On the Combinatorics of Set Families

A thesis submitted to the School of Mathematics at the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Andrew Jonathon Langworthy

March 2018

Abstract

This thesis concerns the combinatorics and algebra of set systems. Let V be a set of size n. We define a vector space M^n with basis the power set of V. This space decomposes into a direct sum of eigenspaces under certain incidence maps. Any collection of k-sets S embeds naturally into this space, and so decomposes as a sum of eigenvectors. The main objects of study are the lengths of these eigenvectors, which we call the shape of S. We prove that the shape of S is a linear transformation of the inner distribution, and show that t-designs have a specific shape. We give some classifications of the shape of collections of k-sets for small k.

Given a permutation group G, we define the subspace M^G of M^n of all vectors fixed by G. We show that this space is spanned by the G-orbits of the power set of V and as a consequence of this, prove the Livingstone-Wagner Theorem. We then give some results about groups that have the same number of orbits on 2-sets and 3-sets.

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Acknowledgements

I would first and foremost like to thank my supervisor Johannes Siemons for his continual support and apparently infinite patience. Without his guidance, both mathematical and sometimes emotional, I have no doubt that this thesis would not have been completed. My thanks also extend to Jennifer Ryan for her help and support.

I would like to thank my wonderful partner Sam, who has been there for me after every long day and late night. I am very lucky to have had such love and encouragement from her throughout the entirety of this PhD programme.

I would also like to thank all the other graduates in the office for making my time so enjoyable. In particular, the (un)official UEA Maths Christmas Team consisting of Tom, Pete, Mike, George, Rob, and Dane. My thanks and apologies also extend to everyone else in the office who put up with the Christmas antics every December with only the most minor of complaints. A special thanks extends to Tom for the Friday drinks and the proofreading. It was very much appreciated!

Finally, I would also like to thank the EPSRC for funding me for 3 years, and my Mum and Dad for all their love and support.

Dedications

For Sam.

1

Introduction

Let V be a set of size n. We consider the power set of V, denoted by L^n and view it as a partially ordered set $(L^n, \subseteq)$ under the containment relation for subsets. In turn, we may partition L^n into the sets L^n_k for $0 \leq k \leq n$, where L^n_k is the set of subsets of V of size k. In future, we call these *k-subsets* of V. The objects of interest in this thesis are subsets S of L^n_k. We call these *k-families*. We study such k-families in the context of the poset L^n.

Many areas of study in combinatorics relate to L^n. One such example are t-designs. A t-(n, k, λ) design is a collection of k-subsets (or *blocks*) of an n-set V such that every t-set of V is contained in precisely λ blocks. Hence a t-design is simply a k-family S contained in L^n where every t-set in L^n is related to exactly λ blocks. More information on t-designs can be found in [Col10, vLW01].

Another example arises from the study of permutation groups. If $G \leq \mathrm{Sym}(V)$ is a permutation group then we may induce an action of G on L^n from the action of G on V. Since the image of a k-set is still a k-set, G acts on each L^n_k. This action partitions each L^n_k into orbits, each of which are k-families.

To study k-families we devise tools to encode the relational structure of L^n into the algebraic structure of an algebra associated to L^n. Similar techniques are used in other areas of algebraic combinatorics. For example, to study graphs we may use techniques from algebraic graph theory and consider their adjacency matrices and spectra. To study association schemes, we may look instead at the *Bose-Mesner algebra*.

However, the subject of this thesis is to introduce a different algebraic approach, as follows.

In Chapter 2 we construct an algebra relating to the Boolean poset L^n. For an n-set V we define L^n to be the set of all subsets of V. For each $0 \le k \le n$ we define L^n_k to be the k-subsets of V. Given a field $\mathbb{F}$ such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$ we construct the vector space $\mathbb{F}L^n = M^n$ of formal sums of elements of L^n with coefficients in $\mathbb{F}$. This space splits naturally into a direct sum

$$M^n = M^n_0 \oplus M^n_1 \oplus \ldots \oplus M^n_n,$$

where M^n_k is the space $\mathbb{F}L^n_k$, the subspace with the k-subsets of V as a basis. The space M^n carries an inner product $\langle\ ,\ \rangle$, and from this we obtain a norm $\|x\| = \sqrt{\langle x, x\rangle}$. We afford the vector space M^n a structure of an algebra with multiplication of two sets being their union and extending linearly.

Further, we want to encode the $\subseteq$ relation from L^n into M^n. To do this we define *incidence maps* derived from this containment relation. In particular, there is a symmetric map $\nu^+_k : M^n_k \to M^n_k$ for each $0 \le k \le n-1$ defined in Equation 2.3. From this map, we get the main theorem of the chapter. Here, $k' = \min\{k, n-k\}$.

Theorem 2.2.3. *Let $0 \le i \le k \le n$. Then ν^+_k has $k'+1$ eigenvalues, written*

$$\lambda_{k,0} > \lambda_{k,1} > \cdots > \lambda_{k,k'}.$$

These are given recursively by $\lambda_{k,i} = \lambda_{k-1,i} + n - 2k$ for $i < k'$ and $\lambda_{k,k'} = n - 2k$. In particular, all eigenvalues are positive integers given by $\lambda_{k,i} = (n-k-i)(k-i+1)$ for $i \le k'$.

With these eigenvalues of ν^+_k, we have the associated eigenspace decomposition of each M^n_k into $k'+1$ eigenspaces. We denote the eigenspace with eigenvalue $\lambda_{k,i}$ by

$E_{k,i}$ for $0 \le i \le k'$. So, we have a further decomposition, given by

$$M_k^n = E_{k,0} \oplus E_{k,1} \oplus \ldots \oplus E_{k,k'}. \tag{1.1}$$

This means that for any $f \in M_k^n$, we may split f into *eigenspace components* as follows:

$$f = f_{k,0} + f_{k,1} + \ldots + f_{k,k'}, \tag{1.2}$$

where $f_{k,i} \in E_{k,i}$. We explicitly define projection maps $\pi_{k,i}$ on page 40 that map $f \mapsto f_{k,i}$. It is this decomposition of f into eigenspace components which is the essential tool for proving the results of this thesis.

Now that the structure on M^n has been developed in Chapter 2, we use this structure to concentrate on k-families in Chapter 3. In particular, we have the embedding of L^n into M^n given by $x \mapsto 1 \cdot x$. This means that if $S \subseteq L_k^n$ is a k-family, then there is a natural embedding of S into M_k^n, given by

$$S \mapsto [S] = \sum_{x \in S} x. \tag{1.3}$$

We call $[S]$ the *characteristic vector* of S in M^n.

Since $[S] \in M_k^n$, we may use the eigenspace decomposition in Equation 1.2 to split $[S]$ into the sum $[S] = [S]_{k,0} + [S]_{k,1} + \ldots + [S]_{k,k'}$, where each $[S]_{k,i} \in E_{k,i}$. To study S, we study these components. In particular, we consider their lengths. We define the *shape* of S to be the $(k'+1)$-tuple

$$\mathrm{sh}(S) = (\mathrm{sh}_0(S), \ldots, \mathrm{sh}_{k'}(S)),$$

where $\mathrm{sh}_i(S) = \left|\left|[S]_{k,i}\right|\right|^2$. This brings us to the following key question.

Question 3.1.3. *Given a k-family S, what combinatorial information about S can we recover from the shape* $\mathrm{sh}(S)$*?*

This will be the motivation for the remainder of the thesis.

To explore this question, we turn to the thesis of Delsarte [Del73]. He defines the *inner distribution* $a(S)$ of a k-family S. We give this in Definition 3.3.1 in terms of the self-intersection properties of S. The main result of Chapter 3 is the connection of this inner distribution of S to the shape $\mathrm{sh}(S)$ of S.

Theorem 3.3.3. *Fix $n \in \mathbb{N}$ and $0 \leq k \leq \frac{n}{2}$. Then there exists an invertible matrix $M \in GL_{k+1}(\mathbb{Q})$ such that*

$$a(S)M = \mathrm{sh}(S)$$

for any k-family S.

As an application, we give a new proof of the famous Erdős-Ko-Rado Theorem [EKR61] of 1961 in terms of the shape of S.

In Chapter 4 we introduce the following definition. We say that a k-family S is *I-free* if for some $I \subseteq \{0, 1, \ldots, k' + 1\}$ we have that the ith shape component of S is zero for all $i \in I$. This leads to the natural problem of classifying the I-free k-families S. To this end, we first note that we have to differentiate between the k-family S over an n-set V, and the same k-family S defined over the $(n+1)$-set $V \cup \{\alpha\}$ for some $\alpha \notin V$. We show that the shape of S is dependent on such a choice of ground set. This is in contrast to the inner distribution.

With this in mind, we then prove the following proposition, originally by Graver and Jurkat [GJ73].

Proposition 4.2.3. *Let $f \in M_k^n$ and $t \leq k$. Then f is t-homogeneous if and only if*

$$f_{k,1} = f_{k,2} = \ldots = f_{k,t} = 0.$$

In particular, if S is a k-family then S is a t-design if and only if

$$[S]_{k,1} = [S]_{k,2} = \ldots = [S]_{k,t} = 0,$$

i.e. if S is $\{1, \ldots, t\}$-free.

We then give some examples of I-free families.

Next we turn our attention to particular values of k. We note that 2-families are simple undirected graphs and so we give the following theorem that gives an explicit formula for the shape of a 2-family.

Theorem 4.4.2. *Let S be a 2-family with shape $\mathrm{sh}(S) = (\mathrm{sh}_0(S), \mathrm{sh}_1(S), \mathrm{sh}_2(S))$. Then we have explicit formulae for $\mathrm{sh}_i(S)$ for $i \in \{0, 1, 2\}$, depending on the degrees of the vertices.*

As a corollary, we classify the 1 and 2-free 2-families. As an application of this theorem we give a bound on the sum of squares of degrees of a graph S.

Lastly we classify the 3-free 3-families. In particular, we have the following theorem.

Theorem 4.5.2. *Let S be a 3-family over a ground set V with $|V| = n$, such that $\mathrm{sh}_3(S) = 0$. Then we have an explicit characterisation of S into one of three types.*

The proof is by induction, and in Appendix B we give a list of 28 3-free 3-families on $n = 6$ points that form the base case. The interest in $\{k'\}$-free set families comes from the theory of permutation groups acting on k-sets, which brings us to the final chapter.

In Chapter 5 we apply some of the ideas of the thesis to the orbits of finite groups. A finite group $G \leq \mathrm{Sym}(V)$ acts on the n-set V, and so also acts on the k-subsets of V. We define the *centralizer algebra*

$$M^G = \{f \in M^n \ : \ f = f^g \ \ \forall g \in G\},$$

and show that it too decomposes into eigenspaces $E^G_{k,i}$ in the same way as the original algebra, as in Equation 1.1.

The orbits of this G-action can be embedded into M^n via Equation 1.3, and we show that the orbits on L^n_k form a basis of M^G_k. We denote the number of these orbits by $\sigma_k(G)$. The Livingstone-Wagner Theorem [LW65] states that if $k \leq \frac{n}{2}$, then $\sigma_k(G) \geq \sigma_{k-1}(G)$. We prove the following theorem that obtains as a corollary the Livingstone-Wagner Theorem.

Theorem 5.2.2. *Let $G \leq \mathrm{Sym}(V)$ and $0 \leq k \leq \frac{n}{2}$. Then we have*

$$M_{k-1}^G \cong M_k^G / E_{k,k}^G.$$

In particular, this gives us that

$$\dim(E_{k,k}^G) = \dim(M_k^G) - \dim(M_{k-1}^G).$$

As another corollary we see that for $t < k \leq \frac{n}{2}$, we have that G satisfies $\sigma_t(G) = \sigma_k(G)$ if and only if all of the k-orbits of G are $\{t+1, \ldots, k\}$-free. We draw attention to the similarities between this theorem and Proposition 4.2.3; t-designs are $\{1, \ldots, t\}$-free, and orbits of G that obtain equality in the Livingstone-Wagner Theorem are $\{t+1, \ldots, k\}$-free.

We go on to use Theorem 4.5.2 to investigate the structure of groups that achieve equality for $k = 2$ and $k = 3$. We prove the following result from Cameron, Neumann, and Saxl [CNS79] in terms of the orbit shape.

Theorem 5.3.10. *If G is a primitive permutation group of degree n and $\sigma_2(G) = \sigma_3(G)$, then G is 3-homogeneous.*

1.1 Notation

In this thesis, groups always act on the right and we write them using exponential notation. Functions are applied on the left. We make a quick note here about k and $k' = \min\{k, n-k\}$. Most of the results in the thesis hold true for k', but there are some that do not follow if we replace k by $n-k$. We will make it clear where important results do not hold for $n-k$ as opposed to k. Table 1.1 below outlines the important notation.

V	A set of size n
α, β, γ	Elements of V
L^n and L^n_k	The power set of V and the set of all k-sets of V respectively
S, T	Subsets of L^n_k
M^n and M^n_k	The vector spaces with bases L^n and L^n_k respectively.
$[S]$	The formal sum $\sum_{x \in S} x \in M^n$
$\mathbf{1}_k$	The formal sum $[L^n_k] = \sum_{x \in L^n_k} x \in M^n$
$\varepsilon, \delta, \nu^+, \nu^-$	Incidence maps on M^n
I	The identity map on M^n
$E_{k,i}$	The i^{th} eigenspace of ν^+ in M^n_k
$\pi_{k,i}$	The projection map from M^n_k onto $E_{k,i}$
$f_{k,i}$	The projection of $f \in M^n_k$ onto $E_{k,i}$
$\mathrm{sh}_i(f)$	$\mathrm{sh}_i(f) = \left\|\left\|f_{k,i}\right\|\right\|^2$
$\mathrm{sh}(f)$	The shape of f, the $(k'+1)$-tuple $(\mathrm{sh}_0(f), \ldots, \mathrm{sh}_{k'}(f))$
k'	$\min\{k, n-k\}$
G	A finite permutation group on V
$\sigma_k(G)$	The number of orbits of G on k-subsets of V
M^G and M^G_k	The centralizer algebra of G on L^n and L^n_k respectively
$E^G_{k,i}$	The intersection of M^G_k and $E_{k,i}$

Table 1.1: Notation

2

The Linear Algebra of a Boolean Lattice

In this chapter we introduce the main object of study in the thesis, the linear algebra of the Boolean lattice, and study the algebraic structure. Many of the ideas are based upon work by Siemons et. al. in [MS96, BJS98, JS01], which dealt with the homology of the Boolean lattice.

We also draw from [Sie13], which laid much of the ground work for this chapter. We will reference any results that were given in this manuscript. All proofs are original to this thesis.

2.1 The Boolean lattice and associated algebra

We begin this chapter with a brief introduction to the Boolean lattice. We follow the text [BS81], Sections I.1, I.3, and IV.1. For a more in-depth introduction to lattice theory, see [DP02]. Let $(P, \leq)$ be a finite poset. Where the context is clear, we will write $(P, \leq)$ as just P. If every pair of elements $x, y \in P$ has a unique supremum and infimum, we call the supremum of x and y the *join* and denote this by $x \vee y$. Similarly, the infimum is called the *meet* and is denoted by $x \wedge y$. Since suprema and infima always exist for every pair $x, y \in P$, then P has a unique upper bound 1 and unique lower bound 0. We call a poset with these properties a *lattice*. Further, if we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{and} \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

i.e. the join and meet operations distribute over each other, then we say that the lattice is *distributive*. Lastly, we say that a distributive lattice is *complemented* if for all $x \in P$ there exists a $y \in P$ such that $x \vee y = 1$ and $x \wedge y = 0$.

If all of the above conditions hold, then we call $(P, \leq)$ a *Boolean lattice*. A Boolean lattice $(P, \leq)$ has a *dual* $P' = (P, \geq)$ given by inverting the order and swapping the roles of the join and meet operations. This means that in P' the infimum of x and y is given by $x \vee y$ and the minimal element of P' is 1.

Now let $V = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of size n. We denote the power set $\mathscr{P}(V)$ by L^n and note that this is a poset under the inclusion relation. If we define $x \vee y = x \cup y$ and $x \wedge y = x \cap y$ then $(L^n, \subseteq)$ is a distributive lattice with upper bound V and lower bound $\emptyset$. In fact, this lattice is also complemented, since if $x \in L^n$ then $V \backslash x \in L^n$ is such that $x \cup (V \backslash x) = V$ and $x \cap (V \backslash x) = \emptyset$. Hence $(L^n, \subseteq)$ is an example of a Boolean lattice. In fact this is no accident: by Stone's representation theorem [HG98, Representation Theorem, p.76] every finite Boolean lattice is isomorphic to the lattice of subsets of an n-set V, also known as a *field of sets*.

We would like to add some extra algebraic structure to the Boolean lattice. Let $\mathbb{F}$ be a field such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$ and consider the vector space $\mathbb{F}L^n$, the vector space of formal sums of elements of L^n over $\mathbb{F}$. We denote this vector space by

$$M^n = \left\{ \sum_{x \in L^n} f_x x \ : \ x \in L^n, \, f_x \in \mathbb{F} \right\}.$$

By construction L^n is a basis of M^n and so the dimension of M^n is 2^n. We identify elements $x \in L^n$ with $1 \cdot x \in M^n$ and so we may think of L^n as a subset of M^n. For example, a typical element of M^n may be of the form $a\{\alpha, \beta\} + b\{\alpha, \gamma\} + c\{\beta, \tau, \eta\}$ with $a, b, c \in \mathbb{F}$. Although we do not use this idea, we note that M^n can also be understood as the vector space of all functions $f : L^n \to \mathbb{F}$, defined by $f(x) = f_x$ for $x \in L^n$.

Following the general structure of the Boolean lattice, we can afford M^n the structure of an algebra by adding a multiplication operation. The natural choice is taking set unions. For $x, y \in L^n$ we say $x \cdot y = x \cup y$ and extend this linearly to M^n so that multiplication distributes over the addition of the vector space. For example,

$$a\{\alpha,\eta\}\cdot(b\{\alpha,\beta\}+c\{\gamma,\tau\}) = ab\{\alpha,\beta,\eta\}+ac\{\alpha,\gamma,\tau,\eta\}$$

for $a, b, c \in \mathbb{F}$. This means that a k-set is the product of its 1-element subsets. This justifies us abusing notation slightly and leaving out the set brackets. For example, if $a = b = 2$ and $c = 1$, the above then becomes $4\alpha\beta\eta + 2\alpha\gamma\tau\eta = 2\alpha\eta(2\beta + \gamma\tau)$.

We quickly draw comparisons between this construction and a group algebra. Let G be a finite group and $\mathbb{F}$ a field. The group algebra $\mathbb{F}[G]$ is the algebra over $\mathbb{F}$ with basis $\{1 \cdot g \ : \ g \in G\}$. A general element of $\mathbb{F}[G]$ is of the form

$$p = \sum_{g\in G} p_g \cdot g,$$

and the algebra multiplication is given by

$$p \circ q = \left(\sum_{g\in G} p_g \cdot g\right) \circ \left(\sum_{g\in G} q_g \cdot g\right) = \sum_{g,h\in G} p_h q_{h^{-1}g} g.$$

Therefore an element of $\mathbb{F}[G]$ is a linear sum of group elements with coefficients in $\mathbb{F}$ and multiplication on basis elements is borrowed from the group structure. This is directly comparable to our construction of M^n. Elements are linear sums of elements in L^n with coefficients in $\mathbb{F}$ and multiplication on basis elements is borrowed from the structure of the Boolean lattice; in this case the union product.

There is another natural way to realise M^n as an algebra. We define $x \cdot y = x \cap y$ for $x, y \in L^n$ and then extend as before. Denote this algebra by $M^{n\prime}$. We can define a map from M^n to $M^{n\prime}$ by mapping $x \mapsto \overline{x}$ for $x \in L^n$, taking complements, and extending linearly. Since this map is its own inverse it is bijective and so is a natural isomorphism between M^n and $M^{n\prime}$ as vector spaces. Since $\overline{x \cup y} = \overline{x} \cap \overline{y}$, the map is an algebra isomorphism as well. If we consider M^n as the algebra associated to the Boolean lattice $(L^n, \subseteq)$, then $M^{n\prime}$ is the algebra associated to the dual $(L^n, \supseteq)$.

However, throughout this thesis M^n denotes the algebra whose multiplication is given by the union of sets, not the intersection. If we require the intersection product, or it is not clear which product we are using, we may write

$$f \cdot g = f \cup g \qquad \text{and} \qquad f \cdot g = f \cap g$$

for the union and intersection products respectively.

We introduce one last piece of structure onto M^n; we give it an inner product. For $x, y \in L^n$ define

$$\langle x, y \rangle = \begin{cases} 1 & \text{if} \quad x = y, \\ 0 & \text{otherwise.} \end{cases}$$

We extend this into M^n linearly in both arguments. Note that this product is positive-definite, and bilinear by construction. It also transforms the basis L^n of M^n into an orthonormal basis. This is useful since if $f \in M^n$ then we may write

$$f = \sum_{x \in L^n} f_x x \quad \text{with} \quad f_x \in \mathbb{F}. \tag{2.1}$$

Here we have $f_x = \langle f, x \rangle$, since

$$\langle f, x \rangle = \left\langle \sum_{y \in L^n} f_y y, x \right\rangle = \sum_{y \in L^n} f_y \langle y, x \rangle = f_x \langle x, x \rangle = f_x.$$

With an inner product we get a natural norm on M^n, defined to be

$$||f||^2 = \langle f, f \rangle.$$

We call $||f||$ the *length* of f. As we mentioned, L^n is an orthonormal basis of M^n since for any $x \in L^n$ we have

$$||x|| = \sqrt{\langle x, x \rangle} = 1.$$

The algebra M^n is also a quotient of a polynomial algebra. Let A be the polynomial algebra $A = \mathbb{F}[X_\alpha, X_\beta, \ldots, X_\omega]$ where the X_ι are indexed by the elements of V, and take I to be the ideal generated by the set

$$\{X_\alpha^2 - X_\alpha \;:\; \alpha \in V\}. \tag{2.2}$$

Proposition 2.1.1. *For A and I as above we have that A/I and M^n are isomorphic as algebras.*

This is a special case of [Hey99, Proposition 2.1]. Under the union multiplication, L^n is a semigroup generated by all subsets of V under the relations that $\{\alpha\} \cdot \{\alpha\} = \{\alpha\}$. Hence [Hey99] tells us that the algebra $\mathbb{F}L^n$ is isomorphic to the polynomial algebra modulo the ideal generated by Equation 2.2. We give our own proof here for completeness.

Proof of Proposition 2.1.1. Let $\overline{X}$ be the residue class $X + I$ of the monomial

$$X = \prod_{\alpha \in V} X_\alpha^{i_\alpha}$$

where $i_\alpha \in \mathbb{N}_0$. Now let $\theta : A/I \to M^n$ be defined on the monomial $\overline{X}$ by

$$\theta(\overline{X}) = \prod_{\alpha \in V} \alpha^{i_\alpha}$$

where the right-hand product is the union product. We extend this linearly and claim that this is an algebra homomorphism. Indeed, it is clear that this commutes with multiplication and so all we need to show is that θ is independent of the choice of residue class representative.

Let f be a representative from the residue class $f + I$ and define g to be the polynomial with the same coefficients as f but in each monomial X^j the power of X_α is 0 if it was 0 in f and 1 otherwise. For example, $X_\alpha^2 X_\beta + 2X_\alpha X_\gamma^3$ becomes $X_\alpha X_\beta + 2X_\alpha X_\gamma$. Since all $\alpha \in L_1^n$ are idempotent in M^n, we have that $\theta(f) = \theta(g)$ and so $g \in f + I$. Clearly such a g is unique. This means that θ is independent of the choice of residue class representative and hence θ is an algebra homomorphism.

Since every residue class contains such a g and this g is unique, the set of residue classes are in bijection with the set of such g. In turn, the set of such g are in bijection with $x \in L^n$, the bijection given by $g \longleftrightarrow \theta(g)$. Hence θ is a bijection and therefore an isomorphism as required. $\square$

2.1.1 Incidence maps

As we did with the meet and join operations, we would like to encode the partial order $\subseteq$ of the Boolean lattice $P = (L^n, \subseteq)$ into the algebra M^n in an algebraic way. To this end we define the map

$$\varepsilon : M^n \to M^n \qquad \text{by} \qquad \varepsilon(x) = \begin{cases} \sum_{y \supset x} y & \text{if} \quad |y| = |x| + 1 \leq n \\ 0 & \text{otherwise} \end{cases}$$

on basis elements $x \in L^n$ and extend linearly. This means that $\langle \varepsilon(x), y \rangle = 1$ if and only if $|y| = |x| + 1$ and $y \supset x$. In general, for $x, y \in L^n$ we have that $y \supset x$ with $|y| = |x| + i$ if and only if

$$\left\langle \varepsilon^i(x), y \right\rangle = i!$$

since $i!$ is the number of paths from x to y in the Boolean lattice.

The dual lattice P' has the partial order $\supseteq$ and so we follow the same method as we did above. We define the map

$$\delta : M^n \to M^n \qquad \text{by} \qquad \delta(x) = \begin{cases} \sum_{y \subset x} y & \text{if} \quad 0 \leq |y| = |x| - 1 \\ 0 & \text{otherwise} \end{cases}$$

on basis elements $x \in L^n$ and extend linearly. This means that $\langle \delta(x), y \rangle = 1$ if and only if $|y| = |x| - 1$ and $y \subset x$. In general, for $x, y \in L^n$ we have that $y \subset x$ with $|y| = |x| - i$ if and only if

$$\left\langle \delta^i(x), y \right\rangle = i!$$

as before. To draw one last parallel between the partial order of the lattice and the ε

and δ maps, note that $\varepsilon(V) = 0$ since V is the maximal element of P. The same is true for $\delta(\emptyset)$. We note that by definition both ε and δ are linear transformations of M^n as a vector space, but are not algebra homomorphisms. However, they do behave nicely with respect to the inner product.

Lemma 2.1.2 ([Sie13]). *If $f_1, f_2 \in M^n$ then $\langle \varepsilon(f_1), f_2\rangle = \langle f_1, \delta(f_2)\rangle$. In particular ε and δ are adjoints of each other.*

Proof. Since $\langle\ ,\ \rangle$ is linear in the first and second variables, it suffices to show this for $f_1, f_2 \in L^n$, which we denote by x and y. Note that

$$\langle \varepsilon(x), y\rangle = \begin{cases} 1 \quad \text{if} \quad x \subset y \quad \text{and} \quad |x|+1 = |y|, \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

since by definition if $x \subset y$, then y is a term in $\varepsilon(x)$. However, this is the same when we look at δ:

$$\langle x, \delta(y)\rangle = \begin{cases} 1 \quad \text{if} \quad y \supset x \quad \text{and} \quad |y|-1 = |x|, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

This gives us equality. □

We now give a small lemma that will be of use later.

Lemma 2.1.3 (Leibniz Rule, [Sie13]). *Let $f, g \in M^n$ such that all sets appearing as summands in f are pairwise disjoint from those appearing in g. Then*

$$\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g)$$

where $\cdot$ is our standard union multiplication.

Proof. Since δ and $\cdot$ are linear it suffices to show this for $x, y \in L^n$ such that $x \cap y = \emptyset$. Consider some z a summand of $\delta(x \cdot y)$ and note that z has coefficient 1, since $x \cdot y$ is a single set in L^n. Now, z must either contain all points in x and all but one points in

y or all points in y and all but one points in x. In the former case, z is a summand in $\delta(y) \cdot x$ and in the latter it is a summand in $\delta(x) \cdot y$. It cannot be both. Therefore z is a summand of $\delta(x) \cdot y + x \cdot \delta(y)$ with coefficient 1. This argument works in reverse, completing the proof. □

Using ε and δ, we now split M^n into subspaces. To do this we partition L^n into $n+1$ parts $L^n = \bigcup_{i=0}^{n} L_i^n$, where $L_k^n \subset L^n$ is the set of all k-sets of L^n. We now define subspaces M_k^n of M^n, where

$$M_k^n = \mathbb{F} L_k^n$$

is the vector space over $\mathbb{F}$ with basis the k-sets of L^n. Here, M^n is isomorphic to the direct sum $M_0^n \oplus M_1^n \oplus \ldots \oplus M_n^n$ and the dimension of M_k^n is $\binom{n}{k}$. With these subspaces, we can restrict our ε and δ-maps to give

$$\varepsilon_k \colon M_k^n \to M_{k+1}^n \qquad \text{and} \qquad \delta_k \colon M_k^n \to M_{k-1}^n.$$

We will not usually use this notation, as most of the time it will be clear what we mean, but it will sometimes be useful for clarification.

Now note that if we compose ε_k with δ_{k+1} we obtain a vector space endomorphism of M_k^n. Formally, we denote

$$\nu_k^+ = \delta_{k+1} \circ \varepsilon_k. \tag{2.3}$$

Here, ν_k^+ is non-zero only if $0 \leq k \leq n-1$. As with ε and δ, we can think of ν_k^+ as the restriction to M_k^n of some linear transformation $\nu^+ \colon M^n \to M^n$. We define a related endomorphism by

$$\nu_k^- = \varepsilon_{k-1} \circ \delta_k.$$

This is non-zero only if $1 \leq k \leq n$, and is the restriction of a map $\nu^- \colon M^n \to M^n$ to M_k^n. Again, these maps are linear transformations, but not algebra homomorphisms.

By Lemma 2.1.2 we know that ε and δ are adjoint to each other and so

$$\left\langle \nu^+(f_1), f_2 \right\rangle = \langle \varepsilon(f_1), \varepsilon(f_2) \rangle = \left\langle f_1, \nu^+(f_2) \right\rangle. \tag{2.4}$$

Hence ν^+ is self-adjoint or *symmetric*, and similarly for ν^-.

A basic property of the Boolean lattice is encapsulated in the next lemma, which will be vital in the proof of Theorem 2.2.3.

Lemma 2.1.4 (A-type Lemma, [Sie13]). *Let* $0 \leq k \leq n$. *Then* $\nu_k^+ - \nu_k^- = (n-2k)I$ *where* I *denotes the identity transformation on* M_k^n.

Proof. Since ν_k^+ and ν_k^- are linear it suffices to prove this for basis elements and the rest follows by linearity. Recall that for any $f \in M_k^n$ we have

$$f = \sum_{x \in L_k^n} \langle f, x \rangle x.$$

Now take some $z \in L_k^n$ and note that

$$\nu_k^+(z) = \sum_{x \in L_k^n} \langle \nu_k^+(z), x \rangle x = \sum_{x \in L_k^n} \langle \varepsilon_k(z), \varepsilon_k(x) \rangle x,$$

as ε_k is adjoint to δ_{k+1}. We can see that

$$\langle \varepsilon_k(z), \varepsilon_k(x) \rangle = \begin{cases} n-k & \text{if} \quad z = x, \\ 1 & \text{if} \quad |z \cup x| = k+1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

Now we do a similar thing with ν_k^-. We have

$$\nu_k^-(z) = \sum_{x \in L_k^n} \langle \delta_k(z), \delta_k(x) \rangle x,$$

and again we look at

$$\langle \delta_k(z), \delta_k(x) \rangle = \begin{cases} k & \text{if} \quad z = x, \\ 1 & \text{if} \quad |z \cap x| = k-1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

Since $|z \cup x| = k+1$ if and only if $|z \cap x| = k-1$, we see that all terms in $(\nu_k^+ - \nu_k^-)z$

are 0 apart from the z term, which has coefficient $n-2k$ as required. □

Note that this lemma only depends on the containment properties of the Boolean lattice; we are using the property that if two k-sets join at a $(k+1)$-set, they also meet in a $(k-1)$-set and vice versa. This property holds for some other posets ordered by inclusion. For example, consider the projective space over some finite field $PG(n,q)$, and form the poset with elements the subspaces of $PG(n,q)$ ordered by inclusion. This has the same property; two k-dimensional subspaces intersect into a $(k-1)$-dimensional space if and only if the subspace generated by their union is $(k+1)$-dimensional.

The Boolean lattice is also the poset of a simplex with the ordering given by inclusion. The symmetry group of the simplex is the Coxeter group of type A, the symmetric group. This justifies the name A-type Lemma. In contrast, another object of study is the poset of the hyperoctahedron. The symmetries here are given by $\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$, the Coxeter group of B type. This was done by Siemons and Summers in [SS17], where a B-type incidence lemma was given.

2.1.2 The G-action on M_k^n

Let $G \leq \mathrm{Sym}(n)$ be a permutation group acting on a ground set V. Then G also acts on L_1^n in the natural way: for $g \in G$ and $\alpha \in V$ the action of G on L_1^n is given by

$$\{\alpha\}^g = \{\alpha^g\}.$$

In a similar way we define the action of G on L_k^n by saying that G acts pointwise on the elements of a k-set. Formally, if $\alpha_1, \dots, \alpha_k \in V$ and $g \in G$ then

$$\{\alpha_1, \alpha_2, \dots, \alpha_k\}^g = \{\alpha_1^g, \alpha_2^g, \dots, \alpha_k^g\}.$$

If we think of L_k^n as a basis of M_k^n and extend the G-action linearly, then this makes M_k^n into a G-module.

This action is very natural, but for it to be of use to us it will need to commute (or at least, work "nicely") with the maps and structure we have already introduced. Fortunately, this is true across the board.

Lemma 2.1.5. *If $f_1, f_2 \in M^n$ and $g \in \mathrm{Sym}(n)$ then we have $\langle f_1, f_2\rangle = \left\langle f_1^g, f_2^g \right\rangle$.*

Proof. For $i = 1, 2$, by Equation 2.1 we have

$$f_i = \sum_{x \in L^n} \langle f_i, x\rangle\, x,$$

and hence

$$\langle f_1, f_2\rangle = \left\langle \sum_{x\in L^n} \langle f_1, x\rangle\, y, \sum_{y\in L^n} \langle f_2, y\rangle\, x \right\rangle = \sum_{x,y\in L^n} \langle f_1, x\rangle \langle f_2, y\rangle \langle x, y\rangle .$$

However, since $\langle x, y\rangle = 0$ if and only if $x \neq y$ and 1 otherwise, this becomes

$$\sum_{x\in L^n} \langle f_1, x\rangle \langle f_2, x\rangle .$$

Similarly we obtain

$$\left\langle f_1^g, f_2^g \right\rangle = \left\langle \sum_{x\in L^n} \langle f_1, x\rangle\, x^g, \sum_{y\in L^n} \langle f_2, y\rangle\, y^g \right\rangle = \sum_{x,y\in L^n} \langle f_1, x\rangle \langle f_2, y\rangle \langle x^g, y^g\rangle .$$

Again, since $\langle x^g, y^g\rangle = 0$ if and only if $x \neq y$ and 1 otherwise, this becomes

$$\sum_{x\in L^n} \langle f_1, x\rangle \langle f_2, x\rangle ,$$

and we have equality. □

Corollary 2.1.6. *If $f \in M^n$ and $g \in \mathrm{Sym}(n)$ then $||f||^2 = ||f^g||^2$.*

Proof. Since $||f||^2 = \langle f, f\rangle$, this follows immediately from Lemma 2.1.5 by setting $f_1 = f_2 = f$. □

Lemma 2.1.7. *Let $f_1, f_2 \in M^n$ and $g \in \mathrm{Sym}(n)$. Then $(f_1 \cdot f_2)^g = f_1^g \cdot f_2^g$.*

Proof. As before, for $i = 1, 2$ we have

$$f_i = \sum_{x \in L^n} \langle f_i, x \rangle \, x$$

and so

$$\begin{aligned}
(f_1 \cdot f_2)^g &= \left(\left(\sum_{x \in L^n} \langle f_1, x \rangle \, x \right) \cdot \left(\sum_{y \in L^n} \langle f_2, y \rangle \, y \right) \right)^g \\
&= \left(\sum_{x, y \in L^n} \langle f_1, x \rangle \langle f_2, y \rangle \, (x \cdot y) \right)^g \\
&= \sum_{x, y \in L^n} \langle f_1, x \rangle \langle f_2, y \rangle \, (x \cdot y)^g.
\end{aligned}$$

Also

$$\begin{aligned}
f_1^g \cdot f_2^g &= \left(\sum_{x \in L^n} \langle f_1, x \rangle \, x \right)^g \cdot \left(\sum_{y \in L^n} \langle f_2, y \rangle \, y \right)^g \\
&= \left(\sum_{x \in L^n} \langle f_1, x \rangle \, x^g \right) \cdot \left(\sum_{x \in L^n} \langle f_2, x \rangle \, x^g \right) \\
&= \sum_{x, y \in L^n} \langle f_1, x \rangle \langle f_2, y \rangle \, (x^g \cdot y^g).
\end{aligned}$$

Now, note that since x, y are subsets of V, then $(x \cup y)^g = x^g \cup y^g$, completing the proof. $\square$

Lemma 2.1.8. *The action of* $\mathrm{Sym}(n)$ *on* M^n *commutes with the* ε *and* δ*-functions. In particular, for* $f \in M^n$ *we have*

$$\varepsilon(f)^g = \varepsilon(f^g) \qquad \textit{and} \qquad \delta(f)^g = \delta(f^g).$$

Proof. Both ε and δ are linear, so it suffices to show the equality for basis elements. So let $x \in L_k^n$ and note that

$$\varepsilon_k(x)^g = \left(\sum_{y \in L_{k+1}^n} \langle \varepsilon_k x, y \rangle \, y \right)^g = \sum_{y \in L_{k+1}^n} \langle \varepsilon_k x, y \rangle \, y^g,$$

and

$$\varepsilon_k(x^g) = \sum_{y \in L_{k+1}^n} \langle \varepsilon_k(x^g), y \rangle \, y.$$

Now choose z such that $\langle \varepsilon_k x, z \rangle = 1$. This means that in the first equation z^g has coefficient 1. Also $x \subset z$ and hence $x^g \subset z^g$, which gives $\langle \varepsilon_k(x^g), z^g \rangle = 1$. Therefore the coefficient of z^g in the second equation is 1. This argument works in reverse, proving equality. The proof for δ is identical. $\square$

Lemma 2.1.8 also tells us that the G-action also commutes with $\nu^+ = \delta\varepsilon$.

2.2 Eigenspace decomposition

We begin this section with the Spectral Theorem. This will be a key theorem for Theorem 2.2.3, which allows us to split M_k^n into a sum of eigenspaces.

Theorem 2.2.1 (Spectral Theorem, [Lan05], XV. Theorem 7.1)**.** *Let V be a non-empty finite-dimensional real vector space with a symmetric positive-definite bilinear form $\langle \ , \ \rangle$, and let ρ be a symmetric linear map. Then V has an orthogonal basis consisting of eigenvectors of ρ.*

Remark. We have already noted that $\langle \ , \ \rangle$ is positive-definite and bilinear, and we showed that ν^+ is symmetric in Equation 2.4. However, the Spectral Theorem requires that the field over the vector space is the real numbers. All we have claimed about our field $\mathbb{F}$ is that it lies between $\mathbb{Q}$ and $\mathbb{R}$. To this end, until the remark on page 32 we will assume that $\mathbb{F} = \mathbb{R}$. This means that we may apply the Spectral Theorem to M_k^n.

We now give a simple lemma, which will help us link the eigenspaces and eigenvalues of ν^+ and ν^- to each other. The lemma concerns the relationship between the eigenvalues and eigenspaces of related linear transformations. In particular, it will show us that ν_k^+ and ν_{k+1}^- share the same non-zero eigenvalues. We include a proof for completeness.

Lemma 2.2.2 ([GR01, Lemma 8.2.4])**.** *Let V and W be $\mathbb{F}$-vector spaces, and let ρ, ψ be two linear transformations $\rho : V \to W$ and $\psi : W \to V$. Then the maps $\rho\psi : W \to W$ and $\psi\rho : V \to V$ have the same non-zero eigenvalues. If $\lambda \neq 0$ is such an eigenvalue and we denote the λ-eigenspaces of V and W as E^V_λ and E^W_λ respectively, then the restrictions*

$$\rho : E^V_\lambda \to E^W_\lambda \qquad \textit{and} \qquad \psi : E^w_\lambda \to E^V_\lambda$$

are isomorphisms.

Proof. We prove first that $\rho\psi$ and $\psi\rho$ have the same non-zero eigenvalues. To see this let $\lambda \neq 0$ be an eigenvalue of $\psi\rho$. Then we have some $v \in V$ such that $\psi\rho(v) = \lambda v$. Now apply ρ to both sides and we see that

$$\rho\psi(\rho(v)) = \lambda\rho(v),$$

and so λ is also an eigenvalue for $\rho\psi$. We now prove injectivity. Suppose that $\rho(x) = \rho(y)$ for $x, y \in E^V_\lambda$. Then applying ψ we see that $\psi\rho(x) = \psi\rho(y)$, which gives $\lambda x = \lambda y$, giving ρ is indeed injective from E^V_λ to E^W_λ. To see that it is surjective, take some $w \in E^W_\lambda$ and consider $v_w = \frac{1}{\lambda}\psi(w)$. Then applying ρ to this we obtain

$$\begin{aligned} \rho\big(\frac{1}{\lambda}\psi(w)\big) &= \frac{1}{\lambda}\rho\psi(w) \\ &= \frac{\lambda}{\lambda}w = w, \end{aligned}$$

showing that ρ defines an isomorphism.

Swapping the order of ρ and ψ gives us that eigenvalues of $\rho\psi$ are also eigenvalues of $\psi\rho$, giving us equality. A symmetry argument shows us that ψ also defines an isomorphism between E^W_λ and E^V_λ. □

In particular, we may take $V = M^n_k$ and $W = M^n_{k+1}$. Then, we take $\rho = \varepsilon_k$ and $\psi = \delta_{k+1}$. This gives us that ε_k and δ_{k+1} are isomorphisms between non-zero eigenspaces of ν^+_k and ν^-_{k+1}, and by Lemma 2.1.4 any eigenvector for ν^-_{k+1} with eigenvalue $\lambda \neq 0$ is also an eigenvector for ν_{k+1}^{+} with eigenvalue $\lambda + n - 2k - 2$. By extension then, ε_k and δ_{k+1} are isomorphisms between eigenspaces (with non-zero eigenvalue) of M_k^n and M_{k+1}^n. This is an important fact that will be used a lot.

However, it is also possible that we have zero as an eigenvalue for ν_k^{+} or ν_{k+1}^{-}. In fact, by dimension arguments, this always happens unless

$$\binom{n}{k} = \dim M_k^n = \dim M_{k+1}^n = \binom{n}{k+1}.$$

If $k < \lfloor \frac{n}{2} \rfloor$ then ν_k^{-} has a zero eigenvalue and if $k > \lfloor \frac{n}{2} \rfloor$ then ν_k^{+} has a zero eigenvalue. This gives us enough information to define all eigenvalues recursively. From now on, we will denote

$$k' = \min\{k, n-k\}.$$

Remark. We make considerable use of k' throughout this thesis, but we note that on a first reading very little is lost by assuming $k \leq \frac{n}{2}$, which means $k' = k$.

We have the following main result on eigenvalues.

Theorem 2.2.3 ([Sie13]). *Let $0 \leq i \leq k \leq n$. Then ν_k^{+} has $k'+1$ eigenvalues, written*

$$\lambda_{k,0} > \lambda_{k,1} > \cdots > \lambda_{k,k'}.$$

These are given recursively by $\lambda_{k,i} = \lambda_{k-1,i} + n - 2k$ for $i < k'$ and $\lambda_{k,k'} = n - 2k$. In particular, all eigenvalues are positive integers given by $\lambda_{k,i} = (n-k-i)(k-i+1)$ for $i \leq k'$.

Proof. The proof is based on Lemma 2.1.4 and induction only. Although we don't expand on this here, this means that the proof applies to the algebra of other posets where Lemma 2.1.4 holds. Now, note that $M_0^n = \langle \emptyset \rangle \cong \mathbb{F}$ and so ν_0^{+} can have at most one eigenvalue. It can be checked that $\nu_0^{+}(\emptyset) = n \cdot \emptyset$ and so $\lambda_{0,0} = n - 2(0) = n$. This completes the base case.

Now let $1 \leq k \leq n$ and consider the eigenvalues of ν_k^{+}. For inductive hypothesis, we assume that ν_{k-1}^{+} has k' eigenvalues $\lambda_{k-1,0},\ldots,\lambda_{k-1,k'-1}$ such that

$$\lambda_{k-1,i} = (n-k-i+1)(k-i).$$

If $k = k'$ then by hypothesis all of these eigenvalues are non-zero, and by Lemma 2.2.2 these eigenvalues are also the non-zero eigenvalues of ν_k^-. We now recall Lemma 2.1.4, which says that $\nu_k^+ = \nu_k^- + (n-2k)I$, and so the eigenvalues of ν^+ are

$$\lambda_{k-1,i} + n - 2k = (n-k-i+1)(k-i) + n - 2k = (n-k-i)(k-i+1) = \lambda_{k,i}$$

for $0 \leq i \leq k-1$. Also, ν_k^- has 0 as an eigenvalue and so ν_k^+ has one last eigenvalue $\lambda_{k,k} = 0 + n - 2k = n - 2k$. This completes the first case.

If $k' = n-k$ and $(k-1)' = n-(k-1)$ we proceed in a similar manner. In this case the eigenvalue $\lambda_{k-1,(k-1)'} = (n-k-(k-1)'+1)(k-(k-1)') = 0$ and so the eigenvalues of ν_k^- are $\lambda_{k-1,0},\ldots,\lambda_{k-1,(k-1)'-1}$. Note that $(k-1)'-1 = n-k+1-1 = n-k = k'$. Hence we have k' non-zero eigenvalues. These are the same eigenvalues as $\nu_k^+ - (n-2k)I$ and so ν_k^+ has k' eigenvalues. As before, by Lemma 2.1.4 these eigenvalues are

$$\lambda_{k-1,i} + n - 2k = (n-k-i+1)(k-i) + n - 2k = (n-k-i)(k-i+1) = \lambda_{k,i}.$$

Note that $\lambda_{k,k'} = 0$ in this case.

The last case to check is when $k' = n-k$ but $(k-1)' = k-1 \neq n-(k-1)$. Note that this is only true when $k' = (k-1)'$ and so $n = 2k+1$. In this case the eigenvalues of ν_{k-1}^+ are $\lambda_{k-1,0},\ldots,\lambda_{k-1,k-1}$, which are all non-zero. Hence the non-zero eigenvalues of ν_k^- are $\lambda_{k,0},\ldots,\lambda_{k,k-1} = \lambda_{k,k'}$. As before, $\lambda_{k,k'} = 0$. This completes the proof. $\square$

Later on, we will also need to use the eigenvalues of ν_k^-. Using Lemma 2.1.4, we can see that these are $\lambda_{k,i} - n + 2k$ for $i \leq k'$. For ease of notation we will denote these eigenvalues by

$$\lambda_{k,i}^- = \lambda_{k,i} - n + 2k = (n-k-i+1)(k-i). \tag{2.5}$$

Remark. Since $\mathbb{R}L_k^n$ has an orthogonal basis of eigenvectors of ν^+, we call ν^+ *diagonalisable* over $\mathbb{R}L_k^n$. This means that the minimum polynomial of ν^+ has distinct roots [KW98, Theorem 12.4]. Furthermore, we know that these roots are integers (they are the eigenvalues of ν^+) and by [KW98, Corollary 12.13] ν^+ is diagonalisable over $\mathbb{F}L_k^n$ for any field $\mathbb{F}$ containing the roots. In particular, $\mathbb{F}L^n$ has an orthogonal basis consisting of eigenvectors of ν^+ for any $\mathbb{F} \supset \mathbb{Z}$. Hence we may now assume $\mathbb{F}$ is any field containing $\mathbb{Q}$ as before.

Using the Spectral Theorem we split M_k^n into a direct sum of eigenspaces with respect to the map ν_k^+. Denote the eigenspace with eigenvalue $\lambda_{k,i}$ as $E_{k,i}$. Then we have for any $0 \leq k \leq n$

$$M_k^n = E_{k,0} \oplus E_{k,1} \oplus \cdots \oplus E_{k,k'}. \tag{2.6}$$

We know by Lemma 2.2.2 that ε and δ are isomorphisms between $E_{k,i}$ and $E_{k+1,i}$ for $0 \leq i \leq \min\{k', (k+1)'\}$ and so we have in general that $E_{k,i} \cong E_{\ell,i}$ for $i \leq \min\{k', \ell'\}$. From this we get that if $k' = k$ then M_{k-1}^n embeds into M_k^n, and if $k' = n-k$ then M_{k+1}^n embeds into M_k^n. We see this in Figure 2.1. It makes it clear precisely how the eigenspaces $E_{k,i}$ relate to each other. In particular, we can see the dimension of each eigenspace clearly.

Lemma 2.2.4. *Let* $0 \leq i \leq k'$. *Then*

$$\dim(E_{k,i}) = \binom{n}{i} - \binom{n}{i-1}.$$

Proof. Recall that since M_k^n has L_k^n as a basis, the dimension of M_k^n is $\binom{n}{k}$. From Figure 2.1, it is clear that we have

$$M_k^n \cong M_{k-1}^n \oplus E_{k,k'}.$$

In particular, this means that $\dim(E_{k,k'}) = \dim(M_k^n) - \dim(M_{k-1}^n)$ and hence

$$\dim(E_{k,k'}) = \binom{n}{k'} - \binom{n}{(k-1)'}.$$

Since $E_{k,i} \cong E_{\ell,i}$ for all $i \leq \max\{k', \ell'\}$, this completes the proof. $\square$

$$
\begin{array}{ccccccccccc}
M_n^n & = & E_{n,0} & & & & & & & & \\
 & & \wr\| & & & & & & & & \\
M_{n-1}^n & = & E_{n-1,0} & \oplus & E_{n-1,1} & & & & & & \\
 & & \wr\| & & \wr\| & & & & & & \\
M_{n-2}^n & = & E_{n-2,0} & \oplus & E_{n-2,1} & \oplus & E_{n-2,2} & & & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
M_k^n & = & E_{k,0} & \oplus & E_{k,1} & \oplus & E_{k,2} & \oplus & \ldots & \oplus & E_{k,k'} \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \iddots & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
M_2^n & = & E_{2,0} & \oplus & E_{2,1} & \oplus & E_{2,2} & & & & \\
 & & \wr\| & & \wr\| & & & & & & \\
M_1^n & = & E_{1,0} & \oplus & E_{1,1} & & & & & & \\
 & & \wr\| & & & & & & & & \\
M_0^n & = & E_{0,0} \cong \mathbb{F} & & & & & & & &
\end{array}
$$

Figure 2.1: Eigenspace decomposition of the linear algebra of L^n

Lemma 2.2.5. *For each $0 \leq k \leq n$ and $0 \leq i \leq k'$, the eigenspaces $E_{k,i}$ are G-invariant for $G \leq \mathrm{Sym}(n)$.*

Proof. Let $f \in E_{k,i}$ for some $0 \leq i \leq k' \leq n$ and let $g \in G$. Then

$$\nu^+(f^g) = \left(\nu^+(f)\right)^g = \lambda_{k,i} f^g.$$

This means that f^g is an eigenvector of ν^+ with eigenvalue $\lambda_{k,i}$ and so $f^g \in E_{k,i}$. Hence the $E_{k,i}$ are G-invariant. $\square$

In fact we can say more.

Proposition 2.2.6 ([Sie13]). *Fix $k \leq n$. Then the eigenspaces $E_{k,i}$ are pairwise non-isomorphic irreducible* Sym(n)*-modules, i.e. $E_{k,i} \cong E_{\ell,j}$ if and only if $i = j$.*

Proof. Firstly, we note that for some $x \in L_k^n$ the stabilizer of x in Sym(n) has $k+1$ orbits on L_k^n, with each orbit given by

$$\mathcal{O}_i = \{y \in L_k^n : |x \cap y| = i\} \qquad \text{with} \qquad 0 \leq i \leq k.$$

This means that the permutation rank of Sym(n) on L_k^n is $k+1$. From the Handbook of Combinatorics [GGL95, Chapter 12, Corollary 6.7] we see that if $k \leq \frac{n}{2}$ this is equal to the inner product $\langle \pi, \pi \rangle$, where π is the permutation character of Sym(n) on M_k^n.

We now proceed by induction on k. We see that $E_{0,0}$ is irreducible since it is one dimensional, showing the base case. For the inductive hypothesis, assume that $E_{k-1,0}, E_{k-1,1}, \ldots, E_{k-1,k-1}$ are irreducible. Since $\varepsilon : E_{k-1,i} \to E_{k,i}$ defines an isomorphism that commutes with the action of Sym(n) we have that $E_{k,i}$ are also irreducible for $0 \leq i \leq k-1$, so it suffices to prove that $E_{k,k}$ is also irreducible.

To this end, assume that $E_{k,k}$ splits into q irreducible submodules $\overline{E}_1, \ldots, \overline{E}_q$. We now define π_i to be the restriction of the permutation character π to the subspace $E_{k,i}$ and $\overline{\pi}_i$ to be the restriction to $\overline{E}_i$. Therefore we get

$$\langle \pi, \pi \rangle = \left\langle \bigoplus_{i=0}^{k} \pi_i \oplus \bigoplus_{j=1}^{q} \overline{\pi}_j, \bigoplus_{i=0}^{k} \pi_i \oplus \bigoplus_{j=1}^{q} \overline{\pi}_j \right\rangle.$$

Since all of these subspaces are irreducible, the inner product of non-equal characters is zero, and so this becomes

$$k+1 = \langle \pi, \pi \rangle = \sum_{i=1}^{k} \langle \pi_i, \pi_i \rangle + \sum_{j=1}^{q} \left\langle \overline{\pi}_j, \overline{\pi}_j \right\rangle.$$

Each $E_{k,i}$ for $i \leq k-1$ is non-zero and irreducible and so $\langle \pi_i, \pi_i \rangle = 1$. This means that the sum of all $\left\langle \overline{\pi}_j, \overline{\pi}_j \right\rangle$ must equal 1 and so at most one of these is non-zero and must be irreducible. Hence without loss of generality $E_{k,k} = \overline{E}_1$.

Note that we only need to do this for $k \leq \lfloor \frac{n}{2} \rfloor$, if $\ell > \lfloor \frac{n}{2} \rfloor$ then $E_{\ell,i} \cong E_{k,i}$ for some $k \leq \lfloor \frac{n}{2} \rfloor$. □

2.2.1 Polytopes spanning $E_{k,i}$

Now that we have split M_k^n into a direct sum of pairwise orthogonal eigenspaces, the next thing to do is to give M_k^n a spanning set of eigenvectors. To this end, we define a *polytope*. This is a new concept to this thesis.

Definition 2.2.7. A *polytope of type* (k, i) is an element of M_k^n of the form

$$(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)\ldots(\alpha_i - \beta_i)(\gamma_1 + \ldots + \gamma_\ell)$$

where all α_r, β_s are pairwise-distinct elements of V and the γ_j's are all the subsets of $V \backslash \{\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i\}$ of size $k - i$.

For brevity, we write this polytope as

$$[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i},$$

and say that $(\alpha_1 - \beta_1)\ldots(\alpha_i - \beta_i)$ is the *head* of the polytope, and that $(\gamma_1 + \ldots + \gamma_s)$ is the *tail*.

For example, if we let $k = i = 1$ then we get polytopes of the form $[\alpha; \beta]_0 = (\alpha - \beta)$. If $k = 2$ and $i = 1$ then $[\alpha, \beta]_1 = (\alpha - \beta)(\gamma_1 + \ldots + \gamma_{n-2})$ where the γ_j are all elements of $V \backslash \{\alpha, \beta\}$. Note that if $i = 0$ the $(k, 0)$ polytope is solely tail and is written for some k as $[-,-]_k$. If $i = k$, the polytope of type (k, k) is head only.

Now, let $i = k' = k$ and consider the head only polytope

$$p = [\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k] = (\alpha_1 - \beta_1)\ldots(\alpha_k - \beta_k).$$

We claim that $p \in E_{k,k}$. For this to be true, we need that $p \in \ker(\delta_k)$. To show this, recall Lemma 2.1.3. Since $(\alpha_i - \beta_i) \Cap (\alpha_j - \beta_j) = 0$ for all $i, j \leq k$ and $\delta(\alpha - \beta) = 0$, we have $\delta(p) = 0$.

Now fix such a polytope p and consider the space $S = \text{span}\{p^g \ : \ g \in \text{Sym}(n)\}$. This is a subspace of $E_{k,k}$ and by construction it is G-invariant. Since $E_{k,k}$ is irreducible $S = E_{k,k}$. This gives us a spanning set for the eigenspaces $E_{i,i}$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The following lemma will help generalise this and give us a spanning set for all eigenspaces $E_{k,i}$.

Lemma 2.2.8. *Fix the polytope* $p = [\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i} \in M_k^n$. *Then*

$$\delta(p) = (n - k - i + 1)[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i-1}.$$

Proof. Recall that by construction the head and tail of the polytope p, written p_h and p_t respectively, are such that $p_h \Cap p_t = 0$. Hence we may use Lemma 2.1.3 again to give

$$\delta(p) = \delta(p_h \cdot p_t) = \delta(p_h) \cdot p_t + p_h \cdot \delta(p_t).$$

As we have just seen, $\delta(p_h) = 0$ and so

$$\delta(p) = p_h \cdot \delta(p_t).$$

Now, the tail of p is the sum of all $(k-i)$-sets of the $(n-2i)$-set $V \setminus \{\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i\}$. Hence $\delta(p_t)$ is the sum of all $(k-i-1)$-sets of the $(n-2i)$-set $V \setminus \{\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i\}$ with coefficient $(n - k - i + 1)$, completing the proof. □

We make a small note here, we refer to the map

$$[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i} \mapsto [\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i-1}$$

as *tail-cutting*. By the above lemma, this is just some multiple of δ. We do not make much use of it in this thesis, but in does come up again briefly in Chapter 5. We now give a vital theorem, giving us a basis for the eigenspaces $E_{k,i}$.

Theorem 2.2.9. *The eigenspaces $E_{k,i}$ are spanned by polytopes of the form*

$$[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i}.$$

Proof. Let $p \in M_k^n$ be such a polytope. Using Lemma 2.2.8 we can apply δ to p and see that $\delta^{k-i}(p) = c(\alpha_1 - \beta_1) \ldots (\alpha_i - \beta_i) \in E_{i,i}$ for some coefficient $c \in \mathbb{F}$. Since δ^{k-i} is an isomorphism between $E_{k,i}$ and $E_{i,i}$, it must be that $p \in E_{k,i}$. Since the set of all head polytopes $[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_0$ span $E_{i,i}$, the preimages under δ^{k-i} of these head polytopes must span $E_{k,i}$. These preimages are the polytopes $[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i}$ as required. □

Since the $E_{k,i}$ are irreducible, by Theorem A.1 these eigenspaces are Specht modules. We give a brief introduction to Specht modules in Appendix A. These modules have a standard basis in terms of polytabloids and we may find a corresponding basis in terms of polytopes. However, this will not been of use in this thesis, and so we consign the discussion to Appendix A.

2.3 Projecting onto $E_{k,i}$

Since M_k^n splits into $k' + 1$ orthogonal eigenspaces of ν_k^+,

$$M_k^n = E_{k,0} \oplus E_{k,1} \oplus \ldots \oplus E_{k,k'},$$

we have that every $f \in M_k^n$ also has a unique decomposition of the form

$$f = f_{k,0} + \ldots + f_{k,k'} \tag{2.7}$$

where $f_{k,i} \in E_{k,i}$. We call this decomposition the *spectral decomposition* of f and call $f_{k,i}$ the i^{th} *spectral component of* f. In order to compute these components we define the projection maps

$$\begin{aligned}\pi_{k,i}\colon M_k^n &\to M_k^n\\ f &\mapsto f_{k,i}.\end{aligned}$$

In particular, this means that $\pi_{k,i}(M_k^n) = E_{k,i}$. These projection maps will be essential in the study of M_k^n. By definition $\pi_{k,i}(h) = h$ if and only if $h \in E_{k,i}$. This means that these maps are idempotent since if $f \in M_k^n$, then

$$\pi_{k,i} \circ \pi_{k,i}(f) = \pi_{k,i}(f_{k,i}) = f_{k,i}.$$

We will now give a few elementary properties of the projection maps. The first is that

$$\pi_{k,i} \circ \pi_{k,j} = \begin{cases} \pi_{k,i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This is true for $i = j$ since the maps are idempotent, and also true for $i \neq j$ since $E_{k,i} \cap E_{k,j} = \emptyset$. The next thing to see is that $\pi_{k,0} + \ldots + \pi_{k,k'} = I$ on M_k^n since if $f \in M_k^n$ we have

$$\left(\pi_{k,0} + \ldots + \pi_{k,k'}\right)(f) = \pi_{k,0}(f) + \ldots + \pi_{k,k'}(f) = f_{k,0} + \ldots + f_{k,k'} = f.$$

Lemma 2.3.1. *We may write ν^+ as a sum of the projection maps π. In particular,*

$$\nu_k^+ = \lambda_{k,0}\pi_{k,0} + \lambda_{k,1}\pi_{k,1} + \cdots + \lambda_{k,k'}\pi_{k,k'}.$$

Proof. Take $f = f_{k,0} + \ldots + f_{k,k'}$. Then

$$\begin{aligned}\nu_k^+(f) &= \nu_k^+ f_{k,0} + \ldots + \nu_k^+ f_{k,k'}\\ &= \lambda_{k,0} f_{k,0} + \ldots + \lambda_{k,k'} f_{k,k'}\\ &= \lambda_{k,0}\pi_{k,0} f + \ldots + \lambda_{k,k'}\pi_{k,k'} f\\ &= \left(\lambda_{k,0}\pi_{k,0} + \ldots + \lambda_{k,k'}\pi_{k,k'}\right) f\end{aligned}$$

as required. □

It is a well-known fact from linear algebra that the projection map $\pi_{k,i}$ takes the form of a polynomial in ν_k^+. To see this explicitly, define the polynomial

$$\mu_{k,i}(x) = (x-\lambda_{k,0})(x-\lambda_{k,1})\dots(x-\lambda_{k,i-1})(x-\lambda_{k,i+1})\dots(x-\lambda_{k,k'}) \qquad (2.8)$$

in $\mathbb{Z}[x]$. Using this definition we can give the following lemma, essential to the study of the eigenspaces $E_{k,i}$. In particular, it tells us that the projections $\pi_{k,i}$ commute with the G-action.

Lemma 2.3.2. *We have*

$$\pi_{k,i} = \frac{1}{\mu_{k,i}(\lambda_{k,i})}\mu_{k,i}(\nu_k^+)$$

for all $0 \le i \le k'$. In particular, π is a polynomial of a linear map and so π is linear itself. Furthermore, the G-action commutes with $\pi_{k,i}$.

Proof. Since both ν_k^+ and multiplication by a scalar are linear, it suffices to show that $\frac{1}{\mu_{k,i}(\lambda_{k,i})}\mu_{k,i}(\nu_k^+)$ has kernel $\bigoplus_{j\neq i} E_{k,j}$ and acts as the identity on $E_{k,i}$. Also note that if $f \in E_{k,\ell}$, then $(\nu_k^+ - \lambda_{k,j})f$ is just a scalar multiple of f for $l \neq j$. Now, assume $f \in E_{k,\ell}$ for some $\ell \neq i$. Then

$$\mu_{k,i}(\nu_k^+)(f) = \prod_{j\neq i}(\lambda_{k,\ell} - \lambda_{k,j}) \cdot f.$$

In particular, since $i \neq l$, there is some j in the product with $j = \ell$ and so one of the product terms is $(\lambda_{k,\ell} - \lambda_{k,\ell}) = 0$. Hence we have $\pi_{k,i}(f) = 0$ for $f \in E_{k,l}$, which gives us that the kernel is $\bigoplus_{j\neq i} E_{k,j}$.

Now assume $f \in E_{k,i}$. Then

$$\mu_{k,i}(\nu_k^+)(f) = \prod_{j\neq i}(\lambda_{k,i} - \lambda_{k,j}) \cdot f = \mu_{k,i}(\lambda_{k,i}) \cdot f$$

and so $\pi_{k,i}(f) = f$, proving that $\pi_{k,i}$ acts as the identity on $E_{k,i}$.

Since the G-action commutes with ν_k^+ by Lemma 2.1.8, it also commutes with polynomials in ν_k^+ and so it commutes with $\pi_{k,i}$ as claimed. $\square$

Now we know what the projection maps look like, we may ask how they interact with our ε and δ-functions. This brings us to the following lemma.

Lemma 2.3.3. *We have*

$$\delta_k \pi_{k,i} = \pi_{k-1,i}\delta_k \qquad \textit{and} \qquad \varepsilon_k \pi_{k,i} = \pi_{k+1,i}\varepsilon_k$$

for all $i \leq k'$.

Proof. Let $f \in M_k^n$. Then note that $\delta_k \pi_{k,i}(f) = \delta_k(f_{k,i})$. However since we have $\delta_k(f_{k,i}) \in E_{k-1,i}$ we must have that $\pi_{k-1,i}$ acts as the identity on it. Hence we know that $\delta_k \pi_{k,i}(f) = \delta_k(f_{k,i}) = \pi_{k-1,i}\delta_k(f_{k,i})$ completing the first claim. The proof of the second is identical. $\square$

Lemma 2.3.4. *For $f_1, f_2 \in M_k^n$ we have*

$$\left\langle \pi_{k,i}(f_1), \pi_{k,i}(f_2) \right\rangle = \left\langle \pi_{k,i}(f_1), f_2 \right\rangle = \left\langle f_1, \pi_{k,i}(f_2) \right\rangle.$$

Proof. By Lemma 2.3.2, $\pi_{k,i}$ is a polynomial in ν^+ and so $\pi_{k,i}$ is symmetric. The fact that $\pi_{k,i}$ is idempotent completes the proof. $\square$

We can use these projection maps to study the length of vectors in M_k^n. The length $||f||^2$ of f is determined from $||f||^2 = \langle f, f \rangle$, and we compute $||f||^2$ in terms of its eigenspace components;

$$\begin{aligned}
\langle f, f \rangle &= \left\langle f_{k,0} + \cdots + f_{k,k'}, f_{k,0} + \cdots + f_{k,k'} \right\rangle \\
&= \sum_{i=0}^{k'} \sum_{j=0}^{k'} \langle f_{k,i}, f_{k,j} \rangle.
\end{aligned}$$

Recalling that $E_{k,i}$ is orthogonal to $E_{k,j}$ for all $i \neq j$ we see that the only non-zero terms are those of the form $\langle f_{k,i}, f_{k,i} \rangle = \left|\left|f_{k,i}\right|\right|^2$. This gives us that

$$||f||^2 = \left|\left|f_{k,0}\right|\right|^2 + \left|\left|f_{k,1}\right|\right|^2 + \ldots + \left|\left|f_{k,k'}\right|\right|^2. \tag{2.9}$$

Since δ and π "commute" (in the sense of Lemma 2.3.3) we can get some nice bounds for the length of f with respect to $\delta(f)$.

Lemma 2.3.5. *Let $f \in M_k^n$. Then we have*

1) $\left|\left|\delta_k f_{k,i}\right|\right|^2 = \lambda_{k,i}^- \left|\left|f_{k,i}\right|\right|^2$,

2) $\left|\left|\varepsilon_k f_{k,i}\right|\right|^2 = \lambda_{k,i} \left|\left|f_{k,i}\right|\right|^2$,

3) $\left|\left|\nu_k^+ f_{k,i}\right|\right|^2 = \lambda_{k,i}^2 \left|\left|f_{k,i}\right|\right|^2$.

Proof. Note first that $\left|\left|\delta_k f_{k,i}\right|\right|^2 = \langle \delta_k f_{k,i}, \delta_k f_{k,i} \rangle$. Since ε_{k-1} is the adjoint of δ_k, the right hand side becomes $\langle f_{k,i}, \nu_k^- f_{k,i} \rangle$. Since $f_{k,i} \in E_{k,i}$, it is an eigenvector of ν_k^+ and by extension an eigenvector of ν_k^-, by Lemma 2.1.4, with eigenvalue $\lambda_{k,i}^-$. This completes the proof of part 1) and the proof of 2) is identical. For part 3), since $\nu^+ = \delta_{k+1}\varepsilon_k$ we just apply the first and second parts, remembering that $\lambda_{k,i} = \lambda_{k+1,i}^-$ by Lemma 2.2.2. □

2.4 Examples

We finish up this chapter with a collection of small examples to illustrate some of the techniques used for working in M_k^n.

Example 2.4.1 ($\alpha \in M_1^n$).

Let V be a set of size n with $\alpha \in V$ and consider $\alpha \in M_1^n$. Since $M_1^n = E_{1,0} \oplus E_{1,1}$, we know that

$$\alpha = \pi_{1,0}(\alpha) + \pi_{1,1}(\alpha).$$

For brevity, we will write $\pi_{1,i}(\alpha) = \alpha_{1,i}$ for $i = 0, 1$. Recall that $E_{1,0}$ is a one-dimensional G-module spanned by the polytope $p = \sum_{\gamma \in L_1^n} \gamma$, which is all tail in the sense of Definition 2.2.7. Similarly, $E_{1,1}$ is $(n-1)$-dimensional and is spanned by the polytopes $(\beta - \gamma)$ for $\beta, \gamma \in L_1^n$. Note that these polytopes have no tail. With this knowledge we can see that

$$\alpha = \frac{1}{n} \sum_{\gamma \in V} \gamma + \sum_{\beta \in V \backslash \alpha} \frac{1}{n}(\alpha - \beta). \tag{2.10}$$

In fact, the polytopes $(\alpha - \beta)$ for $\beta \in V \backslash \alpha$ comprise a basis of $E_{1,1}$, which we prove in Appendix A.

From these, we can work out the length of α and both of its components. Firstly, $\langle \alpha, \alpha \rangle = 1$ and so $||\alpha||^2 = 1$. Since $\alpha_{1,0} = \frac{1}{n} \sum_{\gamma \in V} \gamma$ we have

$$\left|\left|\alpha_{1,0}\right|\right|^2 = \left\langle \frac{1}{n} \sum_{\gamma \in V} \gamma, \frac{1}{n} \sum_{\gamma \in V} \gamma \right\rangle = \frac{1}{n^2} \left\langle \sum_{\gamma \in V} \gamma, \sum_{\gamma \in V} \gamma \right\rangle = \frac{1}{n}.$$

By Equation 2.9 we know that $||\alpha||^2 = \left|\left|\alpha_{1,0}\right|\right|^2 + \left|\left|\alpha_{1,1}\right|\right|^2$ and so $\left|\left|\alpha_{1,1}\right|\right|^2 = \frac{n-1}{n}$.

Example 2.4.2 ($\nu^+(\alpha) \in M_1^n$)**.**

Now we shall compute $\nu^+(\alpha)$. From the definition of ν^+, we obtain

$$\nu^+(\alpha) = \delta(\sum_{\beta \in V \backslash \alpha} \alpha\beta) = (n-1)\alpha + \sum_{\beta \in V \backslash \alpha} \beta.$$

Using Equation 2.10 we can write this as

$$\begin{aligned}
&(n-1)\left(\frac{1}{n}\sum_{\gamma \in V}\gamma + \sum_{\gamma \in V \backslash \alpha} \frac{1}{n}(\alpha - \gamma)\right) + \sum_{\beta \in V \backslash \alpha}\left(\frac{1}{n}\sum_{\gamma \in V}\gamma + \sum_{\gamma \in V \backslash \beta}\frac{1}{n}(\beta - \gamma)\right) \\
&= \frac{2(n-1)}{n}\sum_{\gamma \in V}\gamma + \frac{n-1}{n}\sum_{\gamma \in V \backslash \alpha}(\alpha - \gamma) + \sum_{\beta \in V \backslash \alpha}\frac{1}{n}(\beta - \alpha) \\
&= \frac{2(n-1)}{n}\sum_{\gamma \in V}\gamma + \frac{n-2}{n}\sum_{\gamma \in V \backslash \alpha}(\alpha - \gamma).
\end{aligned}$$

As before, we know that $\left|\left|\nu^+(\alpha)\right|\right|^2 = \left|\left|\nu^+(\alpha)_{1,0}\right|\right|^2 + \left|\left|\nu^+(\alpha)_{1,1}\right|\right|^2$. To calculate

$\left|\left|v^+(\alpha)\right|\right|^2$ we write

$$\left|\left|v^+(\alpha)\right|\right|^2 = \left\langle (n-1)\alpha + \sum_{\beta\in V\setminus\alpha} \beta, (n-1)\alpha + \sum_{\beta\in V\setminus\alpha} \beta \right\rangle = (n-1)^2 + n - 1 = n(n-1).$$

Next we note that

$$\left|\left|v^+(\alpha)_{1,0}\right|\right|^2 = \left\langle \frac{2(n-1)}{n}\sum_{\gamma\in V}\gamma, \frac{2(n-1)}{n}\sum_{\gamma\in V}\gamma \right\rangle = n\frac{4(n-1)^2}{n^2} = \frac{4(n-1)^2}{n}.$$

Finally we can calculate

$$\left|\left|v^+(\alpha)_{1,1}\right|\right|^2 = \left|\left|v^+(\alpha)\right|\right|^2 - \left|\left|v^+(\alpha)_{1,0}\right|\right|^2 = n(n-1) - \frac{4(n-1)^2}{n} = \frac{(n-1)(n-2)^2}{n}.$$

However, there is another, easier, way to calculate these lengths. By Lemma 2.3.5 we know that

$$\left|\left|v^+(\alpha)_{1,i}\right|\right|^2 = \lambda_{1,i}^2 \left|\left|\alpha_{1,i}\right|\right|^2. \tag{2.11}$$

Since $\lambda_{k,i} = (n-k-i)(k-i+1)$ we know that $\lambda_{1,0} = 2(n-1)$ and $\lambda_{1,1} = n-2$. Putting these values into Equation 2.11 we obtain the answers we calculated before, as expected.

Example 2.4.3 ($\alpha\beta \in M_2^n$).

Now consider the set $\alpha\beta \in M_2^n$ with $\alpha, \beta \in V$. Recall that $\alpha\beta$ is shorthand for the 2-set $\{\alpha,\beta\} = \{\alpha\}\cdot\{\beta\}$ under union multiplication. Let us calculate $\pi_{2,0}(\alpha\beta)$. We already know that $E_{2,0}$ is one-dimensional and is spanned by the sum of all 2-sets and so we should have that

$$\pi_{2,0}(\alpha\beta) = c\sum_{x\in L_2^n} x$$

for some $c \in \mathbb{Q}$. Applying v_2^+ to $\alpha\beta$ gives us

$$v_2^+(\alpha\beta) = \delta_3\big(\sum_{\substack{\gamma\in V\\ \gamma\neq\alpha,\beta}} \alpha\beta\gamma\big) = (n-2)\alpha\beta + \sum_{\substack{\eta\in V\\ \eta\neq\alpha,\beta}} \alpha\eta + \sum_{\substack{\tau\in V\\ \tau\neq\alpha,\beta}} \beta\tau.$$

From Lemma 2.3.2 we have that

$$\pi_{2,0} = \frac{1}{\mu_{2,0}(\lambda_{2,0})}\mu_{2,0}(\nu_2^+).$$

Inputting values into Theorem 2.2.3, we know that $\lambda_{2,0} = 3(n-2)$, $\lambda_{2,1} = 2(n-3)$, and $\lambda_{2,2} = n-4$. From Equation 2.8 we know that $\mu_{2,0} = (x-\lambda_{2,1})(x-\lambda_{2,2})$ and so we have $\mu_{2,0}(\lambda_{2,0}) = 2n(n-1)$. This gives

$$\begin{aligned}\pi_{2,0}(\alpha\beta) &= \frac{1}{2n(n-1)}(\nu_2^+ - \lambda_{2,1})(\nu_2^+ - \lambda_{2,2})(\alpha\beta)\\ &= \frac{1}{2n(n-1)}(\nu_2^+ - \lambda_{2,1})\left((n-2-\lambda_{2,2})\alpha\beta + \sum_{\substack{\eta\in V\\ \eta\neq\alpha,\beta}}\alpha\eta + \sum_{\substack{\tau\in V\\ \tau\neq\alpha,\beta}}\beta\tau\right).\end{aligned}$$

The notation gets unwieldy from now on, and so we will continue by case analysis to find the coefficients of each 2-set. We denote the coefficient of each 2-set x in $\pi_{2,0}(\alpha\beta)$ by c_x.

Firstly, we look at the coefficient of $\alpha\beta$. From the first term, $(\nu_2^+ - \lambda_{2,1})(n-2-\lambda_{2,2})\alpha\beta$, we get that $c_{\alpha\beta} = (n-2-\lambda_{2,2})(n-2-\lambda_{2,1})$. From the second term, $(\nu_2^+ - \lambda_{2,1})\left(\sum \alpha\eta\right)$, we obtain $n-2$ lots of $\alpha\beta$, one from each $\nu_2^+(\alpha\eta)$. We get the same amount from the third term $(\nu_2^+ - \lambda_{2,1})\left(\sum \beta\tau\right)$, one from each $\nu_2^+(\beta\tau)$. This gives us that

$$c_{\alpha\beta} = (n-2-\lambda_{2,2})(n-2-\lambda_{2,1}) + 2(n-2) = 4.$$

Next, we consider $\alpha\eta$ for $\eta \neq \alpha, \beta$. This has coefficient of $(n-2-\lambda_{2,2})$ from the first term. From the second term we get $(n-2)-\lambda_{2,1}+(n-3)$ and from the third term we get 1. So

$$c_{\alpha\eta} = n-2-\lambda_{2,2} + n-2-\lambda_{2,1} + n-3+1 = 4.$$

By symmetry, this is also the coefficient of all $\beta\tau$ terms for $\tau \neq \alpha, \beta$.

For the final case, consider the terms of the form $\eta\tau$ for $\eta, \tau \neq \alpha, \beta$. These do not appear in the first term and appear twice in the second and third terms. These come from v_2^+ of $\alpha\eta$, $\alpha\tau$, $\beta\eta$, and $\beta\tau$. This gives

$$c_{\eta\tau} = 4.$$

This gives is that very 2-set has the same coefficient in the projection and so

$$\pi_{2,0}(\alpha\beta) = \frac{4}{2n(n-1)} \sum_{\eta,\tau \in V} \eta\tau = \binom{n}{2}^{-1} \sum_{\eta,\tau \in V} \eta\tau. \tag{2.12}$$

Note that this means that

$$\begin{aligned} ||\alpha\beta_{2,0}||^2 &= \left\langle \binom{n}{2}^{-1} \sum_{\eta,\tau \in V} \eta\tau, \binom{n}{2}^{-1} \sum_{\eta,\tau \in V} \eta\tau \right\rangle \\ &= \binom{n}{2}^{-2} \left\langle \sum_{\eta,\tau \in V} \eta\tau, \sum_{\eta,\tau \in V} \eta\tau \right\rangle = \binom{n}{2}^{-1}. \end{aligned}$$

Doing this again for $\pi_{2,1}(\alpha\beta)$ and $\pi_{2,2}(\alpha\beta)$ is possible, but time consuming. However, we can take a short cut. As before, we write $\pi_{2,i}(\alpha\beta) = \alpha\beta_{2,i}$. Since

$$\delta(\alpha\beta) = \alpha + \beta,$$

then by Lemma 2.3.5 we know that

$$\lambda_{2,1}^- \left|\left|\alpha\beta_{2,1}\right|\right|^2 = \left|\left|\delta(\alpha\beta_{2,1})\right|\right|^2 = \left|\left|(\alpha+\beta)_{1,1}\right|\right|^2, \tag{2.13}$$

with the last equality holding by Lemma 2.3.3. Also,

$$\left|\left|(\alpha+\beta)_{1,0}\right|\right|^2 = \left|\left|\delta(\alpha\beta)_{2,0}\right|\right|^2 = \lambda_{2,0}^- \left|\left|\alpha\beta_{2,0}\right|\right|^2 = \lambda_{2,0}^- \binom{n}{2}^{-1}.$$

By Equation 2.5, we know that $\lambda_{2,0}^- = 2(n-1)$. This means that

$$\left|\left|(\alpha+\beta)_{1,0}\right|\right|^2 = 2(n-1)\binom{n}{2}^{-1} = \frac{4}{n},$$

and so

$$\left|\left|(\alpha+\beta)_{1,1}\right|\right|^2 = ||\alpha+\beta||^2 - \left|\left|(\alpha+\beta)_{1,0}\right|\right|^2 = \frac{2(n-2)}{n}.$$

Using this, we may calculate the value of $\left|\left|\alpha\beta_{2,1}\right|\right|^2$ using Equation 2.13. Since $\lambda^-_{2,1} = n-2$ (again using Equation 2.5) we have that

$$\left|\left|(\alpha\beta)_{2,1}\right|\right|^2 = \frac{1}{\lambda^-_{2,1}} \left|\left|(\alpha+\beta)_{1,1}\right|\right|^2 = \frac{2}{n}.$$

Lastly, we can calculate $\left|\left|\alpha\beta_{2,2}\right|\right|^2$ in the following way.

$$\left|\left|\alpha\beta_{2,2}\right|\right|^2 = ||\alpha\beta||^2 - \left|\left|\alpha\beta_{2,0}\right|\right|^2 - \left|\left|\alpha\beta_{2,1}\right|\right|^2 = 1 - \binom{n}{2}^{-1} - \frac{2}{n} = \frac{n-3}{n-1}.$$

We discuss Example 2.4.1 and Example 2.4.3 in more detail in Subsection 3.2.1.

3

On the Shape of a k-family

As before, let V be a set of size n and fix $k \leq n$ with $k' = \min\{k, n-k\}$. We fix these throughout. Let S be a family of k-element subsets of V. For short, we will call S a *k-family*. Many classical combinatorial structures can be thought of as k-families. For example, we consider simple graphs. A finite simple graph Γ is an ordered pair (V, S) comprising of a set V of vertices and a set S of edges. These edges are 2-sets of V. In this way we see the edges of Γ as a 2-family. Since V is fixed of size n, then Γ is uniquely determined by its edge set S and so we may associated a simple graph with the 2-family S of edges.

Next, let G be a permutation group acting on V. The orbits of this action partition V into sets of points, and so each of these orbits is a 1-family. More generally, this action induces an action on k-sets, and so each of these orbits is a k-family.

For another example, consider t-designs. A t-(n,k,λ) *design* is a collection of k-subsets (also known as *blocks*) of a base set V of size n such that every t-set is contained in exactly λ many k-sets. A t-design therefore is a k-family.

Lastly, there are numerous results regarding the intersection properties of subsets of L^n. In general, given some $T \subseteq L^n$ with certain intersection properties, what are the upper and lower bounds on $|T|$? If T consists only of k-sets, then T is a k-family. One famous example is the Erdős-Ko-Rado Theorem [EKR61]. This states that a family $\mathscr{F} \subseteq L^n_k$ of pairwise intersecting k-sets must have size less than or equal to $\binom{n-1}{k-1}$. This is a special case of the Ray-Chaudhuri-Wilson Theorem [RCW75]. This states that for a family of k-sets $\mathscr{F} \subseteq L_k^n$ and a set $W \subseteq \{0, 1, \ldots, k\}$ of size w, if $|A \cap B| \in W$ for all $A, B \in \mathscr{F}$ then $|\mathscr{F}| \leq \binom{n}{w}$. In fact, many important theorems in combinatorics could be phrased in terms of k-families.

3.1 The spectral shape

Recall (from page 18) that we may consider a subset x of V as an element of M^n via the mapping $x \mapsto 1 \cdot x$. We generalise this idea by defining a map that takes $S \subseteq L^n$ to a corresponding vector $[S] \in M^n$ by

$$S \mapsto [S] = \sum_{x \in S} x \in M^n,$$

i.e. taking the sum of all elements in S. We call $[S]$ the *characteristic vector* of S. If we need to emphasise the ground set V, we may write $[S] = [S]^V$ or $[S]^n$ if we wish to emphasise the size of V.

Remark. This notion generalises naturally to multisets. If S is a multiset with $x \in S$ appearing λ_x times, then we can define $[S] = \sum_{x \in S} \lambda_x x$. For our purposes however, taking $S \subseteq L^n$ will suffice. This means that if S is a set then

$$[S] = \sum_{x \in L^n} \lambda_x x \qquad \text{where} \qquad \lambda_x = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

This is useful as it makes it clear that for a k-family S, the coefficient of all k-sets in $[S]$ is either 0 or 1, a fact that we will use later.

So, we have some $S \subseteq L_k^n$, a k-family. On the other hand, we have its embedding $[S]$ into M_k^n, which has a sophisticated internal structure given by the eigenspace decomposition. The natural question then arises:

Question 3.1.1. *How are the combinatorial attributes of a k-family S reflected in the algebraic properties of $[S]$?*

We may think of this as a sort of "representation theory" of set systems in terms of the linear algebra M^n. Answering this question, or more precisely the related Question 3.1.3, will be the primary motivation of this chapter and Chapter 4.

To begin, we need to be more precise in what we mean by "algebraic properties". Recall that in Equation 2.6 we showed that M_k^n is the orthogonal sum

$$M_k^n = E_{k,0} \oplus E_{k,1} \oplus \ldots \oplus E_{k,k'}$$

of eigenspaces of the map ν^+. Hence, any $f \in M_k^n$ has an spectral decomposition of the form

$$f = f_{k,0} + f_{k,1} + \ldots + f_{k,k'}$$

with $f_{k,i} \in E_{k,i}$. Therefore if S is a k-family, then $[S]$ also has such a decomposition

$$[S] = [S]_{k,0} + [S]_{k,1} + \ldots + [S]_{k,k'},$$

where $[S]_{k,i} \in E_{k,i}$. To study this decomposition we introduce the *shape* of a k-family, a way to measure the size of each $[S]_{k,i}$.

Definition 3.1.2. Let $f \in M_k^n$ with spectral decomposition $f = f_{k,0} + \ldots + f_{k,k'}$. We call the $(k'+1)$-tuple

$$\mathrm{sh}(f) = \left(||f_{k,0}||^2, ||f_{k,1}||^2, \ldots, ||f_{k,k'}||^2\right)$$

the *shape* of f. We denote the i^{th} component by $\mathrm{sh}_i(f)$. In particular, if $f = [S]$ then for brevity we write

$$\mathrm{sh}(S) = \mathrm{sh}([S])$$

and $\mathrm{sh}_i([S]) = \mathrm{sh}_i(S)$. We call $\mathrm{sh}(S)$ the shape of S.

Note that $\mathrm{sh}_i(S) = \left\langle \pi_{k,i}([S]), \pi_{k,i}([S])\right\rangle$. In particular, since $\pi_{k,i}$ is symmetric and idempotent we have

$$\mathrm{sh}_i(S) = \left\langle [S], \pi_{k,i}([S]) \right\rangle = \left\langle \pi_{k,i}([S]), [S] \right\rangle.$$

In future, we will write $\pi_{k,i}[S] = \pi_{k,i}([S])$ to make the notation cleaner. We will also do the same thing with the ε, δ, and ν maps.

Now that we have this definition, we may rephrase Question 3.1.1 in a somewhat more precise form.

Question 3.1.3. *Given a k-family S, what combinatorial information about S can we recover from the shape* $\mathrm{sh}(S)$*?*

3.2 Elementary properties of $\mathrm{sh}(S)$

In this section, we give some elementary properties of the shape $\mathrm{sh}(S)$ of a k-family S. Throughout, we fix S to be a k-family over an n-set V.

Lemma 3.2.1. *We have that* $|S| = ||[S]||^2 = \sum_{i=0}^{k'} \mathrm{sh}_i(S)$.

Proof. To show the first equality, note that

$$||[S]||^2 = \langle [S], [S] \rangle = \sum_{x \in S} \langle x, x \rangle = |S|,$$

as required. To see the second, we have that

$$||[S]||^2 = \langle [S], [S] \rangle = \left\langle \sum_{i=0}^{k'} [S]_{k,i}, \sum_{i=0}^{k'} [S]_{k,i} \right\rangle = \sum_{i,j \le k'} \left\langle [S]_{k,i}, [S]_{k,j} \right\rangle.$$

However, $\left\langle [S]_{k,i}, [S]_{k,j} \right\rangle = 0$ if $i \neq j$ since $E_{k,i}$ and $E_{k,j}$ are orthogonal. Hence

$$\sum_{i,j \le k'} \left\langle [S]_{k,i}, [S]_{k,j} \right\rangle = \sum_{i=0}^{k'} \left\langle [S]_{k,i}, [S]_{k,i} \right\rangle = \sum_{i=0}^{k'} \mathrm{sh}_i(S)$$

as required. $\square$

Next, we show that the shape is G-invariant for any $G \leq \mathrm{Sym}(V)$.

Lemma 3.2.2. *Let $f \in M_k^n$ and $g \in G \leq \mathrm{Sym}(V)$. Then*

$$\mathrm{sh}_i(f) = \mathrm{sh}_i(f^g)$$

for all $0 \leq i \leq k'$. In particular, $\mathrm{sh}(f) = \mathrm{sh}(f^g)$.

Proof. We have already shown that the G-action on M_k^n commutes with the projection maps $\pi_{k,i}$ in Lemma 2.3.2. Since $\langle f_1, f_2 \rangle = \langle f_1^g, f_2^g \rangle$ for any $f_1, f_2 \in M_k^n$ by Lemma 2.1.5, this completes the proof. □

This gives us an important corollary.

Corollary 3.2.3. *If x and y are k-sets, then $\mathrm{sh}(x) = \mathrm{sh}(y)$ and $\mathrm{sh}_i(x) \neq 0$ for all $0 \leq i \leq k'$.*

Proof. The first claim follows immediately from Lemma 3.2.2, since if x and y are k-sets then they belong to the same orbit of $\mathrm{Sym}(V)$. The second claim follows from the fact that the k-sets are a a basis for M_k^n and so if there was an $i \leq k'$ such that $\mathrm{sh}_i(x) = 0$ then $E_{k,i} = 0$ and this cannot happen by dimension arguments. □

Next, we turn our attention to the shape of some simple k-families. We give an explicit formula for $\mathrm{sh}_0(S)$ using the length of S.

Lemma 3.2.4. *If S is a k-family, then*

$$[S]_{k,0} = |S| \binom{n}{k}^{-1} \sum_{x \in L_k^n} x.$$

In particular, this means that $\mathrm{sh}_0(S) = |S|^2 \binom{n}{k}^{-1}$.

Proof. Theorem 2.2.9 says that $E_{k,0} = \left\langle \sum_{x \in L_k^n} x \right\rangle = \langle \mathbf{1}_k \rangle$, the span of $[L_k^n]$. Hence we have that $[S]_{k,0} = q\mathbf{1}_k$ for some $q \in \mathbb{Q}$. Also

$$|S| = \langle \mathbf{1}_k, [S] \rangle = \langle \mathbf{1}_k, [S]_{k,0} \rangle = \langle \mathbf{1}_k, q\mathbf{1}_k \rangle = q\binom{n}{k},$$

hence $q = |S|\binom{n}{k}^{-1}$, proving the first part. The second part follows immediately, since

$$\mathrm{sh}_0(S) = \langle [S]_{k,0}, [S]_{k,0} \rangle = q^2 \langle \mathbf{1}_k, \mathbf{1}_k \rangle = q^2\binom{n}{k},$$

completing the proof. □

Corollary 3.2.5. *For any non-empty k-family S, the spectral distribution* $\mathrm{sh}(S)$ *is non-zero in the first component, i.e.* $\mathrm{sh}_0(S) \neq 0$. □

Lemma 3.2.6. *Let* $[S]$ *and* $[T]$ *be the characteristic vectors of some k-family S and its complement T. Then* $[S]_{k,0} = \mathbf{1}_k - [T]_{k,0}$ *and* $[S]_{k,i} = -[T]_{k,i}$ *for* $1 \leq i \leq k'$. *In particular,* $\mathrm{sh}_0(S) = \binom{n}{k} - \mathrm{sh}_0(T)$ *and* $\mathrm{sh}_i(S) = \mathrm{sh}_i(T)$ *for all* $1 \leq i \leq k'$.

Proof. Since S and T are disjoint, $[S]+[T] = [S \cup T] = [L_k^n] = \mathbf{1}_k$. This is the sum of all k-sets and so by Theorem 2.2.9 is a polytope in $E_{k,0}$ only. Hence $[S]_{k,0}+[T]_{k,0} = \mathbf{1}_k$ and $[S]_{k,i} + [T]_{k,i} = 0$ for all $1 \leq i \leq k'$. □

The first equality $[S]+[T] = [S \cup T]$ only holds true because S and T are disjoint.

Finally we give a key lemma that we will be making much use of, particularly in Chapter 4.

Lemma 3.2.7. *If* $k \leq \frac{n}{2}$ *then any* $f \in M_k^n$ *has* $\mathrm{sh}_k(f) = 0$ *if and only if*

$$f = \sum_{x \in L_{k-1}^n} c_x \varepsilon(x)$$

for some $c_x \in \mathbb{Q}$.

Proof. We know that $M_{k-1}^n = E_{k-1,0} \oplus \ldots \oplus E_{k-1,k-1} \cong E_{k,0} \oplus \ldots \oplus E_{k,k-1}$ with the isomorphism given by ε. Since L_{k-1}^n is a basis for M_{k-1}^n, then $\{\varepsilon(x) : x \in L_{k-1}^n\}$ is a basis for $E_{k,0} \oplus \ldots \oplus E_{k,k-1}$. Note that this means that these coefficients c_x are unique. $\square$

We can see that this can be generalised: for $0 \leq \ell \leq k$, we have $f_{k,\ell} = f_{k,\ell+1} = \ldots = f_{k,k} = 0$ if and only if

$$f = \sum_{x \in L_\ell^n} c_x \varepsilon^{k-\ell}(x) \tag{3.1}$$

with $c_x \in \mathbb{Q}$. Again, we stress the importance of this lemma, as it will be vital to study vectors $f \in M_k^n$ where $\mathrm{sh}_k(f) = 0$.

3.2.1 Examples

In this section, we give some examples of shapes of some small k-families for $k = 1, 2$.

1. A 1-family: Let $x \in L_1^n$ be a single point. Then from Example 2.4.1 we know that

$$\mathrm{sh}(x) = \left(\frac{1}{n}, \frac{n-1}{n}\right).$$

To fully illustrate this, consider the diagram below.

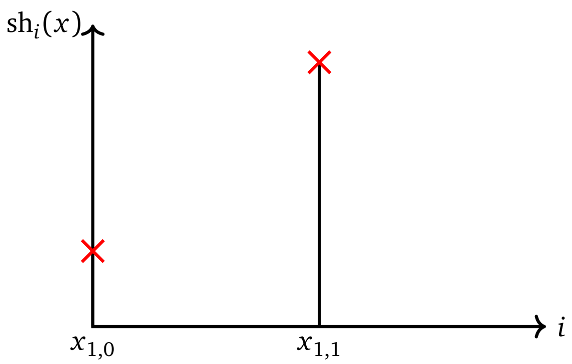


Figure 3.1: Eigenspace decomposition lengths of a 1-set $x \in M_1^n$

In fact, we can say more about the shape of a 1-family.

Proposition 3.2.8. *Let S be a 1-family over an n-set V. Then shape of S is given by*

$$\mathrm{sh}(S) = \left(\frac{|S|^2}{n}, |S| \frac{n - |S|}{n} \right).$$

In particular, $\mathrm{sh}(S) = \mathrm{sh}(T)$ *if and only if* $|S| = |T|$.

Proof. Lemma 3.2.4 gives us $\mathrm{sh}_0(S)$ and by Lemma 3.2.1, we know that

$$\mathrm{sh}_1(S) = |S| - \mathrm{sh}_0(S),$$

completing the proof. □

2. A single 2-set: Let y be a single 2-set over V, where $|V| \geq 4$. The spectral decomposition of y is therefore

$$y = y_{2,0} + y_{2,1} + y_{2,2}.$$

In Example 2.4.3 we calculated the shape of y to be

$$\mathrm{sh}(y) = \left(\binom{n}{2}^{-1}, \frac{2}{n}, \frac{n-3}{n-1} \right). \tag{3.2}$$

As before, we illustrate this with the graph below.

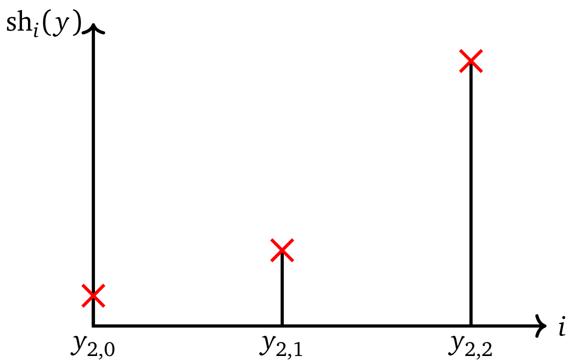


Figure 3.2: Eigenspace decomposition lengths of a single 2-set $y \in M_2^n$

3. The k-family L_k^n: Consider the k-family L_k^n. This has characteristic vector $[L_k^n]$

which we denote by $\mathbf{1}_k$. This is precisely the polytope $[-,-]_k$ and so by Theorem 2.2.9 we have

$$\mathrm{sh}(\mathbf{1}_k) = \left(\binom{n}{k}, 0, 0, \ldots, 0\right). \tag{3.3}$$

4. A cycle graph: Lastly, let $n = 6$ and consider the cycle graph $C_6 \subset L_2^6$. One can calculate that its characteristic vector $[C_6] = 12 + 23 + 34 + 45 + 56 + 16$ has shape

$$\mathrm{sh}(C_6) = \left(\frac{12}{5}, 0, \frac{18}{5}\right). \tag{3.4}$$

We note briefly here that $\mathrm{sh}_1(C_6) = 0$. This is an example of a more general phenomenon that we will discuss in Section 4.2.

The first two examples lead us to the following conjecture.

Conjecture 3.2.9. *Fix n and choose $k \le n$ and $i \le k'$. Let $x \in L_k^n$ be a single k-set. Then*

$$\mathrm{sh}_i(x) = \dim(E_{k,i})\binom{n}{k}^{-1} = \left(\binom{n}{i} - \binom{n}{i-1}\right)\binom{n}{k}^{-1}.$$

In particular, if we denote $x \in L_k^n$ by x^n then we have that the sequence $(\mathrm{sh}_i(x^n))_{i \le k'}$ is strictly increasing and the sequence $(\mathrm{sh}_i(x^n))_{n \ge k}$ is strictly decreasing if $i < k'$ and increasing if $i = k'$.

3.3 The inner distribution of a k-family versus the shape

To motivate this section, we refer back to Question 3.1.3. It asks what combinatorial information about a k-family S we can recover from $\mathrm{sh}(S)$. We have seen in the previous section some examples of shapes, as well as some of the basic properties of shapes, but we have not yet touched upon the vague "combinatorial information". By Lemma 3.2.2 the shape $\mathrm{sh}(S)$ of a k-family S is invariant under the action of $\mathrm{Sym}(n)$, and so we want combinatorial information about S that is also G-invariant. An obvious choice is the intersection properties of S, i.e. the size of the intersections $x \cap y$ for $x, y \in S$. To study these properties, we start this section off with a quick introduction to association schemes. We use [Del73] as a rough guide. For a more complete introduction to association schemes see Bailey's book [Bai04].

Let X be a finite set. We partition $X \times X$ into $k+1$ parts

$$R = R_0 \cup R_1 \cup \cdots \cup R_k$$

such that

1. $R_0 = \{(x,x) \,:\, x \in X\}$.
2. For every $i < k$ there exists some $j < k$ such that $R_j = \{(x,y) \,:\, (y,x) \in R_i\}$.
3. For all $(x,y) \in R_s$, $|\{z \in X \,:\, (x,z) \in R_i,\ (z,y) \in R_j\}| = p^s_{i,j}$. In particular, $p^s_{i,j}$ does not depend on the choice of x and y. We will also assume that $p^s_{i,j} = p^s_{j,i}$.

Note that the superscript s here is not a power. We then define the pair (X,R) to be an *association scheme* and call each R_i a *class* of the association scheme. We call the association scheme *symmetric* if $(x,y) \in R_i$ implies $(y,x) \in R_i$ for every i. If $(x,y) \in R_i$ then we say that x and y are i^{th} *associates*.

We may associate to each R_i its adjacency matrix. This is the matrix $A_i \in GL_{|X|}(\mathbb{C})$ where

$$A_i(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in R_i \\ 0 & \quad \text{otherwise.} \end{cases}$$

Note that $A_0 = I$, the identity matrix, and $\sum A_i = J$ where J is the all-one matrix. We have that the span of all these adjacency matrices form a subalgebra $\mathscr{A}$ of $GL_{|X|}(\mathbb{C})$, as we have the following identity:

$$A_i \cdot A_j = \sum_{s=0}^{k} p^s_{i,j} A_s.$$

All $A \in \mathscr{A}$ are normal matrices; matrices who commute with their conjugate transpose. This algebra $\mathscr{A}$ is known as the *Bose-Mesner algebra* of the association scheme (X,R).

One example of association schemes are the *Hamming schemes*. For some set Ω of size n take $X = \Omega^k$ and make this a metric space by defining the Hamming distance between two k-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ to be

$$d_H(x, y) = \left|\{(x_i, y_i) \in X^2 \;:\; x_i \neq y_i,\; 1 \leq i \leq k\}\right|.$$

We then partition X^2 into $k+1$ subsets $R_i = \{(x, y) \in X^2 : d_H(x, y) = i\}$. This Hamming scheme is denoted $H(k, n)$. This example is used heavily in coding theory. For a brief introduction, see [vL13].

Another collection of examples come in the form of strongly regular graphs. A graph $\Gamma = \Gamma(n, k, \lambda, \mu)$ is *strongly regular* if it has n vertices, each vertex is of degree k, any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. This gives an association scheme (X, R) with $X = V(\Gamma)$ and $R = R_0 \cup R_1 \cup R_2$ where R_0, R_1, R_2 are the relations describing equality, adjacency, and non-adjacency respectively.

The association schemes that are of interest to us in this thesis are the so-called *Johnson schemes*. We take X to be the k-sets of a ground set V of size n and then partition X^2 into $k+1$ subsets

$$R_i = \{(x, y) \;:\; x, y \in X,\; |x \cap y| = k - i\}.$$

We denote such a Johnson scheme by $J(k, n)$. It is easy to see that this is a symmetric association scheme.

We will now define the inner distribution of a k-family, first introduced by Delsarte in his thesis [Del73].

Definition 3.3.1. If (X, R) is an association scheme with $k+1$ classes, the inner distribution of $Y \subseteq X$ is the $(k+1)$-tuple $a(Y) = (a_0, a_1, \ldots, a_k)$ where

$$a_i(Y) = \frac{1}{|Y|}|R_i \cap Y^2|.$$

We call a_i the i^{th} *inner distribution of* Y. Note that since we divide by $|Y|$, this is the average number of i^{th} associates of y over all $y \in Y$.

If $X = L_k^n$ is the Johnson scheme, then we consider the set of ordered pairs of L_k^n and partition this set into $k+1$ subsets

$$L_k^n \times L_k^n = R_0 \cup R_1 \cup \ldots \cup R_k,$$

where we define $R_i := \left\{(x, y) \in L_k^n \times L_k^n \;:\; |x \cap y| = k - i\right\}$. Here, any $S \subseteq L_k^n$ has inner distribution

$$a_i(S) = \left|\{S^2 \cap R_i\}\right| |S|^{-1}.$$

Lemma 3.3.2. *Let S be a k-family with inner distribution $a(S)$. Then $a_0(S) = 1$ and*

$$\sum_{i=0}^{k} a_i = |S|.$$

Proof. Since R_0 is diagonal, $R_0 \cap S$ consists of pairs (x, x) for $x \in S$ and so has $|S|$ elements, proving the first part. The proof of the second part follows immediately from the fact that since all R_i are disjoint, we have that

$$\sum_{i=0}^{k} |S^2 \cap R_i| = |S^2 \cap (L_k^n \times L_k^n)| = |S|^2.$$

□

We now return to the earlier discussion of the shape of a k-family, and link this to the inner distribution. This is the subject of the main theorem of the chapter.

Theorem 3.3.3. *Fix $n \in \mathbb{N}$ and $0 \leq k \leq \frac{n}{2}$. Then there exists an invertible matrix $M \in GL_{k+1}(\mathbb{Q})$ such that*

$$a(S)M = \mathrm{sh}(S)$$

for any k-family S.

To prove this, we will first need the following lemma, giving an explicit description of $\left|\left|\delta^i[S]\right|\right|^2$ in terms of the inner distribution.

Lemma 3.3.4. *Let $x, y \in L_k^n$ with $|x \cap y| = \ell$. Then*

$$\langle \delta^i(x), \delta^i(y) \rangle = \begin{cases} 0 & 0 \leq i < \ell \\ \binom{\ell}{k-i}(i!)^2 & \ell \leq i \leq k \\ 0 & i > k. \end{cases}$$

Proof. This is true for $i < \ell$, since for such i no $(k-i)$-subset of x is also a $(k-i)$-subset of y since this would mean $|x \cap y| > \ell$. It is also true for $i > k$ since then we would have $\delta^i(x) = 0$ for any $x \in L_k^n$. So, we consider the case for $\ell \leq i \leq k$. Take some i in this range and look at $\delta^i(x)$ for some $x \in L_k^n$. This is a sum of subsets of x of size $k - i$. Each subset appears with coefficient equal to the number of ways it can be reached from x by successive removal of i points. This is $i!$ since we have i choices for removing the first point, then $i - 1$ choices for removing the second point and so on. This gives us that

$$\delta^i(x) = \sum_{\substack{z \in L_{k-i}^n \\ z \subset x}} (i!)z.$$

We now count how many of these subsets are also in the sum $\delta^i(y)$. These are precisely the subsets of $x \cap y$ of size $k - i$. There are $\binom{\ell}{k-i}$ of these and so

$$\langle \delta^i(x), \delta^i(y) \rangle = \left\langle \sum_{\substack{z' \in L_{k-i}^n \\ z' \subset y}} (i!)z', \sum_{\substack{z \in L_{k-i}^n \\ z \subset x}} (i!)z \right\rangle = \binom{\ell}{k-i}(i!)^2.$$

$\square$

Proof of Theorem 3.3.3. Choose a k-family S and note that

$$\left|\left|\delta^i[S]\right|\right|^2 = \sum_{x\in S}\left(\langle\delta^i(x),\delta^i(x)\rangle + \sum_{\substack{y\in S\\|x\cap y|=k-1}}\langle\delta^i(x),\delta^i(y)\rangle + \ldots + \sum_{\substack{y\in S\\|x\cap y|=k-i}}\langle\delta^i(x),\delta^i(y)\rangle\right)$$
$$= \sum_{x\in S}\langle\delta^i(x),\delta^i(x)\rangle + \sum_{\substack{x,y\in S\\|x\cap y|=k-1}}\langle\delta^i(x),\delta^i(y)\rangle + \ldots + \sum_{\substack{x,y\in S\\|x\cap y|=k-i}}\langle\delta^i(x),\delta^i(y)\rangle.$$

By definition, $|\{(x,y)\ :\ x,y\in S, |x\cap y| = k-i\}| = |S|a_i$, and so from the above sum and Lemma 3.3.4 we get

$$\left|\left|\delta^i[S]\right|\right|^2 = |S|(i!)^2\left[\binom{k}{k-i}a_0 + \binom{k-1}{k-i}a_1 + \ldots + \binom{k-i}{k-i}a_i\right]. \tag{3.5}$$

Hence, if we write $d(S) = \left(||[S]||^2, ||\delta[S]||^2, \ldots, ||\delta^k[S]||^2\right)$ then we can say

$$a(S)M_1 = d(S),$$

where $M_1 \in GL_{k+1}(\mathbb{Q})$ is the upper triangular matrix where the $(i,j)^{\text{th}}$ entry is the coefficient of a_{i+1} in $||\delta^j[S]||^2$ as in Equation 3.5. Explicitly, M_1 is given by

$$M_1 = \begin{pmatrix} |S| & \binom{k}{k-1}1!^2|S| & \binom{k}{k-2}2!^2|S| & \ldots & \binom{k}{0}k!^2|S| \\ 0 & \binom{k-1}{k-1}1!^2|S| & \binom{k-1}{k-2}2!^2|S| & \ldots & \binom{k-1}{0}k!^2|S| \\ 0 & 0 & \binom{k-2}{k-2}2!^2|S| & \ldots & \binom{k-2}{0}k!^2|S| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & k!^2|S| \end{pmatrix}$$

Now we will follow a similar procedure for $d(S)$ and $\text{sh}(S)$. In doing so we will make liberal use of Lemma 2.3.5, which states that $||\delta[S]_{k,i}||^2 = \lambda^-_{k,i}||[S]_{k,i}||^2$. Applying this multiple times, for $j < k-i$ we have

$$\left|\left|\delta^i[S]_{k,j}\right|\right|^2 = c_{i,j}\left|\left|[S]_{k,j}\right|\right|^2 \qquad \text{where} \qquad c_{i,j} = \prod_{\ell=0}^{i}\lambda^-_{k-\ell,j}.$$

This means that $||\delta^k[S]||^2 = c_{k,0}||[S]_{k,0}||^2$. Doing the same thing for $\left|\left|\delta^{k-1}[S]\right|\right|^2$ we have $||\delta^{k-1}[S]||^2 = c_{k-1,0}||[S]_{k,0}||^2 + c_{k-1,1}||[S]_{k,1}||^2$ and so on. In general, we have

$$\left|\left|\delta^i[S]\right|\right|^2 = \sum_{\ell=0}^{k-i} c_{i,\ell} \left|\left|[S]_{k,\ell}\right|\right|^2. \tag{3.6}$$

We can now follow the same pattern as we did before. We have a matrix M_2 such that

$$\mathrm{sh}(S)M_2 = d(S),$$

where the $(i,j)^{\text{th}}$ entry is 1 when $j = 1$, and $c_{j-1,i-1}$ otherwise. Explicitly, M_2 is given by

$$M_2 = \begin{pmatrix} 1 & \lambda^-_{k,0} & \lambda^-_{k,0}\lambda^-_{k-1,0} & \dots & \lambda^-_{2,0}\cdots\lambda^-_{k,0} & \lambda^-_{1,0}\lambda^-_{2,0}\cdots\lambda^-_{k,0} \\ 1 & \lambda^-_{k,1} & \lambda^-_{k,1}\lambda^-_{k-1,1} & \dots & \lambda^-_{2,1}\cdots\lambda^-_{k,1} & 0 \\ 1 & \lambda^-_{k,2} & \lambda^-_{k-1,2}\lambda^-_{k,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda^-_{k,k-1} & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Putting these two identities together we obtain

$$a(S)M_1 = \mathrm{sh}(S)M_2, \tag{3.7}$$

and since M_2 is an upper triangular matrix with non-zero diagonal entries, M_2 has an inverse, meaning that there exists a matrix $M = M_1M_2^{-1}$ such that

$$a(S)M = \mathrm{sh}(S)$$

as required. □

Since we have explicit formulae for M_1 and M_2 we could calculate M, but a more succinct method is to relate this back to [Del73]. To see this we go back to the definition of a Bose-Mesner algebra on page 57. Note that we can give the adjacency matrices in terms of powers of v^+ maps. Obviously $A_0 = I$ the identity matrix. If x is a k-set then recall from Equation 2.3 that

$$v^+(x) = (n-k)x + \sum y$$

where the sum runs over all y that intersect x in $k-1$ points. Hence we see

$$A_1 = v^+ - (n-k)I.$$

In general, A_i is a polynomial in v^+. This means that the algebra spanned by the maps $(v_k^+)^i$ for $0 \leq i \leq k$ is the Bose-Mesner algebra. We now claim that we can diagonalise this algebra. The diagonalising matrix D is the change of basis matrix from the basis L_k^n to a basis of eigenvectors. We give an explicit basis in Appendix A. This means that we may use Equations 3.8 and 4.34 from Delsarte [Del73], which states that M is the matrix Q where

$$Q(i,j) = \frac{E(j-1,i-1)\left(\binom{n}{i} - \binom{n}{i-1}\right)}{\binom{k}{j}\binom{n-k}{j}}|S|.$$

Here, $E(j,i)$ is the *Eberlein polynomial* (see [GGL95, Chapter 15])

$$E(j,i) = \sum_{r=0}^{j} (-1)^r \binom{i}{r}\binom{k-i}{j-r}\binom{n-k-i}{j-r}. \tag{3.8}$$

Its inverse is the matrix given by

$$P(i,j) = E(i-1,j-1)\frac{1}{|S|}.$$

3.4 The Erdős-Ko-Rado Theorem

At the beginning of the chapter, we briefly mentioned the famous Erdős-Ko-Rado Theorem. This theorem gives a bound on the size of a k-family under certain intersection conditions. There have been many proofs of this theorem given over the years, along with many generalisations and related theorems. For some surveys, see [Bor11, DF83]. Now that we have a link between the shape of k-families and their intersection numbers, we can give a new proof.

Theorem 3.4.1 (Erdős-Ko-Rado, [EKR61]). *Let S be a k-family over a ground set V such that for any $x, y \in S$ we have $x \cap y \neq \emptyset$. Then*

$$|S| \leq \binom{n-1}{k-1}$$

where $n = |V|$. This bound is tight and it is reached when S is the collection of all k-sets containing a single point $\alpha \in V$.

Proof. If any two $x, y \in S$ have non-empty intersection then $a_k(S) = 0$. By using Theorem 3.3.3 we obtain

$$\sum_{i=0}^{k} (-1)^i \binom{n-k-i}{k-i} \left|\left|[S]_{k,i}\right|\right|^2 = 0.$$

This means that

$$\begin{aligned}
\binom{n-k}{k} \left|\left|[S]_{k,0}\right|\right|^2 &= \sum_{i=1}^{k} (-1)^{i-1} \binom{n-k-i}{k-i} \left|\left|[S]_{k,i}\right|\right|^2 \\
&\leq \binom{n-k-1}{k-1} \sum_{i=1}^{k} \left|\left|[S]_{k,i}\right|\right|^2 \\
&= \binom{n-k-1}{k-1} \left(|S| - \binom{n}{k}^{-1} |S|^2 \right)
\end{aligned}$$

where the last equality follows from Lemma 3.2.4. Using this again on the left-hand side we obtain

$$\binom{n-k}{k} \binom{n}{k}^{-1} |S|^2 \leq \binom{n-k-1}{k-1} \left(|S| - \binom{n}{k}^{-1} |S|^2 \right)$$

and hence $\binom{n}{k}^{-1} |S| \leq \frac{k}{n-k} \left(1 - \binom{n}{k}^{-1} |S|\right)$. Hence $\binom{n}{k}^{-1} \frac{n}{n-k} |S| \leq \frac{k}{n-k}$, and so

$$|S| \leq \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$$

as required. This bound is reached when S is the collection of all k-sets containing a single point. $\square$

4

On k-families of Particular Shape

Throughout this chapter, we fix V a set of size n and $k \leq n$. We also repeat the definition of $k' = \min\{k, n-k\}$ as before. Recall from Definition 3.1.2 that the shape of a k-family S is the $(k'+1)$-tuple

$$\mathrm{sh}(S) = (\mathrm{sh}_0(S), \mathrm{sh}_1(S), \dots, \mathrm{sh}_{k'}(S)),$$

where $\mathrm{sh}_i(S) = \left|\left|[S]_{k,i}\right|\right|^2 = \left\langle \pi_{k,i}[S], [S]\right\rangle$. In the previous chapter, we investigated some of the properties of the shape of a k-family S. In this chapter, we consider the converse. Given some $(k+1)$-tuple $\mathbf{a}$, what k-families S satisfy $\mathbf{a} = \mathrm{sh}(S)$? In particular, we are interested in the case where some of these $\mathrm{sh}_i(S)$ are zero. To this end, we give the following definition.

Definition 4.0.1. Let S be a k-family over a ground set V with characteristic vector $[S]$. Given some $I \subseteq \{1, \dots, k'\}$, we say that S (or $[S]$) is *I-free* if $\mathrm{sh}_i(S) = 0$ for all $i \in I$. If S is $\{i\}$-free then we will dispense with the set brackets and just write i-free.

Note that this means that if a set is I-free it is also J-free for any $J \subseteq I$. So, if $\mathrm{sh}_i(S) = 0$ for some $i \leq k'$ then S is i-free. Also note that we need not consider the case where $0 \in I$ since we know that any non-empty k-family S has $\mathrm{sh}_0(S) > 0$ by Lemma 3.2.4.

To help motivate this definition, we give the following proposition that helps us to construct some new I-free families from old.

Proposition 4.0.2. *Let S be a k-family such that $|x \cap y| < k-1$ for all $x, y \in S$. Then $\varepsilon[S]$ and $\delta[S]$ are the characteristic vectors of $(k+1)$ and $(k-1)$-families respectively. Further,*

$$\mathrm{sh}_i(\varepsilon[S]) = \lambda_{k,i}\,\mathrm{sh}_i(S) \qquad \text{and} \qquad \mathrm{sh}_i(\delta[S]) = \lambda^-_{k,i}\,\mathrm{sh}_i(S).$$

In particular, the following are equivalent for $I \subseteq \{1, \ldots, k'\}$:

1. *S is I-free,*
2. *$\varepsilon[S]$ is I-free,*
3. *$\delta[S]$ is I-free.*

Proof. The map ε takes a k-set x to the sum of all $(k+1)$-sets containing it. Consider y a summand in $\varepsilon(x)$. If y is also a summand in $\varepsilon(y)$ for some $x \neq y \in S$, then $|x \cap y| = k-1$, contradicting our choice of S. Hence the coefficient of y in the sum $\varepsilon[S]$ is precisely 1 if it contains some k-family $x \in S$, and 0 otherwise. The proof that $\delta[S]$ is a $(k-1)$-family is identical.

We now use Lemma 2.3.5, which tells us that $\left|\left|\varepsilon[S]_{k,i}\right|\right|^2 = \lambda_{k,i}\left|\left|[S]_{k,i}\right|\right|^2$. By definition of shape, this means that $\mathrm{sh}_i(\varepsilon[S]) = \lambda_{k,i}\,\mathrm{sh}_i(S)$. Again, the proof for $\delta[S]$ is the same. This also proves the last claim, that $\varepsilon[S]$ and $\delta[S]$ are I-free if and only if $[S]$ is I-free.

We quickly note that if $k \leq \frac{n}{2}$ and S is k-free, then $\delta[S]$ is trivially k-free, since in this case $\delta\left(E_{k,k}\right) = 0$. Similarly, if $k \geq \frac{n}{2}$ and S is k'-free, then $\varepsilon[S]$ is k'-free too. □

In Theorem 3.3.3 we showed that the shape of a k-family S is closely related to the inner distribution of S. In this chapter we will highlight a difference between these two distributions. Recall that the inner distribution of S is the $(k+1)$-tuple

$$a(S) = (a_0(S), \ldots, a_k(S)),$$

where $a_i(S) = |S|^{-1}|\{(x, y) \in S^2 \;:\; |x \cap y| = k-i\}|$. Now, take some $S \subseteq L^n_k$ and note that we have a natural inclusion $S \subseteq L_k^m$ for any $m > n$. However, whichever ground set we consider for S the inner distribution is the same; it does not depend on m.

We can now do the same thing for the shape of S. Since $S \subseteq L_k^n$, we know that $[S] \in M_k^n$, but there is a natural inclusion of $S \subseteq L_k^m$ for $m > n$ and so we may also think of $[S] \in M_k^m$. However, the shape $\mathrm{sh}(S)$ differs depending on the ground set V. In particular, the inner distribution of S is independent of the ground set V, whereas the shape of S is very sensitive to changes in the size of V.

We also consider examples of I-free sets where $I = \{1, \ldots, t\}$ and $I = \{t, \ldots, k'\}$. The former are t-(n, k, λ) designs, and we show this in Proposition 4.2.3. For the latter, we are particularly interested in the case where $I = \{k'\}$. The motivation behind this concerns the number of orbits of a permutation group G on k and $(k+1)$-sets. If G has the same number of orbits on k and $(k+1)$-sets then we show that each G-orbit on the $(k+1)$-sets is k-free. We formalise this in Lemma 5.3.1. The main theorems of this chapter are Theorem 4.4.2 and Theorem 4.5.2. These completely classify the shape of a 2-family and the 3-free 3-families.

4.1 The support of a k-family

This section will be fairly technical and is the only section in the thesis where we will consistently need to be referring to the ground set V over which a k-family is defined, since the results in this section are to do with comparing the shape of S over a set V to the shape of S over a set $W \supset V$. To help with this, we introduce a little bit of new notation. If S is a k-family over a ground set V of size n then we may write $[S] = [S]^n = [S]^V$ if we want to emphasise the ground set. Similarly, we will sometimes write $\mathrm{sh}_i(S) = \mathrm{sh}_i^n(S) = \mathrm{sh}_i^V(S)$ to emphasise the ground set. We first define the support of a vector in M_k^n.

Let $f = \sum_{x \in L_k^n} f_x x \in M_k^n$ over a ground set V. Then we define the *support* of f to be the set

$$\mathrm{Supp}(f) = \{\alpha \in V : \exists x \in L_k^n, \alpha \in x, \langle x, f \rangle \neq 0\} = \bigcup_{f_x \neq 0} x.$$

In the case where S is a k-family over V we write $\mathrm{Supp}([S]) = \mathrm{Supp}(S)$. In particular, this means that

$$\mathrm{Supp}(S) = \{\alpha \in V : \exists x \in S, \alpha \in x\} = \bigcup_{x \in S} x.$$

In other words, the support of some $f \in M_k^n$ is the smallest ground set V over which f can be written. For example, $\mathrm{Supp}(\alpha\beta + \frac{1}{2}\beta\gamma) = \{\alpha, \beta, \gamma\}$ and the support of a single k-set always has size k.

As we mentioned earlier in the chapter, S is a k-family over any ground set W with $W \supseteq \mathrm{Supp}(S)$ and so the following natural question arises.

Question 4.1.1. *How does the shape of S change, if at all, when we consider S as a k-family over* $\mathrm{Supp}(S)$ *as opposed to a k-family over some W containing* $\mathrm{Supp}(S)$*?*

We answer part of this question with an example. Consider Equation 3.2. There, we saw that if x is a 2-set over a ground set V of size n, then

$$\mathrm{sh}(x) = \left(\binom{n}{2}^{-1}, \frac{2}{n}, \frac{n-3}{n-1} \right).$$

This means that if $|V| = 4$ then $\mathrm{sh}^4(x) = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$, but if $|V| = 5$, then we have $\mathrm{sh}^5(x) = \left(\frac{1}{10}, \frac{2}{5}, \frac{1}{2}\right)$.

At the beginning of this chapter, we said that we were particularly interested in k-families that are I-free for some $I \subseteq \{1, \ldots, k'\}$ and so we also give an example of this case. Let $S \subseteq L_2^6$ be the cycle graph on 6 points over its support. Then $[S]$ is the sum $12 + 23 + 34 + 45 + 56 + 16$ and by Equation 3.4 we know that $\mathrm{sh}_1(S) = 0$. Recall that this means that S is 1-free.

Now consider this same S over the ground set $W = \{1, \ldots, 7\}$. We can show that $\mathrm{sh}_1^W(S) \neq 0$ by taking the inner product of $[S]$ and some polytope in $E_{2,1}$ and showing that these are non-zero. Recall that a polytope $p \in E_{2,1}$ is a vector of the form

$p = (\alpha - \beta) \sum_{\gamma \in V \setminus \{\alpha,\beta\}} \gamma$. Choosing $\alpha = 1$ and $\beta = 7$ we see that

$$\langle [S], (1-7)(2+3+4+5+6) \rangle = \langle S, 12+13+14+15+16 \rangle = 2 \neq 0.$$

Hence $[S]_{2,1} \neq 0$. In fact, this also follows directly from Proposition 4.2.3.

Hence we see that shape is a property of the set-family S and its embedding into L_k^V for some $V \supseteq \mathrm{Supp}(S)$. As we have mentioned before, this is in contrast to the inner distribution of S, which does not change when we change the ground set. This emphasises the need to know n and k when attempting to find the shape of S using Theorem 3.3.3.

Lemma 4.1.2. *Let S be a k-family over a ground set V, and $0 \leq i \leq k'$. Then $\mathrm{sh}_i^V(S) = 0$ implies that $\mathrm{sh}_i^W(S) = 0$ for all $V \supseteq W \supseteq \mathrm{Supp}(S)$. In particular, if S is I-free over a ground set V, then S is also I-free over any set W with $V \supseteq W \supseteq \mathrm{Supp}(S)$.*

Proof. Note that this is trivially true if $V = \mathrm{Supp}(S)$. Now assume that $[S]_{k,i}^V = 0$ and choose a polytope $p^W := [\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_{k-i} \in E_{k,i}^W \subset M_k^W$. The tail of p^W is composed of the sum of all $(k-i)$-subsets of W not containing any of the $2i$ points of the head. Now consider the k-sets $x \in L_{k-i}^V$ that contain at least one point from $V \setminus W$ but do not contain any α_r, β_s from the head of the polytope p^W. Let t be the sum of all such x, so $t \in M_{k-i}^V$.

Now, note that

$$p = p^{W,V} + (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \ldots (\alpha_i - \beta_i) t$$

is a polytope in $E_{k,i}^V$, where $p^{W,V}$ is the obvious embedding of p^W into M^V. Note that p^W is not a polytope over V (and indeed, it is not even necessarily an element of $E_{k,i}^V$). The polytope p has the same head as p^W, but the tail of p is the sum of $(k-i)$-subsets of the set $V \setminus \{\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i\}$. Since p is a polytope in $E_{k,i}^V$, we have that $\langle [S], p \rangle = 0$. In particular, this means that

$$\left\langle [S], p^{W,V} \right\rangle + \langle [S], (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \ldots (\alpha_i - \beta_i) t \rangle = 0.$$

Now consider the sum $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)\dots(\alpha_i - \beta_i)t$. Every set in this sum will have at least one point from $V\backslash W$ since all sets in t do. Since no sets in $[S]$ have non-empty intersection with $V\backslash W$, this means that

$$\langle [S], (\alpha_1 - \beta_1)(\alpha_2 - \beta_2)\dots(\alpha_i - \beta_i)t\rangle = 0.$$

Hence $\left\langle [S]^V, p^{W,V}\right\rangle = 0$. Now note that the inner product on M_k^W is simply the restriction of the inner product on M_k^V. Therefore the previous equality implies that $\left\langle [S]^W, p^W\right\rangle = 0$ in M_k^W. Since p^W was arbitrary, this means that $[S]_{k,i}^W = 0$ as required. □

We give an example of such a set. Let $V = \{1,2,3,4,5,6,7\}$, and take S to be the 3-family $\{123, 124, 125, 126, 134, 135, 136, 145, 146, 156\} \subset L_3^V$. The support of S is the set $W = \{1,2,3,4,5,6\}$. When defined over V, we can write $[S] = \frac{1}{2}\varepsilon^2(1) - \varepsilon(17)$. Since $[S] = \varepsilon(f)$ for some $f \in M_2^7$, it must be that S is 3-free. However, if we consider S over its support W, we can see that $S = \frac{1}{2}\varepsilon^2(1)$. We can say more about the case where S is defined over a ground set $V \supsetneq \mathrm{Supp}(S)$.

Lemma 4.1.3. *Let S be a k-family over a ground set V such that $|V \setminus \mathrm{Supp}(S)| \geq t$ for some $t \leq k'$. Then $\mathrm{sh}_i \neq 0$ for all $1 \leq i \leq t$.*

Proof. Let $i \leq t$ and assume $\{\beta_1, \dots, \beta_i\} \in V \setminus \mathrm{Supp}(S)$. Now let $x = \{\alpha_1, \dots, \alpha_k\}$ be an element of S. Let p be the polytope $[\alpha_1, \dots, \alpha_i; \beta_1, \dots, \beta_i]_{k-i}$ with tail t. Then

$$\langle [S], p\rangle = \langle [S], \alpha_1\alpha_2\dots\alpha_i t\rangle\,.$$

All terms in the right-hand side are positive and x must appear as a summand. Hence the inner product is non-zero and $[S]_{k,i} \neq 0$ as required. □

As an example of this, consider $S = \{x\}$ for a single k-set x for $k \leq \frac{n}{2}$. Here, the support $\mathrm{Supp}(S) = x$ and so $|V \setminus \mathrm{Supp}(S)| \geq k'$. Hence $\mathrm{sh}_i(S) \neq 0$ for all $0 \leq i \leq k'$, giving an alternative proof to Corollary 3.2.3.

We also have a way of inducing k-free families in larger k, and we give that now after the following important remark.

Remark. For the rest of the section, the results only hold for $k \le \frac{n}{2}$. This is because the proofs all rely on finding a polytope in $E_{k,k}$ with no tail. This only happens when $k \le \frac{n}{2}$.

Lemma 4.1.4. *Let S be a k-free k-family on an n-set V, such that $k \le \frac{n}{2}$. Then the $(k+1)$-family*

$$\alpha S = \{x \cup \alpha \ : \ x \in S\}$$

over the $(n+1)$-set $V \cup \{\alpha\}$ is $(k+1)$-free.

Proof. Since $[S]_{k,k} = 0$ then by Lemma 3.2.7

$$[S] = \sum_{x \in L^n_{k-1}} c_x \varepsilon_{k-1}(x)$$

for some $c_x \in \mathbb{Q}$, where $\varepsilon_{k-1} \colon M^n_{k-1} \to M^n_k$. Since $\alpha \notin x$ for every $x \in S$, the sets αx are all of size $k+1$. Hence $[\alpha S] = \sum c_x \alpha \varepsilon_{k-1}(x)$. We now see that $\alpha \varepsilon_{k-1}(x) = \varepsilon'_k(\alpha x)$, where $\varepsilon'_k \colon M^{n+1}_k \to M^{n+1}_{k+1}$, which completes the proof. $\square$

The next lemma is very similar to Lemma 4.1.4, but we show that we can drop the condition that $\alpha \notin V$.

Lemma 4.1.5. *Let S be a k-free k-family on an n-set V such that $k \le \frac{n}{2}$, and let $\alpha \in V$. Suppose that $\alpha \notin \mathrm{Supp}(S)$. Then the $(k+1)$-family*

$$\alpha S = \{x \cup \alpha \ : \ x \in S\}$$

over V is $(k+1)$-free.

Proof. Consider a polytope $p \in E_{k+1,k+1}$ such that $\alpha \notin \mathrm{Supp}(p)$. Then $\langle [\alpha S], p \rangle = 0$, since every set $x \in \alpha S$ contains α. Therefore, let p be such that $\alpha \in \mathrm{Supp}(p)$ and without loss of generality, let p be of the form $(\alpha - \beta)p'$ for some $\beta \neq \alpha$ and $p' \in E_{k,k}$.

Then

$$\langle [\alpha S], p\rangle = \left\langle [\alpha S], \alpha p'\right\rangle = \left\langle [S], p'\right\rangle = 0$$

by assumption. Hence $[\alpha S]_{k+1,k+1} = 0$. $\square$

We can do a similar thing in the other direction. Suppose we have some $f \in M_k^n$ and $\alpha \in V$ such that for every $x \in L_k^n$ with $\langle x, f\rangle \neq 0$ we have $\alpha \in x$. Then we may write $f = \alpha f_\alpha$ for some $f_\alpha \in M_{k-1}^{n-1}$.

Lemma 4.1.6. *Suppose $f \in M_k^n$ is of the form αf_α and $k \leq \frac{n}{2}$. Then $(\alpha f_\alpha)_{k,k} = 0$ if and only if $(f_\alpha)_{k-1,k-1} = 0$, where we take $f_\alpha \in M_{k-1}^{n-1}$.*

Proof. If $(f_\alpha)_{k-1,k-1} = 0$, then $(\alpha f_\alpha)_{k,k} = 0$ by Lemma 4.1.5. To show the converse direction, let p be a polytope in $E_{k,k}^n$ with α in the first factor, i.e.

$$p = (\alpha - \beta)(\gamma_1 - \beta_1)\dots(\gamma_{k-1} - \beta_{k-1}).$$

Since every k-set in αf_α contains α, we have

$$0 = \langle \alpha f_\alpha, p\rangle = \langle \alpha f_\alpha, \alpha(\gamma_1 - \beta_1)\dots(\gamma_{k-1} - \beta_{k-1})\rangle = \langle f_\alpha, (\gamma_1 - \beta_1)\dots(\gamma_{k-1} - \beta_{k-1})\rangle.$$

But by Theorem 2.2.9 we have that $E_{k-1,k-1}^{n-1}$ is spanned by these polytopes and we are done. $\square$

Lastly, note that given some $f \in M_k^n$ and any $\alpha \in V$ we may factorise f as

$$f = \alpha f_\alpha + f^\alpha \tag{4.1}$$

where we let $f_\alpha = \sum \langle f, x\rangle (x \backslash \{\alpha\})$ where the sum runs over all $x \ni \alpha$, and define $f^\alpha = f - \alpha f_\alpha$. In particular, Supp($f_\alpha$) and Supp($f^\alpha$) are contained in $V \backslash \{\alpha\}$. In other words, we split f into the sum of all those k-sets containing α and the sum of all those not containing α.

In general, we may extend this to any subset $W \subseteq V$. We factorise

$$f = W \cdot f_W + f^W \tag{4.2}$$

where f_W is defined as

$$f_W = \sum_{x \supseteq W} \langle f, x \rangle \, (x \backslash W),$$

and $f^W = f - W \cdot f_W$. If f is the characteristic vector of a k-family S we write $f = [S] = \alpha[S]_\alpha + [S]^\alpha$. We may think of f^α as an element of M_k^n or as an element of M_k^{n-1}, since it contains no sets containing α. Similarly, we may also write

$$[S] = W[S]_W + [S]^W.$$

The following key lemma shows why we use such a decomposition.

Lemma 4.1.7. *Let $f \in M_k^V$ for $k \leq \frac{n}{2}$ such that $\mathrm{sh}_k^V(f) = 0$ and choose $\alpha \in V$. If we write $f = \alpha f_\alpha + f^\alpha$ then $\mathrm{sh}_k^{V \backslash \alpha}(f^\alpha) = 0$, where we take $f^\alpha \in M_k^{V \backslash \alpha}$.*

Proof. Let $p \in E_{k,k}$ be a polytope satisfying $\alpha \notin \mathrm{Supp}(p)$ and note that it has no tail. Then

$$0 = \langle f, p \rangle = \langle \alpha f_\alpha + f^\alpha, p \rangle = \langle f^\alpha, p \rangle$$

where the last equality is due to the fact that no k-set in the sum p contains α, but every k-set in αf_α does. Now note that such p span the eigenspace $E_{k,k}^{n-1} \subset M_k^{n-1}$ and so $f_{k,k}^\alpha = 0$ as required. □

This lemma is very suggestive of induction on n: if we start with some $f \in M^n$ that is k-free, then we obtain $f^\alpha \in M^{n-1}$ that is also k-free. We will see this a little later in Section 4.5. Also note that the converse of Lemma 4.1.7 is not true. As a counterexample, consider the case where $f = x$, a single k-set. If we take some $\alpha \in x$, then $f^\alpha = 0$, and so certainly $\mathrm{sh}_k(f^\alpha) = 0$. However, as we showed in Corollary 3.2.3, $\mathrm{sh}_k(x) \neq 0$ for any k-set x.

4.2 Homogeneity and t-designs

In this section, we examine k-families that are $\{1,\ldots,t\}$-free. To this end, let $t \leq k$ and let $x \in L_k^n$ be a k-set. Then $\delta^{k-t}(x) = (k-t)! \sum y$ where the sum runs over the t-sets $y \in L_t^n$ with $y \subset x$. It follows that

$$\left\langle \delta^{k-t}(x), y \right\rangle = \begin{cases} (k-t)! & \text{if } y \subset x \text{ is a } t\text{-set}, \\ 0 & \text{otherwise.} \end{cases}$$

In a similar vein, if S is a k-family and y a t-set, then

$$\left\langle \delta^{k-t}[S], y \right\rangle = \lambda(k-t)!$$

where λ is the number of $x \in S$ with $y \subset x$. This gives rise to the following definition.

Definition 4.2.1. Consider some $f \in M_k^n$ and fix $t \leq k'$ an integer. Then we call f *t-homogeneous* if

$$\langle \delta^{k-t}(f), x \rangle = \langle \delta^{k-t}(f), y \rangle$$

for all $x, y \in L_t^n$. In particular, if $f = [S]$ is the characteristic vector for some k-family S, then

$$\langle \delta^{k-t}[S], x \rangle = \langle \delta^{k-t}[S], y \rangle$$

for all $x, y \in L_t^n$. Equivalently, we say that S is t-homogeneous if the number of $x \in S$ containing some t-set y is independent of the choice of y.

Remark. Recall that a (simple) t-(n,k,λ) *design* is a collection S of k-sets (or *blocks*) over a ground set V (of *points*) such that any t-set x of V is contained in precisely λ many k-sets in S. This is equivalent to the definition above. Hence S is t-homogeneous if and only if it is a t-design.

This gives us a rich source of examples of t-homogeneous families. For example, the edge set of a d-regular graph on n vertices is a 1-$(n,2,d)$ design, since every 1-set (i.e. vertex) is contained in precisely d edges. Other examples of designs can come from projective spaces. The projective space $PG(n,q)$ is an example of a 2-design of the form 2-$\left(\frac{q^{n+1}-1}{q-1}, q+1, 1\right)$ since any two points define a unique line. Designs with $\lambda = 1$, like the projective space, are known as *Steiner systems* and are well studied. Steiner systems that take the form of 2-$(n,3,1)$ designs and 3-$(n,4,1)$ designs are called *Steiner triple systems* and *Steiner quadruple systems* respectively. For some surveys, see [CR99, LR78].

If f is t-homogeneous but is not a characteristic vector for some k-family S (i.e. does not have coefficients 0 or 1 for each $x \in S$) then we can still view f as a weighted t-design, with each block x having weight f_x. Multiplying by a common denominator c, we can make a t-design cf where all weights are integers. This is called an *integral design*. These are the objects of study by Graver and Jurkat [GJ73], and by Wilson [Wil75].

Lemma 4.2.2. *Let $f \in M_k^n$ be t-homogeneous. Then f is also ℓ-homogeneous for any $1 \leq \ell \leq t$. In particular, this means that any t-design is also an ℓ-design for any $\ell \leq t$.*

Proof. Let $x, y \in L_\ell^n$. Then we have

$$\langle \delta^{k-\ell} f, x - y \rangle = \langle \delta^{k-t} f, \varepsilon^{t-\ell}(x-y) \rangle.$$

Since $\varepsilon^{t-\ell} x$ and $\varepsilon^{t-\ell} y$ are both in M_t^n it follows that this is identically 0, and so we obtain $\langle \delta^{k-\ell} f, x \rangle = \langle \delta^{k-\ell} f, y \rangle$ as required. □

This is exactly what we expect, since a t-design is also an ℓ-design for all $1 \leq \ell \leq t$. Now, if $S \subseteq L_k^n$ is t-homogeneous then we have very strong restrictions on the values of $[S]_{k,i}$. This is shown in the following proposition, originally from Graver and Jurkat in a slightly different form.

Proposition 4.2.3 ([GJ73, Theorem 1.2]). *Let $f \in M_k^n$ and $t \leq k$. Then f is t-homogeneous if and only if*

$$f_{k,1} = f_{k,2} = \ldots = f_{k,t} = 0.$$

In particular, if S is a k-family then S is a t-design if and only if

$$[S]_{k,1} = [S]_{k,2} = \ldots = [S]_{k,t} = 0,$$

i.e. if S is $\{1, \ldots, t\}$-free.

Proof. Let $x, y \in L_t^n$. Then since S is t-homogeneous, by definition we have that $\langle \delta^{k-t}[S], x\rangle = \langle \delta^{k-t}[S], y\rangle$. This means that $\left\langle \varepsilon^{k-t}(x), [S]\right\rangle = \left\langle \varepsilon^{k-t}(y), [S]\right\rangle$ and so

$$\left\langle \varepsilon^{k-t}(x-y), [S]\right\rangle = 0. \tag{4.3}$$

Now note that $\dim(E_{t,0}) = 1$ and by fixing x and choosing y freely we see that $\dim\left\langle x - y | x, y \in L_t^n\right\rangle = \binom{n}{t} - 1$. It is clear that $\langle x - y, \mathbf{1}_t\rangle = 0$, and hence by dimension arguments we have

$$\langle x - y | x, y \in L_t^n\rangle = E_{t,0}^{\perp}.$$

This means that $\{\varepsilon^{k-t}(x-y) \ : \ x, y \in L_t^n\}$ spans $E_{k,1} \oplus \ldots \oplus E_{k,t}$ and so by Equation 4.3, we get $[S]_{k,1} = [S]_{k,2} = \ldots = [S]_{k,t} = 0$, as required.

For the other direction, assume $[S]_{k,1} = \cdots = [S]_{k,t} = 0$. Then $\delta^{k-t}[S] \in E_{t,0}$ and so $\delta^{k-t}[S] \perp x - y$ for any $x, y \in L_t^n$. This means that Equation 4.3 holds and we are done. $\square$

4.3 Examples

In this section, we give some examples of $\{1, \ldots, t\}$-free and $\{t+1, \ldots, k\}$-free k-families. By Lemma 3.2.4 we know that a k-family S is 0-free if and only if $S = \emptyset$.

1. Designs: By Proposition 4.2.3, we know that $\{1,\ldots,t\}$-free k-families are precisely t-designs. Conversely any t-design is $\{1,\ldots,t\}$-free. For a t-(n,k,λ) design to exist it must satisfy certain divisibility conditions, explicitly

$$\binom{k-i}{t-i} \,\Bigg|\, \lambda\binom{n-i}{t-i} \qquad \text{for} \qquad 0 \leq i \leq t-1. \tag{4.4}$$

We get these from [Col10, Theorem 3.3]. The above conditions are necessary, but not sufficient. For example, there is no 3-$(11,5,2)$ design, even though the divisbility conditions hold. Until relatively recently it was not known that non-trivial t-designs existed for $t > 6$. However, in 1987 Teirlinck [Tei87] showed that they must exist. In 2014 Keevash [Kee14] showed that given k, λ, and t there exists some n_0 such that for all $n > n_0$ a t-(n,k,λ) design exists, provided that the divisibility conditions of Equation 4.4 are satisfied. Hence for sufficiently large n there are many k-families such that $[S]_{k,1} = \ldots = [S]_{k,t}$.

We also briefly mention that given some t-homogeneous k-family S, we can recover λ from the shape $\mathrm{sh}(S)$ using the formula

$$\lambda = |S|\binom{k}{t}\binom{n}{t}^{-1} = \binom{k}{t}|S|^{-1}\,\mathrm{sh}_0(S).$$

We have some ways to construct new t-designs from old. For a partial list, once can study [Col10, Remark 3.4]. This notes that a complement of a t-design is also a t-design. This can also been seen from Lemma 3.2.6, since $\mathrm{sh}_i(S) = \mathrm{sh}_i(S^C)$ for all $1 \leq i \leq k'$. We can also use Proposition 4.0.2, we know that if S is a t-design with no two blocks intersecting in $k-1$ points, then $\varepsilon[S]$ and $\delta[S]$ are the characteristic vectors of two new t-designs on $k+1$ and $(k-1)$-sets respectively.

2. Designs with repeated blocks: It is worth giving a quick look here into t-designs with repeated blocks. These generalise simple t-designs by assigning every block B a weight $c_B \in \mathbb{N}$ and requiring $\lambda = \sum_{B \ni x} c_B$ for some t-set x to be independent of choice of t-set x. As we have mentioned before, we call such designs *integral t-designs*, and recover the simple t-design when $c_B = 1$ for all blocks B. Graver and Jurkat [GJ73] showed that designs with repeated blocks exist whenever the divisibility conditions of Equation 4.4 hold. The following proposition is implied by their paper, and shows that for any $t < k \leq \frac{n}{2}$, at least one t-(n,k,λ) design exists.

Proposition 4.3.1. *Given $t < k < \lfloor \frac{n}{2} \rfloor$ there exists a λ such that a t-(n,k,λ) design with repeated blocks exists, where each block appears with multiplicity* 1 *or* 2*.*

Proof. Let t,k,n satisfy the above inequalities. Select some polytope $p \in E_{k,t+1}$ and consider $\mathbf{1}_k + p$. Since p is a polytope it is a sum of k-sets with coefficients 1 or -1. Adding the all-one vector $\mathbf{1}_k$ to this gives a sum of k-sets x with coefficients $c_x \in \{0,1,2\}$. This is the characteristic vector of the multiset whose elements are the k-sets x with weights c_x. By Proposition 4.2.3, this multiset is a t-design. □

3. $\{k+1,\ldots,(2k-1)'\}$**-free families:** The next set of examples that we consider are $\{k+1,\ldots,(2k-1)'\}$-free families. We use Equation 3.1 to construct some such S. Let S be a k-family such that for every $x,y \in S$ we have x and y are disjoint. Then $1 \leq |S| \leq \lfloor \frac{n}{k} \rfloor$. Note that $\varepsilon[S]$ is the sum of $(k+1)$-sets with coefficients 0 or 1 and for any two sets $x \neq y \in \varepsilon[S]$ with coefficient 1, we have $|x \cap y| \leq 1$. We may repeat this until we obtain $\varepsilon^{k-1}[S] \in M^n_{2k-1}$. Each $(2k-1)$-set in this sum will have coefficient 0 or $(k-1)!$, and so the sum

$$[T] = \frac{1}{(k-1)!} \sum_{x \in S} \varepsilon^{k-1}(x)$$

is the characteristic vector of some k-family T. By Equation 3.1, we can see that $[T]$ has $\mathrm{sh}_{2k-1,k+1}(T) = \ldots = \mathrm{sh}_{2k-1,(2k-1)'}(T) = 0$. Since such an S always exists (we can take it to be a single k-set) we can construct such a T for any n and k satisfying $k \leq \frac{n}{2}$.

4. The 3**-orbits of** D_6 **on** 6 **points:** Consider the dihedral group D_6 acting as the symmetry group of a regular 6-gon, with vertices labelled $1,\ldots,6$. Consider the orbits of the 3-sets of the vertices. The orbits are:

$$S_1 = 123 + 126 + 156 + 234 + 345 + 456$$
$$S_2 = 124 + 125 + 134 + 136 + 145 + 146 + 235 + 236 + 245 + 256 + 346 + 356$$
$$S_3 = 135 + 246.$$

Note that D_6 also has three orbits on 2-sets, given by

$$T_1 = 12 + 16 + 23 + 34 + 45 + 56$$
$$T_2 = 13 + 15 + 24 + 26 + 35 + 46$$
$$T_3 = 14 + 25 + 36.$$

As we will see later in Chapter 5, this (relatively rare) phenomenon means that each of S_1, S_2, and S_3 are all 3-free. We come back to this example in Example 5.3.3.

5. A collection of $(k + t)$-free families: Another set of examples we consider are constructed using Lemma 4.1.4. Assume that we have a k-free k-family S with $\alpha_1, \ldots, \alpha_t \notin \mathrm{Supp}(S)$. Then we can inductively construct the $(k+t)$-free $(k+t)$-family $\alpha_1 \ldots \alpha_t S$.

6. k-free k-families: We may use Theorem 3.3.3 to give a necessary and sufficient condition on the inner distribution of k-free k-families for $k \leq \frac{n}{2}$. In particular, we have

$$\mathrm{sh}_k(S) = \sum_{i=0}^{k} a_i \frac{E(k,i)\left(\binom{n}{i} - \binom{n}{i-1}\right)}{\binom{n-k}{k}}, \tag{4.5}$$

where

$$E(k,i) = \sum_{r=0}^{k} (-1)^r \binom{i}{r}\binom{k-i}{k-r}\binom{n-k-i}{k-r}$$

as in Equation 3.8.

Before we finish the section, we give the following theorem, which classifies the k-families S of shape $\mathrm{sh}(S) = (\mathrm{sh}_0(S), \mathrm{sh}_1(S), 0, \ldots, 0)$.

Theorem 4.3.2. *Let S be a $\{2,\ldots,k'\}$-free k-family. Then either S is the empty set, the family of all k-sets containing a single point $\alpha \in V$, or one of their complements.*

Proof. By Lemma 3.2.7 we have $[S] = \sum_{\alpha \in V} c_\alpha \varepsilon^{k-1}(\alpha)$ for $c_\alpha \in \mathbb{Q}$. The coefficient of $x \in [S]$ is therefore

$$d_x = (k-1)! \sum_{\alpha \in x} c_\alpha,$$

and this must be 0 or 1. This factorial term appears because each k-set in $\varepsilon^{k-1}(\alpha)$ appears with coefficient $(k-1)!$.

Since d_x only takes two values, $|\{c_\alpha : \alpha \in V\}| \leq 2$. To see this, consider a $(k-2)$-set y and consider the k-sets $y \cup \{\alpha, \beta\}$, $y \cup \{\alpha, \gamma\}$, and $y \cup \{\beta, \gamma\}$. If c_α, c_β, and c_γ were all distinct, then the three k-sets above would all have distinct coefficients.

If all coefficients are the same, then $[S] \in E_{k,0}$ and so either $S = L_k^n$ or $S = \emptyset$. So, assume that there are two distinct coefficients; that is, $\{c_\alpha : \alpha \in V\} = \{a, b\}$ is of size 2. In this case, the sums $a + a$, $b + b$ and $a + b$ are distinct. By the arguments above this cannot happen. So, without loss of generality we can say that there is only one α such that $c_\alpha = a$. If we consider some $x \in L_k^n$ with $\alpha \notin x$, then for $\beta \in x$ we must have $c_\beta \in \{0, \frac{1}{k!}\}$. If $c_\beta = 0$, then $c_\alpha = \frac{1}{(k-1)!}$ and if $c_\beta = \frac{1}{k!}$, then $c_\alpha = -\frac{k-1}{k!}$.

In the first case, S has characteristic vector $\frac{1}{(k-1)!}\varepsilon(\alpha)$ and in the second case, S is its complement. □

4.4 The shape of a 2-family

As we have mentioned before, if we fix n then a (simple, undirected) graph S is just a collection of edges, which themselves are pairs of vertices. Hence 2-families and graphs are the same thing. In this section we will calculate the shape of a 2-family in graph-theoretic terms. To begin, we give the following lemma, linking the inner distribution of a graph S with the degrees of the vertices of S.

Lemma 4.4.1. *Define $d(\alpha)$ to be the degree of α in the graph S. Then*

$$a_1(S) = \frac{1}{|S|}\sum_{\alpha\in V}\left(d(\alpha)^2 - d(\alpha)\right).$$

Proof. We have that $a_1(S)$ counts the number of pairs $(x,y)\in S^2$ such that x and y have intersection of size exactly 1, and then normalises by dividing by $|S|$. For each point α such that $d(\alpha)\geq 2$, the number of pairs meeting at α is $\binom{d(\alpha)}{2}$. Since a_1 counts all pairs twice we have

$$a_1(S) = \frac{2}{|S|}\sum_{\substack{\alpha\in V\\ d(\alpha)\geq 2}}\binom{d(\alpha)}{2} = \frac{1}{|S|}\sum_{\substack{\alpha\in V\\ d(\alpha)\geq 2}} d(\alpha)^2 - d(\alpha).$$

Now note that if $d(\alpha)\in\{0,1\}$ then $d(\alpha)^2 - d(\alpha) = 0$ and hence

$$a_1(S) = \frac{1}{|S|}\sum_{\alpha\in V}\left(d(\alpha)^2 - d(\alpha)\right).$$

□

With this lemma, we may explicitly calculate the the shape of a graph S in graph theoretic terms. In particular, the following theorem gives the shape of S in terms of the degrees of the vertices and the number of edges.

Theorem 4.4.2. *Let S be a 2-family with shape $\mathrm{sh}(S) = (\mathrm{sh}_0(S), \mathrm{sh}_1(S), \mathrm{sh}_2(S))$. Then*

$$\mathrm{sh}_0(S) = \binom{n}{2}^{-1}|S|^2 \tag{4.6}$$

$$\mathrm{sh}_1(S) = \frac{1}{n-2}\sum_{\alpha\in V}(d(\alpha))^2 - \frac{1}{n}\left(\sum_{\alpha\in V} d(\alpha)\right)^2 \tag{4.7}$$

$$\mathrm{sh}_2(S) = |S|\left(1 + \binom{n-1}{2}^{-1}|S|\right) - \frac{1}{n-2}\sum_{\alpha\in V} d(\alpha)^2 \tag{4.8}$$

Proof. Equation 4.6 was proven in Lemma 3.2.4. To prove the other two, we use

Theorem 3.3.3. Setting $k = 2$, from Equation 3.7 we obtain

$$\begin{pmatrix} |S| & 0 & 0 \\ 2|S| & |S| & 0 \\ 4|S| & 4|S| & 4|S| \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & n-2 & 2(n-1) \\ 0 & 0 & 2n(n-1) \end{pmatrix} \begin{pmatrix} \mathrm{sh}_2(S) \\ \mathrm{sh}_1(S) \\ \mathrm{sh}_0(S) \end{pmatrix}.$$

This gives us that

$$2|S| + |S|a_1 = (n-2)\,\mathrm{sh}_1(S) + 2(n-1)\,\mathrm{sh}_0(S). \tag{4.9}$$

Using Equation 4.6, Lemma 4.4.1, and the fact that $2|S| = \sum_{\alpha \in V} d(\alpha)$, we obtain

$$(n-2)\,\mathrm{sh}_1(S) = \sum_{\alpha \in V} (d(\alpha))^2 - \frac{1}{n}\left(\sum_{\alpha \in V} d(\alpha)\right)^2, \tag{4.10}$$

giving us Equation 4.7.

To obtain Equation 4.8, we substitute $\mathrm{sh}_1(S) = |S| - \mathrm{sh}_0(S) - \mathrm{sh}_2(S)$ into Equation 4.7 and simplify. □

We can make a few remarks about these equalities. Firstly, note that the right-hand side of Equation 4.10 is non-negative; this is precisely the Cauchy-Schwarz inequality. Hence $\mathrm{sh}_1(S)$ can be thought of as a measure of how good a bound Cauchy-Schwarz is with respect to the sum of degrees of a graph S.

Secondly, since $\mathrm{sh}_2(S) \geq 0$, Equation 4.8 gives us that

$$\sum_{\alpha \in V} d(\alpha)^2 \leq |S|\left(n - 2 + \frac{2}{n-1}|S|\right). \tag{4.11}$$

This bound of the sum of squares of degrees of a graph is originally due to de Caen in his much-cited paper [dC98].

Finally for this section, we recall that we have focused our attention on I-free k-families. The following corollary classifies the 1 and 2-free graphs.

Corollary 4.4.3. *Let S be a simple graph on vertex set V of size n. Then*

1. $\mathrm{sh}_1(S) = 0$ *if and only if S is a regular graph.*
2. $\mathrm{sh}_2(S) = 0$ *if and only if S is the empty graph, the star graph $K_{1,n-1}$, or one of their complements.*

Proof. The first claim following from the fact that a graph is regular if and only if $d(\alpha) = d(\beta)$ for all vertices $\alpha, \beta \in V$. In other words, each vertex is contained in the same number of edges, or 2-sets. Hence S is 1-homogeneous and by Proposition 4.2.3, this is true if and only if $\mathrm{sh}_1(S) = 0$.

The second claim is Theorem 4.3.2 in the case where $k = 2$. □

4.5 The shape of a 3-family

In this section we will classify the 3-free 3-families over a ground set V with $|V| \geq 6$. To do this, we need the fact that such a 3-family has the form

$$[S] = \sum_{\beta,\gamma \in V} c_{\beta\gamma}\varepsilon(\beta\gamma)$$

for $c_{\beta\gamma} \in \mathbb{Q}$, which is Equation 3.1.

Remark. It is worth noting that this is a slight abuse of notation. The sum actually runs over $\beta\gamma \in L_2^n$ and not $\beta, \gamma \in V$. However, the former notation is far uglier under a summation sign and so we will continue with the latter.

To make full use of this, we will use it in conjunction with Lemma 4.1.7. If we write $[S] = \alpha[S]_\alpha + [S]^\alpha$ where $[S]_\alpha, [S]^\alpha \in M_3^{n-1}$, then $[S]^\alpha$ is 3-free over the ground set $V \backslash \{\alpha\}$. In particular, this means that

$$[S]^\alpha = \sum_{\beta,\gamma \in V\backslash\{\alpha\}} d_{\beta\gamma}\varepsilon(\beta\gamma)$$

where $d_{\beta\gamma} \in \mathbb{Q}$. Note that ε here is the map $\varepsilon^{\alpha} : M_2^{V\setminus\{\alpha\}} \to M_3^{V\setminus\{\alpha\}}$. Using this notation, we are now ready to give the first lemma of the section.

Lemma 4.5.1. *With the notation above, for every* 2*-set* x *not containing* α *we have* $c_x = d_x$.

Proof. Note that $[S]^{\alpha}$ is a sum of k-sets in f not containing α. In $[S]$, the coefficient of a 3-set y not containing α is $\sum_{\gamma\in y} c_{y\setminus\{\gamma\}}$ and in $[S]^{\alpha}$ it is $\sum_{\gamma\in y} d_{y\setminus\{\gamma\}}$. Hence we may take $d_x = c_x$ and since the set $\{\varepsilon^{\alpha}(x) \,:\, x \in L_{k-1}^{n}\}$ form a basis of $E_{k,0}^{n-1} \oplus \cdots \oplus E_{k,k-1}^{n-1}$ this is the only way to do it. □

We are now ready to give our main theorem of the chapter, classifying 3-free 3-families into three different types.

Theorem 4.5.2. *Let* S *be a* 3*-family over a ground set* V *with* $|V| = n$*, such that* $\mathrm{sh}_3(S) = 0$*. Then* $[S]$ *has exactly one of the three forms*

$$[S] = \frac{1}{2}\sum_{\alpha\in V} c_{\alpha}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\,\varepsilon(\alpha\beta), \qquad \text{(Type 1)}$$

$$[S] = \frac{1}{6}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in V} c_{\alpha}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\,\varepsilon(\alpha\beta), \qquad \text{(Type 2)}$$

$$[S] = \frac{1}{3}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in V} c_{\alpha}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\,\varepsilon(\alpha\beta), \qquad \text{(Type 3)}$$

where in Types 1 and 2, we have $c_{\alpha}, c_{\alpha\beta} \in \{0,1\}$ *for* $\alpha,\beta \in V$*. In Type 3, we have* $c_{\alpha} \in \{0,1\}$ *and* $c_{\alpha\beta} \in \{0,1,2\}$ *for* $\alpha,\beta \in V$.

Proof. First we recall that $[S]$ can be written as

$$[S] = \sum_{\alpha,\beta\in V} d_{\alpha\beta}\,\varepsilon(\alpha\beta).$$

As we have mentioned in Lemma 3.2.7, these coefficients $d_{\alpha,\beta}$ are unique. Also, we may write

$$[S] = \alpha[S]_{\alpha} + [S]^{\alpha}$$

for all $\alpha \in V$. We view $[S]_\alpha$ and $[S]^\alpha$ as set families over $V\backslash\{\alpha\}$, i.e. over $n-1$ points. Hence a natural way to proceed is by induction on n. For the base case of $n=6$, see Appendix B for the list of 3-free 3-families for $n=6$.

By the inductive hypothesis and Lemma 4.1.7 we know that each $[S]^\alpha$ is of Type 1, Type 2, or Type 3. Since $n \geq 7$ there must be at least three of the same type, but in fact they are all of the same type. To see this, take some $\alpha, \beta \in V$ and consider $d_{\alpha\beta} \in \mathbb{Q}$, the coefficient of $\varepsilon(\alpha\beta)$ in $[S]$. By Lemma 4.5.1 we know that this is also the coefficient of $\varepsilon(\alpha\beta)$ in $[S]^\gamma$ for all $\gamma \neq \alpha, \beta$. By the inductive hypothesis, each $[S]^\gamma$ has a representation of Type 1, Type 2, or Type 3, with coefficients $p_\emptyset$, p_α, and $p_{\alpha\beta}$. Hence we know that

$$d_{\alpha\beta} = \frac{1}{3}p_\emptyset + \frac{1}{2}(p_\alpha + p_\beta) - p_{\alpha\beta}.$$

But by hypothesis, since $p_\emptyset \in \{0,1,2\}$ there is only one possibility for $p_\emptyset$. This determines the type of $[S]^\gamma$ and so all $[S]^\gamma$ for $\gamma \neq \alpha, \beta$ are of the same type. We now repeat the process for some other pair $\sigma, \tau \neq \alpha, \beta$ to show that all $[S]^\gamma$ with $\gamma \neq \sigma, \tau$ have the same type. Since $n \geq 7$ there is always some overlap between these two sets and so all $[S]^\gamma$ have the same type for $\gamma \in V$.

Now we consider the coefficient c_τ for each $[S]^\eta$ with $\eta \neq \tau$. Since $c_\tau \in \{0,1\}$ there must be at least three $[S]^\eta$ that agree on this coefficient. Let these be $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$. Choose some $\sigma \neq \alpha, \beta, \gamma, \tau$ and consider the sum

$$\frac{1}{3}c_\emptyset + \frac{1}{2}c_\tau + \frac{1}{2}c_\sigma - c_{\tau\sigma} = d_{\tau\sigma}, \tag{4.12}$$

which holds true in each $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$ by Lemma 4.5.1. Hence the sum $\frac{1}{2}c_\sigma - c_{\tau\sigma}$ is equal in $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$. But the only way this is true is if $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$ agree on both c_σ and $c_{\sigma\tau}$. Since σ was chosen arbitrarily, $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$ agree on every c_σ where $\sigma \in V\backslash\{\alpha,\beta,\gamma\}$. We can now use the same argument to show that they also agree on every $c_{\sigma\tau}$ for $\sigma, \tau \in V\backslash\{\alpha,\beta,\gamma\}$ using Equation 4.12. Therefore $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$ agree on all coefficients that they share; all $c_\sigma, c_{\sigma\tau}$ for

$\sigma, \tau \in V \setminus \{\alpha, \beta, \gamma\}$.

In fact, we can use the same arguments for $[S]^\alpha$ and $[S]^\beta$ and show that they agree on all coefficients not containing α and β. In particular, they agree on all coefficients containing γ. We can do this for any two of $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$.

Now we combine $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$. Consider the sum

$$[S]^\alpha + c_\alpha \frac{1}{2}\varepsilon^2(\alpha) + \sum_{\sigma \in V \setminus \alpha, \beta} c_{\alpha\sigma}\varepsilon(\alpha\sigma) + c_{\alpha\beta}\varepsilon(\alpha\beta) \tag{4.13}$$

where the coefficients c_α and $c_{\alpha\sigma}$ for $\sigma \neq \beta$ are the coefficients in $[S]^\beta$, and the coefficient $c_{\alpha\beta}$ is the coefficient in $[S]^\gamma$. We claim that this sum is equal to $[S]$. Indeed, choose any two $\sigma, \tau \in V$ and consider, as before, the sum

$$d_{\sigma\tau} = \frac{1}{3}c_\emptyset + \frac{1}{2}(c_\sigma + c_\tau) - c_{\sigma\tau}$$

for the coefficients c_i in Equation 4.13. For any choice of σ, τ there is at least one of $[S]^\alpha$, $[S]^\beta$, and $[S]^\gamma$ containing all of the coefficients $c_\emptyset$, c_σ, c_τ, and $c_{\tau\sigma}$, and if two or three of them contain the coefficients, they must agree those coefficients. By Lemma 4.5.1, this $d_{\sigma\tau}$ is the coefficient in $[S]$ of $\varepsilon(\sigma\tau)$ and so this sum is indeed equal to $[S]$. Since all coefficients c_i in Equation 4.13 are those of the type of $[S]^\alpha$, we have $[S]$ is of the same type as $[S]^\alpha$, completing the proof. □

Note that $\frac{1}{6}\varepsilon^3(\emptyset)$ is the sum of all 3-sets of V and $\frac{1}{2}\varepsilon^2(\alpha)$ is the sum of all 3-sets of V containing α. Hence we have shown that all 3-free 3-families are simply integer sums of $\sum_{\alpha,\beta,\gamma \in V} \alpha\beta\gamma$, $\sum_{\beta,\gamma \in V \setminus \{\alpha\}} \alpha\beta\gamma$, and $\sum_{\gamma \in V \setminus \{\alpha,\beta\}} \alpha\beta\gamma$.

Remark. In Lemma 4.5.1 we showed that for a 3-free family S, the coefficients of $\varepsilon(\beta\gamma)$ in $[S]$ and $[S]^\alpha$ are equal for $\beta, \gamma \neq \alpha$. By the same arguments, if $[S]$ is of Type 1, Type 2, or Type 3, then $[S]^\alpha$ is of the same type, and shares all coefficients c_β and $c_{\beta\gamma}$.

Theorem 4.5.2 will prove very useful in the next chapter, when we compare 3-free

3-families to group orbits. However, we will also make use of the following proposition. This states that the type of $[S]$ is unique (which was shown in the proof of Theorem 4.5.2), and there are only at most two ways of writing $[S]$ in terms of its type.

Proposition 4.5.3. *Let S be a 3-free 3-family. Assume that*

$$\begin{aligned}[S] &= p_\emptyset \frac{1}{6}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in V} p_\alpha \varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} p_{\alpha\beta}\varepsilon(\alpha\beta)\\ &= q_\emptyset \frac{1}{6}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in V} q_\alpha \varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} q_{\alpha\beta}\varepsilon(\alpha\beta),\end{aligned}$$

where each of those two representations is of Type 1, Type 2, or Type 3. If there is some i such that $p_i \neq q_i$ then $p_\emptyset = q_\emptyset$ and $p_\alpha = (1-q_\alpha)$ for all $\alpha \in V$. Also

- $q_{\alpha\beta} = p_{\alpha\beta}$ *if* $p_\alpha \neq p_\beta$,
- $q_{\alpha\beta} = p_{\alpha\beta} - 1$ *if* $p_\alpha = p_\beta = 1$,
- $q_{\alpha\beta} = p_{\alpha\beta} + 1$ *if* $p_\alpha = p_\beta = 0$.

Before we begin the proof, we give an example. Let $S \subset L_3^6$ with $[S] = 146 + 156 + 234 + 235 + 345 + 346 + 356 + 456$. Then

$$\begin{aligned}[S] &= \frac{1}{2}\varepsilon^2(3+6) - \varepsilon(13+26+36)\\ &= \frac{1}{2}\varepsilon^2(1+2+4+5) - \varepsilon(12+13+14+15+23+24+25+26+45).\end{aligned}$$

Proof of Proposition 4.5.3. Now, assume that $[S]$ has two different representations as in the statement of the proposition. However, we also know that $[S]$ can be uniquely represented as $[S] = \sum_{\alpha,\beta\in V} c_{\alpha\beta}\varepsilon(\alpha\beta)$ for some $c_x \in \mathbb{Q}$. So, choose some $\alpha, \beta \in V$. Then

$$c_{\alpha\beta} = \frac{1}{3}p_\emptyset + \frac{1}{2}(p_\alpha + p_\beta) - p_{\alpha\beta} = \frac{1}{3}q_\emptyset + \frac{1}{2}(q_\alpha + q_\beta) - q_{\alpha\beta}.$$

Since $p_\emptyset, q_\emptyset \in \{0,1,2\}$ it must be that $p_\emptyset = q_\emptyset$. If $p_\alpha = q_\alpha$ for all $\alpha \in V$ then $p_{\alpha\beta} = q_{\alpha\beta}$ and the representations are the same. So, assume that $p_\alpha \neq q_\alpha$. Then to keep the sums the same, it must be that $p_\beta \neq q_\beta$. If $p_\alpha \neq p_\beta$ then $p_\alpha + p_\beta = q_\alpha + q_\beta$ and so $p_{\alpha\beta} = q_{\alpha\beta}$. If $p_\alpha = p_\beta$ then $\frac{1}{2}(p_\alpha + p_\beta) = \frac{1}{2}(q_\alpha + q_\beta) \pm 1$ and so $p_{\alpha\beta} = q_{\alpha\beta} \pm 1$. □

In other words, Proposition 4.5.3 tells us that there are at most two ways of choosing the coefficients c_x for $[S]$ in Theorem 4.5.2. However, note that this second representation may not always exist. For example, $[S] = \frac{1}{6}\varepsilon^3(\emptyset) - \varepsilon(\alpha\beta)$ does not have a second representation. The last lemma of the chapter tells us how to identify the type of S^C from the type of S.

Lemma 4.5.4. *Let S be a 3-free 3-family of Type 1. Then the complement of S, denoted by S^C, is of Type 2 and vice-versa. If S is of Type 3 then its complement is also of Type 3.*

Proof. Let S be of Type 1. Then

$$[S] = \frac{1}{2}\sum_{\alpha\in W}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\varepsilon(\alpha\beta)$$

for some $W \subseteq V$ and $c_{\alpha\beta} \in \{0,1\}$. Since

$$\frac{1}{2}\sum_{\alpha\in V}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V}\varepsilon(\alpha\beta) = 0,$$

it is clear that

$$[S^C] = \frac{1}{6}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in L_1^n\setminus W}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V}(1-c_{\alpha\beta})\varepsilon(\alpha\beta),$$

which is of Type 2. Clearly then the complement of an S of Type 2 is of Type 1.

Now let S be of Type 3. Then

$$[S] = \frac{1}{3}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in W}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\varepsilon(\alpha\beta)$$

for some $W \subseteq V$ and $c_{\alpha\beta} \in \{0,1,2\}$. We claim that the complement S^C has representation

$$[S^C] = \frac{1}{3}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha\in V\setminus W}\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V}(2-c_{\alpha\beta})\varepsilon(\alpha\beta),$$

which is of Type 3. The sum of these is

$$\frac{2}{3}\varepsilon^3(\emptyset) + \sum_{\alpha\in V}\varepsilon^2(\alpha) - 2\sum_{\alpha,\beta\in V}\varepsilon(\alpha\beta)$$

and since $\sum\varepsilon(\alpha\beta) = \frac{1}{2}\varepsilon^3(\emptyset)$, this sum is simply $\frac{1}{6}\varepsilon^3(\emptyset)$, completing the proof. □

To make use of Theorem 4.5.2, we take a look at 3-free families of Type 1. Let S be such a family, and write

$$[S] = \sum_{\alpha,\beta\in V} d_{\alpha\beta}\varepsilon(\alpha\beta).$$

Since S is of Type 1, then S is also of the form

$$[S] = \frac{1}{2}\sum_{\alpha\in V} c_\alpha\varepsilon^2(\alpha) - \sum_{\alpha,\beta\in V} c_{\alpha\beta}\varepsilon(\alpha\beta),$$

for $c_\alpha, c_{\alpha\beta} \in \{0,1\}$. Let W be the set

$$W = \{\alpha \in V \ : \ c_\alpha = 1\}.$$

Assume $3 \le |W| \le n-3$ and choose $\alpha, \beta \in W$, and $\gamma, \tau \notin W$. Then first we note that $c_{\gamma\tau} = 0$, otherwise the coefficient of some $x \subseteq V\setminus W$ would be negative. Now consider the coefficients $c_{\alpha\gamma}$ and $c_{\alpha\tau}$. Then at most one of these can be 1, since otherwise the coefficient of $\alpha\gamma\tau$ would be too low.

We make a similar argument with $c_{\alpha\gamma}$ and $c_{\beta\gamma}$. At most one of these can be 1, otherwise that forces $c_{\alpha\beta} = 0$, which would mean the coefficient of $\alpha\beta\eta$ would be too high. Finally, consider the coefficient $c_{\alpha\beta}$. If this is 0, then for any other $c_x = 0$ with $x \subset W$, we must have $x \cap \{\alpha,\beta\} = 0$, otherwise $\alpha\beta \cup x$ would have coefficient at least 2.

These are the only restrictions on the choice of coefficient c_x. So, to recap, $[S]$ is of the form

$$[S] = \frac{1}{2}\sum_{\alpha\in W}\varepsilon^2(\alpha) - \left(\sum_{\alpha,\beta\in W}\varepsilon(\alpha\beta) - \sum_{\alpha\beta\in W_1}\varepsilon(\alpha\beta)\right) - \sum_{\alpha\beta\in W_2}\varepsilon(\alpha\beta), \tag{4.14}$$

where W_1 is a collection of disjoint 2-subsets of W, and W_2 is a collection of disjoint 2-sets with one member in W, and one member in $V\backslash W$.

Now we deal with the case where $|W| < 3$. If $|W| = 0$, then $S = \emptyset$. if $W = \{\alpha\}$ is of size 1, then at most one $c_{\alpha\beta} = 1$ and we obtain $[S] = \frac{1}{2}\varepsilon^2(\alpha)$ or $[S] = \frac{1}{2}\varepsilon^2(\alpha) - \varepsilon(\alpha\beta)$. Lastly, if $W = \{\alpha, \beta\}$ is of size 2, then at most one of $c_{\alpha\gamma}$ and $c_{\alpha\eta}$ can be 1, otherwise $\alpha\gamma\eta$ has negative coefficient. This means that $c_{\alpha\beta}$ must be 1. If all other coefficients are zero, this gives $[S] = \frac{1}{2}\varepsilon(\alpha+\beta) - \varepsilon(\alpha\beta)$. We can also choose $c_{\alpha\gamma}$ and $c_{\beta\eta}$ to be 1, provided $\gamma \neq \eta$.

Next, the case where $|V\backslash W| < 3$. As before, we can choose some set of disjoint 2-sets W_1 and set $c_x = 0$ for $x \in W_1$, and $c_y = 1$ otherwise. If $|V\backslash W| = 0$, then S depends entirely on the choice of W_1. In this case, we have

$$[S] = \frac{1}{2}\sum_{\alpha\in V}\varepsilon^2(\alpha) - \left(\sum_{\alpha\beta\in L_k^n}\varepsilon(\alpha\beta) - \sum_{x\in W_1}\varepsilon(x)\right) = \sum_{x\in W_1}\varepsilon(x). \tag{4.15}$$

This will be used in the proof of Theorem 5.3.10.

If $V\backslash W = \{\alpha\}$ then we can now choose at most two $c_{\alpha\beta}$ and $c_{\alpha\gamma}$ to be 1, since in this case we need $c_{\beta\gamma} = 0$. Note that this is different to the general case of Equation 4.14, where we could only have at most one $c_{\alpha\beta} = 1$.

We have a similar result for $V\backslash W = \{\alpha, \beta\}$. In this case, if we take $c_{\alpha\beta} = 1$, then it must be that all $c_{\alpha\gamma}$ and $c_{\beta\eta}$ are 0. Then all $c_{\gamma\eta} = 1$, otherwise $\alpha\gamma\eta$ has coefficient larger than 1. In this case, S is the complement of $\frac{1}{2}\varepsilon^2(\alpha+\beta) - \varepsilon(\alpha\beta)$. If instead we choose $c_{\alpha\beta} = 0$, then we can choose W_1 as before in Equation 4.14. In this case we can choose two $c_{\alpha\gamma}$ and $c_{\alpha\eta}$ to be 1, provided that $\gamma\eta \in W_1$. In this case, we also need $c_{\beta\gamma}$ and $c_{\beta\eta}$ to be 1 too, completing the case for S being Type 1.

5

On the Shape of G-orbits

In this final chapter, we apply some of the ideas from the previous chapters to the orbits of a permutation group on V and the induced action on L^n. Each orbit is a k-family and so embeds into M_k^n via the map $S \mapsto [S]$. From this we can investigate the shape of a G-orbit on k-sets. We give a result on the structure of the space spanned by k-orbits that has the well-known Livingstone-Wagner Theorem as a corollary. We then investigate the case of equality, where a group G has the same number of orbits on k and $(k-1)$-sets.

5.1 The centralizer algebra M^G

Let G be a permutation group on V, i.e. a subgroup of $\mathrm{Sym}(V)$. Then G has an induced action on L_k^n for each $0 \le k \le n$ given by $x^g = \{\alpha^g : \alpha \in x\}$. This can be extended to a linear action of G on M^n; if $f = \sum f_x x$ then $f^g = \sum f_x x^g$. To study this G-action more effectively, we give the following definition.

Definition 5.1.1. For $G \le \mathrm{Sym}(V)$ we define

$$M^G = \{f \in M^n \ : \ f^g = f \ \ \forall g \in G\}$$

to be the *centralizer algebra* of G.

This is a subspace of M^n since the induced group action on $f \in M^n$ is linear over addition. However, it is also a subalgebra of M^n, since by Lemma 2.1.7 the union multiplication commutes with the group action.

Furthermore, for any subspace W of M^n we define

$$W^G = \{f \in W \ : \ f^g = f \ \ \forall g \in G\}$$

to be the *centralizer of G in W*.

Lemma 5.1.2. *For W a subspace of M^n we have $W^G = W \cap M^G$.*

Proof. If $f \in W^G$ then clearly it is also in W, and since it is fixed by all $g \in G$ it is also an element of M^G. Conversely, if $f \in W$ and $f \in M^G$ then f is fixed by all $g \in G$ and so $f \in W^G$. □

As with M^n we have that

$$M^G = \bigoplus_{i=0}^{n} M_k^G$$

where $M_k^G = M^G \cap M_k^n = \{f \in M_k^n : f^g = f \ \ \forall g \in G\}$.

For two short examples, if $G = \{1\}$ the trivial group then $M^G = M^n$ since every set is fixed by G. If $G = \mathrm{Sym}(V)$ then M_k^G is 1-dimensional for each k. To see this, consider some $f \in M_k^G$ and choose an $x \in L_k^n$ such that $\langle f, x\rangle = f_x \neq 0$. For any $y \in L_k^n$ there exists some $g \in G$ such that $x^g = y$ and so since f is fixed by G, it must be that $f_y = f_x$ for every $y \in L_k^n$ and so $f = f_x \mathbf{1}_k$.

We note that M^G inherits the inner product from M^n by restriction, and so we also keep the induced norm. Also, the algebra M^G receives linear transformations from M^n. Since the G-action commutes with ε and δ by Lemma 2.1.8, we may consider their restriction

$$\varepsilon_k^G : M_k^G \to M_{k+1}^G \qquad \text{and} \qquad \delta_{k+1}^G : M_{k+1}^G \to M_k^G.$$

Since ε^G is the restriction of ε, we will write it as ε. This will not cause confusion.

Since we have ε and δ we combine them to make $\delta_k^G \varepsilon_k^G = \nu_k^G$. Again, we will omit the implicit superscript G unless necessary for clarity.

Since we have the ν maps, it seems reasonable that we can emulate the arguments of Theorem 2.2.3 and split M_k^G into the direct sum of eigenspaces of ν^+. To this end we define the subspaces

$$E_{k,i}^G = E_{k,i} \cap M^G,$$

so $f \in E_{k,i}^G$ if and only if f is fixed by G and $\nu^+(f) = \lambda_{k,i} f$. Note that

$$E_{k,i}^G = \{f \in E_{k,i} \ : \ f^g = f \ \ \forall g \in G\}$$

by Lemma 5.1.2. This leads us to the following lemma.

Lemma 5.1.3. *Let $G \leq \mathrm{Sym}(n)$ with M_k^G and $E_{k,i}^G$ as before. Then*

$$M_k^G = E_{k,0}^G \oplus E_{k,1}^G \oplus \ldots \oplus E_{k,k'}^G.$$

Proof. The statement that $M_k^G \supseteq E_{k,0}^G \oplus \ldots \oplus E_{k,k'}^G$ is true, as an element of $E_{k,0}^G \oplus \ldots \oplus E_{k,k'}^G$ is of the form $f = f_{k,0} + f_{k,1} + \ldots + f_{k,k'}$. Obviously this is an element of M_k^n and $f^g = f$.

For the other containment, take some $f \in M_k^G$. This splits into eigenspace components $f = f_{k,0} + \ldots + f_{k,k'}$. Then we need to show that $f_{k,i}^g = f_{k,i}$ for all $g \in G$ and $i \leq k'$. Since the group action commutes with ν^+, it also commutes with polynomials in ν^+, and so commutes with $\pi_{k,i}$ for all $i \leq k'$. Hence

$$f_{k,i}^g = \pi_{k,i}(f)^g = \pi_{k,i}(f^g) = \pi_{k,i}(f) = f_{k,i},$$

completing the proof. □

This means that, as in M_k^n, any $f \in M_k^G$ has a unique spectral decomposition of the form $f = f_{k,0} + \ldots + f_{k,k'}$, and each of the $f_{k,i}$ is fixed by every $g \in G$. Note that the shape $\mathrm{sh}(f)$ of $f \in M_k^G$ is the same whether we think of f as an element of the centralizer algebra or the algebra M_k^n. This means that we may talk about the shape of an orbit of G without confusion.

Next we see that ε and δ are isomorphisms between $E_{k,i}^G$ and $E_{\ell,i}^G$ for $0 \leq k, \ell \leq n$ and $0 \leq i \leq \min\{k', \ell'\}$, in the same way as in M^n.

Lemma 5.1.4. *For $0 \leq k' \leq \ell' \leq n$, the restriction of ε to $E_{k,i}^G$ defines an isomorphism*

$$\varepsilon^{\ell-k} : E_{k,i}^G \to E_{\ell,i}^G$$

for $0 \leq i \leq k'$.

Proof. Without loss of generality, let $k < \frac{n}{2}$ and $\ell = k+1$. Since ε is injective, the restriction of ε to $E_{k,i}^G$ is also injective. To show surjectivity, consider some $f \in E_{k+1,i}^G$. Then there exists a unique $\tilde{f} \in E_{k,i}$ such that $\varepsilon(\tilde{f}) = f$. Let $g \in G$ and note that $f^g = \big(\varepsilon(\tilde{f})\big)^g = \varepsilon(\tilde{f}^g)$. Since $f \in E_{k+1,i}^G$, we have $f^g = f$. Hence $\varepsilon(\tilde{f}^g) = f$ and so $\tilde{f} = \tilde{f}^g$ for all $g \in G$. Hence $\tilde{f} \in E_{k,i}^G$, proving surjectivity. □

The proof that δ is also an isomorphism is identical. This means that we can draw Figure 5.1. This is a figure similar to Figure 2.1, with the appropriate restrictions. In particular, each M_k^G is a direct sum of k' eigenspaces of ν^+, and the first $k-1$ of these eigenspaces are isomorphic to those below them.

It is worth noting that the only important part of this diagram is bottom half. In the same way as in M^n, we have a complement function that takes $f = \sum_{x \in L_k^n} f_x x$ and maps it to its complement $f^C = \sum_{x \in L_k^n} f_x(V \backslash x)$. This commutes with the G-action and so $M_k^G \cong M_{n-k}^G$. We will see this again in Proposition 5.2.1.

$$
\begin{array}{ccccccccccc}
M_n^G & = & E_{n,0}^G & & & & & & & & \\
 & & \wr\| & & & & & & & & \\
M_{n-1}^G & = & E_{n-1,0}^G & \oplus & E_{n-1,1}^G & & & & & & \\
 & & \wr\| & & \wr\| & & & & & & \\
M_{n-2}^G & = & E_{n-2,0}^G & \oplus & E_{n-2,1}^G & \oplus & E_{n-2,2}^G & & & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
M_k^G & = & E_{k,0}^G & \oplus & E_{k,1}^G & \oplus & E_{k,2}^G & \oplus & \dots & \oplus & E_{k,k'}^G \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \iddots & & \\
 & & \wr\| & & \wr\| & & \wr\| & & & & \\
M_2^G & = & E_{2,0}^G & \oplus & E_{2,1}^G & \oplus & E_{2,2}^G & & & & \\
 & & \wr\| & & \wr\| & & & & & & \\
M_1^G & = & E_{1,0}^G & \oplus & E_{1,1}^G & & & & & & \\
 & & \wr\| & & & & & & & & \\
M_0^G & = & E_{0,0}^G \cong \mathbb{F} & & & & & & & &
\end{array}
$$

Figure 5.1: Eigenspace decomposition of the centralizer algebra M^G

5.2 The G-orbits on subsets

The induced action of a permutation group $G \leq \mathrm{Sym}(V)$ partitions L_k^n into orbits for each $0 \leq k \leq n$. As these orbits are set families, then we can embed them into M^n in our standard way. Let x be a k-set and denote the orbit $\{x^g \;:\; g \in G\}$ by x^G. Hence we have

$$\left[x^G\right] = \sum_{y \in x^G} y \in M_k^n.$$

This has shape $\mathrm{sh}(x^G) = \left(\mathrm{sh}_0(x^G), \mathrm{sh}_1(x^G), \dots, \mathrm{sh}_{k'}(x^G)\right)$. As we have seen, the shape of $[x^G]$ gives combinatorial information about the k-family x^G.

In the remainder of this chapter, we will study permutation groups G from their

G-orbits. We denote the number of G-orbits on L_k^n by $\sigma_k(G)$. The numbers $\sigma_k(G)$ are well studied [Sie82, CS83]. The following proposition gives us a link between the numbers $\sigma_k(G)$ and M_k^G.

Proposition 5.2.1. *Let $G \leq \mathrm{Sym}(n)$ and assume that G has ℓ orbits on L_k^n denoted by $S_1, \ldots, S_\ell$. Then M_k^G can take the set $\{[S_1], \ldots, [S_\ell]\}$ as a basis. In particular,* $\dim(M_k^G) = \sigma_k(G)$.

Proof. Note that $[S_i]^g = [S_i]$ for all $g \in G$ and so each $[S_i] \in M_k^G$. Also, these are linearly independent by construction since any k-set appears in exactly one of the S_i's.

Now let $f \in M_k^G$. We proceed to write f in two ways

$$f = \sum_{x \in L_k^n} f_x x \qquad \text{and} \qquad f = \frac{1}{|G|} \sum_{g \in G} f^g.$$

The second equality is true since $f^g = f$. Now, we combine the two to obtain

$$f = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in L_k^n} f_x x^g.$$

This means that the coefficent f_x is given by

$$f_x = \frac{|\mathrm{Stab}_G(x)|}{|G|} \sum_{y \in x^G} f_y.$$

Since $|\mathrm{Stab}_G(x)| = |\mathrm{Stab}_G(y)|$ for any $x, y \in L_k^n$ in the same orbit, for any such x and y we have $f_x = f_y$. This means that f is a sum of orbit sums and the linear independence of orbit sums (since they are disjoint) completes the claim. Since the orbit sums of k-sets of G form a basis of M_k^G, we see that $\dim(M_k^G) = \sigma_k(G)$. $\square$

This leads us to the following important theorem, which is analogous to Lemma 2.2.4. It gives us the Livingstone-Wagner Theorem as a corollary.

Theorem 5.2.2. *Let $G \leq \mathrm{Sym}(V)$ and $0 \leq k \leq \frac{n}{2}$. Then we have*

$$M_{k-1}^G \cong M_k^G / E_{k,k}^G.$$

In particular, this gives us that

$$\dim E_{k,k'}^G = \dim(M_k^G) - \dim(M_{k-1}^G).$$

Proof. By Lemma 5.1.4, we know that $\delta : E_{k,i} \to E_{k,i-1}$ is an isomorphism for all $0 \leq i \leq (k-1)'$. Hence we have

$$E_{k,0}^G \oplus \cdots \oplus E_{k,(k-1)'}^G \cong E_{k-1,0}^G \oplus \cdots \oplus E_{k-1,(k-1)'}^G = M_{k-1}^G.$$

This proves the first claim. The second follows immediately, since by the first claim,

$$M_k^G \cong M_{k-1}^n \oplus E_{k,k'}^G.$$

□

We note that by Lemma 2.2.8, the surjection $\delta : M_k^G \to M_{k-1}^G$ is explicit. We can simply take our orbits as sums of polytopes and then tail-cut and scale by $n-k-i+1$. This means that we have the following lemma, that for $k \leq \frac{n}{2}$, orbits on $(k-1)$-sets are uniquely defined by the orbits on k-sets.

Lemma 5.2.3. *Given some $G \leq \mathrm{Sym}(V)$ and the orbits of G on k-sets for some $k \leq \frac{n}{2}$, the orbits of G on t-sets are uniquely defined for every $0 \leq t \leq k$.*

Proof. This follows from the fact that $\delta^{k-t} : M_k^G \to M_t^G$ is a surjection. □

In fact, this tells us that if we know the orbits of G on $(\lfloor \frac{n}{2} \rfloor)$-sets, we know the orbits of G on t-sets for all $0 \leq t \leq n$.

Now, as a corollary of Theorem 5.2.2, we obtain the following; the famous Livingstone-

Wagner Theorem. This theorem, proved in 1965 by Livingstone and Wagner [LW65], is central to the study of permutation groups on k-sets. It gives us a bound on the number of orbits of G on L_k^n, given the number of orbits on L_{k+1}^n.

Theorem 5.2.4 (Livingstone-Wagner Theorem)**.** *Let $G \leq \mathrm{Sym}(n)$ act on an n-set. If $k \leq \lfloor \frac{n}{2} \rfloor$ then we have that the number of orbits of G on k-sets is greater than or equal to the number of orbits on $(k-1)$-sets. In our notation, for $k \leq \lfloor \frac{n}{2} \rfloor$ we have*

$$\sigma_k(G) \geq \sigma_{k-1}(G).$$

Proof. From Proposition 5.2.1, we have that $\dim(M_k^G) = \sigma_k(G)$. Hence Theorem 5.2.2 says that $0 \leq \dim(E_{k,k'}) = \sigma_k(G) - \sigma_{k-1}(G)$. □

Next, we give another definition. If G has only one orbit on L_k^n then we say that G is *k-homogeneous*, or equivalently that G acts *k-homogeneously*. The Livingstone-Wagner Theorem tells us that a k-homogeneous group is also $(k-1)$-homogeneous.

For example, the symmetric group $\mathrm{Sym}(n)$ is n-homogeneous and a transitive group is simply 1-homogeneous. For a non-trivial example, we have the following. Let q be a prime power, n an natural number, and let W be the vector space of dimension n over the finite field $\mathbb{F}_q$. Then the group $PGL(n,q)$, the projective general linear group of W, acts on the one-dimensional subspaces of W, which we denote by $PG(n-1,q)$. This action is 2-homogeneous since any two points define a line in $PG(n-1,q)$ and any line can be mapped to any other. However, the group is not 3-homogeneous since three collinear points cannot be mapped to three non-collinear points. The definition of k-homogeneity leads us to the following lemma.

Lemma 5.2.5. *G is t-homogeneous if and only if every G-orbit $S \in L_k^n$ is $\{1,\ldots,t\}$-free for every $k \geq t$.*

Proof. Fix $k \geq t$ and let $S \in L_k^n$ be a G-orbit. If G is t-homogeneous, there is only one G-orbit on t-sets. This orbit is L_t^n. By Equation 3.3, this has shape $\left(\binom{n}{t}, 0, 0, \ldots, 0\right)$. By Proposition 5.2.1, we know that $[L_t^n]$ is a basis of M_t^G and hence $E_{t,i}^G = 0$ for

$1 \leq i \leq t$. Now by Lemma 5.1.4, this means that $E^G_{k,i} = 0$ for $1 \leq i \leq t$. Hence S must be $\{1, \ldots, t\}$-free.

In the other direction, if all G-orbits on k-sets are $\{1, \ldots, t\}$-free then $E^G_{k,i} = 0$ for $1 \leq i \leq t$. Since the restriction $\delta^{k-t} : E^G_{k,i} \to E^G_{t,i}$ is an isomorphism, this means that $E^G_{t,i} = 0$ for $1 \leq i \leq t$. Hence $M^G_t = E^G_{t,0}$ and so must be 1-dimensional. Hence it must be spanned by $[L^n_t]$, and so G is t-homogeneous. □

Recall by Proposition 4.2.3 that we call $f \in M^n_k$ *t-homogeneous* if and only if f is $\{1, \ldots, t\}$-free. The above lemma justifies this use: if G is t-homogeneous then all of its orbits must be t-homogeneous and vice versa.

5.3 Equality in the Livingstone-Wagner Theorem

Recall that the Livingstone-Wagner Theorem says that $\sigma_k(G) \geq \sigma_{k-1}(G)$. The question then arises, when does equality occur? There have been many papers on this topic, for example [MS04, BH09]. Cameron [Cam78, Cam81, Cam83] has done a lot of work in the case where G is an infinite permutation group. Some of these papers considered algebraic properties of G to calculate $\sigma_k(G)$. Here we will take a different approach, concentrating instead on the combinatorial properties of the G-orbits on k and $(k+1)$-sets. To study this equality, we use the following key lemma.

Lemma 5.3.1. *For $G \leq \mathrm{Sym}(V)$ and $0 \leq t \leq k \leq \frac{n}{2}$ we have that $\sigma_k(G) = \sigma_t(G)$ if and only if all G-orbits $S \subseteq L^n_k$ are $\{t+1, \ldots, k\}$-free.*

Proof. If all G-orbits $S \subseteq L^n_k$ are $\{t+1, \ldots, k\}$-free, then $E_{k,i} = 0$ for $t+1 \leq i \leq k$. This means that

$$M^G_k = E_{k,0} \oplus E_{k,1} \oplus \ldots \oplus E_{k,t}.$$

This is isomorphic to M^G_t under the map δ^{k-t}, and so $\sigma_k(G) = \sigma_t(G)$ by Proposition 5.2.1.

For the other direction, if $\sigma_t(G) = \sigma_k(G)$, then by Theorem 5.2.2 and a small induction we have that

$$\sum_{i=t+1}^{k} \dim(E^G_{k,i}) = \dim(M^G_k) - \dim(M^G_t) = \sigma_k(G) - \sigma_t(G) = 0,$$

completing the proof. □

Remark. We contrast this result with Proposition 4.2.3. We showed that a k-family S is a (generalised) t-design if and only if S is $\{1,\dots,t\}$-free. The previous result concerns G-orbits that are $\{t+1,\dots,k\}$-free, in some sense the opposite of t-designs.

So, to show that $\sigma_k(G) = \sigma_{k+1}(G)$ we can look at the orbits of G on $(k+1)$-sets and check that the characteristic vector of each orbit is $(k+1)$-free. If this is the case, then G has the same number of orbits on k and $(k+1)$-sets. To show that G does not satisfy this condition, it suffices to find one orbit S of G on t-sets for any $k+1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ such that $[S]_{t,k+1} \neq 0$, since if $E^G_{t,k+1} \neq 0$ then $E^G_{k+1,k+1} \neq 0$.

We now give some examples.

Example 5.3.2 ($G = \mathrm{Sym}(t) \times \mathrm{Sym}(n-t)$)**.** Let $G = \mathrm{Sym}(t) \times \mathrm{Sym}(n-t)$ with $t \leq \frac{n}{2}$ and consider G as a subgroup of $\mathrm{Sym}(n)$. Then G acts on n points with two orbits. We also see that it acts on 2-sets with 3 orbits and in general, for $\ell \leq t$, we have G acts on ℓ-sets with $\ell+1$ orbits. However, once we look at the action of G on $(t+1)$-sets we see that G still has $t+1$ orbits. This means that

$$\dim M^G_t = \dim M^G_{t+1}$$

and so $E^G_{t+1,t+1} = 0$. In particular, if we take some orbit S on $(t+1)$-sets we have $S_{t+1,t+1} = 0$ in M^n_k. In general, we may take G acting on k-sets and for $\frac{n}{2} \geq k > t$ we have that for any orbit S we obtain

$$\mathrm{sh}(S) = (\mathrm{sh}_0(S), \dots, \mathrm{sh}_t(S), 0, \dots, 0).$$

Example 5.3.3 ($G = D_6$)**.** Here we give the small example of $G = D_6$, the dihedral group, acting on 6 points. We use the computer algebra package GAP (see [GAP13]) to calculate the orbits S of k-sets of $\{1,\ldots,6\}$ for $k=\{1,2,3\}$, and we display them in Table 5.1 below. Note that these orbits are discussed in Section 4.3, and this shows that all orbits on 3-sets are 3-free. This is a consequence of Lemma 5.3.1.

k	$\sigma_k(G)$	$\mathrm{sh}(S)$	$a(S)$
1	1	$(6,0)$	$(1,15)$
2	3	$(12/5,0,18/5)$	$(1,2,3)$
		$(12/5,0,18/5)$	$(1,2,3)$
		$(3/5,0,12/5)$	$(1,0,2)$
3	3	$(9/5,0,21/5,0)$	$(1,2,2,1)$
		$(36/5,0,24/5,0)$	$(1,5,5,1)$
		$(1/5,0,9/5,0)$	$(1,0,0,1)$

Table 5.1: The shape and inner distribution of the orbits of D_6

Example 5.3.4 ($G=\mathrm{PSL}(2,11)$)**.** In this example we consider $G=\mathrm{PSL}(2,11)$ acting on the projective line $PG(1,11)$ consisting of twelve points. We use GAP to calculate

k	$\sigma_k(G)$	$\mathrm{sh}(T)$	$a(T)$
1	1	$(12,0)$	$(1,11)$
2	1	$(66,0,0)$	$(1,20,45)$
3	1	$(220,0,0,0)$	$(1,27,108,84)$
4	2	$(220,0,0,0,110)$	$(1,20,114,148,47)$
		$(55,0,0,0,110)$	$(1,8,60,72,24)$
5	2	$(550,0,0,0,110,0)$	$(1,29,174,294,144,18)$
		$(22,0,0,0,110,0)$	$(1,5,30,70,20,6)$
6	6	$(275/21,0,0,0,55/3,0,550/7)$	$(1,0,36,36,36,0,1)$
		$(825/7,0,0,0,165,0,330/7)$	$(1,12,78,148,78,12,1)$
		$(132/7,0,0,0,0,0,792/7)$	$(1,0,45,40,45,0,1)$
		$(132/7,0,0,0,0,0,792/7)$	$(1,0,45,40,45,0,1)$
		$(275/21,0,0,0,55/3,0,550/7)$	$(1,0,36,36,36,0,1)$
		$(275/21,0,0,0,55/3,0,550/7)$	$(1,0,36,36,36,0,1)$

Table 5.2: The shape and inner distribution of the orbits of $\mathrm{PSL}(2,11)$

the orbits T of k-sets of PG$(1,11)$ for $k=1,\dots,6$, and display them in Table 5.2.

Note that the shapes of all orbits are 3-homogeneous. This is because PSL$(2,11)$ acts 3-homogeneously on PG$(1,11)$. Also, all orbits S on $k=5,6$ points are 5-free since $\sigma_4(G)=\sigma_5(G)$. This gives us some examples of 3 and 5-designs. The first orbit on 6-sets is a 3-design, and the third is a 5-design. Note that it is not immediately clear from the inner distributions that they represent designs.

Now that we have seen these examples, we can give a small result classifying groups with $\sigma_1(G)=\sigma_2(G)$. This result has a fairly short proof through simple counting arguments, mentioned in [Cam81], but we use our machinery for a proof.

Proposition 5.3.5. *Let G have exactly t orbits on points and on 2-sets. Then $t \le 2$ and either G is transitive on 2-sets or G fixes a single point α and is transitive on the 2-sets not containing α.*

Proof. Label the orbits of G on 2-sets as $S_1,\dots,S_t$ and their corresponding characteristic vectors in M_2^G as $[S_1],\dots,[S_t]$. Then by Lemma 5.3.1, we know that each $[S_i]\in E_{2,0}^G\oplus E_{2,1}^G \subseteq E_{2,0}\oplus E_{2,1}$. Since each $[S_i]$ is a characteristic vector, we may use Corollary 4.4.3. This gives us three choices for each $[S_i]$, since all orbits must be non-empty.

So, if $S_1=L_2^n$ then G is transitive on 2-sets, and so S_i is the only orbit and $t=1$. If S_1 corresponds to the star with central vertex α, then note that the only other non-intersecting orbit is its complement, and so $t=2$. Here, G is transitive on the set of 2-sets not containing α and so G is transitive on the points $V\backslash\{\alpha\}$. Since the other orbit is the set $\{\alpha\beta \,:\, \alpha\neq\beta\in V\}$, we must have that G fixes α, proving the claim. □

To continue, we need the concept of primitivity. Let G be a permutation group acting on V. We call a non-empty $\Delta\subseteq V$ a *block* for G if for any $g\in G$ we have $\Delta^g=\Delta$ or $\Delta^g\cap\Delta=\emptyset$. We call such a block *trivial* if $|\Delta|=1$, or $|\Delta|=n$. We call G *primitive* if G has no non-trivial blocks. It is clear that the trivial blocks are blocks for any $G \leq \mathrm{Sym}(V)$. The following proposition shows us how strong primitivity is with respect to transitivity. We include the proof for completeness.

Proposition 5.3.6 ([Cam99, Theorem 1.7]). *Let $G \leq \mathrm{Sym}(n)$ be a permutation group. If G is primitive, then it is transitive, and if G is 2-homogeneous then it is primitive.*

Proof. Let $S \subseteq V$ with $|S| > 1$ be fixed by G. Such an S always exists, since we may either take S to be an orbit, or S to be the union of two orbits of length 1. Then since $S^g = S$ for all $g \in G$, we have that S is a block. Hence if G is primitive then $S = V$ and so G is transitive on V.

Now let G be imprimitive with S a non-trivial block. Take $x \in S$ and $y \in V \backslash S$ and note that since G is imprimitive, we have $\{x, y\}^g \nsubseteq S$ since S is a block. Hence G cannot be 2-homogeneous. □

Note that the converse of the above proposition is not true. For counterexamples, consider the dihedral groups D_4 and D_5. The group D_4 acting on 4 points is transitive but imprimitive, and D_5 acting on 5 points is primitive but not 2-homogeneous.

Primitive groups are in some sense the building blocks of permutation groups. To realise this, we first note that any non-transitive group G acting on V has orbits $S_1, \dots, S_t$. We see that any $g \in G$ induces some k-tuple $(g_1, \dots, g_t)$ where g_i is the induced permutation of g acting on S_i. In this way we may consider G as a subgroup of a Cartesian product of groups $G_1 \times \cdots \times G_t$ where each G_i acts transitively on S_i.

So, we can restrict our study of permutation groups to transitive permutation groups, but we can go further. Let $H \leq \mathrm{Sym}(n)$ and K be a finite group. We can define an action ϕ of H on K^n by $\phi(h^{-1})(k_1, \dots, k_n) = (k_{1^h}, \dots, k_{n^h})$, by letting H act on the indices of K^n. We then define the *wreath product* $K \wr H$ to be the semidirect product $K^n \rtimes_\phi H$. Now let G be a transitive imprimitive permutation group with blocks $S_1, \dots, S_t$. Since G is transitive, these must all be the same size and hence G is a subgroup of the wreath product $\mathrm{Sym}(S_1) \wr \mathrm{Sym}(t)$.

5.3.1 Imprimitive groups

If G is an imprimitive group then as we saw earlier, we may consider G as a subgroup of some wreath product of symmetric groups. This is useful to us due to the following lemma, proved by Bundy and Hart [BH09] using a character-theoretic proof. Our proof is new, and uses the machinery built up previously in this chapter.

Lemma 5.3.7 ([BH09, Lemma 1.3])**.** *Let $G \leq \mathrm{Sym}(n)$ and let H be a subgroup of G. Then*

$$\sigma_k(G) - \sigma_{k-1}(G) \leq \sigma_k(H) - \sigma_{k-1}(H).$$

Proof. Recall from Theorem 5.2.2 that $\sigma_k(G) - \sigma_{k-1}(G) = \dim E^G_{k,k'}$. Hence it suffices to prove that $\dim E^G_{k,k'} \leq \dim E^H_{k,k'}$. To do this, consider some $f \in E^G_{k,k}$. Then $f^g = f$ for all $g \in G$. Since $H \leq G$ this is true for all $g \in H$ and so $f \in E^H_{k,k}$. Hence $E^G_{k,k'} \subseteq E^H_{k,k'}$ and the result follows. □

In particular, this means that if an imprimitive group H with r blocks of size s on V is such that $\sigma_k(H) = \sigma_{k-1}(H)$, then it is also true that $G = \mathrm{Sym}(r) \wr \mathrm{Sym}(s)$ satisfies the condition that $\sigma_k(G) = \sigma_{k-1}(G)$.

The following proposition gives some restrictions on imprimitive G that obtain equality in the Livingstone-Wagner Theorem. However, we note that it is not as strong as [BH09, Theorem 3.7], which implies Proposition 5.3.8.

Proposition 5.3.8. *Let G be a transitive imprimitive group acting on an n-set V with a block of size k. If $k < \frac{n}{2}$ then $\sigma_r > \sigma_{r-1}$ for all $2 \leq r \leq k$. If $k = \frac{n}{2}$ then $\sigma_r > \sigma_{r-1}$ for all even r.*

Proof. First note that by the Livingstone-Wagner theorem we know that $\sigma_r \geq \sigma_{r-1}$ for all $r \leq \frac{n}{2}$, so it suffices to show that $\sigma_r \neq \sigma_{r-1}$ for all appropriate r. Now, let B be a block of size k and $S = [B^G]$ the characteristic vector of the orbit of B in M^n_k. The inner distribution of S is $a(S) = \left(1, 0, 0, \ldots, 0, \frac{n-k}{k}\right)$, since no two sets in B^G have non-zero intersection. Now recall that

$$a(S)Q = \mathrm{sh}(S)$$

for the matrix $Q = (Q(i,j))_{0\le i,j\le k}$ where

$$Q(i,j) = \frac{E(j-1,i-1)\left(\binom{n}{i}-\binom{n}{i-1}\right)}{\binom{k}{j}\binom{n-k}{j}}|S|$$

Since $a_i(S) = 0$ for all $1 \le i \le k-1$, this means that

$$\mathrm{sh}_i(S) = Q(i,0) + \frac{n-k}{k}Q(i,k).$$

Inputting the values of $Q(i,0)$ and $Q(i,k)$ we obtain

$$\mathrm{sh}_i(S) = \binom{n}{i} - \binom{n}{i-1} + \frac{n-k}{k}\left((-1)^i\left(\binom{n}{i}-\binom{n}{i-1}\right)\binom{n-k-i}{k-i}\binom{n-k}{k}^{-1}\right),$$

and so if $\mathrm{sh}_i(S) = 0$ then we need that

$$(-1)^i\binom{n-k-i}{k-i}\binom{n-k}{k}^{-1}\frac{n-k}{k} = -1.$$

Simplifying the binomial terms, this means that

$$(-1)^{i-1}\frac{(k-1)!(n-k-i)!}{(n-k-1)!(k-i)!} = 1. \tag{5.1}$$

Firstly, note that this is always true if $i = 1$, which means $\mathrm{sh}_1(S) = 0$. This is exactly what we expect, since we assumed that G was transitive. Also, if $n-k = k$ (i.e. $k = \frac{n}{2}$), then this is true if and only if i is odd. Lastly, assume $n-k \ne k$ and $i \ge 2$. Then since $i < \frac{n}{2}$ we have that

$$\frac{(n-k-i)!}{(k-i)!} < \frac{(n-k-1)!}{(k-1)!}$$

and so Equation 5.1 can never be satisfied.

Now, recall that $\sigma_r = \sigma_{r-1}$ for $r \le \frac{n}{2}$ if and only if $E^G_{r,r} = 0$. Since $\delta : E^G_{r,s} \to E^G_{r-1,s}$ is an isomorphism for $r > s$, then $E^G_{k,i} \neq 0$ implies that $E^G_{i,i} \neq 0$ and so if $\mathrm{sh}_i(S) \neq 0$, then $\sigma_i > \sigma_{i-1}$. This happens when $k < \frac{n}{2}$ and when $k = \frac{n}{2}$ and i is even, as required. $\square$

5.3.2 Primitive groups

We can also study primitive groups that reach equality in the Livingstone-Wagner theorem. These are relatively rare, with a small collection being found in [BH09]. No others are known. To investigate such groups, it will be useful to have a combinatorial consequence for G being primitive. We get this from the Rudio Lemma, originally from 1888. We find this in Wielandt's book [Wie64, Theorem 1.8.1].

Lemma 5.3.9 (Rudio Lemma)**.** *Let G act primitively on a finite set V, and let $W \subset V$ be a non-empty proper subset. Then for any distinct $\alpha, \beta \in W$ there exists some $g \in G$ such that $\alpha \in W^g$ and $\beta \notin W^g$.*

We now give a theorem, originally found in [CNS79]. This proof is new, and uses the classification of Theorem 4.5.2. We give this proof with the hope that a similar method may be used in the case where $\sigma_3(G) = \sigma_4(G)$.

Theorem 5.3.10 ([CNS79, Theorem 5])**.** *If G is a primitive permutation group of degree n and $\sigma_2(G) = \sigma_3(G)$, then G is 3-homogeneous.*

Proof. Let G be a primitive permutation group of degree n with $\sigma_2(G) = \sigma_3(G)$. Then by Theorem 5.2.2, we know that $E^G_{3,3} = 0$. Hence if $S \subseteq L^n_3$ is an orbit of G on 3-sets, then $\mathrm{sh}_3(S) = 0$. By Theorem 4.5.2, this means that S is of Type 1, Type 2, or Type 3. We can write

$$[S] = c_\emptyset \frac{1}{6}\varepsilon^3(\emptyset) + \sum_{\alpha \in V} c_\alpha \frac{1}{2}\varepsilon^2(\alpha) + \sum_{\alpha,\beta \in V} c_{\alpha\beta}\,\varepsilon(\alpha\beta).$$

Since S is an orbit of G, we have $S^g = S$ for all $g \in G$ and so

$$[S] = [S^g] = c_\emptyset \frac{1}{6}\varepsilon^3(\emptyset^g) + \sum_{\alpha \in V} c_\alpha \frac{1}{2}\varepsilon^2(\alpha^g) + \sum_{\alpha,\beta \in V} c_{\alpha\beta}\,\varepsilon(\alpha^g\beta^g)$$

because the G-action is linear and commutes with ε by Lemma 2.1.8.

Since G is primitive, it acts transitively on L_1^n and so if $c_\alpha = 1$, then it must be that $c_\beta = 1$ in some representation of $[S]$ for all $\beta \in V$. However, as we showed in Proposition 4.5.3 there are only at most two representations of S and if $c_\alpha = 1$ in one of them, $c_\alpha = 0$ in the second.

So, assume that S has two representations and consider the set

$$W = \{\alpha \in V \ : \ c_\alpha = 1 \text{ in the first representation}\}.$$

Any $g \in G$ either sends W to itself (if g fixes the first representation), or to $V \backslash W$ (if g sends the first representation to the second). Hence W is a block of G, and since G is primitive $W = \emptyset$ or $W = V$. We can now use Theorem 4.5.2.

First consider shapes of Type 3. If S is an orbit of G of Type 3 then S^C is a union of orbits and by Lemma 4.5.4 they have the same type. This means that one of S or S^C is of the form

$$[T] = \frac{1}{3}\varepsilon^3(\emptyset) + \frac{1}{2}\sum_{\alpha \in L_1^n} \varepsilon^2(\alpha) - \sum_{\alpha\beta \in L_2^n} c_{\alpha\beta}\,\varepsilon(\alpha\beta),$$

i.e. where all $c_\alpha = 1$, or of the form

$$[T] = \frac{1}{3}\varepsilon^3(\emptyset) - \sum_{\alpha\beta \in L_2^n} c_{\alpha\beta}\,\varepsilon(\alpha\beta),$$

i.e. where all $c_\alpha = 0$.

Now we use the remark on page 87. Choose some $W \subset V$ such that $|V \backslash W| = 6$ and consider $[T]^W$, defined in Equation 4.2. Since T is a 3-family, we can always find a W such that $[T]^W \neq 0$. Choose such a W and note that $[T]^W$ is also of Type 3 and by Lemma 4.5.1 has all c_α either 0 or 1. Since it has support of size at most 6, then it must appear (modulo some permutation in Sym(6)) in Table B.2. However, we can see that no 3-family of Type 3 has all $c_\alpha = 1$ or all $c_\alpha = 0$ and so no such T exists. Hence no such S exists.

Now we consider the case where S is of Type 1. Again, all c_α must be 0 or 1, and since S is of Type 1 they must all be 1, since otherwise $S = \emptyset$. We now reference Equation 4.15 that says that an S with this representation must be of the form $\sum \varepsilon(x)$ where the sum runs over some collection W of disjoint 2-subsets of V. In particular, this also means that W is fixed under the action of G. It cannot be a union of orbits because G is transitive, and so W is a single orbit. However, G is primitive and so we may use the Rudio Lemma 5.3.9. Consider some $x \in W$ and apply the Rudio Lemma to the pair $\alpha, \beta \in x$. Since no 2-sets in W intersect, there is no set that contains α but not β and so we have a contradiction. Hence such an S of Type 1 cannot be a G-orbit.

Finally, we consider the case where S is of Type 2. In this case, the complement of S is of Type 1 and is a union of G-orbits. By the above arguments, this complement is of the form $\sum \varepsilon(x)$ where the sum runs over some collection W of disjoint 2-subsets of V. If W is non-empty, then W is a union of G-orbits, and so must be a single orbit. We now use the Rudio Lemma as before to reach a contradiction. If $W = \emptyset$, then G is 3-transitive, completing the proof. $\square$

5.4 Further work

Much of the work done in this thesis could be expanded upon or generalised. For example, we have a combinatorial description of $\{1, \ldots, t\}$-free k-families; they are t-designs. We also have a description of $\{2, \ldots, k\}$-free families by Theorem 4.3.2. However, we do not have a combinatorial description for a k-family S to be I-free for some arbitrary $I \subseteq \{1, \ldots, k'\}$. Having such description would allow us to extend many of the results given. In particular, it would be very nice to have a simpler combinatorial description than Equation 4.5 for a k-family to be k-free. We have one for group orbits, but not in general.

This very general problem seems intractable at present, but for a more achievable goal, we would like to have a classification of 4-free 4-families, analagous to Theorem 4.5.2. In turn, this may allow us to extend the arguments of Theorem 5.3.10 to orbits of primitive groups where $\sigma_3(G) = \sigma_4(G)$.

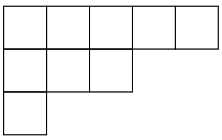
Although we know a lot about the structure of M^n as a vector space, we do not know very much about the properties of the union and intersection products. This is mainly because the union and intersection product do not commute with ε and δ and so the intersection of two set families "spreads out" over many $E_{k,i}$. In particular, we would like to know what $[S] \cdot [T]$ would be when S and T are both orbits of a permutation group G.

Finally, Bundy and Hart [BH09, Conjecture 3.2] conjecture that they have a complete list of all groups of the form $\mathrm{Sym}(r) \wr \mathrm{Sym}(s)$ that obtain equality in the Livingstone-Wagner Theorem for some $k \leq \frac{rs}{2}$. By using the methods of Proposition 5.3.8 it is possible that more effort may result in a verification of their conjecture, or perhaps some asymptotic results.

A

Specht Modules

The following brief introduction to Specht modules follows [Jam06, Chapter 3]. A *partition* of a natural number n is a sequence $\mu = (\mu_1, \ldots, \mu_r)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > 0$ and $\sum_i \mu_i = n$. We write $\mu \vdash n$ for μ a partition of n. We can describe partitions μ using a *Young diagram*, which is a set of n boxes arranged in left-justified rows with the i^{th} row having μ_i boxes. For example, the diagram below is the partition $(5, 3, 1)$.



Now let μ be a partition of n. Then a *Young tableau* t of shape μ is a Young diagram of shape μ with boxes indexed by the numbers $\{1, \ldots, n\}$. A Young tableau is called *standard* if the indices increase along rows and down columns. For example,

1	3	4	6	7
5	2			

is a Young tableau, but is not standard. There is a natural action of $\mathrm{Sym}(n)$ on the set of Young tableaux, where a group element permutes the boxes in the Young diagram by permuting their indices.

From the Young tableau we can define the *tabloid* of t, written $\{t\}$. This is the equivalence class of Young tableaux where t and t' are equivalent if one is obtained from the other by permuting elements in the same row. We write a tabloid as a tableau without vertical lines. For example, the tabloid generated by the tableau above is

$$\begin{array}{ccccc} \hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & & & \\ \cline{1-2} \end{array}.$$

The action of a permutation group is defined to be $\{t\}^g = \{t^g\}$. We define the *polytabloid* of t as a formal sum

$$\overline{\{t\}} = \sum_{g \in C_t} \operatorname{sgn}(g)\{t^g\},$$

where C_t is the *column stabilizer* of t; the subgroup of $\mathrm{Sym}(n)$ whose action is invariant on the columns of t. For example, we can take the tableau

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 \\ \cline{1-2} \end{array}$$

and create the polytabloid

$$\begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & & & \\ \cline{1-2} \end{array} - \begin{array}{ccccc} \hline 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 7 & & & \\ \cline{1-2} \end{array} - \begin{array}{ccccc} \hline 1 & 3 & 4 & 5 & 7 \\ \hline 2 & 6 & & & \\ \cline{1-2} \end{array} + \begin{array}{ccccc} \hline 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 2 & & & \\ \cline{1-2} \end{array}.$$

Notice that we have rearranged the rows since each summand is a tabloid, which means that the order of elements in a row does not matter. Finally, the *Specht module* for the partition μ, denoted by S^μ, is the module generated by all polytabloids $\overline{\{t\}}$ where t is of shape μ. Specht modules are of interest for the following reason:

Theorem A.1 ([Jam06], Theorem 4.12). *Over a field of characteristic zero, in our case $\mathbb{Q}$, the set of Specht modules $\{S^\lambda\}_{\lambda \vdash n}$ is a complete set of irreducible representations of* $\mathrm{Sym}(n)$*.*

They are also free modules and so have bases, which are given below.

Theorem A.2 ([Jam06], Theorem 8.4). *The polytabloids of standard Young tableau of shape μ form a basis for the Specht module S^μ.*

We now give a basis for our eigenspaces $E_{k,i}$ and show explicitly that they are Specht modules by defining an isomorphism from polytopes to polytabloids. Consider the polytope $[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i]_t$ which we denote by $p_{k,i}$ and assign to this element a Young tableau $t(p_{k,i})$ of the form

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\alpha_1 & \alpha_2 & \cdots & \alpha_{i-1} & \alpha_i & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-2i} \\
\hline
\beta_1 & \beta_2 & \cdots & \beta_{i-1} & \beta_i \\
\cline{1-5}
\end{array}
$$

where the γ_j are the elements of $V \backslash \{\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i\}$.

We can now look at the polytabloids associated to each of these tableau. We claim that the map

$$
\begin{aligned}
\rho : \ E_{k,i} &\to S^{(n-i,i)} \\
p_{k,i} &\mapsto \overline{\{t(p_{k,i})\}}
\end{aligned}
$$

is an isomorphism of Sym(n)-modules. Before we show this explicitly, it is worth pointing out that $E_{k,i} \cong E_{k+1,i}$ and so that is why $S^{(n-i,i)}$ is independent of k.

The first thing to note is that this map is bijective. This is not immediately obvious however, since ordering of columns matters for Young tableaux, whereas it does not for polytopes. For example, the polytope $p_1 = [\alpha_1, \alpha_2, \ldots, \alpha_i; \beta_1, \beta_2 \ldots, \beta_i]_t$ is equal to the polytope $p_2 = [\alpha_2, \alpha_1, \ldots, \alpha_i; \beta_2, \beta_1, \ldots, \beta_i]_t$ and yet their associated tableaux are different. However, note that their column stabilizers are equal, and so when we take the polytabloid, we actually get the same image under ρ since we do not care about orderings of rows. So, we have that ρ is injective and surjectivity is obvious by definition of polytabloids since they are defined by tableaux.

Both spaces are free abelian and so all that is left to prove is that the action of Sym(n) commutes with ρ. However, this is easy to see since the action of Sym(n) on a polytabloid is the action on its underlying tableau and the action on tableau is identical to the action on polytopes. This proves the claim.

By Theorem A.2 the polytabloids associated to standard tableaux form a basis for their Specht module, and so it follows that the polytopes associated to standard tableaux form a basis of $E_{k,i}$. To get this basis for our eigenspaces, order V by some total order $<$. We can then define a polytope as a standard polytope if the following conditions hold.

- $\alpha_j < \alpha_k$ and $\beta_j < \beta_k$ $\forall j, k < i$
- $\alpha_j < \gamma$, $\forall \gamma \in V_i$
- $\alpha_j < \beta_j$, $\forall j \leq i$

These polytopes correspond to standard Young tableaux and so form a basis of $E_{k,i}$.

B

The 3-free 3-families on 6 Points

In this appendix we give 28 distinct (up to relabelling) 3-free 3-families that form the base case for Theorem 4.5.2, and show that they are Type 1, Type 2, or Type 3. All calculations were done using GAP.

Since $\dim(M_3^6) = 20$, there are 2^{20} many 3-families on 6 points. However, we only care about 3-families up to relabelling of the vertices, since by Lemma 3.2.2 we know that $\mathrm{sh}(S) = \mathrm{sh}(S^g)$ for all $g \in \mathrm{Sym}(6)$. Therefore we only consider 3-families up to the natural action of Sym(6). So, firstly we obtain a representative of each orbit of Sym(6) on the set of all 3-families. There are 2136 of these, as we can see from OEIS A000665. To each of these representatives, we apply the function $\pi_{3,3}$ and choose those $[S]$ that satisfy $\pi_{3,3}[S] = 0$. This leaves us with 46 representatives. Lastly, by Lemma 4.5.4 we know that if a 3-family S one of the types from Theorem 4.5.2, then the complement S^C also has one of the types from Theorem 4.5.2. Hence it suffices to only consider 3-families S with $|S| \leq 10$. This leaves us with 28 set families to check, which we do by hand. We show this procedure in Figure B.1. We list the 28 families in Table B.1, and their representations in Table B.2. As we see, these are all of Type 1, Type 2, or Type 3.

$$2^{20} \xrightarrow{\text{Action of Sym}(6)} 2136 \xrightarrow{\text{Apply } \pi_{3,3}} 46 \xrightarrow[|S|>10]{\text{Remove } S \text{ with}} 28$$

Figure B.1: Calculating the base case for Theorem 4.5.2

We now make a few remarks. Firstly, we may look at the shape of each of these set families. Since the intersection numbers of a set family do not identify it up to permutations, nor does the shape of a set family. For example, consider rows 3 and 4 in Table B.1 and denote them by S_3 and S_4 respectively. Then $S_3 = \{123, 124, 125, 126\}$ and $S_4 = \{123, 124, 134, 234\}$. We have

$$\mathrm{sh}(S_3) = \mathrm{sh}(S_4) = \left(\frac{4}{5}, 2, \frac{6}{5}\right).$$

This is because each 3-set in S_3 intersects each other 3-set in precisely two points. This is the same in S_4. However, they do not belong in the same orbit of Sym(6) since in S_3 each pair of sets intersect in the same 2-set, while in S_4 each pair of sets intersect in a distinct 2-set. The fact that more than one family shares a shape in Table B.1 is not a rare occurrence. For the 10 3-families of size 10, we only have three unique shapes.

Secondly, there is a 3-free 3-family on $n = 6$ points of size 2: $S = \{123, 456\}$. This is unusual, since there is no 3-free 3-family of size 2 for larger n. To see this, recall that if S is a 3-free 3-family then

$$[S] = \sum_{\alpha,\beta \in V} c_{\alpha\beta}\, \varepsilon(\alpha\beta)$$

for some $c_{\alpha\beta} \in \mathbb{Q}$. Hence we may think of S as a complete graph on n vertices with the edge $\alpha\beta$ having weight $c_{\alpha\beta} \in \mathbb{Q}$. Each triangle in this graph will have the three edge-weights summing to 0 or 1. In 2006, Bendall and Margot [BM06] showed that for $n \geq 7$ the number of triangles of below-average weight in an edge-weighted complete graph is at least $n-2$. In our case, this means that a 3-free 3-family (i.e. an edge-weighted complete graph with each triangle having weight-sum 0 or 1) can have at most $n-2$ many 3-sets not appearing in it. Taking complements, this means that the smallest 3-free 3 family on n points has at least $n-2$ sets. This bound is sharp, we can take the 3-family containing all 3-sets that contain two given points. This has characteristic vector $\varepsilon(\alpha\beta)$ for some $\alpha, \beta \in V$. Indeed, Bendall and Margot also showed that this was the only possibility for such a minimal set. Also, it may be noted that $|S|$ is even for every 3-free S. However, it is not true that 3-free 3-families have even size for every n; for $n = 7$ we have $S = \{123, 124, 125, 126, 127\}$ has size 5 and is clearly 3-free since $[S] = \varepsilon(12)$.

Lastly, it is worth noting that some of the 3-families in Table B.1 are just sums of others. If S_i is the 3-family in the row labelled i in Table B.1, then we note that each S_i is some sum of S_2, S_3, S_4, or one of their images under the action of some $g \in \mathrm{Sym}(6)$.

	$\lvert S\rvert$	123	124	125	126	134	135	136	145	146	156	234	235	236	245	246	256	345	346	356	456
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	4	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	1	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
5	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
6	6	1	1	1	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
7	6	1	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
8	6	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	1	0	0	1	1
9	6	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0	0	0	0	0	0
10	6	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0
11	8	0	0	0	0	0	0	0	0	0	1	1	0	1	0	1	1	0	1	1	1
12	8	0	0	0	0	0	0	1	0	1	1	0	1	0	1	0	1	0	0	1	1
13	8	0	0	0	1	0	0	1	0	1	1	1	1	0	1	0	0	1	0	0	0
14	8	0	0	0	0	0	0	0	1	1	1	0	1	1	0	0	1	0	0	1	1
15	8	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
16	8	0	0	0	0	0	1	1	1	1	0	0	1	1	1	1	0	0	0	0	0
17	8	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1
18	8	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
19	10	0	0	0	0	0	0	1	1	1	1	0	1	1	1	0	1	0	0	1	1
20	10	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1
21	10	1	1	1	0	1	1	0	1	0	0	1	1	0	1	0	0	1	0	0	0
22	10	0	0	0	0	0	0	1	1	0	1	1	0	1	1	0	0	1	1	1	1
23	10	0	0	0	1	0	0	1	1	1	1	1	1	1	1	0	0	1	0	0	0
24	10	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	0	1	1	1
25	10	0	0	0	1	0	1	0	1	1	1	1	1	1	0	1	0	1	0	0	0
26	10	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
27	10	0	0	0	1	0	0	1	0	1	1	0	1	0	1	0	1	1	0	1	1
28	10	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0

Table B.1: The set families to be checked for the base case of Theorem 4.5.2

	Type	Representation of $[S]$
1	1	0
2	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+3)-\varepsilon(12+13+23)$
3	1	$\frac{1}{2}\varepsilon^2(1+2+3+4+5+6)$ $-\varepsilon(13+14+15+16+23+24+25+26+34+35+36+45+46+56)$
4	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(5+6)-\varepsilon(15+25+35+45+56+16+26+36+46)$
5	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+5+6)$ $-\varepsilon(12+13+14+15+15+16+16+23+24+25+25+26+26+34+35+36+45+46+56)$
6	1	$\frac{1}{2}\varepsilon^2(1)-\varepsilon(16)$
7	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+6)$ $-\varepsilon(12+13+14+15+16+23+24+25+25+26+26+35+36+45+46+56)$
8	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+4+6)-\varepsilon(12+13+14+15+16+24+26+34+36+45+46)$
9	1	$\frac{1}{2}\varepsilon^2(1+3+4)-\varepsilon(12+13+14+34+35+46)$
10	1	$\frac{1}{2}\varepsilon^2(2+3+4+5+6)-\varepsilon(12+13+24+25+26+34+35+36+45+46+56)$
11	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}(2+3+4+6)$ $-\varepsilon(12+13+14+16+23+24+25+26+34+35+36+45+46)$
12	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(3+4)-\varepsilon(12+13+14+15+23+24+26+34+34+35+36+45+46)$
13	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+6)-\varepsilon(12+13+14+15+26+36+46+56)$
14	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(5+6)-\varepsilon(12+13+15+16+24+25+26+34+35+36+45+46+56)$
15	1	$\frac{1}{2}\varepsilon^2(3+6)-\varepsilon(13+26+36)$
16	2	$\frac{1}{6}\varepsilon^3(\emptyset)-\varepsilon(12+34+56)$
17	1	$\varepsilon^2(1+2+3+4+5+6)-\varepsilon(12+13+14+15+16+23+24+25+26+34+35+46+56)$
18	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(3+6)-\varepsilon(12+13+26+34+35+36+46+56)$
19	1	$\frac{1}{2}\varepsilon^2(2+4+6)-\varepsilon(12+24+26+34+46)$
20	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(2)-\varepsilon(12+13+24+25+26)$
21	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(6)-\varepsilon(16+26+36+46+56)$
22	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(2+3+4)-\varepsilon(12+13+14+23+24+25+26+34+35+46)$
23	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+3)$ $-\varepsilon(12+12+13+13+14+15+23+24+25+26+34+35+36+46+56)$
24	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+3)$ $-\varepsilon(12+12+13+13+14+15+16+23+24+25+26+34+35+36+45)$
25	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+3+5+6)$ $-\varepsilon(12+12+13+13+14+15+16+23+24+25+26+34+35+36+36+45+46+56+56)$
26	3	$\frac{1}{3}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+5+6)$ $-\varepsilon(12+12+13+14+15+16+25+26+34+35+36+45+46+56+56)$
27	2	$\frac{1}{6}\varepsilon^3(\emptyset)+\frac{1}{2}\varepsilon^2(1+2+3+4+6)-\varepsilon(12+13+14+15+23+24+26+34+36+46)$
28	1	$\frac{1}{2}\varepsilon^2(1)$

Table B.2: The representations of the set families in Table B.1

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