ANALOGUES OF CENTRALIZER SUBALGEBRAS FOR FIAT 2-CATEGORIES AND THEIR 2-REPRESENTATIONS

MARCO MACKAAY, VOLODYMYR MAZORCHUK,
VANESSA MIEMIETZ AND XIAOTING ZHANG

ABSTRACT. The main result of this paper establishes a bijection between the set of equivalence classes of simple transitive 2-representations with a fixed apex $\mathcal{J}$ of a fiat 2-category $\mathscr{C}$ and the set of equivalence classes of faithful simple transitive 2-representations of the fiat 2-subquotient of $\mathscr{C}$ associated with a diagonal $\mathcal{H}$-cell in $\mathcal{J}$. As an application, we classify simple transitive 2-representations of various categories of Soergel bimodules, in particular, completing the classification in types B_3 and B_4.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Let $\Bbbk$ be an algebraically closed field, A a finite dimensional unital $\Bbbk$-algebra and B a subalgebra of A, which is also assumed to be unital, however, whose unit is not necessarily the same as in A. In this generality, there is no obvious relation between the categories A-mod and B-mod of finite dimensional A- and B-modules, respectively. However, if we additionally assume that B is the centralizer subalgebra of an idempotent e in A, that is $B = eAe$, then the functor $Ae \otimes_B -$ provides a full embedding of B-mod into A-mod.

Finitary 2-categories, introduced in [MM1], are higher representation theoretic analogues of finite-dimensional algebras, where "higher representation theory" refers to the 2-upgrade of the classical representation theory which originated in [BFK, CR, KL, Ro]. Foundations of 2-representation theory of finitary 2-categories were laid in the series [MM1, MM2, MM3, MM4, MM5, MM6] and then various further developments appeared in [CM, KMMZ, MMMT, MaMa, MT, MMZ1, MMZ2, MZ1, MZ2, Xa, Zh1, Zh2, Zi1, Zi2], see [Ma1, Ma2] for overviews.

The basic classification problem in higher representation theory is that of *simple transitive* 2-representations of a given 2-category $\mathscr{C}$, as defined in [MM5]. These simple transitive 2-representations are appropriate higher analogues of simple modules over algebras. It turns out that in many cases (see [MM5, MM6, GM1, GM2, Zh2, Zi1, Zi2, MZ1, MZ2, MMZ1, MMZ2] and [Ma2] for an overview) simple transitive 2-representations are exhausted by the class of *cell 2-representations* which were defined already in [MM1] using the intrinsic combinatorial structure of $\mathscr{C}$. However, in many cases, the situation is more complicated or the answer is not known. Some elementary examples of 2-categories which have simple transitive 2-representations that are not cell 2-representations appear already in [MM5]. The first non-elementary example, a certain subquotient of the 2-category of Soergel bimodules (over the coinvariant algebra) of type B_2, was considered and completed in [MaMa]. There, both the construction and the classification rely on a delicate and technically very involved analysis of the behavior of 2-natural transformations and modifications related to a cell 2-representation. The technique was later applied in [KMMZ] to classify simple transitive 2-representations for all *small quotients* of Soergel bimodules associated to finite Coxeter systems. It

turns out that simple transitive 2-representations which are not cell 2-representations appear in all types $I_2(n)$, with $n > 4$ even.

In the present paper we observe that the abstract results of [MMMT] can, in some cases, be applied to relate 2-representations of a 2-finitary category $\mathscr{C}$ and a finitary 2-subcategory $\mathscr{A}$ of $\mathscr{C}$ of a certain kind. Very loosely, $\mathscr{A}$ should be such that it can be considered as a vague analogue of a "centralizer subalgebra" in $\mathscr{C}$. Our analogy goes as follows: in the classical setup eAe-proj is equivalent to the idempotent split additive subcategory $\mathrm{add}(Ae)$ of A-proj. In our setup, $\mathscr{A}$ is a 2-subcategory of $\mathscr{C}$ given as the "additive span" of some indecomposable 1-morphisms in $\mathscr{C}$. One major complication in the 2-setup compared to the classical case is the absence of any kind of general induction procedure. In classical representation theory, one can always use induction given by the tensor product. So far, there is no reasonable general analogue of the tensor product in higher representation theory. We go around this difficulty and use the approach of [MMMT] to 2-representations based on the study of (co)algebra 1-morphisms in the abelianization of $\mathscr{C}$. This approach is based on ideas from [EGNO, ENO, EO, Os]. Our main result is Theorem 15 which can roughly be formulated as follows (we refer the reader to the text for the precise formulation).

Main theorem. For a fiat 2-category $\mathscr{C}$ and its 2-subcategory $\mathscr{A}$ as above, there is a bijection between certain classes of simple transitive 2-representations of $\mathscr{C}$ and $\mathscr{A}$.

On the surface, this result carries a remarkable resemblance with the abstract classification of simple modules over finite semigroups, see [Cl, Mu, Po, GMS].

Our main theorem has various immediate applications which work in both directions (that is in the direction from $\mathscr{C}$ to $\mathscr{A}$ and in the direction from $\mathscr{A}$ to $\mathscr{C}$). In Section 5 we use the direction "from $\mathscr{C}$ to $\mathscr{A}$" to classify simple transitive 2-representations of certain subquotients of Soergel bimodules in finite Coxeter types, both simplifying and extending the results of [MaMa] and [KMMZ]. In Section 6 we use the direction "from $\mathscr{A}$ to $\mathscr{C}$" to classify certain simple transitive 2-representations of Soergel bimodules (over the coinvariant algebra) in finite Coxeter types, extending the results of [KMMZ] and, in particular, completing the classification in types B_3 and B_4. All preliminaries are collected in Section 2. Section 3 contains some basic technical results on comodule categories over coalgebra 1-morphisms and functors between them.

2. Preliminaries in abstract 2-representation theory

2.1. **2-categories and 2-representations.** We refer to [Le, McL] for generalities and terminology on 2-categories. We refer to [MM1, MM2, MM3, MM4, MM5, MM6] for

all terminology concerning 2-representations of finitary 2-categories. We recall some terminology below.

Following [MM1], a *finitary* 2-category $\mathscr{C}$ is a 2-category such that

- it has finitely many objects;

- each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is equivalent to $A_{\mathtt{i},\mathtt{j}}$-proj, for some finite dimensional, unital, associative $\Bbbk$-algebra $A_{\mathtt{i},\mathtt{j}}$;

- all compositions are (bi)additive and (bi)linear whenever applicable;

- all identity 1-morphisms are indecomposable.

Normally, we denote by $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ the horizontal and vertical compositions in $\mathscr{C}$, respectively. However, in Section 3 we will often suppress the notation $\circ_{\mathrm{h}}$ to save space in diagrams. For the same reason, we will deviate from the notation of the [MM1]-[MM6] series and denote by 1_{X} the identity 2-morphism of a 1-morphism X.

A *finitary* 2-representation of $\mathscr{C}$ is a functorial action of $\mathscr{C}$ on categories equivalent to A-proj, where A is a finite dimensional, unital, associative $\Bbbk$-algebra. A typical example of a finitary 2-representation is the *principal* 2-representation $\mathbf{P}_{\mathtt{i}} := \mathscr{C}(\mathtt{i}, \_)$, where $\mathtt{i} \in \mathscr{C}$. An *abelian* 2-representation of $\mathscr{C}$ is a functorial action of $\mathscr{C}$ on categories equivalent to A-mod, where A is a finite dimensional, unital, associative $\Bbbk$-algebra. For simplicity, given any 2-representation $\mathbf{M}$, we will use the action notation $\mathrm{F}\,X$ instead of the representation notation $\mathbf{M}(\mathrm{F})(X)$.

Both finitary and abelian 2-representations of $\mathscr{C}$ form 2-categories, denoted $\mathscr{C}$-afmod and $\mathscr{C}$-mod, respectively, in which 1-morphisms are strong 2-natural transformations and 2-morphisms are modifications, see [MM3] for details. These two 2-categories are connected by the diagrammatic *abelianization* 2-functor $\overline{\cdot} : \mathscr{C}\text{-afmod} \to \mathscr{C}\text{-mod}$, see [MM2, Subsection 4.2].

Let $\mathbf{M} \in \mathscr{C}$-afmod. We say that $\mathbf{M}$ is *transitive*, see [MM5, Subsection 3.1], provided that, for any $\mathtt{i},\mathtt{j} \in \mathscr{C}$ and any indecomposable $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, there exists $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ such that Y is isomorphic to a summand of $\mathrm{F}\,X$. A transitive 2-representation $\mathbf{M}$ is called *simple transitive*, see [MM5, Subsection 3.5], if $\mathbf{M}$ does not have any non-zero proper $\mathscr{C}$-invariant ideals. By [MM5, Section 4], every $\mathbf{M} \in \mathscr{C}$-afmod has a *weak Jordan-Hölder series* with simple transitive subquotients which are defined uniquely up to permutation and equivalence.

If $\mathbf{M}$ is a 2-representation of $\mathscr{C}$, then the *rank* of $\mathbf{M}$ is the number of isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$. Let $X_1, X_2, \ldots, X_n$ be a fixed list of representatives of these isomorphism classes (in particular, n is the rank of $\mathbf{M}$). To each 1-morphism F one associates an $n \times n$-matrix $[\mathrm{F}] = (m_{i,j})$, where $m_{i,j}$ records the multiplicity of X_i as a summand of $\mathrm{F}\,X_j$.

2.2. **Cells and cell 2-representations.** For indecomposable 1-morphisms F and G in $\mathscr{C}$, we write $\mathrm{F} \geq_L \mathrm{G}$ provided that F is isomorphic to a direct summand of HG, for some 1-morphism H. The relation $\geq_L$ is called the *left* pre-order on the (finite) set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$. The *right* pre-order $\geq_R$ and the *two-sided* pre-order $\geq_J$ are defined similarly using GH and $\mathrm{H}_1\mathrm{GH}_2$, respectively. The equivalence classes for these pre-orders are called the respective *cells*. We refer to [MM2, Section 3] for more details.

A two-sided cell is *regular* provided that all left (resp. right) cells inside this two-sided cell are not comparable with respect to the left (resp. right) order. A regular two-sided cell is called *strongly regular* if the intersection of each left and each right cell inside this two-sided cell contains exactly one isomorphism class of indecomposable 1-morphisms.

Given a left cell $\mathcal{L}$ in $\mathscr{C}$, there is $\mathtt{i} \in \mathscr{C}$ such that all 1-morphisms in $\mathcal{L}$ start at $\mathtt{i}$. The additive closure, in $\coprod_{\mathtt{j} \in \mathscr{C}} \mathbf{P}_{\mathtt{i}}(\mathtt{j})$, of all 1-morphisms F satisfying $\mathrm{F} \geq_L \mathcal{L}$ is $\mathscr{C}$-stable and hence gives a 2-representation of $\mathscr{C}$, denoted $\mathbf{N}_{\mathcal{L}}$. The 2-representation $\mathbf{N}_{\mathcal{L}}$ contains a unique ideal, denoted $\mathbf{I}_{\mathcal{L}}$, which is maximal in the set of all proper $\mathscr{C}$-stable ideals of $\mathbf{N}_{\mathcal{L}}$. The quotient $\mathbf{N}_{\mathcal{L}}/\mathbf{I}_{\mathcal{L}}$ is called the cell 2-representation of $\mathscr{C}$ associated to $\mathcal{L}$ and denoted $\mathbf{C}_{\mathcal{L}}$. Indecomposable objects of this 2-representation are in natural bijection with indecomposable 1-morphisms in $\mathcal{L}$. We refer to [MM2, Subsection 6.5] for more details.

If $\mathbf{M}$ is any transitive 2-representation of $\mathscr{C}$, then there is a unique two-sided cell, called the *apex* of $\mathbf{M}$, which is maximal, with respect to the two-sided order, in the set of all two-sided cells whose elements are not annihilated by $\mathbf{M}$, see [CM, Subsection 3.2] for details.

2.3. **Fiat** 2-**categories.** A finitary 2-category $\mathscr{C}$ is called *fiat* provided that it has a weak involutive anti-equivalence $\star$ (reversing the direction of both 1- and 2-morphisms) such that, for every 1-morphism F, the pair $(\mathrm{F}, \mathrm{F}^{\star})$ is an adjoint pair of 1-morphisms. We refer to [MM1, Subsection 2.4] for more details.

2.4. **Special modules for positively based algebras.** Here we assume that our field $\Bbbk$ is a subfield of $\mathbb{C}$. Let A be a $\Bbbk$-algebra of dimension $n < \infty$ with a fixed basis $\mathbf{B} = \{1 = a_1, \ldots, a_n\}$. We say that $\mathbf{B}$ is *positive* if all structure constants with respect to this basis are non-negative real numbers. For $i, j \in \{1, 2, \ldots, n\}$, set $a_i \leq_L a_j$ provided that there is some a_s such that $a_s a_i$ has a non-zero coefficient at a_j. This is the *left* preorder on $\mathbf{B}$. Equivalence classes with respect to it are called *left cells*. The right and two-sided preorders and cells are defined similarly using multiplication on the right and on both sides, respectively.

An A-module V with a fixed basis $\{v_1, \ldots, v_k\}$ is said to be *positively based* provided that the matrix of each a_i in this basis has non-negative real coefficients and the matrix of $a_1 + a_2 + \cdots + a_n$ has positive real coefficients. Each positively based module has a unique *special* simple subquotient, namely the one which contain the Perron-Frobenius eigenvector for $a_1 + a_2 + \cdots + a_n$. Special simple modules for positively based algebras correspond to non-nilpotent two-sided cells and are classified in [KiM]. We refer to [KiM] for more details.

3. Coalgebra 1-morphisms and simple transitivity of 2-representations

3.1. **(Co)algebra 1-morphisms and their (co)module categories.** Let $\mathscr{C}$ be a finitary 2-category. The paper [MMMT] introduces certain, very technical, injective and projective abelianizations $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$, respectively. They have natural structures of 2-categories extending the one on $\mathscr{C}$. The main results of [MMMT] require the assumption that $\mathscr{C}$ is fiat, so we will also assume this from now on. In case $\mathscr{C}$ is fiat, its action on both $\underline{\mathscr{C}}$ and $\overline{\mathscr{C}}$ is given by exact functors. As every object in $\underline{\mathscr{C}}$ (resp. $\overline{\mathscr{C}}$) is isomorphic to a kernel (resp. cokernel) of a morphism between two objects in $\mathscr{C}$, the snake lemma implies that the action of $\underline{\mathscr{C}}$ (resp. $\overline{\mathscr{C}}$) on itself is left (resp. right) exact.

A *coalgebra* 1-*morphism* in $\underline{\mathscr{C}}$ is a triple $(C, \Delta_C, \varepsilon)$, where C is a 1-morphism and $\Delta_C : C \to CC$ and $\varepsilon : C \to \mathbb{1}$ are 2-morphisms which satisfy the usual axioms for the comultiplication and counit of a coalgebra. Each coalgebra 1-morphism C in $\underline{\mathscr{C}}$ comes together with the category $\mathrm{comod}_{\underline{\mathscr{C}}}(C)$ of (right) C-comodule 1-morphisms in $\underline{\mathscr{C}}$ and its full subcategory $\mathrm{inj}_{\underline{\mathscr{C}}}(C)$ of injective objects. The latter is a finitary (left) 2-representation of $\mathscr{C}$. We also have the category $(C)\mathrm{comod}_{\underline{\mathscr{C}}}$ of left C-comodule 1-morphisms in $\underline{\mathscr{C}}$.

Dually one defines *algebra* 1-*morphisms* (A, μ, η) in $\overline{\mathscr{C}}$ and the corresponding categories $\mathrm{mod}_{\overline{\mathscr{C}}}(A)$ and $(A)\mathrm{mod}_{\overline{\mathscr{C}}}$ of A-module 1-morphisms in $\overline{\mathscr{C}}$ and its full subcategory $\mathrm{proj}_{\overline{\mathscr{C}}}(A)$ of projective objects. We refer to [MMMT, EGNO] for further details.

Lemma 1.

(*i*) *The categories* $\mathrm{comod}_{\underline{\mathscr{C}}}(C)$ *and* $(C)\mathrm{comod}_{\underline{\mathscr{C}}}$ *are abelian subcategories of* $\underline{\mathscr{C}}$.

(*ii*) *The categories* $\mathrm{mod}_{\overline{\mathscr{C}}}(A)$ *and* $(A)\mathrm{mod}_{\overline{\mathscr{C}}}$ *are abelian subcategories of* $\overline{\mathscr{C}}$.

Proof. The only non-obvious thing to check is the existence of kernels and cokernels. We do this for claim (i) and the dual construction works for claim (ii). Let (M, δ_M) and (N, δ_N) be two C-comodule 1-morphisms and $\varphi : M \to N$ a homomorphism of comodule 1-morphisms. Then we have the commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{Ker}(\varphi) & \xrightarrow{\ \xi\ } & M & \xrightarrow{\ \varphi\ } & N & \xrightarrow{\ \zeta\ } & \mathrm{Coker}(\varphi) \\
\downarrow & & \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} & & \downarrow \\
\mathrm{Ker}(\varphi)C & \xrightarrow{\ \xi 1_C\ } & MC & \xrightarrow{\ \varphi 1_C\ } & NC & \xrightarrow{\ \zeta 1_C\ } & \mathrm{Coker}(\varphi)C
\end{array}
$$

where the middle square commutes by definition, the upper row is exact and the lower row is left exact, since the action of $\underline{\mathscr{C}}$ is left exact by construction. Using the universal property of kernels and cokernels, we obtain the outer commuting squares that are induced by the middle one. From this diagram it is easy to see that the kernel and the cokernel of φ inherit the structure of C-comodule 1-morphisms, as claimed. □

Theorem 9 in [MMMT] asserts that, for every transitive 2-representation $\mathbf{M}$ of a fiat 2-category $\mathscr{C}$, there is a coalgebra 1-morphism C in $\underline{\mathscr{C}}$ such that $\mathbf{M}$ is equivalent to the 2-representation $\mathrm{inj}_{\underline{\mathscr{C}}}(C)$ of $\mathscr{C}$. Our main aim of this section is to prove that this correspondence matches simple transitive 2-representations and cosimple coalgebra 1-morphisms, see Corollary 12.

3.2. **Simple transitive quotients and embeddings of coalgebra** 1-**morphisms.** Let $\mathscr{C}$ be a fiat 2-category. Let $\mathbf{M}$ be a transitive 2-representation of $\mathscr{C}$ and $\mathbf{N}$ its simple transitive quotient. Denote by π the natural morphism $\mathbf{M} \twoheadrightarrow \mathbf{N}$.

Let X be a non-zero object in $\mathbf{M}(\mathtt{i})$ (hence also, automatically, a non-zero object in $\mathbf{N}(\mathtt{i})$ as well). Denote by $C = (C_X, \Delta_C, \varepsilon_C)$ and $D = (D_X, \Delta_D, \varepsilon_D)$ the corresponding coalgebra 1-morphisms in $\underline{\mathscr{C}}$, given by

$$\mathrm{Hom}_{\underline{\mathbf{M}(\mathtt{i})}}(X, F\,X) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(C, F) \quad \text{and} \quad \mathrm{Hom}_{\underline{\mathbf{N}(\mathtt{i})}}(X, F\,X) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(D, F),$$

respectively, for all 1-morphisms $F \in \underline{\mathscr{C}}$. We denote by $\mathrm{coev}_{X,X}^{\mathbf{M}}$ and $\mathrm{coev}_{X,X}^{\mathbf{N}}$ the image of 1_C and 1_D under the isomorphisms

$$\mathrm{Hom}_{\underline{\mathbf{M}(\mathtt{i})}}(X, C\,X) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(C, C) \quad \text{and} \quad \mathrm{Hom}_{\underline{\mathbf{N}(\mathtt{i})}}(X, D\,X) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(D, D),$$

respectively.

Lemma 2. *There exist 2-morphisms $\alpha\colon D \to C$ and $\beta\colon C \to D$ in $\underline{\mathscr{C}}$ such that the equality $\beta\alpha = 1_D$ holds. In particular, α is a monomorphism.*

Proof. The solid part of the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X, C\,X) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\underline{\mathscr{C}}}(C, C) \\
\downarrow & & \vdots \\
\mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X, C\,X) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\underline{\mathscr{C}}}(D, C)
\end{array}
$$

induces the dashed morphism on the right, and we define α to be the image of 1_C under this morphism.

Similarly, the solid part of the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X, D\,X) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\underline{\mathscr{C}}}(C, D) \\
\downarrow & & \vdots \\
\mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X, D\,X) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\underline{\mathscr{C}}}(D, D)
\end{array}
$$

induces the dashed morphism on the right, and noting that this morphism is necessarily an epimorphism, we can choose a preimage $\beta \in \mathrm{Hom}_{\underline{\mathscr{C}}}(C, D)$ of $1_D \in \mathrm{Hom}_{\underline{\mathscr{C}}}(D, D)$.

To prove $\beta\alpha = 1_D$, consider the cube

which we claim commutes. Indeed, the top and bottom square commute by definition, the front and back arrows by functoriality of the isomorphisms defining the internal hom, and the left side square commutes since π is a morphism of 2-representations and to be precise , one would write $(\mathbf{M}(\beta))_X \circ_v -$ for the $\beta_X \circ_v -$ in the back and $(\mathbf{N}(\beta))_X \circ_v -$ for the one in the front, which equals $\pi(\mathbf{M}(\beta))_X \circ_v -$. Thus, the square on the right hand side commutes as well. Considering the image of 1_C by going both ways around the square, we see that going down, this maps to β and then to 1_D going forwards, by definition of β. On the other hand, going forwards first, 1_C maps to α by definition, and then going down to $\beta\alpha$. This proves $\beta\alpha = 1_D$. $\square$

Lemma 3. *For any 1-morphism $F \in \underline{\mathscr{C}}$, the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathscr{C}}}(C, F) & \xrightarrow{\ -\circ_v \alpha\ } & \mathrm{Hom}_{\underline{\mathscr{C}}}(D, F) \\
\Big\| \sim & & \Big\| \sim \\
\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X, FX) & \xrightarrow{\ \pi\ } & \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X, FX)
\end{array}
$$

commutes.

Proof. Taking any $\gamma \in \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C}, \mathrm{F})$, the first vertical isomorphism sends γ to the 2-morphism $(\mathbf{M}(\gamma))_X \circ_\mathrm{v} \mathrm{coev}^{\mathbf{M}}_{X,X}$ and we have

$$\pi((\mathbf{M}(\gamma))_X \circ_\mathrm{v} \mathrm{coev}^{\mathbf{M}}_{X,X}) = (\mathbf{N}(\gamma))_X \circ_\mathrm{v} \pi(\mathrm{coev}^{\mathbf{M}}_{X,X})$$
$$= (\mathbf{N}(\gamma))_X \circ_\mathrm{v} (\mathbf{N}(\alpha))_X \circ_\mathrm{v} \mathrm{coev}^{\mathbf{N}}_{X,X}$$
$$= (\mathbf{N}(\gamma \circ_\mathrm{v} \alpha))_X \circ_\mathrm{v} \mathrm{coev}^{\mathbf{N}}_{X,X}$$

where the latter coincides with the image of $\gamma \circ_\mathrm{v} \alpha$ under the second vertical isomorphism. $\square$

Proposition 4. *The morphism α defined in Lemma 2 is a morphism of coalgebra 1-morphisms.*

Proof. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C},\mathrm{C}) \xrightarrow{\ -\circ_\mathrm{v}\alpha\ } \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{C}) \underset{\alpha\circ_\mathrm{v}-}{\overset{\beta\circ_\mathrm{v}-}{\rightrightarrows}} \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{D}) \\
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X,\mathrm{C}\,X) \longrightarrow \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X,\mathrm{C}\,X) \underset{\alpha_X\circ_\mathrm{v}-}{\overset{\beta_X\circ_\mathrm{v}-}{\rightrightarrows}} \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X,\mathrm{D}\,X) \\
\end{array}
$$

(left vertical: $(1_\mathrm{C}\mathrm{coev}^{\mathbf{M}}_{X,X})\circ_\mathrm{v}-$, middle: $(1_\mathrm{C}(\alpha_X\circ_\mathrm{v}\mathrm{coev}^{\mathbf{N}}_{X,X}))\circ_\mathrm{v}-$, right: $(1_\mathrm{D}\mathrm{coev}^{\mathbf{N}}_{X,X})\circ_\mathrm{v}-$)

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X,\mathrm{CC}\,X) \longrightarrow \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X,\mathrm{CC}\,X) \underset{((\alpha\alpha)_X)\circ_\mathrm{v}-}{\overset{((\beta\beta)_X)\circ_\mathrm{v}-}{\rightrightarrows}} \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X,\mathrm{DD}\,X) \\
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C},\mathrm{CC}) \xrightarrow{\ -\circ_\mathrm{v}\alpha\ } \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{CC}) \underset{(\alpha\alpha)\circ_\mathrm{v}-}{\overset{(\beta\beta)\circ_\mathrm{v}-}{\rightrightarrows}} \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{DD})
\end{array}
$$

where the epimorphisms are induced by π, the vertical isomorphisms between the first and second, and the third and fourth row are the functorial isomorphisms from the definition of the internal hom.

The top two and the bottom two squares commute by construction (where for the squares on the right, commuting means that any two possible length two paths around the square are equal) and Lemma 3. To see that the left middle square commutes, note that under the epimorphism $\mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X,\mathrm{C}\,X) \twoheadrightarrow \mathrm{Hom}_{\underline{\mathbf{N}(\mathbf{i})}}(X,\mathrm{C}\,X)$ the image of $\mathrm{coev}^{\mathbf{M}}_{X,X}$ is $\alpha_X \circ_\mathrm{v} \mathrm{coev}^{\mathbf{N}}_{X,X}$ and that again π is a morphism of 2-representations. Commutativity of the right middle square (again meaning any paths of length two around the square are equal) is checked by direct computation.

To show α preserves comultiplication, we need to check that $(\alpha \circ_\mathrm{h} \alpha) \circ_\mathrm{v} \Delta_\mathrm{D} = \Delta_\mathrm{C} \circ_\mathrm{v} \alpha$. Equivalently, as β is a left inverse to α, we can show that

$$(1) \qquad \Delta_\mathrm{D} = (\beta \circ_\mathrm{h} \beta) \circ_\mathrm{v} \Delta_\mathrm{C} \circ_\mathrm{v} \alpha.$$

The right hand side of (1) is precisely the image of the 2-morphism $1_\mathrm{C} \in \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C},\mathrm{C})$ in $\mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{DD})$ by going down on the left and then across along the bottom line in the diagram (recall that the comultiplication Δ_C is defined as the image, under the isomorphism with $\mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C},\mathrm{CC})$, of $(1_\mathrm{C}\circ_\mathrm{h}\mathrm{coev}^{\mathbf{M}}_{X,X})\circ_\mathrm{v}\mathrm{coev}^{\mathbf{M}}_{X,X} \in \mathrm{Hom}_{\underline{\mathbf{M}(\mathbf{i})}}(X,\mathrm{CC}\,X)$).

The left hand side of (1) is the image of the 2-morphism $1_\mathrm{D} \in \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{D})$ when mapped to $\mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{D},\mathrm{DD})$ along the right vertical of the diagram, so we are done if we can show that 1_C maps to 1_D along the top of the diagram. However, the maps along the top take 1_C to $\beta\alpha$, which equals 1_D by Lemma 2, and thus α respects comultiplication.

Preservation of the counit simply follows from commutativity of the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{M}}(\mathbf{i})}(X,X) & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_{\mathscr{C}}(\mathrm{C},\mathbb{1}_{\mathbf{i}}) \\
\downarrow & & \downarrow{\scriptstyle -\circ_{\mathrm{v}}\alpha} \\
\mathrm{Hom}_{\underline{\mathbf{N}}(\mathbf{i})}(X,X) & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_{\mathscr{C}}(\mathrm{D},\mathbb{1}_{\mathbf{i}}),
\end{array}
$$

which is obtained by applying Lemma 3 to $F = \mathbb{1}_{\mathbf{i}}$. This completes the proof.   □

3.3. **Cotensor products.** Let $\mathscr{C}$ be a fiat 2-category and $\mathrm{C} = (\mathrm{C}, \tilde{\Delta}, \tilde{\varepsilon})$ a coalgebra 1-morphism in $\underline{\mathscr{C}}$. Let $(\mathrm{M}, \delta_M) \in \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})$ and $(\mathrm{N}, \delta_N) \in (\mathrm{C})\mathrm{comod}_{\underline{\mathscr{C}}}$. In this subsection we adjust the classical construction of the cotensor product $\mathrm{M}\square_{\mathrm{C}}\mathrm{N}$, see [Ta], to our setup.

We define $\mathrm{M}\square_{\mathrm{C}}\mathrm{N}$ as the equalizer of the diagram

$$
(2) \qquad \mathrm{M}\square_{\mathrm{C}}\mathrm{N} \overset{\xi}{\dashrightarrow} \mathrm{MN} \underset{1_{\mathrm{M}}\delta_{\mathrm{N}}}{\overset{\delta_{\mathrm{M}}1_{\mathrm{N}}}{\rightrightarrows}} \mathrm{MCN},
$$

that is, $\mathrm{M}\square_{\mathrm{C}}\mathrm{N}$ is defined as the kernel of the 2-morphism $\delta_{\mathrm{M}}1_{\mathrm{N}} - 1_{\mathrm{M}}\delta_{\mathrm{N}}$. For a fixed M, this is, obviously, covariantly functorial in N, and vice versa. It follows that $\_\square_{\mathrm{C}}\_$ is a bifunctor.

Assume that $\mathrm{D} = (\mathrm{D}, \Delta, \varepsilon)$ is another coalgebra 1-morphism in $\underline{\mathscr{C}}$ and that N is a C-D-bicomodule 1-morphism with the right coaction of D given by $\gamma : \mathrm{N} \to \mathrm{ND}$. Then $\mathrm{M}\square_{\mathrm{C}}\mathrm{N}$ has the natural structure of a right D-comodule 1-morphism, as seen from the diagram

$$
(3) \qquad
\begin{array}{ccccc}
\mathrm{M}\square_{\mathrm{C}}\mathrm{N} & \overset{\xi}{\dashrightarrow} & \mathrm{MN} & \underset{1_{\mathrm{M}}\delta_{\mathrm{N}}}{\overset{\delta_{\mathrm{M}}1_{\mathrm{N}}}{\rightrightarrows}} & \mathrm{MCN} \\
{\scriptstyle\delta_{\mathrm{M}\square_{\mathrm{C}}\mathrm{N}}}\big\downarrow & & {\scriptstyle 1_{\mathrm{M}}\gamma}\big\downarrow & & \big\downarrow{\scriptstyle 1_{\mathrm{M}}1_{\mathrm{C}}\gamma} \\
\mathrm{M}\square_{\mathrm{C}}\mathrm{ND} & \overset{\xi 1_{\mathrm{D}}}{\dashrightarrow} & \mathrm{MND} & \underset{1_{\mathrm{M}}\delta_{\mathrm{N}}1_{\mathrm{D}}}{\overset{\delta_{\mathrm{M}}1_{\mathrm{N}}1_{\mathrm{D}}}{\rightrightarrows}} & \mathrm{MCND}.
\end{array}
$$

Here we have two commutative solid squares: the vertical solid maps form a commutative square with the upper horizontal solid maps by interchange law, and with the lower horizontal solid maps due to the fact that N is a C-D-bicomodule 1-morphism. Therefore we obtain the induced dotted map on the right which defines the necessary structure. Due to functoriality of the construction mentioned above, we thus have a functor

$$
\_\square_{\mathrm{C}}\mathrm{N} : \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C}) \to \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{D}).
$$

The following lemma establishes one of the basic familiar properties of this functor.

Lemma 5. *The functor*

$$
\_\square_{\mathrm{C}}\mathrm{C} : \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C}) \to \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})
$$

is isomorphic to the identity functor on $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})$.

Proof. Let (M, δ) be a C-comodule 1-morphism. Consider the diagram

$$M \xrightarrow{\;\delta\;} MC \underset{1_M\tilde{\Delta}}{\overset{\delta 1_C}{\rightrightarrows}} MCC.$$

The left morphism on this diagram equalizes the right two morphisms due to the coassociativity part of the coaction axiom. By the counitality part of the coaction axiom, the composite

$$M \xrightarrow{\;\delta\;} MC \xrightarrow{\;1_M\tilde{\varepsilon}\;} M.$$

is the identity. Therefore δ is monic. By the universal property of kernels, it follows that δ factors through $M\square_C C$ and the factorization map is a monomorphism. This thus defines a monic natural transformation from the identity functor to $\_\square_C C$ and it remains to verify that it is an isomorphism. For this it is enough to construct a monomorphism from $M\square_C C$ to M.

Consider the diagram

$$
\begin{array}{ccc}
M\square_C C & \xrightarrow{\;\xi\;} & MC \underset{1_M\tilde{\Delta}}{\overset{\delta 1_C}{\rightrightarrows}} MCC \\
& {\scriptstyle 1_M\tilde{\varepsilon}}\downarrow & \qquad\qquad\downarrow{\scriptstyle 1_M 1_C\tilde{\varepsilon}} \\
& M & \xrightarrow{\;\delta\;} MC
\end{array}
$$

With respect to the upper of the two arrows from MC to MCC, we have a commutative square by the interchange law. The composite of the lower of the two arrows with the rightmost vertical arrow is the identity by the counitality axiom. As the arrow starting at $M\square_C C$ equalizes these two arrows, all paths from the north-west $M\square_C C$ to the south-east MC on the diagram are equal to ξ. In particular, $(1_M\tilde{\varepsilon}) \circ_v \xi : M\square_C C \to M$ is a right factor of a monomorphism and hence is monic. The claim follows. $\qquad\square$

3.4. An adjunction using cotensor products.
Let $\mathscr{C}$ be a fiat 2-category. Consider two coalgebra 1-morphisms $C = (C, \tilde{\Delta}, \tilde{\varepsilon})$ and $D = (D, \Delta, \varepsilon)$ in $\underline{\mathscr{C}}$. Consider also a monomorphism $\iota : D \to C$ of coalgebra 1-morphisms. In this situation we have an exact embedding

$$\Psi : \mathrm{comod}_{\underline{\mathscr{C}}}(D) \longrightarrow \mathrm{comod}_{\underline{\mathscr{C}}}(C)$$

where, given a D-comodule 1-morphism (M, δ_M), the C-comodule 1-morphism $\Psi(M)$ is defined to be M with the coaction map given by the composite

$$M \xrightarrow{\;\delta_M\;} MD \xrightarrow{\;1_M\iota\;} MC.$$

The monomorphism ι defines on D the structure of a left C-comodule 1-morphism via

$$D \xrightarrow{\;\Delta_D\;} DD \xrightarrow{\;\iota 1_D\;} CD,$$

which, in fact, turns D into a C-D-bicomodule 1-morphism with the regular right coaction of D. The aim of this subsection is to show that the functor $\Phi := \_\square_C D$ is right adjoint to Ψ and to describe some basic properties of this adjunction. All diagrams in this section will be diagrams in $\underline{\mathscr{C}}$.

For a C-comodule 1-morphism M, we denote by ξ_M the embedding of $M\square_C D$ into MD given by the definition of $\_\square_C D$.

Lemma 6. *Let* (M, δ) *be a* C*-comodule* 1*-morphism. Then the composite*

$$\mathrm{M}\square_{\mathrm{C}}\mathrm{D} \overset{\xi_{\mathrm{M}}}{\longrightarrow} \mathrm{MD} \overset{1_{\mathrm{M}}\varepsilon}{\longrightarrow} \mathrm{M}$$

is a monomorphism.

We denote the composite monomorphism given by Lemma 6 by ζ_{M}.

Proof. Consider the diagram

(4)

Here the upper part commutes by the interchange law and the lower part commutes by the axioms of a coalgebra. Hence this diagram commutes.

Now consider the diagram

(5)

The part of this diagram which can be recognized as (4) commutes by the previous paragraph. The right square commutes by the interchange law. And the map ξ_{M} is, by definition, the equalizer of the middle triangle. Therefore, if we start at $\mathrm{M}\square_{\mathrm{C}}\mathrm{D}$, then the whole diagram commutes.

In particular, the composite $\delta \circ_{\mathrm{v}} (1_{\mathrm{M}}\varepsilon) \circ_{\mathrm{v}} \xi_{\mathrm{M}}$ equals the composite of the monomorphism ξ_{M} and the monomorphism $1_{\mathrm{M}}\iota$. Hence $\delta \circ_{\mathrm{v}} (1_{\mathrm{M}}\varepsilon) \circ_{\mathrm{v}} \xi_{\mathrm{M}}$ is a monomorphism. Consequently, $(1_{\mathrm{M}}\varepsilon) \circ_{\mathrm{v}} \xi_{\mathrm{M}}$ is a monomorphism, as claimed. $\square$

As an immediate corollary from Lemma 6, we have:

Corollary 7. *The functor* $\Phi\Psi$ *is isomorphic to the identity functor on* $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{D})$.

Proof. Let (M, δ) be a D-comodule 1-morphism. Consider the diagram

$$\mathrm{MD} \underset{1_{\mathrm{M}}\Delta}{\overset{\delta 1_{\mathrm{D}}}{\rightrightarrows}} \mathrm{MDD} \overset{1_{\mathrm{M}}\iota 1_{\mathrm{D}}}{\longrightarrow} \mathrm{MCD}$$

By definition, $\mathrm{M}\square_{\mathrm{D}}\mathrm{D}$ is the equalizer of the two morphisms on the left and $\mathrm{M}\square_{\mathrm{C}}\mathrm{D}$ is the equalizer of the two morphisms from the left to the right. As ι is monic and composition in $\underline{\mathscr{C}}$ is left exact by construction, we get that $1_{\mathrm{M}}\iota 1_{\mathrm{D}}$ is monic. Hence $\mathrm{M}\square_{\mathrm{C}}\mathrm{D} \cong \mathrm{M}\square_{\mathrm{D}}\mathrm{D}$ and the latter is isomorphic to M by Lemma 5. The claim follows. $\square$

Lemma 8. *There is a monic natural transformation from* $\Psi\Phi$ *to the identity functor on* $\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})$.

Proof. Let (M, δ) be a C-comodule 1-morphism. Consider the diagram

(6)

$$
\begin{array}{ccc}
& M \xrightarrow{\ \delta\ } MC \\
\Phi(M) \xrightarrow{\ \xi_M\ } MD \xrightarrow{\ \delta 1_D\ } MCD \\
\Phi(M)D \xrightarrow{\ \xi_M 1_D\ } MDD \\
\Phi(M)C \xrightarrow{\ \xi_M 1_C\ } MDC \xrightarrow{\ 1_M \iota 1_C\ } MCC.
\end{array}
$$

with vertical maps $1_M \varepsilon$, $1_M 1_C \varepsilon$, $1_M \Delta$, $1_M \iota 1_D$, $1_M 1_C \iota$, $1_M 1_C \tilde{\varepsilon}$, $1_{\Phi(M)} \iota$, $1_M 1_D \iota$

We claim that everything, except for the triangle in the middle of this diagram, commutes. The top square in the first column commutes by the definition of Φ. The bottom square in the first column commutes by the interchange law. The rightmost triangle commutes as ι is a homomorphism of coalgebra 1-morphisms. The top square in the middle column commutes by the interchange law. The lower quadrilateral in the middle column also commutes by the interchange law.

Starting with ξ_M, the triangle in the middle commutes by the definition of Φ. This means that all paths in (6) starting at $\Phi(M)$ and ending in MC commute.

Next we consider the diagram

(7)

$$
\begin{array}{ccc}
D \xrightarrow{\ \Delta\ } DD \underset{1_D \varepsilon}{\overset{\varepsilon 1_D}{\rightrightarrows}} D \\
\downarrow{\iota} \qquad \downarrow{\iota\iota} \qquad \downarrow{\iota} \\
C \xrightarrow{\ \tilde{\Delta}\ } CC \underset{1_C \tilde{\varepsilon}}{\overset{\tilde{\varepsilon} 1_C}{\rightrightarrows}} C.
\end{array}
$$

This diagram commutes since ι is a homomorphism of coalgebra 1-morphisms (in fact, all horizontal paths from the extreme left all the way to the extreme right are the identities). From (7) it follows that, in the path

$$MD \to MDD \to MDC \to MCC \to MC$$

in (6), the very last step $1_M 1_C \tilde{\varepsilon}$ can be exchanged with $1_M \tilde{\varepsilon} 1_C$ without changing the whole composite.

The latter, in turn, implies that $(1_M \varepsilon) \circ_v \xi_M$ is a natural transformation from $\Psi\Phi$ to the identity functor on $\mathrm{comod}_{\mathscr{C}}(C)$. It is monic by Lemma 6, as claimed. □

Corollary 9. *The functor Φ is right adjoint to Ψ.*

Proof. Let α denote the natural transformation from the identity functor on $\mathrm{comod}_{\mathscr{C}}(D)$ to $\Phi\Psi$ constructed in Corollary 7. Let β denote the natural transformation from $\Psi\Phi$ to the identity functor on $\mathrm{comod}_{\mathscr{C}}(C)$ constructed in Lemma 8.

Let (M, δ_M) be a D-comodule 1-morphism and (N, δ_N) a C-comodule 1-morphism. We have to check that both composites

(8) $\Phi(N) \xrightarrow{\alpha_{\Phi(N)}} \Phi\Psi\Phi(N) \xrightarrow{\Phi(\beta_N)} \Phi(N)$ and $\Psi(M) \xrightarrow{\Psi(\alpha_M)} \Psi\Phi\Psi(M) \xrightarrow{\beta_{\Psi(M)}} \Psi(M)$

are the identities.

By definition of α, we have the commutative diagram

(9)
$$\begin{array}{ccc} M & \xrightarrow{\ \delta_M\ } & MD \\ {\scriptstyle \alpha_M}\downarrow & & \| \\ \Phi\Psi(M) & \xrightarrow{\ \xi_{\Psi(M)}\ } & \Psi(M)D. \end{array}$$

Applying Ψ to the left vertical arrow gives the commutative diagram

(10)
$$\begin{array}{ccc} \Psi(M) & \xrightarrow{\ \delta_M\ } & MD \\ {\scriptstyle \alpha_M=\Psi(\alpha_M)}\downarrow & & \| \\ \Psi\Phi\Psi(M) & \xrightarrow{\ \xi_{\Psi(M)}\ } & \Psi(M)D. \end{array}$$

Similarly, by the definition of β, we have the commutative diagram

(11)
$$\begin{array}{ccc} \Psi\Phi(N) & \xrightarrow{\ \xi_N\ } & ND \\ & {\scriptstyle \beta_N}\searrow & \downarrow{\scriptstyle 1_N\varepsilon} \\ & & N \end{array}$$

For $N = \Psi(M)$, we get the commutative diagram

(12)
$$\begin{array}{ccc} \Psi\Phi\Psi(M) & \xrightarrow{\ \xi_{\Psi(M)}\ } & \Psi(M)D \\ & {\scriptstyle \beta_{\Psi(M)}}\searrow & \downarrow{\scriptstyle 1_{\Psi(M)}\varepsilon} \\ & & \Psi(M) \end{array}$$

Putting (10) on top of (12) and using the counitality axiom for M verifies that the second composite in (8) is the identity.

To verify that the first composite in (8) is the identity, consider the diagram

(13)
$$\begin{array}{ccccc} & & ND & & \\ & {\scriptstyle \xi_N}\nearrow & & \searrow{\scriptstyle 1_N\Delta} & \\ \Phi(N) & \xrightarrow{\ \Phi(\delta_N)\ } & \Phi(N)D & \xrightarrow{\ \xi_N 1_D\ } & NDD \\ {\scriptstyle \alpha_{\Phi(N)}}\downarrow & {\scriptstyle \xi_{\Psi\Phi(N)}}\nearrow & & & \downarrow{\scriptstyle 1_N\varepsilon 1_D} \\ \Phi\Psi\Phi(N) & \xrightarrow{\ \Phi(\beta_N)\ } & \Phi(N) & \xrightarrow{\ \xi_N\ } & ND. \end{array}$$

Here the left triangle is just (9) evaluated at $\Phi(N)$ and hence commutes. Note that $\beta_N := (1_N\varepsilon) \circ_v \xi_N$ by (11). The pentagon commutes due to the definition of how Φ acts on morphisms, see the paragraph just before Lemma 6. Indeed, from the definition of β_N it follows that the composite $\Phi(N)D \to NDD \to ND$ is $\beta_N 1_D$. Applying the definition of how Φ acts on β_N, we get the commutativity of the pentagon. The top part of the diagram commutes by (3). Therefore the whole diagram (13) commutes.

Consequently, we have

$$\xi_N \circ_v \Phi(\beta_N) \circ_v \alpha_{\Phi(N)} = (1_N\varepsilon 1_D) \circ_v (1_N\Delta) \circ_v \xi_N = \xi_N,$$

where the last equality follows from the counitality axiom for D. Using monicity of ξ_N, this verifies that the first composite in (8) is the identity morphism as well. This completes the proof. $\qquad\square$

As an immediate consequence, we may now record:

Corollary 10. *The functor Φ maps injective comodule 1-morphisms to injective co-module 1-morphisms and is full on injective comodule 1-morphisms.*

Proof. By Corollary 9, Φ is right adjoint to an exact functor. Therefore Φ maps injective comodule 1-morphisms to injective comodule 1-morphisms. Let $X, Y \in \mathrm{inj}_{\underline{\mathscr{C}}}(C)$. Let $\varphi : \Phi(X) \to \Phi(Y)$ be a homomorphism. Consider $\Psi(\varphi) : \Psi\Phi(X) \to \Psi\Phi(Y)$. We can now use Lemma 8 to embed $\Psi\Phi(Y)$ into Y and then use the injectivity of Y to lift $\Psi(\varphi)$ to a morphism $\psi : X \to Y$. Then $\Phi(\psi) = \varphi$ by Corollary 7. This completes the proof. $\qquad\square$

Proposition 11. *The functor Φ is a homomorphism of 2-representations of $\mathscr{C}$.*

Proof. Taking into account the exactness of the action of $\mathscr{C}$, the assertion follows from the construction and the fact that Ψ is a strict homomorphism of $\mathscr{C}$-modules. Alternatively, one can argue as [Ke] that Φ has a canonical structure of a homomorphism of $\mathscr{C}$-modules

$$\tag{14} \mathrm{F}\Phi(Y) \cong \Phi\Psi(\mathrm{F}\Phi(Y)) = \Phi(\mathrm{F}\Psi\Phi(Y)) \to \Phi(\mathrm{F}\,Y)$$

given by adjunction and the fact that Ψ is a strict homomorphism of 2-representations of $\mathscr{C}$. Lemma (8) implies that the right map in (14) is monic. For $X \in \mathrm{inj}_{\underline{\mathscr{C}}}(D)$, we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(X, \Phi(\mathrm{F}\,Y)) &\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(\Psi(X), \mathrm{F}\,Y) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(\mathrm{F}^{\star}\,\Psi(X), Y) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(\Psi(\mathrm{F}^{\star}\,X), Y) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(\mathrm{F}^{\star}\,X, \Phi(Y)) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(D)}(X, \mathrm{F}\,\Phi(Y)).
\end{aligned}
$$

This shows that $\mathrm{F}\Phi(Y)$ is isomorphic to $\Phi(\mathrm{F}\,Y)$ and, by finite-dimensionality of morphisms spaces, the monic right map in (14) is, in fact, and isomorphism. The claim follows. $\qquad\square$

3.5. **An alternative description of Φ.** Let $\mathscr{C}$ be a fiat 2-category. Consider two coalgebra 1-morphisms $C = (C, \tilde{\Delta}, \tilde{\varepsilon})$ and $D = (D, \Delta, \varepsilon)$ in $\underline{\mathscr{C}}$. Consider also a monomorphism $\iota : D \to C$ of coalgebra 1-morphisms and let (J, κ) be the cokernel of ι. Then J is a *coideal* of C in the sense that there exist 2-morphisms $\lambda : J \to CJ$ and $\rho : J \to JC$ such that the following diagrams commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\tilde{\Delta}} & CC \\
{\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle 1\kappa} \\
J & \xrightarrow{\lambda} & CJ
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
C & \xrightarrow{\tilde{\Delta}} & CC \\
{\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle \kappa 1} \\
J & \xrightarrow{\rho} & JC.
\end{array}
$$

Note that we have the exact sequence

$$0 \longrightarrow D \xrightarrow{\ \iota\ } C \xrightarrow{\ \kappa\ } J \longrightarrow 0.$$

Let (M, δ_{M}) be a D-comodule 1-morphism and (N, δ_{N}) be a C-comodule 1-morphism. Let $\varphi : M \to N$ be a morphism of C-comodule 1-morphisms from $\Psi(M)$ to N. Consider

the following diagram:

(15)

The solid part of this diagram commutes by our assumptions. We define (K_0, τ_0) as the kernel of $(1_N \kappa) \circ_v \delta_N$ with $(N/K_0, \pi_0)$ being the cokernel of τ_0. This gives the dashed parts of the diagram. As the solid part commutes, the map from M to NJ is zero and hence factors through K_0 by the universal property of kernels. Next we note that the solid sequence

$$ND \hookrightarrow NC \longrightarrow NJ$$

is left exact. Since the map from K_0 to NJ is zero by definition of K_0, this map factors through ND by the universal property of kernels. This gives the dotted part of the diagram and the whole diagram commutes.

We would like to construct a D-subcomodule 1-morphism of N. However, note that K_0 is not yet a D-comodule 1-morphism as we have a map from K_0 to ND, while we need a map from K_0 to $K_0 D$. Therefore we need to do more.

Let (K_1, τ_1) be the kernel of $(\pi_0 1_D) \circ_v \zeta_0$. By commutativity of (15), the map from M to $N/K_0 D$ factors through $K_0 D$ and hence is zero. Therefore ξ_0 factors via K_1 through some map $\xi_1 : M \to K_1$. Moreover, as the sequence

$$K_0 D \hookrightarrow ND \longrightarrow N/K_0 D$$

is left exact, the map from K_1 to ND factors through some $\zeta_1 : K_1 \to K_0 D$. Now we can continue this and construct a sequence of monomorphisms

$$\ldots K_2 \overset{\tau_2}{\hookrightarrow} K_1 \overset{\tau_1}{\hookrightarrow} K_0,$$

with corresponding cokernels $(K_i/K_{i+1}, \pi_i)$ together with factorization maps

$$\xi_i : M \to K_i \qquad \text{and} \qquad \zeta_i : K_i \to K_{i-1} D$$

such that the obtained infinite diagram commutes.

As $\mathscr{C}$ is finitary, this sequence must stabilize, that is $K_i = K_{i+1}$, for some i. In this case K_i becomes a D-comodule 1-morphism by construction. We can now set $\tilde{\Phi}(N) := K_i$ and by functoriality of the above construction this defines a functor from $\mathrm{comod}_{\underline{\mathscr{C}}}(C)$ to $\mathrm{comod}_{\underline{\mathscr{C}}}(D)$. The construction also automatically gives a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{comod}_{\underline{\mathscr{C}}}(C)}\big(\Psi(M), N\big) \cong \mathrm{Hom}_{\mathrm{comod}_{\underline{\mathscr{C}}}(D)}\big(M, \tilde{\Phi}(N)\big)$$

making $(\Psi, \tilde{\Phi})$ into an adjoint pair of functors. By uniqueness of adjoint functors, the functor $\tilde{\Phi}$ defined in this subsection is isomorphic to the functor $\_\square_{\mathrm{C}}\mathrm{D}$ from Subsection 3.4.

3.6. **Cosimple coalgebra 1-morphisms and simple transitive 2-representations.** From the above, we deduce a useful corollary strengthening [MMMT, Corollary 11].

Corollary 12. *Let $\mathbf{M}$ be a transitive 2-representation and $X \in \mathbf{M}(\mathtt{i})$. Consider the coalgebra 1-morphism C associated to $\mathbf{M}$ and X. Then C is cosimple if and only if $\mathbf{M}$ is simple transitive.*

Proof. Assume C is cosimple and $\mathbf{M}$ is not simple transitive. Let $\mathbf{N}$ be the simple transitive quotient of $\mathbf{M}$. Then the natural projection $\mathbf{M} \twoheadrightarrow \mathbf{N}$ is not an equivalence. Therefore the homomorphism between the coalgebra 1-morphisms C and D, for $\mathbf{M}$ and $\mathbf{N}$, respectively, given by Proposition 4 is not an isomorphism. Therefore C is not cosimple, a contradiction.

Assume now that C is not cosimple, then we have a monomorphism

$$(16) \qquad\qquad \iota : \mathrm{D} \hookrightarrow \mathrm{C}$$

for some coalgebra 1-morphism D such that ι is not an isomorphism. Consider the functor $\_\square_{\mathrm{C}}\mathrm{D}$ from Subsection 3.4. As ι is not an isomorphism and, by Corollary 10, $\_\square_{\mathrm{C}}\mathrm{D}$ is full when restricted to injective comodule 1-morphisms, we obtain that $\mathrm{inj}_{\mathscr{C}}(\mathrm{D})$ is a proper quotient of $\mathrm{inj}_{\mathscr{C}}(\mathrm{C})$, and hence $\mathrm{inj}_{\mathscr{C}}(\mathrm{C})$ is not simple transitive. The claim follows. $\qquad\square$

4. Centralizers in fiat 2-categories

4.1. **A general observation.** The following statement is similar in spirit to [MMMT, Corollary 12].

Proposition 13. *Let $\mathscr{C}$ be a fiat 2-category and $\mathscr{A}$ a 2-subcategory of $\mathscr{C}$. We assume that*

(a) *on the 2-level, $\mathscr{A}$ is 2-full;*

(b) *on the 1-level, $\mathscr{A}$ is isomorphism closed and fully additively closed (in the sense of both being additively closed and idempotent complete);*

(c) *$\mathscr{A}$ is stable under the weak involution $\star$ of $\mathscr{C}$.*

Then every simple transitive 2-representation of $\mathscr{A}$ is equivalent to a simple transitive subquotient of the restriction, from $\mathscr{C}$ to $\mathscr{A}$, of a simple transitive 2-representation of $\mathscr{C}$.

Proof. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{A}$. Note that the 2-category $\mathscr{A}$ is fiat due to assumption (c). Therefore, by [MMMT, Theorem 9], there is a coalgebra 1-morphism $\mathrm{C} \in \underline{\mathscr{A}}$ such that $\mathbf{M}$ is equivalent to $\mathrm{inj}_{\mathscr{A}}(\mathrm{C})$. Thanks to assumptions (a) and (b), the 2-category $\underline{\mathscr{A}}$ is, naturally, a full subcategory of $\underline{\mathscr{C}}$. This implies that C is also a coalgebra 1-morphism in $\underline{\mathscr{C}}$ and $\mathrm{inj}_{\mathscr{A}}(\mathrm{C})$ is, naturally, a full subcategory of $\mathrm{inj}_{\mathscr{C}}(\mathrm{C})$. The latter is a finitary 2-representation of $\mathscr{C}$. It has a unique simple transitive subquotient $\mathbf{N}$ which contains $\mathrm{inj}_{\mathscr{A}}(\mathrm{C})$ and in which the objects of $\mathrm{inj}_{\mathscr{A}}(\mathrm{C})$ are non-zero. Therefore $\mathbf{M}$ is equivalent to a simple transitive subquotient of the restriction of $\mathbf{N}$ from $\mathscr{C}$ to $\mathscr{A}$. $\qquad\square$

4.2. **The special case of an $\mathcal{H}$-cell.** In this subsection, we strengthen the claim of Proposition 13 for a very special 2-subcategory $\mathscr{A}$ of $\mathscr{C}$. We assume that $\mathscr{C}$ is a fiat 2-category with a unique maximal two-sided cell $\mathcal{J}$ with respect to the two-sided order. By [KiM, Corollary 19], the two-sided cell $\mathcal{J}$ is regular. Let $\mathcal{L}$ be a left cell in $\mathcal{J}$ and G a Duflo involution in $\mathcal{L}$. Finally, set $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^\star$ (here $\mathcal{L}^\star$ can be alternatively described as the right cell containing G). Let $\mathscr{A} = \mathscr{A}_{\mathcal{H}}$ be the 2-full 2-subcategory of $\mathscr{C}$ generated by all 1-morphisms in $\mathcal{H}$ together with the (unique) relevant identity morphism, that is, the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$, where $\mathtt{i}$ is the domain of all 1-morphisms in $\mathcal{L}$. Then $\mathscr{A}$ is fiat and its indecomposable 1-morphisms are $\mathbb{1}_{\mathtt{i}}$ and those in $\mathcal{H}$. Moreover, $\mathcal{H}$ is a two-sided cell in $\mathscr{A}$.

Proposition 14. *In the above setup, let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}$ with apex $\mathcal{J}$. Then the restriction of $\mathbf{M}$ to $\mathscr{A}$ contains a unique simple transitive subquotient with apex $\mathcal{H}$.*

Proof. Let F be a multiplicity free sum of representatives of isomorphism classes of indecomposable 1-morphisms in $\mathcal{H}$. Note that

$$\mathcal{H}^\star = (\mathcal{L} \cap \mathcal{L}^\star)^\star = \mathcal{L} \cap \mathcal{L}^\star = \mathcal{H},$$

which implies that $\mathrm{F}^\star = \mathrm{F}$.

Consider $\mathbf{M}$ as a 2-representation of $\mathscr{A}$ by restriction. Let $\mathbf{N}$ be the 2-subrepresentation given by the action of $\mathscr{A}$ on the additive closure of all objects of the form $\mathrm{F}\,M$, where $M \in \mathbf{M}(\mathtt{i})$. To prove our proposition, it remains to show that $\mathbf{N}$ is transitive.

First, we observe that $\mathrm{add}(\mathrm{F}\,M) = \mathrm{add}(\mathrm{F}\,M')$, for all $M, M' \in \mathbf{M}(\mathtt{i})$ such that $\mathrm{F}\,M \neq 0$ and $\mathrm{F}\,M' \neq 0$. Indeed, as $\mathrm{F}\,M' \neq 0$ and $\mathbf{M}$ is a transitive 2-representation of $\mathscr{C}$, there is a 1-morphism H in $\mathscr{C}$ such that M belongs to $\mathrm{add}(\mathrm{HF}\,M')$. This means that $\mathrm{add}(\mathrm{F}\,M) \subset \mathrm{add}(\mathrm{FHF}\,M')$. However, FHF is in $\mathrm{add}(\mathrm{F})$ as $\mathcal{J}$ is regular and $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^\star$. Therefore $\mathrm{add}(\mathrm{F}\,M) \subset \mathrm{add}(\mathrm{F}\,M')$. The opposite inclusion follows by symmetry.

Now we argue that the claim of the previous paragraph implies transitivity of $\mathbf{N}$. From the previous paragraph, we have that $\mathbf{N}(\mathtt{i})$ coincides with $\mathrm{add}(\mathrm{F}\,M)$, for any $M \in \mathbf{M}(\mathtt{i})$ such that $\mathrm{F}\,M \neq 0$. Fix M such that $\mathrm{F}\,M \neq 0$ and let $N \in \mathbf{N}(\mathtt{i})$. Then

$$0 \neq \mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(N, \mathrm{F}\,M) \cong \mathrm{Hom}_{\mathbf{M}(\mathtt{i})}(\mathrm{F}\,N, M),$$

as $\mathrm{F} = \mathrm{F}^\star$, which implies $\mathrm{F}\,N \neq 0$. By the previous paragraph, $\mathrm{add}(\mathrm{F}\,N) = \mathbf{N}(\mathtt{i})$ which establishes transitivity of $\mathbf{N}$. $\square$

4.3. **The main result.** Here we continue to work in the setup of Subsection 4.2.

Proposition 14 defines a map, which we call Θ, from the set of equivalence classes of simple transitive 2-representations of $\mathscr{C}$ with apex $\mathcal{J}$ to the set of equivalence classes of simple transitive 2-representations of $\mathscr{A}$ with apex $\mathcal{H}$. Explicitly, we take a simple transitive 2-representation $\mathbf{M}$ of $\mathscr{C}$ and define $\Theta(\mathbf{M})$ to be the simple transitive quotient of the 2-subrepresentation of the restriction of $\mathbf{M}$ to $\mathscr{A}$, given by the action of $\mathscr{A}$ on the additive closure of all objects of the form $\mathrm{F}\,M$, for $M \in \mathbf{M}(\mathtt{i})$.

Proposition 13 defines a map Ω in the opposite direction as follows:

- take a simple transitive 2-representation $\mathbf{N}$ of $\mathscr{A}$ with apex $\mathcal{H}$;

- let X be a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\mathbf{N}(\mathtt{i})$;

- let D_X be the unique (up to isomorphism) coalgebra 1-morphism in $\underline{\mathscr{A}}$ constructed in [MMMT, Lemma 5] such that $\mathbf{N}$ is equivalent to $\mathrm{inj}_{\underline{\mathscr{A}}}(\mathrm{D}_X)$;

- set $\Omega(\mathbf{N})$ to be the unique simple transitive quotient of $\mathrm{inj}_{\mathscr{C}}(\mathrm{D}_X)$ which contains $\mathrm{inj}_{\mathscr{A}}(\mathrm{D}_X)$.

Theorem 15. *The maps Θ and Ω constructed above provide mutually inverse bijections*

$$\Theta\colon \left\{ \begin{array}{c} \text{equivalence classes of simple} \\ \text{transitive 2-representations} \\ \text{of } \mathscr{C} \text{ with apex } \mathcal{J} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{equivalence classes of simple} \\ \text{transitive 2-representations} \\ \text{of } \mathscr{A} \text{ with apex } \mathcal{H} \end{array} \right\} \colon \Omega.$$

To prove Theorem 15, we will need the following lemma.

Lemma 16. *Assume that $\mathscr{C}$ is $\mathcal{J}$-simple. Any non-zero 2-ideal in $\underline{\mathscr{C}}$ contains a non-zero 2-ideal of $\mathscr{C}$, in particular, it contains the identity 2-morphisms for all 1-morphisms in $\mathcal{J}$.*

Proof. Let H and K be two 1-morphisms in $\underline{\mathscr{C}}$ and $\alpha : \mathrm{H} \to \mathrm{K}$ a non-zero 2-morphism contained in our non-zero 2-ideal in $\underline{\mathscr{C}}$. Let G be a multiplicity-free sum of representatives of isomorphism classes of all indecomposable 1-morphisms in $\mathcal{J}$. Consider the 2-morphism

$$\beta := 1_{\mathrm{G}}\alpha 1_{\mathrm{G}} : \mathrm{GHG} \to \mathrm{GKG},$$

which is then also in the 2-ideal.

We claim that β is non-zero and that it induces an isomorphism between some summands of GHG and GKG, moreover, these summands belong both to $\mathscr{C}$ and to $\mathcal{J}$, up to isomorphism.

Let $\mathcal{L}$ be a left cell in $\mathcal{J}$. Consider the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$. As $\mathscr{C}$ is $\mathcal{J}$-simple, this 2-representation is faithful and its abelianization $\underline{\mathbf{C}_{\mathcal{L}}}$ is a faithful 2-representation of $\underline{\mathscr{C}}$ due to faithfulness of representability of functors by bimodules. By [KMMZ, Theorem 2], in this 2-representation, the 1-morphism G is represented by a projective functor. As $\mathbf{C}_{\mathcal{L}}$ is transitive and $\mathscr{C}$ is fiat, this projective functor is an additive generator of the category of all projective endofunctors of $\coprod_{\mathbf{i}\in\mathscr{C}} \mathbf{C}_{\mathcal{L}}(\mathbf{i})$.

Our first claim is that, in the extension of $\underline{\mathbf{C}_{\mathcal{L}}}$ to $\underline{\mathscr{C}}$, both GHG and GKG act by non-zero projective functors. We prove the claim for GHG; for GKG the arguments are similar. Assume H is given by a diagram $\mathrm{H}_1 \xrightarrow{\gamma} \mathrm{H}_2$, where H_1 and H_2 are now 1-morphisms in $\mathscr{C}$. Then GHG is given by the diagram $\mathrm{GH}_1\mathrm{G} \xrightarrow{1_{\mathrm{G}}\gamma 1_{\mathrm{G}}} \mathrm{GH}_2\mathrm{G}$. Here any indecomposable summand of both $\mathrm{GH}_1\mathrm{G}$ and $\mathrm{GH}_2\mathrm{G}$ is in $\mathcal{J}$.

We claim that there are decompositions

(17) $$\mathrm{GH}_1\mathrm{G} \cong \mathrm{X}_1 \oplus \mathrm{X}_2 \quad \text{and} \quad \mathrm{GH}_2\mathrm{G} \cong \mathrm{Y}_1 \oplus \mathrm{Y}_2$$

such that $1_{\mathrm{G}}\gamma 1_{\mathrm{G}}$ annihilates X_1 and induces an isomorphism between X_2 and Y_1. Let B denote the underlying algebra of $\mathbf{C}_{\mathcal{L}}$, that is, $\coprod_{\mathbf{i}\in\mathscr{C}} \mathbf{C}_{\mathcal{L}}(\mathbf{i})$ is equivalent to B-proj. Assume, for simplicity, that G is represented by the B-B-bimodule $Q_1 \otimes_{\Bbbk} P_1$, where Q_1 is a left projective B-module and P_1 is a right projective B-module. Assume, moreover, that H_1 is similarly represented by the B-B-bimodule $Q_2 \otimes_{\Bbbk} P_2$ and that H_2 is represented by the B-B-bimodule $Q_3 \otimes_{\Bbbk} P_3$. Note that, in general, 1-morphisms in $\mathscr{C}$ will be represented by sums of bimodules of such form but, as our arguments extend easily to this general case, we will save the significant notational and technical difficulties involved. Then $1_{\mathrm{G}}\gamma 1_{\mathrm{G}}$ corresponds to a homomorphism of B-B-bimodules

$$Q_1 \otimes_{\Bbbk} P_1 \otimes_B Q_2 \otimes_{\Bbbk} P_2 \otimes_B Q_1 \otimes_{\Bbbk} P_1 \to Q_1 \otimes_{\Bbbk} P_1 \otimes_B Q_3 \otimes_{\Bbbk} P_3 \otimes_B Q_1 \otimes_{\Bbbk} P_1.$$

Both functors are isomorphic to direct sums of copies of $Q_1 \otimes_{\Bbbk} P_1$ with multiplicity vector spaces given by $P_1 \otimes_B Q_2 \otimes_{\Bbbk} P_2 \otimes_B Q_1$ and $P_1 \otimes_B Q_3 \otimes_{\Bbbk} P_3 \otimes_B Q_1$, respectively. The morphism $1_{\mathrm{G}}\gamma 1_{\mathrm{G}}$ defines a linear map between these multiplicity spaces.

Now, as the bimodule representing G is an additive generator for projective B-B-bimodules, it follows that both multiplicity spaces are non-zero and the morphism defined by $1_\mathrm{G}\gamma 1_\mathrm{G}$ is non-zero. By choosing appropriate bases in these multiplicity spaces, we obtain the decomposition (17). The definition of injective abelianization implies that $\mathrm{GH}_1\mathrm{G} \xrightarrow{1_\mathrm{G}\gamma 1_\mathrm{G}} \mathrm{GH}_2\mathrm{G}$ is isomorphic to X_1. The latter is in $\mathscr{C}$ and each of its indecomposable summands is in $\mathcal{J}$.

Now we claim that there are decompositions $\mathrm{GHG} \cong \mathrm{X}_1 \oplus \mathrm{X}_2$ and $\mathrm{GKG} \cong \mathrm{Y}_1 \oplus \mathrm{Y}_2$ such that β annihilates X_1 and induces an isomorphism between X_2 and Y_1. This is proved by the same arguments as above, which also show that, since the bimodule representing G is an additive generator for projective B-B-bimodules, β is non-zero. This implies that X_2 is non-zero and the identity 1-morphism on X_2 belongs to our 2-ideal.

Thus, any non-zero 2-ideal in $\underline{\mathscr{C}}$ contains a non-zero 2-ideal of $\mathscr{C}$. The rest of the claim follows from $\mathcal{J}$-simplicity of $\mathscr{C}$. $\square$

Proof of Theorem 15. It follows immediately from the definitions that $\Theta \circ \Omega$ is the identity.

To prove that $\Omega \circ \Theta$ is the identity, let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}$ with apex $\mathcal{J}$. Without loss of generality we may assume that $\mathscr{C}$ is $\mathcal{J}$-simple. As before, we let F be a multiplicity free sum of all indecomposable 1-morphisms in $\mathcal{H}$. Let $X \in \mathbf{M}(\mathtt{i})$ be a multiplicity free direct sum of representatives of isomorphism classes of all indecomposable objects which appear in $\mathrm{F}\,Y$, where $Y \in \mathbf{M}(\mathtt{i})$ is an additive generator (note that $X \neq 0$ as $0 \neq \mathbf{M}(\mathrm{F})$ and $\mathrm{add}(\mathbf{M}(\mathrm{F})\mathbf{M}(\mathrm{F})) = \mathrm{add}(\mathbf{M}(\mathrm{F}))$). Note also that, due to Proposition 14, our choice of X here fits with the choice of X which was used for the definition of Ω above. By [MMMT, Theorem 9], there exists a coalgebra 1-morphism C_X in $\underline{\mathscr{C}}$ such that $\mathbf{M}$ is equivalent to $\mathrm{inj}_{\underline{\mathscr{C}}}(\mathrm{C}_X)$ and such that, for all H in $\mathscr{C}$, we have

$$\mathrm{Hom}_{\underline{\mathbf{M}}}(X, \mathrm{H}\,X) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(\mathrm{C}_X, \mathrm{H}).$$

The coalgebra object C_X is cosimple by Corollary 12.

Consider the inclusion 2-functor $\iota\colon \mathscr{A} \hookrightarrow \mathscr{C}$. This extends to a 2-functor $\underline{\mathscr{A}} \hookrightarrow \underline{\mathscr{C}}$, whose component functors are left exact and fully faithful, and send injective objects to injective objects. We denote by τ its (component-wise) left adjoint (cf. [Au, KhM]), implying we have adjunction morphisms $\varepsilon\colon \tau\iota \to \mathrm{Id}_{\underline{\mathscr{A}}}$ and $\eta\colon \mathrm{Id}_{\underline{\mathscr{C}}} \to \iota\tau$, the first being an isomorphism. Using these, for any $\mathrm{G}, \mathrm{H} \in \underline{\mathscr{C}}$, we have a morphism

$$\beta_{\mathrm{GH}}\colon \tau(\mathrm{GH}) \to \tau(\iota\tau(\mathrm{G})\iota\tau(\mathrm{H})) = \tau\iota(\tau(\mathrm{G})\tau(\mathrm{H})) \to \tau(\mathrm{G})\tau(\mathrm{H})$$

given by the composition of $\tau(\eta_\mathrm{G} \circ_\mathrm{h} \eta_\mathrm{H})$ with $\varepsilon_{\tau(\mathrm{G})\tau(\mathrm{H})}$, using, in the middle, that ι is a 2-functor, see [Ke] for details. Consequently, τ is an oplax 2-functor. In particular, $\tau(\mathrm{C}_X)$ has the canonical structure of a coalgebra 1-morphism (cf. [Ke]). The counit of $\tau(\mathrm{C}_X)$ is the image of the counit of C_X under τ (note that τ, as a quotient functor, preserves the identity 1-morphism). The comultiplication for $\tau(\mathrm{C}_X)$ is the image of the comultiplication of C_X under τ composed with $\beta_{\mathrm{C}_X\mathrm{C}_X}$.

As ι is a 2-functor, by a similar reasoning $\iota\tau(\mathrm{C}_X)$ has the canonical structure of a coalgebra 1-morphism. Furthermore, from the definitions it follows that the evaluation of η at C_X is a homomorphism of coalgebra 1-morphisms. This homomorphism induces a morphism of 2-representations $\Psi\colon \mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C}_X) \to \mathrm{comod}_{\underline{\mathscr{C}}}(\iota\tau(\mathrm{C}_X))$. Note that Ψ is exact and faithful by construction.

Now we claim that τ is faithful. Indeed, if τ were not faithful, then, by Lemma 16, τ would annihilate all identity 2-morphisms for all 1-morphisms in $\mathcal{J}$. This contradicts the fact that τ is adjoint to ι and that ι is faithful.

The fact that τ is faithful implies that η_{C_X} is monic. Indeed the adjunction axioms (evaluated at C_X) yield that $\tau(\eta_{C_X})$ is monic, meaning τ kills the kernel of η_{C_X}. As τ is faithful, the kernel of η_{C_X} must be zero. Hence, by Subsection 3.4, we also have the adjoint functor Φ which maps $\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))$ to $\mathrm{inj}_{\underline{\mathscr{C}}}(C_X)$ which is full on injective objects.

Now, for $G, H \in \mathcal{L}$, we compute

(18)
$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))}(G\,\iota\tau(C_X), H\,\iota\tau(C_X)) &\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))}(\iota\tau(C_X), G^\star H\,\iota\tau(C_X)) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))}(\iota\tau(C_X), \iota G^\star H\,\tau(C_X)) \\
&\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X))}(\tau(C_X), G^\star H\,\tau(C_X)) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{A}}}(\tau(C_X), G^\star H).
\end{aligned}
$$

Here in the first line we use involutivity of $\star$; in the second line we use that $G^* H \in \mathcal{H}$, for any $G, H \in \mathcal{L}$, and hence $G^* H$ commutes with ι; in the third line we use that ι is full and faithful; and in the fourth line we use [MMMT, Lemma 7].

Similarly, we also have
$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(C_X)}(G\,C_X, H\,C_X) &\cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(C_X)}(C_X, G^\star H\,C_X) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}}(C_X, G^\star H).
\end{aligned}
$$

Now we note that, using adjunction and the fact that $G^\star H \in \mathscr{A}$, we have
$$
\mathrm{Hom}_{\underline{\mathscr{C}}}(C_X, G^\star H) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(C_X, \iota(G^\star H)) \cong \mathrm{Hom}_{\underline{\mathscr{A}}}(\tau(C_X), G^\star H).
$$

Combining this, we obtain
$$
\mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(C_X)}(G\,C_X, H\,C_X) \cong \mathrm{Hom}_{\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))}(G\,\iota\tau(C_X), H\,\iota\tau(C_X)),
$$

for each $G, H \in \mathcal{L}$. Together with the fact that $\mathrm{add}(\{G\,X \,:\, G \in \mathcal{L}\}) = \coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$, this implies that Φ is, in fact, an equivalence and $\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X)) \cong \mathbf{M}$ is simple transitive.

Consequently, the coalgebra 1-morphism $\iota\tau(C_X)$ is cosimple by Corollary 12. As ι is left exact, this implies that the coalgebra 1-morphism $\tau(C_X)$ in $\underline{\mathscr{A}}$ is cosimple. Thus, again by Corollary 12, the 2-representation $\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X))$ is simple transitive.

On the other hand, by construction, $\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X))$, which is given by the additive closure of $\mathrm{F}\,\tau(C_X)$, is equivalent to the transitive 2-subrepresentation $\mathbf{N}$ defined in the proof of Proposition 14 of the restriction of $\mathbf{M} \cong \mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))$ to $\mathscr{A}$. As $\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X))$ is simple transitive, this proves
$$
\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X)) \cong \Theta(\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))) \cong \Theta(\mathbf{M}).
$$

Now by definition, $\Omega(\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X)))$ is the simple transitive quotient of $\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X))$, which is already simple transitive. Thus
$$
\Omega(\Theta(\mathbf{M})) \cong \Omega(\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(C_X)) \cong \mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(C_X)) \cong \mathbf{M}.
$$

This, finally, yields that $\Omega \circ \Theta$ is the identity and the proof is complete. $\qquad\square$

Corollary 17. *The following statements hold.*

(i) *In the construction of $\Omega(\mathbf{N})$, already $\mathrm{inj}_{\underline{\mathscr{C}}}(D_X)$ is simple transitive.*

   (*ii*) *In the construction of* $\Theta(\mathbf{M})$, *we do not need to take any quotients, that is,* $\Theta(\mathbf{M})$ *is a* 2*-subrepresentation.*

Proof. Claim (i) follows from the part of the proof of Theorem 15 which shows that Ψ is an equivalence between $\mathrm{inj}_{\underline{\mathscr{C}}}(\iota\tau(\mathrm{C}_X))$ and the original simple transitive 2-representation $\mathbf{M}$.

Claim (ii) is proved by noting, as in the last paragraph of the proof of Theorem 15, that $\mathrm{inj}_{\underline{\mathscr{A}}}(\tau(\mathrm{C}_X))$ is equivalent to the 2-subrepresentation $\mathbf{N}$ of the restriction of $\mathbf{M}$ to $\mathscr{A}$ constructed in the proof of Proposition 14, and is proved to be simple transitive in the second to last paragraph of the proof of Theorem 15. $\qquad\square$

4.4. **Some consequences.**

Corollary 18. *Let* $\mathscr{C}$ *be as in Subsection 4.3 and* $\mathcal{L}_i$, *where* $i = 1, 2$, *be two different left cells in* $\mathcal{J}$. *Then, for* $\mathcal{H}_i = \mathcal{L}_i \cap \mathcal{L}_i^\star$, *where* $i = 1, 2$, *there is a natural bijection between the set of equivalence classes of simple transitive* 2*-representations of* $\mathscr{A}_{\mathcal{H}_1}$ *with apex* $\mathcal{H}_1$ *and the set of equivalence classes of simple transitive* 2*-representations of* $\mathscr{A}_{\mathcal{H}_2}$ *with apex* $\mathcal{H}_2$.

Proof. This follows by applying Theorem 15 first to the pair $\mathscr{C}$ and $\mathscr{A}_{\mathcal{H}_1}$ and then to the pair $\mathscr{C}$ and $\mathscr{A}_{\mathcal{H}_2}$. $\qquad\square$

Corollary 19. *Let* $\mathscr{C}$ *be a fiat* 2*-category and* $\mathcal{J}$ *a two-sided cell in* $\mathscr{C}$. *Assume that* $\mathcal{J}$ *contains a left cell* $\mathcal{L}$ *such that* $|\mathcal{L} \cap \mathcal{L}^\star| = 1$. *Then* $\mathcal{J}$ *is strongly regular.*

Proof. Due to [KiM, Corollary 19], we already know that $\mathcal{J}$ is regular. Without loss of generality we may assume that $\mathscr{C}$ is as in Subsection 4.2. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^\star$ and $\mathscr{A} = \mathscr{A}_{\mathcal{H}}$. Then the two-sided cell $\mathcal{H}$ in $\mathscr{A}$ is strongly regular and consists of one element. Therefore, due to [MM5, Theorem 18] (in the version of [MM6, Theorem 33]), $\mathscr{A}$ has a unique, up to equivalence, simple transitive 2-representation with apex $\mathcal{H}$.

From Theorem 15 we thus obtain that $\mathscr{C}$ has a unique, up to equivalence, simple transitive 2-representation with apex $\mathcal{J}$. Consequently, for any two left cells $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{J}$, the corresponding cell 2-representations $\mathbf{C}_{\mathcal{L}_1}$ and $\mathbf{C}_{\mathcal{L}_2}$ of $\mathscr{C}$ are equivalent. Let $\Psi : \mathbf{C}_{\mathcal{L}_1} \to \mathbf{C}_{\mathcal{L}_2}$ be an equivalence. Then Ψ induces a bijection from $\mathcal{L}_1$ (which indexes isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{C}_{\mathcal{L}_1}(\mathtt{i})$) to $\mathcal{L}_2$ (which indexes isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{C}_{\mathcal{L}_2}(\mathtt{i})$).

For $j = 1, 2$, let L_{F_j} be the simple object in $\coprod_{\mathtt{i} \in \mathscr{C}} \overline{\mathbf{C}_{\mathcal{L}_j}}(\mathtt{i})$ corresponding to some $\mathrm{F}_j \in \mathcal{L}_j$. By [MM1, Lemma 12], the annihilators of L_{F_1} and L_{F_2} in $\mathscr{C}$ have a chance to coincide only if F_1 and F_2 belong to the same right cell in $\mathcal{J}$. Therefore the bijection from $\mathcal{L}_1$ to $\mathcal{L}_2$ induced by Ψ preserves right cells. In other words, for any right cell $\mathcal{R}$ in $\mathcal{J}$, we have

$$(19) \qquad\qquad |\mathcal{L}_1 \cap \mathcal{R}| = |\mathcal{L}_2 \cap \mathcal{R}|.$$

Using (19), from $|\mathcal{H}| = 1$, we thus get $|\mathcal{L}' \cap \mathcal{L}'^\star| = 1$, for any left cell $\mathcal{L}'$ in $\mathcal{J}$. Applying $\star$, we get $|\mathcal{R} \cap \mathcal{L}| = 1$, for any right cell $\mathcal{R}$ in $\mathcal{J}$. Now, using (19) once more, it follows that $\mathcal{J}$ is strongly regular. $\qquad\square$

5. APPLICATION TO SOERGEL BIMODULES, PART I

5.1. **Soergel bimodules and their small quotients.** Let (W, S) be a finite Coxeter system. Further, let $V_{\mathbb{R}}$ be a reflection faithful W-module, V its complexification and C the corresponding *coinvariant algebra*. Let $\mathscr{S}$ denote the fiat 2-category of Soergel C-C-bimodules associated with (W, S) and V, see [So1, So2, EW]. The unique object in $\mathscr{S}$ is denoted i. For $w \in W$, we denote by θ_w the unique (up to isomorphism) indecomposable Soergel bimodule which corresponds to w. Then the split Grothendieck ring of $\mathscr{S}$ is isomorphic to $\mathbb{Z}[W]$ and this isomorphism sends the class of θ_w to the corresponding element $\underline{H}_w$ of the Kazhdan-Lusztig basis in $\mathbb{Z}[W]$, see [KL, EW]. Consequently, the left, right and two-sided cells in $\mathscr{S}$ are given by the corresponding left, right and two-sided Kazhdan-Lusztig cells in W.

The 2-category $\mathscr{S}$ has a unique minimal (with respect to the two-sided order) two-sided cell, namely, the two-sided cell $\mathcal{J}_e := \{\theta_e\}$. In the set of the remaining two-sided cells there is again a unique minimal (with respect to the two-sided order) two-sided cell which we call $\mathcal{J}$ for the rest of this section. This two-sided cell contains all θ_s, where $s \in S$. According to [KMMZ, Proposition 4], elements of $\mathcal{J}$ can be characterized in many different ways, for example, as those θ_w, where $w \in W$, for which w has a unique reduced expression.

In the set of all 2-ideals of $\mathscr{S}$ which do not contain any 1_{θ_w}, where $\theta_w \in \mathcal{J}$, there is a unique ideal that is maximal with respect to inclusions. The corresponding quotient of $\mathscr{S}$ is denoted by $\mathscr{S}_{\mathrm{sm}}$ and called the *small quotient* of $\mathscr{S}$. We refer the reader to [KMMZ, Section 3] for details (note the change of notation from $\underline{\mathscr{L}}$ to $\mathscr{S}_{\mathrm{sm}}$ to avoid conflict with injective abelianization).

5.2. **Simple transitive 2-representations of $\mathscr{S}_{\mathrm{sm}}$.** The following statement, essentially, summarizes [Zi1, Theorem 6.1], [KMMZ, Theorem 1] and [MT, Theorems I, II, III].

Theorem 20.

(i) If (W, S) is not of type $I_2(n)$ with $n > 4$ even, then every simple transitive 2-representation of $\mathscr{S}_{\mathrm{sm}}$ is equivalent to a cell 2-representation.

(ii) If (W, S) is of type $I_2(n)$ with $n > 4$ even and $n \neq 12, 18, 30$, then, apart from cell 2-representations, $\mathscr{S}_{\mathrm{sm}}$ has exactly two extra equivalence classes of simple transitive 2-representations.

(iii) If (W, S) is of type $I_2(n)$ with $n = 12, 18, 30$, then, apart from cell 2-representations, $\mathscr{S}_{\mathrm{sm}}$ has at least four extra equivalence classes of simple transitive 2-representations. Under the additional assumption of gradability, these exhaust all simple transitive 2-representations.

5.3. **The 2-category $\mathscr{Q}$.** Consider the 2-category $\mathscr{S}_{\mathrm{sm}}$ and fix $s \in S$. Denote by $\mathscr{Q}$ the 2-subcategory $\mathscr{A}_{\mathcal{H}}$ of $\mathscr{S}_{\mathrm{sm}}$, where $\mathcal{H} = \mathcal{L}_s \cap \mathcal{L}_s^{\star}$ with $\mathcal{L}_s$ being the left cell containing θ_s. In the cases when (W, S) is of type $I_2(4)$ or $I_2(5)$, the 2-category $\mathscr{Q}$ was studied in [MaMa] where all simple transitive 2-representations of $\mathscr{Q}$ were classified.

If (W, S) is simply laced, then both two-sided cells of $\mathscr{S}_{\mathrm{sm}}$, and hence also of $\mathscr{Q}$, are strongly regular, see [KMMZ, Corollary 5]. Therefore, in this case, the classification of simple transitive 2-representations for both $\mathscr{S}_{\mathrm{sm}}$ and $\mathscr{Q}$ follows directly from [MM5, Theorem 18] in the version of [MM6, Theorem 33].

In this section we combine Theorems 15 and 20 to classify simple transitive 2-representations of $\mathcal{Q}$.

5.4. **The case of $I_2(n)$ with n odd.** The following result generalizes [MaMa, Theorem 5] where the case $n = 5$ was considered. Our approach is rather different. We denote by $\mathcal{L}$ the left cell of $\mathcal{Q}$ containing θ_s.

Theorem 21. *If (W, S) is of type $I_2(n)$ with n odd, then every simple transitive 2-representation of $\mathcal{Q}$ is equivalent to a cell 2-representation.*

Proof. By Theorem 20, under our assumptions, every simple transitive 2-representation of $\mathscr{S}_{\mathrm{sm}}$ is a cell 2-representation. The 2-category $\mathscr{S}_{\mathrm{sm}}$ has three left cells: the left cell $\mathcal{L}_e$ consisting of θ_e, the left cell $\mathcal{L}_s$ containing θ_s and the left cell $\mathcal{L}_t$ containing θ_t, where $t \in S$ is the other simple reflection.

The cell 2-representation $\mathbf{C}_{\mathcal{L}_e}$ of $\mathscr{S}_{\mathrm{sm}}$ is just an action of $\mathscr{S}_{\mathrm{sm}}$ on complex vector spaces where θ_e acts as the identity and all other θ_w acts as zero. Via the bijection in Theorem 15 (applied to the appropriate quotients of $\mathscr{S}_{\mathrm{sm}}$ and $\mathcal{Q}$) this corresponds to the cell 2-representation $\mathbf{C}_{\mathcal{L}_e}$ of $\mathcal{Q}$. It is also clear that the bijection in Theorem 15 matches the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathcal{Q}$ with the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$ of $\mathscr{S}_{\mathrm{sm}}$.

To complete the proof, it is now left to show that the 2-representations $\mathbf{C}_{\mathcal{L}_s}$ and $\mathbf{C}_{\mathcal{L}_t}$ of $\mathscr{S}_{\mathrm{sm}}$ are equivalent. Indeed, consider the abelianization $\overline{\mathbf{C}_{\mathcal{L}_t}}$. In the category $\overline{\mathbf{C}_{\mathcal{L}_t}}(\mathtt{i})$, consider the simple top $L_{w_0 s}$ of the projective object $0 \to \theta_{w_0 s}$. By the Yoneda Lemma, there is a unique 2-natural transformation Ψ from the principal 2-representation $\mathbf{P}_\mathtt{i}$ to $\overline{\mathbf{C}_{\mathcal{L}_t}}$ which sends $\mathbb{1}_\mathtt{i}$ to $L_{w_0 s}$. The multiplication table of $\mathscr{S}_{\mathrm{sm}}$ is obtained inductively from the following formulae:

$$\theta_s \theta_w = \begin{cases} \theta_s, & w = e; \\ \theta_{st}, & w = t; \\ \theta_{tsw_0}, & w = sw_0; \\ \theta_w \oplus \theta_w, & \ell(sw) < \ell(w); \\ \theta_{sw} \oplus \theta_{tw}, & \text{else.} \end{cases} \qquad \theta_t \theta_w = \begin{cases} \theta_t, & w = e; \\ \theta_{ts}, & w = s; \\ \theta_{stw_0}, & w = tw_0; \\ \theta_w \oplus \theta_w, & \ell(tw) < \ell(w); \\ \theta_{tw} \oplus \theta_{sw}, & \text{else.} \end{cases}$$

We note the symmetry of the above formulae with respect to swapping, for w, the elements s and $w_0 s$, the elements ts and $w_0 ts$, and so on. From this and the explicit description of the endomorphism algebras of the underlying category $\mathbf{C}_{\mathcal{L}_t}(\mathtt{i})$ in [KMMZ, Subsection 7.2], it follows easily that Ψ maps $\theta_w \in \mathcal{L}_s$ to an indecomposable projective object in $\mathbf{C}_{\mathcal{L}_t}(\mathtt{i})$. By comparing the Cartan matrices of the endomorphism algebras of the underlying categories, we see this Ψ induces an equivalence between $\mathbf{C}_{\mathcal{L}_s}$ and $\mathbf{C}_{\mathcal{L}_t}$. The claim of the theorem follows. $\qquad\square$

5.5. **The case of $I_2(4)$.** For $\mathscr{X} \in \{\mathcal{Q}, \mathscr{S}_{\mathrm{sm}}\}$ and $w \in \{e, s, t\}$ (here $w \neq t$ if $\mathscr{X} = \mathcal{Q}$), we denote by $\mathcal{L}_w^{(\mathscr{X})}$ the left cell of $\mathscr{X}$ containing θ_w. The following result significantly simplifies the proof of [MaMa, Theorem 12].

Theorem 22. *If (W, S) is of type $I_2(4)$, then $\mathcal{Q}$ has exactly three equivalence classes of simple transitive 2-representations, namely $\mathbf{C}_{\mathcal{L}_e^{(\mathcal{Q})}}$, $\mathbf{C}_{\mathcal{L}_s^{(\mathcal{Q})}}$ and $\Theta(\mathbf{C}_{\mathcal{L}_t^{(\mathscr{S}_{\mathrm{sm}})}})$.*

We note that the 2-representation $\Theta(\mathbf{C}_{\mathcal{L}_t^{(\mathscr{S}_{\mathrm{sm}})}})$ is not equivalent to a cell 2-representation.

Proof. By [Zi1, Theorem 6.12], up to equivalence, simple transitive 2-representations of $\mathscr{S}_{\mathrm{sm}}$ are exactly $\mathbf{C}_{\mathcal{L}_e^{(\mathscr{S}_{\mathrm{sm}})}}$, $\mathbf{C}_{\mathcal{L}_s^{(\mathscr{S}_{\mathrm{sm}})}}$ and $\mathbf{C}_{\mathcal{L}_t^{(\mathscr{S}_{\mathrm{sm}})}}$. Now the claim follows from Theorem 15 similarly to the first part of the proof of Theorem 21. $\qquad\square$

Remark 23. As already mentioned, the argument above is a significant simplification of [MaMa, Theorem 12]. However, the original proof of [MaMa, Theorem 12] is crucially used in [KMMZ] and hence in the results below.

5.6. **The case of $I_2(n)$ with $n \neq 12, 18, 30$ and $n > 4$ even.**

Theorem 24. *If (W, S) is of type $I_2(n)$ with $n \neq 12, 18, 30$ and $n > 4$ even, then $\mathscr{Q}$ has exactly five equivalence classes of simple transitive 2-representations (the images under Θ of the five equivalence classes of simple transitive 2-representations of $\mathscr{S}_{\mathrm{sm}}$).*

Proof. Similarly to the above, this follows from Theorems 15 and 20. We just note that, for even n, all cell 2-representations of $\mathscr{S}_{\mathrm{sm}}$ are inequivalent. $\qquad\square$

5.7. **The cases of $I_2(12)$, $I_2(18)$ and $I_2(30)$.**

Proposition 25. *If (W, S) is of type $I_2(12)$, $I_2(18)$ or $I_2(30)$, then $\mathscr{Q}$ has at least seven equivalence classes of simple transitive 2-representations (the images under Θ of the seven equivalence classes of simple transitive 2-representations of $\mathscr{S}_{\mathrm{sm}}$). Under the additional assumption of gradability, these exhaust all simple transitive 2-representations of $\mathscr{Q}$.*

Proof. Similarly to the above, this follows from Theorems 15 and 20. $\qquad\square$

5.8. **The cases of rank higher than 2.**

Theorem 26. *If (W, S) is of type H_3 or H_4, then every simple transitive 2-representation of $\mathscr{Q}$ is equivalent to a cell 2-representation.*

Proof. From [KMMZ, Subsection 4.2], the indecomposable 1-morphisms in $\mathscr{Q}$ are θ_e, θ_s and θ_{sts}, where s and t are two simple reflections such that $(st)^5 = e$. The general definition of Soergel bimodules implies that the $\mathcal{H}$-simple quotient of $\mathscr{Q}$ in types H_3 and H_4 is biequivalent to the $\mathcal{H}$-simple quotient of the 2-category $\mathscr{Q}$ in type $I_2(5)$. Therefore the claim follows from Theorem 21 (see also [MaMa, Theorem 5]). $\qquad\square$

Corollary 27. *If (W, S) is of type H_3 or H_4 and $\mathcal{L}_1$ and $\mathcal{L}_2$ are two left cells in $\mathcal{J}$, then the cell 2-representations $\mathbf{C}_{\mathcal{L}_1}$ and $\mathbf{C}_{\mathcal{L}_2}$ of $\mathscr{S}_{\mathrm{sm}}$ are equivalent.*

Proof. This follows from Theorem 26 as $\mathcal{H}$ is the only non-identity two-sided cell in $\mathscr{Q}$. $\qquad\square$

Theorem 28. *If (W, S) is of type F_4 or B_n, for $n > 2$, then $\mathscr{Q}$ has three equivalence classes of simple transitive 2-representations, two of which are cell 2-representations.*

Proof. From [KMMZ, Subsection 4.2], the indecomposable 1-morphisms in $\mathscr{Q}$ are θ_e, θ_s and θ_{sts}, where s and t are two simple reflections such that $(st)^4 = e$. The general definition of Soergel bimodules implies that the $\mathcal{H}$-simple quotient of $\mathscr{Q}$ in types F_4 or B_n, for $n > 2$, is biequivalent to the $\mathcal{H}$-simple quotient of the 2-category $\mathscr{Q}$ in type $I_2(4)$. Therefore the claim follows from Theorem 22 (see also [MaMa, Theorem 12]). $\qquad\square$

Corollary 29. *If (W, S) is of type F_4 or B_n, for $n > 2$, then there are exactly two equivalence classes of cell 2-representations of $\mathscr{S}_{\mathrm{sm}}$ with apex $\mathcal{J}$.*

Proof. This follows from Theorems 20 and 28 as in this case $\mathscr{Q}$ has two equivalence classes of simple transitive 2-representations with apex $\mathcal{H}$. $\qquad\square$

6. Application to Soergel bimodules, part II

6.1. **Nice two-sided cells.** In this section, we let (W, S) be a finite Coxeter system and $\mathscr{S}$ the corresponding 2-category of Soergel bimodules, see Subsection 5.1 for details. Let $\mathcal{J}$ be a two-sided cell of $\mathscr{S}$. We will say that $\mathcal{J}$ is *nice* provided that the following conditions are satisfied.

- For any left cell $\mathcal{L}$ in $\mathcal{J}$ and any right cell $\mathcal{R}$ in $\mathcal{J}$, we have $|\mathcal{L} \cap \mathcal{R}| \leq 2$;

- for any left cell $\mathcal{L}$ in $\mathcal{J}$ there are right cells $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathcal{J}$ such that we have $|\mathcal{L} \cap \mathcal{R}_1| = 1$ and $|\mathcal{L} \cap \mathcal{R}_2| = 2$;

- $\mathcal{J}$ contains the longest element for some parabolic subgroup of W.

For example, from [KMMZ, Subsection 4.2] it follows that the two-sided cell of $\mathscr{S}$ which contains all simple reflections is nice if and only if W is of type F_4 or B_n, for $n \geq 2$.

Example 30. Recall that, for a Coxeter group (W, S), an element $w \in W$ is *fully commutative* provided that any two reduced expressions of w can be obtained from each other only using commutativity of Coxeter generators, see [St]. By [GL, Theorem 3.1.1], in type B_n, the set of all fully commutative elements is a union of two-sided cells. Assume W is of type B_n and let $S' \subset S$ be a non-empty subset such that any pair of elements in S' commutes. Let w_0' be the product of all elements in S' and $\mathcal{J}$ the two-sided cell containing w_0'. Note that w_0' is the longest element of the parabolic subgroup generated by S'. With this in mind, the very last paragraph of [GL], just before the bibliography, implies that $\mathcal{J}$ is nice.

6.2. **Simple transitive 2-representations with nice apex.** The main result of this section is the following theorem.

Theorem 31. *Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{S}$ with a nice apex $\mathcal{J}$. Then $\mathbf{M}$ is equivalent to a cell 2-representation. Moreover, there are exactly two equivalence classes of simple transitive 2-representations of $\mathscr{S}$ with apex $\mathcal{J}$.*

Proof. Let $\mathscr{C}$ be the minimal quotient of $\mathscr{S}$ in which 1-morphisms of $\mathcal{J}$ survive (the $\mathcal{J}$-simple quotient in the terminology of [MM1]). Let w_0' be an element of $\mathcal{J}$ which is the longest element of some parabolic subgroup of W, say W'. Let $\mathcal{L}$ and $\mathcal{R}$ be the left and the right cells containing w_0', respectively. Set $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$. Let $\mathscr{A}$ be the 2-subcategory of $\mathscr{C}$ generated by θ_e and the elements in $\mathcal{H}$.

Note that $\mathcal{R} = \mathcal{L}^\star$. Therefore, if we assume $|\mathcal{H}| = 1$, from Corollary 19 we deduce that $\mathcal{J}$ is strongly regular. This, however, contradicts our assumption that $\mathcal{J}$ is nice. Therefore $|\mathcal{H}| = 1$ is not possible.

The above shows that $|\mathcal{H}| > 1$, that is $|\mathcal{H}| = 2$ as $\mathcal{J}$ is nice. Let θ_w be the element in $\mathcal{H}$ different from $\theta_{w_0'}$. Let $\mathscr{S}\mathscr{S}$ be the 2-category of *singular Soergel bimodules*, see [MM1, Subsection 7.2]. The objects of $\mathscr{S}\mathscr{S}$ correspond to parabolic subgroups of W. If $\mathtt{i}$ is the object corresponding to the trivial subgroup, then the full 2-subcategory of $\mathscr{S}\mathscr{S}$ with object $\mathtt{i}$ is naturally identified with $\mathscr{S}$. We let $\mathtt{j}$ be the object corresponding to W'. Then there are 1-morphisms $\theta^{\mathrm{on}} \in \mathscr{S}\mathscr{S}(\mathtt{i}, \mathtt{j})$ and $\theta^{\mathrm{out}} \in \mathscr{S}\mathscr{S}(\mathtt{j}, \mathtt{i})$ such that

$$\theta_{w_0'} \cong \theta^{\mathrm{out}} \theta^{\mathrm{on}} \quad \text{and} \quad \mathbb{1}_{\mathtt{j}}^{\oplus \mathbf{l}(w_0')} \cong \theta^{\mathrm{on}} \theta^{\mathrm{out}},$$

where $\mathbf{l}$ denotes the length function. We denote by θ the 1-morphism in $\mathscr{S}\mathscr{S}(\mathtt{j}, \mathtt{j})$ such that

$$\theta_w \cong \theta^{\mathrm{out}} \theta \theta^{\mathrm{on}}.$$

Let $\mathscr{SS}_{\mathtt{j}}$ be the full 2-subcategory of $\mathscr{SS}$ with object $\mathtt{j}$. Let $\mathscr{A}^{\vee}$ be the 2-subcategory of $\mathscr{SS}_{\mathtt{j}}$ given by $\mathbb{1}_{\mathtt{j}}$ and θ. Combining Theorem 15 with [MMMT, Corollary 12], we obtain that there is a bijection between simple transitive 2-representations of $\mathscr{S}$ with apex $\mathcal{J}$ and simple transitive 2-representations of $\mathscr{A}^{\vee}$.

Next we claim that $\mathscr{S}$ has at least two equivalence classes of cell 2-representations with apex $\mathcal{J}$. This is similar to the argument in the proof of Corollary 19. As $\mathcal{J}$ is nice, there are left cells $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{J}$ such that $|\mathcal{L}_1 \cap \mathcal{R}| = 1$ and $|\mathcal{L}_2 \cap \mathcal{R}| = 2$. In the proof of Corollary 19, we show that an equivalence between $\mathbf{C}_{\mathcal{L}_1}$ and $\mathbf{C}_{\mathcal{L}_2}$ would induce a bijection between $\mathcal{L}_1 \cap \mathcal{R}$ and $\mathcal{L}_2 \cap \mathcal{R}$. Since these have different cardinalities in our situation, we deduce that $\mathbf{C}_{\mathcal{L}_1}$ and $\mathbf{C}_{\mathcal{L}_2}$ are not equivalent.

To complete the proof of the proposition, it is now sufficient to show that $\mathscr{A}^{\vee}$ has at most two equivalence classes of simple transitive 2-representations. From the previous paragraph, we know that it has at least two, moreover, one of them has rank one and another one has rank two. Note that $\mathbb{1}_{\mathtt{j}}$ and θ belong to the same two-sided cell of $\mathscr{A}^{\vee}$ for otherwise there would exist a 2-representation of $\mathscr{A}^{\vee}$ which annihilates θ but not $\mathbb{1}_{\mathtt{j}}$. This would give, via [MMMT, Corollary 12], a 2-representation of $\mathscr{C}$ which annihilates θ_w but not $\theta_{w_0'}$, a contradiction. Consequently, $\{\mathbb{1}_{\mathtt{j}}, \theta\}$ is both a left, a right and a two-sided cell.

Consider the matrix $[\theta]$ in the cell 2-representation $\mathbf{C}$ of $\mathscr{A}^{\vee}$. Let $P_{\mathbb{1}_{\mathtt{j}}}$ and P_{θ} be the indecomposable projective in $\overline{\mathbf{C}}(\mathtt{j})$ corresponding to $\mathbb{1}_{\mathtt{j}}$ and θ, respectively. Let $L_{\mathbb{1}_{\mathtt{j}}}$ and L_{θ} be their respective simple tops. With respect to the basis $P_{\mathbb{1}_{\mathtt{j}}}$, P_{θ}, the matrix $[\theta]$ has the form

$$\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix},$$

for some non-negative integers a and b. As cell 2-representation is simple transitive, we must have $b > 0$. As $\mathbb{1}_{\mathtt{j}}$ acts on $\mathbf{C}(\mathtt{j})$ as the identity functor, from [KMMZ, Theorem 2] it follows that $\overline{\mathbf{C}}(\mathtt{j})$ is semi-simple. We also have the matrix bookkeeping the composition multiplicities in $\theta\, L_{\mathbb{1}_{\mathtt{j}}}$ and $\theta\, L_{\theta}$. By [AM, Lemma 8], this matrix is

$$\begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}.$$

We know that, on the one hand, $\theta\, P_{\mathbb{1}_{\mathtt{j}}} \cong P_{\theta}$ but, on the other hand L_{θ} has composition multiplicity b in $\theta\, L_{\mathbb{1}_{\mathtt{j}}}$. As $\overline{\mathbf{C}}(\mathtt{j})$ is semisimple, we have $P_{\mathbb{1}_{\mathtt{j}}} \cong L_{\mathbb{1}_{\mathtt{j}}}$ and $P_{\theta} \cong L_{\theta}$. Therefore $b = 1$.

Let $A := \mathbb{C}[x]/(x^2 - ax - 1)$. Then A is positively based with basis $1, x$ (see Subsection 2.4). It has a unique two-sided cell and the special module for this two-sided cell is the one-dimensional A-module on which x acts via $\frac{a+\sqrt{a^2+4}}{2}$. Note that $\frac{a+\sqrt{a^2+4}}{2}$ is rational (in case of a non-negative integer a) if and only if $a = 0$. Therefore, in case A has a one-dimensional module on which x acts as an integer, we must have $a = 0$. Consequently, we have

$$[\theta] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In case $a = 0$, the algebra A has, up to equivalence, exactly two based modules in which $1 + x$ acts as a matrix with positive integral coefficients. These are the modules in which the matrix of x is one of the following two matrices:

(20)
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad (1).$$

Now we can use the usual arguments (like in [MaMa, Subsection 5.10]) to show that the first matrix in (20) corresponds to the cell 2-representation of $\mathscr{A}^{\vee}$. Indeed, let

$\mathbf{K}$ be a simple transitive 2-representation of $\mathscr{A}^{\vee}$ in which $[\theta]$ is the first matrix in (20). Sending $\mathbb{1}_{\mathtt{j}}$ to a simple module L in $\overline{\mathbf{K}}(\mathtt{j})$ induces a homomorphism from the cell 2-representation of $\mathscr{A}^{\vee}$ to $\mathbf{K}$ which is an equivalence as $\overline{\mathbf{K}}(\mathtt{j})$ is semisimple and $L = \mathbb{1}_{\mathtt{j}}(L)$ and $\theta(L)$ are indecomposable projectives which generate $\overline{\mathbf{K}}(\mathtt{j})$. This shows that the cell 2-representation of $\mathscr{A}^{\vee}$ is the only simple transitive 2-representations of $\mathscr{A}^{\vee}$ for which $[\theta]$ is the first matrix in (20).

Clearly, the second matrix in (20) corresponds to a unique simple transitive 2-representation of $\mathscr{A}^{\vee}$ (where both $\mathbb{1}_{\mathtt{j}}$ and θ act as the identity functors and only scalar natural transformations survive). Therefore we obtain that $\mathscr{A}^{\vee}$ has at most two equivalence classes of simple transitive 2-representations, as required. The claim of the theorem follows.   □

6.3. **Large quotients of Soergel bimodules.** The 2-category $\mathscr{S}$ has a unique maximal two-sided cell, namely $\{\theta_{w_0}\}$, where w_0 is the longest element of W. Furthermore, if we remove this two-sided cell, in what remains there is again a unique maximal two-sided cell. In this subsection we denote this two-sided cell by $\mathcal{J}'$. Under the map $w \mapsto ww_0$, which is a bijection reversing the two-sided order, the cell $\mathcal{J}'$ corresponds to the two-sided cell $\mathcal{J}$ of W containing all simple reflections, see [KMMZ, Subsection 4.1]. The minimal quotient of $\mathscr{S}$ in which elements of $\mathcal{J}'$ survive will be called the *large quotient* of $\mathscr{S}$ and denoted $\mathscr{S}^{\vee}$. Our terminology is motivated by the contrast to the terminology in [KMMZ]. From [KMMZ, Subsection 4.1] it follows that, in case (W, S) is simply laced, the unique maximal two-sided cell of $\mathscr{S}^{\vee}$ is strongly regular. Therefore, in the simply laced case, every faithful simple transitive 2-representation of $\mathscr{S}^{\vee}$ is equivalent to a cell 2-representation by [MM5, Theorem 18].

Corollary 32. *Assume that W is of type B_n, for $n \geq 2$. Then every faithful simple transitive 2-representation of $\mathscr{S}^{\vee}$ is equivalent to a cell 2-representation. Moreover, there are exactly two equivalence classes of such 2-representations.*

Proof. We claim that $\mathcal{J}'$ is nice. By [KaLu, Remark 3.3] or [BB, Corollary 6.2.10], the map $w \mapsto ww_0$ sends left cells to left cells, right cells to right cells, and two-sided cells to two-sided cells. Hence the first two conditions from the definition of a nice two-sided cell for $\mathcal{J}'$ follow from the corresponding conditions for $\mathcal{J}$, cf. Subsection 6.1. It remains to prove that $\mathcal{J}'$ contains the longest element in some parabolic subgroup.

Assume that the Coxeter diagram of W is

$$1 \overset{4}{\underline{\quad\quad}} 2 \underline{\quad\quad} 3 \underline{\quad\quad} \ldots \underline{\quad\quad} n \; .$$

We identify simple reflections with the vertices of this diagram. Let W' be the parabolic subgroup generated by the simple reflections 1, 2,..., $n-1$, and w'_0 be the longest element in W'. Then

$$w_0 = (n(n-1)(n-2)\ldots 212 \ldots (n-1)n) \ldots (32123)(212)(1),$$
$$w'_0 = ((n-1)(n-2)(n-3)\ldots 212 \ldots (n-2)(n-1)) \ldots (32123)(212)(1),$$
$$w_0 = (n(n-1)(n-2)\ldots 212 \ldots (n-1)n)w'_0.$$

As the element $n(n-1)(n-2)\ldots 212 \ldots (n-1)n$ belongs to $\mathcal{J}$, see [KMMZ, Subsection 4.2], it follows that $w'_0 \in \mathcal{J}'$.

The above proves that $\mathcal{J}'$ is nice. Now the claim follows directly from Theorem 31.   □

Remark 33. The observations in this remark are based on the lists of two-sided cells in types B_3 and B_4 which were computed by Tobias Kildetoft using GAP and can be found in Section 7 below.

If W is of type B_3, then $\mathscr{S}$ has only two two-sided cells which are not strongly regular, namely, the two-sided cell $\mathcal{J}'$ defined above and the two-sided cell $\mathcal{J}$ which contains all simple reflections. Simple transitive 2-representations of $\mathscr{S}$ with strongly regular apex are equivalent to cell 2-representations by [MM5, Theorem 18]. Simple transitive 2-representations of $\mathscr{S}$ with apex $\mathcal{J}$ are equivalent to cell 2-representations by [KMMZ, Theorem 37]. Simple transitive 2-representations of $\mathscr{S}$ with apex $\mathcal{J}'$ are equivalent to cell 2-representations by Corollary 32. This completes the classification of simple transitive 2-representations of $\mathscr{S}$ in type B_3. In particular, all simple transitive 2-representations of $\mathscr{S}$ in type B_3 are equivalent to cell 2-representations.

Similarly, one can check that all non-strongly regular two-sided cells of type B_4 are nice. In particular, it follows that all simple transitive 2-representations of $\mathscr{S}$ in type B_4 are equivalent to cell 2-representations.

In type B_5 there is a unique two-sided cell (for which the value of Lusztig's $\mathbf{a}$-function from [Lu] is 11) which is neither strongly regular nor nice. The classification of simple transitive 2-representations of $\mathscr{S}$ in type B_5 with apex being this two-sided cell seems to be the smallest open case for the moment.

Remark 34. The maximal two-sided cell for $\mathscr{S}^{\vee}$ in Coxeter types F_4, H_3 and H_4 is not nice and hence the above approach does not work in these cases.

7. Appendix: Cell structure in types B_3 and B_4

This appendix contains lists of Kazhdan-Lusztig cells in types B_3 and B_4. The lists were computed by Tobias Kildetoft using the GAP package. As we could not find any of these lists anywhere, we provide them here. We thank Tobias Kildetoft for his help with these lists.

7.1. **Cells in type** B_3**.** We consider the Coxeter graph

$$1 \underline{\qquad 4 \qquad} 2 \underline{\qquad\qquad} 3$$
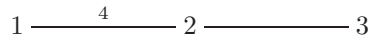
of type B_3 and the corresponding Coxeter group (W, S), where $S = \{s_1, s_2, s_3\}$. Each element $w \in W$ will be given by one of its reduced expressions. For simplicity, we denote the simple reflection s_i by i. For example, 1213 denotes the element $s_1 s_2 s_1 s_3$.

Boxes correspond to two-sided cells, given with respect to the increasing two-sided order (which is linear in this case). Rows are right cells and columns are left cells. For each two-sided cell we also give the value of Lusztig's $\mathbf{a}$-function on this cell, cf. [Lu]. Elements given in bold are longest elements of parabolic subgroups.

The first two-sided cell. Its $\mathbf{a}$-value is 0.

e

The second two-sided cell. Its $\mathbf{a}$-value is 1.

1	12	123
121		
21	**2**	23
	212	2123
321	32	**3**
	3212	32123

The third two-sided cell. Its **a**-value is 2.

121321	1213	12132
1321	**13**	132
21321	213	2132

The fourth two-sided cell. Its **a**-value is 3.

12321	1232	123212
2321	**232**	23212
212321	21232	2123212

The fifth two-sided. Its **a**-value is 4.

1212	12123	121232
12123212	1212321	
13212	132123	1232123
1213212	12132123	
213212	2132123	232123
		21232123

The sixth two-sided. Its **a**-value is 9.

$$\boxed{\textbf{121232123}}$$

7.2. **Cells in type** B_4. Here we use the same conventions as in Subsection 7.1 and consider the Coxeter graph

$$1 \overset{4}{\rule{2cm}{0.4pt}} 2 \rule{2.5cm}{0.4pt} 3 \rule{2.5cm}{0.4pt} 4$$

Two-sided cells are again listed with respect to the increasing two-sided order. This order is linear with one exception that the two two-sided cells with the same **a**-value are not comparable.

The first two-sided cell. Its **a**-value is 0.

$$\boxed{\textbf{e}}$$

The second two-sided cell. Its **a**-value is 1.

1	12	123	1234
121			
21	**2**	23	234
	212	2123	21234
321	32	**3**	34
	3212	32123	321234
4321	432	43	**4**
	43212	432123	4321234

The third two-sided cell. Its **a**-value is 2.

3243 321243	324 32124	321432	3214	32143	3214321
243 21243	**24** 2124	21432	214	2143	214321
213243	21324	2132 21321432	2134 213214	213 2132143	21321 213214321
1243	124	1432 121432	**14** 1214	143 12143	14321 1214321
13243	1324	132 1321432	134 13214	**13** 132143	1321 13214321
1213243	121324	12132 121321432	12134 1213214	1213 12132143	121321 1213214321

The fourth two-sided cell. Its **a**-value is 3.

32123432123	3212343	321243212	3212343212	32123432	3212432	321234321	32124321
3432123	**343**	3243212	343212	3432	32432	34321	324321
212321243	2123243	2123212	21232124	212324	21232	2123214	212321
2123432123	212343	21243212	212343212	2123432	212432	21234321	2124321
23432123	2343	243212	2343212	23432	2432	234321	24321
2321243	23243	23212	232124	2324	**232**	23214	2321
123432123	12343	1243212	12343212	123432	12432	1234321	124321
12321243	123243	123212	1232124	12324	1232	123214	12321

The fifth two-sided cell. Its **a**-value is 4.

212321243212	2123212432	212321432	21232143	2123214321	21232124321
2321243212	23212432	2321432	232143	23214321	232124321
213243212	2132432	213432	21343	2134321	21324321
13243212	132432	13432	**1343**	134321	1324321
1213243212	12132432	1213432	121343	12134321	121324321
12321243212	123212432	12321432	1232143	123214321	1232124321

The sixth two-sided cell. Its **a**-value is 4.

321234321234 34321234	32124321234 324321234	3212432123 32432123	3214321234	321432123	32143212
21234321234 234321234	2124321234 24321234	212432123 2432123	214321234	21432123	2143212
2123212343 23212343	212321234 2321234	21232123 232123	21321234	2132123	213212
1234321234	124321234	12432123	14321234 1214321234	1432123 121432123	143212 12143212
123212343	12321234	1232123	1321234 121321234	132123 12132123	13212 1213212
12123243	1212324	121232	121234 12123214	12123 1212321	**1212** 12123212

The seventh two-sided cell. Its **a**-value is 5.

21324321234	2132432123	2134321234	213432123	21343212	213214321234	21321432123	2132143212
2132123432	213212432	213212343	21321243	2132124	21321234321	2132124321	21321243212
1324321234	132432123	134321234	13432123	1343212	13214321234	1321432123	132143212
132123432	13212432	13212343	1321243	**132124**	1321234321	132124321	1321243212
12123432	1212432	1212343	121243	**12124**	121234321	12124321	121243212
121324321234	12132432123	12134321234	1213432123	121343212	1213214321234	121321432123	12132143212
12132123432	1213212432	1213212343	121321243	12132124	121321234321	12132124321	121321243212
1212321432	12123212432	121232143	1212321243	121232124	12123214321	121232124321	1212321243212

The eights two-sided cell. Its **a**-value is 6.

21232123432123 232123432123	2123212343212 23212343212	21232123432 232123432	2132123432123	213212343212	212321234321 2321234321
2123212432123 21232432123	21232124321234 2123243212	212324321234 21232432	212321432123	21232143212	2123214321234 212324321
232432123 23212432123	232124321234 23243212	2324321234 **232432**	2321432123	232143212	23214321234 2324321
1232123432123	123212343212	1232123432	132123432123 12132123432123	13212343212 1213212343212	12321234321
121232432123	12123243212	121232432	12123432123 1212321432123	1212343212 121232143212	1212324321
123212432123 1232432123	1232124321234 123243212	12324321234 1232432	12321432123	1232143212	123214321234 12324321

The ninth two-sided cell. Its **a**-value is 9.

212321234321234 2321234321234	21321234321234	2132124321234	213212432123
12321234321234	1321234321234 121321234321234	132124321234 12132124321234	13212432123 1213212432123
1212324321234	121234321234 12123214321234	12124321234 121232124321234	1212432123 12123212432123
121232123432	12123212343 1212321234321	1212321234 12123212343212	**121232123** 121232123432123

The tenth two-sided cell. Its **a**-value is 16.

$$\boxed{\mathbf{1212321234321234}}$$

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M. M.: Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, PORTUGAL & Departamento de Matemática, FCT, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, PORTUGAL, email: `mmackaay@ualg.pt`

Vo. Ma.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: `mazor@math.uu.se`

Va. Mi.: School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK, `v.miemietz@uea.ac.uk`

X. Z.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: `xiaoting.zhang@math.uu.se`