This importance of the previous engagement of the learner with similar situations is illustrated in Lavie et al. (2018). There the discussion about routines and tasks is interrelated with the precedent situations. Lavie and colleagues provide a new definition for routines which is "tied to a particular task situation and to a particular person" (Lavie et al., 2018, p. 9). They then turn to discuss what happens when a learner is asked to engage with a new task.

> "if a person who finds herself in a new task situation is actually able to act, it is mainly to her previous experience. More specifically, she can perform because the current task situation harks back to precedents – to past situations which she interprets as sufficiently similar to the present one to justify repeating what was done then, whether it was done by herself of by another person." (Lavie et al., 2018, p. 8)

Finally, the idea of the inter and intra conflict discussed in Nardi (1999) can also be seen using the commognitive theory as horizontal meta-level learning and vertical meta-level learning (Tabach and Nachlieli, 2016). The analysis of students' solutions allowed for a discussion of these shifts in students' discourses.

Having discussed the contribution of my thesis to the research of University Mathematics Education, the implications of my results to practice, and the contribution to the commognitive theory. I am now turning to the advantages and challenges of using commognition.

9.3 Advantages and challenges of using commognition

In this section, I discuss the advantages and challenges that I was faced with when engaging with the theory of commognition.

One of the advantages in using the commognitive framework as an analytical tool in my study was that it offered a rich description of the discourses that are at play when discussing the question itself and also when examining in detail the students' solutions. The analytical framework that Morgan and Sfard (2016) introduced provided a guide for analysis concerning the mathematical and subjectifying aspects of examination discourses at the secondary school level. The framework (also presented in 12.5) offers a classification of aspects of the discourse, questions that guide the analysis and textual indicators which were helpful during the analysis process. These questions and textual indicators which were designed for the examination questions were the ones that guided, my adaptation of the analytical framework (as described in section 4.5.4) for the students' scripts and allowed characterisation of these aspects of the discourses present in students' scripts.

The depth of the analytical framework allowed a characterisation of the mathematical discourses that the students are expected to engage with and the students' actual engagement. This analysis highlighted the connections between the mathematical discourses that are intertwined in the questions but also in students' responses. Additionally, it allowed to examine in detail the connection between recall and substantiation routines (e.g., section 6.3.1), the procedures expected, and the ones enacted by the students (e.g., sections 6.3.2, 6.3.3, and 8.3.2), and the degree that the closing conditions were not met in students' solutions (e.g., sections 6.3.4 and 8.3.4). Finally, it allowed discussions in students' use of symbols and words that were incompatible with the lecturers' expectations (e.g., 6.2.2, 6.2.3, 6.2.5, 8.2.1, and 8.2.3). I should note that the word use available from the theory allows for this kind of detailed analysis.

In the frameworks discussed in detail in section 2.1, there is the characterisation of some tasks asking students to engage in the recall of theorems or facts about mathematical objects. However, using this framework, the intricate nature of recall routine was further examined in the students' responses. Sfard discusses that engagement in a recall routine provides information as to how this recalled narrative was initially memorised by the learner (Sfard, 2008, p. 234). The reproduction of a definition or a theorem is an exploration. It is an act of recall, but it is an exploration in the sense that the student can recall part of the definition or the theorem and have to explore how these objects that are part of the definition are connected and what these relationships are.

However, I should also note some challenges that I was faced with when using the commognitive framework. The data of my study are mainly documents (either tasks or answers to these) which are the product either of lecturers' engagement with the assessment discourse or students' engagement with mathematical discourses. The nature of the data makes it harder to gauge whether the distinctions between deeds, rituals, and explorations are the appropriate ones. Specifically, the characterisations of the routines of the tasks were based on the lecture notes and the model solutions of the lectures. However, the characterisations were not made by the lecturers themselves. It is challenging to distinguish whether a task is an exploration, ritual or a deed for a student just by examining the written solution. Lecturers' perspectives, the context of the module and the model solutions from the lecturers, were the ones that assisted in the characterisation of the routines in the tasks. However, in my study I was not focusing on the development of discourses, this can be considered by examining additional material produced by the students during the academic year (e.g., Ioannou, 2012). My data examined snapshots of the students' activity which of course was missing the dynamic environment. However, it allowed for the discussion, which is the focus of my thesis, on expected and actual student participation to the university mathematical discourses.

Commenting on the mathematical discourses at play, was another challenge that I was faced with when analysing students' scripts. The mathematical discourses are highly intertwined, others are connected horizontally and others vertically. This relationship between the various discourses made the analysis very intricate and time-consuming. Aiming to ensure that the appropriate depth was reached, I shared initial findings of my analysis at various conferences (see section 11 for the publications of preliminary results of my analysis) and discussed these with colleagues who were (or not) users of the commognitive framework. Other researchers have identified these challenges when using the theory of commognition, and there is a current project lead by Sfard in which tools and methods of how to deploy the commognitive framework will be discussed. This project is currently in development[1]

Since 2008 when the theoretical framework was originally presented in Sfard's book, many researchers have deployed it and developed it further. However, as Nardi and colleagues note with their review on studies using the commognitive framework the studies "merely scratch the surface of the potentialities within the commognitive framework" (Nardi et al., 2014, p. 195). The potency of the framework is also noted by Presmeg (2016). She notes that the framework allows a discussion of the "details of mathematical discourse (taken broadly), that have significant explanatory value" (Presmeg, 2016, p. 429) and that "the commognitive theoretical framework still has much unrealized potential to be useful in mathematics education research at all levels" (Presmeg, 2016, p. 430). Commognition is still a recently introduced theory, and more uses of the theory are needed to illustrate the full potential and also elaborate further the notions described in the original book by Sfard (2008).

Recent articles aim to further the elaboration of the theory in more contexts and the elaboration of the theoretical constructs providing a very fruitful direction for research using commognition (e.g., Heyd-Metzuyanim et al., 2018; Lavie et al., 2018; Nachlieli & Tabach, 2018; Viirman & Nardi, 2018). In these articles, aspects of the theoretical framework are further elaborated and exemplified in various educational contexts. As the studies reviewed in Nardi et al (2014) my study also aims to "assist towards the 'reification' of the framework's potent analytical procedures into tools that can generate grander and broader analyses (Nardi et al., 2014, p. 195). My study aims to contribute in that emergent field with the contribution regarding the notion of commognitive conflict, the discussion of the discontinuity between discourses, the application of the Morgan and Sfard

[1] Sfard discusses this project at Research Gate https://www.researchgate.net/project/Guide-for-a-perplexed-researcher-OR-How-to-do-discursive-research

(2016) analytical framework at the university level, and the extension of the framework to include students' scripts (section 4.5.4).

Having discussed the advantages and challenges of my use of the commognitive theory, I now turn to the limitations of my study.

9.4 Limitations

In section 4.4.5, I discuss the methodological limitations of my study. Here, I turn to the limitations concerning the study. The nature of my research is qualitative and the results from a qualitative study are not generalizable, as the context of the study plays a huge role in the results of the analysis.

My study provides an adaptation of the Morgan and Sfard (2016) framework by adding another layer in the framework looking at the students' scripts aiming to identify unresolved commognitive conflicts. This framework can be used to analyse qualitative data and can be adapted in other contexts. Further categories might be needed to address data coming from other modules offered in other years of study.

Additionally, in the methodology section (4.5.4), I briefly discuss some of the categories of the original Morgan and Sfard (2016) framework which I did not examine in my analysis of the tasks. These aspects (e.g., objectification of discourse) provide insight into the objectification and specialisation of the discourse. However, the contribution of these can be highlighted when tasks from various modules from the same university or when tasks from similar modules coming from various institutions are compared. This could be addressed by further research aiming to examine the transitional modules offered at various institutions or aiming to characterise the level of objectification and specialisation of the tasks from the same module over a particular period of time. There are studies that show how institutions offer “bridging courses” to assist in the transition from school to university (Kayander and Lovric 2005; Biehler et al. 2011). These courses could be analysed and the themes reported above could be examined to highlight the level of objectification and specialisation of the discourse expected from the students.

Another limitation of the study is that the characterisations of the routines of tasks were based on the lecture notes and the model solutions of the lectures and they were not characterised by the lecturers themselves. The characterisation by the lecturers has been done by researchers such as Schoenfeld and Herrmann's (1982) and Bergqvist (2012). In the next section, I discuss this idea as I would like to examine the potentiality of the analytical framework further and examine lecturers' views on this categorisation. Similarly, the characterisation of the sub-tasks in rituals and substantiation routines is providing a limited picture of the routines the students are expected to engage with. In every sub-task, more than one routine are required, and these routines are interwoven as mentioned in the analysis chapters. It is challenging to distinguish whether a task is an exploration, ritual or a deed for a student just by examining the written solution. Lecturers' perspectives, the context of the module and the model solutions from the lecturers, were the ones that assisted in the characterisation of the routines in the tasks.

The study aimed to investigate students' participation in university mathematics discourse, as this can be seen in their final year examination scripts. I note that focusing on final year examination scripts is evidence to what these students have been doing the whole year. However, in order to examine further this transition, the analytical framework could be used earlier to characterise the tasks which the students see during the module either as part of their lectures, or the exercise sheets and the coursework. A similar analysis of the students' scripts to the exercise sheets and coursework would provide insight as to how these commognitive conflicts can be raised earlier and thus be avoided at the final examination stage.

The literature on examination tasks the research examines a large number of tasks (e.g., Tallman et al., 2016). However, in my analysis, I am using the tasks and the lecturers' perspectives to gain insight in the expected engagement by the first-year undergraduate students, and then I focus on their actual engagement by looking at their written responses to these tasks.

Also, this study reflects the UK context. It is based in a well-recognized UK institution, and the focus was on a first-year module. The results might not have been the same if the study was conducted in a different country as the examinations both at university and at secondary school are very specific to the context.

From the analysed data there is a visible difference between the Sets, Numbers and Proofs optional tasks being solved and the ones from Probability. Many students did not engage with the tasks from the Probability part of the module. The lecturer from the module also recognized that this is the case from the previous years. The lecturer as an attempt tried to work with the students in showing them the applicability side of the context of the Probability in real life.

Finally, from the analysis, there are some differences between the assessment practices of the two lecturers in this first-year module. Some of these are due to the nature of the different parts of the module. However, the sources of these differences were not examined further with follow up interviews, as the lecturer data in this study is mostly contributing in the characterisation of the tasks and students' expected engagement in mathematical discourses. In the next section, I discuss suggestions for further research these arise from the limitations of this study and reflection on the analysis of the data.

9.5 Further ideas for research

In this section, I describe the next stages after the completion of the thesis. My study, investigates students' participation to university mathematical discourse, focusing on one first-year module both in terms of what is expected (looking at the lecturers' perspectives on assessment and the examination tasks) and the actual participation (investigating the (un)resolved commognitive conflicts in students' examination scripts). Following up on the nuanced analysis of the students' scripts and the examination tasks from this module, I am planning to examine further the other sets of data from the other two modules which are not taught in the first year. These two modules are in different mathematical areas and they are in a different year of study. So, it would be interesting to see the transitions there and the varying discourses that come into play.

Following, my analysis of students' examination scripts, I found that the markers endorsement routines are different. Specifically, the markers decide differently as to how many marks to deduct when there is evidence of unresolved commognitive conflicts. Also, it would be interesting to examine whether there are different endorsement routines for the same marker across different students or different tasks.

Revisiting, Schoenfeld and Herrmann's study (1982), using commognitive theory, I posit that the lecturers classified the tasks according to the rules of the discourse and students categorised the same tasks according to the objects of the discourse. I plan to visit the data from the second interview with the students and examine students' perspectives on the examination tasks. This way, I add another layer of analysis to the current one and examine how students classify, or which aspects of the tasks are discussed by the students.

The analytical tool used in this study allows for a nuanced analysis of the unresolved commognitive conflicts evidenced in students' examination scripts. This also allowed a characterisation of the different discourses coming into play in students' solutions either at a word, visual mediator use layer or students' engagement with routines of proof or recall first and then substantiation. If the lecturers are aware of the type of engagement asked from the students and the transition needed for a smooth shift from the school discourse to university mathematical discourse, they can facilitate this transition and make it smoother, either by stressing this more during the lectures or by creating coursework tasks that have this transition in mind. This would not only be helpful for the first-year students but also for students studying in further years as the analysis also highlights the different university mathematical discourses and the inter (horizontal meta-level learning) or intra (vertical meta-level learning) shifts (Nardi, 1999; Tabach & Nachlieli, 2016).

Another aspect which I aim to explore further is students' engagement with recall routines and the implications on their engagement with substantiation routines. We report findings in Thoma and Nardi (2018b, submitted to PME 42) on students' scripts to the third task from the Sets, Numbers and Proofs part of the module with the functions and the characterisations as injective or surjective. Students experience difficulties in recalling definitions, axioms, theorems and propositions and using appropriately universal, existential and logical expressions. In the first year of their studies, the students are asked to use these in a variety of different mathematical contexts. However, the students do not necessarily have clear meaning-making about these symbols. This can be supported by the analysis in chapters 6 and 8.

Additionally, as mentioned in the limitations (section 9.4), the data analysis would enrich by allowing students to discuss and comment on their written work. This was not possible due to timing difficulties and students' availability. However, an analysis of the exercises and the coursework given to the students during the year might provide a chance for students and researchers to discuss further the nature of (un)resolved commognitive conflicts and provide insight into students' discourse development during the academic year. These results can then be used to inform the analysis of the students' final year examination scripts and offer awareness as to the mistakes that were made due to the pressure and stress of the examinations.

It is crucial to mention the importance of contextualising the results of the study. The transition for the students is deeply embedded in the context of secondary and university level. Similarities and differences in this transition can be observed over the years of study but also in the contexts of different countries, as the requirements to enter the mathematics department vary from country to country as well as their secondary school curriculum. In attempting to view the participation of Greek students in mathematical discourse at secondary level, examinations for entrance at university level, considering the examination tasks during a decade (2006 – 2015) examining the complexity in terms of procedure to be followed by the students we observed an increase in directions and in the independence of the subtasks. A decrease is observed in the visual mediators being asked to be produced by the students, as the time passes the visual mediators are mostly given to the students instead of asking them to produce these (Thoma & Nardi, 2015). The investigation in students' participation in the discourse in different countries would provide further insight into the transition and illustrate further the complexity of the said transition.

Another issue that might be interesting to discuss is the use of the framework for the creation of intervention tasks and modules that might assist students for a smoother transition during the year. Apart from the shifts of the discourses between the secondary mathematical discourse and the tertiary mathematical discourse, there are also shifts between the mathematical discourses that the students are asked to engage with either within the first year of their studies but also the later stages of their study too. Creating resources in terms of tasks or specific workshops that will address these shifts and aim to facilitate students' transition from the different discourses might be something that this framework could be utilised in. The framework address and recognizes the different mathematical discourses and the analysis allows to characterise the type of routines and practices that the students are being asked to engage in. In the first year of mathematical studies, the students are asked to engage in practices of proving, practices of defining and engage with strict symbolism and formal mathematical terminology. Creating situations where these shifts are visible might be something that could be done during the year by the lecturers in the lectures, or being addressed in the lecture notes, or assisting in the creation of specific type of resources, tasks and situations where the solutions of the students or model solutions of the tasks are being shown to the students so that they compare their own engagement with the discourse. Potentially, the analysis of students' solutions and their own analysis of their peers' solutions could highlight commognitive conflicts during the academic year. These resources can be used in the sessions during the year as a way of raising awareness between the different ways that visual mediators and word use is being used in the community.

9.6 Reflections on the journey as a researcher in University Mathematics Education

My study focuses on students' transition from secondary school to university. Using the commognitive framework, I focus on the shifts in their discourses. However, during my postgraduate studies, I also found myself using the commognitive framework to view my own trajectory of participation both in terms of the university mathematics education research discourse but also in the university mathematics discourse. Sfard (2008) discusses the role of the commognitive researcher both as an outsider and an insider of the discourse to be studied (p. 280). In this closing section, I discuss my position as a commognitive researcher regarding the university mathematical discourse but also my development as a participant in the community of university mathematics education research.

I started my postgraduate studies in the United Kingdom immediately after I finished my undergraduate degree in mathematics at a Greek university. I was introduced to the research in mathematics education and the community practices initially from my master's course. I experienced various shifts in my discourse in mathematics education while trying to become more familiar with the practices of the mathematics education research community (Nardi, 2015). These shifts were both in terms of the rules of the discourse but also in the objects of the new, to me, discourse. This development of my personal discourse is visible in the early publications reporting initial findings of my thesis (Thoma & Iannone, 2015; Thoma & Nardi, 2015; Thoma, 2016; Thoma & Nardi, 2016; 2017).

From the first year of my postgraduate research studies, I became an associate in various research projects. This experience assisted in my development as a researcher. Although the projects were not focusing on university mathematics education, working with more experienced researchers and engaging in discussions about methodology, analysis, the creation of analytical frameworks and reporting the findings in publications in journals or conferences had a considerable influence in my development as a researcher. This was accompanied by a variety of presentations of the early stages of my work in national and international conferences (see section 11 for a list of publications and presentations produced from this study) and the monthly Research in Mathematics Education group meetings in my university. My participation in these meetings and conferences as participant, presenter, and reviewer assisted in the development of my discourse both regarding routines (e.g., the process of reviewing a conference or a journal article) but also concerning word use (e.g., the word "discourse", "routines").

However, as mentioned at the start of this section, I also found myself taking the position of an insider and outsider regarding university mathematics discourse. Being able to move flexibly between the two positions was helpful in the interviews with the lecturers and the analysis of my data. As a recent graduate from university, I was very close to the position of the undergraduate students who were newcomers themselves to the university mathematical community. This allowed me to be able to position myself as an insider when looking at the students' participation as seen in their examination scripts. Furthermore, this position as an outsider of the specific educational context, allowed me to have fruitful discussions with the lecturers of the modules and question further how the examination tasks were created. At the same being an insider of the mathematical community myself, allowed them to refer to the rules and terminology of their practice and provided further insight into their assessment practices and their expectations from the students' engagement with the mathematics discourse.

Additionally, the change in language and context also assisted in allowing me to take the position of the outsider when studying university mathematical discourses. I graduated from a Greek mathematics department and had a different experience participating in the university mathematics discourse from the students who were studying in the mathematics department in the UK. This was also due to the various educational systems prior to coming to the university. For example, engaging with rigorous forms of proofs starts at the secondary school in Greece. I was, thus, able to position myself as an outsider in terms of the UK university mathematics and therefore examine further the discourse used and at the same time as an insider knowing the rules that determine the participation to university mathematics and the endorsed narratives of the community.

Finally, during my postgraduate studies, I delivered outreach sessions and worked as a Learning Enhancement tutor of Mathematics and Statistics at the Student Support Services of the university. Both experiences greatly influenced my position as an outsider as well as an insider in the mathematics discourse. As part of a larger programme of the university's outreach to the local community, I collaborated with colleagues in delivering an outreach session which aimed to introduce ideas presented in university mathematics to early secondary school students. Additionally, since September 2015, I was employed as a Learning Enhancement Tutor supporting students who did not major in mathematics but had a substantial component of mathematics in their undergraduate studies (e.g., students from Biology, Economics). In these two roles, I found myself continually moving from the position of an insider to the position of an outsider trying to facilitate students' engagement with the mathematical discourse. This constant shift influenced my own engagement with the mathematical discourses greatly as it raised my awareness of the shifts between those and the potential commognitive conflicts that students might experience, as they go through these shifts.

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11 Declaration of published work

Parts of the research work for this thesis have been presented in national and international conferences and published in a research journal and conference proceedings. The publications in the conference proceedings are reports of the ongoing analysis. I would like to thank my co-authors for their collaboration in these publications.

Research journal article:

Thoma, A., & Nardi, E. (2018). Transition from School to University Mathematics: Manifestations of Unresolved Commognitive Conflict in First Year Students' Examination Scripts. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 161-180.

Peer-reviewed conference papers:

Thoma, A., & Nardi, E. (2018b). Conflating discourses: University students' exam responses on injective and surjective functions. Submitted to the 42 Conference of the International Group for the Psychology of Mathematics Education (PME).

Thoma, A., & Nardi, E. (2017). Discursive shifts from school to university mathematics and lecturer assessment practices: Commognitive conflicts regarding variables. In Dooley, T. & Gueudet, G. (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (CERME10, February 1 – 5, 2017). Dublin, Ireland. 2266-2273.

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Thoma, A. & Iannone, P. (2015) Analysing university closed book examination using two frameworks. In Vondrova, N. & Krainer, K. (Eds.), Proceedings of the 9th *Congress of European Research in*

Mathematics Education. 4-8 February 2014, Prague, Czech Republic. 2256-2262

Thoma, A. & Nardi, E. (2015). Student participation in the mathematical discourse of the Panhellenic exam: Autonomy, complexity and use of visual mediators [Η συμμετοχή των μαθητών στο μαθηματικό λόγο των Πανελλαδικών: αυτονομία, πολυπλοκότητα και χρήση οπτικών διαμεσολαβητών]. *Proceedings of the 6th Conference of the Greek Association of Research in Mathematics Education*. Thessaloniki, Greece: ENEDIM.

Oral Presentations without publication:

Thoma, A. & Nardi, E. (2016, June). *Engagement with mathematical discourse in closed-book examinations: Lecturers' perspectives and student responses to a Probability question*. British Society for Research into Learning Mathematics (BSRLM) Conference, Loughborough University, UK.

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Poster presentation:

Thoma, A. (2016). Lecturers' pedagogical routines and expectations on students' engagement in closed book examinations. Presented in the *13th International Congress in Mathematics Education* (ICME 13), Hamburg, Germany.

12 Appendices

12.1 Information sheets

In the following sections, I provide the information sheets given to the lecturers (12.1.1 and 12.1.2) and the student participants (12.1.3 and 12.1.4)

12.1.1 Information Sheet – Lecturers - 1

[for lecturers – materials and interview]

Title: Participating in the discourse of university mathematics – analysing closed-book examination tasks

Researcher: Athina Thoma

Supervisor: Elena Nardi

I would like to invite you to take part in my research and I need your signed consent if you agree to participate. To facilitate your decision, I would like to explain to you your involvement and the nature of my study. Please take the time to read this information carefully to help you decide whether or not you would like to take part. Please contact me if there is anything that is not clear or if you would like more information. Thank you for reading this.

What is this study about?

My study aims to examine the closed-book examination tasks of undergraduate mathematics modules and their relationship with the students' approaches to learning and their views of mathematics. Additionally, I will investigate the students' and lecturers' perspectives on the examination tasks.

How will you be involved?

I will ask you to provide the teaching material used in your module, the coursework tasks, the current and past examination tasks, the model solutions given to the markers, and the students' solutions of the current examination tasks. I will interview you for approximately 1 hour, at a time that is agreeable to you, and the interview will be audio recorded. The interview will focus on your perspectives of the examination tasks and will take place after the examination of your module. I will also be interviewing some of your students on their approaches to learning and their views of mathematics. I will additionally be observing them solving a mathematical task and interviewing them on their perspectives on that task.

Who will have the access to the research information (data)?

Data management will follow the procedures laid down by the 1988 Data Protection Act. I will not keep information about you that could identify you to someone else. All the names of the individuals taking part in the research and the school(s) will be anonymised to preserve confidentiality. The data will be stored safely and used in an anonymised format for the purposes of my research and further academic publications.

Who has reviewed the study?

The research study has been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

Who do I speak to if problems arise?

If there is a problem please let me know. You can contact me via the University at the following address:

Athina Thoma

School of Education and Lifelong Learning

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a.thoma@uea.ac.uk

If you would like to speak to someone else you can contact my supervisor:

Elena Nardi

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NORWICH NR4 7TJ

e.nardi@uea.ac.uk

Tel: + 44 (0) 1603 59 2631

If you have any complaints about the research, please contact the Head of the School of Education and Lifelong Learning, Dr Nalini Boodhoo, at n.boodhoo@uea.ac.uk.

OK, I want to take part – what do I do next?

You need to fill in one copy of the consent form and return it to me. Please keep the information sheet and the 2nd copy of the consent form for your records.

Can I change my mind?

Yes. Your participation in my research is completely voluntary and you have the right to withdraw from the research at any time.

Thank you very much for your time.

12.1.2 Information Sheet – Lecturers - 2

[for lecturer participants – materials, lecture observation and interview]

Title: Participating in the discourse of university mathematics – analysing closed-book examination tasks

Researcher: Athina Thoma

Supervisor: Elena Nardi

I would like to invite you to take part in my research and I need your signed consent if you agree to participate. To facilitate your decision, I would like to explain to you your involvement and the nature of my study. Please take the time to read this information carefully to help you decide whether or not you would like to take part. Please contact me if there is anything that is not clear or if you would like more information. Thank you for reading this.

What is this study about?

My study aims to examine the closed-book examination tasks of undergraduate mathematics modules and their relationship with the students' approaches to learning and their views of mathematics. Additionally, I will investigate the students' and lecturers' perspectives on the examination tasks.

How will you be involved?

I will ask you to provide the teaching material used in your module, the coursework tasks, the current and past examination tasks, the model solutions given to the markers and the students' solutions of the current examination tasks. I will interview you for approximately 1 hour, at a time that is agreeable to you, and the interview will be audio recorded. The interview will focus on your perspectives of the examination tasks and will take place after the examination of your module. I will be observing your lectures and noting the mathematical tasks and their solutions that are presented in those lectures. Additionally, I will be interviewing some of your students on their approaches to learning and their views of mathematics. Finally, I will be observing them solving a mathematical task and interviewing them on their perspectives on that task.

Who will have the access to the research information (data)?

Data management will follow the procedures laid down by the 1988 Data Protection Act. I will not keep information about you that could identify you to someone else. All the names of the individuals taking part in the research and the school(s) will be anonymised to preserve confidentiality. The data will be stored safely and used in an anonymised format for the purposes of my research and further academic publications.

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Can I change my mind?

Yes. Your participation in my research is completely voluntary and you have the right to withdraw from the research at any time.

Thank you very much for your time.

12.1.3 Information Sheet – Students - 1

[for student participants - interviews]

Title: Participating in the discourse of university mathematics – analysing closed-book examination tasks

Researcher: Athina Thoma

Supervisor: Elena Nardi

I would like to invite you to take part in my research and I need your signed consent if you agree to participate. To facilitate your decision, I would like to explain to you your involvement and the nature of my study. Please take the time to read this information carefully to help you decide whether or not you would like to take part. Please contact me if there is anything that is not clear or if you would like more information. Thank you for reading this.

What is this study about?

My study aims to examine the closed-book examination tasks of undergraduate mathematics modules and their relationship with the students' approaches to learning and their views of mathematics. Additionally, I will investigate the students' and lecturers' perspectives on the examination tasks.

How will you be involved?

I will interview you for approximately 20 to 30 minutes, at a time that is agreeable to you, and the interview will be audio recorded. The interview will focus on your approaches to learning and your views of mathematics.

Who will have the access to the research information (data)?

Data management will follow the procedures laid down by the 1988 Data Protection Act. I will not keep information about you that could identify you to someone else. All the names of the individuals taking part in the research and the school(s) will be anonymised to preserve confidentiality. The data will be stored safely and used in an anonymised format for the purposes of my research and further academic publications. Additionally, I will share part of the analysed data in a completely anonymised form with your lecturers.

Who has reviewed the study?

The research study has been approved under the regulations of the University of East Anglia's School of Education and Lifelong Learning Research Ethics Committee.

Who do I speak to if problems arise?

If there is a problem please let me know. You can contact me via the University at the following address:

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You need to fill in one copy of the consent form and return it to me. Please keep the information sheet and the 2nd copy of the consent form for your records.

Can I change my mind?

Yes. Your participation in my research is completely voluntary and you have the right to withdraw from the research at any time.

Thank you very much for your time.

12.1.4 Information Sheet – Students - 2

[for student participants - observations and interviews]

Title: Participating in the discourse of university mathematics – analysing closed-book examination tasks

Researcher: Athina Thoma

Supervisor: Elena Nardi

I would like to invite you to take part in my research and I need your signed consent if you agree to participate. To facilitate your decision, I would like to explain to you your involvement and the nature of my study. Please take the time to read this information carefully to help you decide whether or not you would like to take part. Please contact me if there is anything that is not clear or if you would like more information. Thank you for reading this.

What is this study about?

My study aims to examine the closed-book examination tasks of undergraduate mathematics modules and their relationship with the students' approaches to learning and their views of mathematics. Additionally, I will investigate the students' and lecturers' perspectives on the examination tasks.

How will you be involved?

I will ask you to choose one or more mathematical tasks from the past examinations of [name of module]. I will observe you while you are solving the task. Furthermore, I will ask you to report your thinking while you are solving it, and this will be audio-recorded. After this I will interview you for approximately 1 hour and the interview will be audio recorded.

Who will have the access to the research information (data)?

Data management will follow the procedures laid down by the 1988 Data Protection Act. I will not keep information about you that could identify you to someone else. All the names of the individuals taking part in the research and the school(s) will be anonymised to preserve confidentiality. The data will be stored safely and used in an anonymised format for the purposes of my research and further academic publications. Additionally, I will share part of the analysed data in a completely anonymised form with your lecturers.

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OK, I want to take part – what do I do next?

You need to fill in one copy of the consent form and return it to me. Please keep the information sheet and the 2nd copy of the consent form for your records.

Can I change my mind?

Yes. Your participation in my research is completely voluntary and you have the right to withdraw from the research at any time.

Thank you very much for your time.

12.2 Consent forms

(1ST COPY FOR RETURN TO RESEARCHER)

Participating in the discourse of university mathematics – analysing closed-book examination tasks

I have read the information about the study.

Please tick the relevant box.

I am willing to take part in the study.

I am willing to be audio recorded as part of the study.

Your Name: ……………………………………

Your Signature: ……………………………………………………….

Date: ……………………………………………..

CONSENT FORM

(2ND COPY FOR YOUR RECORDS)

Participating in the discourse of university mathematics – analysing closed-book examination tasks

I have read the information about the study.

Please tick the relevant box.

I am willing to take part in the study.

I am willing to be audio recorded as part of the study.

Your Name: ……………………………………

Your Signature: …………………………………………………………

Date: ……………………………………………

12.3 Sets, Numbers and Probability Examination Tasks

12.3.1. Compulsory task from Sets, Numbers and Proofs

(i) Prove by induction that for all natural numbers n,

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

[6 marks]

(ii)

(a) Suppose a, b, d, m, n are integers. Give the definition of what is meant by saying that d is a divisor of a. Using this, prove that if d is a divisor of a and d is a divisor of b, then d is a divisor of $ma + nb$.

(b) Use the Euclidean algorithm to find the greatest common divisor d of 123 and 45. Hence (or otherwise) find integers m, n with $123m + 45n = d$.

(c) Do there exist integers s, t such that $123s + 45t = 7$? Explain your answer carefully.

[14 marks]

12.3.2. Compulsory task from Probability

(i) In the framework of the modern probability, give the definition of two disjoint events and state the three Kolmogorov's axioms; then use them to demonstrate the following two propositions:

(a) For any event $A = \emptyset$, prove that $P(A) = 0$.

You may assume Proposition 2, that is

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1 and A_2 are disjoint events.

(b) For any events A and B such that $A \subseteq \mathrm{B}$, prove that $P(A) \leq P(\mathrm{B})$.

[12 marks]

(ii) Let A and B be two events, with $P(A) = \frac{2}{5}$, $P(B|A) = \frac{5}{8}$ and $P(A \cup B) = p$.

(a) Show that $(A \cap B) = \frac{1}{4}$.

(b) Find $P(B)$ and the range of possible values for the parameter p.

(c) Find $P(B^c|A)$ and $P(A \cap B^c)$.

[8 marks]

12.3.3. First Optional task from Sets, Numbers and Proofs

(i) Prove carefully that if A, B and C are sets then
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Give an example of sets A, B and C such that
$A \cap (B \cup C) \neq (A \cap B) \cup C$.

[10 marks]

(ii) Suppose that A is a non-empty set and $\sim$ is a relation on A. Give the definitions of what is meant by saying that $\sim$ is *reflexive, symmetric* and *transitive*. In each of the following cases, decide which (if any) of these properties the given relation has. Give reasons for your answers.

(a) $A = \mathbb{Z}$ and $a \sim b \Leftrightarrow |a - b| \leq 10$ (for a, $b \in \mathbb{Z}$).

(b) $A = \mathbb{R}$ and $a \sim b \Leftrightarrow a - b \in \mathbb{Q}$ (for a, $b \in \mathbb{R}$).

[10 marks]

12.3.4. Second Optional task from Sets, Numbers and Proofs

(i) Suppose A and B are sets and $f: A \to B$ is a function. Define what is meant by f being *surjective* and what is meant by f being *injective*.

For each of the following functions decide whether it is injective, surjective (or both, or neither). Give brief reasons for your answers.

(a) $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = 1/(1 + sin^2(x))$ for $x \in \mathbb{R}$.

(b) $h: \mathbb{Z} \to \mathbb{Z}$ where $h(n) = 3n$ for $n \in \mathbb{Z}$.

[10 marks]

(ii)

(a) State (but do not prove) Fermat's Little Theorem.

(b) Compute the remainder when 27^{313} is divided by 11.

(c) Find an integer $x \in \mathbb{Z}$ such that $19x \equiv 1\ (mod\ 36)$

[10 marks]

12.3.5. First Optional task from Probability

(i) Let X be a Poisson random variable with parameter λ having probability mass function $P(X = x) = \frac{\lambda^k e^{-\lambda}}{k!}$

(a) Show that

$$\sum_{k=0}^{\infty} P(X = x) = 1$$

(b) By assuming the validity of the relation in (a), calculate $E(X)$.

[8 marks]

(ii) Students travelling to the city centre arrive at the (name of the university) bus stop according to a Poisson process of intensity 15 per 10 minutes between 5pm and 7 pm, and of intensity 4 per 15 minutes during the rest of the day.

(a) What is the probability that at least 15 students arrive at the bus stop between 5pm and 5.10pm?

(b) What is the probability that at most 10 students arrive at the bus stop between 9am and 9.30am?

(c) Suppose that no students are at the bus stop at 10.30am. What is the probability that the bus stop will remain empty for a further 6 minutes?

(d) What is the most probable event between: the event A describing 15 students arriving between 5.30pm and 5.40pm; and the event B describing 4 students arriving between 10am and 10.15am?

[12 marks]

12.3.6. Second Optional task from Probability

(i) Define expectation $E(X)$ and variance $V(X)$ of a continuous random variable X.

(ii) A random variable X is said to have a normal $N(\mu, \sigma^2)$ distribution with mean μ and variance σ^2 if its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(a) Show that the probability density function $f(x)$ satisfies the first Kolmogorov axiom of modern probability.

(b) By rigorously evaluating the expectation $E(X)$, prove that it is equal to the mean μ.

You may use the result $\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}$.

[10 marks]

(iii) The *standard normal* random variable Z is a particular case of *normal* random variable having mean $\mu = 0$ and variance $\sigma^2 = 1$. Its cumulative density function is defined by

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$$

and its values are computed numerically and tabulated in the statistical tables.

(a) Give the definition of cumulative distribution function $F(X)$ of the *normal* random variable X having $f(x)$ as a probability density function. Then, explain why the following relation holds

$F(X) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, that is $z \equiv \frac{x-\mu}{\sigma}$.

(b) Consider the *normal* random variable T which has mean $\mu = 50$ and variance $\sigma^2 = 64$.

Find $P(T \leq 26)$ and $P(\mathrm{T} < 130 | \mathrm{T} > 90)$.

[10 marks]

12.4 Model solutions to Sets, Numbers and Probability examination tasks

12.4.1 Model solution to compulsory task from Sets, Numbers and Proofs

Solution 1: (i) Let $P(n)$ be the statement

$$P(n): \qquad 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

We proceed by induction on n:

Base Step $P(1)$ says

$$P(1): \qquad 2^1 = 2^{1+1} - 2$$

which is certainly true since both sides are 2.

[2 marks]

Inductive Step Suppose k is a natural number for which $P(k)$ is true– so

$$2^1 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 2 \qquad (\dagger)$$

We want to deduce that $P(k+1)$ is true. Adding 2^{k+1} to both sides of ($\dagger$), and doing some rearranging, we get:

$$\begin{aligned} 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2.2^{k+1} - 2 \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2, \end{aligned}$$

so $P(k+1)$ is also true, completing the inductive step. Hence $P(n)$ is true for all $n \in \mathbb{N}$, by induction. **[4 marks]**

(ii)(a) d is a *divisor* of a means that there is $k \in \mathbb{Z}$ with $a = kd$. **[2 marks]**

If d is a divisor of a and of b then there exist $k, l \in \mathbb{Z}$ with $a = kd$ and $b = ld$. Then for all $m, n \in \mathbb{Z}$ we have

$$ma + nb = m(kd) + n(ld) = (mk + nl)d.$$

As $(mk + nl) \in \mathbb{Z}$, it follows that d divides $ma + nb$. **[2 marks]**

(ii)(b) Following the method in lectures, let $a = 123$ and $b = 45$. Carrying out the Euclidean algorithm we obtain:

$$\begin{array}{rcr|rcl} a & = & 123 & & & \\ 2b & = & 90 & 45 & = & b \\ \hline a - 2b & = & 33 & 33 & = & a - 2b \\ \hline 6b - 2a & = & 24 & 12 & = & 3b - a \\ \hline 3a - 8b & = & 9 & 9 & = & 3a - 8b \\ \hline & & 9 & 3 & = & 11b - 4a \\ \hline & & 0 & & & \end{array}$$

We conclude that $\gcd(123, 45) = 3$ and that

$$3 = 11b - 4a = 11 \cdot 45 + (-4) \cdot 123 = 123m + 45n$$

where $m = -4$ and $n = 11$. **[8 marks]**

(ii)(c) No. Since 3 divides both 123 and 45 it follows from (ii)(a) that 3 divides $123s + 45t$ for all $s, t \in \mathbb{Z}$, but 3 does not divide 7. **[2 marks]**

Figure 12.4.1: Model solution to compulsory task from Sets, Numbers and Proofs

12.4.2 Model solution to compulsory task from Probability

2. (i) Two events A and B are disjoint if $A \cap B = \phi$.

Suppose that $A \subseteq S$ is an event of the sample space S. The K-axioms are

(A1) $P(A) \geq 0$;

(A2) $P(S) = 1$;

(A3) For any collection of countable pairwise disjoint events $A_1, A_2, \ldots$, it holds

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i)$$

(a) Let us consider the countable collection of pairwise disjoint events $A_i = \phi \ \forall i \geq 1$. Then by (A3)

$$P(\phi) = \sum_{i=1}^{\infty} P(\phi) = \lim_{n\to\infty} nP(\phi)$$

$$\Rightarrow \lim_{n\to\infty} (n-1)P(\phi) = 0 \Rightarrow P(\phi) = 0 .$$

(b) As $A \subseteq B$ then we can write $B = \underbrace{\overset{A}{(A\cap B)} \cup (A^c \cap B)}_{\text{disjoint}}$. Thus by Prop. 2

$P(B) = P(A) + P(A^c \cap B)$.

As $P(A^c \cap B) \geq 0$ by (A1) $\Rightarrow P(B) \geq P(A)$.

(ii) (a) By making use of the multiplication rule, or equivalently of the definition of conditional probability, it results

$P(A\cap B) = P(B|A) \times P(A) = \frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$.

(b) For any events A and B it is valid

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

then

$P(B) = P(A\cup B) + P(A\cap B) - P(A) =$

$= p + \frac{1}{4} - \frac{2}{5} = p + \frac{5-8}{20} = p - \frac{3}{20}$.

Thus, as the probability is always non-negative and $P(S)=1$, $\frac{3}{20} \leq p \leq \frac{23}{20}$ should hold.

(c) As the conditional probability is a good probability measure then it holds

$P(B^c|A) = 1 - P(B|A) = 1 - \frac{5}{8} = \frac{3}{8}$.

By using the multiplication rule we have

$P(A\cap B^c) = P(B^c|A)P(A) = \frac{3}{\cancel{8}_4} \cdot \frac{\cancel{2}^1}{5} = \frac{3}{20}$.

Figure 12.4.2a: Model solution to compulsory task from Probability – First Version

$\boxed{Q2}$ (i)

$P(A) \geq 0$ — 2

$P(S) = 1$ — 2

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, iff $A_i \cap A_j = \phi$, $i \neq j$ — 2

6 MARKS

(i) (a) $A_i = \phi$ — 1

Use (A3) — 2

3 MARKS

(ii) (b) $B = (A \cap B) \cup (A^c \cap B)$ — 2

'' A as $A \subseteq B$

Use (A1) — 1

3 MARKS

(ii) (a) $P(A \cap B) = P(B|A)\,P(A)$ — 2

(ii) (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ — 1

Use (A1) and (A2) $\frac{3}{20} \leq p \leq \frac{20}{20}$ — 1

not ok if $\frac{23}{20}$

(ii) (c) $P(B^c|A) = 1 - P(B|A)$ — 2

$P(A \cap B^c) = P(B^c|A)\,P(A)$ — 2

8 MARKS

Figure 12.4.2b: Model solution to compulsory task from Probability – Second Version

12.4.3 Model solution to first optional task from Sets, Numbers and Proofs

Solution 3:

(i) First we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. So suppose $x \in A \cap (B \cup C)$; that is, $x \in A$ and $x \in B \cup C$. The latter means that $x \in B$ or $x \in C$.

- If $x \in B$, then $x \in A$ and $x \in B$ so $x \in A \cap B$;
- If $x \in C$, then $x \in A$ and $x \in C$ so $x \in A \cap C$.

So $x \in A \cap B$ or $x \in A \cap C$ – that is, $x \in (A \cap B) \cup (A \cap C)$. **[4 marks]**

Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. So suppose $y \in (A \cap B) \cup (A \cap C)$; that is, $y \in A \cap B$ or $y \in A \cap C$.

- If $y \in A \cap B$ then $y \in A$ and $y \in B$, so $y \in B \cup C$;
- If $y \in A \cap C$ then $y \in A$ and $y \in C$, so $y \in B \cup C$.

Hence in both cases $y \in A$ and $y \in B \cup C$ – that is, $y \in A \cap (B \cup C)$. **[4 marks]**

For the last part, we can take $A = \varnothing$, $B = C = \{1\}$. Then

$$A \cap (B \cup C) = \varnothing \cap \{1\} = \varnothing,$$

while

$$(A \cap B) \cup C = \varnothing \cup \{1\} = \{1\}.$$

[2 marks]

(ii) Let A be a set and $\sim$ a relation on A. The relation $\sim$ is *reflexive* if for all $a \in A$ we have $a \sim a$. The relation $\sim$ is *symmetric* if for all $a, b \in A$, whenever $a \sim b$ then $b \sim a$. The relation $\sim$ is *transitive* if for all $a, b, c \in A$, whenever $a \sim b$ and $b \sim c$ then $a \sim c$. **[4 marks]**

(ii)(a) This is reflexive as $|a - a| = 0 \leq 10$ for all $a \in \mathbb{Z}$. It is symmetric as $|a - b| = |b - a|$. It is not transitive: for example $1 \sim 9$ and $9 \sim 18$ but $|1 - 18| = 17 > 10$, so $1 \not\sim 18$. **[3 marks]**

(ii)(b) The relation is reflexive as $(a - a) = 0 \in \mathbb{Q}$. It is symmetric: if $a \sim b$ then $a - b \in \mathbb{Q}$, so $(b - a) = -(a - b) \in \mathbb{Q}$. It is transitive, because if $a \sim b$ and $b \sim c$ then $(a - b), (b - c) \in \mathbb{Q}$ so their sum $(a - c)$ is in $\mathbb{Q}$. **[3 marks]**

Comments: 3(i) Proof seen in lectures. Moderately difficult. 3(ii) Standard definitions. Easy. 3(ii)(a) and (b). Two standard relations questions. Moderate.

Figure 12.4.3: Model solution to first optional task from Sets, Numbers and Proofs

12.4.4 Model solution to second optional task from Sets, Numbers and Proofs

Solution 4:

(i) A function $f : A \to B$ is

- *surjective* if, for every $b \in B$ there exists $a \in A$ with $f(a) = b$;
- *injective* if, whenever $a, a' \in A$ and $a \neq a'$, then $f(a) \neq f(a')$ (so f sends distinct elements of A to distinct elements of B).

[2 marks]

(i)(a) g is not injective since $g(0) = 1/(1+0) = g(\pi)$ with $0 \neq \pi$. **[2 marks]**

g is not surjective. Indeed, since $0 \leq \sin^2(x) \leq 1$ for all $x \in \mathbb{R}$ it follows that

$$1/2 \leq 1/(1+\sin^2(x)) \leq 1$$

for all $x \in \mathbb{R}$. Thus, for example, $2 \in \mathbb{R}$ but $g(x) \neq 2$ for all $x \in \mathbb{R}$. **[2 marks]**

(i)(b) h is not surjective. For example, $1 \in \mathbb{Z}$ but there is no integer $n \in \mathbb{Z}$ such that $1 = h(n) = 3n$.

[2 marks]

h is injective, since for all $a, b \in \mathbb{Z}$ we have $h(a) = h(b) \Rightarrow 3a = 3b \Rightarrow a = b$. **[2 marks]**

(ii)(a) FLT: "If p is prime and $a \in \mathbb{Z}$ is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$". **[2 marks]**

(ii)(b) By FLT since 11 is a prime and 27 is not divisible by 11 it follows that $27^{10} \equiv 1 \pmod{11}$. Therefore $27^{313} = 27^{310}27^3 \equiv 27^3 \equiv 5^3 \pmod{11}$. Now $5^2 = 25 \equiv 3 \pmod{11}$. So $5^3 = 5.5^2 \equiv 5.3 \equiv 15 \equiv 4 \pmod{11}$. So the remainder on dividing 27^{313} by 11 is 4. **[4 marks]**

(ii)(c) Applying the Euclidean Algorithm with $a = 36$ and $b = 19$ gives:

$$\begin{array}{rcr|rcl}
a & = & 36 & & & \\
b & = & 19 & 19 & = & b \\
\hline
a-b & = & 17 & 17 & = & a-b \\
\hline
16b-8a & = & 16 & 2 & = & 2b-a \\
\hline
9a-17b & = & 1 & 2 & & \\
\hline
 & & & 0 & &
\end{array}$$

We conclude that $\gcd(36, 19) = 1$ and that

$$1 = 9a - 17b = 9 \cdot 36 + (-17) \cdot 19.$$

Reducing modulo 36 then gives

$$1 \equiv (-17) \cdot 19 \pmod{36}.$$

So $x = -17$ would do. (Equivalently we could take $x = 19 \equiv -17 \pmod{36}$. So in fact, as it turns out, $19 \cdot 19 \equiv 1 \pmod{36}$.) **[4 marks]**

Comments: 4(i) Standard definitions. Easy. 4(i)(a) Similar to tutorial question. Easy. 4(i)(b) Similar to coursework question. Moderate. 4(ii)(a) Standard result from lectures. Easy. 4(ii)(b) Computation. Moderate. 4(ii)(c) Computation, similar to tutorial sheet questions. Moderate.

Figure 12.4.4: Model solution to second optional task from Sets, Numbers and Proofs

12.4.5 Model solution to first optional task from Probability

5. (i) (a) For any Poisson distribution with intensity λ it holds

$$\sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} \frac{\ell^k e^{-\ell}}{k!} =$$

$$= \underbrace{e^{-\ell} \sum_{k=0}^{\infty} \frac{\ell^k}{k!}}_{} = e^{-\ell} \cdot e^{\ell} = 1$$

BECAUSE MACLAURIN EXPANSION OF EXP.

(b)

$$E(X) = \sum_{k=0}^{\infty} k\, P(X=k) = \sum_{k=0}^{\infty} k \frac{\ell^k e^{-\ell}}{k!}$$

$$= \underbrace{0 \cdot \cancel{\frac{\ell^0 e^{-\ell}}{0!}}}_{k=0} + \underbrace{\sum_{k=1}^{\infty} \cancel{k} \frac{\ell^k e^{-\ell}}{\cancel{k!}\,(k-1)!}}_{k \geq 1}, \text{ set } k' = k-1$$

$$= \sum_{k'=0}^{\infty} \frac{\ell^{k'+1} e^{-\ell}}{k'!} = \ell \sum_{k'=0}^{\infty} \frac{\ell^{k'} e^{-\ell}}{k'!} = \ell$$

(ii) Let us define two intensity parameters

$\alpha_1 = \frac{15}{10 \text{ mins}} = \frac{15}{\frac{1}{6} h} = \frac{90}{h}$ between 5pm – 7pm

$\alpha_2 = \frac{4}{15 \text{ mins}} = \frac{4}{\frac{1}{4} h} = \frac{16}{h}$ for the other times

(a) We use α_1 as intensity, $\ell = \frac{15}{10 \text{ mins}} \cdot 30 \text{ mins} = 45$

$$P(X \geq 15) = 1 - P(X < 14)$$

$$= 1 - F(14) = 1 - \cancel{0.6694}\ 0.4657 = \cancel{0.3306}\ 0.5343$$

(b) We use α_2 as intensity, $\ell = \frac{4}{15 \text{ mins}} \cdot 30 \text{ mins} = 8$

$$P(X \leq 10) = F(10) = 0.8159$$

(c) We need to find the probability of no students arriving between 10.30 and 10.36 am.

We use α_2, $\ell = \frac{4}{\cancel{15}_{5} \text{ mins}} \cdot \cancel{6}^{2} \text{ mins} = \frac{8}{5} = 1.6$

$$P(X=0) = F(0) = 0.2019$$

Otherwise let us calculate it directly

$$P(X=0) = \frac{1.6^0 e^{-1.6}}{0!} = e^{-1.6} = 0{,}2019$$

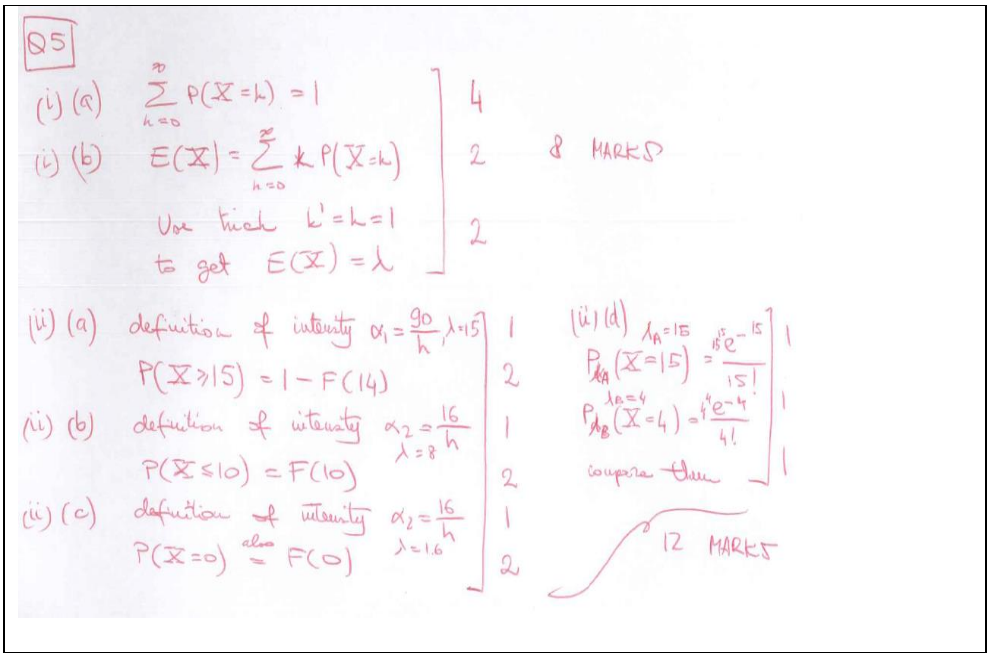
(d) In both cases we have $\ell = 1$ but

$$P(A) = P(X=15) = \frac{15^{15} e^{-15}}{15!} = \cancel{\frac{e^{-1}}{15!}} = 0.1024$$

$$P(B) = P(X=4) = \frac{4^4 e^{-4}}{4!} = \cancel{\frac{e^{-1}}{4!} > \frac{e^{-1}}{15!}} = 0.1954$$

Thus the second event B is more probable.

Figure 12.4.5a: Model solution to first optional task from Probability – First Version



Q5

(i) (a) $\sum_{k=0}^{\infty} P(X=k) = 1$ 4

(i) (b) $E(X) = \sum_{k=0}^{\infty} k\, P(X=k)$ 2 8 MARKS

Use trick $k' = k-1$
to get $E(X) = \lambda$ 2

(ii) (a) definition of intensity $\alpha_1 = \frac{90}{h}, \lambda = 15$ 1
$P(X \geq 15) = 1 - F(14)$ 2

(ii) (b) definition of intensity $\alpha_2 = \frac{16}{h}$, $\lambda = 8$ 1
$P(X \leq 10) = F(10)$ 2

(ii) (c) definition of intensity $\alpha_2 = \frac{16}{h}$, $\lambda = 1.6$ 1
$P(X=0) \overset{\text{also}}{=} F(0)$ 2

(ii) (d) $\lambda_A = 15$
$P_{\lambda_A}(X=15) = \frac{15^{15} e^{-15}}{15!}$ 1
$\lambda_B = 4$
$P_{\lambda_B}(X=4) = \frac{4^{4} e^{-4}}{4!}$ 1
compare them 1

12 MARKS

Figure 12.4.5b: Model solution to first optional task from Probability – Second Version

12.4.6 Model solution to second optional task from Probability

6. (i) The expectation and variance of a continuous random variable X are

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad , \text{ and}$$

$$V(X) = \int_{-\infty}^{\infty} \left[x - E(X) \right]^2 f(x)\, dx \; ,$$

given $f(x)$ the probability density function.

(ii) (a) Let us consider the generic event
$A = \{ a \leq x \leq b \}$, given $a \leq b$. Then

$$P(A) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$P(A)$ satisfies (A1) because it is an integral of a positive-defined function $f(x)$.

(b)
$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let us define $y = \frac{x-\mu}{\sigma}$, $dy = \frac{dx}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \sigma \int_{-\infty}^{\infty} (y+\mu) e^{-\frac{y^2}{2\sigma^2}} dy =$$

$$= \frac{1}{\sqrt{2\pi}} \Big(\mu \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy}_{= \sqrt{2\pi}} + \underbrace{\int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy}_{=0 \text{ because odd function}} \Big) = \mu \; .$$

(iii) (a) The cumulative distribution function of the random variable X having $f(x)$ as probability density function is

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

As

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

we can make the substitution $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$ so that

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz = \Phi(z) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

(b)
$$P(T \leq 26) = \Phi\left(\frac{t-\mu}{\sigma}\right) = \Phi\left(\frac{26-50}{8}\right)$$

$$= \Phi(-3) = 0.0013$$

$$P(T < 130 \mid T > 90) = \frac{P(T < 130 \cap T > 90)}{P(T > 90)}$$

$$= \frac{P(90 < T < 130)}{1 - P(T < 90)} = \frac{\Phi\left(\frac{130-50}{8}\right) - \Phi\left(\frac{90-50}{8}\right)}{1 - \Phi\left(\frac{90-50}{8}\right)}$$

$$= \frac{\Phi(10) - \Phi(5)}{1 - \Phi(5)}$$

Figure 12.4.6a: Model solution to second optional task from Probability – First Version

$\boxed{Q6}$ (i) $E(X) = \int x\, f(x)\, dx$ 2 — 4 MARKS

(−1 PER WRONG LIMITS)

$V(X) = \int [x - E(X)]^2 f(x)\, dx$ 2

(−1 PER WRONG LIMITS)

(ii) (a) any event $A = \{a \le x \le b\}$ 2

$P(A) = \int_a^b f(x)\, dx \geq 0$ as $f(x) \geq 0$ (1 if only say $f(x) \geq 0$) — 6 MARKS

(ii) (b) $\mu = E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \times e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ 1

$y = \frac{x-\mu}{\sigma}$ change of variable 1

$E(X) = \mu$ (if mistake in final calculation remove 1 point) 2

(iii) (a) $F(x) = \int_{-\infty}^{x} f(x)\, dx$ (if no interval remove 1 point) 2 — 4 MARKS

$P(X \le x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$z = \frac{x-\mu}{\sigma}$ so $P(X \le x) = \Phi(z)$ 2

iii (b) $P(T \le 26) = \Phi\left(\frac{26-50}{8}\right)$ 1 — 6 MARKS

$= \Phi(-3)$ 1

$P(T < 130 \mid T > 90) = \frac{P(T<130 \cap T>90)}{P(T>90)}$ 2

$= \frac{P(90 < T < 130)}{P(T>90)} = \frac{\Phi(10) - \Phi(5)}{1 - \Phi(5)}$ 2

Figure 12.4.6b: Model solution to second optional task from Probability – Second Version

12.5 Morgan and Sfard (2016) framework

Table 2: Analytic framework for mathematising aspects of examination discourse – as presented in Morgan & Sfard (2016, p. 106-107)

I. Aspects of the discourse	II. Questions guiding the analysis	III. Textual indicators
Vocabulary and syntax (lexico-grammatical aspects)		
A. specialisation	To what extent is specialised mathematical language used?	➢ lexical items used in accordance with mathematical definitions, considered at the level of: ○ vocabulary ○ sentence ○ text unit ➢ extra-mathematical context ○ depth of engagement with context
B. objectification of the discourse	To what extent does the discourse speak of properties of objects and relations between them rather than of processes?	➢ nominalisation: use of a 'grammatical metaphor', converting a process (verb, e.g. *rotate*) into an object (noun, e.g. *rotation*) ➢ the use of specialised mathematical nouns such as *function*, *sequence* which encapsulate processes into an object ➢ complexity of compound nominal groups
C. logical complexity	What kinds of logical relationships are present and how explicit are they?	➢ the types and frequencies of conjunctions, disjunctions, implications, negations and quantifiers
Visual mediators		
D. the presence of multiple visual mediators	To what extent does the discourse make use of specialised mathematical modes?	➢ presence of tables, diagrams, algebraic notation, etc.
	How are multiple visual mediators incorporated into the discourse?	➢ provided in the text or to be produced by the student ➢ linguistic, visual and/or spatial relationships between modes

E. transitions between visual mediators	What transformations need to be made between different modes?	➢ presence of or demand for two or more modes of communicating 'equivalent' information, e.g. an equation formed from a word problem; a unit of text that involves table, graph and algebraic expressions corresponding to the same function
	How are transformations indicated in the discourse?	➢ provided in the text or to be produced by the student ➢ explicit linguistic or visual links between modes
Routines		
F. the types of action demanded of students	What areas of mathematics are involved?	➢ topics
	What are the characteristics of the routine procedures?	➢ algorithmic or heuristic? ➢ complexity ➢ explicitly hinted at?
Endorsed narratives		
G. the origin of mathematical knowledge	What is the degree of alienation of the discourse?	➢ mathematical objects as agents in processes ➢ agency obscured by: ○ non-finite verb forms ○ passive voice
	To what extent is mathematics construed as involving material action or as atemporal objects and their properties?	➢ mathematical objects involved in: ○ material processes ○ relational or existential processes
	To what extent is mathematics presented as a human activity?	➢ human agents in mathematical processes ○ thinking ○ scribbling
H. the status of mathematical knowledge as	To what extent does the text indicate that decisions or choices are possible during mathematical activity?	➢ modifiers indicating degree of certainty (e.g. may, can, will …) ➢ conditional clauses (e.g. if … or when …)

absolute or contingent		➢ explicit decisions have been or need to be made

Table 3: Analytic scheme for subjectifying aspects of examination discourse – as presented in Morgan & Sfard (2016, p. 108)

I. Aspects of the discourse	**II. Questions guiding the analysis**	**III. Textual indicators**
A. student–author relationship	What kind of relationship is constructed between the student and a mathematical community?	➢ use of personal pronouns ○ inclusive or exclusive we ○ other personal pronouns
	Is the student given instructions or invited to consider mathematical questions?	➢ interrogative (questions) ➢ imperative (instructions)
B. student autonomy	In responding to an examination question, how many independent decisions is the student allowed/required to make in: ➢ designing the path to follow?	➢ the grain size of the task
	➢ interpreting the task?	➢ complexity of utterances ○ lengths of a sentence ○ grammatical complexity: the depth of 'nesting' of subordinate clauses and phrases ○ logical complexity
	➢ choosing the form of the 'answer'?	➢ the layout ○ the physical size of the answer ○ the space provided for the work to be done on the way toward solution ○ format of the answer (units, precision, no. of solutions)

		○ modality of the answer (graph? algebraic expression?)
	➢ choosing/constructing the mode of response?	➢ visual mediators: verbal, symbolic, or graphic: supplied or to be produced?