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Unsteady motion of circular cylinder under ice cover

V K Kostikov[1,2], **N I Makarenko**[1,2] **and A A Korobkin**[3]

[1] Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia
[2] Novosibirsk State University, Novosibirsk, Russia
[2] University of East Anglia, Norwich, UK

E-mail: makarenko@hydro.nsc.ru

Abstract. An interaction of the moving submerged circular cylinder with elastic solid ice cover is investigated. The method of reduction to integral-differential system of equations is extended to the case of elastic ice sheet floating on the fluid surface. The elastic properties of the ice are accounted for by the Euler's beam theory. A special iterative procedure is developed to construct small-time asymptotic solution for the cylinder moving with constant acceleration from rest. Elastic-gravity waves in the ice cover are studied in a wide range of parameters.

1. Introduction

In this paper we study analytically the deflection of an elastic ice-plate floating on the surface of ideal fluid caused by unsteady motion of submerged circular cylinder. The most recent theories and data related to the problem of external loading on the surface of the ice were discussed in [1]. The problem of unsteady motion of submerged body under ice cover in homogenous or stratified fluids was considered in linear statement in [2]. The linearised problem on hydroelastic behaviour of the floating plate of finite size was investigated in [3]. Our analysis uses reduction of the Euler equations to the nonlinear boundary integral-differential system of equations for the wave elevation together with normal and tangential fluid velocities [4]-[6]. In this method the key role is played by the integral equation for a normal fluid velocity that discribes interaction of the cylinder with the floating ice-cover.

2. Formulation of the problem

The plane irrotational flow of a heavy inviscid deep fluid is considered in the coordinate system Oxy with a vertical y-axis. The circular cylinder moves totally submerged under ice cover. Trajectory of the cylinder center $(x_c(t), y_c(t))$ is known at any time moment $t > 0$. The ice cover is modeled by an elastic solid plate with constant small thickness h and density ρ_{ice} floating freely on the free surface. The shape of the plate $y = \eta(x,t)$ is unknown and to be determined during solution of the problem. Initially, the fluid, ice plate and cylinder are at rest. The line $y = 0$ corresponds to undisturbed contact surface between ice and fluid (Figure 1). Let h_0, u_0, and ρ be initial submergence depth of the cylinder center, characteristic speed of the cylinder, and fluid density, respectively. The dimensionless variables use the depth h_0 as a length scale, the speed u_0 as a velocity scale, the quantity ρu_0^2 as a pressure unit, and the ratio h_0/u_0 as a time unit. Unsteady flow of incompressible fluid under ice cover is described by the Euler equations for the velocity of the flow $\mathbf{u} = (U, V)$ and hydrodynamic pressure p

$$\begin{cases} U_t + UU_x + VU_y + p_x = 0, \\ V_t + UV_x + VV_y + p_y = -\lambda, \\ U_x + V_y = 0, \quad U_y - V_x = 0, \end{cases} \tag{1}$$

where $\lambda = gh_0/u_0^2$ is the square of the inverse Froude number and g is the gravity acceleration.

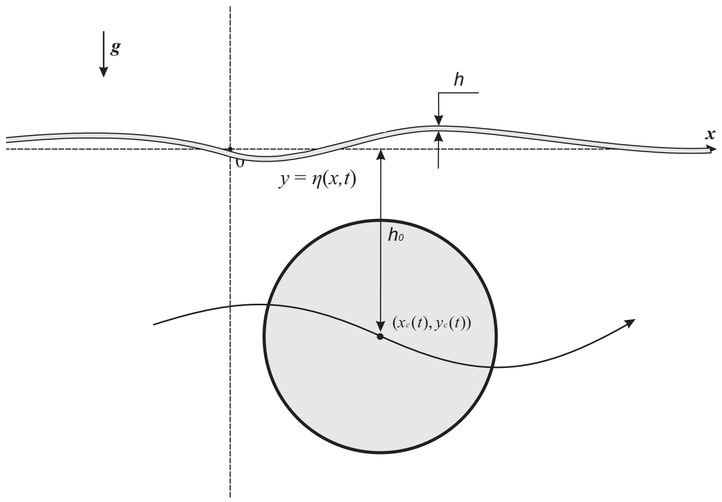

Figure 1. Scheme of motion.

We assume in our considerations that fluid remains to be in contact with an ice cover. Moreover, in contrast to the free surface flow, the pressure at the ice-covered surface depend on the deflection of the ice-plate. Therefore the following boundary kinematic and dynamic conditions should be satisfied

$$\eta_t + U\eta_x = V, \quad p = \alpha\eta_{tt} + \beta\eta_{xxxx} \qquad (y = \eta(x,t)), \tag{2}$$

where $\alpha = \rho_{ice}h/(\rho h_0)$ and $\beta = Eh^3/(12\rho h_0^3 u_0^2)$ are dimensionless parameters responsible for inertia and elasticity of the ice cover respectively. Here E is Young modulus. The boundary condition at the moving cylinder surface has the form

$$(\mathbf{u} - \mathbf{u_{cyl}}) \cdot \mathbf{n} = 0 \qquad \Big((x - x_c(t))^2 + (y - y_c(t))^2 = r^2\Big), \tag{3}$$

where $\mathbf{n}$ is the unit normal and r is non-dimensional radius of the cylinder. We assume that the fluid is at rest at infinity, so we have $U, V, \eta \to 0$ as $|\mathbf{x}| \to \infty$. The initial velocity field $\mathbf{u}_{|t=0} = \mathbf{u}_0$ should be irrotational

$$U_{0x} + V_{0y} = 0, \quad U_{0y} - V_{0x} = 0,$$

and also it should satisfy the compatibility condition

$$(\mathbf{u}_0 - \mathbf{u_{cyl}}(0)) \cdot \mathbf{n}_0 = 0 \qquad \Big((x - x_c(0))^2 + (y - y_c(0))^2 = r^2\Big).$$

These relations are satisfied, in particular, when the cylinder starts moving smoothly from rest.

3. Integro-differential equations on the free surface

We introduce the tangential velocity $u(x,t) = (U + \eta_x V)_{|y=\eta}$ and normal velocity $v(x,t) = (V - \eta_x U)_{|y=\eta}$ of the fluid particles at the interface $y = \eta(x,t)$ between ice and fluid, and reduce the basic equations (1)-(2) to an equivalent system of boundary integral-differential equations for the unknown functions u, v, η

$$\eta_t = v, \qquad u_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{u^2 - 2\eta_x uv - v^2}{1+\eta_x^2}\right) + \lambda\eta_x + \alpha\eta_{ttx} + \beta\eta_{xxxxx} = 0. \tag{4}$$

$$[I + A(\eta) + r^2 A_r(\eta)]v = [B(\eta) + r^2 B_r(\eta)]u + v_d(\eta). \tag{5}$$

The integral operators A and B are given by

$$\begin{aligned} A(\eta)u(x) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta(x)-\eta(s)-(x-s)\eta'(x)}{(x-s)^2+[\eta(x)-\eta(s)]^2}u(s)ds, \\ B(\eta)u(x) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x-s+[\eta(x)-\eta(s)]\eta'(x)}{(x-s)^2+[\eta(x)-\eta(s)]^2}u(s)ds, \end{aligned} \tag{6}$$

The operators A_r and B_r act by the formulae

$$\begin{aligned} A_r(\eta)u(x) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{[\eta(s)-y_c-r^2P(x)]Q'(x)-[s-x_c-r^2Q(x)]P'(x)}{[\eta(s)-y_c-r^2P(x)]^2+[s-x_c-r^2Q(x)]^2}u(s)ds, \\ B_r(\eta)u(x) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{[\eta(s)-y_c-r^2P(x)]P'(x)+[s-x_c-r^2Q(x)]Q'(x)}{[\eta(s)-y_c-r^2P(x)]^2+[s-x_c-r^2Q(x)]^2}u(s)ds, \end{aligned} \tag{7}$$

with Poisson kernels

$$P(x) = \frac{\eta(x)-y_c}{(x-x_c)^2+(\eta(x)-y_c)^2}, \quad Q(x) = \frac{x-x_c}{(x-x_c)^2+(\eta(x)-y_c)^2}.$$

Functions v_d in (5) is defined by relation

$$v_d(\eta) = 2r^2\Big(\dot{y}_c Q(x) - \dot{x}_c P(x)\Big)'. \tag{8}$$

Prime superscript $(\dots)'$ in formulae (6)–(8) denotes partial differentiation with respect to the spatial variable x. Respectively, the time variable t was omitted from the functions η, u, v in formulae (5)–(8) for simplicity since it appears there only as a parameter.

4. Small-time asymptotic solution

We consider the unsteady flow which starts from rest, $\eta(x,0) = u(x,0) = v(x,0) = 0$, and is caused by two-dimensional motion of the cylinder with constant acceleration. The trajectory of the cylinder can be taken in dimensionless form as $x_c(t) = t^2\cos\theta$, $y_c(t) = -1 + t^2\sin\theta$. The solution of system (4)-(5) is derived by the small-time expansion method in the form

$$\begin{aligned} \eta(x,t) &= t^2\eta_2(x) + t^3\eta_3(x) + t^4\eta_4(x) + \dots, \\ u(x,t) &= tu_1(x) + t^2u_2(x) + t^3u_3(x)\dots, \\ v(x,t) &= tv_1(x) + t^2v_2(x) + t^3v_3(x) + \dots. \end{aligned}$$

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where the coefficients η_n and u_n may be evaluated via v_n by formulae following from differential equations (4)

$$\begin{aligned}
&\eta_{n+1} = \frac{v_n}{n+1} \quad (n \geq 1), \\
&u_1 = -\alpha v_1', \quad u_2 = 0, \quad u_3 = \frac{1}{6}[v_1^2 - \lambda v_1 - \beta v_1^{(4)}]' - \alpha v_3', \\
&u_4 = 0, \quad u_5 = \frac{1}{20}[4v_1 v_3 - \lambda v_3 - \beta v_3^{(4)}]' - \alpha v_5', \quad \dots
\end{aligned} \tag{9}$$

Using the power expansion of the free surface elevation η we can determine similar power series for the integral operators defined by formulae (6) and (7)

$$\begin{aligned}
&A(\eta) = t^2 A^{(2)} + t^3 A^{(3)} + \dots, \quad B(\eta) = B^{(0)} + t^2 B^{(2)} + \dots, \\
&A_r(\eta) = A_r^{(0)} + t^2 A_r^{(2)} + t^3 A_r^{(3)} + \dots, \quad B_r(\eta) = B_r^{(0)} + t^2 B_r^{(2)} + \dots
\end{aligned}$$

Here the leading-order coefficients of the operators A and B have the form:

$$A^{(2)}u(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\eta_2(x) - \eta_2(s)}{(x-s)^2} u(s)ds - \eta_2'(x)\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{u(s)ds}{x-s}, \quad B^{(0)}u(x) = Hu(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{u(s)ds}{x-s},$$

where H denotes the Hilbert transform. Operators $A_r^{(0)}$ and $B_r^{(0)}$ can be evaluated as follows

$$A_r^{(0)}v(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{(1 - r^2 p(x))q'(x) - (s - r^2 q(x))p'(x)}{(1 - r^2 p(x))^2 + (r^2 q(x) - s)^2} v(s)ds,$$

$$B_r^{(0)}u(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{(1 - r^2 p(x))p'(x) + (s - r^2 q(x))q'(x)}{(1 - r^2 p(x))^2 + (r^2 q(x) - s)^2} u(s)ds,$$

with functions

$$p(x) = P(x)|_{\{\eta=0,\, x_c=0,\, y_c=-1\}} = \frac{1}{1+x^2}, \quad q(x) = Q(x)|_{\{\eta=0,\, x_c=0,\, y_c=-1\}} = \frac{x}{1+x^2}. \tag{10}$$

The dipole term v_d has the power expansion $v_d(x,t) = t v_d^{(1)}(x) + t^3 v_d^{(3)}(x) + \dots$ with coefficients

$$\begin{aligned}
&v_d^{(1)}(x) = 4r^2(q'(x)\sin\theta - p'(x)\cos\theta), \\
&v_d^{(3)}(x) = 4r^2(p''(x)\cos 2\theta - q''(x)\sin 2\theta) + 4r^2(\eta_2(x)[p'(x)\sin\theta + q'(x)\cos\theta])'.
\end{aligned} \tag{11}$$

Thus, integral equation (5) for normal fluid velocity v leads to the set of equations for coefficients v_n

$$[I + r^2 A_r(0)]v_n = \varphi_n, \quad (n \geqslant 1)$$

where functions φ_n can be evaluated explicitly due to relations (9) via the coefficients v_1, v_2, $\dots$, v_n by the formulae

$$\begin{aligned}
&\varphi_1 = v_d^{(1)} + Hu_1 + r^2 B_r^{(0)} u_1, \quad \varphi_2 = 0, \\
&\varphi_3 = v_d^{(3)} + Hu_3 + r^2\left(B_r^{(0)}u_3 - A_r^{(2)}v_1\right) - A^{(2)}v_1, \quad \varphi_4 = 0, \quad \dots
\end{aligned} \tag{12}$$

The main difficulty created by the presence of an ice cover is that small-time expansion for the tangential velocity $u(x,t)$ starts with the first order of time variable t, which is due to the presence of the terms η_{ttx} in differential equation (4). This fact prevents us from formulating explicit formulae expressing v_n via the previous coefficients $v_1,\dots,v_{n-1}$ as it was made in [6] in case of the free surface problem. Instead of that we have series of integro-differential equations

$$\begin{aligned}
&v_1+\alpha(H+r^2B_r^{(0)})\partial_x v_1=v_d^{(1)},\quad v_2=0,\\
&v_3+\alpha(H+r^2B_r^{0})\partial_x v_3=v_d^{(3)}+\frac{1}{6}H\partial_x(v_1^2-\lambda v_1)+\\
&+r^2\left(\frac{1}{6}B_r^{(0)}\partial_x(v_1^2-\lambda v_1)'-A_r^{(2)}v_1\right)-\frac{\beta}{6}\left(H+r^2B_r^{(0)}\right)\partial_x^5 v_1,
\end{aligned}\tag{13}$$

To solve these equations we apply the method of successive approximations. By this approach we suppose $\alpha=\beta=0$ (i.e. we ignore the presence of ice cover in the first order) and calculate v_i from equations (13). Coefficients obtained by this way correspond to the problem on free surface waves caused by circular cylinder. Expansion for the wave elevation $\eta(x,t)$ can be found from the first relation in formulae (9) as follows

$$\begin{aligned}
&I_0(\eta_2(x))=2r^2\Big(1-\frac{r^2}{4}\Big)\Big(q'(x)\sin\theta-p'(x)\cos\theta\Big)+O(r^6),\quad \eta_3(x)=0,\\
&I_0(\eta_4(x))=r^2\Big(1-\frac{r^2}{4}\Big)\Big(p''(x)\Big[\cos2\theta+\frac{\lambda}{6}\sin\theta\Big]+\Big[\frac{\lambda}{6}\cos\theta-\sin2\theta\Big]q''(x)\Big)+\\
&\quad+\frac{r^4}{4}\Big(p'(x)\Big[\sin2\theta-\frac{\lambda}{3}\cos\theta\Big]+\Big[\frac{\lambda}{3}\sin\theta+\cos2\theta\Big]q'(x)\Big)+\\
&\quad+\frac{r^4}{9}\Big(p''''(x)\cos2\theta-q''''(x)\sin2\theta\Big)+\frac{r^4}{3}\Big(p''(x)-\frac{7}{4}q'(x)\Big)+O(r^6).
\end{aligned}\tag{14}$$

Operator I_0 means that η_2 and η_4 were found at the zeroth step. Then we substitute obtained coefficients (14) into equations (13), which get the following form

$$\begin{aligned}
&I_1(v_1)+\alpha(H+r^2B_r^{(0)})\partial_x I_0(v_1)=I_0(v_1),\quad v_2=0,\\
&I_1(v_3)+\alpha(H+r^2B_r^{(0)})\partial_x I_0(v_3)=I_0(v_3)-\frac{\beta}{6}\left(H+r^2B_r^{(0)}\right)\partial_x^5 I_0(v_1),
\end{aligned}\tag{15}$$

The values of coefficients η_2 and η_4 at the next itaration step can be evaluated directly from equations (15):

$$\begin{aligned}
I_1(\eta_2(x))=&\frac{r^2}{2}\Big(4+r^2(\alpha-1)\Big)\Big(q'(x)\sin\theta-p'(x)\cos\theta\Big)+\\
&+\frac{\alpha r^2}{2}(4-r^2)\left(q''(x)\cos\theta+p''(x)\sin\theta\right),
\end{aligned}\tag{16}$$

$$\begin{aligned}
I_1(\eta_4(x))=&r^2\Big(2+\frac{\alpha r^2}{16}[\lambda+8]-\frac{r^2}{12}[\lambda+6+45\beta]\Big)\left(p'(x)\cos\theta-q'(x)\sin\theta\right)+\\
&+3r^4\left(1-\frac{3\alpha}{2}\right)\left(p'(x)\sin2\theta+q'(x)\cos2\theta\right)+\\
&+r^2\left(\frac{\lambda}{6}\left[1-\frac{r^2}{4}\right]+\alpha\left[\frac{r^2}{2}\left(1+\frac{\lambda}{6}\right)-1\right]\right)(p''(x)\sin\theta+q''(x)\cos\theta)+\\
&+r^2\left(1+\frac{r^2}{4}[\alpha-1]\right)(p''(x)\cos2\theta-q''(x)\sin2\theta)+
\end{aligned}\tag{17}$$

$$+\frac{\lambda\alpha r^2}{6}\left(1-\frac{r^2}{4}\right)(p'''(x)\cos\theta - q'''(x)\sin\theta) - \alpha r^2\left(1-\frac{r^2}{4}\right)(p'''(x)\sin 2\theta - q'''(x)\cos 2\theta)+$$
$$+\frac{r^4}{9}\Big([p^{(4)}(x)-\alpha q^{(5)}(x)]\cos 2\theta - [q^{(4)}(x)+\alpha p^{(5)}(x)]\sin 2\theta\Big)+$$
$$+\frac{2\beta r^2}{3}\left(1-\frac{r^2}{4}\right)(p^{(6)}(x)\sin\theta + q^{(6)}(x)\cos\theta) - \frac{r^4}{12}\Big(7q'(x)+[7\alpha-4]p''(x)+\frac{\alpha}{4}q'''(x)\Big)$$

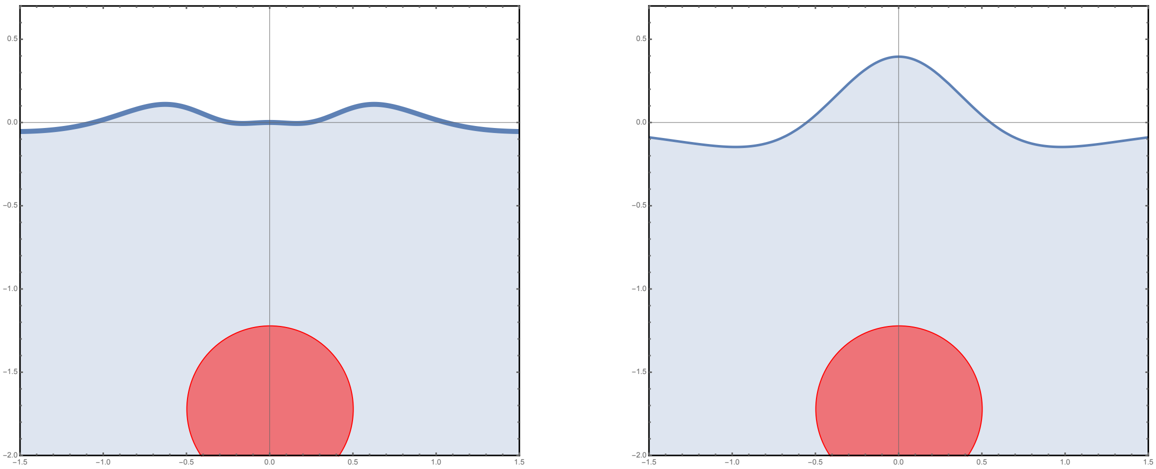
Consequently, repeating this procedure we can calculate ice deflection profile coefficients $I_n(\eta_2)$ and $I_n(\eta_4)$ with any demanded accuracy with respect to parameters α and β from the following formulae

$$I_{n+1}(\eta_2) = I_n(\eta_2) - \alpha(H + r^2 B_r^{(0)})\partial_x I_n(\eta_2),$$
$$I_{n+1}(\eta_4) = I_n(\eta_4) - \alpha(H + r^2 B_r^{(0)})\partial_x I_n(\eta_4) - \frac{\beta}{3}(H + r^2 B_r^{(0)})\partial_x^5 I_n(\eta_2). \quad (18)$$

Here $I_n(\eta)$ denotes the value of η calculated at the n-th step of iterative procedure.

Having the explicit expression for ice deflection $\eta(x,t)$ we can determine pressure p in the fluid under the ice plate from dynamic boundary condition (2). If we look for a solution in the form of power series then the pressure p at the n-th step of iterative procedure takes the form

$$I_n(p(x,t)) = 2\alpha I_n(\eta_2(x)) + \beta\partial_x^2 I_n(\eta_2(x))t^2 + (12\alpha I_n(\eta_4(x)) + \beta\partial_x^4 I_n(\eta_4(x)))t^4, \quad (19)$$



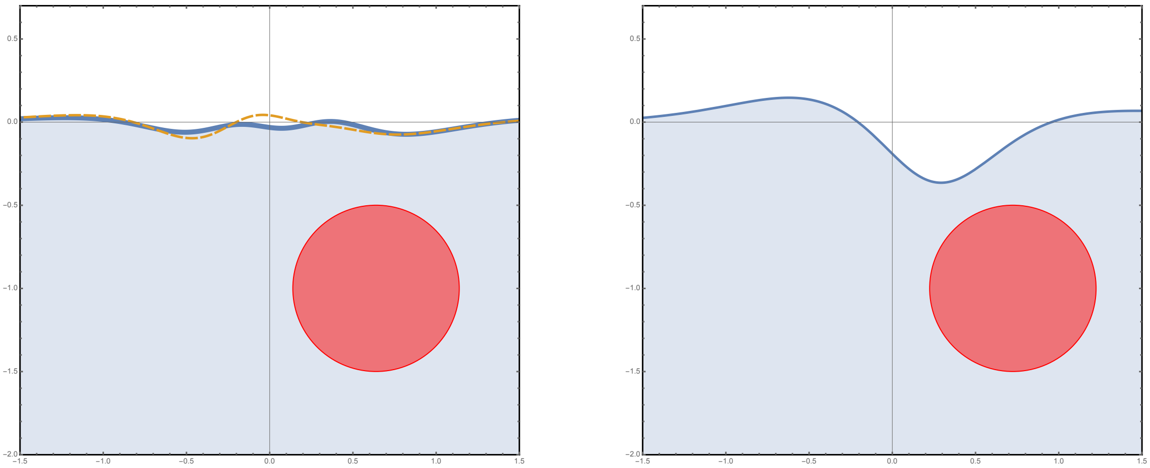
a. *Ice plate* ($\alpha = 0.4$, $\beta = 0.001$) b. *Free surfac.* ($\alpha = 0$, $\beta = 0$)

Figure 2. Deformation of the boundary $y = \eta(x,t)$ at the time $t = 0.85$ caused by vertical sumbersion of the cylinder of radius $r = 0.5$ with parameter $\lambda = 10$.

5. Analysis of the solution

Coefficients (16) and (17) determine the deformation of the boundary $y = \eta(x,t)$ in wide range of paramters: acceleration rate of the cylinder (parameter λ), direction of motion (angle θ), size of the cylinder r, mass and elastic constant of the ice cover (parameters α and β). In particularly, when $\alpha = \beta = 0$ we have the case of the free surface flow; when $\beta = 0$ we have the case of the flow under broken ice.

Figures 2 and 3 demonstrate the differences in the flow patterns generated by the cylinder moving under ice plate and free surface. Indeed, in case of the free surface flow, the vertically submerging cylinder ($\theta = -\frac{\pi}{2}$) causes inertial deflection of free boundary with subsequent formation of intense vertical splash jet. In contrast, two additional deflection zones appear symetrically above the cylinder submerging under elastic ice plate. Similarly, for horizontal motion of the cylinder ($\theta = 0$), the floating ice cover supress essentially impulsive response of the fluid, but small-amplitude elastic waves are generated in this case. Figure 4 shows the pressure distribution in the fluid under the ice plate. In accordance with this picture, the elastic plate experiences the highest loads in two symetric zones of intense deflection above the cylinder. Therefore we can predict that ice is more likely to break in these zones.



a. Ice plate ($\alpha = 0.3$, $\beta = 0.002$) *b. Free surface* ($\alpha = 0$, $\beta = 0$)

Figure 3. Deformation of the boundary $y = \eta(x, t)$ at the time $t = 0.85$ caused by horizontal motion of the cylinder of radius $r = 0.5$ with parameter $\lambda = 10$.

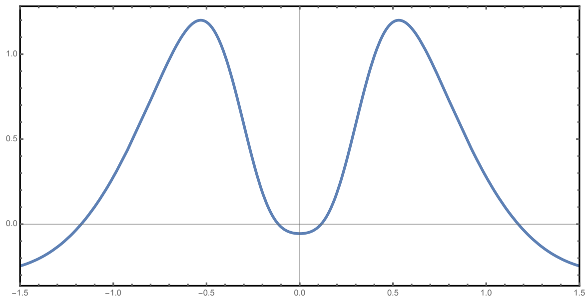


Figure 4. Pressure distribution in the fluid under the ice plate at the time $t = 0.85$ for submerging cylinder of radius $r = 0.5$; $\lambda = 10$, $\alpha = 0.4$, $\beta = 0.001$.

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